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Weighted Besov and Triebel spaces: Interpolation by the real method

Dedicated to Professor M. Ohtsuka on the occasion of his 60th birthday

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Let w be in the class A_∞ of Muckenhoupt and $0<p,\ q\leq\infty$. Our aim is to give a study of weighted Besov and Triebel spaces $B^{s;w}_{p,q}$ and $F^{s;w}_{p,q}$ with an emphasis on interpolation properties of these spaces. This study is partially motivated by the recent interest shown in the theory of weighted Hardy spaces where many results for H^p are seen to be true also for H^p_w. Though the investigation of Besov and Triebel spaces in the case $w=1$ is rather extensive, and, as far as general theory is concerned, exhaustive (see e.g., [19], [20], [24], [25], [26]), there is no comprehensive treatment for the case $w\neq 1$; there is a note of Kokilašvili [15] where maximal inequalities and Fourier multipliers are observed for weighted homogeneous Triebel spaces. As for other types of weight functions, there are results of Löfström [16] and Triebel [24] for weighted Besov spaces in the case $1\leq p,\ q\leq\infty$. While their methods are based on some Fourier multipliers for weighted L^p-spaces, our study relies heavily on the technique of maximal functions developed by Fefferman-Stein [9], and Peetre [19]; other main sources of reference are [20], [25] and [26].

The plan of the paper is as follows. §1 is used to fix notation and to recall results on weight functions needed later; we also give in this section a summary of results on weighted vector-valued Hardy spaces. §2 is devoted to the study of fundamental properties of weighted Besov and Triebel spaces, and these include maximal inequalities, Fourier multipliers, embedding theorems, etc. §3 can be considered as the main part of our paper where we give interpolation formulas for weighted vector-valued Hardy spaces, and then use these to duduce interpolation theorems for weighted Besov and Triebel spaces. Finally, in the Appendix (§4), a reproduction of [5], we prove results on weighted Hardy spaces used in previous sections. In particular, we show that weighted Hardy spaces in the present context are special cases of weighted Triebel spaces (Littlewood-Paley characterization); thus, this section, besides being of self-interest, also serves as one of the bases for our study.

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§1. Notation and preliminaries

All functions and distributions are assumed to be defined on the n-dimensional Euclidean space R^n; $\mathscr{S}$ is the Schwartz class of rapidly decreasing functions and $\mathscr{S}'$, its dual, is the space of tempered distributions. The Fourier transform is defined by

$$\mathscr{F} f(x) = \hat{f}(x) = \int e^{-2\pi i x\cdot y} f(y)dy, \quad f\in\mathscr{S};$$

$\mathscr{F}$ is extended to $\mathscr{S}'$ by duality.

Hereafter, we shall always assume that w is in the *class* A_∞ of Muckenhoupt, i.e., w is a locally integrable function, $w(x)>0$ for almost every x and

$$(A_\infty) \qquad |E| \leq \lambda|I| \quad \text{implies} \quad w(E) \leq C\lambda^{1/r}w(I)$$

for any cube I (with sides parallel to axes) and any (Lebesgue) measurable subset E of I, with constants $C>0$, $r\geq 1$, independent of I and E. Here $|E|$ denotes the Lebesgue measure of E and $w(E)=\int_E w(x)dx$. In the rest of this paper, unimportant constants are denoted by C, C_1, c,...; they might be different from one occurrence to the next. It is known that if $w\in A_\infty$, then $\int w(x)dx=\infty$, and $w\in A_p$ for some p, $1<p<\infty$, i.e.,

$$(A_p) \qquad \left\{\frac{1}{|I|}\int_I w(x)dx\right\}^{1/p} \left\{\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}dx\right\}^{1/p'} \leq C$$

for all cubes I, where $1/p+1/p'=1$. The (A_p)-condition then implies the following (B_p)-condition:

$$(B_p) \quad \int (t+|x-y|)^{-np}w(y)dy \leq ct^{-np}\int_{|x-y|<t} w(y)dy, \quad x\in R^n,\ t>0.$$

For these properties of weight functions and related facts, we refer to the paper of Coifman and Fefferman [7] and references given there. We also let

$$L^p_w = \left\{f;\ \|f\|_{p,w} = \left(\int |f(x)|^p w(x)dx\right)^{1/p} < \infty\right\},\ 0<p<\infty,$$

$$L^\infty_w = L^\infty \quad \text{and} \quad \|f\|_{\infty,w} = \|f\|_\infty,$$

where L^p, $0<p\leq\infty$, are the Lebesgue spaces. For a locally integrable function g, let Mg denote the *Hardy maximal function* of g, i.e.,

$$Mg(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)} |g(y)|dy, \quad x\in R^n,$$

$$B(x, r) = \{y; |x-y| < r\}.$$

The following weighted version of an inequality of Fefferman and Stein is useful for our purpose.

LEMMA 1.1 ([1; Theorem 3.1], [15; Theorem 1]). *If* $1<p, q<\infty$, $w \in A_p$ *and* $\{f_j\}$ *is a sequence in* L^p_w, *then*

$$\|(\textstyle\sum_j |Mf_j|^q)^{1/q}\|_{p,w} \leq C\|(\textstyle\sum_j |f_j|^q)^{1/q}\|_{p,w}.$$

Let ψ be a non-negative function in $\mathscr{S}$ such that $\operatorname{supp}\psi=\{1/2\leq|x|\leq 2\}$, $\psi(x)>0$ for $1/2<|x|<2$ and $\sum_{j=-\infty}^{\infty}\psi(2^{-j}x)=1$ for $|x|\neq 0$. The existence of such a function ψ is well-known; see, e.g., [2; Lemma 6.1.7]. Let ψ_j, $j=0, \pm 1, \pm 2,\dots$, and Ψ be functions in $\mathscr{S}$ given by

$$\hat{\psi}_j(x) = \psi(2^{-j}x), \qquad \hat{\Psi}(x) = 1 - \textstyle\sum_{j=1}^{\infty}\hat{\psi}_j(x).$$

We define *weighted Besov and Triebel spaces* as follows.

$$\begin{aligned} B^{s,w}_{p,q} &= \{f \in \mathscr{S}'; \|f\|_{B(s,w;p,q)} = \|\Psi * f\|_{p,w} + \|\{\psi_j * f\}_{j=1}^{\infty}\|_{p,w;q,s} \\ &= \|\Psi * f\|_{p,w} + (\textstyle\sum_{j=1}^{\infty}(2^{js}\|\psi_j * f\|_{p,w})^q)^{1/q} < \infty\}, \\ \dot{B}^{s,w}_{p,q} &= \{f \in \mathscr{S}'; \|f\|_{\dot{B}(s,w;p,q)} = \|\{\psi_j * f\}_{j=-\infty}^{\infty}\|_{p,w;q,s} < \infty\}, \end{aligned}$$

where $-\infty<s<\infty$ and $0<p, q\leq\infty$. Here $(\sum_j(2^{js}\|\psi_j * f\|_{p,w})^q)^{1/q}$ is interpreted as $\sup_j(2^{js}\|\psi_j * f\|_{p,w})$ if $q=\infty$.

$$\begin{aligned} F^{s,w}_{p,q} &= \{f \in \mathscr{S}'; \|f\|_{F(s,w;p,q)} = \|\Psi * f\|_{p,w} + \|\{\psi_j * f\}_{j=1}^{\infty}\|_{q,s;p,w} \\ &= \|\Psi * f\|_{p,w} + \|(\textstyle\sum_{j=1}^{\infty}(2^{js}|\psi_j * f(x)|)^q)^{1/q}\|_{p,w} < \infty\}, \\ \dot{F}^{s,w}_{p,q} &= \{f \in \mathscr{S}'; \|f\|_{\dot{F}(s,w;p,q)} = \|\{\psi_j * f\}_{j=-\infty}^{\infty}\|_{q,s;p,w} < \infty, \end{aligned}$$

where $-\infty<s<\infty$, $0<p<\infty$ and $0<q\leq\infty$. We notice that when dealing with homogeneous spaces (spaces denoted with a dot) we shall make calculus modulo polynomials. It is useful in our study to introduce *vector-valued Hardy spaces*. Let $0<p<\infty$ and $1\leq q\leq\infty$. Define

$$\begin{aligned} H^p_w(l_q) = H^p_{w,q} &= \{f = \{f_j\} \subset \mathscr{S}'; \|f\|_{H(p,w;q)} \\ &= \|\sup\nolimits_{0<t<\infty}(\textstyle\sum_j |\phi_t * f_j(x)|^q)^{1/q}\|_{p,w} < \infty\}, \\ h^p_w(l_q) = h^p_{w,q} &= \{f=\{f_j\} \subset \mathscr{S}'; \|f\|_{h(p,w;q)} \\ &= \|\sup\nolimits_{0<t<1}(\textstyle\sum_j |\phi_t * f_j(x)|^q)^{1/q}\|_{p,w} < \infty\}, \end{aligned}$$

where $\phi \in \mathscr{S}$ with $\int\phi(x)dx=1$, and $\phi_t(x)=t^{-n}\phi(x/t)$. Similarly, we can define $H^p_w(l^s_q)$ and $h^p_w(l^s_q)$, $-\infty<s<\infty$; we consider here the case $s=0$ for the sake of simplicity and observe that all results stated for l_q-valued spaces are also valid for l_q^s-valued spaces. We write H_w^p and h_w^p for scalar-valued Hardy spaces (cf. [3], [10]). An important fact in the real variable theory of Hardy spaces is that the definitions of $H_{w,q}^p$ and $h_{w,q}^p$ are independent of the particular function $\phi \in \mathcal{S}$ entering in their definitions, and this is given by the following theorem:

THEOREM 1.2 (cf. [3], [9], [11]). *Let* $0<h\leq\infty$, $0<p<\infty$ *and* $1\leq q\leq\infty$. *Then the following statements* (A), (B) *and* (C) *and equivalent for an* $f=\{f_j\}\subset\mathcal{S}'$.

(A) $N^+f(x)=\sup_{0<t<h}(\sum_j|\phi_t*f_j(x)|^q)^{1/q}\in L_w^p$ *for some* $\phi\in\mathcal{S}$ *with* $\int\phi(x)dx=1$.

(B) $Nf(x)=\sup_{|x-y|<\beta t<\beta h}(\sum_j|\phi_t*f_j(y)|^q)^{1/q}\in L_w^p$ *for some* $\beta>0$ *and some* ϕ *as above.*

(C) $N^*f(x)=\sup_{|x-y|<\beta t<\beta h,\Phi\in\mathcal{A}_N}(\sum_j|\Phi_t^j*f_j(y)|^q)^{1/q}\in L_w^p$ *for some* $\beta>0$ *and a sufficiently large N, where*

$$\mathcal{A}_N = \{\Phi = \{\Phi^j\} \subset \mathcal{S};\ \sup_{x,j,|\alpha|\leq N}(1+|x|)^N|D^\alpha\Phi^j(x)|\leq 1\}.$$

Furthermore, the L_w^p*-quasi-norms of the functions* N^+f, Nf *and* N^*f *are equivalent to each other.*

PROOF. The proof of the equivalence between (B) and (C) can be done by an argument similar to that in the scalar-valued case ([9]), [11], [3]); note that it is not a corollary of the result in the latter case.

To prove the implication (A)$\Rightarrow$(B), it suffices, on account of monotone convergence theorem, to show

$$\|Nf\|_{p,w}\leq C\|N^+f\|_{p,w} \tag{1}$$

for some constant C (depending on ϕ, n, p, q, h and β) and for all f such that $f_j=0$ except for a finite number of j's. This last assumption on f implies that $Nf\in L_w^p$ since $Nf\leq\sum_j Nf_j$ (finite sum), and each $Nf_j\in L_w^p$ by the corresponding result in the scalar-valued case. Observing that $u(x,t)=(\sum_j|\phi_t*f_j(x)|^q)^{1/q}$ is continuous on R_+^{n+1}, we see that

$$\|N^\lambda u\|_{p,w}\leq C\|Nf\|_{p,w}, \tag{2}$$

where

$$N^\lambda u(x) = \sup_{y,0<t<h}u(y,t)\{t/(t+|x-y|)\}^\lambda,$$

$$\lambda > nr_0/p,\ r_0 = \inf\{r;\ w\in A_r\}\ (<\infty)$$

(cf. [3; Lemma 4.1]). Letting $\phi_i=\partial\phi/\partial x_i=D_i\phi$, $i=1,\dots,n$, and

$$N_\mu u_i(x) = \sup_{\Gamma_\mu(x)}(\sum_j|\phi_{i,t}*f_j(y)|^q)^{1/q},\ \mu>\beta,$$

where $\Gamma_\mu(x)=\{(y, t); |y-x|<\mu t<\mu h\}$, we derive from the equivalence between (B) and (C) ($\{c_\phi \Phi^j\} \in \mathscr{A}_N$, $\Phi^j=\phi_i$ for all j), and (2) that

$$\|N_\mu u_i\|_{p,w} \le C_\phi \|Nf\|_{p,w}. \tag{3}$$

(Note that $\phi_{i,t}=(D_i\phi)_t=tD_i(\phi_t)$.) Let (y, t) be an arbitrary point in $\Gamma_\beta(x)$, fixed for a moment. Then, there exists $\delta>0$ (depending on β and μ) such that $\{(z, t); |z-y|\le\delta t\}\subset\Gamma_\mu(x)$. The mean-value theorem of calculus (for mappings between normed vector spaces) implies for such y and z,

$$\begin{aligned} u(y, t)^r &\le \{u(z, t) + \delta \sup_{|z'-y|\le\delta t}(\textstyle\sum_{i=1}^n (\sum_j |\phi_{i,t}*f_j(z')|^q)^{1/q})\}^r \\ &\le N^+f(z)^r + \delta^r \textstyle\sum_i [N_\mu u_i(x)]^r, \end{aligned}$$

where $0<r\le 1$ and $p/r>r_0$. Integrating both sides of the above inequality over $|z-y|\le\delta t$ with respect to the z-variable, and then taking the supremum over all $(y, t)\in\Gamma_\beta(x)$, we obtain

$$Nf(x)^r \le (1+1/\delta)^n M((N^+f)^r)(x) + \delta^r \textstyle\sum_{i=1}^n [N_\mu u_i(x)]^r.$$

Next, taking the $L_w^{p/r}$-norm of both sides of the above, and using (3) and the weighted estimate for the Hardy maximal function (cf. Lemma 1.1), we see that

$$\|Nf\|_{p,w}^r \le C(1+1/\delta)^n \|N^+f\|_{p,w}^r + c'_\phi \delta^r \|Nf\|_{p,w}^r.$$

Since $\|Nf\|_{p,w}<\infty$, we obtain (1) by choosing δ so small that $c'_\phi\delta^r\le 1/2$. Since the implication (B)$\Rightarrow$(A) is trivial, the proof of the theorem is complete.

REMARK 1.3. (i) We note that the equivalence between (B) and (C) is also true for $0<q<1$. The difficulty with the implication (A)$\Rightarrow$(B) is that we have used a mean-value theorem for mappings between normed vector spaces, and $(\sum_{j=1}^k |x_j|^q)^{1/q}$ is not a norm on R^k if $0<q<1$.

(ii) The spaces $H_{w,q}^p$ and $h_{w,q}^p$ are non-decreasing in q.

(iii) Notice that we take $h=1$ (resp. $h=\infty$) when we deal with $h_{w,q}^p$ (resp. $H_{w,q}^p$).

We summarize properties of Hardy spaces, which will be needed later, in the next theorem. Let $\hat{\mathcal{O}}_0$ denote the subset of functions in $\mathscr{S}$ whose Fourier transforms have compact supports not containing the origin. Let $\mathscr{S}'_\infty$ be the space of $f=\{f_j\}\subset\mathscr{S}'$ equipped with the following topology: $f^\mu=\{f_j^\mu\}\to 0$ in $\mathscr{S}'_\infty$ if $f_j^\mu(\phi)\to 0$ for any $\phi\in\mathscr{S}$ and any j; $\mathscr{S}'_\infty$ is then a Hausdorff topological vector space.

THEOREM 1.4. (i) *$H_{w,q}^p$ and $h_{w,q}^p$ are quasi-Banach spaces with quasi-norms $\|\cdot\|_{H(p,w;q)}$ and $\|\cdot\|_{h(p,w;q)}$, respectively. Moreover, we have the following continuous embeddings:*

$$H^p_{w,q} \subset h^p_{w,q} \subset \mathscr{S}'_\infty.$$

(The symbol "$\subset$" will denote continuous embedding hereafter).

(ii) *Let* $\Psi \in \mathscr{S}$ *be such that* $\int \Psi(x)dx=1$ *and* $\int x^\alpha \Psi(x)dx=0$ *for* $0<|\alpha|\leq k$, *and* $f=\{f_j\} \in h^p_{w,q}$. *If* k *is sufficiently large, then* $g=\{f_j-\Psi * f_j\} \in H^p_{w,q}$ *and*

$$\|g\|_{H(p,w;q)} \leq C\|f\|_{h(p,w;q)}.$$

(iii) *The set of all* $f=\{f_j\} \in h^p_{w,q}$ (*resp.* $H^p_{w,q}$) *such that* $\{f_j\} \subset \mathscr{S}$ (*resp.* $\hat{\mathscr{O}}_0$) *is dense in* $h^p_{w,q}$ (resp. $H^p_{w,q}$).

(vi) $\dot{F}^{0,w}_{p,2} = H^p_w$ (*modulo polynomials*),

$F^{0,w}_{p,2} = h^p_w$.

PROOF. The assertion (i) follows easily from Theorem 1.2, while (ii) can be proved as in the scalar-valued case (cf. [11], [3]). The assertion (iii) is derived from the corresponding result in the scalar-valued case; the proofs of the latter fact and (iv) are given in [5] and will be reproduced in Appendix §4 for reader's convenience.

Before going to the main part of our paper, we make the following conventions on the range of the parameters: $-\infty < s, s_0, s_1 < \infty$, $0 < p, p_0, p_1, q, q_0, q_1 \leq \infty$. When considering Hardy and Triebel spaces, we also assume $0<p, p_0, p_1<\infty$. We also let $r_0=\inf\{r; w \in A_r\}$ $(<\infty)$. Furthermore, since we shall explicitly deal only with non-homogeneous spaces, we let ψ_0 denote the function Ψ used in the definitions of Besov and Triebel spaces; ψ_j, $j=1, 2,\dots$, are the same as before. Thus, we now have $\sum_{j=0}^\infty \hat{\psi}_j(x)=1$ for all x. We shall retain these conventions and notation in the rest of this paper.

§2. Fundamental properties

Hereafter, we shall state results for non-homogeneous spaces and make remarks in the homogeneous case only if there are differences either in the results or proofs; thus, without remarks, it will mean that the results, after appropriately rephrasing, are valid also for homogeneous spaces. Since the proofs of the results in §2 are modelled after those given by Peetre ([19]), [20]) and Triebel ([25], [26]) in the case $w=1$, we shall not go into details but only indicate when there are simplifications or some technical difficulties.

2.1. Maximal inequalities

The following simple lemma is useful in proving maximal inequalities for Besov and Triebel spaces.

LEMMA 2.1. *If $g \in \mathcal{S}'$ is a function in L^p_w and* supp $\hat{g}$ *is compact, then*

$$\|g^*_\lambda\|_{p,w} \le C\|g\|_{p,w},$$

where

$$\lambda > nr_0/p \qquad \text{and} \quad g^*_\lambda(x) = \sup_y |g(x-y)|(1+|y|)^{-\lambda},$$

and C is a constant that might depend on the diameter of the support of $\hat{g}$. Consequently, for all such g, $g^*_\lambda(x) < \infty$ for almost every x.

PROOF. Let Φ be a function in $\mathcal{S}$ such that $\hat{\Phi}=1$ on a neighbourhood of supp $\hat{g}$. Then, by using the relation $g=\Phi * g$ and an argument similar to the one given by Peetre [19; pp. 125–127], we obtain

$$g^*_\lambda(x) \le c\delta^{-n/r}(M(|g|^r)(x))^{1/r} + c\delta g^*_\lambda(x), \tag{4}$$

where $0<r\le 1$ satisfies $p/r>r_0$, and $0<\delta\le 1$ will be chosen later. Since the lemma is obvious if $p=\infty$, we consider only the case $p<\infty$. Assume first that $g^*_\lambda \in L^p_w$, so that $g^*_\lambda(x)<\infty$ for almost every x. Choose δ so small that $c\delta \le 1/2$. Then, the desired result in this case follows from (4) and the weighted estimate for the Hardy maximal function (note that $w \in A_{p/r}$). To prove the lemma for arbitrary g, let $\phi \in \mathcal{S}$ be such that $\phi(0)=1$ and supp $\hat{\phi} \subset \{|x| \le 1\}$. It is obvious that for any x and z,

$$\begin{aligned} |g(x-z)|(1+|z|)^{-\lambda} &= \lim_{t\to 0} |\phi(t(x-z))g(x-z)|(1+|z|)^{-\lambda} \\ &\le \liminf_{t\to 0} (\phi(t\cdot)g)^*_\lambda(x). \end{aligned}$$

Since g is a C^∞-function of polynomial growth, $\phi(t\cdot)g \in \mathcal{S}$ for each $t>0$ and thus, $(\phi(t\cdot)g)^*_\lambda \in L^p_w$. Hence, the lemma follows from Fatou's lemma and the result just proved in the case $g^*_\lambda \in L^p_w$. The proof of the lemma is now complete.

Let ϕ_j, $j=0, 1, 2,\dots$, be functions in $\mathcal{S}$ satisfying the following assumptions: supp $\hat{\phi}_0 \subset \{|x| \le 2^k\}$, supp $\hat{\phi}_j \subset \{2^{j-k} \le |x| \le 2^{j+k}\}$, $j=1, 2,\dots$, and $|D^\alpha \hat{\phi}_j(x)| \le C_\alpha 2^{-j|\alpha|}$, $j=0, 1, 2,\dots$, where k is a positive integer. Define

$$\phi^*_{j\lambda} f(x) = \sup_y |\phi_j * f(x-y)|(1+2^j|y|)^{-\lambda}, f \in \mathcal{S}', \lambda > 0.$$

THEOREM 2.2. *The following inequalities hold*:

$$\|\{\phi^*_{j\lambda} f\}\|_{p,w;q,s} \le cC(\phi)\|f\|_{B(s,w;p,q)}, \quad \lambda > nr_0/p,$$

$$\|\{\phi^*_{j\lambda} f\}\|_{q,s;p,w} \le cC(\phi)\|f\|_{F(s,w;p,q)}, \quad \lambda > \max(nr_0/p, n/q),$$

where $C(\phi)=\max_{|\alpha|\le N} C_\alpha$, N being sufficiently large, and c might depend on n, p, q, s and k.

PROOF. The arguments of Peetre [19; pp. 126–127] and Triebel [25; 2.3.1] give

$$\|\{\phi_{j\lambda}^* f\}\|_{q,s;p,w} \le cC(\phi)\|\{\psi_{j\lambda}^* f\}\|_{q,s;p,w},$$
$$\|\{\phi_{j\lambda}^* f\}\|_{p,w;q,s} \le cC(\phi)\|\{\psi_{j\lambda}^* f\}\|_{p,w;q,s},$$

and also

$$\psi_{j\lambda}^* f(x) \le c\delta^{-n/r}(M(|\psi_j * f|^r)(x))^{1/r} + c\delta\psi_{j\lambda}^* f(x)$$

for all j and x, where $r=n/\lambda$ and $0<\delta\le 1$. Now, Lemma 2.1 implies that $\psi_{j\lambda}^* f(x) <\infty$ for almost every x, and thus, by choosing δ so small that $c\delta\le 1/2$ in the above, we obtain

$$\psi_{j\lambda}^* f(x) \le C(M(|\psi_j * f|^r)(x))^{1/r} \tag{5}$$

for all j and for almost every x. If $0<p,\ q<\infty$, then, by noting that $r<\min(p, q)$ and $w\in A_{p/r}$, we derive the result for the F-space case by using (5) and Lemma 1.1. The case $q=\infty$ for F-space and the B-space case can be similarly deduced from (5) and the weighted estimate for the maximal function (we need not use the vector-valued version in the last two cases). We note that the pointwise estimate (5) helps us to simplify a limit argument used by both Peetre [19] and Triebel [25].

COROLLARY 2.3. *If $p<\infty$, then we can replace the $l_q^s(L_w^p)$-quasi-norm (resp. $L_w^p(l_q^s)$-quasi-norm, $1\le q\le\infty$) by the $l_q^s(h_w^p)$-quasi-norm (resp. $h_w^p(l_q^s)$-quasi-norm) in the definition of $B_{p,q}^{s,w}$ (resp. $F_{p,q}^{s,w}$).*

PROOF. Let $\phi\in\mathcal{S}$ be such that $\int\phi(x)dx=1$ and $\operatorname{supp}\hat{\phi}\subset\{|x|\le 1\}$. We observe that for each $j=1, 2,\ldots$, $\phi_t*\psi_j*f=0$ unless $t\le 2^{1-j}$. Thus,

$$|\phi_t*\psi_j*f(x)| \le C\psi_{j\lambda}^* f(x)\int|\phi(y)|(1+|y|)^\lambda dy \le C\psi_{j\lambda}^* f(x), \qquad j=1, 2,\ldots.$$

Since the above inequality obviously holds for $0<t\le 1$ and $j=0$, we see that

$$\sup_{0<t<1}|\phi_t*\psi_j*f(x)| \le C\psi_{j\lambda}^* f(x),\ j=0, 1, 2,\ldots,$$
$$\sup_{0<t<1}\left(\textstyle\sum_j(2^{js}|\phi_t*\psi_j*f(x)|)^q\right)^{1/q} \le \left(\textstyle\sum_j(2^{js}\psi_{j\lambda}^* f(x))^q\right)^{1/q},$$

and hence the corollary follows from Theorem 2.2.

We are now ready to list elementary properties of weighted Besov and Triebel spaces. They are consequences of either obvious computations or Theorem 2.2.

THEOREM 2.4. (i) *$B_{p,q}^{s,w}$ (resp. $F_{p,q}^{s,w}$) is a quasi-Banach space with quasi-norm $\|\cdot\|_{B(s,w;p,q)}$ (resp. $\|\cdot\|_{F(s,w;p,q)}$). Furthermore, we have the following continuous embeddings:*

$$\mathscr{S} \subset B^{s,w}_{p,q} \subset \mathscr{S}', \quad \mathscr{S} \subset F^{s,w}_{p,q} \subset \mathscr{S}'.$$

(ii) *$\mathscr{S}$ is dense in both $B^{s,w}_{p,q}$ and $F^{s,w}_{p,q}$ if $p, q<\infty$.*

(iii) *If σ is a tempered distribution such that*

$$C_\sigma = \sup_j \int |\psi_j * \sigma(y)|(1+2^j|y|)^\lambda dy < \infty,$$

then

$$\|\sigma * f\|_{B(s,w;p,q)} \leq cC_\sigma \|f\|_{B(s,w;p,q)}, \quad \lambda > nr_0/p,$$

$$\|\sigma * f\|_{F(s,w;p,q)} \leq cC_\sigma \|f\|_{F(s,w;p,q)}, \quad \lambda > \max(nr_0/p, n/q).$$

(It is well-known that $C_\sigma \leq C(\max_\alpha C_\alpha)$ if $|D^\alpha \hat{\sigma}(x)| \leq C_\alpha(1+|x|)^{-|\alpha|}$ for all $|\alpha| \leq \lambda + n/2+1$ (Bernstein's theorem).)

Before going to the next subsection, let us remark that Theorem 2.2 gives us much flexibility in choosing the sequence $\{\psi_j\}$ entering in the definitions of Besov and Triebel spaces. We refer to Peetre ([19], [20]) for many choices of $\{\psi_j\}$ satisfying rather weak conditions (cf. also [25], [26]).

2.2. Embedding theorems

First, we give a weighted version of Plancherel-Polya's inequality for entire functions of exponential type.

LEMMA 2.5. *Let $f \in L^p_w \cap \mathscr{S}'$ $(0<p<\infty)$ be such that* $\operatorname{supp} \hat{f} \subset \{|x| \leq t\}$, $t>0$. *Assume that $w(B(x, 1/t)) \geq ct^{-d}$ for some $d>0$ and all x. Then*

$$\|f\|_{p_1,w} \leq Ct^{d(1/p-1/p_1)}\|f\|_{p,w}, \; 0<p<p_1 \leq \infty.$$

PROOF. It suffices to prove the lemma only in the case $p_1=\infty$. Assume first that $f \in L^\infty_w = L^\infty$. Let $0<q<1$ be such that $r=p/q>r_0$ (thus, $w \in A_r$). Take $\phi \in \mathscr{S}$ such that $\hat{\phi}=1$ on $\{|x| \leq 1\}$. Then $\hat{\phi}_s=1$ on $\{|x| \leq t\}$, $s=1/t$. This property of ϕ_s, $(B_{r'})$ and (A_r) imply

$$\begin{aligned} |f(x)| &\leq \int |\phi_s(x-y)| \, |f(y)|^{1-q} |f(y)|^q w(y)^{1/r} w(y)^{-1/r} dy \\ &\leq C\|f\|_\infty^{1-q} \|f\|^q_{p,w} \, t^n \left(\int w(y)^{-r'/r}(s/(s+|x-y|))^{nr'} dy\right)^{1/r'} \\ &\leq C\|f\|_\infty^{1-q}\|f\|^q_{p,w} \left(\int_{|x-y|<s} w(y)dy\right)^{-q/p} \leq Ct^{dq/p}\|f\|_\infty^{1-q}\|f\|^q_{p,w}. \end{aligned}$$

Since we know a priori that $\|f\|_\infty<\infty$, we obtain the desired result in this case. For arbitrary f, use a limit argument similar to the proof of Lemma 2.1. We note that if $w=1$, then we can take $d=n$, and the lemma is just the already known Plancherel-Polya's inequality (cf. [20], [25]).

We say that the weight function w is in the class $\mathscr{M}_d$ $(d>0)$ if $w(B(x, t))\geq ct^d$ for all x and $0<t\leq 1$.

THEOREM 2.6. (i) *If* $-\infty<s_1\leq s_0<\infty$, *then*

$$B^{s_0,w}_{p,q}\subset B^{s_1,w}_{p,q},\qquad F^{s_0,w}_{p,q}\subset F^{s_1,w}_{p,q}.$$

(ii) $B^{s,w}_{p,r}\subset F^{s,w}_{p,q}\subset B^{s,w}_{p,t}$, $p<\infty$, $r=\min(p, q)$, $t=\max(p, q)$.

(iii) *If* $0<q_0\leq q_1\leq\infty$, *then*

$$B^{s,w}_{p,q_0}\subset B^{s,w}_{p,q_1},\qquad F^{s,w}_{p,q_0}\subset F^{s,w}_{p,q_1}.$$

(iv) *If* $w\in\mathscr{M}_d$, $d>0$, $-\infty<s_1\leq s_0<\infty$, $0<p_0\leq p_1\leq\infty$ *and* $s_0-d/p_0=s_1-d/p_1$, *then*

$$B^{s_0,w}_{p_0,q}\subset B^{s_1,w}_{p_1,q}.$$

(v) *If* $w\in\mathscr{M}_d$, $d>0$, $-\infty<s_1<s_0<\infty$, $0<p_0<p_1<\infty$ *and* $s_0-d/p_0=s_1-d/p_1$, *then*

$$F^{s_0,w}_{p_0,\infty}\subset F^{s_1,w}_{p_1,q},\qquad F^{s_0,w}_{p_0,\infty}\subset B^{s_1,w}_{p_1,p_0}.$$

PROOF. The proofs of (i), (ii) and (iii) are obvious by the monotone character of l_q-spaces and Minkowski's inequality. The assertion (iv) follows immediately from Lemma 2.5, while the first assertion of (v) is derived from Lemma 2.5 and an argument similar to the one given in the case $w=1$ [26; pp. 101–103]. As for the second assertion of (v), we observe from the first and (ii) that $F^{s_0,w}_{p_0,\infty}\subset B^{s_1,w}_{p_1,\infty}$. The result then follows from Theorem 3.3 (i) and Theorem 3.5 (i) (cf. [13] for the case $w=1$).

REMARK 2.7. We note that (i) is not true for homogeneous spaces. As for (iv) and (v), we must assume $w\in\dot{\mathscr{M}}_d$, i.e., $w(B(x, t))\geq ct^d$ for all x and $t>0$.

2.3. Lifting properties, potential spaces, and dual spaces

In the rest of this section, we shall list some other properties of Besov and Triebel spaces; again, their proof are quite similar to those for the case $w=1$ by using results from previous subsections, and we do not go into details here (cf. [26]). We let $P^{s,w}_p=F^{s,w}_{p,2}$. Since $F^{0,w}_{p,2}=h^p_w$ (see §4), we see that $P^{0,w}_p=h^p_w$.

THEOREM 2.8. (i) *Let* $\hat{J}_\sigma(x)=(1+4\pi^2|x|^2)^{-\sigma/2}$, $-\infty<\sigma<\infty$. *Then* J_σ *is an isomorphism of* $B^{s,w}_{p,q}$ *(resp.* $F^{s,w}_{p,q}$*) onto* $B^{s+\sigma,w}_{p,q}$ *(resp.* $F^{s+\sigma,w}_{p,q}$*)*.

(ii) *If* $0<p<\infty$ *and* $m=1, 2,\ldots$, *then*

$$P_p^{m,w} = W_p^{m,w} = \{f \in \mathscr{S}';\ \|f\|_{W(m;p,w)} = \textstyle\sum_{|\alpha|\le m} \|D^\alpha f\|_{h(p,w)} < \infty\}$$
$$= \{f \in \mathscr{S}';\ \|f\|_{h(p,w)} + \textstyle\sum_{j=1}^n \|D_j^m f\|_{h(p,w)} < \infty\}.$$

We just note that the proof of the above theorem is done by considering certain Fourier multipliers and using Theorem 2.4 (iii).

REMARK 2.9. For homogeneous spaces one should replace J_σ by I_σ, where $\hat{I}_\sigma(x) = (2\pi|x|)^{-\sigma}$.

In order to describe dual spaces, we need some terminologies. We let $\mathscr{U}$ denote the set of all sequences $\{\phi_j\} \subset \mathscr{S}$ satisfying the following properties: supp $\hat{\phi}_0 \subset \{|x| \le 2\}$, supp $\hat{\phi}_j \subset \{2^{j-1} \le |x| \le 2^{j+1}\}$, $j = 1, 2, \ldots$, and $\sum_j \hat{\phi}_j(x) = 1$ for all x. In the rest of this section (only), we use the following notation: $q' = p' = \infty$ if p, $q < 1$, $w'(x) = w(x)^{1-p'}$ if $p' < \infty$, $w'(x) = w(x)$ if $p' = \infty$,

$$\|\{f_j\}\|_{\infty, w'; q', s} = (\textstyle\sum_j (2^{js}\|f_j/w'\|_\infty)^{q'})^{1/q'},$$
$$\|\{f_j\}\|_{q', s; \infty, w'} = \|(\textstyle\sum_j (2^{js}|f_j/w'|)^{q'})^{1/q'}\|_\infty.$$

We denote by $\mathscr{B}_{p',q'}^{s,w'}$ (resp. $\mathscr{F}_{p',q}^{s,w'}$) the space of those tempered distributions f for which there exist $\{\phi_j\} \in \mathscr{U}$, $\{f_j\} \in l_{q'}^s(L_{w'}^{p'})$ (resp. $L_{q'}^{p'}(l_{q'}^s)$) such that $f = \sum_j \phi_j * f_j$ in $\mathscr{S}'$. We define

$$\|f\|_{\mathscr{B}(s,w';p',q')} = \inf \|\{f_j\}\|_{p',w';q',s},$$

where the infimum is taken over all possible representations of f; similarly, we define $\|f\|_{\mathscr{F}(s,w';p',q')}$. If $1 < p < \infty$ and $w \in A_p$ (hence $w' \in A_{p'}$), then it follows from a weighted estimate for singlular integrals [7] that $\mathscr{B}_{p',q'}^{s,w'} = B_{p',q'}^{s,w'}$; as for the $\mathscr{F}$-space, we see that $\mathscr{F}_{p',q'}^{s,w'} = F_{p',q'}^{s,w'}$ if in addition, $1 < q' < \infty$ (in this later case we must use a weighted estimate for vector-valued singluar integrals in [1]).

THEOREM 2.10. *If* $1 \le p < \infty$, *then*

$$(B_{p,q}^{s,w})' = \mathscr{B}_{p',q'}^{-s,w'}, \qquad 0 < q < \infty,$$
$$(F_{p,q}^{s,w})' = \mathscr{F}_{p',q'}^{-s,w'}, \qquad 1 \le q < \infty,$$

where E' *is the space of continuous linear functionals on the quasi-nomed vector space* E.

In concluding this section, we should remark that there are important properties such as traces, dual spaces for $p < 1$, etc., that we are not able yet to establish in the weighted case. This is mainly due to our inability in finding a good substitute for Lemma 8 in [20; Chap. 11].

§3. Interpolation theorems

3.1. Retracts and coretracts

We first give a quick review of interpolation theory for the sake of easy reference. Let A_0 and A_1 be two quasi-Banach spaces which are linear subspaces of a Hausdorff topological vector space E, and such that the corresponding inclusions are continuous; we call (A_0, A_1) a compatible couple. We denote by A_0+A_1 the vector sum of A_0 and A_1. For $a\in A_0+A_1$ and $t>0$, we let

$$K(t, a; A_0, A_1) = \inf_{a=a_0+a_1, a_i\in A_i, i=0,1} (\|a_0\|_0+t\|a_1\|_1),$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ are the quasi-norm on A_0 and A_1, respectively. By $(A_0, A_1)_{\theta,q}=A_{\theta,q}$, where $0<\theta<1$, $0<q\le\infty$ (or $\theta=0$, $q=\infty$ or $\theta=1$, $q=\infty$), we mean the space of all $a\in A_0+A_1$ for which

$$\|a\|_{A_{\theta,q}} = \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta}K(t, a; A_0, A_1))^q t^{-1}dt\right)^{1/q} < \infty.$$

We notice that $A_{\theta,q}$ is then a quasi-Banach space with respect to the quasi-norm $\|\cdot\|_{\theta,q}$. The following theorem is useful in computing interpolation spaces.

REITERATION THEOREM. *If $\theta=(1-\lambda)\theta_0+\lambda\theta_1$, $0<\lambda<1$ and $0\le\theta_0\ne\theta_1\le 1$, then*

$$(A_{\theta_0,q_0}, A_{\theta_1,q_1})_{\lambda,q} = A_{\theta,q},$$
$$(A_{\theta_0,q_0}, A_1)_{\lambda,q} = A_{\theta,q},\ \theta_1 = 1$$

for any q_0, q_1 and q.

For these results and related facts, we refer to [2] and [21]. Another fact on interpolation theory that we shall use is the following:

Let $A=(A_0, A_1)$ and $B=(B_0, B_1)$ be two compatible couples of quasi-Banach spaces. Assume that there exist linear mappings R and S, $R\colon A_0+A_1\to B_0+B_1$, $S\colon B_0+B_1\to A_0+A_1$, such that $R_i=R|_{A_i}$ (resp. $S_i=S|_{B_i}$) is a continuous linear map from A_i (resp. B_i) into B_i (resp. A_i), and $R_i\circ S_i$ is an identity map, $i=0, 1$. Then

$$\|b\|_{B_{\theta,q}} \approx \|Sb\|_{A_{\theta,q}}.$$

Here "$\approx$" means the equivalence. When the above assumptions hold, we say that (B_0, B_1) is a *retract* of (A_0, A_1), and (A_0, A_1) is a *coretract* of (B_0, B_1) (we also say that B_i is a *retract* of A_i, and A_i is a *coretract* of B_i).

In the rest, let Φ be a function in $\mathscr{S}$ such that $\operatorname{supp}\Phi\subset\{1/3\le|x|\le 3\}$, $\Phi=1$

on $\{1/2 \le |x| \le 2\}$. Let $\Phi_0, \Phi_1, \Phi_2, \ldots$ be functions in $\mathscr{S}$ satisfying the following properties: $\hat{\Phi}_0 = 1$ on $\operatorname{supp} \hat{\psi}_0$, $\operatorname{supp} \hat{\Phi}_0 \subset \{|x| \le 4\}$, and $\hat{\Phi}_j(x) = \Phi(2^{-j}x)$, $j = 1, 2, \ldots$. We let $Sf = \{\psi_j * f\}_{j=0}^{\infty}$ and $R(\{f_j\}) = \sum_{j=0}^{\infty} \Phi_j * f_j$ whenever the latter has a meaning in $\mathscr{S}'$.

LEMMA 3.1. *R is a bounded linear operator from $l_q^s(h_w^p)$ (resp. $h_w^p(l_q^s)$, $1 < q \le \infty$) into $B_{p,q}^{s,w}$ (resp. $F_{p,q}^{s,w}$), and S is a bounded linear operator from $B_{p,q}^{s,w}$ (resp. $F_{p,q}^{s,w}$, $1 < q \le \infty$) into $l_q^s(h_w^p)$ (resp. $h_w^p(l_q^s)$). Furthermore, $R \circ S$ is the identity map.*

PROOF. The conclusion on S is just Corollary 2.3. As for the proof of the assertion on R, we assume $s = 0$ for the sake of simplicity. It can be seen from maximal inequality for weighted Hardy spaces that $\sum_j \Phi_j * f_j$ exists in $\mathscr{S}'$ for $\{f_j\} \in l_\infty(h_w^p)$ (which contains $l_q(h_w^p)$ and $h_w^p(l_q)$ for any q). Now, a consideration of the supports of $\hat{\psi}_k$ and $\hat{\Phi}_j$ implies that

$$\psi_k * R(\{f_j\}) = \sum_{|j-k| \le 3} \psi_k * \Phi_j * f_j, \quad k = 0, 1, 2, \ldots$$

Hence the result for the B-space case follows easily from multiplier criterion for Hardy spaces (see §4). As for the F-space case, we see that for any m,

$$\|(\sum_{k \le m} |\psi_k * R(\{f_j\})|^q)^{1/q}\|_{p,w} \le \|\{\sum_{|j-k| \le 3} \psi_k * \Phi_j * f_j\}_{k=0}^{m}\|_{h(p,w;q)}.$$

The last term is dominated by a constant multiple of a finite sum of $h_{w,q}^p$-quasinorms of elements of the form $\{\psi_k * \Phi_{k-j} * f_{k-j}\}$, $|j| \le 3$, $j \le k$. Since for such j,

$$\|\{\psi_k * \Phi_{k-j} * f_{k-j}\}_k\|_{h(p,w;q)} \le C \|\{f_k\}\|_{h(p,w;q)} \tag{6}$$

by an l_q-valued analogue of multiplier results in §4, we obtain the desired result by letting $m \to \infty$ (monotone convergence theorem). Here, we must assume $1 < q \le \infty$ since in the proof of (6) we use the inequality

$$\|(\sum_k |\psi_k * \Phi_{k-j} * f_{k-j}|^q)^{1/q}\|_{r,w} \le C \|(\sum_k |f_k|^q)^{1/q}\|_{r,w}$$

for $w \in A_r$, $1 < r < \infty$. To prove this last inequality, we observe that

$$(\sum_k |\psi_k * \Phi_{k-j} * f_{k-j}|^q)^{1/q} \le C(\sum_k (M f_{k-j})^q)^{1/q}$$

and then appeal to Lemma 1.1 if $1 < q < \infty$; for $q = \infty$, just note that

$$\sup_k M f_{k-j} \le M(\sup_k f_{k-j}).$$

The proof of Lemma 3.1 is thus complete.

3.2. Interpolation of weighted vector-valued Hardy spaces

Our aim in this subsection is to prove the following theorem:

THEOREM 3.2. *If* $0<p_0<\infty$, $0<\theta<1$, $1/p=(1-\theta)/p_0$ *and* $1\leq q\leq\infty$, *then*

$$(H_w^{p_0}(l_q^s),\ L^\infty(l_q^s))_{\theta,p} = H_w^p(l_q^s), \tag{7}$$

$$(h_w^{p_0}(l_q^s),\ L^\infty(l_q^s))_{\theta,p} = h_w^p(l_q^s). \tag{8}$$

Consequently, if $0<p_0$, $p_1<\infty$, $0<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$, *then*

$$(H_w^{p_0}(l_q^s),\ H_w^{p_1}(l_q^s))_{\theta,p} = H_w^p(l_q^s), \tag{9}$$

$$(h_w^{p_0}(l_q^s),\ h_w^{p_1}(l_q^s))_{\theta,p} = h_w^p(l_q^s) \tag{10}$$

by Reiteration theorem.

PROOF. As before, we assume $s=0$ for the sake of simplicity. The proof of the $H_{w,q}^p$-case can be done by adopting the arguments in the scalar-valued case given in [8] for $w=1$. We just note that the assumption $1\leq q\leq\infty$ is used in an essential way to interchange the order of summation and integration (Minkowski's inequality); in the course of the proof, we also use Theorem 1.4 (iii) and the (B_r)-condition, $r>r_0$.

Next, we turn to the proof of the $h_{w,q}^p$-case. Fix a non-negative function $\phi\in\mathscr{S}$ with $\int\phi(x)dx=1$ and supp ϕ is contained in the unit cube with center at the origin. One direction is easy. Consider the map: $f=\{f_j\}\mapsto N^+f=\sup_{0<t<1}(\sum|\phi_t*f_j(\cdot)|^q)^{1/q}$. Then N^+ is a sublinear operator and maps $h_{w,q}^{p_0}$ into $L^{p_0}(\mu)$, $d\mu=wdx$. On the other hand, since

$$(\sum|\phi_t*f_j(x)|^q)^{1/q} \leq (\sum\phi_t*|f_j|^q(x))^{1/q} \leq \|\sum|f_j|^q\|_\infty^{1/q}$$

by Hölder's inequality ($q\geq 1$), N^+ maps $L^\infty(l_q)$ into $L^\infty(\mu)=L^\infty$. Thus, we conclude from a general version of Marcinkiewicz interpolation theorem that N^+ maps $(h_w^{p_0}(l_q),\ L^\infty(l_q))_{\theta,p}$ into $(L^{p_0}(\mu),\ L^\infty(\mu))_{\theta,p}=L^p(\mu)=L_w^p$, i.e., $(h_w^{p_0}(l_q),\ L^\infty(l_q))_{\theta,p}\subset h_{w,q}^p=h_w^p(l_q)$. We note that even in this easy direction, we have used the assumption $q\geq 1$. To prove the converse, let $f=\{f_j\}\in h_{w,q}^p$, and Ψ be as in Theorem 1.4 (ii). Then, it follows from this theorem and (7) that $f-\Psi*f\in H_{w,q}^p\subset(h_{w,q}^{p_0},\ L^\infty(l_q))_{\theta,p}$. Thus, it suffices to show that $\Psi*f\in(h_{w,q}^{p_0},\ L^\infty(l_q))_{\theta,p}$. For any $\gamma>0$, we assert that there exists a decomposition $\Psi*f=g+b$ into "*good part*" and "*bad part*" satisfying the following:

$$\|g\|_{L^\infty(l_q)}\leq\gamma, \tag{11}$$

$$\|b\|_{h(p_0,w;q)}^{p_0}\leq C\int_{\{N^*f>c\gamma\}}N^*f(x)^{p_0}w(x)dx. \tag{12}$$

For this purpose, let $\{I_k\}$ be a decomposition of R^n into half-closed (disjoint) unit cubes, and let $f^k=\{f_j^k\}$, $f_j^k=(\Psi*f_j)\chi_{I_k}$, χ_{I_k} being the characteristic function of I_k. Then, for each j,

$$f_j = \sum_k f_j^k \text{ in } \mathscr{S}' \qquad (\text{in fact in } h_w^p).$$

We define $g=\sum_{k\in E_1} f^k$, where $E_1=\{k;\ \|(\sum_j |f_j^k|^q)^{1/q}\|_\infty \le \gamma\}$. Since $\{I_k\}$ is a disjoint family, we see that

$$\|g\|_{L^\infty(l_q)} \le \gamma,$$

and obtain the estimate on the good part. Denote by I_k^* the closed cube with the same center as I_k and with length of sides 2, and let $E_2=\{k;\ k\notin E_1\}$, i.e., $k\in E_2$ if and only if $\|\{f_j^k\}_j\|_{L^\infty(l_q)}>\gamma$. By writing $\Psi * f_j$ as $\eta_{1/2} * f_j$, and noting that $c\eta\in\mathscr{A}_N$ and $I_k\times\{1/2\}\subset\Gamma_\beta(x)=\{(y,t);\ |x-y|<\beta t<\beta\}$ for some $\beta>0$ and for any $x\in I_k^*$, we see that

$$N^*f \ge c\|\{f_j^k\}_j\|_{L^\infty(l_q)} \ge c\gamma \quad \text{on } I_k^*,\ k\in E_2. \tag{13}$$

The estimate (12) on the $h_{w,q}^{p_0}$-quasi-norm of b would follow from (13) if we can show that

$$\|b\|_{h(p_0,w;q)}^{p_0} \le C \sum_{k\in E_2} \|\{f_j^k\}\|_{L^\infty(l_q)}^{p_0} w(I_k). \tag{14}$$

We observe that

$$\phi_t * b_j(x) = \sum_{k\in E_2} \phi_t * f_j^k(x) = 0$$

for $x\notin \cup_{k\in E_2} I_k^* = \Omega^*$ and $0<t<1$. Thus, we obtain

$$\begin{aligned}
\|b\|_{h(p_0,w;q)}^{p_0} &= \int_{\Omega^*} \sup_{0<t<1} \left(\sum_j |\phi_t * (\textstyle\sum_{m\in E_2} f_j^m)(x)|^q\right)^{p_0/q} w(x)\,dx \\
&\le \sum_{k\in E_2} \int_{I_k^*} \left(\sup_{0<t<1} \textstyle\sum_j (\phi_t * |\sum_{m\in E_{2,k}} f_j^m|^q)(x)\right)^{p_0/q} w(x)\,dx \\
&\le C \sum_{k\in E_2} \int_{I_k^*} \textstyle\sum_{m\in E_{2,k}} \sup_{0<t<1} \left((\phi_t * (\sum_j |f_j^m|^q))(x)\right)^{p_0/q} w(x)\,dx \\
&\le C \sum_{k\in E_2} \textstyle\sum_{m\in E_{2,k}} \|\{f_j^m\}\|_{L^\infty(l_q)}^{p_0} w(I_m^*) \le C \sum_{k\in E_2} \|\{f_j^k\}\|_{L^\infty(l_q)}^{p_0} w(I_k).
\end{aligned}$$

Here, we let $E_{2,k}=\{m\in E_2;\ I_m^*\cap I_k^*\neq\emptyset\}$; note that $w(I_m^*)\le Cw(I_k)$ for $m\in E_{2,k}$, and $\sharp(E_{2,k})\le\gamma_n$ (γ_n depending only on the dimension n). We have thus obtained (14), and completed the proofs of (11) and (12). The proof of (8) is then finished in the same way as in [8]; we only write it down here for the sake of easy reference. Let $\gamma=f^*(t^{p_0})$ in the above decomposition, where f^* denotes the non-increasing rearrangement of N^*f with respect to the measure wdx. Putting $u=t^{p_0}$, we obtain

$$\Psi * f = b_t + g_t, \qquad \|g_t\|_{L^\infty(l_q)} \le f^*(u),$$

$$\|b_t\|_{h(p_0,w;q)}^{p_0} \le C\int_{\{N^*f\ge cf^*(u)\}} N^*f(x)^{p_0} w(x)\,dx \le C\int_0^u f^*(s)^{p_0}\,ds.$$

Since

$$K(t, \Psi * f; h^{p_0}_{w,q}, L^\infty(l_q)) \leq \|b_t\|_{h(p_0,w;q)} + t\|g_t\|_{L^\infty(l_q)},$$

we see that

$$\int_0^\infty (t^{-\theta}K(t, \Psi*f; h^{p_0}_{w,q}, L^\infty(l_q)))^p t^{-1}dt$$
$$\leq C\left\{\int_0^\infty t^{-\theta p}\left(\int_0^u f^*(s)^{p_0}ds\right)^{p/p_0} t^{-1}dt + \int_0^\infty (t^{(1-\theta)}f^*(u))^p t^{-1}dt\right\}$$
$$\leq C\left\{\int_0^\infty (t^{(1-\theta)/p_0}f^*(t))^p t^{-1}dt + \int_0^\infty (t^{1/p}f^*(t))^p t^{-1}dt\right\} \leq C\|f\|^p_{h(p,w;q)}$$

by Hardy's inequality. The proof of the theorem is thus complete.

3.3. Interpolation of weighted Besov and Triebel spaces

Before proceeding on, we recall that by our definition $B^{s,w}_{\infty,q} = B^s_{\infty,q}$. We note also that $B^{s,w}_{p,q}$ (resp. $F^{s,w}_{p,q}$, $1<q\leq\infty$) is a retract of $l^s_q(h^p_w)$ (resp. $h^p_w(l^s_q)$) for $0<p<\infty$ (Lemma 3.1), while it is known that $B^s_{\infty,q}$ is a retract of $l^s_q(L^\infty)$ (see e.g., [2]). We let $P^{s,w}_\infty = P^s_\infty = \{f;\ J^{-s}f \in L^\infty\}$.

THEOREM 3.3. *If* $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$, $s_0\neq s_1$, $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$, *then the following interpolation formulas hold*:

(i) $(B^{s_0,w}_{p,q_0}, B^{s_1,w}_{p,q_1})_{\theta,r} = B^{s,w}_{p,r}$.

(ii) $(B^{s,w}_{p,q_0}, B^{s,w}_{p,q_1})_{\theta,q} = B^{s,w}_{p,q}$.

(iii) $(B^{s_0,w}_{p_0,q_0}, B^{s_1,w}_{p_1,q_1})_{\theta,p} = B^{s,w}_{p,p}$, $\quad p=q$.

(iv) $(P^{s_0,w}_p, P^{s_1,w}_p)_{\theta,r} = B^{s,w}_{p,r}$, $\quad p<\infty$.

(v) $(P^{s,w}_{p_0}, P^{s,w}_{p_1})_{\theta,p} = P^{s,w}_p$, $\quad p<\infty$.

PROOF. We first notice the following well-known result: *Let A be a quasi-Banach space and (A_0, A_1) be a couple of compatible quasi-Banach spaces. Then, we have*

$$(l^{s_0}_{q_0}(A_0), l^{s_1}_{q_1}(A_1))_{\theta,q} = l^s_q((A_0,A_1)_{\theta,q}),$$
$$(l^s_{q_0}(A), l^s_{q_1}(A))_{\theta,q} = l^s_q(A),$$
$$(l^{s_0}_{q_0}(A), l^{s_1}_{q_1}(A))_{\theta,r} = l^s_r(A),$$

where the parameters are the same as in Theorem 3.3 (cf. [2], [20]). The assertions (i), (ii) and (iii) then follow from this, the result on retracts given in 3.1, the observation before Theorem 3.3 and interpolation formulas for weighted Hardy spaces proved in 3.2 (cf. also [2], [20] for the case $w=1$).

The proofs of (iv) and (v), which are also routine, are given here for the sake of easy reference. We need the following result in interpolation theory: *If* $a=\sum_{j=-\infty}^{\infty} a_j$ *in* A_0+A_1, $a_j \in A_0 \cap A_1$ *for each* j, $1 \neq \lambda > 0$ *and*

$$\|a\|_J = (\textstyle\sum_{j=-\infty}^{\infty} (\lambda^{-j\theta} \max(\|a_j\|_0, \lambda^j \|a_j\|_1))^r)^{1/r} < \infty,$$

then $a \in (A_0, A_1)_{\theta,r}$ *and* $\|a\|_{\theta,r} \leq C\|a\|_J$ (cf. [2], [21]). Let $\{\Phi_k\}$ be the sequence given in 3.1. Since for each $k=0, 1, 2,\ldots$ and $f \in \mathscr{S}'$, we have

$$J^s(\psi_k * f) = (J^s \Phi_k) * (\psi_k * f),$$

$$|D^\alpha \mathscr{F}(J^s \Phi_k)(x)| \leq C_\alpha 2^{-ks}(1+|x|)^{-|\alpha|},$$

we see that

$$\|J^s(\psi_k * f)\|_{h(p,w)} \leq C 2^{-ks} \|\psi_k * f\|_{h(p,w)}$$

by a multiplier result on weighted Hardy spaces (see §4). Hence, it follows that

$$\begin{aligned}
&2^{k(s-s_0)} \max(\|\psi_k * f\|_{P(s_0;p,w)}, 2^{k(s_0-s_1)}\|\psi_k * f\|_{P(s_1;p,w)}) \\
&\approx 2^{k(s-s_0)} \max(\|J^{-s_0}(\psi_k * f)\|_{h(p,w)}, 2^{k(s_0-s_1)}\|J^{-s_1}(\psi_k * f)\|_{h(p,w)}) \\
&\leq C 2^{ks}\|\psi_k * f\|_{h(p,w)}.
\end{aligned}$$

Thus, it follows from Corollary 2.3 that if $f \in B_{p,r}^{s,w}$, then $f \in (P_p^{s_0,w}, P_p^{s_1,w})_{\theta,r}$, provided we can show that $f=\sum_k \psi_k * f$ in $P_p^{s_0,w}+P_p^{s_1,w}$. This latter fact can be easily seen by observing from maximal inequality that $f=\sum_k \psi_k * f$ in $B_{p,r}^{s,w}$. For the other direction of (iv), let $f \in (P_p^{s_0,w}, P_p^{s_1,w})_{\theta,r}$. For $f_i \in P_p^{s_i,w}$ such that $f=f_0+f_1$ (as elements of $\mathscr{S}'$), we see that, for $k=0, 1, 2,\ldots$,

$$\|\psi_k * f\|_{h(p,w)} \leq C(2^{-ks_0}\|f_0\|_{P(s_0;p,w)} + 2^{-ks_1}\|f_1\|_{P(s_1;p,w)}).$$

Thus,

$$\|\psi_k * f\|_{h(p,w)} \leq C 2^{-ks_0} K(2^{k(s_0-s_1)}, f; P_p^{s_0,w}, P_p^{s_1,w}),$$

and the converse inclusion follows. The proof of (v) can be done in the same spirit by using J^{-s} to lift $P_p^{s,w}$ to h_w^p and then applying the results on weighted Hardy spaces obtained in 3.2.

REMARK 3.4. (i) We note that (iii) and (v) give interpolation relations between some weighted and non-weighted spaces.

(ii) If one introduces Besov and Hardy spaces based on Lorentz spaces with respect to the measure wdx, then interpolation formulas can be derived also for other values of q and r.

THEOREM 3.5. *If* $0<p_0$, $p_1<\infty$, $0<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$, $s=(1-\theta)s_0+\theta s_1$, $s_0 \neq s_1$, *then the following interpolation formulas hold*:

(i) $$(F^{s,w}_{p_0,q}, F^{s,w}_{p_1,q})_{\theta,p} = F^{s,w}_{p,q}, \quad q>1.$$

(ii) $$(F^{s_0,w}_{p_0,q_0}, F^{s_1,w}_{p_1,q_1})_{\theta,p} = F^{s,w}_{p,p}\ (= B^{s,w}_{p,p}),\ p_i \le q_i.$$

(iii) $$(F^{s_0,w}_{p,q_0}, F^{s_1,w}_{p,q_1})_{\theta,r} = B^{s,w}_{p,r}.$$

PROOF. The assertion (i) follows from results on retracts and Theorem 3.2. For the proof of (ii), let $f \in (F^{s_0,w}_{p_0,q_0}, F^{s_1,w}_{p_1,q_1})_{\theta,p}$. For any $\lambda>0$ (fixed for a moment), there exist decompositions $f=f^0_k+f^1_k$, $f^i_k \in F^{s_i,w}_{p_i,q_i}=A_i$, $k=0, \pm 1, \pm 2, \ldots$, such that

$$(\textstyle\sum_{k=-\infty}^{\infty}(2^{-k\theta}(\|f^0_k\|_{F(s_0,w;p_0,q_0)} + 2^k\|f^1_k\|_{F(s_1,w;p_1,q_1)}))^p)^{1/p} \le \|f\|_{(A_0,A_1)_{\theta,p}} + \lambda.$$

Since for each x, j and k,

$$\psi_j * f(x) = \psi_j * f^0_k(x) + \psi_j * f^1_k(x),$$

and $l^s_p = (l^{s_0}_{q_0}, l^{s_1}_{q_1})_{\theta,p}$, we derive from a result of Holmstedt [12; p. 193] that

$$\begin{aligned}\|\{\psi_j*f(x)\}_j\|_{l(p,s)} &\le C\{(\textstyle\sum_{k=-\infty}^{\infty}(2^{-k\theta}\|\{\psi_j*f^0_k(x)\}\|_{l(q_0,s_0)})^{p_0})^{(1-\theta)/p_0}\\ &\quad\times(\textstyle\sum_{k=-\infty}^{\infty}(2^{k(1-\theta)}\|\{\psi_j*f^1_k(x)\}\|_{l(q_1,s_1)})^{p_1})^{\theta/p_1}\}\end{aligned}$$

for almost every x. Hence, it follows from Hölder's inequality and the simple inequality $a^\eta b^{1-\eta} \le a+b$ ($a, b \ge 0$ and $0<\eta\le 1$) that

$$\begin{aligned}&\|f\|_{F(s,w;p,p)}\\ &= \|\{\psi_j*f\}\|_{p,s;p,w} \le C\Big\{\int[(\textstyle\sum_{k=-\infty}^{\infty}(2^{-k\theta}\|\{\psi_j*f^0_k(x)\}\|_{l(q_0,s_0)})^{p_0})^{(1-\theta)p/p_0}\\ &\times(\textstyle\sum_{k=-\infty}^{\infty}(2^{k(1-\theta)}\|\{\psi_j*f^1_k(x)\}\|_{l(q_1,s_1)})^{p_1})^{\theta p/p_1}]w(x)dx\Big\}^{1/p}\\ &\le C\Big\{\Big[\int(\textstyle\sum_{k=-\infty}^{\infty}(2^{-k\theta}\|\{\psi_j*f^0_k(x)\}\|_{l(q_0,s_0)})^{p_0})w(x)dx\Big]^{(1-\theta)p/p_0}\\ &\times\Big[\int(\textstyle\sum_{k=-\infty}^{\infty}(2^{k(1-\theta)}\|\{\psi_j*f^1_k(x)\}\|_{l(q_1,s_1)})^{p_1})w(x)dx\Big]^{\theta p/p_1}\Big\}^{1/p}\\ &\le C(\textstyle\sum_{k=-\infty}^{\infty}(2^{-k\theta}\|f^0_k\|_{F(s_0,w;p_0,q_0)}+2^{k(1-\theta)}\|f^1_k\|_{F(s_1,w;p_1,q_1)})^p)^{1/p}\\ &\le C(\|f\|_{(A_0,A_1)_{\theta,p}}+\lambda).\end{aligned}$$

Since λ is arbitrary, we obtain one direction of (ii); note that we do not use the assumption $p_i \le q_i$ in the proof of this direction. For the other direction, we observe from the assumption $p_i \le q_i$ that $B^{s_i,w}_{p_i,p_i} \subset F^{s_i,w}_{p_i,q_i}$ by Theorem 2.6 (ii). Thus, the converse inclusion follows from Theorem 3.3 (iii). The last assertion (iii) is derived again from Theorem 3.3 (i) by observing that $B^{s_i,w}_{p,u_i} \subset F^{s_i,w}_{p,q_i} \subset B^{s_i,w}_{p,v_i}$, where $u_i = \min(p, q_i)$ and $v_i = \max(p, q_i)$.

3.4. Remarks

In concluding this section, we make a number of remarks.

(a) We note that if $1<p_i<\infty$, $0<q_i\leq\infty$ (resp. $1<q_i\leq\infty$), $\lambda\leq\min(p_0, p_1)$, and $w_0, w_1\in A_\lambda$, then $B^{s_i,w_i}_{p_i,q_i}$ (resp. $F^{s_i,w_i}_{p_i,q_i}$) is a retract of $l^{s_i}_{q_i}(L^{p_i}_{w_i})$ (resp. $L^{p_i}_{w_i}(l^{s_i}_{q_i})$). Thus, one can extend many interpolation formulas given in [24] to $(B^{s_0,w_0}_{p_0,q_0}, B^{s_1,w_1}_{p_1,q_1})_{\theta,r}$ and $(F^{s_0,w_0}_{p_0,q_0}, F^{s_1,w_1}_{p_1,q_1})_{\theta,r}$. It remains, however, the problem of describing interpolation spaces when either p_i, q_i or w_i does not satisfy the above assumptions.

(b) While real interpolation results for weighted Besov spaces are parallel to the case $w=1$, those for Triebel spaces are still incomplete; we note that some parts of Theorem 3.5 (i) and (ii) seem new even for the case $w=1$. Further, we have not given any results on interpolation by the complex method. For the case $w=1$, there are results of Triebel [27] for both Besov and Triebel spaces (cf. also [17]), and of Calderón-Torchinsky [6; II] for parabolic H^p-spaces; moreover, in the announcement of Strömberg-Torchinsky [23], they indicated that complex interpolation for H^p_w $(p\geq1)$ can be carried out by using atoms. We hope to return to these subjects as well as others, such as traces,..., at a later occasion.

(c) Lastly, we state a result on pseudo-differential operators which is of interest.

Let $\sigma(x, \xi)$ be a continuous and bounded function defined on $R^n\times R^n$ which is infinitely differentiable with respect to ξ. Let $\sigma(D)=\sigma(x, D)$ be the pseudo-differential operator whose symbol is σ. Then, the following two propositions hold:

(i) *If $\rho>\rho_B=\max(s, nr_0/p-s)$ and $\|D^\beta_\xi\sigma(\cdot, \xi)\|_{B(\rho;\infty,\infty)}\leq C_\beta(1+|\xi|)^{-|\beta|}$, then $\sigma(D)$ is bounded on $B^{s,w}_{p,q}$.*

(ii) *If $\rho>\rho_F=\max(s, \max(nr_0/p, n/q)-s)$ and $\|D^\beta_\xi\sigma(\cdot, \xi)\|_{B(\rho;\infty,\infty)}\leq C_\beta(1+|\xi|)^{-|\beta|}$, then $\sigma(D)$ is bounded on $F^{s,w}_{p,q}$.*

In particular, if σ is a classical symbol in the class $S^0_{1,0}$, then $\sigma(D)$ is bounded on $B^{s,w}_{p,q}$ and $F^{s,w}_{p,q}$.

The proof of the above result can be done in a way similar to the case $w=1$ given in the proof of Theorem 3 in [4]. In contrast to the case $w=1$ where we can also prove (i) without using interpolation theorem, our proof of the weighted case relies on Theorem 3.3 in an essential way. We notice that the above result immediately implies a regularity theorem for elliptic partial differential equations in terms of weighted spaces; we refer to [20; Appendix D] for a discussion in the case $w=1$.

In connection with (ii), we notice that in the announcement [23] there is a result which states that pseudo-differential operators map H^p_w boundedly into

L^p_w, $p \geq 1$. We can see from (ii) and the relation $h^p_w = F^{0,w}_{p,2}$ (see §4) that if σ satisfies the assumptions of (ii) with $s=0$ and $q=2$ (in particular, if $\sigma \in S^0_{1,0}$), then $\sigma(D)$ can be interpreted as a bounded operator from h^p_w ($\supset H^p_w$) into L^p_w.

§4. Appendix

Our aim is to prove the following identities:

$$\dot{F}^{0,w}_{p,2} = H^p_w \quad \text{(mod. polynomials)}, \tag{15}$$

$$F^{0,w}_{p,2} = h^p_w. \tag{16}$$

The proof is a reproduction of [5] which we give here for reader's convenience.

LEMMA 4.1. *The Schwartz class $\mathscr{S}$ is dense in h^p_w.*

PROOF. Let $f \in h^p_w$ and $\Psi \in \mathscr{S}$ be such that $\hat{\Psi}=1$ on a neighbourhood of the origin. Then $g = f - \Psi * f \in H^p_w$ (cf. Theorem 1.4 (ii)). Since $\hat{g}=0$ on a neighbourhood of the origin, the Poisson integral $u = K_t * g$ is well-defined on R^+_{n+1}, and it can be proved that $\|g\|_{H(p,w)} \approx \|\sup_{|x-y|<t} |u(y, t)|\|_{p,w} \approx \|\sup_{0<t<\infty} |u(x, t)|\|_{p,w}$. (Here $K_t(x) = K(x, t) = \Gamma((n+1)/2)\pi^{-(n+1)/2} t(t^2 + |x|^2)^{-(n+1)/2}$.) The desired result then follows by an argument similar to the case $w=1$ ([11; pp. 35–36], [9; Corollary to Theorem 10]) by observing that

$$u_\delta = u(\cdot, \delta) \longrightarrow g \quad \text{in} \quad H^p_w \qquad \text{as} \quad \delta \longrightarrow 0.$$

This last assertion is derived from the following two facts:

$$|u(x, t)| \leq C t^{-\beta} \|g\|_{H(p,w)} (1 + |x|)^\gamma, \quad x \in R^n,\ t > 0$$

for some $\beta > 0$, $\gamma > 0$, by a "*sub-mean-value*" property of $|u|^p$, and

$$\lim_{t \to 0} u(x, t) \qquad \text{exists for almost every} \quad x$$

by a well-known result of Calderón.

REMARK 4.2. It can be seen from the above lemma and the embedding $\mathscr{S} \subset h^p_w$ that the space of functions in $\mathscr{S}$ whose Fourier transforms have compact supports is dense in h^p_w.

LEMMA 4.3. *Let $f \in H^p_w$, and η be a function in $\mathscr{S}$ such that* supp $\hat{\eta}$ *is compact. Then*

$$\eta_t * f \longrightarrow 0 \qquad in \quad H^p_w \qquad as \quad t \longrightarrow \infty.$$

PROOF. Take another function Ψ in $\mathscr{S}$ with $\int \Psi(x)dx = 1$. Then

$$|\Psi_s*\eta_t*f(x)| \leq \sup_{y,t}\{|\eta_t*f(y)|[t/(t+|x-y|]^\lambda\}\int|\Psi_s(x-y)|(1+|x-y|/t)^\lambda dy$$

$$= N_\eta^\lambda f(x)\int s^{-n}|\Psi((x-y)/s)|(1+|x-y|/t)^\lambda dy \leq C_\Psi N_\eta^\lambda f(x) \text{ for } s\leq t.$$

Similarly, we see that

$$|\Psi_s*\eta_t*f(x)| \leq C_\eta N_\Psi^\lambda f(x) \qquad \text{for} \quad s>t.$$

Thus,

$$\sup_{0<s<\infty}|\Psi_s*\eta_t*f(x)| \leq C\max(N_\Psi^\lambda f(x),\ N_\eta^\lambda f(x)).$$

Since $N_\Psi^\lambda f$ and $N_\eta^\lambda f$ are in L_w^p for $\lambda>nr_0/p$, to complete the proof of the lemma, we need only show that

$$\text{(17)} \qquad \sup_{0<s<\infty}|\Psi_s*\eta_t*f(x)| \longrightarrow 0 \qquad \text{as} \quad t\longrightarrow\infty$$

for almost every x. For this purpose, take a function $\phi\in\mathscr{S}$ such that $\phi(0)=1$ and $\operatorname{supp}\hat{\phi}\subset\{|y|\leq 1\}$. Then

$$\operatorname{supp}\mathscr{F}(\phi(\cdot/t)(\Psi_s*\eta_t*f)) \subset \{|y|\leq c/t\}.$$

Let $x\in R^n$ be fixed. The observation on the support just given and Plancherel-Polya's inequality for entire functions of exponential type [20; Chapter 11, Lemma 1] (cf. also Lemma 2.5) imply that

$$\begin{aligned}|\phi(x/t)(\Psi_s*\eta_t*f)(x)| &\leq \|\phi(\cdot/t)(\Psi_s*\eta_t*f)\|_\infty \\ &\leq Ct^{-n/q}\Big(\int|\phi(y/t)|^q|(\Psi_s*\eta_t*f)(y)|^q w(y)^{q/p}w(y)^{-q/p}dy\Big)^{1/q},\end{aligned}$$

where $0<q<p$ be such that $r=p/q>r_0$. Hence, it follows that

$$|\phi(x/t)(\Psi_s*\eta_t*f)(x)| \leq Ct^{-n/q}\|\eta_t*f\|_{H(p,w)}\Big\{\int w(y)^{-r'/r}\{t/(t+|y|)\}^{nr'}dy\Big\}^{1/qr'}$$

$$\leq C\|f\|_{H(p,w)}\Big\{t^{-nr'}\int_{|y|<t}w(y)^{-r'/r}dy\Big\}^{1/qr'} \leq C\|f\|_{H(p,w)}\Big\{\int_{|y|<t}w(y)dy\Big\}^{-1/qr}.$$

The last two inequalities are consequences of the $(B_{r'})$-condition and (A_r)-condition, respectively. Noting that the last term, being independent of s, tends to 0 as t tends to ∞ (since $w(R^n)=\infty$), and $|\phi(x/t)|\geq 1/2$ for large t, we obtain (17). The proof of the lemma is hence complete.

LEMMA 4.4. *The space $\hat{\mathscr{O}}_0$ is dense in H_w^p.*

PROOF. Let $f\in H_w^p$. Then, it follows from Lemma 4.1 (Remark 4.2) that there exists a sequence $\{f_j\}\subset\mathcal{S}$ such that each $\hat{f}_j$ has compact support and $f_j\to f$ in h_w^p as $j\to\infty$. Consequently, $f_j-\Psi*f_j\to f-\Psi*f$ in H_w^p by Theorem 1.4 (ii), where $\Psi\in\mathcal{S}$ be such that $\hat{\Psi}=1$ in a neighbourhood of the origin and supp $\hat{\Psi}$ is compact. The desired result is then deduced from Lemma 4.3.

REMARK 4.5. (i) If $1<p<\infty$ and $w\in A_p$, then $H_w^p=h_w^p=L_w^p$. In fact, the inclusion $h_w^p\subset L_w^p$ follows easily from Lemma 4.1. For the inclusion $L_w^p\subset H_w^p$, we note that for $f\in L_w^p$ and $\phi\in\mathcal{S}$,

$$\sup_{0<t<\infty}|\phi_t*f(x)|\le C_\phi Mf(x),$$

and then use the weighted estimate for the Hardy maximal function.

(ii) It can be seen from Lemma 4.2 (resp. Lemma 4.4) that if $f\in h_w^p$ (resp. H_w^p), and $\Psi\in\mathcal{S}$ with $\int\Psi(x)dx=1$, then $\Psi_t*f\to f$ in h_w^p (resp. H_w^p) as $t\to 0$.

LEMMA 4.6. *Let K be a tempered distribution such that $\hat{K}$ is a bounded function, K is of class C^N outside the origin and*

$$|D^\alpha K(x)|\le C_\alpha|x|^{-|\alpha|-n},\quad |\alpha|\le N$$

for a sufficiently large N. Then

$$\|K*f\|_{H(p,w)}\le C\|f\|_{H(p,w)},\quad f\in\hat{\mathcal{O}}_0.$$

And thus, the operator $Tf=K*f$, initially defined on $\hat{\mathcal{O}}_0$, can be extended to a bounded operator on H_w^p.

PROOF. The proof is similar to that given for the case $w=1$ in [9], so that we only sketch necessary modifications. Notation in [9] is retained. The first modification we need is the estimate for $\tilde{f}$.

$$\begin{aligned}w(\{\sup_{\varepsilon>0}|K_{\varepsilon M}*\tilde{f}|>\lambda\})&\le\lambda^{-q}\|\sup_{\varepsilon>0}|\varphi_\varepsilon*K_M*\tilde{f}|\|_{q,w}^q\\&\approx\lambda^{-q}\|K_M*\tilde{f}\|_{H(q,w)}^q\le C\lambda^{-q}\int_{R^n\setminus\Omega}|f(y)|^q w(y)dy+Cw(\Omega),\end{aligned}$$

where $\max(p,r_0)<q<\infty$. Note here that we have used the identity $H_w^q=L_w^q$ (Remark 4.5 (i)) and the boundedness of the operator $g\mapsto K_M*g$ on L_w^q ([7]). Secondly, the weighted estimate for the Marcinkiewicz integral ([1]), [14]) gives

$$w(\{\sup_{\varepsilon>0}|K_{\varepsilon M}*(f-\tilde{f})|>\lambda\})\le Cw(\Omega).$$

Remaining detailed arguments are similar to [9].

REMARK 4.7. By using the technique in [9], one can see that $Tf=\lim_{\varepsilon\to 0,M\to\infty}K_{\varepsilon M}*f$ for any $f\in H_w^p$.

LEMMA 4.8. *Let m be an infinitely differentiable function such that*

$$|D^\alpha m(x)| \leq C_\alpha(1+|x|)^{-|\alpha|} \qquad \textit{for any multi-index} \quad \alpha,$$

and $\hat{K}=m$. Then $f\mapsto K*f$ defines a bounded operator on h_w^p.

PROOF. We need only show that

$$\|K*f\|_{h(p,w)} \leq C\|f\|_{h(p,w)}, \quad f\in\mathcal{S}.$$

Let $\Psi\in\mathcal{S}$ such that $\hat{\Psi}=1$ in a neighbourhood of the origin. Then $g=f-\Psi*f\in H_w^p$. Since K satisfies the assumptions of Lemma 4.6 by the well-known technique of estimating a kernel from its symbol (Bernstein's theorem), we see that

$$\|K*g\|_{H(p,w)} \leq C\|g\|_{H(p,w)} \leq C\|f\|_{h(p,w)}.$$

Thus, it remains only to show that

$$\|K*\Psi*f\|_{h(p,w)} \leq C\|f\|_{h(p,w)},$$

which is obvious since $K*\Psi\in\mathcal{S}$ (see the proof of Lemma 4.3). Note that the various constants C appearing in the proof depend on $\{C_\alpha\}$, p, w and Ψ.

PROOFS OF (15) AND (16) (cf. [18], [26]). With all the hard preparations having been done, we are now ready to prove our results. We only give a proof for (16) since the other assertion can be similarly verified. By Theorem 2.4 (ii) and Remark 4.2, it suffices to show that

$$\|f\|_{h(p,w)} \approx \|f\|_{F(0,w;p,2)} \tag{18}$$

for any $f\in\mathcal{S}$ such that $\operatorname{supp}\hat{f}$ is compact. Let r_j, $j=0, 1, 2,\ldots$, be Rademacher functions ([22]). Noting that for any such f, $\psi_j*f=0$ except for a finite number of j's, and using an inequality in [22; Appendix D], we see that

$$\begin{aligned}\int(\textstyle\sum_j |\psi_j*f(x)|^2)^{p/2}w(x)\,dx &\leq C\int\left\{\int_0^1 |(\textstyle\sum_j r_j(t)\psi_j)*f(x)|^p dt\right\}w(x)\,dx\\ &\leq C\int_0^1 \|(\textstyle\sum_j r_j(t)\psi_j)*f\|_{h(p,w)}^p dt \leq C\|f\|_{h(p,w)}^p.\end{aligned}$$

The last inequality follows from the fact that $\sum_{j=0}^k r_j(t)\psi_j$ satisfies the assumptions of Lemma 4.8 with constants $\{C_\alpha\}$ independent of k and t. Thus, we obtain one direction of (18). For the other direction, let $\{\Phi_j\}$ be the sequence given in 3.1. Then, by an l_2-valued analogue of Lemma 4.8 (or 4.7), we obtain

$$\begin{aligned}\|f\|_{h(p,w)} &= \|\textstyle\sum_j \Phi_j*\psi_j*f\|_{h(p,w)} \leq C\|\{\psi_j*f\}\|_{h(p,w;2)}\\ &\leq C\|(\textstyle\sum_j \sup_{0<t<1} |\phi_t*\psi_j*f|^2)^{1/2}\|_{p,w},\end{aligned}$$

where $\phi \in \mathscr{S}$, $\int \phi(x)dx = 1$ and supp $\hat{\phi} \subset \{|x| \leq 1\}$. The last term is then dominated by $C\|f\|_{F(0,w;p,2)}$ by an argument similar to the proof of Corollary 2.3. We note that in the proof of the first inequality in the above, we have used the inequality

$$\|\sum_j K_j * g_j\|_{r,w} \leq C\|(\sum_j |g_j|^2)^{1/2}\|_{r,w}$$

for $w \in A_r$, $1 < r < \infty$, and for a suitable sequence of kernels $\{K_j\}$. This last inequality is well-known for $w = 1$ (cf. e.g., [24]). For the case $w \in A_r$, just modify the arguments given by Coifman-Fefferman [7] in the scalar-valued case (cf. also [1]). The proof of (16) is thus complete.

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