Geometry & Topology 23 (2019) 2397–2474

Homological stability of topological moduli spaces

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Given a graded E_1–module over an E_2–algebra in spaces, we construct an augmented semi-simplicial space up to higher coherent homotopy over it, called its canonical resolution, whose graded connectivity yields homological stability for the graded pieces of the module with respect to constant and abelian coefficients. We furthermore introduce a notion of coefficient systems of finite degree in this context and show that, without further assumptions, the corresponding twisted homology groups stabilise as well. This generalises a framework of Randal-Williams and Wahl for families of discrete groups.

In many examples, the canonical resolution recovers geometric resolutions with known connectivity bounds. As a consequence, we derive new twisted homological stability results for various examples including moduli spaces of high-dimensional manifolds, unordered configuration spaces of manifolds with labels in a fibration, and moduli spaces of manifolds equipped with unordered embedded discs. This in turn implies representation stability for the ordered variants of the latter examples.

55P48, 55R40, 55R80, 57R19, 57R50

Published: 13 October 2019 DOI: 10.2140/gt.2019.23.2397

A sequence of spaces

$$\cdots \to \mathcal{M}_{n-1} \to \mathcal{M}_n \to \mathcal{M}_{n+1} \to \cdots$$

is said to satisfy *homological stability* if the induced maps in homology are isomorphisms in degrees that are small relative to n. There is a well-established strategy for proving homological stability that traces back to an argument by Quillen for the classifying spaces of a sequence of inclusions of groups G_n. Given simplicial complexes whose connectivity increases with n and on which the groups G_n act simplicially, transitively on simplices, and with stabilisers isomorphic to groups G_{n-k} prior in the sequence, stability can often be derived by employing a spectral sequence relating the different stabilisers. Randal-Williams and Wahl [61] axiomatised this strategy of proof, resulting in a convenient categorical framework for proving homological stability for families of discrete groups that form a braided monoidal groupoid. Their work unifies and improves many classical stability results and has led to a number of applications since its introduction; see Friedrich [25], Gandini and Wahl [30], Patzt and Wu [53], Randal-Williams [60], and Szymik and Wahl [69].

However, homological stability phenomena have been proved to occur not only in the context of discrete groups, but also in numerous nonaspherical situations, many of them of a moduli space flavour, such as unordered configuration spaces of manifolds (McDuff [46] and Segal [65; 67]), the most classical example, or moduli spaces of high-dimensional manifolds (Galatius and Randal-Williams [27; 28]) to emphasise a more recent one. The majority of the stability proofs in this context resemble the original line of argument for discrete groups, and one of the objectives of the present work is to provide a conceptualisation of this pattern.

Instead of considering the single spaces $\mathcal{M}_n$ and the maps $\mathcal{M}_n \to \mathcal{M}_{n+1}$ between them one at a time, it is beneficial to treat them as a single space $\mathcal{M} = \coprod_{n\geq 0} \mathcal{M}_n$ together with a *grading* $g_{\mathcal{M}}\colon \mathcal{M} \to \mathbb{N}_0$ to the nonnegative integers, capturing the decomposition of $\mathcal{M}$ into the pieces $\mathcal{M}_n$, and a *stabilisation map* $s\colon \mathcal{M} \to \mathcal{M}$ that restricts to the maps $\mathcal{M}_n \to \mathcal{M}_{n+1}$, so it increases the degree by one. From the perspective of homotopy theory, such $\mathcal{M}$ that result from families $\mathcal{M}_n$ that are known to stabilise homologically usually share the characteristic of forming a *(graded) E_1–module over an E_2–algebra* — the homotopy-theoretical analogue of a module over a braided monoidal category. This observation is the driving force behind the present work.

Referring to Section 2.1 for a precise definition, we encourage the reader to think of a graded E_1–module $\mathcal{M}$ over an E_2–algebra $\mathcal{A}$ as a pair of spaces $(\mathcal{M}, \mathcal{A})$ together with gradings $g_{\mathcal{M}}\colon \mathcal{M}\to\mathbb{N}_0$ and $g_{\mathcal{A}}\colon \mathcal{A}\to\mathbb{N}_0$, a homotopy-commutative multiplication $\oplus\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$, and a homotopy-associative action-map $\oplus\colon \mathcal{M}\times\mathcal{A}\to\mathcal{M}$. These are required to satisfy various axioms, among them additivity with respect to $g_{\mathcal{M}}$ and $g_{\mathcal{A}}$; see Definition 2.3. Given such $\mathcal{M}$ and $\mathcal{A}$, the choice of a *stabilising object* $X\in\mathcal{A}$, meaning an element of degree 1, results in a *stabilisation map*

$$s := (-\oplus X)\colon \mathcal{M}\to\mathcal{M}$$

that increases the degree by 1 and hence gives rise to a sequence

$$\cdots\to\mathcal{M}_{n-1}\xrightarrow{s}\mathcal{M}_n\xrightarrow{s}\mathcal{M}_{n+1}\to\cdots$$

of the subspaces $\mathcal{M}_n = g_{\mathcal{M}}^{-1}(n)$ of a fixed degree. The sequences of spaces arising in this fashion are the ones whose homological stability behaviour the present work is concerned with.

The key construction of this work is introduced in Section 2.2. We assign to $\mathcal{M}$ its *canonical resolution*

$$R_\bullet(\mathcal{M})\to\mathcal{M}, \tag{1}$$

which is an augmented semi-simplicial space up to higher coherent homotopy — a notion made precise in Section 1.5, but which can be thought of as an augmented semi-simplicial space in the usual sense. The fibre $W_\bullet(A)$ of the canonical resolution at a point $A\in\mathcal{M}$ is an analogue of the simplicial complex in Quillen's argument; it is a semi-simplicial space up to higher coherent homotopy whose space of p–simplices $W_p(A)$ is the homotopy fibre at A of the $(p+1)^{\text{st}}$ iterated stabilisation map $s^{p+1}\colon \mathcal{M}\to\mathcal{M}$. Thus $W_\bullet(A)$ should be thought of as the *space of destabilisations of A* — a terminology that suggests that the canonical resolution controls the stability behaviour of $\mathcal{M}$, justified by Theorems A and C.

To state our main theorems, we call the canonical resolution of $\mathcal{M}$ *graded $\varphi(g_{\mathcal{M}})$–connected in degrees $\geq m$* for a function $\varphi\colon\mathbb{N}_0\to\mathbb{Q}$ if the restriction $|R_\bullet(\mathcal{M})|_n\to\mathcal{M}_n$ of the geometric realisation of (1) to the preimage of $\mathcal{M}_n$ is $\lfloor\varphi(n)\rfloor$–connected in the usual sense for all $n\geq m$. The first theorem, proved in Section 3, treats homological stability with constant and abelian coefficients, the latter being local systems on which the commutator subgroups of the fundamental groups at all basepoints act trivially.

Theorem A *Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra with stabilising object X and L a local system on $\mathcal{M}$. If the canonical resolution of $\mathcal{M}$ is graded*

$\frac{1}{k}(g_{\mathcal{M}}-2+k)$–connected in degrees ≥ 1 for some $k \geq 2$, then

$$s_*\colon \mathrm{H}_i(\mathcal{M}_n; s^*L) \to \mathrm{H}_i(\mathcal{M}_{n+1}; L)$$

is

(i) *an isomorphism for $i \leq \frac{n-1}{k}$ and an epimorphism for $i \leq \frac{n-2+k}{k}$ if L is constant,*

(ii) *an isomorphism for $i \leq \frac{n+1-k}{k}$ and an epimorphism for $i \leq \frac{n}{k}$ if L is abelian and $k \geq 3$.*

Remark In certain cases, discussed in Remark 3.3, the ranges of Theorem A can be improved marginally.

Restricting to homological degree 0, the theorem has the following cancellation result as a consequence.

Corollary B *Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra with stabilising object X. If the connectivity assumption of Theorem A is satisfied, then the fundamental groupoid of $\mathcal{M}$ is X–cancellative for objects of positive degree, ie for objects A and A' of $\mathcal{M}$ of positive degree, $A \oplus X \cong A' \oplus X$ in $\Pi(\mathcal{M})$ implies $A \cong A'$.*

To cover more general coefficients, we note that the fundamental groupoid of an E_2–algebra $\mathcal{A}$ naturally carries the structure of a braided monoidal category $(\Pi(\mathcal{A}), \oplus, b, 0)$ and the fundamental groupoid of an E_1–module $\mathcal{M}$ over $\mathcal{A}$ becomes a right-module $(\Pi(\mathcal{M}), \oplus)$ over it; see Section 2.1. In terms of this, we define in Section 4.1 a *coefficient system F for $\mathcal{M}$* with stabilising object X as an abelian group-valued functor F on $\Pi(\mathcal{M})$, together with a natural transformation $\sigma^F\colon F \to F(- \oplus X)$ for which the image of the canonical morphism $B_m \to \mathrm{Aut}_{\mathcal{A}}(X^{\oplus m})$ from the braid group on m strands acts trivially on the image of $(\sigma^F)^m\colon F \to F(- \oplus X^{\oplus m})$ for all n and m. Such a coefficient system enhances the stabilisation map to a map of spaces with local systems

$$(s; \sigma^F)\colon (\mathcal{M}_n; F) \to (\mathcal{M}_{n+1}; F)$$

by restricting F to subspaces of homogeneous degree. A coefficient system F induces a new one $\Sigma F = F(- \oplus X)$, called its *suspension*, which comes with a morphism $F \to \Sigma F$, named the *suspension map*; see Definition 4.3. The coefficient system F is inductively said to be *of degree r* if the kernel of the suspension map vanishes and the cokernel has degree $r-1$; the zero coefficient system having degree -1. In fact, we define a more general notion of being of *(split) degree r at N* such that F is of degree r in the sense just described if it is of degree r at 0; see Definition 4.1. This notion of a coefficient system of finite (split) degree generalises the one introduced by Randal-Williams and Wahl [61] for braided monoidal groupoids (see Remarks 4.11 and 4.12), which was itself inspired by work of Dwyer [20] and van der Kallen [41] on general linear groups, and work of Ivanov [38] on mapping class groups of surfaces.

Remark There is an alternative point of view on coefficient systems for $\mathcal{M}$, namely as abelian group-valued functors on a category $\langle\mathcal{M},\mathcal{B}\rangle$ constructed from the action of $\Pi(\mathcal{A})$ on $\Pi(\mathcal{M})$; see Remark 4.12.

Our second main theorem, demonstrated in Section 4.2, addresses homological stability of $\mathcal{M}$ with coefficients in a coefficient system of finite degree.

Theorem C *Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra with stabilising object X and F a coefficient system for $\mathcal{M}$ of degree r at $N \geq 0$. If the canonical resolution of $\mathcal{M}$ is graded $\frac{1}{k}(g_{\mathcal{M}}-2+k)$–connected in degrees ≥ 1 for some $k \geq 2$, then the map induced by stabilisation*

$$(s;\sigma^F)_*\colon \mathrm{H}_i(\mathcal{M}_n;F) \to \mathrm{H}_i(\mathcal{M}_{n+1};F)$$

is an isomorphism for $i \leq \frac{n-rk-k}{k}$ and an epimorphism for $i \leq \frac{n-rk}{k}$ when $n > N$. If F is of split degree r at $N \geq 0$ then $(s;\sigma^F)_*$ is an isomorphism for $i \leq \frac{n-r-k}{k}$ and an epimorphism for $i \leq \frac{n-r}{k}$ when $n > N$.

As a proof of concept, we apply the developed theory to three main classes of examples, to which we devote the remainder of this introduction.

Configuration spaces

The *unordered configuration space* $C_n^\pi(W)$ of a manifold with boundary W with labels in a Serre fibration $\pi\colon E \to W$ is the quotient of the *ordered configuration space*

$$F_n^\pi(W) = \{(e_1,\dots,e_n) \in E^n \mid \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j \text{ and } \pi(e_i) \in W \setminus \partial W\}$$

by the apparent action of the symmetric group Σ_n. If W is of dimension $d \geq 2$ and has nonempty boundary, then the union of its configuration spaces $\mathcal{M} = \coprod_{n\geq 0} C_n^\pi(W)$ admits the structure of an E_1–module over the E_2–algebra $\mathcal{A} = \coprod_{n\geq 0} C_n(D^d)$ of configurations in a d–disc, graded by the number of points; see Lemma 5.1. In Section 5.1, we identify its canonical resolution with the *resolution by arcs* — an augmented semi-simplicial space of geometric nature that has already been considered in the context of homological stability (see eg Kupers and Miller [43] and Miller and Wilson [48]) and is known to be sufficiently connected to apply Theorems A and C.

Theorem D *Let W be a connected manifold of dimension at least 2 with nonempty boundary and let $\pi\colon E \to W$ be a Serre fibration with path-connected fibres.*

(i) *For a local system L on $C^{\pi}_{n+1}(W)$, the stabilisation map*

$$s_*\colon \mathrm{H}_i(C^{\pi}_n(W); s^*L) \to \mathrm{H}_i(C^{\pi}_{n+1}(W); L)$$

is an isomorphism for $i \le \frac{1}{2}(n-1)$ and an epimorphism for $i \le \frac{1}{2}n$ if L is constant. It is an isomorphism for $i \le \frac{1}{3}(n-2)$ and an epimorphism for $i \le \frac{1}{3}n$ if L is abelian.

(ii) *If F is a coefficient system of degree r at $N \ge 0$, then the stabilisation map*

$$(s;\sigma^F)_*\colon \mathrm{H}_i(C^{\pi}_n(W); F) \to \mathrm{H}_i(C^{\pi}_{n+1}(W); F)$$

is an isomorphism for $i \le \frac{1}{2}(n-2r-2)$ and an epimorphism for $i \le \frac{1}{2}(n-2r)$ when $n > N$. If F is of split degree r at $N \ge 0$, then it is an isomorphism for $i \le \frac{1}{2}(n-r-2)$ and an epimorphism for $i \le \frac{1}{2}(n-r)$ when $n > N$.

Remark Employing the improvement of Remark 3.3, one obtains for constant coefficients a slightly better isomorphism range of $i \le \frac{1}{2}n$ than the one stated in Theorem D.

Configuration spaces have a longstanding history in the context of homological stability, starting with work of Arnold [1], who established stability for $C_n(D^2)$ with constant coefficients. McDuff [46] and Segal [65; 67] observed that this behaviour is not restricted to the 2–disc and proved stability for more general $C^{\pi}_n(W)$ with constant coefficients and $\pi = \mathrm{id}_W$, which can be extended to general π, eg by adapting the proof for a trivial fibration presented by Randal-Williams in [58]; see Cantero and Palmer [10] and Kupers and Miller [43] for alternative proofs.

As proved for example in [58], the stabilisation map for configuration spaces is in fact split injective in homology with constant coefficients in all degrees — a phenomenon special to configuration spaces, not captured by our general approach.

For a trivial fibration, stability of $C^{\pi}_n(W)$ with respect to a nontrivial coefficient system F was studied by Palmer [52], building on work of Betley [4] on symmetric groups. The second part of Theorem D extends his result to nontrivial fibrations and a significantly larger class of coefficient systems, partly conjectured by Palmer [52, Remark 1.5]; see Remark 5.12 for a more detailed comparison to his work. In the case of surfaces and a trivial fibration, a result similar to Theorem D, but with respect to a slightly smaller class of coefficient systems, is contained in work by Randal-Williams and Wahl [61, Theorem D].

In Section 5.2, we provide a discussion of coefficient systems for configuration spaces by relating them, for instance, to the theory of $\mathcal{FI}$–modules as introduced by Church, Ellenberg, and Farb [14] or to coefficient systems studied in [61]. These considerations provide numerous nontrivial coefficient systems F with respect to which the homology of $C_n^\pi(W)$ stabilises.

To our knowledge, stability with abelian coefficients for configuration spaces of manifolds of dimensions greater than two has not been considered so far. We next discuss a direct consequence of stability with respect to this class of coefficients as the first item in a series of applications exploiting Theorem D.

Oriented configuration spaces The *oriented configuration space* $C_n^{\pi,\mathrm{or}}(W)$ with labels in a Serre fibration π over W is the double cover of $C_n^\pi(W)$ given as the quotient of the ordered configuration space $F_n^\pi(W)$ by the action of the alternating group A_n, or equivalently, the space of labelled configurations ordered up to even permutations. By the space version of Shapiro's lemma, the homology of $C_n^{\pi,\mathrm{or}}(W)$ is isomorphic to $\mathrm{H}_*(C_n^\pi(W);\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}])$, with the action of $\pi_1(C_n^\pi(W))$ on the group ring $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ being induced by the composition of the sign homomorphism with the morphism $\pi_1(C_n^\pi(W)) \to \Sigma_n$, obtained by choosing an ordering of a basepoint. These local systems are abelian and are preserved by pulling back along the stabilisation map, hence homological stability for $C_n^{\pi,\mathrm{or}}(W)$ follows as a byproduct of Theorem D.

Corollary E *Let W and π be as in Theorem D. The map induced by stabilisation*

$$s_*\colon \mathrm{H}_i(C_n^{\pi,\mathrm{or}}(W);\mathbb{Z}) \to \mathrm{H}_i(C_{n+1}^{\pi,\mathrm{or}}(W);\mathbb{Z})$$

is an isomorphism for $i \le \frac{1}{3}(n-2)$ and an epimorphism for $i \le \frac{1}{3}n$.

Stability for oriented configuration spaces of connected orientable surfaces with nonempty boundary and without labels was proved by Guest, Kozlowsky, and Yamaguchi [33] using computations due to Bödigheimer, Cohen, Milgram, and Taylor [7; 8]. Palmer [51] extended this to manifolds of higher dimensions with nonempty boundary and labels in a trivial fibration. Corollary E gives an alternative proof of his result and enhances it by means of general labels and an improved stability range.

Configuration spaces of embedded discs The *configuration space* $C_n^k(W)$ *of unordered* k*–discs* in a connected d–manifold W is the quotient by the action of Σ_n on the *configuration space of ordered* k*–discs*

$$F_n^k(W) = \mathrm{Emb}\big(\textstyle\coprod^n D^k, W\backslash\partial W\big),$$

equipped with the $\mathcal{C}^\infty$–topology. For $k = d$ and oriented W, there are variants $F_n^{d,+}(W)$ and $C_n^{d,+}(W)$ by restricting to orientation-preserving embeddings. Mapping an embedding of a k–disc to its centre point, labelled with the k–frame induced by standard framing of D^k at the origin, results in a map $C_n^k(W) \to C_n^{\pi_k}(W)$, where π_k is the bundle of k–frames in M. This map can be seen to be a weak equivalence by choosing a metric and exponentiating frames. For $k < d$, the fibre of π_k is path-connected, so the homological stability results of Theorem D carry over to $C_n^k(W)$, comprising part of Corollary F below. Using the bundle π_d^+ of oriented d–framings, the argument for $C_n^{d,+}(W)$ is analogous, since the orientability condition ensures that the fibres of π_k^+ are path-connected.

The topological group of diffeomorphisms $\mathrm{Diff}_\partial(W)$ fixing a neighbourhood of the boundary in the $\mathcal{C}^\infty$–topology naturally acts on the configuration spaces $F_n^k(W)$ and $C_n^k(W)$, and the homotopy quotients $F_n^k(W)/\!\!/\mathrm{Diff}_\partial(W)$ and $C_n^k(W)/\!\!/\mathrm{Diff}_\partial(W)$ model the classifying spaces of the subgroups

$$\mathrm{PDiff}_{\partial,n}^k(W) \subseteq \mathrm{Diff}_{\partial,n}^k(W) \subseteq \mathrm{Diff}_\partial(W),$$

where $\mathrm{PDiff}_{\partial,n}^k(W)$ are the diffeomorphisms that fix n chosen embedded k–discs in W and $\mathrm{Diff}_{\partial,n}^k(W)$ are the ones permuting them; see Lemma 5.13. If W is orientable, the (sub)groups of orientation-preserving diffeomorphisms are denoted with a $+$ superscript. In Example 2.21, we explain how the canonical resolution of a graded E_1–module $\mathcal{M}$ over an E_2–algebra $\mathcal{A}$ relates to that of the E_1–module $EG \times_G \mathcal{M}$ over $\mathcal{A}$ in the presence of a graded action of a group G on $\mathcal{M}$ that commutes with the action of $\mathcal{A}$. An application of this consideration to the situation at hand implies the following, carried out in Section 5.3.1.

Corollary F *Let* W *be a* d*–dimensional manifold as in Theorem D and let* $0 \le k < d$.

(i) *For a local system* L, *the stabilisation maps*

$$\mathrm{H}_i(C_n^k(W); s^*L) \to \mathrm{H}_i(C_{n+1}^k(W); L),$$
$$\mathrm{H}_i(\mathrm{BDiff}_{\partial,n}^k(W); s^*L) \to \mathrm{H}_i(\mathrm{BDiff}_{\partial,n+1}^k(W); L)$$

are isomorphisms for $i \leq \frac{1}{2}(n-1)$ and epimorphisms for $i \leq \frac{1}{2}n$ if L is constant. If L is abelian, then they are isomorphisms for $i \leq \frac{1}{3}(n-2)$ and epimorphisms for $i \leq \frac{1}{3}n$.

(ii) *If F is a coefficient system of degree r at $N \geq 0$, then the maps*

$$\mathrm{H}_i(C_n^k(W); F) \to \mathrm{H}_i(C_{n+1}^k(W); F),$$

$$\mathrm{H}_i(\mathrm{BDiff}_{\partial,n}^k(W); F) \to \mathrm{H}_i(\mathrm{BDiff}_{\partial,n+1}^k(W); F)$$

induced by the stabilisation $(s; \sigma^F)$ are isomorphisms for $i \leq \frac{1}{2}(n-2r-2)$ and epimorphisms for $i \leq \frac{1}{2}(n-2r)$ when $n > N$. If F is of split degree r at $N \geq 0$, then they are isomorphisms for $i \leq \frac{1}{2}(n-r-2)$ and epimorphisms for $i \leq \frac{1}{2}(n-r)$ when $n > N$.

If W is oriented, then the analogous statements hold for $C_n^{d,+}(W)$ and $\mathrm{BDiff}_{\partial,n}^{d,+}(W)$.

Remark The isomorphism range for constant coefficients in the previous theorem can be improved to $i \leq \frac{1}{2}n$ by virtue of Remark 3.3.

For compact manifolds W, Tillmann [70] has proved homological stability with constant coefficients for variants of $\mathrm{BDiff}_{\partial,n}^0(W)$ and $\mathrm{BDiff}_{\partial,n}^{d,+}(W)$ involving diffeomorphisms that are only required to fix a disc in the boundary instead of the whole boundary. A Serre spectral sequence argument shows that stability for these variants follows from stability of the spaces $\mathrm{BDiff}_{\partial,n}^0(W)$ and $\mathrm{BDiff}_{\partial,n}^{d,+}(W)$. Hatcher and Wahl [36, Proposition 1.5] have shown stability with constant coefficients for the mapping class groups $\pi_0(\mathrm{Diff}_{\partial,n}^0(W))$, which can be seen to be equivalent to $\mathrm{Diff}_{\partial,n}^0(W)$ for compact 2–dimensional W as a result of the homotopy discreteness of the space of diffeomorphisms of a compact surface; see Earle and Eells [21] and Gramain [31]. In this case, stability with respect to some of the twisted coefficient systems Corollary F deals with is contained in work by Randal-Williams and Wahl [61, Theorem 5.22].

Representation stability The first rational homology group of the ordered configuration space of the 2–disc

$$\mathrm{H}_1(F_n(D^2); \mathbb{Q}) \cong \mathbb{Q}^{\binom{n}{2}},$$

as computed for example by Arnold [2], exemplifies that — in contrast to unordered configuration spaces — the homology of the ordered variant does not stabilise. However, by incorporating the action of the symmetric groups Σ_n, it does stabilise in a more refined, representation-theoretic sense. To make this precise, recall the correspondence between irreducible representations of Σ_n and partitions of n; see [26, Chapter 4]. We denote the irreducible $\Sigma_{|\lambda|}$–module corresponding to a partition $\lambda=(\lambda_1\geq\cdots\geq\lambda_k)\vdash|\lambda|$ of $|\lambda|$ by V_λ and define for $n\geq|\lambda|+\lambda_1$, the *padded partition* $\lambda[n]=(n-|\lambda|\geq\lambda_1\geq\cdots\geq\lambda_k)\vdash n$. Using the Totaro spectral sequence [72], Church [12] has shown that for a connected orientable manifold of dimension at least two with finite-dimensional rational cohomology, the groups $\mathrm{H}^i(F_n(W);\mathbb{Q})$ are *uniformly representation stable* — a concept introduced by Church and Farb [16]. This implies the existence of a constant $N(i)$, depending solely on i, such that the multiplicity of $V_{\lambda[n]}$ in the Σ_n–module $\mathrm{H}^i(F_n(W);\mathbb{Q})$ is independent of n for $n\geq N(i)$. Church's result has been extended in several directions; see Church, Ellenberg, and Farb [14], Miller and Wilson [48], Petersen [56], and Tosteson [71].

A twisted Serre spectral sequence argument (see Lemma 5.14) shows that the multiplicity of an irreducible Σ_n–module V_λ in $\mathrm{H}^i(F_n^\pi(W);\mathbb{Q})$ agrees with the dimension of $\mathrm{H}_i(C_n^\pi(W);V_\lambda)$, where $\pi_1(C_n^\pi(W))$ acts on V_λ via the morphism $\pi_1(C_n^\pi(W))\to\Sigma_n$. This fact allows us to derive the stability of these multiplicities from Theorem D, at least for all manifolds to which the latter theorem applies; see Section 5.3.2.

Corollary G *Let W and π be as in Theorem D and let Z_n be one of the following sequences of Σ_n–spaces:*

(i) $F_n^\pi(W)$.

(ii) $F_n^k(W)$ *for* $0\leq k<d$.

(iii) $F_n^{d,+}(W)$ *if W is oriented.*

(iv) $\mathrm{BPDiff}^k_{\partial,n}(W)$ *for* $0\leq k<d$.

(v) $\mathrm{BPDiff}^{d,+}_{\partial,n}(W)$ *if W is orientable.*

The $V_{\lambda[n]}$–multiplicity in $\mathrm{H}^i(Z_n;\mathbb{Q})$ for a fixed partition λ is independent of n for n large relative to i.

In Remark 5.16, we discuss explicit ranges for Corollary G and compare them to Church's. Let us at this juncture record that our approach leads to ranges that depend on $|\lambda|$, so we do not recover *uniform* representation stability. On the other hand, in contrast to Church's result, we neither require W to be orientable nor to have finite-dimensional rational cohomology or π to be the identity.

Jiménez Rolland [39; 40] has shown uniform representation stability for the cohomology groups $\mathrm{H}^i(\mathrm{BPDiff}^0_{\partial,n}(W);\mathbb{Q})$ for compact orientable surfaces and for compact connected manifolds W of dimension $d \geq 3$, assuming that $\mathrm{BDiff}_\partial(W)$ has the homotopy type of a CW–complex of finite type. Furthermore, she proved uniform representation stability for $\pi_0(\mathrm{PDiff}^0_{\partial,n}(W))$ for compact orientable surfaces, as well as for higher-dimensional manifolds under some further assumptions.

Moduli spaces of manifolds

The moduli space $\mathcal{M}$ of compact d–dimensional smooth manifolds with a fixed boundary P forms an E_1–module over the E_d–algebra $\mathcal{A}$ given by the moduli space of compact d–manifolds with a sphere as boundary; see Lemma 6.1. The homotopy types of $\mathcal{M}$ and $\mathcal{A}$ are

$$\mathcal{M} \simeq \coprod_{[W]} \mathrm{BDiff}_\partial(W) \quad \text{and} \quad \mathcal{A} \simeq \coprod_{[N]} \mathrm{BDiff}_\partial(N),$$

where $[W]$ runs over diffeomorphism classes relative to P of compact d–manifolds with P–boundary and $[N]$ over the ones of compact d–manifolds with a sphere as boundary. Acting with a manifold $X \in \mathcal{A}$ on $\mathcal{M}$ corresponds to taking the boundary connected sum $(- \natural X)$ with X, so the resulting stabilisation map thus restricts on path components to a map of the form

$$s\colon \mathrm{BDiff}_\partial(W) \to \mathrm{BDiff}_\partial(W \natural X), \tag{2}$$

which models the map induced by extending diffeomorphisms by the identity.

As shown in Section 6.1, the canonical resolution of $\mathcal{M}$ with respect to a choice of a stabilising manifold X is equivalent to the *resolution by embeddings* — an augmented semi-simplicial space of submanifolds $W \in \mathcal{M}$, together with embeddings of X with a fixed behaviour near their boundary. For specific manifolds X and W, this resolution and its connectivity has been studied to prove homological stability of (2), first by Galatius and Randal-Williams [28] for $X \cong D^{2p} \sharp (S^p \times S^p)$ and simply connected $2p$–dimensional W with $p \geq 3$. Their results extend the classical stability result for mapping class groups of surfaces by Harer [34] to higher dimensions. As in Harer's theorem, the known connectivity of the resolution by embeddings, and hence the resulting stability ranges, depend on the *X–genus* of W,

$$g^X(W) = \max\{k \geq 0 \mid \text{there exists } M \in \mathcal{M} \text{ such that } M \natural X^{\natural k} \cong W \text{ relative to } P\},$$

which incidentally provides a method of grading E_1–modules $\mathcal{M}$ in general; see Section 2.3. Perlmutter [54] succeeded in carrying out this strategy in the case $X \cong D^{p+q} \sharp (S^p \times S^q)$ with certain $p \neq q$ depending on which W is required to satisfy a connectivity assumption. Recently, Friedrich [25] extended the work of Galatius and Randal-Williams to manifolds W with nontrivial fundamental group in terms of the *unitary stable rank* (see [49, Definition 6.3]) of the group ring $\mathbb{Z}[\pi_1(W)]$. These connectivity results can be restated in our context as graded connectivity for the canonical resolution of $\mathcal{M}$ with respect to different gradings (see Corollary 6.7), allowing us to apply Theorems A and C.

Employing the improvement of Remark 3.3, the ranges with constant and abelian coefficients obtained from Theorem A agree with the ones established in [25; 28; 54] (after extending [54] to abelian coefficients by adapting the methods of [28]). The cancellation result for connected sums of manifolds that we derive from Corollary B coincides with their cancellation results as well. Our main contribution with respect moduli spaces of manifolds lies in the application of Theorem C, ie homological stability with respect to a large class of nontrivial coefficient systems, which has not yet been considered in the context of moduli spaces of high-dimensional manifolds. On path components, it reads as follows.

Theorem H *Let W be a compact $(p+q)$–manifold with nonempty boundary and F a coefficient system of degree r. Denote by $g(W)$ the $(S^p\times S^q)$–genus of W, and set u to be 1 if W is simply connected and to be the unitary stable rank of $\mathbb{Z}[\pi_1(W)]$ otherwise. The stabilisation map*

$$(s,\sigma^F)_*\colon \mathrm{H}_i(\mathrm{BDiff}_\partial(W);F)\to \mathrm{H}_i(\mathrm{BDiff}_\partial(W\sharp(S^p\times S^q));F)$$

is

(i) *an isomorphism for $i\le \frac{1}{2}(g(W)-2r-u-3)$ and an epimorphism for $i\le \frac{1}{2}(g(W)-2r-u-1)$ if $p=q\ge 3$, and*

(ii) *an isomorphism for $i\le \frac{1}{2}(g(W)-2r-m-4)$ and an epimorphism for $i\le \frac{1}{2}(g(W)-2r-m-2)$ if W is $(q-p+2)$–connected and $0<p<q<2p-2$, where $m=\min\{i\in\mathbb{N}_0\mid \text{there exists an epimorphism } \mathbb{Z}^i\to\pi_q(S^p)\}$.*

If F is (split) of degree r at a number $N\ge 0$, the ranges in Theorem H change as per Theorem C.

Remark The unitary stable rank [49, Definition 6.3] of a group ring $\mathbb{Z}[G]$ need not be finite. To provide a class of examples of finite unitary stable rank, recall that G is called *virtually polycyclic* if there is a series $1=G_0\subseteq G_1\subseteq\cdots\subseteq G_n=G$ such that G_i is normal in G_{i+1} and the quotients G_{i+1}/G_i are either finite or cyclic. Its *Hirsch length*

$h(G)$ is the number of infinite cyclic factors. Crowley and Sixt [18, Theorem 7.3] showed $\mathrm{usr}(\mathbb{Z}[G]) \leq h(G)+3$ for virtually polycyclic groups G. In particular, we have $\mathrm{usr}(\mathbb{Z}[G]) \leq 3$ for finite groups and $\mathrm{usr}(\mathbb{Z}[G]) \leq \mathrm{rank}(G)+3$ for finitely generated abelian groups.

In Remark 6.8, we briefly elaborate on how to include the case of orientable surfaces in this picture by utilising high-connectivity of the *complex of tethered chains* — a result of Hatcher and Vogtmann [35]. For constant coefficients, this implies Harer's classical stability theorem [34] with a better, but not optimal range; see Boldsen [9] and Randal-Williams [59]. For twisted coefficients, it extends a result by Ivanov [38] to more general coefficient systems. However, in the case of surfaces, stability with respect to most of these coefficient systems was already known by Randal-Williams and Wahl [61].

In Section 6.2, we show that coefficient systems for $\mathcal{M}$ are equivalent to certain families of modules over the mapping class groups $\pi_0(\mathrm{Diff}_\partial(W)) \cong \pi_1(\mathrm{BDiff}_\partial(W))$ and explain how the action of the mapping class groups on the homology of the manifolds gives rise to a coefficient system of degree 1 for $\mathcal{M}$. This yields the following corollary.

Corollary I *Let W be a compact $(p+q)$–manifold with nonempty boundary and $k \geq 0$. The stabilisation*

$$\mathrm{H}_i(\mathrm{BDiff}_\partial(W);\mathrm{H}_k(W)) \to \mathrm{H}_i\big(\mathrm{BDiff}_\partial(W \sharp (S^p \times S^q));\mathrm{H}_k(W \sharp (S^p \times S^q))\big)$$

is an epimorphism and isomorphism for the same W as in Theorem H and with the same ranges, after replacing r by 1.

Furthermore, in Section 6.3, we provide a short discussion of how our methods can be applied to the case of certain stably parallelisable $(2n-1)$–connected $(4n+1)$–manifolds X and 2–connected W, extending stability results by Perlmutter [55]. Similarly, we also briefly explain how to enhance work of Kupers [42] on homeomorphisms of topological manifolds and automorphisms of piecewise linear manifolds.

Modules over braided monoidal categories

We close in Section 7 by explaining applicability of our results to discrete situations, such as groups or monoids, and by drawing a comparison to [61].

The classifying space $\mathrm{B}\mathcal{M}$ of a graded module $\mathcal{M}$ over a braided monoidal category is a graded E_1–module over an E_2–algebra (see Lemma 7.2), so forms a suitable input for Theorems A and C. In Lemma 7.6, we identify the space of destabilisations $W_\bullet(A)$ of $A \in \mathcal{M}$ with a semi-simplicial set $W_\bullet^{\mathrm{RW}}(A)$ in the case of $\mathcal{M}$ being a groupoid satisfying an injectivity condition. This identification gives rise to a framework for homological stability for modules over braided monoidal categories, phrased entirely in terms of $\mathcal{M}$ and semi-simplicial sets instead of semi-simplicial spaces up to higher coherent homotopy; see Remark 7.8.

Using this, it can, for instance, be concluded that work of Hepworth [37] on homological stability for Coxeter groups with constant coefficients implies their stability with respect to a large class of nontrivial coefficient systems without further effort, as well as stability of their commutator subgroups.

In the case of a braided monoidal groupoid acting on itself, the semi-simplicial sets $W_\bullet^{\mathrm{RW}}(A)$ were introduced by Randal-Williams and Wahl in [61] as part of their stability results for the automorphisms of a braided monoidal groupoid, which this work enhances in various ways. We generalise from braided monoidal groupoids to modules over such, remove all hypotheses on the categories they impose, improve the stability ranges in certain cases (see Remark 7.10), and enlarge the class of coefficient systems (see Remark 7.9). We refer to Section 7.3 for a more detailed comparison of our results in the discrete setting to [61] and also for an analysis of their assumptions on the braided monoidal groupoid.

Acknowledgements I wish to express my gratitude to Nathalie Wahl for her overall support, to Oscar Randal-Williams for enlightening discussions and his hospitality at the University of Cambridge, and to Søren Galatius as well as to Alexander Kupers for valuable comments.

I was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92) and by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 682922).

1 Preliminaries

This section is devoted to fixing conventions and collecting general techniques. We work in the category of compactly generated spaces, use Moore paths throughout, and denote the endpoint of a path μ by $\omega(\mu)$.

1.1 Graded spaces and categories

We denote by $(\overline{\mathbb{N}},+)$ the discrete abelian monoid obtained by extending the nonnegative integers $(\mathbb{N}_0,+)$ by an element ∞ satisfying $k+\infty=\infty$ for all $k\geq 0$.

A *graded space* is a space X together with a continuous map $g_X\colon X \to \overline{\mathbb{N}}$. A *map of graded spaces* is a map that preserves the grading and a *map of degree k between graded spaces* for a number $k \geq 0$ is a map that increases the degree by k. The category of graded spaces is symmetric monoidal, with the monoidal product of two graded spaces (X, g_X) and (Y, g_Y) given by $(X \times Y, g_X + g_Y)$. The subspace of elements of degree $n \in \overline{\mathbb{N}}$ is denoted by $X_n = g_X^{-1}(\{n\}) \subseteq X$. By restricting the grading, subspaces of graded spaces are implicitly considered as being graded. A graded space (X, g_X), or a map $(Y, g_Y) \to (X, g_X)$ of graded spaces, is *$\varphi(g_X)$–connected in degrees $\geq m$* for a function $\varphi\colon \overline{\mathbb{N}} \to \mathbb{Q} \cup \{\infty\}$ satisfying $\phi(\infty) = \infty$ if X_n or $Y_n \to Z_n$, respectively, is $\lfloor \varphi(n) \rfloor$–connected for all $m \leq n < \infty$ in the usual sense. Note that we do not require anything on X_∞ or $Y_\infty \to X_\infty$.

A *graded set X* is a graded space that is discrete. A *graded category $\mathcal{C}$* is a category internal to graded sets, ie a category $\mathcal{C}$ with a function $g_{\mathcal{C}}\colon \operatorname{ob}\mathcal{C} \to \overline{\mathbb{N}}$ whose value on objects that are connected by morphisms is constant. This is equivalent to a grading on the classifying space $\mathrm{B}\mathcal{C}$. A *graded monoidal category* is a monoid internal to graded categories with the monoidal product $(\mathcal{C}, g_{\mathcal{C}}) \times (\mathcal{D}, g_{\mathcal{D}}) = (\mathcal{C} \times \mathcal{D}, g_{\mathcal{C}} + g_{\mathcal{D}})$, ie a monoidal category $(\mathcal{A}, \oplus, 0)$ together with a grading $g_{\mathcal{A}}$ on $\mathcal{A}$ that satisfies $g_{\mathcal{A}}(0) = 0$ and $g_{\mathcal{A}}(X \oplus Y) = g_{\mathcal{A}}(X) + g_{\mathcal{A}}(Y)$. A *graded right-module* $(\mathcal{M}, \oplus)$ over a graded monoidal category $(\mathcal{A}, \oplus, 0)$ is a graded category $\mathcal{M}$ together with a right-action of $(\mathcal{A}, \oplus, 0)$ on $\mathcal{D}$ internal to graded categories, ie a functor $\oplus\colon \mathcal{M} \times \mathcal{A} \to \mathcal{M}$ which is unital and associative up to coherent isomorphisms, and satisfies $g_{\mathcal{M}}(A \oplus X) = g_{\mathcal{M}}(A) + g_{\mathcal{M}}(X)$.

1.2 Homology with local coefficients

We adopt the convention of [76, Chapter VI]: for points x and y in a space X, a morphism in the fundamental groupoid $\Pi(X)$ from x to y is a homotopy class of paths from y to x, resulting in the fundamental group $\pi_1(X, x)$ being a subgroupoid of $\Pi(X)$. A *local system* on a pair of spaces (X, A) with $A \subseteq X$ is a functor F from the fundamental groupoid $\Pi(X)$ of X to the category of abelian groups. It is *constant* if it is constant as a functor. For a path-connected space X, local systems can equivalently be described as modules over $\pi_1(X, x)$, since the fundamental groupoid $\Pi(X)$ is equivalent to the one-object groupoid $\pi_1(X, x)$. Subspaces of spaces with local systems are implicitly equipped with the local system obtained by restriction along the inclusion. When we write (X, A) for a map $A \to X$ that is not necessarily an inclusion, we implicitly replace X by the mapping cylinder of $A \to X$. A *morphism $(f; \eta)$ between pairs with local systems* $(X, A; F)$ and $(Y, B; G)$ is a map of pairs

$f\colon (X,A)\to(Y,B)$ with a natural transformation $\eta\colon F\to f^*G$ of functors on $\Pi(X)$. A *homotopy* between $(f_0;\eta_0)$ and $(f_1;\eta_1)$ from $(X,A;F)$ to $(Y,B;G)$ consists of a homotopy of pairs $H_t\colon (X,A)\to(Y,B)$ from f_0 and f_1 such that

(3)
$$\begin{tikzcd} & G(f_1(-)) \arrow[d, "G(H_t(-))"] \\ F(-) \arrow[ur, "\eta_1"] \arrow[dr, "\eta_0"'] & \\ & G(f_0(-)) \end{tikzcd}$$

commutes. Taking singular chains with coefficients in a local system provides a homotopy-invariant functor $\mathrm{C}_*(-)$ from pairs with local systems to chain complexes. The homology $\mathrm{H}_*(X,A;F)$ of $\mathrm{C}_*(X,A;F)$ is the *homology of the pair* (X,A) *with coefficients in the local system* F. A grading on X results in an additional grading $\bigoplus_{n\in\overline{\mathbb{N}}}\mathrm{H}_*(X_n,A_n;F)$ on $\mathrm{H}_*(X,A;F)$. For a morphism $(X,A;F)\to(Y,B;G)$, the homology of the mapping cone of $\mathrm{C}_*(X,A;F)\to\mathrm{C}_*(Y,B;G)$ is denoted by $\mathrm{H}_*((Y,B;G),(X,A;F))$. If X and Y are graded and the underlying map $X\to Y$ is of degree k, then $\mathrm{H}_*((Y,B;G),(X,A;F))$ inherits an extra grading

$$\mathrm{H}_*((Y,B;G),(X,A;F))=\bigoplus_{n\in\overline{\mathbb{N}}}\mathrm{H}_*((Y_{n+k},B_{n+k};G),(X_n,A_n;F)).$$

We refer to [76, Chapter VI] for more details on homology with local coefficients.

1.3 Augmented semi-simplicial spaces

Denoting by $[p]$ the ordered set $\{0,1,\dots,p\}$, the *semi-simplicial category* is the category Δ_{inj} with objects $[0],[1],\dots$ and order-preserving injections between them. A *semi-simplicial space* $X_\bullet$ is a space-valued functor on $\Delta_{\mathrm{inj}}^{\mathrm{op}}$, or equivalently, a collection of spaces X_p for $p\geq 0$, together with *face maps* $d_i\colon X_p\to X_{p-1}$ for $0\leq i\leq p$ that satisfy the *face relations* $d_id_j=d_{j-1}d_i$ for $i<j$. An *augmented semi-simplicial space* $X_\bullet\to X_{-1}$ is a semi-simplicial space $X_\bullet$ with maps $X_p\to X_{-1}$ for $p\geq 0$ that commute with the face maps. As for simplicial spaces, augmented semi-simplicial spaces $X_\bullet\to X_{-1}$ have a *geometric realisation*: a space over X_{-1}, denoted by $|X_\bullet|\to X_{-1}$; see [22, Section 1.2].

Given an augmented semi-simplicial space $X_\bullet\to X_{-1}$ and a local system F on X_{-1}, we obtain local systems on the spaces of p–simplices X_p and on the realisation $|X_\bullet|$ by pulling back F along the augmentation. Filtering $|X_\bullet|$ by skeleta induces a strongly convergent homologically graded spectral sequence

$$E^1_{p,q} \cong \mathrm{H}_q(X_p; F) \Rightarrow \mathrm{H}_{p+q+1}(X_{-1}, |X_\bullet|; F) \tag{4}$$

defined for $q \geq 0$ and $p \geq -1$; see [22, Section 1.4] and [47, Lemma 2.7]. The differential $d^1\colon \mathrm{H}_q(X_p; F) \to \mathrm{H}_q(X_{p-1}; F)$ is the alternating sum $\sum_{i=0}^{p}(-1)^i(d_i; \mathrm{id})_*$ of the morphisms induced by the face maps for $p > 0$, and induced by the augmentation for $p = 0$. Given a morphism of augmented semi-simplicial spaces

$$(f_\bullet, f_{-1})\colon (X_\bullet \to X_{-1}) \to (Y_\bullet \to Y_{-1}), \tag{5}$$

local systems F on X_{-1} and G on Y_{-1}, and a morphism of local systems $F \to {f_{-1}}^*G$, we obtain a morphism of augmented semi-simplicial objects in spaces with local systems, resulting in a relative version of the spectral sequence (4),

$$E^1_{p,q} \cong \mathrm{H}_q((Y_p; G), (X_p; F)) \Rightarrow \mathrm{H}_{p+q+1}((Y_{-1}, |Y_\bullet|; G), (X_{-1}, |X_\bullet|; F)). \tag{6}$$

If X_{-1} is graded, all spaces X_p and $|X_\bullet|$ inherit a grading by pulling back $g_{X_{-1}}$ along the augmentation. This results in a third grading of the spectral sequence (4), but since the differentials preserve the additional grading, it is just a sum of spectral sequences, one for each $n \in \overline{\mathbb{N}}$. Analogously, if the map f_{-1} of (5) is a map of degree k for gradings on X_{-1} and Y_{-1}, the spectral sequence (6) splits as a sum, and the n^{th} summand of the E_1–page being $E^1_{p,q,n} \cong \mathrm{H}_q((Y_{p,n+k}; G), (X_{p,n}; F))$.

1.4 $\mathcal{C}$–spaces and their rectification

We set up an ad hoc theory of spaces parametrised by a topologically enriched category, serving us as a convenient language in the body of this work.

We call an enriched space-valued functor $X_\bullet$ on a topologically enriched category $\mathcal{C}$ a $\mathcal{C}$–*space*, and write X_C for its value at an object C. An *augmentation* $f_\bullet\colon X_\bullet \to X_{-1}$ of a $\mathcal{C}$–space $X_\bullet$ over a space X_{-1} is a lift of $X_\bullet$ to a functor with values in the overcategory $\mathcal{T}op/X_{-1}$, and an *augmented* $\mathcal{C}$*–space* is a $\mathcal{C}$–space together with an augmentation. We denote the value of an augmented $\mathcal{C}$–space $f_\bullet\colon X_\bullet \to X_{-1}$ at an object C by $f_C\colon X_C \to X_{-1}$. A *morphism of augmented* $\mathcal{C}$*–spaces* is a natural transformation of functors $\mathcal{C} \to \mathcal{T}op/X_{-1}$, and it is called a *weak equivalence* if it is a weak equivalence objectwise. A morphism between a $\mathcal{C}$–space $X_\bullet$ augmented over X_{-1} and a $\mathcal{C}$–space $Y_\bullet$ over Y_{-1} consists of a map $h\colon X_{-1} \to Y_{-1}$ and a morphism $h_\bullet\colon h_*(X_\bullet) \to Y_\bullet$ of $\mathcal{C}$–spaces augmented over Y_{-1}, where $h_*(X_\bullet)$ denotes $X_\bullet$ considered augmented over Y_{-1} via h. Such a morphism is a *weak equivalence* if h is a weak equivalence of spaces and $h_\bullet$ is one of $\mathcal{C}$–spaces over Y_{-1}. An augmented $\mathcal{C}$–space $f_\bullet$ is *fibrant* if all maps f_C are Serre fibrations.

Example 1.1 For $\mathcal{C}$ the opposite of the semi-simplicial category, the notion of a $\mathcal{C}$–space agrees with the one of a semi-simplicial spaces; see Section 1.3. This example motivated our choice of notation.

Definition 1.2 The *fibrant replacement* of an augmented $\mathcal{C}$–space $X_\bullet \to X_{-1}$ is the augmented $\mathcal{C}$–space $X_\bullet^{\mathrm{fib}} \to X_{-1}$ obtained by applying the path-space construction objectwise,

$$X_C^{\mathrm{fib}} = \{(x,\mu) \in X_C \times \operatorname{Path} X_{-1} \mid \omega(\mu) = f_C(x)\},$$

considered as a space over X_{-1} by evaluating paths at zero. It is fibrant and admits a canonical weak equivalence $X_\bullet \to X_\bullet^{\mathrm{fib}}$ of augmented $\mathcal{C}$–spaces, given by mapping $x \in X_C$ to $(x, \mathrm{const}_{f_C(x)}) \in X_C^{\mathrm{fib}}$.

The *fibre* $X_{x,\bullet}$ of an augmented $\mathcal{C}$–space $f_\bullet\colon X_\bullet \to X_{-1}$ at $x \in X_{-1}$ is the $\mathcal{C}$–space that assigns to an object C the fibre $X_{x,C} = f_C^{-1}(x)$. Its *homotopy fibre* $\mathrm{hofib}_x(X_\bullet)$ at x is the fibre of $X_\bullet^{\mathrm{fib}} \to X_{-1}$ at x. If $X_\bullet \to X_{-1}$ is fibrant, then the weak equivalence $X_\bullet \to X_\bullet^{\mathrm{fib}}$ induces a weak equivalence $X_{x,\bullet} \to \mathrm{hofib}_x(X_\bullet)$.

Definition 1.3 Let $\mathcal{C}$ be a small topologically enriched category.

(i) The *bar construction* $\mathrm{B}(Y_\bullet, \mathcal{C}, X_\bullet)$ of a pair of $\mathcal{C}$–spaces $(X_\bullet, Y_\bullet)$, where $X_\bullet$ is co- and $Y_\bullet$ is contravariant, is the realisation of the semi-simplicial space $\mathrm{B}_\blacksquare(Y_\bullet, \mathcal{C}, X_\bullet)$ with p–simplices

$$\textstyle\coprod_{C_0,\dots,C_p \in \mathrm{ob}\,\mathcal{C}} X_{C_0} \times \mathcal{C}(C_0,C_1) \times \cdots \times \mathcal{C}(C_{p-1},C_p) \times Y_{C_p}.$$

The i^{th} face map is induced by composing morphisms in $\mathcal{C}(C_{i-1},C_i)$ and $\mathcal{C}(C_i,C_{i+1})$ for $1 \le i \le p-1$, and by the evaluations $X_{C_0} \times \mathcal{C}(C_0,C_1) \to X_{C_1}$ and $\mathcal{C}(C_{p-1},C_p) \times Y_{C_p} \to X_{C_{p-1}}$ for $i = p-1$ and $i = p$, respectively. An augmentation $X_\bullet \to X_{-1}$ naturally induces a map $\mathrm{B}(Y_\bullet, \mathcal{C}, X_\bullet) \to X_{-1}$.

(ii) The *homotopy colimit*

$$\operatorname{hocolim}_{\mathcal{C}} X_\bullet \to X_{-1}$$

of an augmented $\mathcal{C}$–space $X_\bullet \to X_{-1}$ is the bar construction $\mathrm{B}(*, \mathcal{C}, X_\bullet) \to X_{-1}$.

A $\mathcal{C}$–space is *k–connected* for a number $k \ge 0$ if its homotopy colimit is so. If the base X_{-1} of an augmented $\mathcal{C}$–space $X_\bullet \to X_{-1}$ is graded, then its values X_C and its homotopy colimit inherit gradings by pulling back $g_{X_{-1}}$ from X_{-1}. It is *graded $\varphi(g_{X_{-1}})$–connected in degrees $\geq m$* for $\varphi\colon \overline{\mathbb{N}} \to \mathbb{Q} \cup \{\infty\}$ if $\mathrm{hocolim}_{\mathcal{C}} X_\bullet \to X_{-1}$ is. A functor between topologically enriched categories is a *weak equivalence* if it induces weak equivalences on morphism spaces and a bijection on the set of objects. Note that this notion of weak equivalence is slightly stronger than the usual one. With this choice, it is immediate to see that the map on bar constructions induced by a weak equivalence $(X_\bullet, \mathcal{C}, Y_\bullet) \to (X'_\bullet, \mathcal{C}', Y'_\bullet)$ of triples, defined in the appropriate sense, is a weak equivalence, since levelwise weak equivalences of semi-simplicial spaces realise to weak equivalences; see eg [22, Theorem 2.2]. In particular, taking homotopy colimits turns weak equivalences of $\mathcal{C}$–spaces augmented over X_{-1} into weak equivalences of spaces over X_{-1}.

Lemma 1.4 *Let $X_\bullet \to X_{-1}$ be an augmented $\mathcal{C}$–space and $x \in X_{-1}$. The canonical map*

$$\mathrm{hocolim}_{\mathcal{C}}(\mathrm{hofib}_x(X_\bullet \to X_{-1})) \to \mathrm{hofib}_x(\mathrm{hocolim}_{\mathcal{C}} X_\bullet \to X_{-1})$$

is a weak equivalence.

Proof We show that the map in consideration is even a homeomorphism, provided X_{-1} is a weak Hausdorff space. This implies the claim, since the two functors in comparison both preserve weak equivalences of augmented $\mathcal{C}$–spaces and every augmented $\mathcal{C}$–space $X_\bullet \to X_{-1}$ can be replaced, up to weak equivalence, by one over a weak Hausdorff space, for instance by pulling back the fibrant replacement of $X_\bullet \to X_{-1}$ along a CW–approximation of X_{-1}. We have

$$\begin{aligned} \mathrm{hocolim}_{\mathcal{C}}(\mathrm{hofib}_x(X_\bullet \to X_{-1})) &= |\mathrm{B}_\blacksquare(*, \mathcal{C}, \mathrm{hofib}_x(X_\bullet \to X_{-1}))|, \\ \mathrm{hofib}_x(\mathrm{hocolim}_{\mathcal{C}} X_\bullet \to X_{-1}) &= \mathrm{hofib}_x(|\mathrm{B}_\blacksquare(*, \mathcal{C}, X_\bullet \to X_{-1})|), \end{aligned}$$

so the statement follows from proving that both the bar construction $\mathrm{B}_\blacksquare(*, \mathcal{C}, -)$ as well as the geometric realisation $|-|$ commute with taking homotopy fibres $\mathrm{hofib}_x(-)$. Unwrapping the definitions of $\mathrm{B}_\blacksquare(*, \mathcal{C}, -)$ and $|-|$, these two claims are implied by the fact that the functor $\mathrm{hofib}_x(-)\colon \mathcal{T}\!op/X_{-1} \to \mathcal{T}\!op$ commutes with colimits and also with taking products $- \times Z$ with a fixed space Z. The latter is clear, and the former follows from the fact that the functor $\omega^*\colon \mathcal{T}\!op/X_{-1} \to \mathcal{T}\!op/(\mathrm{Path}_x X_{-1})$ given by pulling back the path fibration $\omega\colon \mathrm{Path}_x X_{-1} \to X_{-1}$ is a left adjoint [45, Proposition 2.1.3], so preserves colimits, together with the observation that the forgetful functor from $\mathcal{T}\!op/X_{-1}$ to $\mathcal{T}\!op$ is colimit-preserving as well. □

For an augmented $\mathcal{C}$–space $X_\bullet \to X_{-1}$, the composition in $\mathcal{C}$ and the evaluation maps $X'_C \times \mathcal{C}(C', C) \to X_C$ combine to augmentations $B_\bullet(\mathcal{C}(\blacksquare, C), \mathcal{C}, X_\blacksquare) \to X_C$ for each C in $\mathcal{C}$, which realise to weak equivalences as they admit extra degeneracies by inserting the identity; see eg [22, Theorem 2.2]. These equivalences are natural in C and compatible with the augmentation to X_{-1}, so assemble to a weak equivalence

$$B(\mathcal{C}(\blacksquare, \bullet), \mathcal{C}, X_\blacksquare) \to X_\bullet$$

of augmented $\mathcal{C}$–spaces — the *bar resolution* of $X_\bullet \to X_{-1}$.

Lemma 1.5 *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a weak equivalence of topologically enriched categories. There is a functor*

$$p_*\colon (\mathcal{T}op/X_{-1})^{\mathcal{C}} \to (\mathcal{T}op/X_{-1})^{\mathcal{D}}$$

that fits into a zigzag

$$p^* p_* \leftarrow \cdot \to \mathrm{id}_{(\mathcal{T}op/X_{-1})^{\mathcal{C}}}$$

of natural transformations between endofunctors on $(\mathcal{T}op/X_{-1})^{\mathcal{C}}$, where the map $p^*\colon (\mathcal{T}op/X_{-1})^{\mathcal{D}} \to (\mathcal{T}op/X_{-1})^{\mathcal{C}}$ is induced by precomposition with p. When evaluated at an augmented $\mathcal{C}$–space, the zigzag consists of weak equivalences.

Proof The value $p_*X_\bullet$ for $X_\bullet \in (\mathcal{T}op/X_{-1})^{\mathcal{C}}$ is the homotopy left Kan-extension of $X_\bullet$ along p, mapping an object D in $\mathcal{D}$ to $\mathrm{B}(\mathcal{D}(p(\blacksquare), D), \mathcal{C}, X_\blacksquare)$. Its pullback $p^* p_* X_\bullet$ fits into a zigzag of augmented $\mathcal{C}$–spaces

$$p^* p_* X_\bullet = \mathrm{B}(\mathcal{D}(p(\blacksquare), p(\bullet)), \mathcal{C}, X_\blacksquare) \leftarrow \mathrm{B}(\mathcal{C}(\blacksquare, \bullet), \mathcal{C}, X_\blacksquare) \to X_\bullet,$$

in which the left arrow is induced by p and the right one is the bar resolution of $X_\bullet$, so both are weak equivalences and compatible with the augmentation. As the zigzag is natural in $X_\bullet$, the claim follows. □

Lemma 1.6 *The homotopy colimit of an augmented semi-simplicial space $X_\bullet \to X_{-1}$ and its geometric realisation are weakly equivalent as spaces over X_{-1}.*

Proof The classifying space of the overcategory $\Delta_{\mathrm{inj}}/[p]$ is isomorphic to the p^{th} topological standard simplex Δ^p, since the nerve of $\Delta_{\mathrm{inj}}/[p]$ is the barycentric subdivision of the p^{th} simplicial standard simplex. This extends to an isomorphism $\Delta^\bullet \cong B(\Delta_{\mathrm{inj}}/\bullet)$ of co-semi-simplicial spaces from which [62, Theorem 6.6.1] implies that, given an augmented semi-simplicial space $X_\bullet \to X_{-1}$, the thin realisation (see [22, Section 1.2]) of $\mathrm{B}_\bullet(*, \Delta_{\mathrm{inj}}^{\mathrm{op}}, X_\bullet)$, considered as a simplicial space, is homeomorphic over X_{-1} to the realisation of $X_\bullet$. But for augmented $\mathcal{C}$–spaces $X_\bullet \to X_{-1}$ on a discrete category $\mathcal{C}$, the fat and the thin geometric realisation of $\mathrm{B}_\bullet(*, \mathcal{C}, X)$ are weakly equivalent over X_{-1}, because $\mathrm{B}_\bullet(*, \mathcal{C}, X)$ is *good* in the sense of [66, Proposition A.1]. $\square$

1.5 Semi-simplicial spaces up to higher coherent homotopy

In the course of this work, a number of constructions that are key to the theory require choices of contractible ambiguity. To deal with such, we are led to consider objects that are as good as semi-simplicial spaces, but only in a homotopical sense. To model those, let us define an *(augmented) semi-simplicial space up to higher coherent homotopy* as an (augmented) $\widetilde{\Delta}_{\mathrm{inj}}$–space X, defined on any topologically enriched category $\widetilde{\Delta}_{\mathrm{inj}}$ that comes with a weak equivalence $\widetilde{\Delta}_{\mathrm{inj}} \to \Delta_{\mathrm{inj}}$. Roughly speaking, these are categories with the same objects as Δ_{inj} and a (weakly) contractible space of choices for all morphisms in Δ_{inj}. In particular, a $\widetilde{\Delta}_{\mathrm{inj}}$–space $X_\bullet$ includes spaces X_p for $p \geq 0$, together with face maps $\tilde{d}_i\colon X_p \to X_{p-1}$, unique up to homotopy. By precomposing with $\widetilde{\Delta}_{\mathrm{inj}} \to \Delta_{\mathrm{inj}}$, every semi-simplicial space is a $\widetilde{\Delta}_{\mathrm{inj}}$–space, and in light of Lemma 1.5, every $\widetilde{\Delta}_{\mathrm{inj}}$–space is equivalent to one arising in this way. By virtue of this rectification result and Lemma 1.6, all homotopy-invariant constructions for semi-simplicial spaces carry over to $\widetilde{\Delta}_{\mathrm{inj}}$–spaces, so in particular, we have analogues of the spectral sequences (4) and (6), the differentials being the alternating sum $\sum_{i=0}^{p}(-1)^i(\tilde{d}_i)_*$ of morphisms induced by (weakly) contractible choices $\tilde{d}_i$ of face maps. A $\widetilde{\Delta}_{\mathrm{inj}}$–space $X_\bullet$ induces a simplicial set $\pi_0(X_\bullet)$ by taking path components, together with a morphism $X_\bullet \to \pi_0(X_\bullet)$ of $\widetilde{\Delta}_{\mathrm{inj}}$–spaces, which is a weak equivalence if and only if $X_\bullet$ is homotopy discrete, ie takes values in homotopy discrete spaces. To emphasise similarities and by abuse of notation, justified by Lemma 1.6, we call the homotopy colimit of an augmented $\widetilde{\Delta}_{\mathrm{inj}}$–space $X_\bullet \to X_{-1}$ its *realisation*, and denote it by $|X_\bullet| \to X_{-1}$, as in the strict case.

2 The canonical resolution of an E_1–module over an E_2–algebra

2.1 E_1–modules over E_n–algebras and their fundamental groupoids

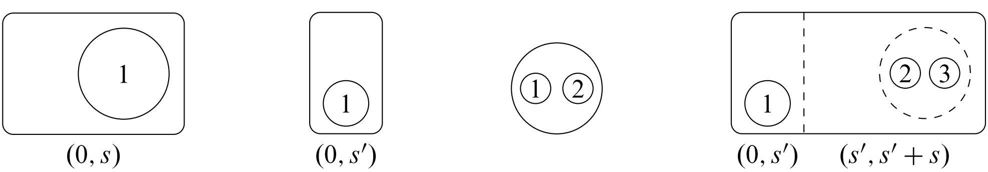
We recall the notion of an E_1–module over an E_n–algebra and explain its relation to modules over monoidal categories.

By an *operad*, we mean a symmetric coloured operad in spaces, and an *algebra* over such is understood in the usual sense; see eg [3, Section 1.1]. For a subspace $X \subseteq \mathbb{R}^n$, we let $\mathcal{D}^k(X)$ be the space of tuples of k embeddings of the closed disc D^n into X that have disjoint interiors and are compositions of scalings and translations. Recall the one-coloured operad $\mathcal{D}^\bullet(D^n)$ of little n–discs [6; 44] with k–operations $\mathcal{D}^k(D^n)$ and operadic composition induced by composing embeddings.

Definition 2.1 Let $\mathcal{SC}_n$ be the coloured operad with colours $\mathfrak{m}$ and $\mathfrak{a}$ whose space of operations $\mathcal{SC}_n(\mathfrak{m}^k, \mathfrak{a}^l; \mathfrak{m})$ is empty for $k \neq 1$ and for $k = 1$ is the space of pairs $(s, \phi) \in [0, \infty) \times \mathcal{D}^l(\mathbb{R}^n)$ such that $\phi \in \mathcal{D}^l((0, s) \times (-1, 1)^{n-1})$, allowing $(0, \varnothing) \in [0, \infty) \times \mathcal{D}^0(\mathbb{R}^n)$ as a valid element of $\mathcal{SC}_n(\mathfrak{m}^k, \mathfrak{a}^0; \mathfrak{m})$. The space $\mathcal{SC}_n(\mathfrak{m}^k, \mathfrak{a}^l, \mathfrak{a})$ is empty for $k \neq 0$ and equals $\mathcal{D}^l(D^n)$ otherwise. The composition restricted to the $\mathfrak{a}$–colour is given by the composition in $\mathcal{D}^\bullet(D^n)$, and the composition

$$\gamma\colon \mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m}) \times (\mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \times \mathcal{SC}_n(\mathfrak{a}^{i_1}; \mathfrak{a}) \times \cdots \times \mathcal{SC}_n(\mathfrak{a}^{i_l}; \mathfrak{a})) \to \mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^{k+i}; \mathfrak{m})$$

for $i = \sum_j i_j$ is given by mapping an element $\big((s, \phi), ((s', \psi), (\varphi^1, \dots, \varphi^l))\big)$ in the codomain to $(s' + s, (\psi, (\phi_1 \circ \varphi^1) + s', \dots, (\phi_l \circ \varphi^l) + s')) \in \mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^{k+i}; \mathfrak{m})$, where $(- + s')$ denotes the translation by s' in the first coordinate. In words, it is defined by adding the parameters, putting the discs of $\mathcal{SC}_n(m, a^k; m)$ to the left of the ones of $\mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m})$, and composing the embeddings of discs of the $\mathcal{SC}_n(\mathfrak{a}^{i_j}; \mathfrak{a})$–factors with the ones of $\mathcal{SC}_n(\mathfrak{m}, \mathfrak{a}^l; \mathfrak{m})$ as in the operad of little n–discs; see Figure 1.



$e \in \mathcal{SC}_2(\mathfrak{m}, \mathfrak{a}; \mathfrak{m})$ $\quad$ $d \in \mathcal{SC}_2(\mathfrak{m}, \mathfrak{a}; \mathfrak{m})$ $\quad$ $f \in \mathcal{SC}_2(\mathfrak{a}^2; \mathfrak{a})$ $\quad$ $\gamma(e; d, f) \in \mathcal{SC}_2(\mathfrak{m}, \mathfrak{a}^3; \mathfrak{m})$

Figure 1: The operadic composition of $\mathcal{SC}_n$.

The canonical embedding $\mathcal{D}^\bullet(D^n) \to \mathcal{D}^\bullet(D^{n+1})$ of little discs operads (see eg [24, Section 4.1.5]), extends to an embedding of two-coloured operads $\mathcal{SC}_n \to \mathcal{SC}_{n+1}$ by taking products with $(-1, 1)$ from the right. Consequently, any algebra over $\mathcal{SC}_{n+1}$ is also one over $\mathcal{SC}_n$.

We call two coloured operads *weakly equivalent* if there is a zigzag between them that consists of morphisms of operads that are weak homotopy equivalences on all spaces of operations.

Remark 2.2 The operad $\mathcal{SC}_n$ is weakly equivalent to a suboperad of the n–dimensional version of the Swiss-cheese operad of [73], motivating the notation.

Definition 2.3 An $E_{1,n}$*–operad* is an operad $\mathcal{O}$ that is weakly equivalent to $\mathcal{SC}_n$. A graded E_1*–module* $\mathcal{M}$ *over an* E_n*–algebra* $\mathcal{A}$ is an algebra $(\mathcal{M},\mathcal{A})$ over an $E_{1,n}$–operad $\mathcal{O}$, considered as an operad in graded spaces, where $\mathcal{M}$ corresponds to the $\mathfrak{m}$–and $\mathcal{A}$ to the $\mathfrak{a}$–colour. That is, it consists of two graded spaces $(\mathcal{M}, g_{\mathcal{M}})$ and $(\mathcal{A}, g_{\mathcal{A}})$, together with multiplication maps for $l \geq 0$ of the form

$$\theta\colon \mathcal{O}(\mathfrak{m},\mathfrak{a}^l;\mathfrak{m})\times\mathcal{M}\times\mathcal{A}^l\to\mathcal{M} \quad \text{and} \quad \theta\colon \mathcal{O}(\mathfrak{a}^l;\mathfrak{a})\times\mathcal{A}^l\to\mathcal{A},$$

which are graded, where $\mathcal{O}(\mathfrak{m},\mathfrak{a}^l;\mathfrak{m})$ and $\mathcal{O}(\mathfrak{a}^l;\mathfrak{a})$ are equipped with the grading that is constantly 0, ie the degree of a multiplication of points is the sum of their degrees. These structure maps are required to satisfy the usual associativity, unitality, and equivariance axioms for an algebra over a coloured operad.

The fundamental groupoid of an algebra over the little 2–discs operad has a braided monoidal groupoid structure; the multiplication is induced by the choice of a 2–operation [24, Chapters 5–6]. Similarly, for a graded algebra $(\mathcal{M},\mathcal{A})$ over an $E_{1,2}$–operad $\mathcal{O}$ and operations $c\in\mathcal{O}(\mathfrak{m},\mathfrak{a};\mathfrak{m})$ and $d\in\mathcal{O}(\mathfrak{a}^2;\mathfrak{a})$, the fundamental groupoid $\Pi(\mathcal{A})$ is a graded braided monoidal groupoid with multiplication induced by d, and $\Pi(\mathcal{M})$ becomes a graded right-module over $\Pi(\mathcal{A})$ with the action induced by c. In other words, the functor $\oplus\colon \Pi(\mathcal{M})\times\Pi(\mathcal{A})\to\Pi(\mathcal{M})$ induced by $\theta(c;-,-)$ is associative, unital up to coherent natural isomorphisms, and compatible with the grading on $\Pi(\mathcal{M})$ and $\Pi(\mathcal{A})$ induced by the grading on $\mathcal{M}$ and $\mathcal{A}$.

Remark 2.4 Since the path components of a space coincide with the path components of its fundamental groupoid in the categorical sense, a grading on an E_1–module over an E_n–algebra is equivalent to a grading of the induced right-module $(\Pi(\mathcal{M}),\oplus)$ over the braided monoidal groupoid $(\Pi(\mathcal{A}),\oplus,b,0)$.

2.2 The canonical resolution

Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra $\mathcal{A}$ with underlying $E_{1,2}$–operad $\mathcal{O}$ and structure maps θ. We call a point $X\in\mathcal{A}$ of degree 1 a *stabilising object* for $\mathcal{M}$, and define the *stabilisation map* with respect to a stabilising point X,

$$s\colon \mathcal{M}\to\mathcal{M},$$

as the multiplication $\theta(c;-,X)$ by X, using an operation $c \in \mathcal{O}(\mathfrak{m},\mathfrak{a},\mathfrak{m})$, which we fix once and for all. As X has degree 1, so does the stabilisation map, which hence restricts to maps $s\colon \mathcal{M}_n \to \mathcal{M}_{n+1}$ between the subspaces of consecutive degrees for all $n \geq 0$. It will be convenient to denote the stabilisation map also by $(-\oplus X)\colon \mathcal{M} \to \mathcal{M}$, and we use the two notations interchangeably.

Remark 2.5 We chose to restrict to stabilising objects of degree 1 to simplify the exposition. However, by keeping track of the gradings, the developed theory generalises to stabilising objects of arbitrary degree.

In the following, we assign to a graded E_1–module $\mathcal{M}$ over an E_2–algebra with stabilising object X an augmented semi-simplicial space $R_\bullet(\mathcal{M}) \to \mathcal{M}$ up to higher coherent homotopy, called the *canonical resolution*. It will be defined as an augmented $\widetilde{\Delta}_{\mathrm{inj}}$–space for a topologically enriched category $\widetilde{\Delta}_{\mathrm{inj}}$ weakly equivalent to Δ_{inj}, constructed from the underlying $E_{1,2}$–operad $\mathcal{O}$. We begin by recalling the braided analogue of the category of finite sets and injections, as introduced in [61].

Definition 2.6 Define the category $U\mathcal{B}$ with objects $[0],[1],\dots$ as in Δ_{inj}, no morphisms from $[q]$ to $[p]$ for $q>p$, and $U\mathcal{B}([q],[p])$ for $q \leq p$ given by the cosets B_{p+1}/B_{p-q}, where B_i denotes the braid group on i strands and B_{p-q} acts on B_{p+1} from the right as the first $p-q$ strands. The composition is defined as

$$U\mathcal{B}([l],[q]) \times U\mathcal{B}([q],[p]) \to U\mathcal{B}([l],[p]), \quad ([b],[b']) \mapsto [b'(1^{p-q} \oplus b)],$$

where $1^{p-q} \oplus b$ is the braid obtained by inserting $p-q$ trivial strands to the left of b, as illustrated in Figure 2.

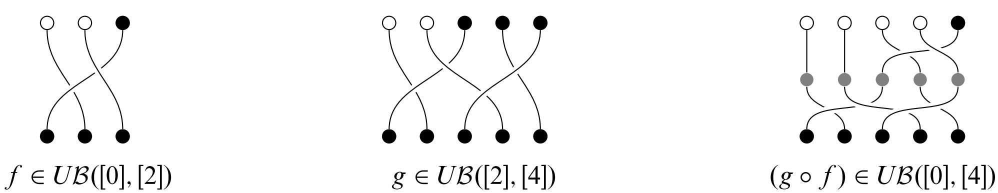


Figure 2: The categorical composition of $U\mathcal{B}$.

The category $U\mathcal{B}$ admits a canonical functor to the category $\mathcal{FI}$ of finite sets and injections by sending a class in B_{p+1}/B_{p-q} to the injection obtained by following the last $q+1$ strands of a representing braid. Visualising $U\mathcal{B}$ as indicated by Figure 2, two braids represent the same morphism if and only if they differ by a braid of the $\circ$–ends. Following the braids of the upper $\bullet$–ends to the lower ends gives the induced injections. This functor admits a section on the subcategory $\Delta_{\mathrm{inj}} \subseteq \mathcal{FI}$, as shown by the next lemma, for the statement of which we consider the braid groups $\coprod_{n\geq 0} B_n$ as the free braided monoidal category on one object X.

Lemma 2.7 *There is a unique functor $\Delta_{\mathrm{inj}} \to U\mathcal{B}$ that maps the face map $d_i \in \Delta_{\mathrm{inj}}([p-1],[p])$ to*

$$[b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}] \in U\mathcal{B}([p-1],[p]).$$

The composition of this functor with the functor $U\mathcal{B} \to \mathcal{FI}$ described above agrees with the inclusion $\Delta_{\mathrm{inj}} \subseteq \mathcal{FI}$.

Proof To prove the first part, it is sufficient to check the face relations

$$[b^{-1}_{X^{\oplus j},X} \oplus X^{\oplus p+1-j}] \circ [b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}] \\ = [b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p+1-i}] \circ [b^{-1}_{X^{\oplus j-1},X} \oplus X^{\oplus p-j+1}]$$

for $i < j$ in $U\mathcal{B}([p-1, p+1])$. The left-hand side agrees with the class of the braid

$$(b^{-1}_{X^{\oplus j},X} \oplus X^{\oplus p+1-j})(X \oplus b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}),$$

which, by applying braid relations, can be seen to agree with the braid

$$(b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p+1-i})(X \oplus b^{-1}_{X^{\oplus j-1},X} \oplus X^{\oplus p-j+1})(b^{-1}_{X,X} \oplus X^{\oplus p}),$$

whose class in $U\mathcal{B}([p-1, p+1]) = B_{p+2}/B_2$ coincides with the right-hand side of the claimed equation. The proof is concluded by observing that the two functors $\Delta_{\mathrm{inj}} \to \mathcal{FI}$ in question agree on the face maps by construction, and thus on all of Δ_{inj}. $\square$

Remark 2.8 In the language of [61], the category $U\mathcal{B}$ is the *free pre-braided monoidal category on one object* [61, Section 1.2]. Unwinding the definitions, their semi-simplicial set $W_n(A, X)$ associated to objects A and X of a pre-braided monoidal category $\mathcal{C}$ (see [61, Section 2]) agrees with the composition

$$\Delta^{\mathrm{op}}_{\mathrm{inj}} \to U\mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \to \mathcal{S}ets,$$

in which the first arrow is the described section, the second is induced by X, and the third is $\mathcal{C}(-, A \oplus X^{\oplus n})$.

In the following, we introduce topological analogues of $U\mathcal{B}$ and Δ_{inj} for any $E_{1,2}$–operad $\mathcal{O}$. To that end, we denote by $\mathcal{O}(k)$ the space obtained from $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m})$ by quotienting out the action of the symmetric group Σ_k on the $\mathfrak{a}$–inputs. To simplify the construction, we assume that the quotient maps $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \to \mathcal{O}(k)$ are covering spaces, although this is not strictly necessary; see Remark 2.22. As the operadic composition γ on $\mathcal{O}$ is equivariant, it induces composition maps

$$\gamma(-;-,1_{\mathfrak{a}}^k)\colon \mathcal{O}(k) \times \mathcal{O}(l) \to \mathcal{O}(k+l).$$

The fixed operation $c \in \mathcal{O}(1)$, used to define the stabilisation map, yields iterated operations $c_k \in \mathcal{O}(k)$ by setting c_0 as the unit $1_{\mathfrak{m}}$ and c_{k+1} inductively as $\gamma(c; c_k, 1_{\mathfrak{a}})$. As a last preparatory step before defining the category $U\mathcal{B}$, we recall that we denote the endpoint of a Moore path μ by $\omega(\mu)$.

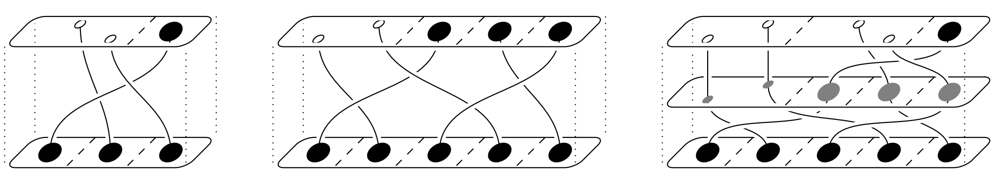
Definition 2.9 Define a topologically enriched category $U\mathcal{O} = U(\mathcal{O}, c)$ with objects $[0], [1], \ldots$ and

$$U\mathcal{O}([q],[p]) = \{(d,\mu) \in \mathcal{O}(p-q) \times \mathrm{Path}_{c_{p+1}}\,\mathcal{O}(p+1) \mid \omega(\mu) = \gamma(c_{q+1}; d, 1_{\mathfrak{a}}^{q+1})\},$$

where $\mathrm{Path}_{c_{p+1}}\mathcal{O}(p+1)$ is the space of Moore paths in $\mathcal{O}(p+1)$ starting at c_{p+1}. The composition is

$$\begin{aligned} U\mathcal{O}([l],[q]) \times U\mathcal{O}([q],[p]) &\to U\mathcal{O}([l],[p]), \\ ((e,\zeta),(d,\mu)) &\mapsto (\gamma(e; d, 1_{\mathfrak{a}}^{q-l}), \mu \cdot \gamma(\zeta; d, 1_{\mathfrak{a}}^{q+1})), \end{aligned}$$

as visualised by Figure 3. Since we are using Moore paths, associativity and unitality follow from the respective properties of the operadic composition.



$(e,\zeta) \in U\mathcal{O}([0],[2])$ $(d,\mu) \in U\mathcal{O}([2],[4])$ $(d,\mu)(e,\zeta) \in U\mathcal{O}([0],[4])$

Figure 3: The categorical composition of $U\mathcal{O}$.

The construction $U(-)$ is functorial in $(\mathcal{O}, c)$ and preserves weak equivalences, since $U\mathcal{O}([q],[p])$ agrees with the homotopy fibre at c_{p+1} of the map

$$\gamma(c_{q+1}; -, 1_{\mathfrak{a}}^{q+1})\colon \mathcal{O}(p-q) \to \mathcal{O}(p+1).$$

Remark 2.10 Using Quillen's bracket construction $\langle -,-\rangle$ for modules over monoidal categories (see [32, page 219]), the category $U\mathcal{B}$ is given by $\langle \mathcal{B},\mathcal{B}\rangle$, where $\mathcal{B} = \coprod_{n\geq 0} B_n$ is the free braided monoidal category acting on itself. Similarly, $U\mathcal{O}$ can be obtained via an analogue of Quillen's construction for monoidal categories internal to spaces, applied to the path-category of the monoid $\coprod_{n\geq 0}\mathcal{O}(n)$.

Lemma 2.11 *The category $U\mathcal{O}$ is homotopy discrete and satisfies $\pi_0(U\mathcal{O})\cong U\mathcal{B}$.*

Before turning to the proof of Lemma 2.11, we encourage the reader to compare Figure 2 with Figure 3.

Proof As $U(-)$ preserves weak equivalences, it suffices to prove the claim for $\mathcal{O} = \mathcal{SC}_2$. Mapping embeddings of discs to their centre yields a homotopy equivalence from the space of operations $\mathcal{SC}_2(n)$ to the unordered configuration space $C_n(\mathbb{R}^2)$ of the plane, which is an Eilenberg–Mac Lane space $K(B_n,1)$ for the braid group B_n. On fundamental groups, the map $\gamma(c_{q+1};-,1_{\mathfrak{a}}^{q+1})\colon \mathcal{O}(p-q)\to\mathcal{O}(p+1)$ is injective, since it is given by including B_{p-q} in B_{p+1} as the first $(p-q)$ strands. From this, one concludes that its homotopy fibre $\mathrm{hofib}_{c_{p+1}}(\gamma(c_{q+1};-,1_{\mathfrak{a}}^{q+1})) = U\mathcal{O}([q],[p])$ is homotopy discrete with path components B_{p+1}/B_{p-q} and that, via this equivalence, the composition coincides with that of $U\mathcal{B}$, proving the claim. □

Equipped with Lemma 2.11, we fix an isomorphism $\pi_0(U\mathcal{O})\cong U\mathcal{B}$ once and for all, which we use, for instance, to identify $\pi_1(\mathcal{O}(p+1),c_{p+1})\cong\pi_0(U\mathcal{O}([p],[p]))$ with the braid group B_{p+1}.

Definition 2.12 The *thickening of the semi-simplicial category* associated to an $E_{1,2}$–operad $\mathcal{O}$ is the subcategory $\widetilde{\Delta}_{\mathrm{inj}}\subseteq U\mathcal{O}$ obtained by restricting $U\mathcal{O}$ to the path components hit by the section $\Delta_{\mathrm{inj}}\to U\mathcal{B}\cong\pi_0(U\mathcal{O})$ of Lemma 2.7. It comes with a weak equivalence to Δ_{inj}, induced by the functor $U\mathcal{O}\to\mathcal{FI}$.

Before proceeding to the central definitions, we remind the reader of the theory of augmented $\mathcal{C}$–spaces for a topologically enriched category $\mathcal{C}$, set up in Section 1.4.

Definition 2.13 Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra with structure maps θ and stabilising object X. Define the contravariant $U\mathcal{O}$–space $B_\bullet(\mathcal{M})$ by sending $[p]$ to the path-space construction of s^{p+1},

$$B_p(\mathcal{M}) = \{(A,\zeta)\in\mathcal{M}\times\mathrm{Path}\,\mathcal{M}\mid \omega(\zeta)=s^{p+1}(A)\},$$

and by

$$UO([q],[p]) \times B_p(\mathcal{M}) \to B_q(\mathcal{M}),$$
$$((d,\mu),(A,\zeta)) \mapsto (\theta(d;A,X^{p-q}),\zeta\cdot\theta(\mu;A,X^{p+1})).$$

Functoriality follows from the associativity of the module structure θ and the composition of Moore paths. Evaluating paths at zero defines an augmentation $B_\bullet(\mathcal{M}) \to \mathcal{M}$, which is a levelwise fibration.

Definition 2.14 Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra with stabilising object X.

(i) The *canonical resolution* of $\mathcal{M}$ is the fibrant augmented $\widetilde{\Delta}_{\mathrm{inj}}$–space

$$R_\bullet(\mathcal{M}) \to \mathcal{M}$$

obtained by restricting the augmented UO–space $B_\bullet(\mathcal{M})$ to the semi-simplicial thickening $\widetilde{\Delta}_{\mathrm{inj}} \subseteq UO$.

(ii) The *space of destabilisations* of a point $A \in \mathcal{M}$ is the $\widetilde{\Delta}_{\mathrm{inj}}$–space $W_\bullet(A)$ defined as the fibre of the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ at A.

Unwrapping the definition, the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ is an augmented semi-simplicial space up to higher coherent homotopy with p–simplices

$$R_p(\mathcal{M}) = \{(A,\zeta) \in \mathcal{M} \times \mathrm{Path}\,\mathcal{M} \mid \omega(\zeta) = s^{p+1}(A)\},$$

augmented over $\mathcal{M}$ by evaluating paths at zero. There is a contractible space of i^{th} face maps, but the following lemma provides a particularly convenient one after choosing a loop $\mu_i \in \Omega_{c_{p+1}}\mathcal{O}(p+1)$ corresponding to the braid $b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}$ via the fixed isomorphism $B_{p+1} \cong \pi_1(\mathcal{O}(p+1),c_{p+1})$.

Lemma 2.15 *The morphism $(c,\mu_i) \in UO([p-1],[p])$ lies in the component of the image of the i^{th} face map $d_i \in \Delta_{\mathrm{inj}}([p-1],[p])$ in $U\mathcal{B}([p-1],[p]) \cong \pi_0 UO([p-1],[p])$, via the section of Lemma 2.7, so the map*

$$\tilde{d}_i \colon R_p(\mathcal{M}) \to R_{p-1}(\mathcal{M}), \quad (A,\zeta) \mapsto (s(A),\zeta\cdot\theta(\mu_i;A,X^{p+1})),$$

is an i^{th} face map of the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$.

Proof The choice of μ_i ensures that, via the isomorphism $\pi_0(UO([p-1],[p])) \cong U\mathcal{B}([p-1],[p])$, the element (c,μ_i) is in the component of $[b^{-1}_{X^{\oplus i},X} \oplus X^{\oplus p-i}]$ in

$U\mathcal{B}([p-1],[p])$. This is exactly the image of $d_i \in \Delta_{\mathrm{inj}}([p-1],[p])$ in $U\mathcal{B}([p-1],[p])$, as claimed. □

Remark 2.16 We borrowed the term *space of destabilisations* from [61], where it stands for certain semi-simplicial sets $W_n(A,X)$ associated to a braided monoidal groupoid. In Section 7.3, it is explained that these semi-simplicial sets are special cases of the spaces of destabilisations in our sense.

Remark 2.17 As $R_\bullet(\mathcal{M}) \to \mathcal{M}$ is fibrant, its fibre $W_\bullet(A)$ is equivalent to its homotopy fibre $\mathrm{hofib}_A(R_\bullet(\mathcal{M}))$, so by virtue of Lemma 1.4, the homotopy fibre at A of the realisation $|R_\bullet(\mathcal{M})| \to \mathcal{M}$ is equivalent to $|W_\bullet(A)|$. In particular, the canonical resolution of $\mathcal{M}$ is graded $\varphi(g_{\mathcal{M}})$–connected in degree $\geq m$ for a function $\varphi\colon \overline{\mathbb{N}} \to \mathbb{Q} \cup \{\infty\}$ satisfying $\phi(\infty) = \infty$ if and only if the spaces of destabilisations $W_\bullet(A)$ are $(\lfloor \varphi(g_{\mathcal{M}}(A)) \rfloor - 1)$–connected for all points $A \in \mathcal{M}$ with finite degree $g_{\mathcal{M}}(A) \geq m$. As points in the same component have equivalent homotopy fibres, it is sufficient to check one point in each component.

Example 2.18 Recall the *free E_2–algebra on a point* $\mathcal{O}^{\mathfrak{a}} = \coprod_{n\geq 0} \mathcal{O}(\mathfrak{a}^n;\mathfrak{a})/\Sigma_n$, graded in the evident way, with the *free E_1–module on a point*

$$\mathcal{O}^{\mathfrak{m}} = \coprod\nolimits_{n\geq 0} \mathcal{O}(\mathfrak{m};\mathfrak{a}^n;\mathfrak{m})/\Sigma_n$$

as a graded E_1–module over it. Choosing the unit $1_{\mathfrak{a}} \in \mathcal{O}(\mathfrak{a};\mathfrak{a})$ as the stabilising object, the space of destabilisations $W_\bullet(c_{p+1})$ is the $\widetilde{\Delta}_{\mathrm{inj}}$–space obtained by restricting the $U\mathcal{O}$–space $U\mathcal{O}(\bullet,[p])$ to $\widetilde{\Delta}_{\mathrm{inj}}$. The category $U\mathcal{O}$ is homotopy discrete with $\pi_0(U\mathcal{O}) \cong U\mathcal{B}$ by Lemma 2.11, so the $\widetilde{\Delta}_{\mathrm{inj}}$–space $W_\bullet(c_{p+1})$ is equivalent to the semi-simplicial set given as the composition of the section $\Delta_{\mathrm{inj}}^{\mathrm{op}} \to U\mathcal{B}^{\mathrm{op}}$ of Lemma 2.7 with $U\mathcal{B}(\bullet,[p])$. Using [35, Proposition 3.2], the realisation of this semi-simplicial set can be seen to be contractible, but we do not go into details, since the consequences of Theorems A and C regarding the twisted homological stability of $K(B_n,1) \simeq \mathcal{O}(\mathfrak{m};\mathfrak{a}^n;\mathfrak{m})/\Sigma_n$ correspond to the case $M = D^2$ and $\pi = \mathrm{id}$ of Theorem D, which is proved in Section 5.

Remark 2.19 The choice of a stabilising object $X \in \mathcal{A}$ for a graded E_1–module $\mathcal{M}$ over an E_2–algebra $\mathcal{A}$ induces a graded E_1–module structure on $\mathcal{M}$ over $\mathcal{O}^{\mathfrak{a}}$. The two canonical resolutions of $\mathcal{M}$ when considered as an module over $\mathcal{A}$ or over $\mathcal{O}^{\mathfrak{a}}$ are identical. In fact, all our constructions and results solely depend on the induced module structure of $\mathcal{M}$ over $\mathcal{O}^{\mathfrak{a}}$ and are in that sense independent of $\mathcal{A}$.

Remark 2.20 Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra with stabilising object X and consider $\mathcal{M}$ as a graded E_1–module over $\mathcal{O}^{\mathfrak{a}}$; see Remark 2.19. For a union of path components $\mathcal{M}' \subseteq \mathcal{M}$ that is closed under the multiplication by X, we define a new grading on $\mathcal{M}$ as an E_1–module over $\mathcal{O}^{\mathfrak{a}}$ by modifying the original grading on $\mathcal{M}'$ by assigning the complement of $\mathcal{M}'$ degree ∞ and leaving the grading on $\mathcal{M}'$ unchanged. We call $\mathcal{M}$ with this new grading the *localisation at* $\mathcal{M}'$. An example for such a subspace $\mathcal{M}'$ is given by the *objects stably isomorphic to an object* $A \in \mathcal{M}$, by which we mean the union of the path components of objects B for which $B \oplus X^{\oplus n}$ is in the component of $A \oplus X^{\oplus m}$ for some $n, m \geq 0$.

Example 2.21 Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra $\mathcal{A}$ and let G be a group acting on $\mathcal{M}$, preserving the grading. If the actions of $\mathcal{A}$ and G on $\mathcal{M}$ commute, then the Borel construction $EG \times_G \mathcal{M}$ inherits a graded E_1–module structure. The choice of a point in EG induces a morphism

$$\begin{tikzcd} R_\bullet(\mathcal{M}) \arrow[r] \arrow[d] & R_\bullet(EG \times_G \mathcal{M}) \arrow[d] \\ \mathcal{M} \arrow[r] & EG \times_G \mathcal{M} \end{tikzcd}$$

of augmented $\widetilde{\Delta}_{\mathrm{inj}}$–spaces, which induces weak equivalences on homotopy fibres. An application of Lemma 1.4 implies that the respective canonical resolutions have the same connectivity.

Remark 2.22 Some constructions of this section work in greater generality. The category $U\mathcal{O}$ and the augmented $U\mathcal{O}$–space $B_\bullet(\mathcal{M})$ can be defined for any coloured operad. $U\mathcal{O}$ then still admits a functor to $\mathcal{FI}$, but might not be homotopy discrete or admit a section on $\Delta_{\mathrm{inj}} \subseteq \mathcal{FI}$. The point-set assumption on the action of Σ_k on $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m})$ can be avoided by constructing $U\mathcal{O}$ using $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m})$ instead of $\mathcal{O}(k)$, which involves taking care of permutations corresponding to preimages of the quotient map $\mathcal{O}(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \to \mathcal{O}(k)$.

2.3 The stable genus

We extend the notion of the *stable genus* of a manifold as introduced in [28] to our context, providing us with a general way of grading modules over braided monoidal categories and by Remark 2.4 also of grading E_1–modules over E_2–algebras.

Let $(\mathcal{M}, \oplus)$ be a right-module over a braided monoidal category $(\mathcal{A}, \oplus, b, 0)$. Recall the free braided monoidal category on one object $\mathcal{B} = \coprod_{n \geq 0} B_n$, consisting of the braid groups B_n. A choice of an object X in $\mathcal{A}$ induces a functor $\mathcal{B} \to \mathcal{A}$ and hence a right-module structure on $\mathcal{M}$ over $\mathcal{B}$. With respect to this module structure, a grading of $\mathcal{M}$ that is compatible with the canonical grading on $\mathcal{B}$ is equivalent to a grading $g_{\mathcal{M}}$ on $\mathcal{M}$ as a category such that $g_{\mathcal{M}}(A \oplus X) = g_{\mathcal{M}}(A) + 1$ holds for all objects A in $\mathcal{M}$.

Definition 2.23 Let X be an object of $\mathcal{A}$ and A an object of $\mathcal{M}$.

(i) The X–*genus* of A is defined as

$$g^X(A) = \sup\{k \geq 0 \mid \text{there exists an object } B \text{ in } \mathcal{M} \text{ with } B \oplus X^{\oplus k} \cong A\} \in \overline{\mathbb{N}}.$$

(ii) The *stable* X–*genus* of A is defined as

$$\overline{g}^X(A) = \sup\{g^X(A \oplus X^{\oplus k}) - k \mid k \geq 0\} \in \overline{\mathbb{N}}.$$

As $\overline{g}^X(A \oplus X) = \overline{g}^X(A) + 1$ holds by definition, the stable X–genus provides a grading of $\mathcal{M}$ when considered as a module over $\mathcal{B}$ via X. This stands in contrast with the (unstable) X–genus, which does in general not define a grading, because the inequality $g^X(A) + 1 \leq g^X(A \oplus X)$ might be strict. For an E_1–module $\mathcal{M}$ over an E_2–algebra $\mathcal{A}$, the choice of a point $X \in \mathcal{A}$ induces an E_1–module structure on $\mathcal{M}$ over the free E_2–algebra on a point $\mathcal{O}^{\mathfrak{a}}$; see Remark 2.19. After taking fundamental groupoids, this results in the module structure of $\Pi(\mathcal{M})$ over $\mathcal{B}$ discussed above, so the stable X–genus provides a grading for $\mathcal{M}$ as an E_1–module over $\mathcal{O}^{\mathfrak{a}}$.

Remark 2.24 If the connectivity assumption of Theorem A is satisfied for an E_1–module $\mathcal{M}$, graded with the stable X–genus, then the cancellation result Corollary B implies $g^X(A \oplus X) = g^X(A) + 1$ for objects A of positive stable genus, which, in turn, implies that for such A, the genus and the stable genus coincide.

3 Stability with constant and abelian coefficients

Let $\mathcal{M}$ be a graded E_1–module over an E_2–algebra with stabilising object X and structure maps θ. We prove Theorem A via a spectral sequence obtained from the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$. All spaces $R_p(\mathcal{M})$ and $|R_\bullet(\mathcal{M})|$ are considered graded by pulling back the grading from $\mathcal{M}$ along the augmentation.

3.1 The spectral sequence

Given a local system L on $\mathcal{M}$, the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ (see Section 2.2) gives rise to a trigraded spectral sequence

$$E^1_{p,q,n} \Rightarrow \mathrm{H}_{p+q+1}(\mathcal{M}_n, |R_\bullet(\mathcal{M})|_n; L), \tag{7}$$

where

$$E^1_{p,q,n} \cong \begin{cases} \mathrm{H}_q(R_p(\mathcal{M})_n; L) & \text{if } p \geq 0, \\ \mathrm{H}_q(\mathcal{M}_n; L) & \text{if } p = -1, \end{cases}$$

with differential $d^1\colon E^1_{p,q,n} \to E^1_{p-1,q,n}$ induced by the augmentation for $p=0$ and the alternating sum $\sum_{i=0}^p (-1)^i (\widetilde{d}_i; \mathrm{id})_*$ for $p>0$, where $\widetilde{d}_i$ is any choice of i^{th} face map of $R_\bullet(\mathcal{M})$; see Sections 1.3 and 1.5. As the differentials do not change the n–grading, it is a sum of spectral sequences, one for each $n \in \overline{\mathbb{N}}$. To identify the E^1–page in terms of the stabilisation $s\colon \mathcal{M} \to \mathcal{M}$, recall from Section 2.1 that the fundamental groupoid $(\Pi(\mathcal{M}), \oplus)$ is a graded module over the graded braided monoidal groupoid $(\Pi(\mathcal{A}), \oplus, b, 0)$.

Lemma 3.1 *There exists an isomorphism $E^1_{p,q,n+1} \cong \mathrm{H}_q(\mathcal{M}_{n-p}; (s^{p+1})^* L)$ with respect to which $d^1\colon E^1_{p,q,n+1} \to E^1_{p-1,q,n+1}$ corresponds to*

$$\sum_{i=0}^{p} (-1)^i (s; \eta_i)_*\colon \mathrm{H}_q(\mathcal{M}_{n-p}; (s^{p+1})^* L) \to \mathrm{H}_q(\mathcal{M}_{n-p+1}; (s^p)^* L),$$

where η_i denotes the natural transformation

$$L(- \oplus b_{X^{\oplus i}, X} \oplus X^{\oplus p-i})\colon L(- \oplus X^{\oplus p+1}) \to L(- \oplus X^{\oplus p+1}).$$

In particular, d^1 agrees for $p=0$ with the stabilisation $(s; \mathrm{id})_*\colon \mathrm{H}_q(\mathcal{M}_n; s^* L) \to \mathrm{H}_q(\mathcal{M}_{n+1}; L)$. Thus, if L is constant, d^1 corresponds to $s_*\colon \mathrm{H}_q(\mathcal{M}_{n-p}; L) \to \mathrm{H}_q(\mathcal{M}_{n-p+1}; L)$ for p even and vanishes for p odd.

Proof Using the choice of face maps $\widetilde{d}_i\colon R_p(\mathcal{M})_{n+1} \to R_{p-1}(\mathcal{M})_{n+1}$ of Lemma 2.15 we consider the square

$$\begin{tikzcd} (\mathcal{M}_{n-p}; (s^{p+1})^* L) \arrow[r, "(\iota_p;\mathrm{id})"] \arrow[d, "(s;\eta_i)"'] & (R_p(\mathcal{M})_{n+1}; L) \arrow[d, "(\widetilde{d}_i;\mathrm{id})"] \\ (\mathcal{M}_{n-p+1}; (s^p)^* L) \arrow[r, "(\iota_{p-1};\mathrm{id})"] & (R_{p-1}(\mathcal{M})_{n+1}; L) \end{tikzcd}$$

where ι_q denotes the canonical equivalence $\mathcal{M}_{n-q} \to R_q(\mathcal{M})_{n+1}$ mapping A to $(A, \mathrm{const}_{s^{q+1}(A)})$. A point $A \in \mathcal{M}_{n-p}$ is mapped by the two compositions in the square to $(s(A), \theta(\mu_i; A, X^{p+1}))$ and $(s(A), \mathrm{const}_{s^{p+1}(A)})$, respectively, which are connected by a preferred homotopy following the path μ_i chosen in Lemma 2.15 to its endpoint. The commutativity of the triangle (3) ensuring that this homotopy extends to one of spaces with local systems (see Section 1.2) is equivalent to the equality $L(\theta(\mu_i; -, X^{p+1}))\eta_i = \mathrm{id}$. But, by the choice of μ_i, the path $\theta(\mu_i; -, X^{p+1})$ corresponds to the braid $- \oplus b^{-1}_{X^{\oplus i}, X} \oplus X^{\oplus p-i}$, so the required relation holds and the square commutes up to homotopy. Taking vertical mapping cones and homology results in the claimed identification. If L is constant, the η_i coincide for all i, so the terms in the alternating sum cancel out. □

Lemma 3.2 *If the local system L is abelian, then the compositions*

$$(\mathcal{M}; (s^{p+2})^* L) \xrightarrow{(s;\mathrm{id})} (\mathcal{M}; (s^{p+1})^* L) \xrightarrow{(s;\eta_i)} (\mathcal{M}; (s^{p})^* L)$$

are for $0 \le i \le p$ all homotopic.

The proof of Lemma 3.2 uses a self-homotopy of $s^2 \colon \mathcal{M} \to \mathcal{M}$ which is crucial for various other arguments. Using the notation of Section 2.2, it is given by

$$[0,1] \times \mathcal{M} \to \mathcal{M}, \quad (t, A) \mapsto \theta(\mu(t); A, X^2), \tag{8}$$

where μ is a choice of loop of length 1, based at $c_2 \in \mathcal{O}(2)$, such that $[(1_{\mathfrak{m}}, \mu)] \in \pi_0(U\mathcal{O}([1],[1]))$ corresponds to the class $[b^{-1}_{X,X}] \in U\mathcal{B}([1],[1])$ via the isomorphism $\pi_0(U\mathcal{O}) \cong U\mathcal{B}$ fixed in Section 2.2. Since μ is unique up to homotopy, this describes the homotopy of s^2 uniquely up to homotopy of homotopies.

Proof of Lemma 3.2 By Section 1.2, the selfhomotopy (8) of s^2 extends to a homotopy of maps of spaces with local systems between the i^{th} and $(i+1)^{\text{st}}$ composition in question if the triangle

$$\begin{tikzcd}
L(-\oplus X^{\oplus p+2}) \arrow[rr, "L(-\oplus X\oplus b_{X^{\oplus i+1},X}\oplus X^{\oplus p-i-1})"] \arrow[drr, "L(-\oplus X\oplus b_{X^{\oplus i},X}\oplus X^{\oplus p-i})"'] & & L(-\oplus X^{\oplus p+2}) \arrow[d, "L(-\oplus b^{-1}_{X,X}\oplus X^{\oplus p})"] \\
& & L(-\oplus X^{\oplus p+2})
\end{tikzcd}$$

commutes. The braid relations give

$$(-\oplus X \oplus b_{X^{\oplus i},X} \oplus X^{\oplus p-i}) = (-\oplus b^{-1}_{X,X} \oplus X^{\oplus p+1})(-\oplus b_{X^{\oplus i+1},X} \oplus X^{\oplus p-i}),$$

so the claim follows if we show that $[b_{X^{\oplus i+1},X}\oplus X]=[X\oplus b_{X^{\oplus i+1},X}]$ holds in the abelianisation. But the braid relation $(b_{X,X}\oplus X)(X\oplus b_{X,X})(b_{X,X}\oplus X)=(X\oplus b_{X,X})(b_{X,X}\oplus X)(X\oplus b_{X,X})$ abelianises to $[b_{X,X}\oplus X]=[X\oplus b_{X,X}]$, from which the claimed identity follows by induction on i. $\square$

3.2 The proof of Theorem A

We prove Theorem A by induction on n, using the spectral sequence (7). Since, by assumption, $|R_\bullet(\mathcal{M})|_{n+1}\to\mathcal{M}_{n+1}$ is $\left(\frac{n-1+k}{k}\right)$–connected for some $k\geq 2$ in the constant or some $k\geq 3$ in the abelian coefficients case, the summand of degree $n+1$ of the spectral sequence converges to zero in the range $p+q\leq\frac{n-1}{k}$. By Lemma 3.1, the differential $d^1\colon E^1_{0,i,n+1}\to E^1_{-1,i,n+1}$ agrees with the stabilisation map $(s;\mathrm{id})_*\colon \mathrm{H}_i(\mathcal{M}_n;s^*L)\to\mathrm{H}_i(\mathcal{M}_{n+1};L)$. Since there are no differentials targeting $E^k_{-1,0,n+1}$ for $k\geq 1$, the stabilisation has to be surjective for $i=0$ if $E^\infty_{-1,0,n+1}$ vanishes, which is the case, since we have $-1\leq\frac{n-1}{k}$ for all $n\geq 0$. In particular, this implies the case $n=0$ for both constant and abelian coefficients, because the isomorphism claims for $n=0$ are vacuous.

Proof for constant coefficients Assume the claim for constant coefficients holds in degrees smaller than n. By Lemma 3.1, the differential $d^1\colon E^1_{p,q,n+1}\to E^1_{p-1,q,n+1}$ corresponds to $s_*\colon \mathrm{H}_q(\mathcal{M}_{n-p};L)\to\mathrm{H}_q(\mathcal{M}_{n-p+1};L)$ for p even, and is zero for p odd. From the induction assumption, we conclude that $E^2_{p,q,n+1}$ vanishes for (p,q) if p is even with $0<p\leq n$ and $q\leq\frac{n-p-1}{k}$, and for (p,q) if p is odd with $0\leq p<n$ and $q\leq\frac{n-p-3+k}{k}$. So in particular, $E^2_{p,q,n+1}$ vanishes if $0<p<n$ and $q\leq\frac{n-p-1}{k}$. As $d^1\colon E^1_{1,i,n+1}\to E^1_{0,i,n+1}$ is zero for all i, $s_*\colon \mathrm{H}_i(\mathcal{M}_n;L)\to\mathrm{H}_i(\mathcal{M}_{n+1};L)$ is injective if both $E^\infty_{0,i,n+1}=0$ and $E^2_{p,q,n+1}=0$ hold for $p+q=i+1$ with $q<i$. This is the case for $i\leq\frac{n-1}{k}$, as claimed, which follows from the established vanishing ranges of E^∞ and E^2. Similarly, the map in question is surjective in degree i if $E^\infty_{-1,i,n+1}=0$ and $E^2_{p,q,n+1}=0$ hold for $p+q=i$ with $q<i$, which is true for $i\leq\frac{n-1+k}{k}$. $\square$

Proof for abelian coefficients Assume the statement holds for degrees smaller than n. The differential $d^1\colon E^1_{p,q,n+1}\to E^1_{p-1,q,n+1}$ corresponds to

$$\sum_i(-1)^i(s,\eta_i)_*\colon \mathrm{H}_q(\mathcal{M}_{n-p};(s^{p+1})^*L)\to\mathrm{H}_q(\mathcal{M}_{n-p+1};(s^p)^*L).$$

In the range where $(s,\mathrm{id})_*\colon \mathrm{H}_q(\mathcal{M}_{n-p-1};(s^{p+2})^*L)\to\mathrm{H}_q(\mathcal{M}_{n-p};(s^{p+1})^*L)$ is surjective, d^1 agrees by Lemma 3.2 with the stabilisation map

$$(s,\mathrm{id})_*\colon \mathrm{H}_q(\mathcal{M}_{n-p};(s^{p+1})^*L)\to\mathrm{H}_q(\mathcal{M}_{n-p+1};(s^p)^*L)$$

for p even, and vanishes for p odd. By induction, this happens for (p,q) such that $0 \le p \le n-1$ and $q \le \frac{n-p-1}{k}$, so by using the induction hypothesis once more, we conclude that $E^2_{p,q,n+1} = 0$ for (p,q) with p even satisfying $0 < p \le n-1$ and $q \le \frac{n-p+1-k}{k}$, and for (p,q) with p odd satisfying $0 \le p < n-1$ and $q \le \frac{n-p-2}{k}$. The rest of the argument proceeds as in the constant case, adapting the ranges and using that $d^1\colon E^1_{1,i,n+1} \to E^1_{0,i,n+1}$ is zero for $i \le \frac{n-1}{k}$. □

Remark 3.3 If $g_{\mathcal{M}}$ is a grading of $\mathcal{M}$, then so is $g_{\mathcal{M}}+m$ for any fixed number $m \ge 0$. Consequently, if the canonical resolution of $\mathcal{M}$ is graded $\bigl(\frac{1}{k}(g_{\mathcal{M}}-m+k)\bigr)$–connected for an $m \ge 2$, then we can apply Theorems A and C to $\mathcal{M}$, graded by $g_{\mathcal{M}} + (m-2)$, which results in a shift in the stability range. By adapting the ranges in the previous proof appropriately and requiring more specific connectivity assumptions, the stability ranges in Theorem A can be improved as follows:

(i) If the canonical resolution is graded $\bigl(\frac{1}{k}(g_{\mathcal{M}}-m+k)\bigr)$–connected for an $m \ge 3$, the surjectivity range in Theorem A for constant coefficients can be improved from $i \le \frac{n-m+k}{k}$ to $i \le \frac{n-m+k+1}{k}$, and the one for abelian coefficients from $i \le \frac{n-m+2}{k}$ to $i \le \frac{n-m+3}{k}$.

(ii) If the canonical resolution is graded $(g_{\mathcal{M}}-1)$–connected in degrees ≥ 1, then the isomorphism range in Theorem A for constant coefficients can be improved from $i \le \frac{n-1}{2}$ to $i \le \frac{n}{2}$, similar to the proof of [58, Theorem 5.1].

4 Stability with twisted coefficients

This section serves to introduce a notion of twisted coefficient systems and to prove Theorem C. Many ideas in this section are inspired by [61, Section 4], which is itself a generalisation of work by Dwyer [20], van der Kallen [41], and Ivanov [38]. We use similar notation to [61] to emphasise analogies, and refer to Remarks 4.11 and 4.12 for a comparison of their notion of coefficient systems to ours.

4.1 Coefficient systems of finite degree

We define coefficient systems of finite degree for graded modules over graded braided monoidal categories, such as fundamental groupoids of graded E_1–modules over E_2–algebras, as described in Section 2.1.

Let $(\mathcal{M},\oplus)$ be a graded right-module over a braided monoidal category $(\mathcal{A},\oplus,b,0)$ in the sense of Section 1.1. We fix a *stabilising object* X, ie an object of $\mathcal{A}$ of degree 1, and recall the free braided monoidal category $\mathcal{B}=\coprod_{n\geq 0} B_n$ on one object, built from the braid groups B_n. The choice of X induces a functor $\mathcal{B}\to\mathcal{A}$, so in particular homomorphisms $B_n\to\mathrm{Aut}_{\mathcal{A}}(X^{\oplus n})$ and a module-structure on $\mathcal{M}$ over $\mathcal{B}$.

Definition 4.1 A *coefficient system* F for $\mathcal{M}$ is a functor

$$F\colon \mathcal{M}\to \mathcal{Ab}$$

to the category of abelian groups, together with a natural transformation

$$\sigma^F\colon F\to F(-\oplus X),$$

called the *structure map of* F, such the image of the canonical morphism $B_m\to\mathrm{Aut}_{\mathcal{A}}(X^{\oplus m})$ acts trivially on the image of $(\sigma^F)^m\colon F\to F(-\oplus X^{\oplus m})$ for all $m\geq 0$. A *morphism between coefficient systems* F and G for $\mathcal{M}$ is a natural transformation $F\to G$ that commutes with the structure maps σ^F and σ^G.

Remark 4.2 The category of coefficient systems for $\mathcal{M}$ is abelian, so in particular has (co)kernels. More concretely, it is a category of abelian group-valued functors on a category $\langle\mathcal{M},\mathcal{B}\rangle$; see Remark 4.12.

Definition 4.3 Define the *suspension* ΣF of a coefficient system F for $\mathcal{M}$ as

$$\Sigma F=F(-\oplus X),$$

together with the structure map $\sigma^{\Sigma F}\colon \Sigma F\to\Sigma F(-\oplus X)$, defined as the composition

$$\Sigma F=F(-\oplus X)\xrightarrow{\sigma^F(-\oplus X)}F(-\oplus X^{\oplus 2})\xrightarrow{F(-\oplus b_{X,X})}F(-\oplus X^{\oplus 2})=\Sigma F(-\oplus X).$$

The structure map σ^F of F induces a morphism $F\to\Sigma F$ of coefficient systems, called the *suspension map*, whose kernel (resp. cokernel) is the *kernel* $\ker(F)$ (resp. *cokernel* $\operatorname{coker}(F)$) of F. We call F *split* if the suspension map is split injective in the category of coefficient systems.

Lemma 4.4 *The suspension ΣF and the suspension map $F\to\Sigma F$ are well-defined.*

Proof The triviality condition for ΣF is implied by the one for F, since $(\sigma^{\Sigma F})^m$ agrees with $F(-\oplus b_{X^{\oplus m},X})\sigma^F(-\oplus X)^m$, which follows by induction on m, using the braid relation $(X\oplus b_{X,X^{\oplus m-1}})(b_{X,X}\oplus X^{\oplus m-1})=b_{X^{\oplus m},X}$. The fact that the suspension map is a morphism of coefficient systems is a consequence of the triviality condition on F, more specifically of $F(-\oplus b_{X,X})(\sigma^F)^2 = (\sigma^F)^2$. $\square$

Remark 4.5 The suspension map gives rise to a natural transformation $\mathrm{id} \to \Sigma$ of endofunctors on the category of coefficient systems for $\mathcal{M}$.

For the remainder of the section, we fix a coefficient system F for the module $\mathcal{M}$.

Definition 4.6 We denote by F_n for $n \geq 0$ the restriction of F to the full subcategory $\mathcal{M}_n \subseteq \mathcal{M}$ of objects of degree n, and define the *degree* and *split degree* of F at an integer N inductively by saying that F has

(i) (split) degree ≤ -1 at N if $F_n = 0$ for $n \geq N$,

(ii) degree r at N for a $r \geq 0$ if $\ker(F)$ has degree -1 at N and $\operatorname{coker}(F)$ has degree $r-1$ at $N-1$, and

(iii) split degree r at N for a $r \geq 0$ if F is split and $\operatorname{coker}(F)$ is of split degree $r-1$ at $N-1$.

Remark 4.7 For all $N \leq 0$, F is of (split) degree r at 0 if and only if it is of (split) degree r at N, and the property of being of (split) degree r at 0 is independent of the chosen grading. However, being of degree r at N depends on the grading if N is positive. If $g_{\mathcal{M}}$ is a grading for $\mathcal{M}$, then so is $g_{\mathcal{M}}+k$ for any $k \geq 0$, and by induction on r one proves that for $k \geq 0$, F is of (split) degree r at N with respect to a grading $g_{\mathcal{M}}$ if and only if it is of (split) degree r at $N-k$ with respect to the grading $g_{\mathcal{M}}+k$.

Lemma 4.8 *The iterated suspension $\Sigma^i F$ for $i \geq 0$ is given by $\Sigma^i F = F(-\oplus X^{\oplus i})$ with structure map*

$$\Sigma^i F = F(-\oplus X^{\oplus i}) \xrightarrow{\sigma^F(-\oplus X^{\oplus i})} F(-\oplus X^{\oplus i} \oplus X) \xrightarrow{F(-\oplus b_{X^{\oplus i},X})} F(-\oplus X \oplus X^{\oplus i}) = \Sigma^i F(-\oplus X).$$

Proof This follows by induction on i, using $(b_{X,X} \oplus X^{\oplus i})(X \oplus b_{X^{\oplus i},X}) = b_{X^{\oplus i}\oplus X,X}$. $\square$

Lemma 4.9 *Let F be a coefficient system for $\mathcal{M}$.*

(i) *For all $i \geq 0$, $\Sigma^i(\ker(F))$ and $\Sigma^i(\operatorname{coker}(F))$ are isomorphic to $\ker(\Sigma^i F)$ and $\operatorname{coker}(\Sigma^i F)$, respectively.*

(ii) *If F is split, then $\Sigma^i F$ is split for all $i \geq 0$.*

(iii) *If F is of (split) degree r at N, then the iterated suspension $\Sigma^i F$ is of (split) degree r at $N-i$ for $i \geq 0$.*

Proof Using Lemma 4.8 and $(X \oplus b^{-1}_{X^{\oplus i},X})(b_{X^{\oplus i+1},X}) = (b_{X^{\oplus i+1},X})(b^{-1}_{X^{\oplus i},X} \oplus X)$, the natural transformation

$$\Sigma^{i+1}F(-) = F(- \oplus X^{\oplus i+1}) \xrightarrow{F(-\oplus b^{-1}_{X^{\oplus i},X})} F(- \oplus X^{\oplus i+1}) = \Sigma^{i+1}F(-)$$

can be seen to commute with the structure map of $\Sigma^{i+1}F$, so defines an automorphism $\Phi\colon \Sigma^{i+1}F \to \Sigma^{i+1}F$. Lemma 4.8 also implies the relation $\Sigma^i(\sigma^F) = \Phi\sigma^{\Sigma^i F}$ and therefore $\Sigma^i((\mathrm{co})\ker(\sigma^F)) = (\mathrm{co})\ker(\Phi\sigma^{\Sigma^i F})$. Hence, the coefficient systems in comparison are (co)kernels of morphisms that differ by an automorphism. This proves the first claim. Given a splitting $s\colon \Sigma F \to F$ for F, the composition $\Sigma^i(s)\Phi$ splits $\Sigma^i F$, which shows the second. Finally, the third follows from the first two by induction on r. □

Remark 4.10 If $\mathcal{M}$ is a groupoid such that all subcategories $\mathcal{M}_n$ are connected, then a coefficient system for $\mathcal{M}$ is equivalently given as a sequence of $\mathrm{Aut}(A \oplus X^{\oplus n-g(A)})$–modules F_n for an element A of minimal degree $g(A)$, together with $(-\oplus X)$–equivariant morphisms $F_n \to F_{n+1}$ such that the image of B_m in $\mathrm{Aut}(X^{\oplus m})$ acts via $(A \oplus X^{\oplus n-g(A)} \oplus -)$ trivially on the image of F_n in F_{n+m} for all n and m.

Remark 4.11 A *pre-braided* monoidal category in the sense of [61] is a monoidal category $(\mathcal{C}, \oplus, b, 0)$ whose unit 0 is initial and whose underlying groupoid $\mathcal{C}^\sim$ is braided monoidal satisfying a certain condition; see [61, Definition 1.5]. In that work, a *coefficient system* for $\mathcal{C}$ at a pair of objects (A, X) is an abelian group-valued functor F^{RW} defined on the full subcategory $\mathcal{C}_{A,X} \subseteq \mathcal{C}$ generated by $A \oplus X^{\oplus n}$ for $n \geq 0$. Considering $\mathcal{C}^\sim_{A,X}$ as a module over the braided monoidal groupoid $\mathcal{C}^\sim_{0,X}$, such a functor F^{RW} gives a coefficient system F in our sense by restricting F^{RW} to $\mathcal{C}^\sim_{A,X}$ and defining the structure map as $\sigma^F(-) := F^{\mathrm{RW}}(- \oplus \iota_X)$, where $\iota_X\colon 0 \to X$ is the unique morphism. In [61], the transformation $- \oplus \iota_X\colon \mathrm{id}_{\mathcal{C}} \to - \oplus X$ is denoted by σ^X, so we have the suggestive identity $F^{\mathrm{RW}}(\sigma^X) = \sigma^F$. Assigning a coefficient system for $\mathcal{C}$ at (A, X) in the sense of [61] to one for $\mathcal{C}^\sim_{A,X}$ in our sense yields a functor between the respective categories of coefficient systems, which can be seen to preserve the *suspension* and *degree* in the sense of [61] and in ours, at least up to isomorphism. See Section 7.3 for a general comparison between [61] and our work.

Remark 4.12 The category of coefficient systems for $\mathcal{M}$ is isomorphic to the category of abelian group-valued functors on a category $\langle\mathcal{M},\mathcal{B}\rangle$. To construct this category, recall *Quillen's bracket construction* $\langle\mathcal{E},\mathcal{F}\rangle$ of a monoidal category $\mathcal{F}$ that acts via $\oplus\colon \mathcal{E}\times\mathcal{F}\to\mathcal{E}$ on a category $\mathcal{E}$; see [32, page 219]. It has the same objects as $\mathcal{E}$, and a morphism from C to C' is an equivalence class of pairs (D, f) with $D\in \operatorname{ob}\mathcal{F}$ and $f\in\mathcal{E}(C\oplus D, C')$, where (D, f) and (D', f') are equivalent if there is an isomorphism $g\in\mathcal{F}(D, D')$ satisfying $f'=f(C\oplus g)$. Using this construction, we obtain the category $\langle\mathcal{M},\mathcal{B}\rangle$ encoding coefficient systems by letting the free braided monoidal category on one object $\mathcal{B}$ act on $\mathcal{M}$ via the functor $\mathcal{B}\to\mathcal{A}$ induced by X, followed by the action of $\mathcal{A}$ on $\mathcal{M}$. The multiplication by X on $\mathcal{M}$ induces an endofunctor $\Sigma\colon \langle\mathcal{M},\mathcal{B}\rangle\to\langle\mathcal{M},\mathcal{B}\rangle$ by mapping a morphism $[D, f]\colon C\to C'$ to $[D, (f\oplus X)(C\oplus b_{X,D})]\colon C\oplus X\to C'\oplus X$. This functor comes together with a natural transformation $\sigma\colon \mathrm{id}\to\Sigma$, given by $[X,\mathrm{id}]$, such that the suspension of a coefficient system F, seen as a functor on $\langle\mathcal{M},\mathcal{B}\rangle$, is the composition $(F\circ\Sigma)$, and its suspension map is $F(\sigma)\colon F\to(F\circ\Sigma)$. From this point of view and using the notation of the previous remark, the functor from coefficient systems in the sense of [61] to ones in ours, described in the previous remark, is given by precomposition with a functor $\langle\mathcal{C}^{\sim}_{A,X},\mathcal{B}\rangle\to\mathcal{C}_{A,X}$ that is the identity on objects and maps a morphism $[X^{\oplus k}, f]$ in $\langle\mathcal{C}^{\sim}_{A,X},\mathcal{B}\rangle$ from C to C' to $f(C\oplus\iota_{X^{\oplus k}})$.

4.2 Twisted stability of E_1–modules over E_2–algebras

We fix a graded E_1–module $\mathcal{M}$ over an E_2–algebra $\mathcal{A}$ with stabilising object X for the rest of the section. Recall from Section 2.1 that its fundamental groupoid $(\Pi(\mathcal{M}),\oplus)$ is a graded right-module over the graded braided monoidal category $(\Pi(\mathcal{A}),\oplus,b,0)$.

Definition 4.13 A *coefficient system* for $\mathcal{M}$ is a coefficient system for $\Pi(\mathcal{M})$ in the sense of Definition 4.1.

The structure map of a coefficient system F for $\mathcal{M}$ enhances the stabilisation map $s\colon \mathcal{M}\to\mathcal{M}$ to a map

$$(s;\sigma^F)\colon (\mathcal{M};F)\to(\mathcal{M};F)$$

of spaces with local systems, which stabilises homologically by Theorem C if the canonical resolution is sufficiently connected and F is of finite degree. The remainder of this section is devoted to the proof.

Remark 4.14 In the course of the proof of Theorem C, it will be convenient to have fixed a notion of a *homotopy-commutative square of spaces with local systems*, by which we mean a square

$$\begin{tikzcd} (X;F) \arrow[r] \arrow[d] & (Y;G) \arrow[d] \\ (X';F') \arrow[r] & (Y';G') \end{tikzcd}$$

together with a specified homotopy between the two compositions, which might be nontrivial, even if the diagram is strictly commutative. Taking singular chains results in a homotopy-commutative square of chain complexes (see Section 1.2) and taking vertical mapping cones of the square induces a morphism

$$\mathrm{H}_*((X';F'),(X;F)) \to \mathrm{H}_*((Y';G'),(Y;G)), \tag{9}$$

which depends on the homotopy. However, homotopies that are homotopic as homotopies give homotopic morphisms on mapping cones, hence they induce the same morphism (9). Horizontal composition of such squares, including the homotopies, induces the respective composition of (9). Even though (9) depends on the homotopy, the long exact sequences of the mapping cones still fit into a commutative ladder.

We denote by $\mathrm{Rel}_*(F) = \mathrm{H}_*((\mathcal{M};F),(\mathcal{M};F))$ the relative groups with respect to the stabilisation $(s;\sigma^F)$, equipped with the additional grading

$$\mathrm{Rel}_*(F) = \bigoplus_{n\in\overline{\mathbb{N}}} \mathrm{H}_*((\mathcal{M}_{n+1};F),(\mathcal{M}_n;F)).$$

Although the square

$$\begin{tikzcd} (\mathcal{M};F) \arrow[r, "(s;\sigma^F)"] \arrow[d, "(s;\sigma^F)"'] & (\mathcal{M};F) \arrow[d, "(s;\sigma^F)"] \\ (\mathcal{M};F) \arrow[r, "(s;\sigma^F)"] & (\mathcal{M};F) \end{tikzcd} \tag{10}$$

commutes strictly, we consider it as homotopy commutative via the homotopy (8) of Section 3.1, which extends to one of spaces with local systems (see Section 1.2), since the triviality condition on coefficient systems gives $F(-\oplus b_{X,X}^{-1})(\sigma^F)^2 = (\sigma^F)^2$. This homotopy-commutative square induces a relative stabilisation

$$(s;\sigma^F)_*^{\sim}\colon \mathrm{Rel}_*(F) \to \mathrm{Rel}_*(F)$$

of degree 1, where the superscript $\sim$ indicates the twist by the homotopy. The homotopy-commutative square (10) factors as a composition of homotopy-commutative squares

$$\begin{tikzcd}
(\mathcal{M};F) \arrow[r, "(\mathrm{id};\sigma^F)"] \arrow[d, "(s;\sigma^F)"'] & (\mathcal{M};\Sigma F) \arrow[r, "(s;\mathrm{id})"] \arrow[d, "(s;\sigma^{\Sigma F})"] & (\mathcal{M};F) \arrow[d, "(s;\sigma^F)"] \\
(\mathcal{M};F) \arrow[r, "(\mathrm{id};\sigma^F)"] & (\mathcal{M};\Sigma F) \arrow[r, "(s;\mathrm{id})"] & (\mathcal{M};F))
\end{tikzcd}$$

in which the square on the left strictly commutes because of the triviality condition, and we equip it with the trivial homotopy. The square on the right is homotopy commutative using the same homotopy as for (10). This induces a factorisation of the relative stabilisation map as

$$\mathrm{Rel}_*(F) \xrightarrow{(\mathrm{id};\sigma^F)_*} \mathrm{Rel}_*(\Sigma F) \xrightarrow{(s;\mathrm{id})^{\sim}_*} \mathrm{Rel}_*(F), \tag{11}$$

where the first map is of degree 0 and the second of degree 1.

Lemma 4.15 *The composition* $\mathrm{Rel}_*(F) \xrightarrow{(s;\sigma^F)^{\sim}_*} \mathrm{Rel}_*(F) \xrightarrow{(s;\sigma^F)^{\sim}_*} \mathrm{Rel}_*(F)$ *is trivial.*

Proof The mapping cones defining $\mathrm{Rel}_*(F)$ induce a commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_*(\mathcal{M};F) \arrow[r] \arrow[d] & \mathrm{H}_*(\mathcal{M};F) \arrow[r] \arrow[d] & \mathrm{Rel}_*(F) \arrow[r, "h_3"] \arrow[d, "h_4"] & \mathrm{H}_{*-1}(\mathcal{M};F) \arrow[r, "h_1"] \arrow[d, "h_2"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_*(\mathcal{M};F) \arrow[r] \arrow[d] & \mathrm{H}_*(\mathcal{M};F) \arrow[r, "h_6"] \arrow[d, "h_8"] & \mathrm{Rel}_*(F) \arrow[r, "h_5"] \arrow[d, "h_7"] & \mathrm{H}_{*-1}(\mathcal{M};F) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_*(\mathcal{M};F) \arrow[r, "h_{10}"] & \mathrm{H}_*(\mathcal{M};F) \arrow[r, "h_9"] & \mathrm{Rel}_*(F) \arrow[r] & \mathrm{H}_{*-1}(\mathcal{M};F) \arrow[r] & \cdots
\end{tikzcd}$$

in which h_7h_4 agrees with the composition in consideration. As h_1 and h_2 both equal $(s;\sigma^F)_*$, we conclude $0 = h_1h_3 = h_2h_3 = h_5h_4$, so the image of h_4 is in the kernel of h_5, which is the image of h_6. Hence it is enough to show $h_7h_6 = 0$. Since $h_8 = h_{10}$ for the same reason as $h_1 = h_2$, the claim follows from the identity $h_7h_6 = h_9h_8 = h_9h_{10} = 0$. $\square$

4.3 The relative spectral sequence

We prove Theorem C via a relative analogue of the spectral sequence (7) of Section 3.1, which we derive from a map of augmented $\widetilde{\Delta}_{\mathrm{inj}}$–spaces

$$\begin{tikzcd} R_\bullet(\mathcal{M}) \arrow[r, dotted] \arrow[d] & R_\bullet(\mathcal{M}) \arrow[d] \\ \mathcal{M} \arrow[r, "s"] & \mathcal{M} \end{tikzcd}$$

covering the stabilisation map s. Indicated by the dotted arrow, this morphism will only be defined up to higher coherent homotopy; we obtain it from replacing the canonical resolution $R_\bullet(\mathcal{M})$ with an equivalent bar construction $B(U\mathcal{B}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M}))$ which admits a strict morphism of the desired form.

To this end, recall from Section 2.2 the homotopy discrete category $U\mathcal{O}$, the isomorphism $\pi_0(U\mathcal{O}) \cong U\mathcal{B}$, and the augmented $U\mathcal{O}$–space $B_\bullet(\mathcal{M})$ whose restriction to the subcategory $\widetilde{\Delta}_{\mathrm{inj}} \subseteq U\mathcal{O}$ is $R_\bullet(\mathcal{M})$, where $\widetilde{\Delta}_{\mathrm{inj}}$ is the union of components hit by the section $\Delta_{\mathrm{inj}} \to U\mathcal{B}$. Define the $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{O})$–space $U\mathcal{O}(\bullet,\blacksquare)$ and the $(\Delta_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{B})$–space $U\mathcal{B}(\bullet,\blacksquare)$ by restricting the hom-functors of $U\mathcal{O}$ and $U\mathcal{B}$ appropriately. Taking components gives a weak equivalence $\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{O} \to \Delta_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{B}$ of enriched categories and one $U\mathcal{O}(\bullet,\blacksquare) \to U\mathcal{B}(\bullet,\blacksquare)$ of $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{O})$–spaces, which fits into a chain of weak equivalences of $\widetilde{\Delta}_{\mathrm{inj}}$–spaces

$$R_\bullet(\mathcal{M}) \xleftarrow{\simeq} B(U\mathcal{O}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})) \xrightarrow{\simeq} B(U\mathcal{B}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})), \tag{12}$$

augmented over $\mathcal{M}$, the left arrow being the restriction of the bar resolution of $B_\bullet(\mathcal{M})$ to $\widetilde{\Delta}_{\mathrm{inj}}$; see Section 1.4. Consider the functor $t\colon U\mathcal{O} \to U\mathcal{O}$ that maps $[p]$ to $[p+1]$ and is defined on morphisms as

$$t\colon U\mathcal{O}([q],[p]) \to U\mathcal{O}(t([q]), t([p])), \quad (d,\mu) \mapsto (d, \gamma(c;\mu, 1_{\mathfrak{a}})),$$

using the operadic composition γ and the element c with which we defined the iterated operations $c_p \in \mathcal{O}(p)$ in Section 2.2. Accompanying this functor, there is a morphism of augmented $U\mathcal{O}$–spaces

$$\begin{tikzcd} B_\blacksquare(\mathcal{M}) \arrow[r] \arrow[d] & B_{t(\blacksquare)}(\mathcal{M}) \arrow[d] \\ \mathcal{M} \arrow[r, "s"] & \mathcal{M} \end{tikzcd} \tag{13}$$

defined by making use of the module structure θ of $\mathcal{M}$ to assign to a p–simplex (A,ζ) in $B_p(\mathcal{M})$ the element $(A, \theta(c;\zeta, X))$ in $B_{p+1}(\mathcal{M})$. Last but not least, we define a morphism of $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}\times U\mathcal{O})$–spaces

$$UB(\bullet,\blacksquare)\to UB(\bullet,t(\blacksquare)) \tag{14}$$

by considering the braid groups $\coprod_{n\geq 0}B_n$ as the free braided monoidal category in one object X to define

$$\begin{aligned} U\mathcal{B}([q],[p]) = B_{p+1}/B_{p-q} \to B_{p+2}/B_{p-q+1} &= U\mathcal{B}([q],t([p])),\\ [b]\mapsto [(b\oplus X)(X^{\oplus p-q}&\oplus b^{-1}_{X^{\oplus p+1},X})]. \end{aligned}$$

Lemma 4.16 *The assignment (14) indeed defines a morphism of $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}\times U\mathcal{O})$–spaces.*

Proof Recall that $U\mathcal{B}(\bullet,\blacksquare)$ is induced from a $(\Delta^{\mathrm{op}}_{\mathrm{inj}}\times U\mathcal{B})$–space via the equivalence $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}\times U\mathcal{O}\to\Delta^{\mathrm{op}}_{\mathrm{inj}}\times U\mathcal{B}$. The semi-simplicial direction of $U\mathcal{B}(\bullet,\blacksquare)$ comes from the section $\Delta_{\mathrm{inj}}\to U\mathcal{B}$ of Lemma 2.7, which maps a face map d_i in $\Delta_{\mathrm{inj}}([q-1],[q])$ to the class $[b^{-1}_{X^{\oplus i},X}\oplus X^{\oplus q-i}]$ in $U\mathcal{B}([p-1],[p])$, so (14) is natural in the semi-simplicial direction if the two braids

$$(X^{\oplus p-q}\oplus b^{-1}_{X^{\oplus i},X}\oplus X^{\oplus q-i}\oplus X)(X^{\oplus p-q+1}\oplus b^{-1}_{X^{\oplus q},X})$$

and

$$(X^{\oplus p-q}\oplus b^{-1}_{X^{\oplus q+1},X})(X^{\oplus p-q+1}\oplus b^{-1}_{X^{\oplus i},X}\oplus X^{\oplus q-i})$$

define the same class in $U\mathcal{B}([q-1],[p+1])=B_{p+2}/B_{p-q+2}$. Braid relations show that these two braids agree up to right-multiplication with $(X^{\oplus p-q}\oplus b^{-1}_{X,X}\oplus X^{q})$, so coincide in B_{p+2}/B_{p-q+2}, which proves the claim since the naturality in the $U\mathcal{B}$–direction is immediate. □

The functor $t\colon U\mathcal{O}\to U\mathcal{O}$, together with the morphisms (13) and (14), induces a map

$$\begin{array}{ccc} B(U\mathcal{B}(\bullet,\blacksquare),U\mathcal{O},B_\blacksquare(\mathcal{M})) & \xrightarrow{\ t\ } & B(U\mathcal{B}(\bullet,\blacksquare),U\mathcal{O},B_\blacksquare(\mathcal{M}))\\ \big\downarrow & & \big\downarrow\\ \mathcal{M} & \xrightarrow{\qquad s\qquad} & \mathcal{M} \end{array}$$

of augmented $\widetilde{\Delta}_{\mathrm{inj}}$–spaces. Pulling back a coefficient system F for the graded module $\mathcal{M}$ along the augmentations, this morphism enhances to one of graded $\widetilde{\Delta}_{\mathrm{inj}}$–spaces with local coefficients that covers the stabilisation map $(s;\sigma^F)\colon(\mathcal{M};F)\to(\mathcal{M};F)$. Identifying $R_\bullet(\mathcal{M})$ with $B(U\mathcal{B}(\bullet,\blacksquare),U\mathcal{O},B_\blacksquare(\mathcal{M}))$ via the zigzag (12) by abuse of notation, we get a trigraded spectral sequence of the form

$$E^1_{p,q,n}\Rightarrow \mathrm{H}_{p+q+1}\big((\mathcal{M}_{n+1},|R_\bullet(\mathcal{M})|_{n+1};F),(\mathcal{M}_n,|R_\bullet(\mathcal{M})|_n;F)\big), \tag{15}$$

where
$$E^1_{p,q,n} \cong \begin{cases} \mathrm{H}_q\big((R_p(\mathcal{M})_{n+1};F),(R_p(\mathcal{M})_n;F)\big) & \text{if } p\geq 0,\\ \mathrm{H}_q((\mathcal{M}_{n+1};F),(\mathcal{M}_n;F)) & \text{if } p=-1,\end{cases}$$

which is a sum of spectral sequences, one for each $n\in\overline{\mathbb{N}}$; see Sections 1.3 and 1.5. Using Lemma 4.8 and

$$(b^{-1}_{X,X}\oplus X^{\oplus p})(X\oplus b_{X^{\oplus i},X}\oplus X^{\oplus p-i})(b_{X^{\oplus p+1},X}) = (X\oplus b_{X^{\oplus p},X})(b_{X^{\oplus i},X}\oplus X^{\oplus p-i+1}),$$

one checks that selfhomotopy (8) of s^2 witnesses homotopy commutativity of

$$\begin{tikzcd} (\mathcal{M};\Sigma^{p+1}F) \arrow[r, "(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus p-i}))"] \arrow[d, "(s;\sigma^{\Sigma^{p+1}F})"'] & (\mathcal{M};\Sigma^p F) \arrow[d, "(s;\sigma^{\Sigma^p F})"] \\ (\mathcal{M};\Sigma^{p+1}F) \arrow[r, "(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus p-i}))"] & (\mathcal{M};\Sigma^p F) \end{tikzcd}$$

which thus induces a morphism $(s;\eta_i)^{\sim}_*\colon \mathrm{Rel}_*(\Sigma^{p+1}F)\to\mathrm{Rel}_*(\Sigma^p F)$ of degree 1; the superscript $\sim$ indicates the twist by the homotopy (8). This morphism serves us to identify the spectral sequence (15) as follows.

Lemma 4.17 *There exists an isomorphism $E^1_{p,q,n+1}\cong\mathrm{Rel}_q(\Sigma^{p+1}F)_{n-p}$ with respect to which the d^1–differential agrees with*

$$\sum_{i=0}^{p}(-1)^i(s;\eta_i)^{\sim}_*\colon \mathrm{Rel}_q(\Sigma^{p+1}F)_{n-p}\to\mathrm{Rel}_q(\Sigma^p F)_{n-p+1}.$$

In particular, the differential $d^1\colon E^1_{0,*,n+1}\to E^1_{-1,*,n+1}$ corresponds to the second map of (11) in degree n.

Proof On p–simplices, the first equivalence of (12) has a preferred homotopy inverse induced by the extra degeneracy given by inserting the identity of $U\mathcal{O}([p],[p])$; see Section 1.4. Composing it with the second equivalence of (12) yields an equivalence that forms the vertical arrows of a square

$$\begin{tikzcd} R_p(\mathcal{M}) \arrow[r, "\tilde{t}"] \arrow[d, "\simeq"'] & R_p(\mathcal{M}) \arrow[d, "\simeq"] \\ B(U\mathcal{B}([p],\bullet),U\mathcal{O},B_\bullet(\mathcal{M})) \arrow[r, "t"] & B(U\mathcal{B}([p],\bullet),U\mathcal{O},B_\bullet(\mathcal{M})) \end{tikzcd}$$

where $\tilde{t}$ is defined by mapping (A,ζ) to $(s(A),s(\zeta)\cdot\theta(\alpha_p;A,X^{p+2}))$. Here $\alpha_p\in\Omega_{c_{p+2}}\mathcal{O}(p+2)$ is any loop that corresponds to $b^{-1}_{X^{\oplus p+1},X}$ under the equivalence

$\pi_1(\mathcal{O}(p+2);c_{p+2}) \cong B_{p+2}$; see Section 2.2. This choice of α_p guarantees that the previous square commutes, which is why it is sufficient to show that

$$\begin{tikzcd}
(\mathcal{M}_{n-p};\Sigma^{p+1}F) \arrow[r, "(s;\sigma^{\Sigma^{p+1}F})"] \arrow[d, "(\iota;\mathrm{id})"'] & (\mathcal{M}_{n-p+1};\Sigma^{p+1}F) \arrow[d, "(\iota;\mathrm{id})"] \\
(R_p(\mathcal{M})_{n+1};F) \arrow[r, "(\tilde{t};\sigma^F)"] & (R_p(\mathcal{M})_{n+2};F)
\end{tikzcd}$$

homotopy commutes in order to prove $E^1_{p,q,n+1} \cong \mathrm{Rel}_q(\Sigma^{p+1}F)_{n-p}$, where ι denotes the canonical equivalence mapping A to $(A,\mathrm{const}_{s^{p+1}(A)})$. On the space level, the two compositions are given by assigning to $A \in \mathcal{M}_{n-p}$ the elements $(s(A),\mathrm{const}_{s^{p+2}(A)})$ and $(s(A),\theta(\alpha_p;A,X^{p+2}))$, respectively. As we have

$$\sigma^{\Sigma^{p+1}F}(-) = F(-\oplus X^{\oplus p+1})\sigma^F(-\oplus X^{\oplus p+1})$$

by Lemma 4.8, the homotopy induced by following α_p to its endpoint is one of maps with local coefficients, which implies the first claim. A relative version of the proof of Lemma 3.1 establishes the identification of the differentials and finishes the proof. □

Lemma 4.18 *The following composition is zero for $n \geq 1$:*

$$\mathrm{Rel}_*(F)_{n-1} \xrightarrow{(\mathrm{id};\sigma^F)} \mathrm{Rel}_*(\Sigma F)_{n-1} \xrightarrow{(\mathrm{id};\sigma^{\Sigma F})} \mathrm{Rel}_*(\Sigma^2 F)_{n-1} \cong E^1_{1,*,n+1} \xrightarrow{d^1} E^1_{0,*,n+1} \cong \mathrm{Rel}_*(\Sigma F)_n.$$

Proof Using Lemma 4.17, the composition in question is the difference between the morphisms in degree $n-1$ induced by the compositions of the two homotopy-commutative squares

$$\begin{tikzcd}
(\mathcal{M};F) \arrow[r, "(\mathrm{id};\sigma^{\Sigma F}\sigma^F)"] \arrow[d, "(s;\sigma^F)"'] & (\mathcal{M};\Sigma^2 F) \arrow[r, "(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus 1-i}))"] \arrow[d, "(s;\sigma^{\Sigma^2 F})"'] & (\mathcal{M};\Sigma F) \arrow[d, "(s;\sigma^{\Sigma F})"'] \\
(\mathcal{M};F) \arrow[r, "(\mathrm{id};\sigma^{\Sigma F}\sigma^F)"] & (\mathcal{M};\Sigma^2 F) \arrow[r, "(s;F(-\oplus b_{X^{\oplus i},X}\oplus X^{\oplus 1-i}))"] & (\mathcal{M};\Sigma F)
\end{tikzcd}$$

for $i=0$ and $i=1$, where the homotopy of the left square is trivial. The nontrivial homotopy of the right square becomes trivial after composing with the left square by the triviality condition for coefficient systems, so the composition in question is the difference of the morphisms induced by the two strictly commutative outer squares. But, again by the triviality condition, we have $F(-\oplus b_{X,X})\sigma^{\Sigma F}\sigma^F = \sigma^{\Sigma F}\sigma^F$, so the two outer squares coincide and the difference of the induced morphisms vanishes. □

4.4 The proof of Theorem C

Given the long exact sequence

$$\cdots \to \mathrm{Rel}_{*+1}(F) \to \mathrm{H}_*(\mathcal{M};F) \xrightarrow{(s;\sigma^F)_*} \mathrm{H}_*(\mathcal{M};F) \to \mathrm{Rel}_*(F) \to \mathrm{H}_{*-1}(\mathcal{M};F) \to \cdots,$$

Theorem C follows as a consequence of the next result.

Theorem 4.19 *Let F be a coefficient system for $\mathcal{M}$ of degree r at $N \geq 0$. If the canonical resolution of $\mathcal{M}$ is graded $((g_{\mathcal{M}}-2+k)/k)$–connected in degrees ≥ 1 for some $k \geq 2$, then*

- (i) *the group $\mathrm{Rel}_i(F)_n$ vanishes for $n \geq \max(N+1, k(i+r))$, and*
- (ii) *if F is of split degree r at $N \geq 0$, then the group $\mathrm{Rel}_i(F)_n$ vanishes for $n \geq \max(N+1, ki+r)$.*

We prove Theorem 4.19 via a double induction on r and $i \geq 0$ by considering the following statement:

($\boldsymbol{H}_{r,i}$) *The vanishing ranges of Theorem 4.19 hold for all F of degree $< r$ at any $N \geq 0$ in all homological degrees i, and for all F of degree r at any $N \geq 0$ in homological degrees $< i$.*

The claim ($\boldsymbol{H}_{r,i}$) holds trivially if $r < 0$ or if $(r,i) = (0,0)$. If ($\boldsymbol{H}_{r,i}$) holds for a fixed r and all i, then ($\boldsymbol{H}_{r+1,0}$) follows, since there is no requirement on coefficient systems of degree $r+1$. Hence, to prove the theorem, it is sufficient to show that ($\boldsymbol{H}_{r,i}$) implies ($\boldsymbol{H}_{r,i+1}$) for $i, r \geq 0$. As the composition

$$\mathrm{Rel}_i(F)_n \xrightarrow{(s;\sigma^F)_*^{\sim}} \mathrm{Rel}_i(F)_{n+1} \xrightarrow{(s;\sigma^F)_*^{\sim}} \mathrm{Rel}_i(F)_{n+2}$$

is zero by Lemma 4.15, it is enough to show injectivity of both maps in the claimed range. Using the factorisation (11), this is implied by the following lemma.

Lemma 4.20 *Let $r \geq 0$ and $i \geq 0$ be such that ($\boldsymbol{H}_{r,i}$) is satisfied, and let F be of degree r at some $N \geq 0$.*

(i) *The morphism*

$$(\mathrm{id}, \sigma_X)_*\colon \mathrm{Rel}_*(F)_n \to \mathrm{Rel}_*(\Sigma F)_n$$

is injective for $n \geq \max(N, k(i+r))$ and surjective for $n \geq \max(N, k(i+r-1))$. If F is of split degree r at $N \geq 0$, then the map is split injective for all n and surjective for $n \geq \max(N, ki+r-1)$.

(ii) *The morphism*

$$(s,\mathrm{id})^{\sim}_{*}\colon \mathrm{Rel}_i(\Sigma F)_n \to \mathrm{Rel}_i(F)_{n+1}$$

is injective in degrees $n \geq \max(N+1, k(i+r))$. *If* F *is of split degree* r *at* $N \geq 0$, *then the map is injective for* $n \geq \max(N+1, ki+r)$.

Proof We begin by proving the first part of the statement. As $\mathrm{Rel}_*(-)$ is functorial in the coefficient system, injectivity of the split case is clear. The remaining claims of the first statement follow from the long exact sequences in $\mathrm{Rel}_*(-)$ induced by the short exact sequences

$$0 \to \ker(F) \to F \to \mathrm{im}(F \to \Sigma F) \to 0$$

and

$$0 \to \mathrm{im}(F \to \Sigma F) \to \Sigma F \to \mathrm{coker}(F) \to 0$$

by applying ($\boldsymbol{H}_{r,i}$), using that $\ker(F)$ has degree -1 at N and that $\mathrm{coker}(F)$ has (split) degree $r-1$ at $N-1$. To prove (ii), we use the spectral sequence (15) and Lemma 4.17. Since $|R_\bullet(\mathcal{M})|_m \to \mathcal{M}_m$ is assumed to be $\left(\frac{m-2+k}{k}\right)$–connected for $m \geq 1$, the groups $\mathrm{H}_*(\mathcal{M}_m, |R_\bullet(\mathcal{M})|_m; F)$ vanish for $* \leq \frac{m-2}{k}$, from which we conclude $E^\infty_{p,q,n+1} = 0$ for $p+q \leq \frac{n}{k}$. We claim that the differential $E^1_{1,i,n+1} \to E^1_{0,i,n+1}$ vanishes for $n \geq \max(N+1, k(i+r))$ in the nonsplit case, and for $n \geq \max(N+1, ki+r)$ in the split one. By Lemma 4.18, this is the case if the maps $\mathrm{Rel}_*(F)_{n-1} \to \mathrm{Rel}_*(\Sigma F)_{n-1} \to \mathrm{Rel}_*(\Sigma^2 F)_{n-1}$ are surjective in that range, which holds by (i). Since the map we want to prove injectivity of agrees with the differential $E^1_{0,i,n+1} \to E^1_{-1,i,n+1}$ by Lemma 4.17, it is therefore enough to show that, in the ranges of the statement, $E^\infty_{0,i,n+1} = 0$ and $E^2_{p,q,n+1} = 0$ for (p,q) with $p+q = i+1$ and $q < i$. By the vanishing range of E^∞ noted above, we have $E^\infty_{0,i,n+1} = 0$ in the required range. The claimed vanishing of E^2 follows from the vanishing even on the E^1–page, which is proved by observing that, by ($\boldsymbol{H}_{r,i}$) and Lemma 4.17, the groups $E^1_{p,q,n+1} \cong \mathrm{Rel}_q(\Sigma^{p+1}F)_{n-p}$ vanish for (p,q) with $q < i$ and $n \geq \max(N-p, k(q+r))$ in the nonsplit case, and for (p,q) satisfying $q < i$ and $n \geq \max(N-p, kq+r)$ in the split case, since $\Sigma^{p+1}F$ has (split) degree r at $N-p-1$ by Lemma 4.9. □

5 Configuration spaces

The *ordered configuration space* of a manifold W with labels in a Serre fibration $\pi\colon E \to W$ is given by

$$F^\pi_n(W) = \{(e_1,\dots,e_n) \in E^n \mid \pi(e_i) \neq \pi(e_j) \text{ for } i \neq j \text{ and } \pi(e_i) \in W \backslash \partial W\},$$

and the *unordered configuration space* is the quotient by the canonical action of the symmetric group,

$$C_n^\pi(W) = F_n^\pi(W)/\Sigma_n.$$

To establish an E_1–module structure on the unordered configuration spaces of W, we assume that W has nonempty boundary, fix a collar $(-\infty, 0] \times \partial W \to W$, and attach an infinite cylinder to the boundary,

$$\widetilde{W} = W \cup_{\{0\}\times\partial W} [0,\infty) \times \partial W.$$

Collar and cylinder assemble to an embedding $\mathbb{R} \times \partial W \subseteq \widetilde{W}$, of which we make frequent use henceforth. We extend the fibration π over $\widetilde{W}$ by pulling it back along the retraction $\widetilde{W} \to W$ and define the space

$$\widetilde{C}_n^\pi(W) = \{(s,e) \in [0,\infty) \times C_n^\pi(\widetilde{W}) \mid \pi(e) \subseteq W \cup (-\infty, s) \times \partial W\},$$

which is an equivalent model for $C_n^\pi(W)$, since the inclusion in $\widetilde{C}_n^\pi(W)$ as the subspace with $s = 0$ can be seen to be an equivalence by choosing an isotopy of $\widetilde{W}$ that pushes $[0,\infty) \times \partial W$ into $(-\infty, 0) \times \partial W$. We furthermore fix an embedded cube $(-1,1)^{d-1} \subseteq \partial W$ of codimension 0, together with a section $l\colon (-1,1)^{d-1} \to E$ of π, which we extend canonically to a section l on $[0,\infty) \times (-1,1)^{d-1} \subseteq \mathbb{R} \times \partial W \subseteq \widetilde{W}$.

Lemma 5.1 *Configurations $\coprod_{n\geq 0} C_n(D^d)$ in a disc form a graded E_d–algebra, and configurations $\coprod_{n\geq 0} \widetilde{C}_n^\pi(W)$ in a d–manifold W with nonempty boundary form an E_1–module over it, both graded by the number of points.*

Proof The operad $\mathcal{D}^\bullet(D^d)$ of little d–discs acts on $\coprod_{n\geq 0} C_n(D^d)$ by

$$\begin{aligned}\theta\colon \mathcal{D}^k(D^d) \times \big(\textstyle\coprod_{n\geq 0} C_n(D^d)\big)^k &\to \textstyle\coprod_{n\geq 0} C_n(D^d),\\ ((\phi_1,\ldots,\phi_k),(\{d_i^1\},\ldots,\{d_i^k\})) &\mapsto \textstyle\bigcup_{j=1}^k \phi_j(\{d_i^j\}),\end{aligned}$$

and this action extends to one of $\mathcal{SC}_d$ (see Definition 2.1) on the pair of spaces $\big(\coprod_{n\geq 0} \widetilde{C}_n^\pi(W), \coprod_{n\geq 0} C_n(D^d)\big)$ via

$$\theta\colon \mathcal{SC}_d(\mathfrak{m}, \mathfrak{a}^k; \mathfrak{m}) \times \textstyle\coprod_{n\geq 0} \widetilde{C}_n^\pi(W) \times \big(\coprod_{n\geq 0} C_n(D^d)\big)^k \to \coprod_{n\geq 0} \widetilde{C}_n^\pi(W),$$

defined as

$$((s,\phi_1,\ldots,\phi_k),(s',\{e_i\}),\{d_i^1\},\ldots,\{d_i^k\}) \mapsto \big(s'+s, \{e_i\} \cup \big(\textstyle\bigcup_{j=1}^k l(\phi_j(\{d_i^j\})+s')\big)\big),$$

using the section l and the translation $(-+s')$ by s' in the $[0,\infty)$–coordinate, as illustrated in Figure 4. □

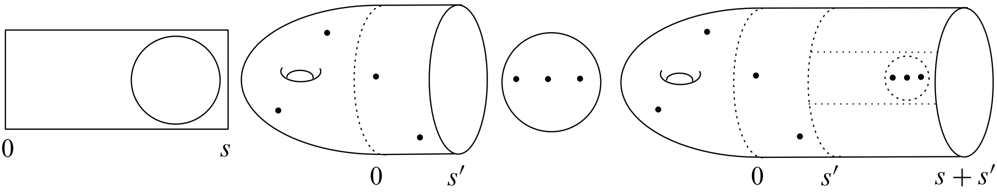


Figure 4: The E_1–module structure on unordered configuration spaces.

5.1 The resolution by arcs

Let W be a smooth connected manifold of dimension $d \geq 2$ with nonempty boundary and $\pi\colon E \to W$ a Serre fibration with path-connected fibre. By Lemma 5.1, the configuration spaces $\mathcal{M} = \coprod_{n \geq 0} \widetilde{C}^{\pi}_n(W)$ form a graded E_1–module over $\mathcal{A} = \coprod_{n \geq 0} C_n(D^d)$ considered as an E_2–algebra via the canonical morphism $\mathcal{SC}_2 \to \mathcal{SC}_d$; see Section 2.1. The stabilisation map $s\colon \mathcal{M} \to \mathcal{M}$ with respect to a chosen stabilising object $X \in C_1(D^d)$, restricted to the subspace of elements of degree n, has the form

$$s\colon \widetilde{C}^{\pi}_n(W) \to \widetilde{C}^{\pi}_{n+1}(W).$$

Remark 5.2 With respect to the described equivalence $C^{\pi}_n(W) \simeq \widetilde{C}^{\pi}_n(W)$, the stabilisation map corresponds to the map $C^{\pi}_n(W) \to C^{\pi}_{n+1}(W)$ that adds a point "near infinity"; see [46; 67].

We prove high-connectivity of the canonical resolution of $\mathcal{M}$ (see Section 2.2) by identifying it with the *resolution by arcs* — an augmented semi-simplicial space of geometric nature, known to be highly connected.

Definition 5.3 The *resolution by arcs* is the augmented semi-simplicial space

$$R^{\bullet-}_{\bullet}(\mathcal{M}) \to \mathcal{M}$$

with

$$R^{\bullet-}_p(\mathcal{M}) \subseteq \mathcal{M} \times (\mathrm{Emb}([-1,0], \widetilde{W}) \times \mathrm{Maps}([-1,0], E))^{p+1},$$

consisting of tuples $((s, \{e_i\}), (\varphi_0, \eta_0), \dots, (\varphi_p, \eta_p))$ such that

(i) the arcs φ_i are pairwise disjoint and connect points in the configuration $\varphi_i(-1) \in \pi(\{e_i\}) \subseteq \widetilde{W}$ to points $\varphi(0) \in \{s\} \times (-1,1) \times \{0\}^{d-2} \subseteq [0,\infty) \times \partial W$ in the order $\varphi_0(0) < \dots < \varphi_p(0)$,

(ii) the interiors of the arcs lie in $W \cup [0,s) \times \partial W$ and are disjoint from the configuration $\pi(\{e_i\})$,

(iii) the path of labels η_i satisfies $(\pi \circ \eta_i) = \varphi_i$ and connects the label of $\varphi_i(-1) \in \pi(\{e_i\})$ to $\eta_i(0) = l(\varphi_i(0))$, and

(iv) there exists $\varepsilon \in (0,1)$ such that

$$\varphi_i(t) = (s+t, \varphi_i(0), 0, \dots, 0) \in (-\infty, s] \times (-1,1)^{d-1} \subseteq \widetilde{W} \quad \text{for } t \in (-\varepsilon, 0].$$

$R_p^{\bullet-}(\mathcal{M})$ is topologised using the compact–open topology on $\mathrm{Maps}([-1,0], E)$ and the $\mathcal{C}^\infty$–topology on $\mathrm{Emb}([-1,0], \widetilde{W})$. The i^{th} face map forgets (φ_i, η_i). The rightmost subfigure of Figure 5 depicts an example.

Theorem 5.4 *The resolution by arcs $R_\bullet^{\bullet-}(\mathcal{M}) \to \mathcal{M}$ is graded $(g_{\mathcal{M}}-1)$–connected.*

Proof Setting $s=0$ in the definition of $R_p^{\bullet-}(\mathcal{M})$ yields a semi-simplicial subspace $\bar{R}_\bullet^{\bullet-}(\mathcal{M}) \subseteq R_\bullet^{\bullet-}(\mathcal{M})$, augmented over $\bar{\mathcal{M}} = \coprod_{n\geq 0} C_n^\pi(W)$. As the inclusion is a weak equivalence by the same argument as for $C_n^\pi(W) \subseteq \widetilde{C}_n^\pi(W)$, the augmented semi-simplicial space $R_\bullet^{\bullet-}(\mathcal{M}) \to \mathcal{M}$ is as connected as $\bar{R}_\bullet^{\bullet-}(\mathcal{M}) \to \bar{\mathcal{M}}$ is. The latter is the standard resolution by arcs for configurations of unordered points, which is known to have the claimed connectivity; see eg the proof of [43, Theorem A.1]. □

Theorem 5.5 *The canonical resolution and the one by arcs are weakly equivalent as augmented $\widetilde{\Delta}_{\mathrm{inj}}$–spaces.*

Assuming Theorem 5.5, Theorem 5.4 ensures graded $(g_{\mathcal{M}}-1)$–connectivity of the canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ (see Section 2.2), which in turn implies Theorem D by an application of Theorems A and C.

We prove Theorem 5.5 via a zigzag of weak equivalences of augmented $\widetilde{\Delta}_{\mathrm{inj}}$–spaces

$$(16)\quad R_\bullet(\mathcal{M}) \xleftarrow{①} \mathrm{B}(U\mathcal{O}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})) \overset{②}{\simeq} \mathrm{B}(U\mathcal{O}^{\bullet-}_{\bullet,\blacksquare}, U\mathcal{O}, B_\blacksquare(\mathcal{M})) \xrightarrow{③} R_\bullet^{\bullet-}(\mathcal{M})^{\mathrm{fib}}$$

between the canonical resolution $R_\bullet(\mathcal{M})$ and the fibrant replacement $R_\bullet^{\bullet-}(\mathcal{M})^{\mathrm{fib}}$ of the resolution by arcs, which is weakly equivalent to the resolution by arcs itself; see Section 1.4. The remainder of this subsection serves to explain the weak equivalences ①–③. We abbreviate the $E_{1,2}$–operad $\mathcal{SC}_2$ by $\mathcal{O}$.

① Recall from Section 2.2 the category $U\mathcal{O}$ and the contravariant $U\mathcal{O}$–space $B_\bullet(\mathcal{M})$ over $\mathcal{M}$ whose restriction to the subcategory $\widetilde{\Delta}_{\mathrm{inj}} \subseteq U\mathcal{O}$ is $R_\bullet(\mathcal{M})$. Using the $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{O})$–space $U\mathcal{O}(\bullet, \blacksquare)$ obtained by restricting the hom-functor of $U\mathcal{O}$, the equivalence ① is the restriction of the bar resolution of $B_\bullet(\mathcal{M})$ to $\widetilde{\Delta}_{\mathrm{inj}}$; see Section 1.4.

For the other parts of the zigzag (16), we define an analogue of the resolution by arcs for the free graded E_1–module $\mathcal{O}^{\mathfrak{m}} = \coprod_{n \geq 0} \mathcal{O}(\mathfrak{m}, \mathfrak{a}^n; \mathfrak{m})/\Sigma_n$; see Example 2.18. For simplification, we choose the centre $X = \{0\} \in C_1(D^d)$ as stabilising object and write s_d for the parameter of elements $d = (s_d, \{\phi_i\}) \in \mathcal{O}^{\mathfrak{m}}$ and $g(d)$ for their degree, ie the cardinality of the set of embeddings $\{\phi_i\}$.

Definition 5.6 Define the augmented semi-simplicial space $R_\bullet^{\bullet-}(\mathcal{O}^{\mathfrak{m}}) \to \mathcal{O}^{\mathfrak{m}}$ with p–simplices

$$R_p^{\bullet-}(\mathcal{O}^{\mathfrak{m}}) \subseteq \mathcal{O}^{\mathfrak{m}} \times \mathrm{Emb}([-1,0], (0,\infty) \times (-1,1))^{p+1},$$

consisting of tuples $((s, \{\phi_j\}), \varphi_0, \dots, \varphi_p)$ such that

(i) the arcs φ_i are pairwise disjoint and connect centre points $\varphi_i(-1) \in \{\phi_j(0)\}$ of the discs to $\varphi_i(0) \in \{s\} \times (-1,1)$ in the order $\varphi_0(0) < \cdots < \varphi_p(0)$,

(ii) the interiors of the arcs lie in $(0,s) \times (-1,1)$ and are disjoint from the centre points $\{\phi_j(0)\}$, and

(iii) there exists an $\varepsilon \in (0,s)$ such that $\varphi_i(t) = (s+t, \varphi_i(0)) \in (0,s] \times (-1,1)$ holds for all $t \in (-\varepsilon, 0]$.

The third subfigure of Figure 5 exemplifies a 0–simplex in $R_\bullet^{\bullet-}(\mathcal{O}^{\mathfrak{m}})$.

② To explain the second equivalence of (16), we note that $\mathcal{O}^{\mathfrak{m}}$ becomes a topological monoid by multiplying elements d and e in $\mathcal{O}^{\mathfrak{m}}$ by $\gamma(e; d, 1^{g(e)})$. The multiplication map is covered by a simplicial action

$$\begin{aligned} \Psi\colon \mathcal{O}^{\mathfrak{m}} \times R_\bullet^{\bullet-}(\mathcal{O}^{\mathfrak{m}}) &\to R_\bullet^{\bullet-}(\mathcal{O}^{\mathfrak{m}}), \\ (d, (e, \varphi_0, \dots, \varphi_p)) &\mapsto (\gamma(e; d, 1^{g(e)}), \varphi_0 + s_d, \dots, \varphi_p + s_d), \end{aligned}$$

where $(- + s_d)$ is the translation in the $(0,\infty)$–coordinate. This action leads to a $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{O})$–space $U\mathcal{O}_{\bullet,\blacksquare}^{\bullet-}$, serving us as mediator between the canonical resolution and the one by arcs. On objects $([p],[k])$, it is

$$\begin{aligned} U\mathcal{O}_{p,k}^{\bullet-} &= \mathrm{hofib}_{c_{k+1}}(R_p^{\bullet-}(\mathcal{O}^{\mathfrak{m}}) \to \mathcal{O}^{\mathfrak{m}}) \\ &= \{(e, \varphi_0, \dots, \varphi_p, \mu) \in R_p^{\bullet-}(\mathcal{O}^{\mathfrak{m}}) \times \mathrm{Path}_{c_{k+1}} \mathcal{O}^{\mathfrak{m}} \mid \omega(\mu) = e\}, \end{aligned}$$

where $\omega(-)$ denotes the endpoint of a Moore path and the elements $c_i \in \mathcal{O}^{\mathfrak{m}}$ are defined as in Section 2.2. The $\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$–direction of $U\mathcal{O}_{\bullet,\blacksquare}^{\bullet-}$ is induced by the semi-simplicial structure of $R_{\bullet}^{\bullet-}(\mathcal{O}^{\mathrm{m}})$ via the functor $\widetilde{\Delta}_{\mathrm{inj}} \to \Delta_{\mathrm{inj}}$; see Section 2.2. The $U\mathcal{O}$–direction is defined by

$$\begin{gathered} U\mathcal{O}([k],[l]) \times U\mathcal{O}^{\bullet-}_{\bullet,k} \to U\mathcal{O}^{\bullet-}_{\bullet,l}, \\ ((d,\mu),(e,\varphi_0,\dots,\varphi_p,\zeta)) \mapsto (\Psi(d,(e,\varphi_0,\dots,\varphi_p)), \mu\cdot\gamma(\zeta;d,1^{k+1})). \end{gathered}$$

The claimed functoriality of $\mathcal{O}^{\bullet-}_{\bullet,\blacksquare}$ follows directly from the associativity of the operadic composition γ. Having introduced the objects involved, the following lemma provides the weak equivalence 2.

Lemma 5.7 *The $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O})$–spaces $U\mathcal{O}^{\bullet-}_{\bullet,\blacksquare}$ and $U\mathcal{O}(\bullet,\blacksquare)$ are weakly equivalent.*

Proof Choose arcs $\varphi^p = (\varphi^p_0,\dots,\varphi^p_p) \in \mathrm{Emb}([-1,0],(0,\infty)\times(-1,1))^{p+1}$ such that (c_{p+1},φ^p) forms an element of $R_p^{\bullet-}(\mathcal{O}^{\mathrm{m}})$ for which the order of the embeddings $\{\phi_i\}$ in $c_{p+1} = (s_{c_{p+1}},\{\phi_i\})$, induced by the order of the arcs φ^p_i they are connected to, agrees with the order of $\{\phi_i\}$ induced by the $(0,\infty)$–coordinate. Acting on $(c_{p+1},\varphi^p,\mathrm{const}_{c_{p+1}}) \in U\mathcal{O}^{\bullet-}_{p,p}$, the $(\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \times U\mathcal{O})$–space $U\mathcal{O}^{\bullet-}_{\bullet,\blacksquare}$ induces a morphism

$$U\mathcal{O}([p],\blacksquare) \to U\mathcal{O}^{\bullet-}_{p,\blacksquare} \tag{17}$$

of $U\mathcal{O}$–spaces, which agrees on $[k] \in \mathrm{ob}(U\mathcal{O})$ with the induced map on diagonal homotopy fibres of the commuting triangle

$$\begin{tikzcd} \mathcal{O}^{\mathrm{m}} \arrow[rr] \arrow[dr, "\gamma(c_{p+1};-,1^{p+1})"'] & & R_p^{\bullet-}(\mathcal{O}^{\mathrm{m}}) \arrow[dl] \\ & \mathcal{O}^{\mathrm{m}} & \end{tikzcd}$$

at c_{k+1}, where the right diagonal map is the augmentation and the horizontal arrow is given by acting on (c_{p+1},φ^p) via Ψ. There is a map $R_p^{\bullet-}(\mathcal{O}^{\mathrm{m}}) \to \mathcal{O}^{\mathrm{m}}$ that forgets the arcs and the discs attached to them, using which the horizontal map can be seen to be an equivalence by following discs along arcs they are attached to. Hence, (17) is an equivalence of $U\mathcal{O}$–spaces, which in particular shows that $U\mathcal{O}^{\bullet-}_{\bullet,\blacksquare}$ is homotopy discrete, as $U\mathcal{O}(\bullet,\blacksquare)$ is so by Lemma 2.11. Therefore, to prove the claim, it is sufficient to show that the equivalence (17) is natural in $[p]$ up to homotopy, which would follow from the homotopy commutativity of

$$\begin{tikzcd} U\mathcal{O}([p],[k]) \arrow[r] \arrow[d, "(\tilde{d}_i)_*"'] & U\mathcal{O}^{\bullet-}_{p,k} \arrow[d, "(d_i)_*"] \\ U\mathcal{O}([p-1],[k]) \arrow[r] & U\mathcal{O}^{\bullet-}_{p-1,k} \end{tikzcd}$$

using the choice of face maps $\widetilde{d}_i = (c, \mu_i) \in \widetilde{\Delta}_{\mathrm{inj}}([p-1],[p])$ provided by Lemma 2.15. The two compositions of the latter diagram map an element (d,ζ) in $U\mathcal{O}([p],[k])$ to

$$(\Psi(d,(c_{p+1},\varphi_0^p,\ldots,\widehat{\varphi_i^p},\ldots,\varphi_p^p)),\zeta)$$

and

$$(\Psi(d,(c_{p+1},\varphi_0^{p-1}+s_c,\ldots,\varphi_{p-1}^{p-1}+s_c)),\zeta\cdot\gamma(\mu_i;d,1^{p+1})),$$

respectively, where $\widehat{(-)}$ indicates that the element is omitted. Recalling that, via the isomorphism $\pi_1(\mathcal{O}^{\mathrm{m}},c_{p+1})\cong B_{p+1}$ fixed in Section 2.2, the loop $\mu_i\in\Omega_{c_{p+1}}\mathcal{O}^{\mathrm{m}}$ corresponds to the braid $b^{-1}_{X^{\oplus i},X}\oplus X^{\oplus p-i}$ in B_{p+1}, we see that our choice of the arcs φ_i^j ensures the existence of a path in $R^{\bullet\!-}_{p-1}(\mathcal{O}^{\mathrm{m}})$ between

$$(c_{p+1},\varphi_0^p,\ldots,\widehat{\varphi_i^p},\ldots,\varphi_p^p) \quad\text{and}\quad (c_{p+1},\varphi_0^{p-1}+s_c,\ldots,\varphi_{p-1}^{p-1}+s_c)$$

that maps, via the augmentation $R^{\bullet\!-}_{p-1}(\mathcal{O}^{\mathrm{m}})\to\mathcal{O}^{\mathrm{m}}$, to μ_i, or at least to its homotopy class. Such a path induces a homotopy between the two compositions of the square, which finishes the proof. $\square$

③ For the rightmost equivalence of (16), we use the module structure θ to define the simplicial map

$$\Phi\colon R^{\bullet\!-}_{\bullet}(\mathcal{O}^{\mathrm{m}})\times\mathcal{M}\to R^{\bullet\!-}_{\bullet}(\mathcal{M}) \tag{18}$$

by the assignment

$$((e,\varphi_0,\ldots,\varphi_p),A)\mapsto\big(\theta(e,A,1^{g(e)}),(\varphi_0+s_A,l(\varphi_0+s_A)),\ldots,(\varphi_0+s_A,l(\varphi_p+s_A))\big)$$

using the embedding $[0,\infty)\times(-1,1)\times\{0\}^{d-2}\subseteq[0,\infty)\times\partial W$, the translation in the $[0,\infty)$–coordinate, and the section $l\colon[0,\infty)\times(-1,1)^{d-1}\to E$, as illustrated in Figure 5. This yields simplicial maps for $k\ge 0$,

$$\begin{gathered}U\mathcal{O}^{\bullet\!-}_{\bullet,k}\times B_k(\mathcal{M})\to R^{\bullet\!-}_{\bullet}(\mathcal{M})^{\mathrm{fib}},\\ ((e,\varphi_0,\ldots,\varphi_p,\mu),(A,\zeta))\mapsto\big(\Phi((e,\varphi_0,\ldots,\varphi_p),A),\zeta\cdot\theta(\mu;A,X^{k+1})\big),\end{gathered}\tag{19}$$

which induce a morphism $\mathrm{B}(U\mathcal{O}^{\bullet\!-}_{\bullet,\blacksquare},U\mathcal{O},B_{\blacksquare}(\mathcal{M}))\to R^{\bullet\!-}_{\bullet}(\mathcal{M})^{\mathrm{fib}}$, since they equalise the diagram

$$\coprod_{f\in U\mathcal{O}([k],[l])}U\mathcal{O}^{\bullet\!-}_{\bullet,k}\times B_l(\mathcal{M})\ \underset{f_*\times\mathrm{id}}{\overset{\mathrm{id}\times f^*}{\rightrightarrows}}\ \coprod_{[k]}U\mathcal{O}^{\bullet\!-}_{\bullet,k}\times B_k(\mathcal{M}).$$

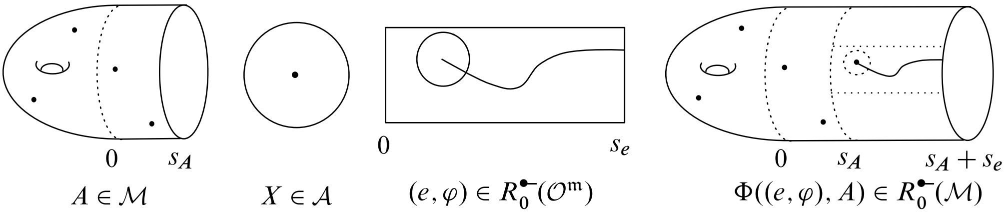


Figure 5: The resolution by arcs and the map Φ.

This explains the morphism (3), which is a weak equivalence by the following lemma that completes the proof of Theorem 5.5, as the morphisms (1)–(3) are all compatible with the augmentation to $\mathcal{M}$.

Lemma 5.8 *The morphism (3) is a weak equivalence.*

Proof On p–simplices, the weak equivalence (1) and the morphism (3) fit into a commutative square

$$\begin{tikzcd}
\mathrm{B}(U\mathcal{O}([p],\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})) \arrow[r, "(1)"] \arrow[d, "\simeq"'] & R_p(\mathcal{M}) \arrow[d, "\simeq"] \\
\mathrm{B}(U\mathcal{O}^{\bullet-}_{p,\blacksquare}, U\mathcal{O}, B_\blacksquare(\mathcal{M})) \arrow[r, "(3)"] & R_p^{\bullet-}(\mathcal{M})^{\mathrm{fib}}
\end{tikzcd}$$

in which the left morphism is induced by the equivalence (17), which has the form $U\mathcal{O}([p],\blacksquare) \to U\mathcal{O}^{\bullet-}_{p,\blacksquare}$ and was defined via the action of $U\mathcal{O}^{\bullet-}_{\bullet,\blacksquare}$ on a certain element $(c_{p+1}, \varphi^p, \mathrm{const}_{c_{p+1}}) \in U\mathcal{O}^{\bullet-}_{p,p}$. The right map is induced by the same element, but using (19), and it is a weak equivalence by an analogous argument as for (17). Consequently, the arrow (3) is a weak equivalence as well and we conclude the assertion. □

Remark 5.9 As demonstrated in the previous proof, the right vertical arrow of the preceding square is a weak equivalence for all $p \geq 0$. However, it does not form a (strict) morphism of $\widetilde{\Delta}_{\mathrm{inj}}$–spaces between $R_\bullet(\mathcal{M})$ and $R_\bullet^{\bullet-}(\mathcal{M})^{\mathrm{fib}}$. There is an alternative proof of Theorem 5.5 by showing that these weak equivalences can be enhanced to a morphism up to higher coherent homotopy between $R_\bullet(\mathcal{M})$ and $R_\bullet^{\bullet-}(\mathcal{M})^{\mathrm{fib}}$.

5.2 Coefficient systems for configuration spaces

Recall from Section 2.1 that the E_1–module structure on $\mathcal{M} = \coprod_{n\geq 0} \widetilde{C}_n^{\pi}(W)$ over $\mathcal{A} = \coprod_{n\geq 0} C_n(D^d)$ induces a right-module structure $\oplus$ on the fundamental groupoid $\Pi(\mathcal{M})$ over the braided monoidal category $(\Pi(\mathcal{A}), \oplus, b, 0)$ and hence, after fixing a stabilising object $X \in C_1(D^n)$, also one over the free braided monoidal category $\mathcal{B}$ on one object. Denoting by $A \in C_0^\pi(W)$ the empty configuration, a coefficient system for $\coprod_{n\geq 0} \widetilde{C}_n^\pi(W)$ is, by Remark 4.10, specified by

(i) a $\pi_1(\widetilde{C}_n^\pi(W), A\oplus X^{\oplus n})$–module M_n for each $n \geq 0$, together with

(ii) $(-\oplus X)$–equivariant morphisms $\sigma\colon M_n \to M_{n+1}$ such that B_m acts trivially via $(A \oplus X^{\oplus n} \oplus -)$ on the image of $\sigma^m\colon M_n \to M_{n+m}$.

Equivalently, a coefficient system is an abelian group-valued functor on Quillen's bracket construction

$$\mathcal{C}^\pi(W) := \langle \textstyle\coprod_{n\geq 0} \pi_1(\widetilde{C}_n^\pi(W)), \mathcal{B}\rangle;$$

compare with Remark 4.12. Using the ordering of $A \oplus X^{\oplus n}$ induced by the $[0,\infty)$–coordinate, a loop γ in $\widetilde{C}_n^\pi(W)$ induces a permutation in n letters, as well as n ordered loops in E by connecting the paths in E forming γ to a basepoint in E via paths in the image of the section $l\colon [0,\infty) \times (-1,1)^{d-1} \to E$. This induces a morphism

$$\pi_1(\widetilde{C}_n^\pi(W)) \to \pi_1(E) \wr \Sigma_n \tag{20}$$

to the wreath product, which we use to relate $\mathcal{C}^\pi(W)$ to other categories via a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^\pi(W) \arrow[r] \arrow[d] & \langle \pi_1(E) \wr \Sigma, \Sigma\rangle \arrow[d] & \\
\mathcal{B}^\pi(W) \arrow[r] \arrow[d, phantom, "\cap" description, "\mathrm{I}"] & \mathcal{FI}_{\pi_1(E)} \arrow[r] \arrow[d, phantom, "\cap" description] & \mathcal{FI} \arrow[d, phantom, "\cap" description] \\
\mathcal{B}^\pi(W)^\sharp \arrow[r] & \mathcal{FI}^\sharp_{\pi_1(E)} \arrow[r] & \mathcal{FI}^\sharp
\end{tikzcd} \tag{21}$$

on which we elaborate in the following.

The category $\langle \pi_1(E)\wr\Sigma, \Sigma\rangle$ results from the action of $\Sigma = \coprod_{n\geq 0}\Sigma_n$ on $\pi_1(E)\wr\Sigma = \coprod_{n\geq 0}\pi_1(E)\wr\Sigma_n$. It receives a functor from $\mathcal{C}^\pi(W)$, induced by the morphisms (20). The category $\mathcal{FI}_{\pi_1(E)}$ of finite sets and injective $\pi_1(E)$–maps [11; 29; 57; 63] is isomorphic to $\langle \pi_1(E)\wr\Sigma, \pi_1(E)\wr\Sigma\rangle$, so is the target of a functor from $\langle \pi_1(E)\wr\Sigma, \Sigma\rangle$, induced by the inclusion $\Sigma \subseteq \pi_1(E)\wr\Sigma$. By forgetting $\pi_1(E)$, the category $\mathcal{FI}_{\pi_1(E)}$ maps to the category $\mathcal{FI}$ of finite sets and injections, on which functors are studied in the context of representation stability; see eg [14; 15]. Both $\mathcal{FI}$ and $\mathcal{FI}_{\pi_1(E)}$ are subcategories of larger categories $\mathcal{FI}^\sharp$ and $\mathcal{FI}^\sharp_{\pi_1(E)}$ of partially defined injections or $\pi_1(E)$–injections [14; 63]. The category of *partial braids* $\mathcal{B}^\pi(W)^\sharp$ has the nonnegative integers as its objects and a morphism from n to m is a pair (k,μ) with $k \leq \min(n,m)$

and μ a morphism in $\Pi(\widetilde{C}_k^\pi(W))$ from a subset of $A\oplus X^{\oplus n}$ to one of $A\oplus X^{\oplus m}$. For trivial π, the category $\mathcal{B}^\pi(W)^\sharp$ was studied by Palmer [52], who also introduced the subcategory $\mathcal{B}^\pi(W)\subseteq\mathcal{B}^\pi(W)^\sharp$ of *full braids*, consisting of morphisms $(k,\mu)\colon n\to m$ with $k=n$. There is a functor $\mathcal{C}^\pi(W)\to\mathcal{B}^\pi(W)$ which is the identity on objects and maps a morphism

$$[\gamma]\in\mathcal{C}^\pi(W)(n,m)=\pi_1(\widetilde{C}_m^\pi(W),A\oplus X^{\oplus m})/B_{m-n}$$

to the path in $\widetilde{C}_n^\pi(W)$ that forms the first n paths in E of γ, ie the ones starting at $A\oplus X^{\oplus n}\subseteq A\oplus X^{\oplus m}$. For $W=D^2$ and $\pi=\mathrm{id}_{D^2}$, the category $\mathcal{B}^\pi(W)$ was considered by Schlichtkrull and Solberg [64].

Remark 5.10 If W is of dimension $d\geq 3$, then the morphisms (20) are isomorphisms [70, Lemma 4.1], from which it follows that the three left horizontal functors in the diagram (21) are isomorphisms. If E is in addition simply connected, then all functors except for the lower vertical inclusions are isomorphisms.

We call an abelian group-valued functor on a category $\mathcal{C}$ of the diagram (21) a *coefficient system* on $\mathcal{C}$. There is a notion of being of *(split) degree* r at an integer N for coefficient systems on any of the categories $\mathcal{C}$, defined analogously to Definition 4.6 by using an endofunctor Σ on $\mathcal{C}$ together with a natural transformation $\sigma\colon\mathrm{id}\to\Sigma$, similar to $\mathcal{C}^\pi(W)$; see Remark 4.12. Most categories of the diagram are of the form $\langle\mathcal{N},\mathcal{G}\rangle$ for a braided monoidal groupoid $\mathcal{G}$ acting on a category $\mathcal{N}$ and for such categories, Σ and σ are defined as in Remark 4.12. For $\mathcal{B}^\pi(W)^\sharp$, the functor Σ maps a morphism (k,μ) to $(k+1,s(\mu))$ using the stabilisation, and σ consists of the constant paths at $A\oplus X^{\oplus n}$. For $\mathcal{B}^\pi(W)$, we obtain Σ and σ by restriction from $\mathcal{B}^\pi(W)^\sharp$. For $\mathcal{FI}^\sharp$ and $\mathcal{FI}^\sharp_{\pi_1(E)}$, the definition is analogous. Note that the morphisms σ of the categories with a $\sharp$ superscript admit left inverses, so all coefficient systems on them are split.

As all functors in the diagram are compatible with Σ and σ, the property of being of (split) degree r at N is preserved by pulling back coefficient systems along them. In conclusion, by pulling back to $\mathcal{C}^\pi(W)$, all coefficient systems of finite degree on any of the categories in the diagram induce coefficient systems for which the homology of $\widetilde{C}_n^\pi(W)$ stabilises by Theorem D. The degree of coefficient systems on some of the categories has been examined before, providing us with a wealth of examples.

Example 5.11 (i) The (split) degree of coefficient systems on *pre-braided monoidal categories* was introduced in [61]. This includes the categories $\langle\pi_1(E)\wr\Sigma,\Sigma\rangle$, $\mathcal{FI}_{\pi_1(E)}$, $\mathcal{FI}^\sharp_{\pi_1(E)}$, $\mathcal{FI}$, and $\mathcal{FI}^\sharp$.

(ii) A *finitely generated* coefficient system F on $\mathcal{FI}_{\pi_1(E)}$ in the sense of [63] is of finite degree provided that $\pi_1(E)$ is finite; see [63, Proposition 3.4.2]. By [63, Remark 3.4.3], this implication remains valid if $\pi_1(E)$ is virtually polycyclic (see the introduction for a definition) and even holds for arbitrary $\pi_1(E)$ if F is *presented in finite degree* or if F extends to $\mathcal{FI}^{\sharp}_{\pi_1(E)}$.

(iii) More quantitatively, coefficient systems on $\mathcal{FI}$ that are *generated in degree* $\leq k$ and *related in degree* $\leq d$, as defined in [13, Definition 4.1], are of degree k at $d+\min(k,d)$ by [61, Proposition 4.18].

(iv) The degree of a coefficient system on $\mathcal{B}^{\pi}(W)^{\sharp}$ has been studied by Palmer [52], who also provides examples of finite degree coefficient systems on $\mathcal{FI}^{\sharp}$; see [52, Section 4]. Note that the degree and the split degree of coefficient systems on these categories coincide.

(v) For $W=D^2$ and $\pi=\mathrm{id}_{D^2}$, the category $\mathcal{C}^{\pi}(W)$ is isomorphic to the category $U\mathcal{B}$ as recalled in Definition 2.6. The *Burau representation* gives rise to an example of a coefficient system of degree 1 at 0 on $U\mathcal{B}$ [61, Example 3.14]. On the basis of this example, Soulié [68] has constructed coefficient systems on $U\mathcal{B}$ of arbitrary degree, using the so-called *Long–Moody construction*.

Remark 5.12 Inspired by work of Betley [4], Palmer [52] proved homological stability for $C_n^{\pi}(W)$ for trivial fibrations π and coefficient systems of finite degree on $\mathcal{B}^{\pi}(W)^{\sharp}$. His surjectivity range agrees with ours, but his result includes split injectivity in all degrees — a phenomenon special to configuration spaces and not captured by our general approach. In his Remark 1.13, Palmer suspects stability for coefficient systems of finite degree on $\mathcal{B}^{\pi}(W)$. Theorem D confirms this and extends his result to a larger class of coefficient systems and nontrivial labels.

5.3 Applications

We complete the proofs of Corollaries F and G sketched in the introduction. Unless stated otherwise, W denotes a manifold satisfying the assumptions of Theorem D.

5.3.1 Configuration spaces of embedded discs Recall from the introduction the configuration spaces of (un)ordered k–discs $C_n^k(W)$ and $F_n^k(W)$ of W, the related subgroups $\mathrm{PDiff}^k_{\partial,n}(W)\subseteq\mathrm{Diff}^k_{\partial,n}(W)\subseteq\mathrm{Diff}_{\partial}(W)$ of diffeomorphisms fixing or permuting n chosen k–discs in W, respectively, and the orientation-preserving variants denoted with a $+$ superscript for $k=d$ and oriented W. The action of $\mathrm{Diff}_{\partial}(W)$ on

$C_n^{\pi_k}(W)$ extends to one on $\coprod_{n\geq 0}\widetilde{C}_n^{\pi_k}(W)$ by extending diffeomorphisms of W to $\widetilde{W}$ via the identity. This action commutes with the E_d–action of $\coprod_n C_n(D^d)$, so the Borel construction $\mathrm{EDiff}_\partial(W)\times_{\mathrm{Diff}_\partial(W)}\mathcal{M}$ inherits a graded E_1–module structure whose canonical resolution is highly connected by Example 2.21. Consequently, Theorems A and C imply (twisted) stability for $\mathrm{EDiff}_\partial(W)\times_{\mathrm{Diff}_\partial(W)}\widetilde{C}_n^{\pi_k}(W)$ for $k<d$ and, as the equivalence $C_n^k(W)\to C_n^{\pi_k}(W)\subseteq\widetilde{C}_n^{\pi_k}(W)$ (see the introduction for the first map) is equivariant, also for $\mathrm{EDiff}_\partial(W)\times_{\mathrm{Diff}_\partial(W)}C_n^k(W)$. The same argument applies to $\mathrm{EDiff}_\partial^+(W)\times_{\mathrm{Diff}_\partial^+(W)}C_n^{d,+}(W)$. As announced in the introduction, we identify these homotopy quotients with classifying spaces of certain diffeomorphism groups. This proves Corollary F.

Lemma 5.13 *For $k<d$, the Borel constructions $\mathrm{EDiff}_\partial(W)\times_{\mathrm{Diff}_\partial(W)}F_n^k(W)$ and $\mathrm{EDiff}_\partial(W)\times_{\mathrm{Diff}_\partial(W)}C_n^k(W)$ are models for the classifying spaces $\mathrm{BPDiff}_{\partial,n}^k(W)$ and $\mathrm{BDiff}_{\partial,n}^k(W)$, respectively. For $k=d$ and oriented W, the analogous identifications for the variants with $+$ superscripts hold.*

Proof It suffices to show that $\mathrm{Diff}_\partial(W)$ acts transitively on $F_n^k(W)$ and $C_n^k(W)$, since the stabilisers of these actions are precisely the subgroups $\mathrm{PDiff}_{\partial,n}^k(W)$ and $\mathrm{Diff}_{\partial,n}^k(W)$, respectively. The required transitivity follows from the fact that the map $\mathrm{Diff}_\partial(W)\to\mathrm{Emb}\big(\coprod^n D^k, W\backslash\partial W\big)$ given by acting on n fixed disjoint parametrised k–discs is, by [50], a fibre bundle with path-connected base space

$$\mathrm{Emb}\big(\textstyle\coprod^n D^k, W\setminus\partial W\big)\simeq F_n^{\pi_k}(W).$$

This same argument applies to $\mathrm{PDiff}_{\partial,n}^{d,+}(W)$ and $\mathrm{Diff}_{\partial,n}^{d,+}(W)$ by using orientation-preserving diffeomorphisms and embeddings, as the fibre of the bundle π_d^+ of oriented d–frames is path-connected. □

5.3.2 Representation stability We prove Corollary G, using the notation of the introduction.

Lemma 5.14 *Let W and π be as in Theorem D, and let $\lambda\vdash n$ be a partition. The V_λ–multiplicity in $\mathrm{H}^i(F_n^\pi(W);\mathbb{Q})$ is the dimension of $\mathrm{H}_i(C_n^\pi(W);V_\lambda)$, where $\pi_1(C_n^\pi(W))$ acts on V_λ via the morphism $\pi_1(C_n^\pi(W))\to\Sigma_n$.*

Proof Delooping the covering space $\Sigma_n\to F_n^\pi(W)\to C_n^\pi(W)$ once results in a fibration sequence with base space $\mathrm{B}\Sigma_n$. We consider the induced Serre spectral sequence, twisted by the local system V_λ on $\mathrm{B}\Sigma_n$,

$$E_{p,q}^2\cong\mathrm{H}_p\big(\mathrm{B}\Sigma_n;\mathrm{H}_q(F_n^\pi(W);V_\lambda)\big)\Rightarrow\mathrm{H}_{p+q}(C_n^\pi(W);V_\lambda).$$

Since the action of $\pi_1(F_n^\pi(W))$ on V_λ is trivial, we conclude

$$\mathrm{H}_p\big(\mathrm{B}\Sigma_n; \mathrm{H}_q(F_n^\pi(W); V_\lambda)\big) \cong \mathrm{H}_p\big(\mathrm{B}\Sigma_n; \mathrm{H}_q(F_n^\pi(W); \mathbb{Q}) \otimes V_\lambda\big).$$

These groups vanishes for $p \neq 0$ as Σ_n has no rational cohomology in positive degree. Hence, the E_2–page is trivial, except for the 0^{th} column, which is isomorphic to the coinvariants $\big(\mathrm{H}_q(F_n^\pi(W); \mathbb{Q}) \otimes V_\lambda\big)_{\Sigma_n}$, which are in turn isomorphic to the invariants $\big(\mathrm{H}_q(F_n^\pi(W); \mathbb{Q}) \otimes V_\lambda\big)^{\Sigma_n}$. As a result of this, the spectral sequence collapses and we can identify $\mathrm{H}_q(C_n^\pi(W); V_\lambda)$ with $\big(\mathrm{H}_q(F_n^\pi(W); \mathbb{Q}) \otimes V_\lambda\big)^{\Sigma_n}$, whose dimension equals the V_λ–multiplicity in $\mathrm{H}_q(F_n^\pi(W); \mathbb{Q})$, since $V_\mu \otimes V_\lambda$ for a partition $\mu \vdash n$ contains a trivial representation if and only if $\mu = \lambda$ and, in that case, it is 1–dimensional; see [26, Example 4.51]. This proves the claim, because the V_λ–multiplicity in $\mathrm{H}^i(F_n^\pi(W); \mathbb{Q})$ equals that in $\mathrm{H}_i(F_n^\pi(W); \mathbb{Q})$. □

Corollary 5.15 *For W and π as in Theorem D and large n relative to i, the $V_{\lambda[n]}$–multiplicity in $\mathrm{H}^i(F_n^\pi(W); \mathbb{Q})$ is independent of n.*

Proof By [14, Proposition 3.4.1], the Σ_n–representations $V_{\lambda[n]}$ assemble into a finitely generated $\mathcal{FI}$–module $V(\lambda)$ with $V(\lambda)_n \cong V_{\lambda[n]}$, which pulls back along $\mathcal{C}^\pi(W) \to \mathcal{FI}$ of (21) to a coefficient system of finite degree for $\coprod_{n\geq 0} \widetilde{C}_n(M)$ by Example 5.11(ii). Combining Theorem C with Lemma 5.14 gives the claim. □

Proof of Corollary G Corollary 5.15 settles the statement for $F_n^\pi(W)$. To derive the claim about $F_n^k(W)$, observe that the equivalence $C_n^k(W) \to C_n^{\pi_k}(W)$ discussed in the introduction is covered by a Σ_n–equivariant equivalence $F_n^k(W) \to F_n^{\pi_k}(W)$, so we have $\mathrm{H}^i(F_n^k(W); \mathbb{Q}) \cong \mathrm{H}^i(F_n^{\pi_k}(W); \mathbb{Q})$ as Σ_n–modules. The remaining part concerning $\mathrm{BPDiff}_{\partial,n}^k(W)$ is shown by using the model

$$\mathrm{BPDiff}_{\partial,n}^k(W) \simeq \mathrm{EDiff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} F_n^k(W)$$

provided by Lemma 5.13, and adapting the argument of Lemmas 5.14 and 5.15 by replacing the covering space $\Sigma_n \to F_n^\pi(W) \to C_n^\pi(W)$ with

$$\Sigma_n \to \mathrm{EDiff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} F_n^k(W) \to \mathrm{EDiff}_\partial(W) \times_{\mathrm{Diff}_\partial(W)} C_n^k(W).$$

The statements about the variants $F_n^{d,+}(W)$ and $\mathrm{PDiff}_n^{d,+}(W)$ are proved similarly. □

The following ranges resulted from a discussion with Peter Patzt, whom the author would like to thank.

Remark 5.16 To obtain explicit ranges for Corollary G, one can show that the $\mathcal{FI}$–module $V(\lambda)$, used in the proof of Corollary 5.15, is generated in degree $|\lambda|+\lambda_1$ and related in degree $|\lambda|+\lambda_1+1$, so the corresponding coefficient system has degree $|\lambda|+\lambda_1$ at $2|\lambda|+2\lambda_1+1$, by Example 5.11(iii). Consequently, one deduces that the $V_{\lambda[n]}$–multiplicities in the cohomology groups of Corollary G are constant for $i \le \frac{1}{2}n-(|\lambda|+\lambda_1+1)$. Note that our range is not uniform, ie depends on the partition. In contrast, the range obtained by Church [12] is $i \le \frac{1}{2}n$ if the dimension is $d \ge 3$, and $i \le \frac{1}{4}n$ for $d=2$, at least for the manifolds to which his result applies.

6 Moduli spaces of manifolds

Throughout the section, we fix a closed manifold P of dimension $d-1$ together with an embedding
$$P \subseteq \mathbb{R}^{d-1} \times \mathbb{R}^\infty$$
which contains the open unit cube $(-1,1)^{d-1} \times \{0\} \subseteq \mathbb{R}^{d-1} \times \mathbb{R}^\infty$ and satisfies $P \subseteq \mathbb{R}^{d-1} \times [0,\infty)^\infty$. We consider compact manifolds W with a specified identification $\partial W = P$ and denote by $\mathrm{Diff}_\partial(W)$ the group of diffeomorphisms fixing a neighbourhood of the boundary, equipped with the C^∞–topology. To construct our preferred model of its classifying space, we choose a collar $c\colon (-\infty,0] \times P \to W$ and denote by $\mathrm{Emb}_\varepsilon(W, (-\infty,0] \times \mathbb{R}^d \times \mathbb{R}^\infty)$ for $\varepsilon > 0$ the space of embeddings e satisfying the equation $(e \circ c)(t,x) = (t,x)$ for $t \in (-\varepsilon, 0]$, using the C^∞–topology.

We define the *moduli space of W–manifolds* $\mathcal{M}(W)$ as the space of submanifolds
$$W' \subseteq (-\infty,0] \times \mathbb{R}^{d-1} \times \mathbb{R}^\infty$$
such that

(i) there is an $\varepsilon > 0$ with $W' \cap (-\varepsilon,0] \times \mathbb{R}^{d-1} \times \mathbb{R}^\infty = (-\varepsilon,0] \times P$, and

(ii) there is a diffeomorphism $\phi\colon W \to W'$ that satisfies $\phi \circ c|_{(-\varepsilon,0]\times P} = \mathrm{inc}_{(-\varepsilon,0]\times P}$,

where inc denotes the inclusion ensured by (i). The space $\mathcal{M}(W)$ is topologised as the quotient of
$$\mathrm{Emb}_\partial(W, (-\infty,0] \times \mathbb{R}^{d-1} \times \mathbb{R}^\infty) = \operatorname*{colim}_{\varepsilon \to 0} \mathrm{Emb}_\varepsilon(W, (-\infty,0] \times \mathbb{R}^{d-1} \times \mathbb{R}^\infty)$$
by the action of the group $\mathrm{Diff}_\partial(W)$ via precomposition. The embedding space $\mathrm{Emb}_\partial(W, (-\infty,0] \times \mathbb{R}^{d-1} \times \mathbb{R}^\infty)$ is weakly contractible by Whitney's embedding theorem, and as the action of $\mathrm{Diff}_\partial(W)$ is free and admits slices by [5], the moduli

space $\mathcal{M}(W)$ provides a model for the classifying space $\mathrm{BDiff}_\partial(W)$. In the case of P being the sphere S^{d-1}, we define a weakly equivalent variant $\mathcal{M}^s(W)$ of $\mathcal{M}(W)$, consisting of submanifolds

$$W' \subseteq D^d \times \mathbb{R}^\infty$$

such that

(i) the interior of W' lies in $(D^d \backslash \partial D^d) \times (-\infty, 0]^\infty$,

(ii) there exists an $\varepsilon > 0$ for which the map $c'\colon (-\varepsilon, 0] \times S^{d-1} \to W'$ sending (t, x) to $((1+t)x, 0)$ is a collar, and

(iii) there is a diffeomorphism $\phi\colon W \to W'$ satisfying $\phi \circ c|_{(-\varepsilon,0]\times P} = c'|_{(-\varepsilon,0]\times P}$.

We call $\mathcal{M} = \coprod_{[W]} \mathcal{M}(W)$ the *moduli space of manifolds with P–boundary*, the union taken over compact manifolds W with an identification $\partial W = P$, one in each diffeomorphism class relative to P. Analogously, the *moduli space of manifolds with sphere boundary* is $\mathcal{A} = \coprod_{[N]} \mathcal{M}^s(N)$ for N with $\partial N = S^{d-1}$.

Lemma 6.1 *The moduli space $\mathcal{A}$ of manifolds with sphere boundary forms an E_d–algebra, and the moduli space $\mathcal{M}$ of manifolds with P–boundary is an E_1–module over it.*

Proof The operad $\mathcal{D}^\bullet(D^d)$ of little d–discs acts on $\coprod_{[N]} \mathcal{M}^s(N)$ by gluing manifolds along their sphere boundary into a disc, instructed by little d–discs. Formally, define

$$\begin{aligned} \theta\colon \mathcal{D}^k(D^d) \times \Big(\textstyle\coprod_{[N]} \mathcal{M}^s(N)\Big)^k &\to \textstyle\coprod_{[N]} \mathcal{M}^s(N), \\ ((\phi_1, \dots, \phi_k), (N_1, \dots, N_k)) &\mapsto \Big(\big(D^d \backslash \textstyle\bigcup_{i=1}^k \operatorname{im} \phi_i\big) \times \{0\}\Big) \cup \Big(\textstyle\bigcup_{i=1}^k r_i N_i + b_i\Big), \end{aligned}$$

where r_i is the radius and b_i the centre of $\phi_i\colon D^d \to D^d$, and $r_i N_i + b_i$ is obtained from N_i by scaling by r_i and translating by b_i, both in the D^d–coordinate. The conditions (i) and (ii) ensure that θ is well-defined. This action extends to an action of $\mathcal{SC}_d$ (see Section 2.1) on $\big(\coprod_{[W]} \mathcal{M}(W), \coprod_{[N]} \mathcal{M}^s(N)\big)$ via

$$\theta\colon \mathcal{SC}_d(\mathfrak{m}, \mathfrak{a}^k, \mathfrak{m}) \times \textstyle\coprod_{[W]} \mathcal{M}(W) \times \Big(\textstyle\coprod_{[N]} \mathcal{M}^s(N)\Big)^k \to \textstyle\coprod_{[W]} \mathcal{M}(W)$$

which maps $((s, \phi_1, \dots, \phi_k), M, N_1, \dots, N_k)$ to the submanifold obtained from

$$M \cup \Big(([0,s] \times P) \backslash \big(\textstyle\bigcup_{i=1}^k \operatorname{im} \phi_i \times \{0\}\big)\Big) \cup \Big(\textstyle\bigcup_{i=1}^k r_i N_i + b_i\Big) \tag{22}$$

by translating in the first coordinate by s to the left, where $r_i N_i + b_i$ is obtained from N_i by scaling by the radius r_i of $\phi_i\colon D^d \to (0,1) \times (-1,1)^{d-1}$ and translating by the centre b_i of ϕ_i, both in the $\mathbb{R}^d$–coordinate. Loosely speaking, we attach a cylinder to the boundary of W, glue in the N_i via the little d–discs, and shift everything to the left, as in Figure 6. This indeed yields a smooth submanifold, since the threefold union (22) is one: our conditions on $P \subseteq \mathbb{R}^{d-1} \times [0,\infty)^\infty$ and on the manifolds $N_i \subseteq D^d \times (-\infty, 0]^\infty$ in $\mathcal{M}^s(N)$ ensure that $[0,s] \times P$ and $\bigcup_{i=1}^k r_i N_i + b_i$ intersect only in $(0,s) \times (-1,1)^{d-1} \times \{0\}$, which, together with properties (i)–(ii) of $\mathcal{M}^s(N)$, implies that the second union of (22) is a smooth submanifold. This manifold intersects with M only in $\{0\} \times P$, so the whole union (22) forms, by property (i) of $\mathcal{M}(W)$ and property (ii) of $\mathcal{M}^s(N)$, a smooth submanifold as well. □

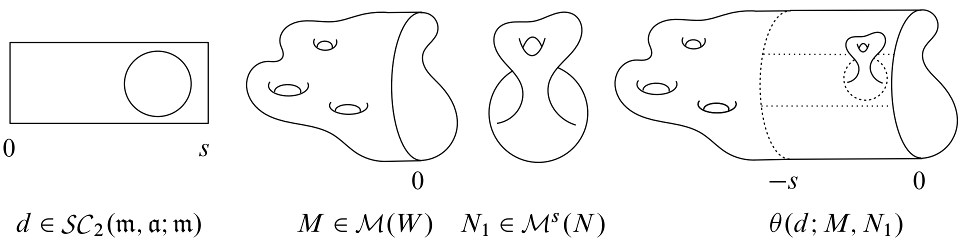


Figure 6: The E_1–module structure on the moduli space of manifolds.

6.1 The resolution by embeddings

By virtue of Lemma 6.1, the moduli space $\mathcal{M}$ of manifolds with P–boundary forms, in dimensions $d \geq 2$, an E_1–module over the moduli space of manifolds with sphere boundary $\mathcal{A}$, considered as an E_2–algebra via the embedding $\mathcal{SC}_2 \to \mathcal{SC}_d$ of Section 2.1. For $A \in \mathcal{M}$ and $X \in \mathcal{A}$, the stabilised manifold $A \oplus X$ is a model for the boundary connected sum $A \natural X$ of A and X. But, in contrast to the usual construction of the boundary connected sum, the manifold $A \oplus X$ contains A as a canonically embedded submanifold, and the boundary of $A \oplus X$ is canonically identified with the boundary of X; see Figure 6. On components, the stabilisation takes the form $s\colon \mathcal{M}(A) \to \mathcal{M}(A \natural X)$, modelling the map

$$s\colon \mathrm{BDiff}_\partial(A) \to \mathrm{BDiff}_\partial(A \natural X)$$

induced by extending diffeomorphisms by the identity.

As we did for configuration spaces in Section 5.1, we identify the canonical resolution of $\mathcal{M}$ with an augmented semi-simplicial space $R_\bullet^{\circ\!\!-}(\mathcal{M})$ of geometric nature, which is a generalisation of one introduced by Galatius and Randal-Williams in [28]. To that end, denote by H_X for $X \in \mathcal{A}$ the manifold obtained from X by gluing in $[-1,0] \times D^{d-1}$ along the embedding

$$\{-1\} \times D^{d-1} \to \partial X = S^{d-1}, \quad x \mapsto (\sqrt{1-|x|}, x).$$

The resulting manifold is, after smoothing corners, diffeomorphic to X, but contains a canonically embedded strip $[-1,0] \times D^{d-1} \subseteq H_X$. When considering embeddings of H_X into a manifold with boundary, we always implicitly require that $\{0\} \times D^d$ be sent to the boundary and the rest of H_X to the interior.

Definition 6.2 Let W be a d–manifold, which is, for some $\varepsilon > 0$, equipped with an embedding $e\colon (-\varepsilon, 0] \times \mathbb{R}^{d-1} \to W$ satisfying $e^{-1}(\partial W) = \{0\} \times \mathbb{R}^{d-1}$. Define a semi-simplicial space $K^X_\bullet(W)$ with the space of p–simplices given by tuples $((\varphi_0, t_0), \dots, (\varphi_p, t_p)) \in (\mathrm{Emb}(H_X, W) \times \mathbb{R})^{p+1}$ such that

(i) the embeddings φ_i are pairwise disjoint,

(ii) there exists an $\delta \in (0, \varepsilon)$ such that $\phi_i(s,p) = e(s, p + t_i e_1)$ holds for $(s,p) \in (-\delta, 0] \times D^{d-1} \subseteq H_X$, where $e_1 \in \mathbb{R}^{d-1}$ is the first basis vector, and

(iii) the parameters are ordered by $t_0 < \cdots < t_p$.

The embedding space is topologised in the C^∞–topology. The semi-simplicial structure is given by forgetting the (φ_i, t_i).

For submanifolds $W \in \mathcal{M}$, we use the embedding $e\colon (-\varepsilon, 0] \times \mathbb{R}^{d-1} \to W$ that is obtained from the canonically embedded cube $(-\varepsilon, 0] \times (-1,1)^{d-1} \subseteq (-\varepsilon, 0] \times P \subseteq W$ by use of the diffeomorphism

$$\mathbb{R} \to (-1,1), \quad x \mapsto \tfrac{2}{\pi} \arctan(x). \tag{23}$$

The group $\mathrm{Diff}_\partial(W)$ acts simplicially on $K^X_\bullet(W)$ by precomposition, so the levelwise Borel construction results in an augmented semi-simplicial space

$$\mathrm{Emb}_\partial(W, (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty) \times_{\mathrm{Diff}_\partial(W)} K^X_\bullet(W) \to \mathcal{M}(W), \tag{24}$$

in terms of which we define the *resolution by embeddings* as the augmented semi-simplicial space

$$R^{\multimapinv}_\bullet(\mathcal{M}) \to \mathcal{M}$$

obtained by taking coproducts of the semi-simplicial spaces (24) over compact manifolds W with P–boundary, one in each diffeomorphism class relative P. This is the analogue of the resolution by arcs for configuration spaces. A point in $R_\bullet^{\multimap}(\mathcal{M})$ consists of a manifold $W \in \mathcal{M}$ and $p+1$ embeddings of H_X into W that form an element of $K_p^X(W)$; see the rightmost subfigure of Figure 7 for an example. Since the augmentation is by construction a levelwise fibre bundle, the resolution by embeddings is fibrant. In particular, its fibre $K_\bullet^X(A)$ at $A \in \mathcal{M}$ is equivalent to the corresponding homotopy fibre.

Theorem 6.3 *The canonical resolution and the resolution by embeddings are weakly equivalent as augmented $\widetilde{\Delta}_{\mathrm{inj}}$–spaces. In particular, $K_\bullet^X(A)$ for $A \in \mathcal{M}$ is weakly equivalent to the space of destabilisations $W_\bullet(A)$ of A.*

To prove Theorem 6.3, we closely follow the proof of Theorem 5.5 for configuration spaces, adopting the notation of Section 5.1. More specifically, we construct a zigzag of weak equivalences

$$R_\bullet(\mathcal{M}) \xleftarrow{①} \mathrm{B}(U\mathcal{O}(\bullet,\blacksquare), U\mathcal{O}, B_\blacksquare(\mathcal{M})) \overset{②}{\simeq} \mathrm{B}(U\mathcal{O}^{\multimap}_{\bullet,\blacksquare}, U\mathcal{O}, B_\blacksquare(\mathcal{M})) \xrightarrow{③} R_\bullet^{\multimap}(\mathcal{M})^{\mathrm{fib}} \tag{25}$$

of augmented $\widetilde{\Delta}_{\mathrm{inj}}$–spaces between the canonical resolution and the fibrant replacement of the resolution by embeddings — analogous to the one for configuration spaces, labelled by (16). The first equivalence ① of (16) carries over to (25) verbatim. To construct ②, we replace the semi-simplicial space $R_\bullet^{\bullet\!-}(\mathcal{O}^{\mathrm{m}})$ with an equivalent variant $R_\bullet^{\multimap}(\mathcal{O}^{\mathrm{m}})$, essentially by including a contractible choice of tubular neighbourhoods of the arcs. To this end, consider for $s>0$ the simplicial space $K_\bullet^{D^d}((0,s]\times(-1,1)^{d-1})$ for which we use the embedding $e\colon (-s,0]\times\mathbb{R}^{d-1}\to(0,s]\times(-1,1)^{d-1}$ obtained from (23) and the translation by s. Call a 0–simplex (φ,t) therein a *little d–disc with thickened tether* if $\varphi|_{D^d}\colon D^d\to(0,s)\times(-1,1)^{d-1}$ is a composition of a scaling and a translation. The embedding $\varphi\colon H_{D^d}\to(0,s]\times(-1,1)^{d-1}$ induces an arc

$$\varphi^{\bullet\!-} := \varphi|_{[-1,0]\times\{0\}}\colon [-1,0]\to(0,s)\times(-1,1)^{d-1},$$

called the *tether* of φ, which connects the little d–disc to the boundary. The embedding φ furthermore induces a trivialisation of the normal bundle of the tether, which we consider as a map $[-1,0]\to V_{d-1}(\mathbb{R}^d)$ to the space of $(d-1)$–frames in $\mathbb{R}^d$. We call a little d–disc with thickened tether (φ,t) *two-dimensional* if

(i) the little d–disc $\varphi|_{D^d}$ is the image of a little 2–disc in $(0,s)\times(-1,1)$ under $\mathcal{SC}_2\to\mathcal{SC}_d$ (see Section 2.1),

(ii) the induced tether $\varphi^{\bullet\!-}$ lies in the slice $(0,s)\times(-1,1)\times\{0\}^{d-2}$, and

(iii) the induced trivialisation $[-1,0]\to V_{d-1}(\mathbb{R}^d)$ equals, up to scaling by a smooth function $[-1,0]\to(0,\infty)$, the parallel transport of the frame $(e_2,\dots,e_d)\in$

$V_{d-1}(\mathbb{R}^d)$ at $\varphi^{\bullet-}(0)$ along the tether $\varphi^{\bullet-}$, where $e_i \in \mathbb{R}^d$ denotes the i^{th} vector in the standard basis.

Definition 6.4 Define the augmented semi-simplicial space $R_\bullet^{\multimap}(\mathcal{O}^{\mathrm{m}}) \to \mathcal{O}^{\mathrm{m}}$ with p–simplices

$$R_p^{\multimap}(\mathcal{O}^{\mathrm{m}}) \subseteq \mathcal{O}^{\mathrm{m}} \times \big(\mathrm{Emb}(H_{D^d}, (0,\infty)\times(-1,1)) \times \mathbb{R}\big)^{p+1}$$

consisting of tuples

$$((s, \{\phi_j\}), (\varphi_0, t_0), \ldots, (\varphi_p, t_p))$$

such that $(\varphi_i, t_i) \in K_p^{D^d}((0,s] \times (-1,1)^{d-1})$ and all (φ_i, t_i) are two-dimensional little d–discs with thickened tethers whose induced little 2–disc is one of the ϕ_j. The third graphic of Figure 7 illustrates a 0–simplex of this semi-simplicial space.

As a two-dimensional little 2–disc with thickened tether is, up to a contractible choice of a thickening, determined by the associated little 2–disc and its tether, the $\widetilde{\Delta}_{\mathrm{inj}}$–spaces $R_\bullet^{\multimap}(\mathcal{O}^{\mathrm{m}})$ and $R_\bullet^{\bullet-}(\mathcal{O}^{\mathrm{m}})$ are weakly equivalent. The $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{O})$–space $U\mathcal{O}_{\bullet,\blacksquare}^{\multimap}$ in (25) is defined in the same way as $U\mathcal{O}_{\bullet,\blacksquare}^{\bullet-}$, but using $R_\bullet^{\multimap}(\mathcal{O}^{\mathrm{m}})$ instead of $R_\bullet^{\bullet-}(\mathcal{O}^{\mathrm{m}})$. Making use of the equivalence between $R_\bullet^{\multimap}(\mathcal{O}^{\mathrm{m}})$ and $R_\bullet^{\bullet-}(\mathcal{O}^{\mathrm{m}})$, the proof of Lemma 5.7 carries over to the manifold case and shows that $U\mathcal{O}_{\bullet,\blacksquare}^{\multimap}$ and $U\mathcal{O}(\bullet,\blacksquare)$ are weakly equivalent $(\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \times U\mathcal{O})$–spaces, which establishes the equivalence ②. Finally, we construct the remaining equivalence ③ via an analogue

$$\Phi\colon \mathcal{M} \times R_\bullet^{\multimap}(\mathcal{O}^{\mathrm{m}}) \to R_\bullet^{\multimap}(\mathcal{M}) \tag{26}$$

of the simplicial map (18), mapping $\big(A, (e, (\varphi_0, t_0), \ldots, (\varphi_p, t_p))\big)$ to the manifold $\theta(e; A, X^{p+1})$ equipped with the embeddings of H_X obtained from the φ_i by replacing D^d by X; see Figure 7. Using (26) instead of (18) in the definition of (19), we obtain simplicial maps $U\mathcal{O}_{\bullet,k}^{\multimap} \times B_k(\mathcal{M}) \to R_\bullet^{\multimap}(\mathcal{M})^{\mathrm{fib}}$, which induce a morphism of the form $\mathrm{B}(U\mathcal{O}_{\bullet,\blacksquare}^{\multimap}, U\mathcal{O}, B_\blacksquare(\mathcal{M})) \to R_\bullet^{\multimap}(\mathcal{M})^{\mathrm{fib}}$, as in the case of configuration spaces. This is the last morphism ③ in the zigzag (25), and it is a weak equivalence by the argument of the proof of Lemma 5.8, minorly modified using the following lemma, which completes the proof of Theorem 6.3.

Lemma 6.5 *For all $p \geq 0$ and elements of the form $(c_{p+1}, (\varphi_0, t_0 \ldots, \varphi_p, t_p)) \in R_p^{\multimap}(\mathcal{O}^{\mathrm{m}})$, the simplicial map Φ induces a weak equivalence $\mathcal{M} \to R_p^{\multimap}(\mathcal{M})$.*

Proof This is a verbatim generalisation of the line of argument explained in [28, Lemma 6.10] for $X = D^{2p} \sharp S^p \times S^p$. $\square$

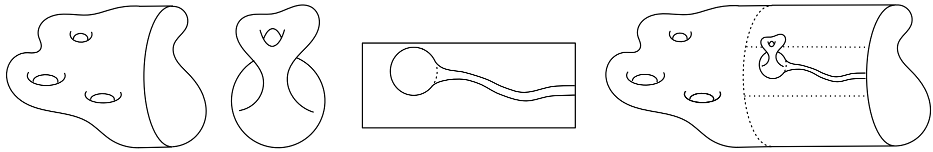


Figure 7: The resolution by embeddings and the map Φ.

Galatius and Randal-Williams [28] proved high-connectivity of $K_\bullet^X(A)$ if A is simply connected and $X \cong D^{2p} \sharp (S^p \times S^p)$ for $p \geq 3$. On the basis of this, Friedrich [25] and Perlmutter [54] proved connectivity results for other choices of A and X. To state their results, recall the stable X–genus $\overline{g}^X$, as introduced in Section 2.3, and let $\mathrm{usr}(\mathbb{Z}[G])$ for a group G denote the *unitary stable rank* [49, Definition 6.3] of its group ring $\mathbb{Z}[G]$, considered as a ring with an anti-involution.

Theorem 6.6 *The realisation of $K_\bullet^X(A)$ for a connected manifold $A \in \mathcal{M}$ is*

(i) $\frac{1}{2}(\overline{g}^X(A)-4)$*–connected if $X \cong D^{2p} \sharp (S^p \times S^p)$, $p \geq 3$, and A is simply connected,*

(ii) $\frac{1}{2}\big(\overline{g}^X(A)-\mathrm{usr}(\mathbb{Z}[\pi_1(A)])-3\big)$*–connected if $X \cong D^{2p} \sharp (S^p \times S^p)$ and $p \geq 3$,*

(iii) $\frac{1}{2}(\overline{g}^X(A)-4-m)$*–connected if $X \cong D^{p+q} \sharp (S^p \times S^q)$, $0 < p < q < 2p-2$, and A is $(q-p+2)$–connected, where m is the smallest number such that there exists an epimorphism of the form $\mathbb{Z}^m \to \pi_q(S^p)$.*

Proof The first two parts are [25, Theorem 4.7; 28, Corollary 5.10]. Corollary 7.3.1 of [54] proves the third claim for the genus $g^X(B)$ instead of its stable variant $\overline{g}^X(B)$. However, the proof given there goes through for $\overline{g}^X(B)$ if one replaces the relation between the genus of a manifold B satisfying the assumption in (ii) and the rank of its associated Wall form (see [54, Proposition 6.1]) with the analogous statement relating the stable genus to the stable rank. □

For a manifold $A \in \mathcal{M}$, we denote by $\overline{g}_A^X$ the grading of $\mathcal{M}$ obtained by localising the stable X–genus at objects stably isomorphic to A; see Remark 2.20. Combining Theorems 6.3 and 6.6 implies the following.

Corollary 6.7 *The canonical resolution $R_\bullet(\mathcal{M}) \to \mathcal{M}$ is*

(i) *graded* $\frac{1}{2}(\overline{g}_A^X-2)$*–connected for $X \cong D^{2p} \sharp (S^p \times S^p)$, $p \geq 3$, and any simply connected $A \in \mathcal{M}$,*

(ii) *graded* $\frac{1}{2}\big(\overline{g}^X_A - \mathrm{usr}(\mathbb{Z}[\pi_1(A)]) - 1\big)$*–connected for* $X \cong D^{2p} \sharp (S^p \times S^p)$, $p \geq 3$, *and any connected* $A \in \mathcal{M}$, *and*

(iii) *graded* $\frac{1}{2}(\overline{g}^X_A - 2 - m)$*–connected for* $X \cong D^{p+q} \sharp (S^p \times S^q)$, $0 < p < q < 2p-2$, *and any* $(q-p+2)$*–connected* $A \in \mathcal{M}$ *with* m *defined as in Theorem 6.6.*

Remark 6.8 In the case $d = 2$, one can use [35, Proposition 5.1] to show that $K^X_\bullet(A)$ is $\frac{1}{2}(\overline{g}^X(A) - 3)$–connected for $X \cong D^2 \sharp (S^1 \times S^1)$ and $A \in \mathcal{M}$ an orientable surface, which implies stability results for diffeomorphism groups of surfaces. Their homotopy discreteness [21; 31] ensures their equivalence to their mapping class group, for which stability has a longstanding history, going back to a result by Harer [34], which has been improved in manifold ways since then; see for instance [9; 17; 38; 59; 61; 74].

By Remark 3.3, Theorems A and C apply to $\mathcal{M}$ when graded by

$$\overline{g}^X_A + 2, \quad \overline{g}^X_A + \mathrm{usr}(\mathbb{Z}[\pi_1(A)]) + 1 \quad \text{or} \quad \overline{g}^X_A + m + 2$$

for X and A as in the respective three cases of Corollary 6.7. On path components, this implies Theorem H, noting that in the relevant ranges, the genus and the stable genus agree; see Remark 2.24.

6.2 Coefficient systems for moduli spaces of manifolds

Recall from Section 4.2 that coefficient systems for the moduli space of manifolds with P–boundary $\mathcal{M}$ are defined in terms of the module structure of the fundamental groupoid $(\Pi(\mathcal{M}), \oplus)$ over the braided monoidal category $(\Pi(\mathcal{A}), \oplus, b, 0)$, induced by the E_1–module structure of $\mathcal{M}$ over the moduli space of manifolds with sphere boundary $\mathcal{A}$. In the following, we provide an alternative description for the fundamental groupoids $\Pi(\mathcal{M})$ and $\Pi(\mathcal{A})$, more suitable for constructing coefficient systems on $\mathcal{M}$.

Define the categories $\mathrm{mcg}(\mathcal{M})$ and $\mathrm{mcg}(\mathcal{A})$ to have the same objects as $\Pi(\mathcal{M})$ and $\Pi(\mathcal{A})$, respectively, and the set of mapping classes $\pi_0(\mathrm{Diff}_\partial(M, N))$ as morphisms between M and N, where $\mathrm{Diff}_\partial(M, N)$ is the space of diffeomorphisms that preserve a germ of the canonical collars of M and N ensured by condition (i) in the definition of $\mathcal{M}(W)$. The composition in $\mathrm{mcg}(\mathcal{M})$ and $\mathrm{mcg}(\mathcal{A})$ is the evident one.

Lemma 6.9 *The category* $\mathrm{mcg}(\mathcal{M})$ *is isomorphic to* $\Pi(\mathcal{M})$, *and* $\mathrm{mcg}(\mathcal{A})$ *to* $\Pi(\mathcal{M})$.

Proof Recall the fibre bundle

$$\mathrm{Diff}_\partial(A) \to \mathrm{Emb}_\partial(A, (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty) \to \mathcal{M}(A)$$

from the construction of $\mathcal{M}(\mathcal{A})$ in the beginning of the section. Lifting a path from A to B in $\mathcal{M}$ to a path in the total space starting at the inclusion $A \subseteq (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty$ gives a path of embeddings that ends at an embedding with image B, and hence provides a diffeomorphism from A to B by restricting to the image. This provides a functor from $\operatorname{mcg}(\mathcal{M})$ to $\Pi(\mathcal{M})$, whose inverse is induced by considering a diffeomorphism as an embedding, choosing a path in the contractible space $\operatorname{Emb}_\partial(A, (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty)$ from the inclusion $A \subseteq (-\infty, 0] \times \mathbb{R}^d \times \mathbb{R}^\infty$ to the embedding obtained from the diffeomorphism, and mapping this path to $\mathcal{M}(A)$. The argument for the isomorphism $\operatorname{mcg}(\mathcal{A}) \cong \Pi(\mathcal{A})$ is analogous. □

The module structure of $\Pi(\mathcal{M})$ over $\Pi(\mathcal{A})$ can be transported via the identification of the preceding lemma to one of $\operatorname{mcg}(\mathcal{M})$ over $\operatorname{mcg}(\mathcal{A})$, considered as a braided monoidal category by making use of the isomorphism $\operatorname{mcg}(\mathcal{A}) \cong \Pi(\mathcal{A})$. In concrete terms, the monoidal structure on $\operatorname{mcg}(\mathcal{A})$ is given on objects by that of $\Pi(\mathcal{A})$ induced by the E_2–multiplication, and on morphisms by multiplying $f \in \operatorname{Diff}_\partial(A, B)$ and $g \in \operatorname{Diff}_\partial(A', X')$ as $f \oplus g \in \operatorname{Diff}_\partial(A \oplus A', B \oplus B')$, defined by extending f and g via the identity. The description of the module structure on $\operatorname{mcg}(\mathcal{M})$ is analogous. Coefficient systems for $\mathcal{M}$ are then given by coefficient systems for the module $\operatorname{mcg}(\mathcal{M})$ over $\operatorname{mcg}(\mathcal{A})$ in the sense of Definition 4.1.

To illustrate how this identification can be used to construct coefficient systems on $\mathcal{M}$, we discuss one example in detail. Consider for $i \geq 0$ the functor $\mathrm{H}_i\colon \operatorname{mcg}(\mathcal{M}) \to \mathcal{Ab}$ that assigns a manifold $A \in \mathcal{M}$ its i^{th} singular homology group $\mathrm{H}_i(A)$. The inclusions $A \subseteq A \oplus X$ induce a natural transformation $\sigma^{\mathrm{H}_i}\colon \mathrm{H}_i(-) \to \mathrm{H}_i(- \oplus X)$ that satisfies the triviality condition for coefficient systems; see Definition 4.1. To calculate the degree of H_i, we consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_i(A) \arrow[r] \arrow[d, "\sigma^{\mathrm{H}_i}"] & \mathrm{H}_i(A \oplus X) \arrow[r] \arrow[d, "\sigma^{\Sigma \mathrm{H}_i}"] & \widetilde{\mathrm{H}}_i(\operatorname{Cone}(P) \natural X) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathrm{H}_i(A \oplus X) \arrow[r] & \mathrm{H}_i(A \oplus X \oplus X) \arrow[r] & \widetilde{\mathrm{H}}_i(\operatorname{Cone}(P) \natural X) \arrow[r] & 0
\end{tikzcd}$$

induced by the long exact sequence of pairs together with the equivalences

$$\mathrm{H}_i(A \oplus X^{\oplus k}, A \oplus X^{\oplus k-1}) \cong \widetilde{\mathrm{H}}_i(\operatorname{Cone}(P) \natural X)$$

obtained by collapsing $A \oplus X^{\oplus k-1}$. The leftmost vertical map is induced by the inclusion, and the second one by the inclusion followed by $A \oplus b_{X,X}$. Naturality of the diagram in A implies triviality of the kernel of H_i, and also that its cokernel is constant, thus of degree 0 if $\widetilde{\mathrm{H}}_i(\mathrm{Cone}(P) \natural X) \neq 0$ and of degree -1 elsewise. Hence, H_i is of degree 1 at 0 if $\widetilde{\mathrm{H}}_i(\mathrm{Cone}(P) \natural X) \neq 0$ and of degree 0 at 0 if $\widetilde{\mathrm{H}}_i(\mathrm{Cone}(P) \natural X) = 0$, from which Corollary I is implied by an application of Theorem H.

6.3 Extensions

6.3.1 Stabilisation by $(2n-1)$–connected $(4n+1)$–manifolds In [55], Perlmutter established high-connectivity of the semi-simplicial spaces $K_\bullet^X(A)$ for 2–connected manifolds A of dimension $4n+1$ with $n \geq 2$, and certain specific $(2n-1)$–connected stably parallelisable manifolds X with finite $\mathrm{H}_{2n}(X;\mathbb{Z})$ and trivial $\mathrm{H}_{2n}(X,\mathbb{Z}/2\mathbb{Z})$. From this, he derived homological stability with constant coefficients of the map

$$\mathrm{BDiff}_\partial(A) \to \mathrm{BDiff}_\partial(A \natural X) \tag{27}$$

for these specific A and X. By using classification results of closed $(2n-1)$–connected stably parallelisable $(4n+1)$–manifolds due to Wall [75] and De Sapio [19], he furthermore concluded that (27) stabilises in fact for all X having the properties described above and not just the specific ones considered before. The methods of this section can be used to extend his homological stability result to abelian coefficients and coefficient systems of finite degree.

6.3.2 Automorphisms of topological and piecewise linear manifolds Kupers [42] adapted the methods of Galatius and Randal-Williams [28] to prove high-connectivity of the relevant semi-simplicial spaces of locally flat embeddings in order to prove homological stability for classifying spaces of homeomorphisms of topological manifolds and PL-automorphisms of piecewise linear manifolds. By extending the ideas of this section, our framework applies also to these examples, resulting in an extension of Kupers' stability results to coefficient systems of finite degree.

7 Homological stability for modules over braided monoidal categories

We explain the applicability of our framework to modules over braided monoidal categories, and compare it with the theory for braided monoidal groupoids developed by Randal-Williams and Wahl [61].

7.1 E_1–modules over E_2–algebras from modules over braided monoidal categories

Recall the categorical *operad of coloured braids* $\mathcal{C}o\mathcal{B}$; see eg [24, Chapter 5]. The category of n–operations is the groupoid $\mathcal{C}o\mathcal{B}(n)$ with linear orderings of $\{1,\dots,n\}$ as objects, and braids connecting the spots as prescribed by the orderings as morphisms. The operadic composition is given by "cabling". Algebras over $\mathcal{C}o\mathcal{B}$ are exactly strict braided monoidal categories, and the topological operad obtained by taking classifying spaces is E_2; see eg [24, Theorem 5.2.12] and [23, Chapter 8]. Extending this, we construct a two-coloured operad whose algebras are modules over braided monoidal categories and whose classifying space is $E_{1,2}$; see Section 2.1.

Definition 7.1 Define a categorical operad $\mathcal{C}o\mathcal{BM}$ with colours $\mathfrak{m}$ and $\mathfrak{a}$ whose operations $\mathcal{C}o\mathcal{BM}(\mathfrak{m}^k,\mathfrak{a}^l;\mathfrak{m})$ are empty for $k\neq 1$ and equal to $\mathcal{C}o\mathcal{B}(l)$ otherwise. The operations $\mathcal{C}o\mathcal{BM}(\mathfrak{m}^k,\mathfrak{a}^l;\mathfrak{a})$ are empty for $k\neq 0$ and equal to $\mathcal{C}o\mathcal{B}(l)$ elsewise. Restricted to the $\mathfrak{a}$–colour, $\mathcal{C}o\mathcal{BM}$ is defined as $\mathcal{C}o\mathcal{B}$. Requiring commutativity of

$$\begin{tikzcd}
\mathcal{C}o\mathcal{BM}(\mathfrak{m},\mathfrak{a}^l;\mathfrak{m})\times\mathcal{C}o\mathcal{BM}(\mathfrak{m},\mathfrak{a}^k;\mathfrak{m})\times\left(\bigtimes_{j=1}^{l}\mathcal{C}o\mathcal{BM}(\mathfrak{a}^{i_j};\mathfrak{a})\right) \arrow[rr, "\gamma_{\mathcal{C}o\mathcal{BM}}"] \arrow[d, "\tau"'] & & \mathcal{C}o\mathcal{BM}(\mathfrak{m},\mathfrak{a}^{k+i};\mathfrak{m}) \arrow[d, equal] \\
\mathcal{C}o\mathcal{B}(k)\times\mathcal{C}o\mathcal{B}(l)\times\left(\bigtimes_{j=1}^{l}\mathcal{C}o\mathcal{B}(i_j)\right) \arrow[r, "\mathrm{id}\times\gamma_{\mathcal{C}o\mathcal{B}}"] & \mathcal{C}o\mathcal{B}(k)\times\mathcal{C}o\mathcal{B}(i) \arrow[r, "\oplus"] & \mathcal{C}o\mathcal{B}(k+i)
\end{tikzcd}$$

defines the remaining composition $\gamma_{\mathcal{C}o\mathcal{BM}}$, where $i=\sum_{j=1}^{l} i_j$, the map τ interchanges the first two factors, $\gamma_{\mathcal{C}o\mathcal{B}}$ is the composition of $\mathcal{C}o\mathcal{B}$ and $\oplus$ is $\gamma_{\mathcal{C}o\mathcal{B}}(\mathrm{id}_{\{1<2\}};-,-)$, ie puts braids next to each other. See Figure 8 for an example.

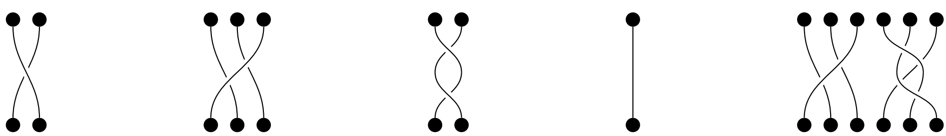


Figure 8: The operadic composition in $\mathcal{C}o\mathcal{BM}$. From left to right: $d\in\mathcal{C}o\mathcal{BM}(\mathfrak{m},\mathfrak{a}^2;\mathfrak{m})$, $e\in\mathcal{C}o\mathcal{BM}(\mathfrak{m},\mathfrak{a}^3;\mathfrak{m})$, $f\in\mathcal{C}o\mathcal{BM}(\mathfrak{a}^2;\mathfrak{a})$, $g\in\mathcal{C}o\mathcal{BM}(\mathfrak{a};\mathfrak{a})$, and $\gamma(d;e,f,g)\in\mathcal{C}o\mathcal{BM}(\mathfrak{m},\mathfrak{a}^6;\mathfrak{m})$.

Recall the notion of a right-module $(\mathcal{M},\oplus)$ over a monoidal category $(\mathcal{A},\oplus,0)$: a category $\mathcal{M}$ with a functor $\oplus\colon\mathcal{M}\times\mathcal{A}\to\mathcal{M}$ that is unital and associative up to coherent isomorphism; see Section 1.1.

Lemma 7.2 *The structure of a (graded) $\mathcal{CoBM}$–algebra on a pair of categories $(\mathcal{M}, \mathcal{A})$ is equivalent to a strict (graded) braided monoidal structure on $\mathcal{A}$ and a strict (graded) right-module structure on $\mathcal{M}$ over it. Furthermore, the topological operad obtained from $\mathcal{CoBM}$ by taking levelwise classifying spaces is $E_{1,2}$.*

Proof The proof of the corresponding result for $\mathcal{CoB}$ in [24, Chapter 5] carries over mutatis mutandis. □

As a consequence of the previous lemma, the classifying space of a graded module over a braided monoidal category carries the structure of a graded E_1–module over an E_2–algebra.

Remark 7.3 The operad of *parenthesised coloured braids* encodes nonstrict braided monoidal categories, and its classifying space operad is E_2 as well [24, Chapter 6]. By considering a parenthesised version of $\mathcal{CoBM}$, this extends in a similar fashion to nonstrict right-modules over nonstrict braided monoidal categories, whose classifying spaces hence also give E_1–modules over E_2–algebras.

7.2 Homological stability for groups and monoids

Let $(\mathcal{M}, \oplus)$ be a graded right-module over a braided monoidal category $(\mathcal{A}, \oplus, b, 0)$ with a stabilising object X, ie an object of $\mathcal{A}$ of degree 1. Taking classifying spaces results by Lemma 7.2 in a graded E_1–module $\mathrm{B}\mathcal{M}$ over the E_2–algebra $\mathrm{B}\mathcal{A}$ with stabilising object $X \in \mathrm{B}\mathcal{A}$, hence provides a suitable input for Theorems A and C. In the following, we introduce a condition on $\mathcal{M}$ that ensures a simplification of the canonical resolution of $\mathrm{B}\mathcal{M}$.

Definition 7.4 The module $(\mathcal{M}, \oplus)$ is called *injective* at an object A of $\mathcal{M}$ if the stabilisation

$$(- \oplus X^{\oplus p+1})\colon \mathrm{Aut}(B) \to \mathrm{Aut}(B \oplus X^{\oplus p})$$

is injective for all objects B for which $B \oplus X^{\oplus p}$ is isomorphic to A for a $p \geq 0$.

Definition 7.5 For an object A of $\mathcal{M}$, define a semi-simplicial set $W_\bullet^{\mathrm{RW}}(A)$, whose p–simplices are given as equivalence classes of pairs (B, f) of an object B of $\mathcal{M}$ and a morphism $f \in \mathcal{M}(B \oplus X^{\oplus p+1}, A)$, where (B, f) and (B', f') are equivalent if there is an isomorphism $g \in \mathcal{M}(B, B')$ satisfying $f' \circ (g \oplus X^{\oplus p+1}) = f$. The i^{th} face of a p–simplex $[B, f]$ is defined to be $[B \oplus X, f \circ (B \oplus b^{-1}_{X^{\oplus i}, X} \oplus X^{\oplus p-i})]$.

Recall the spaces of destabilisations $W_\bullet(A)$, ie the fibres of the canonical resolution; see Definition 2.14.

Lemma 7.6 *If $\mathcal{M}$ is a groupoid, then the semi-simplicial set of path components $\pi_0(W_\bullet(A))$ for an object A of $\mathcal{M}$ is isomorphic to $W_\bullet^{\mathrm{RW}}(A)$. Moreover, $W_\bullet(A)$ is homotopy discrete if and only if $\mathcal{M}$ is injective at A.*

Proof The inclusion of the 0–simplices $\operatorname{ob}\mathcal{M} \subseteq \mathrm{B}\mathcal{M}$, together with the natural map $\operatorname{mor}\mathcal{M} \to \operatorname{Path}\mathcal{M}$, induces a preferred bijection $W_p^{\mathrm{RW}}(A) \to \pi_0(W_p(A))$ for all $p \geq 0$, since every path in $\mathrm{B}\mathcal{M}$ between 0–simplices is homotopic relative to its endpoints to a 1–simplex, ie to a path in the image of $\operatorname{mor}\mathcal{M} \to \operatorname{Path}\mathcal{M}$. By the definition of the respective face maps, these bijections assemble to an isomorphism of simplicial sets, which proves the first claim. The homotopy fibre $W_p(A)$ of the map $\mathrm{B}(-\oplus X^{\oplus p+1})\colon \mathrm{B}\mathcal{M} \to \mathrm{B}\mathcal{M}$ at A is homotopy discrete if and only if the induced morphisms on π_1 based at all objects B with $B \oplus X^{\oplus p+1} \cong A$ for $p \geq 0$ are injective, which is clearly equivalent to $\mathcal{M}$ being locally injective at A. □

Remark 7.7 If $\mathcal{M}$ and $\mathcal{A}$ are groupoids, then $\mathcal{M} \simeq \Pi(\mathrm{B}\mathcal{M})$ holds naturally as a module over $\mathcal{A} \simeq \Pi(\mathrm{B}\mathcal{A})$, so coefficient systems for $\mathrm{B}\mathcal{M}$ (see Definition 4.13) are coefficient systems for $\mathcal{M}$ as in Section 4.1.

Remark 7.8 Since the connectivity of the canonical resolution can be tested on the spaces of destabilisations $W_\bullet(A)$ (see Remark 2.17), Lemmas 7.6 and 7.7 imply a version of Theorems A and C that is phrased entirely in terms of (discrete) categories and semi-simplicial sets. This provides a simplified toolkit for proving homological stability for graded modules over braided monoidal categories with a stabilising object X for which the multiplication $(-\oplus X)\colon \operatorname{Aut}(B) \to \operatorname{Aut}(B \oplus X)$ is injective for all objects B of finite degree.

7.3 Comparison with the work of Randal-Williams and Wahl

Let $(\mathcal{G}, \oplus, b, 0)$ be a braided monoidal groupoid. In [61] it is shown that for objects A and X in $\mathcal{G}$, the maps

$$\mathrm{B}(-\oplus X)\colon \mathrm{B}\operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n}) \to \mathrm{B}\operatorname{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n+1}) \tag{28}$$

satisfy homological stability with constant, abelian, and a class of coefficient systems, if a certain family of associated semi-simplicial sets $W_n(A, X)_\bullet$ (see [61, Definition 2.1]) is sufficiently connected and $\mathcal{G}$ satisfies

(i) injectivity of the stabilisation $(-\oplus X)\colon \mathrm{Aut}_{\mathcal{G}}(A\oplus X^{\oplus n}) \to \mathrm{Aut}_{\mathcal{G}}(A\oplus X^{\oplus n+1})$ for all $n\geq 0$,

(ii) *local cancellation at* (A,X), ie $Y\oplus X^{\oplus m}\cong A\oplus X^{\oplus n}$ for $Y\in\mathcal{G}$ and $1\leq m\leq n$ implies $Y\cong A\oplus X^{\oplus m-n}$,

(iii) no zero-divisors, ie $U\oplus V\cong 0$ implies $U\cong 0$, and

(iv) the unit 0 has no nontrivial automorphisms.

As indicated by our choice of notation, if we consider $\mathcal{G}$ as a module over itself, the simplicial set $W_n(A,X)_\bullet$ of [61] equals $W_\bullet^{\mathrm{RW}}(A\oplus X^{\oplus n})$ as specified in Definition 7.5. To compare [61] with our work, define the module $\mathcal{G}_{A,X}=\coprod_{n\geq 0}\mathrm{Aut}_{\mathcal{G}}(A\oplus X^{\oplus n})$ over the braided monoidal category $\mathcal{G}_X=\coprod_{n\geq 0}\mathrm{Aut}_{\mathcal{G}}(X^{\oplus n})$, both graded in the evident way. By Theorems A and C, the maps (28) stabilise homologically — without assumptions on $\mathcal{G}$ — if the canonical resolution of $\mathrm{B}\mathcal{G}_{A,X}$ is sufficiently connected, or equivalently, if the spaces of destabilisations $W_\bullet(A\oplus X^{\oplus n})$ associated to $\mathrm{B}\mathcal{G}_{A,X}$ are; see Remark 2.17.

The semi-simplicial sets $W_n(A,X)_\bullet$ of [61] are equivalent to the spaces of destabilisations $W_\bullet(A\oplus X^{\oplus n})$ of $\mathrm{B}\mathcal{G}_{A,X}$ if (i) and (ii) hold. Indeed, assumption (ii) implies that $W_n(A,X)_\bullet$ agrees with the semi-simplicial set $W_\bullet^{\mathrm{RW}}(A\oplus X^{\oplus n})$ associated to $\mathcal{G}_{A,X}$ and hence also with $\pi_0(W_\bullet(A\oplus X^{\oplus n}))$ by Lemma 7.6. The first condition imposes injectivity of $\mathcal{G}_{A,X}$ at all objects $A\oplus X^{\oplus n}$, which by Lemma 7.6 is equivalent to the homotopy discreteness of the space of destabilisations $W_\bullet(A\oplus X^{\oplus n})$ of $\mathrm{B}\mathcal{G}_{A,X}$.

Hence, if one prefers to work in a discrete setting as in [61], ie using semi-simplicial sets, condition (i) is necessary. Condition (ii) ensures that the semi-simplicial sets of [61] agree with our spaces of destabilisations $W_\bullet(A\oplus X^{\oplus n})$, whose high-connectivity always imply stability by Theorems A and C. The last two conditions are redundant, ie imposing (i) and (ii) already implies (twisted) homological stability of (28) under the connectivity assumptions of [61]. The presence of these additional assumptions in [61] is due to their usage of Quillen's construction $\langle\mathcal{G},\mathcal{G}\rangle$, since conditions (iii) and (iv) guarantee that the automorphism groups $\mathrm{Aut}_{\langle\mathcal{G},\mathcal{G}\rangle}(A\oplus X^{\oplus n})$ and $\mathrm{Aut}_{\mathcal{G}}(A\oplus X^{\oplus n})$ coincide. If (i)–(iii) are satisfied and the $W_n(A,X)$ are highly connected, then [61] implies stability for $\mathrm{Aut}_{\langle\mathcal{G},\mathcal{G}\rangle}(A\oplus X^{\oplus n})$. Hence, in this case, high-connectivity of $W_n(A,X)$ shows stability for both $\mathrm{Aut}_{\mathcal{G}}(A\oplus X^{\oplus n})$ and $\mathrm{Aut}_{\langle\mathcal{G},\mathcal{G}\rangle}(A\oplus X^{\oplus n})$. The reason for this is that, although these automorphism groups might differ, their quotients $\mathrm{Aut}(A\oplus X^n)/\,\mathrm{Aut}(A\oplus X^{\oplus n-p-1})\cong W_p^{\mathrm{RW}}(A\oplus X^{\oplus n})$, forming the corresponding semi-simplicial sets, agree.

Remark 7.9 The coefficient systems that [61] deals with are functors of finite degree on the subcategory $\mathcal{C}_{A,X} \subseteq \langle \mathcal{G}, \mathcal{G} \rangle$ generated by the objects $A \oplus X^{\oplus n}$. In contrast, Theorem C is applicable to functors of finite degree on $\langle \mathcal{G}_{A,X}, \mathcal{B} \rangle$; see Remarks 4.12 and 7.7. As the canonical functors $\mathcal{G}_{A,X} \to \mathcal{G}$ and $\mathcal{B} \to \mathcal{G}$ induce $\langle \mathcal{G}_{A,X}, \mathcal{B} \rangle \to \mathcal{C}_{A,X}$, every coefficient system of [61] gives one in ours; see Remarks 4.11 and 4.12.

Remark 7.10 The ranges for coefficient systems of finite degree ensured by Theorem C agree with the ones of [61] in the situations in which their work is applicable. The ranges for abelian coefficients of Theorem A improve the ones of [61] marginally, and so does the surjectivity range for constant coefficients in the case $k > 2$. These ranges can in some cases be further improved; see Remark 3.3.

References

[1] **V I Arnold**, *Braids of algebraic functions and cohomologies of swallowtails*, Uspehi Mat. Nauk 23 (1968) 247–248 MR In Russian

[2] **V I Arnold**, *The cohomology ring of the colored braid group*, Mat. Zametki 5 (1969) 227–231 MR In Russian; translated in Math. Notes 5 (1969) 138–140

[3] **C Berger**, **I Moerdijk**, *Resolution of coloured operads and rectification of homotopy algebras*, from "Categories in algebra, geometry and mathematical physics" (A Davydov, M Batanin, M Johnson, S Lack, A Neeman, editors), Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007) 31–58 MR

[4] **S Betley**, *Twisted homology of symmetric groups*, Proc. Amer. Math. Soc. 130 (2002) 3439–3445 MR

[5] **E Binz**, **H R Fischer**, *The manifold of embeddings of a closed manifold*, from "Differential geometric methods in mathematical physics" (H-D Doebner, editor), Lecture Notes in Phys. 139, Springer (1981) 310–329 MR

[6] **J M Boardman**, **R M Vogt**, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math. 347, Springer (1973) MR

[7] **C-F Bödigheimer**, **F R Cohen**, **R J Milgram**, *Truncated symmetric products and configuration spaces*, Math. Z. 214 (1993) 179–216 MR

[8] **C-F Bödigheimer**, **F Cohen**, **L Taylor**, *On the homology of configuration spaces*, Topology 28 (1989) 111–123 MR

[9] **S K Boldsen**, *Improved homological stability for the mapping class group with integral or twisted coefficients*, Math. Z. 270 (2012) 297–329 MR

[10] **F Cantero**, **M Palmer**, *On homological stability for configuration spaces on closed background manifolds*, Doc. Math. 188 (2015) 753–805 MR

[11] **K Casto**, FI_G*–modules, orbit configuration spaces, and complex reflection groups*, preprint (2016) arXiv

[12] **T Church**, *Homological stability for configuration spaces of manifolds*, Invent. Math. 188 (2012) 465–504 MR

[13] **T Church**, **J S Ellenberg**, *Homology of FI–modules*, Geom. Topol. 21 (2017) 2373–2418 MR

[14] **T Church**, **J S Ellenberg**, **B Farb**, *FI–modules and stability for representations of symmetric groups*, Duke Math. J. 164 (2015) 1833–1910 MR

[15] **T Church**, **J S Ellenberg**, **B Farb**, **R Nagpal**, *FI–modules over Noetherian rings*, Geom. Topol. 18 (2014) 2951–2984 MR

[16] **T Church**, **B Farb**, *Representation theory and homological stability*, Adv. Math. 245 (2013) 250–314 MR

[17] **R L Cohen**, **I Madsen**, *Surfaces in a background space and the homology of mapping class groups*, from "Algebraic geometry, I: Seattle 2005" (D Abramovich, A Bertram, L Katzarkov, R Pandharipande, M Thaddeus, editors), Proc. Sympos. Pure Math. 80, Amer. Math. Soc., Providence, RI (2009) 43–76 MR

[18] **D Crowley**, **J Sixt**, *Stably diffeomorphic manifolds and* $l_{2q+1}(\mathbb{Z}[\pi])$, Forum Math. 23 (2011) 483–538 MR

[19] **R De Sapio**, *On* $(k-1)$*–connected* $(2k+1)$*–manifolds*, Math. Scand. 25 (1970) 181–189 MR

[20] **W G Dwyer**, *Twisted homological stability for general linear groups*, Ann. of Math. 111 (1980) 239–251 MR

[21] **C J Earle**, **J Eells**, *The diffeomorphism group of a compact Riemann surface*, Bull. Amer. Math. Soc. 73 (1967) 557–559 MR

[22] **J Ebert**, **O Randal-Williams**, *Semisimplicial spaces*, Algebr. Geom. Topol. 19 (2019) 2099–2150

[23] **Z Fiedorowicz**, **M Stelzer**, **R M Vogt**, *Homotopy colimits of algebras over* $\mathscr{C}at$*–operads and iterated loop spaces*, Adv. Math. 248 (2013) 1089–1155 MR

[24] **B Fresse**, *Homotopy of operads and Grothendieck–Teichmüller groups, I: The algebraic theory and its topological background*, Mathematical Surveys and Monographs 217, Amer. Math. Soc., Providence, RI (2017) MR

[25] **N Friedrich**, *Homological stability of automorphism groups of quadratic modules and manifolds*, Doc. Math. 22 (2017) 1729–1774 MR

[26] **W Fulton**, **J Harris**, *Representation theory*, Graduate Texts in Math. 129, Springer (1991) MR

[27] **S Galatius**, **O Randal-Williams**, *Homological stability for moduli spaces of high dimensional manifolds, II*, Ann. of Math. 186 (2017) 127–204 MR

[28] **S Galatius**, **O Randal-Williams**, *Homological stability for moduli spaces of high dimensional manifolds, I*, J. Amer. Math. Soc. 31 (2018) 215–264 MR

[29] **W L Gan**, **L Li**, *Coinduction functor in representation stability theory*, J. Lond. Math. Soc. 92 (2015) 689–711 MR

[30] **G Gandini**, **N Wahl**, *Homological stability for automorphism groups of RAAGs*, Algebr. Geom. Topol. 16 (2016) 2421–2441 MR

[31] **A Gramain**, *Le type d'homotopie du groupe des difféomorphismes d'une surface compacte*, Ann. Sci. École Norm. Sup. 6 (1973) 53–66 MR

[32] **D Grayson**, *Higher algebraic K–theory, II (after Daniel Quillen)*, from "Algebraic K–theory" (A Dold, B Eckmann, editors), Lecture Notes in Math. 551, Springer (1976) 217–240 MR

[33] **M A Guest**, **A Kozlowsky**, **K Yamaguchi**, *Homological stability of oriented configuration spaces*, J. Math. Kyoto Univ. 36 (1996) 809–814 MR

[34] **J L Harer**, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. 121 (1985) 215–249 MR

[35] **A Hatcher**, **K Vogtmann**, *Tethers and homology stability for surfaces*, Algebr. Geom. Topol. 17 (2017) 1871–1916 MR

[36] **A Hatcher**, **N Wahl**, *Stabilization for mapping class groups of 3–manifolds*, Duke Math. J. 155 (2010) 205–269 MR

[37] **R Hepworth**, *Homological stability for families of Coxeter groups*, Algebr. Geom. Topol. 16 (2016) 2779–2811 MR

[38] **N V Ivanov**, *On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients*, from "Mapping class groups and moduli spaces of Riemann surfaces" (C-F Bödigheimer, R M Hain, editors), Contemp. Math. 150, Amer. Math. Soc., Providence, RI (1993) 149–194 MR

[39] **R Jimenez Rolland**, *Representation stability for the cohomology of the moduli space $\mathcal{M}_g^n$*, Algebr. Geom. Topol. 11 (2011) 3011–3041 MR

[40] **R Jiménez Rolland**, *On the cohomology of pure mapping class groups as FI–modules*, J. Homotopy Relat. Struct. 10 (2015) 401–424 MR

[41] **W van der Kallen**, *Homology stability for linear groups*, Invent. Math. 60 (1980) 269–295 MR

[42] **A Kupers**, *Proving homological stability for homeomorphisms of manifolds*, preprint (2015) arXiv

[43] **A Kupers**, **J Miller**, *E_n–cell attachments and a local-to-global principle for homological stability*, Math. Ann. 370 (2018) 209–269 MR

[44] **J P May**, *The geometry of iterated loop spaces*, Lectures Notes in Math. 271, Springer (1972) MR

[45] **J P May**, **J Sigurdsson**, *Parametrized homotopy theory*, Mathematical Surveys and Monographs 132, Amer. Math. Soc., Providence, RI (2006) MR

[46] **D McDuff**, *Configuration spaces of positive and negative particles*, Topology 14 (1975) 91–107 MR

[47] **J Miller**, **M Palmer**, *A twisted homology fibration criterion and the twisted group-completion theorem*, Q. J. Math. 66 (2015) 265–284 MR

[48] **J Miller**, **J C H Wilson**, *Higher-order representation stability and ordered configuration spaces of manifolds*, Geom. Topol. 23 (2019) 2519–2591

[49] **B Mirzaii**, **W van der Kallen**, *Homology stability for unitary groups*, Doc. Math. 7 (2002) 143–166 MR

[50] **R S Palais**, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. 34 (1960) 305–312 MR

[51] **M Palmer**, *Homological stability for oriented configuration spaces*, Trans. Amer. Math. Soc. 365 (2013) 3675–3711 MR

[52] **M Palmer**, *Twisted homological stability for configuration spaces*, Homology Homotopy Appl. 20 (2018) 145–178 MR

[53] **P Patzt**, **X Wu**, *Stability results for Houghton groups*, Algebr. Geom. Topol. 16 (2016) 2365–2377 MR

[54] **N Perlmutter**, *Homological stability for the moduli spaces of products of spheres*, Trans. Amer. Math. Soc. 368 (2016) 5197–5228 MR

[55] **N Perlmutter**, *Linking forms and stabilization of diffeomorphism groups of manifolds of dimension $4n+1$*, J. Topol. 9 (2016) 552–606 MR

[56] **D Petersen**, *A spectral sequence for stratified spaces and configuration spaces of points*, Geom. Topol. 21 (2017) 2527–2555 MR

[57] **E Ramos**, *On the degree-wise coherence of FI_G–modules*, New York J. Math. 23 (2017) 873–895 MR

[58] **O Randal-Williams**, *Homological stability for unordered configuration spaces*, Q. J. Math. 64 (2013) 303–326 MR

[59] **O Randal-Williams**, *Resolutions of moduli spaces and homological stability*, J. Eur. Math. Soc. 18 (2016) 1–81 MR

[60] **O Randal-Williams**, *Cohomology of automorphism groups of free groups with twisted coefficients*, Selecta Math. 24 (2018) 1453–1478 MR

[61] **O Randal-Williams**, **N Wahl**, *Homological stability for automorphism groups*, Adv. Math. 318 (2017) 534–626 MR

[62] **E Riehl**, *Categorical homotopy theory*, New Mathematical Monographs 24, Cambridge Univ. Press (2014) MR

[63] **S V Sam**, **A Snowden**, *Representations of categories of G–maps*, J. Reine Angew. Math. 750 (2019) 197–226 MR

[64] **C Schlichtkrull**, **M Solberg**, *Braided injections and double loop spaces*, Trans. Amer. Math. Soc. 368 (2016) 7305–7338 MR

[65] **G Segal**, *Configuration-spaces and iterated loop-spaces*, Invent. Math. 21 (1973) 213–221 MR

[66] **G Segal**, *Categories and cohomology theories*, Topology 13 (1974) 293–312 MR

[67] **G Segal**, *The topology of spaces of rational functions*, Acta Math. 143 (1979) 39–72

[68] **A Soulié**, *The Long–Moody construction and polynomial functors*, Ann. Inst. Fourier (Grenoble) (online publication May 2019)

[69] **M Szymik**, **N Wahl**, *The homology of the Higman–Thompson groups*, Invent. Math. 216 (2019) 445–518 MR

[70] **U Tillmann**, *Homology stability for symmetric diffeomorphism and mapping class groups*, Math. Proc. Cambridge Philos. Soc. 160 (2016) 121–139 MR

[71] **P Tosteson**, *Lattice spectral sequences and cohomology of configuration spaces*, preprint (2016) arXiv

[72] **B Totaro**, *Configuration spaces of algebraic varieties*, Topology 35 (1996) 1057–1067 MR

[73] **A A Voronov**, *The Swiss-cheese operad*, from "Homotopy invariant algebraic structures" (J-P Meyer, J Morava, W S Wilson, editors), Contemp. Math. 239, Amer. Math. Soc., Providence, RI (1999) 365–373 MR

[74] **N Wahl**, *Homological stability for the mapping class groups of non-orientable surfaces*, Invent. Math. 171 (2008) 389–424 MR

[75] **C T C Wall**, *Classification problems in differential topology, VI: Classification of $(s-1)$–connected $(2s+1)$–manifolds*, Topology 6 (1967) 273–296 MR

[76] **G W Whitehead**, *Elements of homotopy theory*, Graduate Texts in Math. 61, Springer (1978) MR

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Received: 29 January 2018
Revised: 18 September 2018