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Birational models of moduli spaces of coherent sheaves on the projective plane

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We study the birational geometry of moduli spaces of semistable sheaves on the projective plane via Bridgeland stability conditions. We show that the entire MMP of their moduli spaces can be run via wall-crossing. Via a description of the walls, we give a numerical description of their movable cones, along with its chamber decomposition corresponding to minimal models. As an application, we show that for primitive vectors, all birational models corresponding to open chambers in the movable cone are smooth and irreducible.

14D20; 14E30

Introduction

The birational geometry of moduli spaces of sheaves on surfaces has been studied a lot in recent years; see for instance Arcara, Bertram, Coskun and Huizenga [2], Bayer and Macrì [6; 7], Bertram, Martinez and Wang [9], Coskun and Huizenga [13; 14; 12], Coskun, Huizenga and Woolf [15], Li and Zhao [26] and Woolf [33]. The milestone work in [6; 7] completes the whole picture for K3 surfaces. In this paper, we give a complete description of the minimal model program of the moduli space of semistable sheaves on the projective plane via wall-crossings in the space of Bridgeland stability conditions. As a consequence, we deduce a description of their nef cone, movable cone and the chamber decomposition of their minimal models.

Geometric stability conditions on $\mathbb{P}^2$ The notion of *stability condition* on a $\mathbb{C}$–linear triangulated category was first introduced by Bridgeland [10]. A stability condition consists of a slicing $\mathcal{P}$ of semistable objects in the triangulated category and a central charge Z on the Grothendieck group which is compatible with the slicing. In particular, in this paper we consider the bounded derived category of coherent sheaves on the projective plane. A stability condition $\sigma = (Z, \mathcal{P})$ is called *geometric* if it satisfies the support property and all skyscraper sheaves are σ–stable with the same phase; see Definition 1.12.

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The Grothendieck group $\mathrm{K}(\mathbb{P}^2)$ of $\mathrm{D}^b(\mathbb{P}^2)$ is of rank 3 and $\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2)(=\mathrm{K}(\mathbb{P}^2)\otimes\mathbb{R})$ is spanned by the Chern characters ch_0, ch_1 and ch_2. Due to the work of Drezet and Le Potier, there is a *Le Potier cone* (see Figure 2) in the space $\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2)$, such that there exist slope-stable coherent sheaves with character $w=(\mathrm{ch}_0(>0),\mathrm{ch}_1,\mathrm{ch}_2)\in\mathrm{K}(\mathbb{P}^2)$ if and only if either w is the character of an exceptional bundle, or it is not inside the Le Potier cone. As we will show in Proposition 1.16, by taking the kernel of the central charge, the space of all geometric stability conditions can be realized as a principal $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$–bundle over $\mathrm{Geo}_{\mathrm{LP}}$, which is an open region above the *Le Potier curve*. Note that the $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$–action does not affect the stability of objects. We will write a geometric stability condition as $\sigma_{s,q}$ with $(1,s,q)\in\mathrm{Geo}_{\mathrm{LP}}\subset\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2)$ indicating the kernel of its central charge. Denote the heart of the stability condition $\sigma_{s,q}$ by $\mathrm{Coh}_{\#s}$, and let $\mathfrak{M}^{\mathrm{s(ss)}}_{\sigma_{s,q}}(w)$ be the moduli space of $\sigma_{s,q}$–(semi)stable objects in $\mathrm{Coh}_{\#s}$ with character $w\in\mathrm{K}(\mathbb{P}^2)$. We address the following questions:

(1) For a Chern character w and a geometric stability condition $\sigma_{s,q}$, when is $\mathfrak{M}^{\mathrm{ss}}_{\sigma_{s,q}}(w)$ nonempty?

(2) How does $\mathfrak{M}^{\mathrm{ss}}_{\sigma_{s,q}}(w)$ change when $\sigma_{s,q}$ varies in the space $\mathrm{Geo}_{\mathrm{LP}}$ of geometric stability conditions?

The first question is answered step-by-step in several parts of the paper. Generalizing the result of Drezet and Le Potier for stable sheaves, when the character w is inside the Le Potier cone and not exceptional (see Corollary 1.22), there is no $\sigma_{s,q}$–semistable object with character w (or $-w$) for any geometric stability condition $\sigma_{s,q}$. In other words, $\mathfrak{M}^{\mathrm{ss}}_{\sigma_{s,q}}(w)$ is always empty. When the character w is proportional to an exceptional character, the first question is answered in Corollary 1.22 and Proposition 1.30.

The main case of the first question is when the character is not inside the Le Potier cone.

Theorem A (Lemma 3.12 and Theorem 3.14) *Let $w\in\mathrm{K}(\mathbb{P}^2)$ be a character which is not inside the Le Potier cone. Then $\mathfrak{M}^{\mathrm{ss}}_{\sigma_{s,q}}(w)$ is empty if and only if $\sigma_{s,q}$ is below L^{last}_w or $L^{\text{right-last}}_w$ in* $\mathrm{Geo}_{\mathrm{LP}}$.

The notation L^{last}_w and $L^{\text{right-last}}_w$ is defined in Definition 3.11. As we will see, the description for the last wall is equivalent to that for the boundary of the effective cone of the moduli space. This theorem is first proved in Coskun, Huizenga and Woolf [15] in the $\mathrm{ch}_0\geq 1$ case and in Woolf [33] in the torsion case. In this paper, we provide another proof for these results in our setup. Several results in this proof are needed in the proof of our main theorem on the actual walls.

Let $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$ be the moduli space of stable sheaves of character w. Assume that w is outside the Le Potier cone. It was proved in [15] that $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$ is a Mori dream space of Picard rank 2. In particular, if we consider the stable base locus decomposition of the effective cone of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$, there are only finitely many chambers, and each chamber corresponds to one minimal model of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$. When varying the line bundle in the effective cone and considering the corresponding minimal model, this induces the minimal model program of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$. The following result realizes this via wall-crossing, and in particular answers the second question.

Theorem B (Theorems 2.19 and 2.24) *Let w be a primitive character. The moduli space $\mathfrak{M}^{\mathrm{s}}_{\sigma_{s,q}}(w)$ is smooth and connected for any generic geometric stability condition $\sigma_{s,q}$ when it is nonempty. Any two nonempty moduli spaces $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w)$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma'}(w)$ are birational to each other. The actual walls (resp. chambers) are in one-to-one correspondence with the stable base locus decomposition walls (resp. chambers) of the effective cone of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$. In particular, one can run the whole minimal model program for $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$ via wall-crossing in the space of geometric stability conditions.*

The smoothness result can be proved easily for moduli of stable sheaves by applying Serre duality $\mathrm{Ext}^2(E,E) = \mathrm{Hom}(E,E(-3))^*$ and observing that $E(-3)$ is stable of a smaller slope. However, for Bridgeland stable objects, they may not remain stable after twisting by $\mathcal{O}(-3)$. The key point is to develop a method to compare slopes with respect to different Bridgeland stability conditions in order to conclude the vanishing of the Hom group. This is achieved first in Li and Zhao [26], and generalized to the current situation in Section 2. The following consequence seems new to the theory of MMP of moduli of sheaves.

Theorem C *Let $w \in \mathrm{K}(\mathbb{P}^2)$ be a primitive character not inside the Le Potier cone. Then for each chamber in the movable cone of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$, the corresponding minimal model of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$ is smooth.*

Based on the explicit correspondence between walls in the space of stability conditions and walls in the divisor cone, we may describe all stable base locus walls (including the boundaries of the nef cone, effective cone and movable cone) as actual walls in the space of stability conditions. Here an actual wall for a Chern character w is a wall such that curves are contracted from either side of $\mathfrak{M}^{\mathrm{ss}}_{\sigma_\pm}(w) \dashrightarrow \mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$. So it becomes an important question to ask when a wall is an actual wall. In Section 3, we give a numerical criterion on actual walls, which depends on only the character w, and provides an effective algorithm to compute all actual walls for w.

We introduce some notation first. The notation $l_{\sigma w}$ (defined in Remark 1.5) stands for the line segment between $\operatorname{Ker} Z_\sigma$ and w (viewed as points in the real projective space of $\mathrm{P}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2))$). The notation TR_{wE} (defined in Definition 3.15) stands for a small triangular area determined by w and an exceptional character E. A destabilizing character v for w cannot appear in the triangle TR_{wE}, because otherwise $\mathfrak{M}^{\mathrm{ss}}_\sigma(v)$ would be empty by Theorem A. Now we can state the criterion on actual walls:

Theorem D (Theorem 3.16) *Let $w \in \mathrm{K}(\mathbb{P}^2)$ be a Chern character with $\mathrm{ch}_0(w) \geq 0$ not inside the Le Potier cone. For any stability condition σ inside the Bogomolov cone between the last wall L^{last}_w and the vertical wall $L_{w\pm}$, the stability condition σ is on an actual wall for w if and only if there exists a Chern character $v \in \mathrm{K}(\mathbb{P}^2)$ on the line segment $l_{\sigma w}$ such that $\mathrm{ch}_0(v) > 0$, $\frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)} > \frac{\mathrm{ch}_1(v)}{\mathrm{ch}_0(v)}$, the characters v and $w - v$ are either exceptional or not inside the Le Potier cone, and both of them are not in TR_{wE} for any exceptional bundle E.*

On an actual wall, the Chern character w can always be written as the sum of two proper Chern characters w' and $w - w'$ satisfying the conditions in the theorem. The key is to prove the inverse direction: for two such characters, there exist stable objects of character w, which are given as the extensions of stable objects of characters w' and $w - w'$, and hence destabilized on the wall. Roughly speaking, the three main steps are to show:

(1) $\mathfrak{M}^{\mathrm{s}}_\sigma(w')$ and $\mathfrak{M}^{\mathrm{s}}_\sigma(w - w')$ are nonempty.

(2) The extension group Ext^1 between two generic objects in the moduli spaces $\mathfrak{M}^{\mathrm{s}}_\sigma(w')$ and $\mathfrak{M}^{\mathrm{s}}_\sigma(w - w')$ is nonzero.

(3) The extensions of two stable objects produce σ_+ or σ_- stable objects with character w.

The conditions in the theorem are mainly used in step (1), and step (3) follows from general computations. For step (2), since we are only making use of numerical characters, we can only aim to show $\chi(w - w', w') < 0$. However, one may wonder about the case that on an actual wall, generic objects in $\mathfrak{M}^{\mathrm{s}}_\sigma(w')$ and $\mathfrak{M}^{\mathrm{s}}_\sigma(w - w')$ do not have nontrivial extensions, but objects on some jumping loci extend to $\sigma_\pm$–stable objects. Should this happen, $\chi(w - w', w') \geq 0$ but objects in $\mathfrak{M}^{\mathrm{s}}_\sigma(w')$ and $\mathfrak{M}^{\mathrm{s}}_\sigma(w - w')$ still extend to $\sigma_\pm$–stable objects. From this point of view, it is a bit surprising to have a numerical criterion for actual walls. The main point in Theorem D is to rule out this possibility, and this is done by gaining a good understanding of the last wall and the extension groups.

Moreover, the criterion decides all the actual walls effectively, in the sense that it involves only finitely many steps of computations, and one may write a computer program to output all the actual walls with a given Chern character w as the input. We compute the example for $w=(4,0,-15)$ by hand to show some details of this computation; see Figure 1.

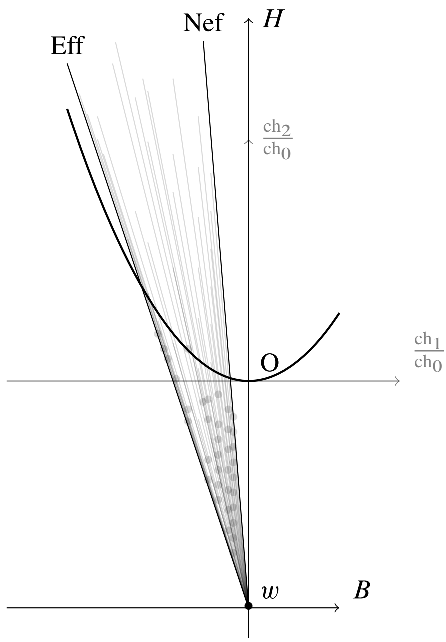

Figure 1: A picture for the actual walls in the case of $w=(4,0,-15)$

As two quick applications of Theorem D, we decide the boundary of the nef cone and the movable cone of $\mathfrak{M}^{s}_{\mathrm{GM}}(w)$ for a primitive character $w=(\mathrm{ch}_0,\mathrm{ch}_1,\mathrm{ch}_2)$. We work out the boundary to the left of the vertical wall; the other side is determined by applying the dualizing functor and considering the character $w'=(\mathrm{ch}_0,-\mathrm{ch}_1,\mathrm{ch}_2)$.

Theorem E (Theorem 4.3, the movable cone) *Let w be a primitive Chern character with $\mathrm{ch}_0(w)\geq 0$ not inside the Le Potier cone. When $\chi(E,w)\neq 0$ for all exceptional bundles E with $\frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)}<\frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)}$, the movable cone boundary on the primitive side coincides with the effective cone boundary.*

When $\chi(E_\gamma,w)=0$ for an exceptional bundle E_γ with $\frac{\mathrm{ch}_1(E_\gamma)}{\mathrm{ch}_0(E_\gamma)}<\frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)}$, the bundle E_γ can be canonically extended to an exceptional collection E_α, E_γ, E_β; see the paragraph above Lemma 4.1 for more details. Denote the characters of $E_\beta\otimes\mathcal{O}(-3)$,

E_α and E_γ by $e_{\beta-3}$, e_α and e_γ, respectively. Then w can be uniquely written as $n_2e_\alpha - n_1e_{\beta-3}$ for some positive integers n_1 and n_2. Define the character P accordingly, as follows:

$$P := \begin{cases} e_\gamma - (3\operatorname{ch}_0(E_\beta) - n_2)e_\alpha & \text{if } 1 \leq n_2 < 3\operatorname{ch}_0(E_\beta), \\ e_\gamma & \text{if } n_2 \geq 3\operatorname{ch}_0(E_\beta). \end{cases}$$

Then the wall L_{Pw} corresponds to the boundary of the movable cone of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$.

Theorem F (Theorem 4.6, the nef cone) *Let w be a primitive Chern character with $\operatorname{ch}_0(w) > 0$ and*

$$\overline{\Delta}(w) := \frac{1}{2}\left(\frac{\operatorname{ch}_1(w)}{\operatorname{ch}_0(w)}\right)^2 - \frac{\operatorname{ch}_2(w)}{\operatorname{ch}_0(w)} \geq 10.$$

*Then the first actual wall L_{vw} to the left of vertical wall (equivalently, the **nef cone boundary** for $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$) is determined by the character v satisfying:*

(1) *$\frac{\operatorname{ch}_1(v)}{\operatorname{ch}_0(v)}$ is the greatest rational number less than $\frac{\operatorname{ch}_1(w)}{\operatorname{ch}_0(w)}$ with $0 < \operatorname{ch}_0(v) \leq \operatorname{ch}_0(w)$.*

(2) *Given the first condition, if $\operatorname{ch}_1(v)$ is even (resp. odd), then $\operatorname{ch}_2(v)$ is the greatest integer (resp. $2\operatorname{ch}_2(v)$ is the greatest odd integer) such that the point v is either an exceptional character, or not inside the Le Potier cone.*

The result on the nef cone was first proved in Coskun and Huizenga [14] when Δ is large enough with respect to ch_0 and $\frac{\operatorname{ch}_1}{\operatorname{ch}_0}$ (see Remark 8.7 in [14] for a lower bound). In Theorem F the bound is explicitly given as $\overline{\Delta} \geq 10$. This bound on $\overline{\Delta}(w)$ is used to show that the first wall is not of higher rank. The rest of the proof is a direct application of our criterion on actual walls.

The result on the movable cone is more subtle. When the Chern character w is right orthogonal to an exceptional bundle E_γ, the jumping locus

$$\{[F] \in \mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w) \mid \operatorname{Hom}(E_\gamma, F) \neq 0\}$$

has codimension 1 and is the exceptional divisor that contracted on the movable cone boundary. However, the wall L_{we_γ} for $\operatorname{Hom}(E_\gamma, F) \neq 0$ may not always be the wall for this contraction. In the case when $n_2 < 3\operatorname{ch}_0(E_\beta)$, the exceptional divisor is already contracted at a wall prior to the wall L_{we_γ}. One simple example of such w is when $(\operatorname{ch}_0, \operatorname{ch}_1, \operatorname{ch}_2) = (1, 0, -4)$; in other words, the ideal sheaf of four points. The Chern character w is right orthogonal to the cotangent bundle Ω. The other exceptional bundles E_α and E_β are $\mathcal{O}(-2)$ and $\mathcal{O}(-1)$, respectively, and w can be written as $2[\mathcal{O}(-2)] - [\mathcal{O}(-4)]$. The jumping locus of $\operatorname{Hom}(\Omega, w) \neq 0$ is the exceptional divisor, and it is the same as the jumping locus $\mathrm{Hom}(\mathcal{I}_1(-1), w) \neq 0$, where $\mathcal{I}_1(-1)$ stands for the ideal sheaf of one point tensored by $\mathcal{O}(-1)$. Since the wall $L_{\mathcal{I}_1(-1)w}$ is between $L_{\Omega w}$ and $L_{w\pm}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, the boundary of the movable cone should be given by $L_{\mathcal{I}_1(-1)w}$. Geometrically, the exceptional locus is where any three points are collinear.

Related work There are several papers studying the birational geometry of moduli of sheaves on the projective plane via wall-crossing; see, as mentioned earlier, the work of Arcara, Bertram, Coskun and Huizenga [2], Bertram, Martinez and Wang [9], Coskun and Huizenga [14; 13; 12], Coskun, Huizenga and Woolf [15], Li and Zhao [26] and Woolf [33].

The study of the Hilbert scheme of points on $\mathbb{P}^2$ first appears in [2], and the wall-crossing behavior is explicitly carried out for small numbers of points. It is also first suggested in [2] that there is a correspondence between the wall-crossing picture in the Bridgeland space and the minimal model program of the moduli space. In [12], the correspondence between walls in the Bridgeland space and stable locus decomposition walls in MMP is established for monomial schemes in the plane. In [26], we proved the full correspondence for Hilbert schemes of points, by establishing similar results as in Section 2 of this paper, and further generalized this correspondence to deformations of Hilbert schemes, which are constructed as Hilbert schemes of noncommutative $\mathbb{P}^2$.

For moduli of torsion sheaves, the effective cone and the nef cone are computed in [33]. For general moduli of sheaves on $\mathbb{P}^2$, the theory is built up in [9]. Among other results, the projectivity of moduli of Bridgeland stable objects is proved in [9]. The effective cone and the ample cone are computed in [14] and [15], respectively. Also, [15] gives the criterion for when the movable cone coincides with the effective cone. We refer to the beautiful lecture notes [13] for details of these results.

Compared with these papers, the smoothness and irreducibility of moduli of Bridgeland stable objects with primitive characters are only proved in this paper. Combined with results in variation of GIT, this allows us to deduce the equivalence between wall-crossing and MMP for moduli of sheaves on $\mathbb{P}^2$ suggested in [2]. We include a different proof for the effective cone result in [15], as our argument on the effective cone is also closely related to the proof of the criterion on actual walls. The numerical criterion on actual walls and the result on the movable cone are new. Our result on the nef cone (Theorem F) follows from the numerical criterion. The nef cone was first computed in [14] when Δ is large enough with respect to ch_0 and $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$ (for a lower bound see [14, Remark 8.7]); the bound in Theorem F is explicitly given by $\overline{\Delta} \geq 10$. Moreover, as a benefit of our setup, most of the paper can treat the torsion case and the positive-rank case uniformly. We make careful remarks on this throughout the paper. Another important application of the wall-crossing machinery is towards the Le Potier strange duality conjecture. A special case is studied in Abe [1].

Organization In Section 1.1, we review some classical work by Drezet and Le Potier on stable sheaves on the projective plane. We prove some useful lemmas by visualizing the geometric stability conditions in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane in Section 1.3. These properties will play a crucial role in the arguments in the paper. In Section 2, we prove that the moduli space $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w)$ is smooth and irreducible for generic σ and primitive w. After recalling some results in variation of GIT, we show that one can run the minimal model program for $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$ by wall-crossing. In Section 3, we first compute the last wall, and then prove the main theorem of the paper: a criterion for actual walls of $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w)$. In Section 4, we compute the nef and movable cone boundary as an application of the criterion for actual walls. Moreover, in Section 4.3, we work out the particular example of the Chern character $(4, 0, -15)$.

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1 Stability conditions on $\mathrm{D}^b(\mathbb{P}^2)$

In this section, we recall some properties of the bounded derived category of coherent sheaves on the projective plane, and the construction of stability conditions on it. In Section 1.1, we explain the structure of $\mathrm{D}^b(\mathbb{P}^2)$ given by exceptional triples, and the numerical criterion on the existence of stable sheaves. A slice of the space of geometric stability conditions is discussed in Section 1.2, and the wall-chamber structure on it is studied in Section 1.3. In Section 1.4, we study the algebraic stability conditions, in other words, the stability conditions associated to the exceptional triples. We also explain how they are glued to the slice of geometric stability conditions. In Section 1.5, we explain in detail the difference and advantage of our setup over the one used in other papers. Finally in Section 1.6, we derive some easy numerical conditions on the existence of stable objects.

1.1 Review: exceptional objects, triples and the Le Potier curve

Let $\mathcal{T}$ be a $\mathbb{C}$–linear triangulated category of finite type. In this article, $\mathcal{T}$ will always be $\mathrm{D}^b(\mathbb{P}^2)$: the bounded derived category of coherent sheaves on the projective plane over $\mathbb{C}$. We first recall the following definitions from [3; 20; 31].

Definition 1.1 An object E in $\mathcal{T}$ is called *exceptional* if

$$\begin{cases} \mathrm{Hom}(E, E[i]) = 0 & \text{for } i \neq 0, \\ \mathrm{Hom}(E, E) = \mathbb{C}. \end{cases}$$

An ordered collection of exceptional objects $\mathcal{E} = \{E_0, \dots, E_m\}$ is called an *exceptional collection* if

$$\mathrm{Hom}(E_i, E_j[k]) = 0 \quad \text{for } i > j \text{ and all } k.$$

Definition 1.2 Let $\mathcal{E} = \{E_0, \dots, E_n\}$ be an exceptional collection. We say this collection $\mathcal{E}$ is *strong* if

$$\mathrm{Hom}(E_i, E_j[q]) = 0 \quad \text{for all } i, j \text{ and for } q \neq 0.$$

This collection $\mathcal{E}$ is called *full* if $\mathcal{E}$ generates $\mathcal{T}$ under homological shifts, cones and direct sums.

An exceptional coherent sheaf on $\mathbb{P}^2$ is locally free. We summarize some results on the classification of exceptional bundles on $\mathbb{P}^2$ and introduce some notation, for details we refer to [18; 20; 24].

The Picard group of $\mathbb{P}^2$ is of rank one with generator $H = [\mathcal{O}(1)]$, and we will, by abuse of notation, identify the i^{th} Chern character ch_i with its degree $H^{2-i}\,\mathrm{ch}_i$. There is a one-to-one correspondence between exceptional bundles and dyadic integers

$$\left\{ \frac{p}{2^m} \;\middle|\; p \in \mathbb{Z},\ m \in \mathbb{Z}_{\geq 0},\ p \text{ is odd when } m \neq 0 \right\}.$$

Denote the exceptional bundle corresponding to $\frac{p}{2^m}$ by $E_{(\frac{p}{2^m})}$. These exceptional bundles are inductively (on m) given as follows:

- $E_{(n)} = \mathcal{O}(n)$ for $n \in \mathbb{Z}$.

- When $m>0$ and $p\equiv 3(\mathrm{mod}\ 4)$, the exceptional bundle $E_{\left(\frac{p}{2^m}\right)}$ is given as the right mutation $R_{E_{\left(\frac{p-1}{2^m}\right)}}E_{\left(\frac{p-3}{2^m}\right)}$. In particular, there is a short exact sequence:
$$E_{\left(\frac{p-3}{2^m}\right)}\to E_{\left(\frac{p-1}{2^m}\right)}\otimes \operatorname{Hom}(E_{\left(\frac{p-3}{2^m}\right)},E_{\left(\frac{p-1}{2^m}\right)})^*\to E_{\left(\frac{p}{2^m}\right)}.$$
- When $m>0$ and $p\equiv 1(\mathrm{mod}\ 4)$, the exceptional bundle $E_{\left(\frac{p}{2^m}\right)}$ is given as the left mutation $L_{E_{\left(\frac{p+1}{2^m}\right)}}E_{\left(\frac{p+3}{2^m}\right)}$. In particular, there is a short exact sequence:
$$E_{\left(\frac{p}{2^m}\right)}\to E_{\left(\frac{p+1}{2^m}\right)}\otimes \operatorname{Hom}(E_{\left(\frac{p+1}{2^m}\right)},E_{\left(\frac{p+3}{2^m}\right)})\to E_{\left(\frac{p+3}{2^m}\right)}.$$

We write the Chern characters of $E_{\left(\frac{p}{2^m}\right)}$ as
$$\tilde{v}\left(\tfrac{p}{2^m}\right):=\tilde{v}(E_{\left(\frac{p}{2^m}\right)})=\left(\mathrm{ch}_0(E_{\left(\frac{p}{2^m}\right)}),\mathrm{ch}_1(E_{\left(\frac{p}{2^m}\right)}),\mathrm{ch}_2(E_{\left(\frac{p}{2^m}\right)})\right).$$
They are inductively (on m) given by the formulas
$$\begin{cases} \tilde{v}(n)=\left(1,n,\frac{n^2}{2}\right) & \text{for } n\in\mathbb{Z},\\ \tilde{v}\left(\frac{p}{2^m}\right)=3\,\mathrm{ch}_0(E_{\left(\frac{p+1}{2^m}\right)})\tilde{v}\left(\frac{p-1}{2^m}\right)-\tilde{v}\left(\frac{p-3}{2^m}\right) & \text{when } m>0,\ p\equiv 3 \pmod 4,\\ \tilde{v}\left(\frac{p}{2^m}\right)=3\,\mathrm{ch}_0(E_{\left(\frac{p-1}{2^m}\right)})\tilde{v}\left(\frac{p+1}{2^m}\right)-\tilde{v}\left(\frac{p+3}{2^m}\right) & \text{when } m>0,\ p\equiv 1 \pmod 4.\end{cases}$$

Remark 1.3 Here are some observations from the definition:

(1) $\tilde{v}\left(\frac{3}{2}\right)$ is the character of the tangent sheaf $\mathcal{T}_{\mathbb{P}^2}=E_{\left(\frac{3}{2}\right)}$.

(2) The exceptional bundle $E_{\left(\frac{p}{2^m}+n\right)}$ is $E_{\left(\frac{p}{2^m}\right)}\otimes\mathcal{O}(n)$ for any $n\in\mathbb{Z}$.

(3) $\frac{\mathrm{ch}_1(E_{(a)})}{\mathrm{ch}_0(E_{(a)})}<\frac{\mathrm{ch}_1(E_{(b)})}{\mathrm{ch}_0(E_{(b)})}$ if and only if $a<b$.

For the rest of this section, we recall the construction of the Le Potier curve C_{LP}, which is closely related to the existence of semistable sheaves.

The Grothendieck group $\mathrm{K}(\mathbb{P}^2)$ has rank 3. We write $\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2):=\mathrm{K}(\mathbb{P}^2)\otimes\mathbb{R}$. Consider the real projective space $\mathrm{P}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2))$ with homogeneous coordinate $[\mathrm{ch}_0,\mathrm{ch}_1,\mathrm{ch}_2]$; we view the locus $\mathrm{ch}_0=0$ as the line at infinity. The complement forms an affine real plane, which is referred to as the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane in this paper. We call $\mathrm{P}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2))$ the projective $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. For any object F in $\mathrm{D}^b(\mathbb{P}^2)$, we write
$$\tilde{v}(F):=(\mathrm{ch}_0(F),\mathrm{ch}_1(F),\mathrm{ch}_2(F))$$
for the (degrees of) Chern characters of F. When $\tilde{v}(F)\neq 0$, we use $v(F)$ to denote the corresponding point in the projective $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. Note that when $\mathrm{ch}_0(F)\neq 0$, the point $v(F)$ is in the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane .

Remark 1.4 In this paper, in all arguments on the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, we assume the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–axis to be horizontal and the $\frac{\mathrm{ch}_2}{\mathrm{ch}_0}$–axis to be vertical. The term "above" means "the $\frac{\mathrm{ch}_2}{\mathrm{ch}_0}$ coordinate is greater than". Other terms such as "below", "to the right" and "to the left" are understood in a similar way.

Denote $\nu(E_{(\frac{p}{2^m})})$ by $e(\frac{p}{2^m})$. To define the Le Potier curve, we need to associate to $E_{(\frac{p}{2^m})}$ three more points $e^+(\frac{p}{2^m})$, $e^l(\frac{p}{2^m})$ and $e^r(\frac{p}{2^m})$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. These will be the vertices of the Le Potier curve in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. The coordinate of $e^+(\frac{p}{2^m})$ is given by

$$e^+\left(\tfrac{p}{2^m}\right) := e_{(\frac{p}{2^m})} - \left(0, 0, \frac{1}{\left(\mathrm{ch}_0(E_{(\frac{p}{2^m})})\right)^2}\right).$$

For any real number a, let $\overline{\Delta}_a$ be the parabola

$$\left\{\left(1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right) \,\middle|\, \overline{\Delta} := \frac{1}{2}\left(\frac{\mathrm{ch}_1}{\mathrm{ch}_0}\right)^2 - \frac{\mathrm{ch}_2}{\mathrm{ch}_0} = a\right\}$$

in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. The point $e^l(\frac{p}{2^m})$ is defined to be the intersection of $\overline{\Delta}_{\frac{1}{2}}$ and the line segment $l_{e^+(\frac{p}{2^m})e(\frac{p-1}{2^m})}$, and $e^r(\frac{p}{2^m})$ is defined to be the intersection of $\overline{\Delta}_{\frac{1}{2}}$ and the line segment $l_{e^+(\frac{p}{2^m})e(\frac{p+1}{2^m})}$.

Remark 1.5 In this paper we always use l_{PQ} to denote the line segment with endpoints P and Q, and use L_{PQ} to denote the line through P and Q in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. When three points P, Q and R are collinear, we may also write L_{PQR} for the line to indicate that these three points are collinear.

In fact, the definition of e^l and e^r has concrete geometric meaning. Let E be the exceptional bundle $E_{(\frac{p}{2^m})}$. A character w on the line $L_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})e(\frac{p-1}{2^m})}$ satisfies the equation

$$\chi(E, w) = \chi(w, E(-3)) = 0.$$

Symmetrically, the line $L_{e^+(\frac{p}{2^m})e^r(\frac{p}{2^m})e(\frac{p+1}{2^m})}$ is given by the equation

$$\chi(w, E) = \chi(E(3), w) = 0$$

in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. As we will see later, these lines detect the existence of morphisms between E and objects of character w.

In the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, consider the open region below all the line segments $l_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})}$, $l_{e^r(\frac{p}{2^m})e^+(\frac{p}{2^m})}$ and the curve $\overline{\Delta}_{\frac{1}{2}}$. The boundary of this open region is a fractal curve consisting of line segments $l_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})}$ and $l_{e^r(\frac{p}{2^m})e^+(\frac{p}{2^m})}$ for all dyadic numbers $\frac{p}{2^m}$ and fractal pieces of points on $\bar{\Delta}_{\frac{1}{2}}$. This curve is in the region between $\bar{\Delta}_{\frac{1}{2}}$ and $\bar{\Delta}_1$.

Definition 1.6 We call this boundary curve the *Le Potier curve* in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, and denote it by C_{LP}.

The cone in $\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2)$ spanned by the origin and C_{LP} is defined to be the *Le Potier cone*, denoted by $\mathrm{Cone}_{\mathrm{LP}}$.

We say a character $v \in \mathrm{K}(\mathbb{P}^2)$ is *not inside* $\mathrm{Cone}_{\mathrm{LP}}$ if either $\mathrm{ch}_0(v) \neq 0$ and the corresponding point $\tilde{v}$ is not above C_{LP} in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, or $\mathrm{ch}_0(v) = 0$ and $\mathrm{ch}_1 > 0$.

Remark 1.7 The line segments $l_{e^+(\frac{p}{2^m})e^l(\frac{p}{2^m})}$ and $l_{e^r(\frac{p}{2^m})e^+(\frac{p}{2^m})}$ do not cover the whole C_{LP}; the complement forms a Cantor set on $\bar{\Delta}_{\frac{1}{2}}$.

The picture for C_{LP} in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane is shown in Figure 2.

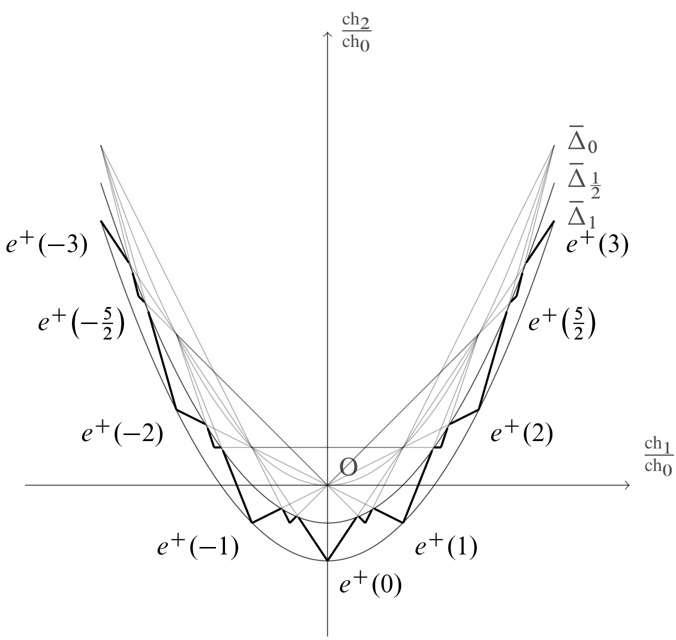

Figure 2: The Le Potier curve C_{LP}

Given the Le Potier curve, we can now state the numerical condition on the existence of stable sheaves.

Theorem 1.8 (Drezet, Le Potier) *There exists a slope-semistable coherent sheaf with character* $w = (\mathrm{ch}_0(>0), \mathrm{ch}_1, \mathrm{ch}_2) \in \mathrm{K}(\mathbb{P}^2)$ *if and only if one of the following two conditions hold:*

(1) w *is proportional to an exceptional character.*

(2) *The point* $\left(1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right)$ *is on or below* C_{LP} *in the* $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$*–plane.*

1.2 Geometric stability conditions

In this section, we follow [5; 11] and recall that the space of geometric stability conditions on $\mathbb{P}^2$ is a $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$–principal bundle over a subspace $\mathrm{Geo}_{\mathrm{LP}}$ of the $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. Another very good reference with more details is [29].

Definition 1.9 A *stability condition* σ on $\mathrm{D}^b(\mathbb{P}^2)$ consists of a pair $(Z, \mathcal{A})$ such that:

- $\mathcal{A}$ is the heart of a t–structure on $\mathrm{D}^b(\mathbb{P}^2)$.
- The central charge Z is a linear map $Z\colon K_{\mathrm{num}}(\mathrm{D}^b(\mathbb{P}^2)) \to \mathbb{C}$.
- $Z(E) \in \{\rho^{\pi i \phi} \mid \rho > 0,\ 0 < \phi \leq 1\}$ for all nonzero $E \in \mathcal{A}$.
- Every nonzero object $E \in \mathcal{A}$ admits a finite *Harder–Narasimhan* filtration
$$0 \subset E_0 \subset E_1 \cdots \subset E_n = E,$$
uniquely determined by the property that each factor $F_i := E_i/E_{i-1}$ is semistable and $\mathrm{Arg}(Z(F_1)) > \mathrm{Arg}(Z(F_2)) > \cdots > \mathrm{Arg}(Z(F_n))$.

Definition 1.10 We say a stability condition $\sigma = (Z, \mathcal{A})$ satisfies the *support property* if there exists a quadratic form on the vector space $K_{\mathrm{num}}(\mathrm{D}^b(\mathbb{P}^2)) \otimes \mathbb{R}$ such that:

- The kernel of Z in $K_{\mathrm{num}}(\mathrm{D}^b(\mathbb{P}^2)) \otimes \mathbb{R}$ is negative definite with respect to Q.
- For any σ–semistable object $E \in \mathcal{A}$, we have $Q([E]) \geq 0$.

Remark 1.11 There are several equivalent definitions of the support property; we refer to Appendix A in [8] and Section 4 in [11] for more details. The support property allows the space of stability conditions to have nice topology and wall-crossing structures.

In some of the literature, different from our terminology, a stability condition is assumed to satisfy the support property, otherwise it is called a prestability condition.

In applications to geometry, the following type of stability condition is always most useful.

Definition 1.12 A stability condition σ on $\mathrm{D}^b(\mathbb{P}^2)$ is called *geometric* if it satisfies the support property and all skyscraper sheaves $k(x)$ are σ–stable of the same phase. We denote the set of all geometric stability conditions by $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$.

In order to construct geometric stability conditions, we want to first introduce the appropriate t–structure. Fix a real number s. A torsion pair of coherent sheaves on $\mathbb{P}^2$ is given by:

- $\mathrm{Coh}_{\leq s}$, the subcategory of $\mathrm{Coh}(\mathbb{P}^2)$ generated (in the sense of extension) by semistable sheaves of slope $\leq s$.
- $\mathrm{Coh}_{>s}$, the subcategory of $\mathrm{Coh}(\mathbb{P}^2)$ generated by semistable sheaves of slope $> s$ and torsion sheaves.
- $\mathrm{Coh}_{\# s} := \langle \mathrm{Coh}_{\leq s}[1], \mathrm{Coh}_{>s} \rangle$.

We define the *geometric area* in the $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane to be the open set

$$\mathrm{Geo}_{\mathrm{LP}} := \{(1,a,b) \mid (1,a,b) \text{ is above } C_{\mathrm{LP}} \text{ and not on } l_{ee^+} \text{ for any exceptional } e\}.$$

Proposition–Definition 1.13 *For a point* $(1,s,q) \in \mathrm{Geo}_{\mathrm{LP}}$, *there exists a geometric stability condition* $\sigma_{s,q} := (Z_{s,q}, \mathrm{Coh}_{\# s})$ *on* $\mathrm{D}^b(\mathbb{P}^2)$ *where the central charge is*

$$Z_{s,q}(E) := (-\mathrm{ch}_2(E) + q \cdot \mathrm{ch}_0(E)) + i(\mathrm{ch}_1(E) - s \cdot \mathrm{ch}_0(E)).$$

In this case, $\mathrm{Ker}(Z_{s,q})$ *consists of the characters corresponding to the point* $(1,s,q)$. *We write* $\phi_{\sigma_{s,q}}$ *or* $\phi_{s,q}$ *for the phase function of* $\sigma_{s,q}$.

For the proof that $\sigma_{s,q}$ is indeed a geometric stability condition, we refer to [5, Corollary 4.6; 11], which also work well for $\mathbb{P}^2$. Here the phase function $\phi_{s,q}$ can also be defined for objects in $\mathrm{Coh}_{\# s}$ as

$$\phi_{s,q}(E) := \tfrac{1}{\pi} \mathrm{Arg}(Z_{s,q}(E)).$$

It is well-defined in the sense that it coincides with the phase function on $\sigma_{s,q}$–semistable objects.

Remark 1.14 The definition of $\sigma_{s,q}$ here is different from the usual one as in [2], which is given as $(Z'_{s,t}, \mathcal{P}_s)$ (see Section 1.5 for the explicit formulas). When $q > \frac{s^2}{2}$, the central charge $Z_{s,q}$ has the same kernel as $Z'_{s,q-\frac{s^2}{2}}$. Their formulas are slightly different. The imaginary parts are the same, but the real parts differ by a multiple of the imaginary part. We would like to use the version here because $q - \frac{s^2}{2}$ is allowed to be negative, and the kernel of the central charge on the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane is clearly $(1, s, q)$.

Remark 1.15 Given a point $P = (1, s, q)$ in $\mathrm{Geo}_{\mathrm{LP}}$, we will also write $\sigma_{\boldsymbol{P}}$, $\phi_{\boldsymbol{P}}$, $\mathrm{Coh}_{\boldsymbol{P}}(\mathbb{P}^2)$ and $Z_{\boldsymbol{P}}$ for the stability condition $\sigma_{s,q}$, the phase function $\phi_{s,q}$, the tilt heart $\mathrm{Coh}_{\#s}(\mathbb{P}^2)$ and the central charge $Z_{s,q}$, respectively.

Up to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$–action, geometric stability conditions are all of the form given in Proposition–Definition 1.13.

Proposition 1.16 [11, Proposition 10.3; 5, Section 3] *Let $\sigma = (Z, \mathcal{P}((0,1]))$ be a geometric stability condition such that all skyscraper sheaves $k(x)$ are contained in $\mathcal{P}(1)$. Then the heart $\mathcal{P}((0,1])$ is $\mathrm{Coh}_{\#s}$ for some real number s. The central charge Z can be written in the form*

$$-\mathrm{ch}_2 + a \cdot \mathrm{ch}_1 + b \cdot \mathrm{ch}_0,$$

where $a, b \in \mathbb{C}$ satisfy the following conditions:

- $\Im a > 0$ *and* $\Im b / \Im a = s$.
- $(1, \Im b/\Im a, (\Re a \Im b/\Im a) + \Re b)$ *is in* $\mathrm{Geo}_{\mathrm{LP}}$.

Thanks to the classification of characters of semistable sheaves on $\mathbb{P}^2$ in [18], this property is proved in the same way as in cases of local $\mathbb{P}^2$ in [5] and K3 surfaces in [11]. Since all discussions in this paper are invariant under the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$–action, geometric stability conditions will be identified with the corresponding points in $\mathrm{Geo}_{\mathrm{LP}}$. We will always visualize $\mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)$ as $\mathrm{Geo}_{\mathrm{LP}}$ in this paper.

1.3 Potential walls and phases

In this section we collect some well-known and useful results about the potential walls. Since our setup is slightly different from the usual one (see Remark 1.14), we give statements and proofs for completeness. We hope this can also illustrate the advantage of our setup.

Definition 1.17 A stability condition is said to be *nondegenerate* if it satisfies the support property and the image of its central charge is not contained in any real line in $\mathbb{C}$. We write $\mathrm{Stab}^{\mathrm{nd}}$ for the space of nondegenerate stability conditions.

The kernel map for the central charges is well-defined on $\mathrm{Stab}^{\mathrm{nd}}$:

$$\mathrm{Ker}\colon \mathrm{Stab}^{\mathrm{nd}} \to \mathrm{P}_{\mathbb{R}}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2)).$$

Lemma 1.18 [10, Theorem 1.2] $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ *acts freely on* $\mathrm{Stab}^{\mathrm{nd}}$ *with closed orbits, and*

$$\mathrm{Ker}\colon \mathrm{Stab}^{\mathrm{nd}}/\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \to \mathrm{P}_{\mathbb{R}}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2))$$

is a local homeomorphism.

Proof By [10, Theorem 1.2], $\mathrm{Stab}^{\mathrm{nd}} \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\mathbb{P}^2),\mathbb{C})$ is a local homeomorphism, whose image lies in the subspace of nondegenerate morphisms in $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\mathbb{P}^2),\mathbb{C})$. When taking the quotient by $\mathrm{GL}^+(2,\mathbb{R})$, the space $\mathrm{Hom}^{\mathrm{nd}}_{\mathbb{Z}}(\mathrm{K}(\mathbb{P}^2),\mathbb{C})/\mathrm{GL}^+(2,\mathbb{R})$ can be identified with the quotient Grassmannian $\mathrm{Gr}_2(3) \cong \mathrm{P}_{\mathbb{R}}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2))$ as a topological space. The statement clearly follows. □

We have the following description of the potential wall, ie the locus of stability conditions for which two given characters are of the same slope.

Lemma 1.19 (potential walls) *Let* $P=(1,s,q)$ *be a point in* $\mathrm{Geo}_{\mathrm{LP}}$*, and let* E *and* F *be two objects in* $\mathrm{Coh}_P(\mathbb{P}^2)$ *whose respective Chern characters* v *and* w *are not zero. Then*

$$Z_P(E) \text{ and } Z_P(F) \text{ are on the same ray}$$

if and only if v*,* w *and* P *are collinear in the projective* $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$*–plane.*

Proof $Z(v)$ and $Z(w)$ are on the same ray if and only if $Z(av-bw)=0$ for some $a,b\in\mathbb{R}_+$. This happens only when v, w and $\mathrm{Ker}\,Z$ are collinear in the projective $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. □

Note that the statement holds even when v and w are torsion, ie when

$$\mathrm{ch}_0(v)=\mathrm{ch}_0(w)=0.$$

We introduce some notation for lines and rays on the (projective) $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. Consider objects E and F such that $v(E)$ and $v(F)$ are not zero, and let $\sigma_{s,q}=\sigma_P$ be a geometric stability condition. Let L_{EF} be the straight line on the projective $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane across $v(E)$ and $v(F)$. Write L_{EP}, as well as $L_{E\sigma}$, for the line across $v(E)$ and P. When $v(E)$ and $v(F)$ are not on the line at infinity, l_{EF} is the line segment from $v(E)$ to $v(F)$ on the $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. $\mathcal{H}_P$ is the right half-plane with either $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}>s$, or $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}=s$ and $\frac{\mathrm{ch}_2}{\mathrm{ch}_0}>q$. Write l^+_{PE} for the ray along L_{PE} which starts from P and is completely contained in $\mathcal{H}_P$. Let $L_{E\pm}$ be the vertical wall $L_{E(0,0,1)}$. Let l_{E+} be the vertical ray along $L_{E(0,0,1)}$ from E going upward, and l_{E-} the vertical ray along $L_{E(0,0,-1)}$ from E going downward.

The following lemma translates the comparison of slopes into a geometric comparison of the positions of two rays. This simplifies a lot of computations and will be used throughout the paper.

Lemma 1.20 *Let $P=(1,s,q)$ be a point in $\mathrm{Geo}_{\mathrm{LP}}$, and let E and F be two objects in $\mathrm{Coh}_{\#s}$. The inequality*

$$\phi_{s,q}(E)>\phi_{s,q}(F)$$

holds if and only if the ray l^+_{PE} is above l^+_{PF}.

Proof By the formula of $Z_{s,q}$, the angle between the rays l^+_{PE} and l_{P-} at the point P is $\pi\phi_{s,q}(E)$. The statement follows from this observation. □

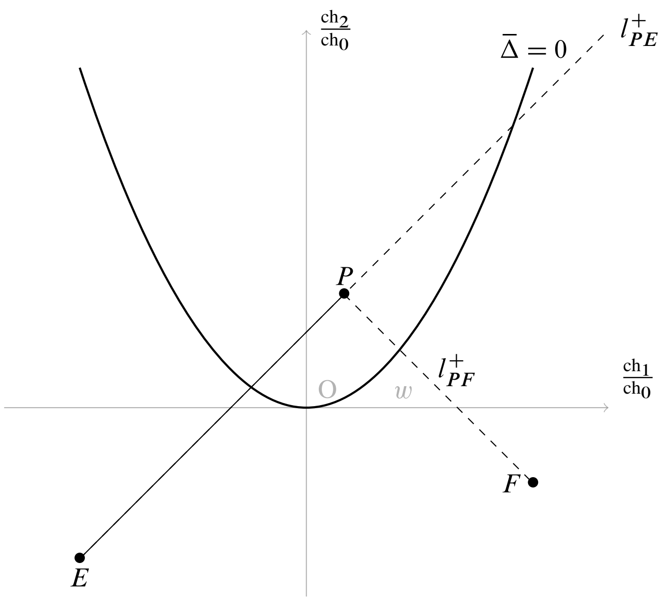

Figure 3: Comparing the slopes at P

An important problem is to study the existence of stable objects with respect to given stability condition and character. This will be solved in several steps in this paper. Now we can make the first observation.

Proposition 1.21 *Let $E\in\mathrm{Coh}_{\#s}$ be a $\sigma_{s,q}$–stable object. Then one of the following cases holds:*

(1) *The rank satisfies $\mathrm{ch}_0(E)=0$, or the point $v(E)$ is not in $\mathrm{Geo}_{\mathrm{LP}}$.*

(2) *There exists a slope-semistable sheaf F such that in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, the point $v(F)$ is above L_{EP} and between the vertical walls l_{E+} and l_{P+}.*

In either case, the line segment l_{EP} ***is not entirely contained inside*** $\mathrm{Geo}_{\mathrm{LP}}$*. In particular, at least one of $v(E)$ and $(1,s,q)$ is outside the negative discriminant area $\overline{\Delta}_{<0}$.*

Proof Assume that case (1) does not hold; in other words, $\mathrm{ch}_0(E) \neq 0$ and $v(E)$ is inside $\mathrm{Geo}_{\mathrm{LP}}$. We need to show that case (2) must happen. When $\mathrm{ch}_0(E) > 0$, we have that $\mathrm{H}^0(E)$ is nonzero. Let $F = \mathrm{H}^0(E)_{\min}$ be the last quotient factor in the Harder–Narasimhan filtration of $\mathrm{H}^0(E)$. Then F is a slope-semistable sheaf, so $v(F)$ is outside $\mathrm{Geo}_{\mathrm{LP}}$. Let D be $\mathrm{H}^{-1}(E)$ and G be the kernel of $\mathrm{H}^0(E) \to F$. We may compare the slopes of E and F:

$$\frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} = \frac{\mathrm{ch}_1(F)+\mathrm{ch}_1(G)-\mathrm{ch}_1(D)}{\mathrm{ch}_0(F)+\mathrm{ch}_0(G)-\mathrm{ch}_0(D)} \geq \frac{\mathrm{ch}_1(F)}{\mathrm{ch}_0(F)}.$$

The inequality holds because when D and G are nonzero, we have

$$\frac{\mathrm{ch}_1(D)}{\mathrm{ch}_0(D)} < \frac{\mathrm{ch}_1(F)}{\mathrm{ch}_0(F)} < \frac{\mathrm{ch}_1(G)}{\mathrm{ch}_0(G)}.$$

Note here that the equality

$$\frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} = \frac{\mathrm{ch}_1(F)}{\mathrm{ch}_0(F)}$$

holds only when D and G are both zero. In this case, $v(E) = v(F)$, hence $v(F)$ is inside $\mathrm{Geo}_{\mathrm{LP}}$, which contradicts our assumption. Therefore, we have a strict inequality, ie $v(F)$ is to the left of $v(E)$. As $F \in \mathrm{Coh}_{>s}$, we have that P is to the left of $v(F)$. In addition, as $\phi_{s,q}(E) < \phi_{s,q}(F)$, by Lemma 1.20 F is above l_{PE}, so case (2) holds.

When $\mathrm{ch}_0(E) < 0$, let $F = \mathrm{H}^{-1}(E)_{\max}$ be the subsheaf of $\mathrm{H}^{-1}(E)$ with the maximum slope $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$. By the same argument, $v(F)$ is to the right of $v(E)$. As $F \in \mathrm{Coh}_{\leq s}$, we have that v is to the left of $L_{P\pm}$ or on the ray l_{P-}. In addition, $\phi_{s,q}(F[1]) < \phi_{s,q}(E)$, so by Lemma 1.20, F is above l_{EP}. As l_{F-} does not intersect $\mathrm{Geo}_{\mathrm{LP}}$, it follows that F is to the left of $L_{P\pm}$.

For the last statement, the region $\overline{\Delta}_{<0}$ is convex, so for any $v(E)$ and $P = (1,s,q)$ that are both in $\overline{\Delta}_{<0}$, the line segment l_{EP} is also in $\overline{\Delta}_{<0}$, which is contained in $\mathrm{Geo}_{\mathrm{LP}}$. □

This induces several useful corollaries. First we get the stability of exceptional bundles for some stability conditions.

Corollary 1.22 *Let E be an exceptional bundle, and $P=(1,s,q)$ be a point in* $\mathrm{Geo}_{\mathrm{LP}}$*. Then E is $\sigma_{s,q}$–stable if $s<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E)$ and l_{EP} is contained in* $\mathrm{Geo}_{\mathrm{LP}}$ *(not including the endpoints). In the homological shifted case, $E[1]$ is $\sigma_{s,q}$–stable if $s\geq\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E)$ and l_{EP} is contained in* $\mathrm{Geo}_{\mathrm{LP}}$*.*

Proof We will prove the first statement. If E is not $\sigma_{s,q}$–stable, then there is a $\sigma_{s,q}$–stable object F destabilizing E. We have the exact sequence

$$0\to \mathrm{H}^{-1}(F)\to \mathrm{H}^{-1}(E)\to \mathrm{H}^{-1}(E/F)\to \mathrm{H}^{0}(F)\to \mathrm{H}^{0}(E)\to \mathrm{H}^{0}(E/F)\to 0.$$

Since $\mathrm{H}^{-1}(E)=0$, we see that $\mathrm{H}^{-1}(F)=0$ and $v(F)$ lies between the vertical lines $L_{P\pm}$ and $L_{E\pm}$. Since $\phi_{s,q}(F)>\phi_{s,q}(E)$, by Lemma 1.20 $v(F)$ is in the region bounded by l_{P+}, l_{PE} and l_{E+}. As l_{EP} is contained in $\mathrm{Geo}_{\mathrm{LP}}$, the line l_{FP} is also contained in $\mathrm{Geo}_{\mathrm{LP}}$. By Proposition 1.21, F is not $\sigma_{s,q}$–stable, which is a contradiction. The second statement can be proved similarly. □

Remark 1.23 The condition that l_{EP} be contained in $\mathrm{Geo}_{\mathrm{LP}}$ is also necessary. Any ray starting from $v(E)$ may only intersect C_{LP} at most once, and only intersect with finitely many l_{ee+} segments. Suppose that $s<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E)$, and l_{EP} intersects some l_{ee+} segments; we may choose the one with minimum $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–coordinate, and denote it by F. The segment l_{FP} is contained in $\mathrm{Geo}_{\mathrm{LP}}$, and $\phi_{s,q}(F)>\phi_{s,q}(E)$. By Corollary 1.22, F is $\sigma_{s,q}$–stable. By [20], $\mathrm{Hom}(F,E)\neq 0$ when $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(F)<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E)$. This shows that E is not $\sigma_{s,q}$–stable.

The second corollary roughly says that when we vary the stability condition in $\mathrm{Geo}_{\mathrm{LP}}$, stable objects remain stable if the slopes do not change.

Corollary 1.24 *Let $\sigma_{s,q}$ be a geometric stability condition and F a σ–stable object. Then for any geometric stability condition τ on the line $L_{F\sigma}$ such that $l_{\tau\sigma}$ is contained in* $\mathrm{Geo}_{\mathrm{LP}}$*, the object F is also τ–stable.*

1.4 Algebraic stability conditions

The structure of $\mathrm{D}^b(\mathbb{P}^2)$ can be studied via full strong exceptional collections. First recall the following definition.

Definition 1.25 An ordered set $\mathcal{E}=\{E_1,E_2,E_3\}$ is an *exceptional triple* in $\mathrm{D}^b(\mathbb{P}^2)$ if $\mathcal{E}$ is a full strong exceptional collection of coherent sheaves in $\mathrm{D}^b(\mathbb{P}^2)$.

Remark 1.26 The exceptional triples in $\mathrm{D}^b(\mathbb{P}^2)$ have been classified in [20] by Gorodentsev and Rudakov. In particular, up to a cohomological shift, each collection consists of exceptional bundles on $\mathbb{P}^2$. In terms of dyadic numbers, their labels are given by one of the following three cases (p is an odd integer when $m \neq 0$):

$$\left\{\frac{p-1}{2^m}, \frac{p}{2^m}, \frac{p+1}{2^m}\right\}, \quad \left\{\frac{p}{2^m}, \frac{p+1}{2^m}, \frac{p-1}{2^m}+3\right\}, \quad \left\{\frac{p+1}{2^m}-3, \frac{p-1}{2^m}, \frac{p}{2^m}\right\}.$$

We recall the construction of algebraic stability conditions associated to an exceptional triple.

Proposition 1.27 [28, Section 3] *Let $\mathcal{E}$ be an exceptional triple in $\mathrm{D}^b(\mathbb{P}^2)$. For any positive real numbers m_1, m_2, m_3 and real numbers ϕ_1, ϕ_2, ϕ_3 such that*

$$\phi_1 < \phi_2 < \phi_3 \quad \text{and} \quad \phi_1 + 1 < \phi_3,$$

there exists a unique stability condition $\sigma = (Z, \mathcal{P})$ such that

(1) *the E_j are σ–stable of phase ϕ_j;*

(2) $Z(E_j) = m_j e^{\pi i \phi_j}$.

Definition 1.28 Let $\mathcal{E}$ be an exceptional triple $\{E_1, E_2, E_3\}$ in $\mathrm{D}^b(\mathbb{P}^2)$, and write $\Theta_{\mathcal{E}}$ for the space of all stability conditions in Proposition 1.27. $\Theta_{\mathcal{E}}$ is parametrized by

$$\{(m_1, m_2, m_3, \phi_1, \phi_2, \phi_3) \in (\mathbb{R}_{>0})^3 \times \mathbb{R}^3 \mid \phi_1 < \phi_2 < \phi_3,\ \phi_1 + 1 < \phi_3\}.$$

We consider the following two subsets of $\Theta_{\mathcal{E}}$:

- $\Theta_{\mathcal{E}}^{\nabla} := \{\sigma \in \Theta_{\mathcal{E}} \mid \phi_2 - \phi_1 < 1,\ \phi_3 - \phi_2 < 1\}$.
- $\Theta_{\mathcal{E}}^{\mathrm{Geo}} := \Theta_{\mathcal{E}} \cap \mathrm{Stab}^{\mathrm{Geo}}$.

We denote by $\mathrm{Stab}^{\mathrm{Alg}}$ the union of $\Theta_{\mathcal{E}}$ for all exceptional triples in $\mathrm{D}^b(\mathbb{P}^2)$. A stability condition in $\mathrm{Stab}^{\mathrm{Alg}}$ is called an *algebraic stability condition*.

Let $\mathcal{E} = \{E_1, E_2, E_3\}$ be an exceptional triple. We write e_i for the points $v(E_i)$ on the $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. Let $\mathrm{TR}_{\mathcal{E}}$ be the inner points in the triangle bounded by $l_{e_1e_2}$, $l_{e_2e_3}$ and $l_{e_3e_1}$ in the $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. Let e_i^* be the points associated to e_i defined in Section 1.1, where $i = 1, 2, 3$ and $*$ could be $+$, l or r. The points e_1^+, e_1^r, e_2 and e_3 are on the line $\chi(-, E_1) = 0$, and e_3^+, e_3^l, e_2 and e_1 are on the line $\chi(E_3, -) = 0$. Let $\mathrm{MZ}_{\mathcal{E}}$ be the inner points of the region bounded by the line segments $l_{e_1e_1^+}$, $l_{e_1^+e_2}$, $l_{e_2e_3^+}$, $l_{e_3^+e_3}$ and $l_{e_3e_1}$.

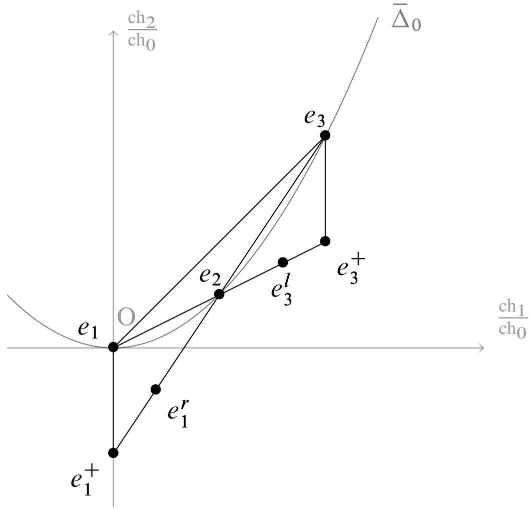

Figure 4: $\mathrm{TR}_{\mathcal{E}}$ and $\mathrm{MZ}_{\mathcal{E}}$

The next proposition explains how the algebraic part $\Theta_{\mathcal{E}}$ "glues" onto the geometric part $\mathrm{Stab}^{\mathrm{Geo}}$.

Proposition 1.29 *Let $\mathcal{E}$ be an exceptional triple. Then we have:*

(1) $\Theta_{\mathcal{E}}^{\nabla} = \widetilde{\mathrm{GL}^+}(2,\mathbb{R})\cdot\{\sigma_{s,q}\in \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)\mid (1,s,q)\in \mathrm{TR}_{\mathcal{E}}\}$.

(2) $\Theta_{\mathcal{E}}^{\mathrm{Geo}} = \widetilde{\mathrm{GL}^+}(2,\mathbb{R})\cdot\{\sigma_{s,q}\in \mathrm{Stab}^{\mathrm{Geo}}(\mathbb{P}^2)\mid (1,s,q)\in \mathrm{MZ}_{\mathcal{E}}\}$.

In particular, $\Theta_{\mathcal{E}}^{\nabla}$ is contained in $\Theta_{\mathcal{E}}^{\mathrm{Geo}}$.

Proof We will first prove the second statement. As $\mathrm{MZ}_{\mathcal{E}}$ is contained in $\mathrm{Geo}_{\mathrm{LP}}$, by Corollary 1.22 E_2 or $E_2[1]$ is $\sigma_{s,q}$–stable for any point $(1,s,q)$ in $\mathrm{MZ}_{\mathcal{E}}$. As e_1^+, e_1^r, e_2 and e_3 are collinear on the line of $\chi(-,E_1)=0$, for any point P in $\mathrm{MZ}_{\mathcal{E}}$, we have that l_{E_3P} is contained in $\mathrm{Geo}_{\mathrm{LP}}$. By Corollary 1.22, E_3 is stable for any stability conditions in $\mathrm{MZ}_{\mathcal{E}}$. For the same reason, $E_1[1]$ is stable for any stability conditions in $\mathrm{MZ}_{\mathcal{E}}$.

For any $(1,s,q)$ in $\mathrm{MZ}_{\mathcal{E}}$, E_3 and $E_1[1]$ are in the heart $\mathrm{Coh}_{\#s}$. By Lemma 1.20, $\phi_{s,q}(E_1[1])<\phi_{s,q}(E_3)$, hence

$$\phi_{s,q}(E_3)-\phi_{s,q}(E_1)>1.$$

When $s\geq \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E_2)$, we have that E_3 and $E_2[1]$ are in the heart $\mathrm{Coh}_{\#s}$, and

$$\phi_{s,q}(E_3)-\phi_{s,q}(E_2)>0.$$

As $(1,s,q)$ is above $L_{e_1e_2}$, by Lemma 1.20 we also have

$$\phi_{s,q}(E_2)-\phi_{s,q}(E_1)>0.$$

When $s<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E_2)$, a similar argument yields the same inequalities for the $\phi_{s,q}(E_i)$. By Proposition 1.27, we get the embedding

$$\mathrm{Ker}^{-1}(\mathrm{MZ}_{\mathcal{E}})\cap\mathrm{Stab}^{\mathrm{Geo}}\hookrightarrow\Theta_{\mathcal{E}}\cap\mathrm{Stab}^{\mathrm{nd}}\xrightarrow{\mathrm{Ker}}\mathrm{P}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2)).$$

For $(1,s,q)$ outside the area $\mathrm{MZ}_{\mathcal{E}}$, by Lemma 1.20 at least one of the inequalities

$$\phi_{s,q}(E_2)\le\phi_{s,q}(E_1),\quad \phi_{s,q}(E_3)\le\phi_{s,q}(E_2)\quad\text{or}\quad \phi_{s,q}(E_3)-\phi_{s,q}(E_1)\le 1$$

holds. Hence $\sigma_{s,q}$ is not contained in $\Theta_{\mathcal{E}}$, the second statement of the proposition holds.

For statement (1), as $\phi_2-\phi_1$ is not an integer, $\Theta_{\mathcal{E}}^{\nabla}\in\mathrm{Stab}^{\mathrm{nd}}$. The image of $\mathrm{Ker}(\Theta_{\mathcal{E}}^{\nabla})$ is in $\mathrm{TR}_{\mathcal{E}}$. By the previous argument, we also have the embedding

$$(\mathrm{Ker}^{-1}(\mathrm{TR}_{\mathcal{E}})\cap\mathrm{Stab}^{\mathrm{Geo}})/\widetilde{\mathrm{GL}}^+(2,\mathbb{R})\hookrightarrow\Theta_{\mathcal{E}}^{\nabla}/\widetilde{\mathrm{GL}}^+(2,\mathbb{R})\xrightarrow{\mathrm{Ker}}\mathrm{TR}_{\mathcal{E}}\subset\mathrm{P}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2)).$$

The map Ker is a local homeomorphism and the composition is an isomorphism. Since $\Theta_{\mathcal{E}}^{\nabla}$ is path-connected, the two maps are both isomorphisms. Thus we get the first statement of the proposition. □

1.5 Remarks on the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane

In this section, we want to summarize some properties of our $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane from previous sections. The aim is to help the readers gain a better understanding, especially those who are already familiar with the classical (s,t)–upper half-plane model.

The setup of the space of stability conditions in the paper is different from the classical (s,t)–upper half-plane model. Recall that we visualize a geometric stability condition as the kernel of its central charge in $\mathrm{K}(\mathbb{P}^2)\otimes\mathbb{R}$. In particular, when the central charge is nondegenerate, which is always the case for geometric stability conditions, the kernel is a straight line in $\mathrm{K}(\mathbb{P}^2)\otimes\mathbb{R}$. Further taking the projectivization of $\mathrm{K}(\mathbb{P}^2)\otimes\mathbb{R}$, the kernel of the central charge is a point on $\mathrm{P}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2))$. For a geometric stability condition to satisfy the Harder–Narasimhan condition, the kernel of the central charge has to be separate and away from all the slope-stable characters and torsion characters. In particular, the kernel can only be in the area $\mathrm{Geo}_{\mathrm{LP}}$ bounded by the Le Potier curve. The $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ action does not affect the kernel of the central charge, and the space of the geometric stability condition is realized as a $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$–principal bundle over $\mathrm{Geo}_{\mathrm{LP}}$.

For a point in $\mathrm{Geo}_{\mathrm{LP}}$ with coordinate $(1,s,q)$, we may write down a stability condition $\sigma_{s,q} = (Z_{s,q}, \mathcal{P}_s)$ with heart $\mathcal{P}_s((0,1]) = \mathrm{Coh}_{\sharp s}$ and central charge

$$Z_{s,q} = -(\mathrm{ch}_2 - q\cdot \mathrm{ch}_0) + i(\mathrm{ch}_1 - s\cdot \mathrm{ch}_1)$$

as in Proposition–Definition 1.13. In many other papers, a family of geometric stability conditions is parametrized by (s,t) on the upper half-plane $\mathbb{H}$ via a map $\sigma'_{s,t} = (Z'_{s,t}, \mathcal{P}_s)$, with the same heart $\mathcal{P}_s((0,1]) = \mathrm{Coh}_{\sharp s}$ and a different central charge

$$Z'_{s,t} = -\left(\mathrm{ch}_2^{\mathrm{s}} + \tfrac{1}{2}t^2\cdot \mathrm{ch}_0\right) + i\,t\,\mathrm{ch}_1^{\mathrm{s}}\,.$$

Up to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ action, $\sigma'_{s,t}$ is the same as $\sigma_{s,\frac{1}{2}(s^2+t^2)}$. Note that under this correspondence, the (s,t)–upper half-plane $\mathbb{H}$ is mapped to $\Delta_{<0}$ in the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane in $\mathrm{P}(\mathrm{K}_{\mathbb{R}}(\mathbb{P}^2))$.

Since this different convention may upset some readers, we want to briefly illustrate some advantages of our approach, which will become more clear later in the paper. One most important benefit is that the characters and the stability conditions are on the same space. As seen in Section 1.3, the potential wall of w and another Chern character v is the straight line across these two points on the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane, or strictly speaking, the line segment in $\mathrm{Geo}_{\mathrm{LP}}$. On the usual (s,t)–upper half-plane, the potential wall is the semicircle with two endpoints $L_{vw}\cap\overline{\Delta}_0$. Let σ_P be a stability condition and w a Chern character on the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane; the argument of $Z_P(w)$ is the angle bounded by L_{P-} and l_{Pw}. We may compare the slopes of different Chern characters by their positions on the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane, and this reduces a huge amount of computation. This allows us to deal with several Chern characters and stability conditions simultaneously.

Moreover, in our setup, the divisor cone can be identified with the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. For a Chern character w, one may draw its $\mathrm{Pic}_{\mathbb{R}}(\mathfrak{M}^{\mathrm{s}}_{\sigma}(w))$ as an HB–coordinate $\left(H\text{ vertical axis; }B\text{ with slope }\frac{\mathrm{ch}_1}{\mathrm{ch}_0}\right)$ with origin at w on the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane; the actual walls are the base locus decomposition walls. The Donaldson morphism identifies $w^{\perp}$ with the divisor cone of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$. Let v belong to $w^{\perp}$. Then the divisor given by v via the Donaldson morphism corresponds to the wall $\chi(-,v)=0$ on the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. The picture for the Chern character $(4,0,-15)$ in Figure 1 can now be interpreted from this new viewpoint.

Another advantage of the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane picture is that the space $\mathrm{Geo}_{\mathrm{LP}}$ is larger than the usual upper half-plane. As explained previously, up to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ action, $\mathrm{Geo}_{\mathrm{LP}}$ is the whole space of geometric stability conditions. The algebraic stability conditions (quiver regions) for exceptional triples are also easier to understand on the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane than on the upper half-plane. The quiver region with heart $\langle E_1[2], E_2[1], E_3\rangle$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane is the area that is below $l_{e_1e_3}$ and above $l_{e_2e_1^+}$ and $l_{e_2e_3^+}$. Since Chern characters of exceptional bundles are usually not on the parabola $\overline{\Delta}_0$ (this is the case only for line bundles), the endpoints of the semicircular potential walls of them involve complicated computation. On the (s,t)–upper half-plane, only quiver regions for heart $\langle \mathcal{O}(k-1)[2], \mathcal{O}(k)[1], \mathcal{O}(k+1)\rangle$ can be neatly described. In this paper, we need the general quiver regions (eg for heart $\langle \mathcal{O}(1)[2], \mathcal{T}[1], \mathcal{O}(2)\rangle$), which are important to decide the stable area for exceptional characters, and are useful for understanding the effective and movable cone boundary of the $\mathfrak{M}^{s}_{\sigma}(w)$. So the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane seems to be a suitable choice.

1.6 First constraint on the last wall

For a character, it is important to study the set of stability conditions for which there exist stable objects of the given character. We call this set the *stable area* of the character. In this section, we give a first constraint on the stable area.

Proposition 1.30 *Let w be a Chern character such that $\mathrm{ch}_0(w) > 0$ and $\overline{\Delta}(w) > 0$, and let E be an exceptional bundle such that $\frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)} < \frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)}$. Suppose w is above the line $L_{e^le^+}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. Then for any point $P \in \mathrm{Geo}_{\mathrm{LP}}$ below L_{wE} and to the left of $L_{E\pm}$, there is no σ_P–semistable object F with Chern character w.*

Remark 1.31 When $\mathrm{ch}_0(w) = 0$, there is a similar statement. The conditions are replaced by "$\mathrm{ch}_1(w) > 0$" and "$\frac{\mathrm{ch}_2}{\mathrm{ch}_1}(w)$ is greater than the slope of $L_{e^le^+}$". The proof is similar and left to the reader.

Proof By the assumptions and Corollary 1.24, we may assume that P is in $\mathrm{MZ}_{\mathcal{E}}$ for an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$ such that $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E_3) \leq \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E)$ and e_3 is above l_{Pw}. By an easy geometric property of C_{LP}, the character w is also above $L_{e_3^le_3^+}$. Now E_3 satisfies the assumptions; without loss of generality, we may assume that $E_3 = E$.

We argue by contradiction. Assume F is a σ_P–stable object with Chern character w. As σ_P is below L_{wE}, by Lemma 1.20 we have

$$\phi_P(F) < \phi_P(E).$$

Since E and F are both σ_P–semistable, we have

$$\operatorname{Hom}(E, F) = 0.$$

On the other hand, since P is in $\mathrm{MZ}_{\mathcal{E}}$, it is to the right of $L_{E(-3)\pm}$. Therefore, $E(-3)[1]$ and F are in the same heart, and we have

$$(\operatorname{Hom}(E, F[2]))^* = \operatorname{Hom}(F, E(-3)) = \operatorname{Hom}(F, E(-3)[1][-1]) = 0.$$

The vanishing of the two Homs implies $\chi(E, F) \leq 0$. But by the assumptions that $\mathrm{ch}_0(F) \geq 0$ and that w is above the line $L_{e^l e^+}$, which is given by $\chi(E, -) = 0$, we have $\chi(E, F) > 0$. This leads to a contradiction. □

Remark 1.32 The symmetric statement for w with $\mathrm{ch}_0(w) < 0$ above $L_{e^+ e^r}$ and for E with larger $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$ can be proved in the same way.

We have the following result on characters of Bridgeland stable objects, generalizing Theorem 1.8.

Corollary 1.33 *Fix a character w. Suppose that there exist $\sigma_{s,q}$–semistable objects of character w for some geometric stability condition $\sigma_{s,q}$. Then w either does not lie inside $\mathrm{Cone}_{\mathrm{LP}}$, or is proportional to an exceptional character.*

Proof Suppose w is inside $\mathrm{Cone}_{\mathrm{LP}}$ and not proportional to any exceptional character. By Proposition 1.21 there is an exceptional character e such that e is above the line segment $l_{w\sigma}$ and between vertical walls $L_{w\pm}$ and $L_{\sigma\pm}$. We may assume that $\frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)} > s$; then w is above $L_{e^l e^+}$. Now by Proposition 1.30, since σ is below L_{we} and to the left of $L_{e\pm}$, there is no σ–semistable object of character w, which is a contradiction. □

We also want to introduce the following important notion.

Definition 1.34 Let L be a straight line in the $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. Suppose that L intersects $\bar{\Delta}_{\leq 0}$ along a line segment with two endpoints $(1, f_1, g_1)$ and $(1, f_2, g_2)$. The $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$*–length* of $L \cap \bar{\Delta}_{\leq 0}$ is defined to be $|f_1 - f_2|$.

In the (s,t)–upper half-plane model in [2], the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{EF} \cap \bar{\Delta}_{\leq 0}$ is the diameter of the semicircular potential wall of E and F. This is a measure of the size of the wall, and we have the following result, which says that for walls of small length, there exists no stable object.

Corollary 1.35 *Let $w \in \mathrm{K}(\mathbb{P}^2)$ be a nonzero character not inside the Le Potier cone $\mathrm{Cone}_{\mathrm{LP}}$, and let σ be a geometric stability condition inside the cone $\overline{\Delta}_{<0}$. When the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{w\sigma} \cap \overline{\Delta}_{\leq 0}$ is less than or equal to 1, there is no σ–stable object F of character w.*

Proof We show the case when $\mathrm{ch}_0(w) \geq 0$; the other case can be proved similarly. Among all integers $k \leq \frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)}$, let c be the largest one such that w is strictly above the line $L_{\mathcal{O}(c-1)\mathcal{O}(c)}$. Note that $\mathcal{O}(c+1)^+$ is on the line $L_{\mathcal{O}(c-1)\mathcal{O}(c)}$. Since w is not inside the Le Potier cone, we have $\frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)} \geq c+1$. Now w is not above the line $L_{\mathcal{O}(c)\mathcal{O}(c+1)}$, so the segment $L_{\mathcal{O}(c+1)w} \cap \overline{\Delta}_{\leq 0}$ has $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length greater than or equal to that of $L_{\mathcal{O}(c)\mathcal{O}(c+1)}$, which is 1. By assumption, σ is inside the cone $\overline{\Delta}_{<0}$, and it must lie on or below the line $L_{\mathcal{O}(c+1)w}$ and to the left of $L_{\mathcal{O}(c+1)\pm}$. Note that $L_{\mathcal{O}(c-1)\mathcal{O}(c)}$ is just $L_{\mathcal{O}(c+1)^+\mathcal{O}(c)^l}$. By Proposition 1.30, there is no σ–stable object of character w. □

Remark 1.36 If F is σ–stable, in the proof we can see that below $L_{\sigma F}$ there exist characters of at least two line bundles $\mathcal{O}(c-1)$ and $\mathcal{O}(c)$.

2 Wall-crossing and canonical line bundles

In this section, we prove our first main theorem: the wall-crossing in stability-condition space induces the MMP for moduli of sheaves on $\mathbb{P}^2$. In Section 2.1, we review the construction of the moduli space of semistable objects as moduli of quiver representations. In Section 2.2, we prove the main technical result on vanishing of certain Ext^2. In Section 2.3, the generic stability of extension objects is proved. This will be used in the proof of the irreducibility of moduli of stable objects, which occupies Section 2.4. We rephrase some results from variation of GIT in our situation in Section 2.5, and use this to prove our main theorem in Section 2.6.

2.1 Construction of the moduli space

In this section, we review the construction of the moduli space of semistable objects on $\mathbb{P}^2$ with a given character via geometric invariant theory. Let w be a Chern character and $\sigma_{s,q}$ be a geometric stability condition; we write $\mathfrak{M}^{s(ss)}_{\sigma_{s,q}}(w)$ for the moduli space of $\sigma_{s,q}$–(semi)stable objects in $\mathrm{Coh}_{\#s}$ with character w. The line $L_{w\sigma_{s,q}}$ passes through $\mathrm{MZ}_{\mathcal{E}}$ for some exceptional triple $\mathcal{E}$. We may choose a point P in $\mathrm{MZ}_{\mathcal{E}}$ for some $\mathcal{E}$ such that the line segment $l_{P_{\sigma_{s,q}}}$ is contained in $\mathrm{Geo_{LP}}$. By Corollary 1.24, the moduli space $\mathfrak{M}^{ss}_{\sigma_{s,q}}(w)$ is the same as $\mathfrak{M}^{ss}_{\sigma_P}(w)$.

Let $\mathcal{E}$ be the exceptional triple consisting of E_1, E_2 and E_3, and let $\mathcal{A}_{\mathcal{E}}$ be the heart $\langle E_1[2],\ E_2[1],\ E_3\rangle$. By Proposition 1.29, E_1, E_2 and E_3 are σ_P–stable. We write the phase $\phi_P(E_i)$ of E_i at σ_P as ϕ_i; then $\phi_1<\phi_2<\phi_3$ and $-1<\phi_1<\phi_3-1<0$. There is a real number t, $0<t<1$, such that $-2<\phi_1-t<-1<\phi_2-t<0<\phi_3-t<1$. Let the heart $\mathrm{Coh}_P[t]$ be generated by σ_P–stable objects with phases in $(t,t+1]$. Then $\mathrm{Coh}_P[t]$ contains σ_P–stable objects $E_1[2]$, $E_2[1]$ and E_3. By Lemma 3.16 in [27], $\mathrm{Coh}_P[t]=\mathcal{A}_{\mathcal{E}}$. For any σ_P–stable object F in Coh_P of character w, the phase $\phi_P(F)$ only depends on w, and is denoted by $\phi_P(w)$. When $\phi_P(w)-t>0$, we have that F is an object in $\mathcal{A}_{\mathcal{E}}$. In particular, when F is a coherent sheaf, there is a "resolution" for F given as

$$0\to E_1^{\oplus n_1}\to E_2^{\oplus n_2}\to E_3^{\oplus n_3}\to F\to 0.$$

The character $\vec{n}=(n_1,n_2,n_3)$ is the unique triple such that

$$n_1\tilde{v}(E_1)-n_2\tilde{v}(E_2)+n_3\tilde{v}(E_3)=w.$$

When $\phi_P(w)-t\le 0$, we have that $F[1]$ is an object in $\mathcal{A}_{\mathcal{E}}$. When F is a coherent sheaf, it appears as the cohomological sheaf at the middle term of

$$E_1^{\oplus n_1}\to E_2^{\oplus n_2}\to E_3^{\oplus n_3}.$$

The character $\vec{n}=(n_1,n_2,n_3)$ is the unique triple such that

$$n_1\tilde{v}(E_1)-n_2\tilde{v}(E_2)+n_3\tilde{v}(E_3)=-w.$$

The following easy lemma is useful to determine whether F or $F[1]$ is in $\mathcal{A}_{\mathcal{E}}$.

Lemma 2.1 *Let P be a point in $\mathrm{MZ}_{\mathcal{E}}$, and F be a σ_P–stable object in $\mathrm{Coh}_{\#P}$. If L_{PF} is above e_3, then F is in the heart $\mathcal{A}_{\mathcal{E}}$. If L_{PF} is above e_1, then $F[1]$ is in the heart $\mathcal{A}_{\mathcal{E}}$.*

Proof By Lemma 1.20, when L_{PF} is above e_3 we have $\phi_P(F)\ge\phi_P(E_3)$. Therefore, $\phi_P(F)-t>0$ and F is in $\mathcal{A}_{\mathcal{E}}$. When L_{PF} is above e_1, we have $\phi_P(F)<\phi_P(E_1[1])$. Therefore, $\phi_P(F)-t<\phi_P(E_1[1])-t<0$ and F is in $\mathcal{A}_{\mathcal{E}}[-1]$. □

Remark 2.2 The case when P is in $\mathrm{TR}_{\mathcal{E}}$ and L_{PF} is below both e_1 and e_3 seems to be missing from the lemma. However, in this case, by Proposition 1.30, F is not σ_P–stable.

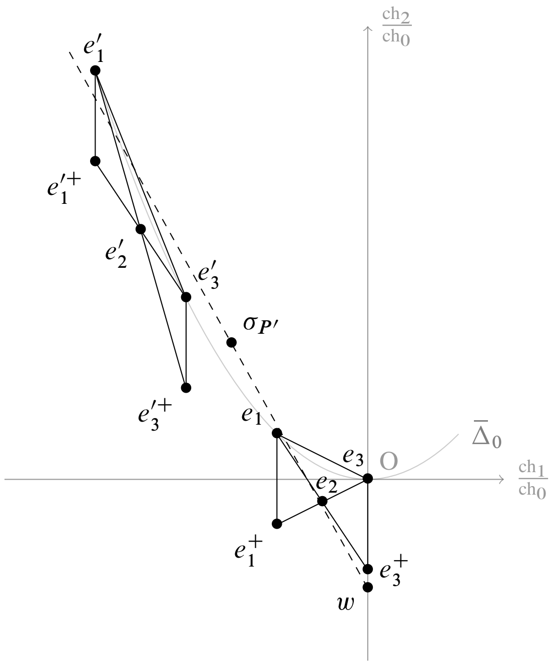

Figure 5: $F[1]$ is in $\mathcal{A}_{\mathcal{E}}$ and F is in $\mathcal{A}_{\mathcal{E}'}$.

We define $Q_{\mathcal{E}} = (Q_0, Q_1)$ to be the quiver associated to the exceptional triple $\mathcal{E}$. The set Q_0 has three vertices v_1, v_2 and v_3. The arrow set Q_1 consists of $\hom(E_1, E_2)$ arrows from v_1 to v_2 and $\hom(E_2, E_3)$ arrows from v_2 to v_3. Let $\vec{n} = (n_1, n_2, n_3)$ be a dimension character for $Q_{\mathcal{E}}$, and H_k be a complex linear space of dimension k. Then the representation space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n})$ can be identified with

$$\{(I, J) \mid I \in \mathrm{Hom}(H_{n_1}, H_{n_2}) \otimes \mathrm{Hom}(E_1, E_2),\ J \in \mathrm{Hom}(H_{n_2}, H_{n_3}) \otimes \mathrm{Hom}(E_2, E_3)\}.$$

We denote the composition map between the E_i by

$$\alpha_{\mathcal{E}} \colon \mathrm{Hom}(E_1, E_2) \otimes \mathrm{Hom}(E_2, E_3) \to \mathrm{Hom}(E_1, E_3).$$

This gives a relation of the quiver $Q_{\mathcal{E}}$, and we have the space of quiver representations with relation

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}}) := \{(I, J) \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}) \mid J \circ I \in \mathrm{Hom}(H_{n_1}, H_{n_3}) \otimes \ker \alpha_{\mathcal{E}}\}.$$

As a subvariety of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n})$, the space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$ is determined by $JI = 0$, which contains $n_1 n_3 \hom(E_1, E_3)$ equations.

The category $\mathcal{A}_{\mathcal{E}}$ is equivalent to the category of finite-dimensional modules over the path algebra $(Q_{\mathcal{E}}, \alpha_{\mathcal{E}})$. Any object F in $\mathcal{A}_{\mathcal{E}}$ with character

$$n_1\widetilde{v}(E_1) - n_2\widetilde{v}(E_2) + n_3\widetilde{v}(E_3)$$

can be written as a representation $\boldsymbol{K}_F$ (unique up to the $G_{\vec{n}}$–action) in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$.

Definition 2.3 Let $\boldsymbol{K} = (I, J)$ and $\boldsymbol{K}' = (I', J')$ be two objects in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$ and $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}', \alpha_{\mathcal{E}})$, respectively. We introduce notation for the following sets of homomorphisms:

$$\mathrm{Hom}^i(\boldsymbol{K}, \boldsymbol{K}') := \bigoplus_j \mathrm{Hom}_{\mathcal{O}}(H_{n_j} \otimes E_j, H_{n'_{j+i}} \otimes E_{j+i}).$$

Here H_{n_i} and $H_{n'_i}$ are defined to be the zero space when $i \neq 0, 1, 2$. The derivatives d^0 and d^1 are linear maps defined by

$$d^0\colon \mathrm{Hom}^0(\boldsymbol{K}, \boldsymbol{K}') \to \mathrm{Hom}^1(\boldsymbol{K}, \boldsymbol{K}'), \quad (f_0, f_1, f_2) \mapsto (I' \circ f_0 - f_1 \circ I, J' \circ f_1 - f_2 \circ J),$$

$$d^1\colon \mathrm{Hom}^1(\boldsymbol{K}, \boldsymbol{K}') \to \mathrm{Hom}^2(\boldsymbol{K}, \boldsymbol{K}'), \qquad (g_1, g_2) \mapsto (J' \circ g_1 + g_2 \circ I).$$

Let F and G be two objects in $\mathcal{A}_{\mathcal{E}}$, and let $\boldsymbol{K}_F$ and $\boldsymbol{K}_G$ be their representations in $\mathbf{Rep}(Q_{\mathcal{E}}, \alpha_{\mathcal{E}})$. The Ext^i groups of F and G can be computed via $\boldsymbol{K}_F$ and $\boldsymbol{K}_G$.

Lemma 2.4 *The* $\mathrm{Ext}^*(F, G)$ *groups are the cohomology of the complex*

$$\mathrm{Hom}^0(\boldsymbol{K}_F, \boldsymbol{K}_G) \xrightarrow{d^0} \mathrm{Hom}^1(\boldsymbol{K}_F, \boldsymbol{K}_G) \xrightarrow{d^1} \mathrm{Hom}^2(\boldsymbol{K}_F, \boldsymbol{K}_G).$$

In particular,

$$\ker d^0 \simeq \mathrm{Hom}(F, G), \quad \mathrm{Hom}^2(\boldsymbol{K}_F, \boldsymbol{K}_G)/\mathrm{im}\, d^1 \simeq \mathrm{Ext}^2(F, G).$$

Let $\vec{\rho}$ be a weight character for objects in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n})$; in particular, $\vec{n} \cdot \vec{\rho} = 0$. An object $\boldsymbol{K}$ in $\mathbf{Rep}(Q, \vec{n})$ is $\vec{\rho}$–(semi)stable if and only if for any nonzero proper subrepresentation $\boldsymbol{K}'$ of $\boldsymbol{K}$ with dimension character $\vec{n}' < \vec{n}$, we have $\vec{n}' \cdot \vec{\rho} < (\leq)\, 0$.

Now we want to relate Bridgeland stability of objects to King stability of quiver representations. Let L be a line on the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane not at infinity. Suppose L intersects $l_{e_1e_3}$ for an exceptional triple $\mathcal{E} = \{E_1, E_2, E_3\}$. Let f be a linear function with variables ch_0, ch_1 and ch_2 such that the zero locus of f is L. Moreover we assume that $f(\widetilde{v}(E_1))$ is positive. The weight character $\vec{\rho}_{L,\mathcal{E}}$ is given, up to a positive scalar, by

$$\bigl(f(\widetilde{v}(E_1)), -f(\widetilde{v}(E_2)), f(\widetilde{v}(E_3))\bigr).$$

Lemma 2.5 *Let F be an object in $\mathcal{A}_{\mathcal{E}}$ and P be a point in $\mathrm{MZ}_{\mathcal{E}}$ such that L_{FP} intersects $l_{e_1e_3}$. Then F (or $F[-1]$) is σ_P–(semi)stable if and only if $\boldsymbol{K}_F$ is $\vec{\rho}_{L_{FP},\mathcal{E}}$–(semi)stable.*

Proof First we want to modify the stability condition in a way that the central charges of the exceptional bundles are better behaved, and the weight character remains the same. Since L_{FP} intersects $l_{e_1e_3}$, by Corollary 1.24 we may assume P is in the triangle area $\mathrm{TR}_{\mathcal{E}}$. The central charges of objects $E_1[2]$, $E_2[1]$ and E_3 are

$$\begin{aligned} Z_P(E_1[2]) &= -\mathrm{ch}_2(E_1)+q\,\mathrm{ch}_0(E_1)+(\mathrm{ch}_1(E_1)-s\,\mathrm{ch}_0(E_1))i,\\ Z_P(E_2[1]) &= \mathrm{ch}_2(E_2)-q\,\mathrm{ch}_0(E_2)-(\mathrm{ch}_1(E_2)-s\,\mathrm{ch}_0(E_2))i,\\ Z_P(E_3) &= -\mathrm{ch}_2(E_3)+q\,\mathrm{ch}_0(E_3)+(\mathrm{ch}_1(E_3)-s\,\mathrm{ch}_0(E_3))i. \end{aligned}$$

There is a suitable real number $0<t<1$ such that the new central charge $Z^\bullet_P:=e^{i\pi t}Z_P$ maps $E_1[2]$, $E_2[1]$ and E_3 to the upper half-plane in $\mathbb{C}$.

Now we can rewrite the stability condition in terms of the weight character. Write

$$\vec{Z}^\bullet_P := \big(Z^\bullet_P(E_1[2]), Z^\bullet_P(E_2[1]), Z^\bullet_P(E_3)\big) = \vec{a}^\bullet + \vec{b}^\bullet i$$

for two real vectors $\vec{a}^\bullet$ and $\vec{b}^\bullet$. The object F is $Z^\bullet_P$–(semi)stable if and only if for any nonzero proper subobject F' in $\mathcal{A}_{\mathcal{E}}$,

$$\mathrm{Arg}\, Z^\bullet_P(F') < (\le)\ \mathrm{Arg}\, Z^\bullet_P(F).$$

In other words, supposing the dimension vector of $\boldsymbol{K}_F$ is $\vec{n}=(n_1,n_2,n_3)$, then for any nonzero proper subrepresentation $\boldsymbol{K}_{F'}$ with dimension vector $\vec{n}'$,

$$\mathrm{Arg}\,\vec{n}'\cdot\vec{Z}^\bullet_P < (\le)\ \mathrm{Arg}\,\vec{n}\cdot\vec{Z}^\bullet_P. \tag{2-1}$$

Let $\vec{\rho}^\bullet$ be the vector

$$-(\vec{b}^\bullet\cdot\vec{n})\vec{a}^\bullet + (\vec{a}^\bullet\cdot\vec{n})\vec{b}^\bullet.$$

Since each factor of $\vec{b}^\bullet$ is nonnegative, the inequality (2-1) holds if and only if

$$\frac{\vec{n}\cdot\vec{a}^\bullet}{\vec{n}\cdot\vec{b}^\bullet} < (\le)\ \frac{\vec{n}'\cdot\vec{a}^\bullet}{\vec{n}'\cdot\vec{b}^\bullet},$$

if and only if $\vec{n}'\cdot\vec{\rho}^\bullet < (\le)\ 0$.

We can also write $\vec{Z}_P := \big(Z_P(E_1[2]), Z_P(E_2[1]), Z_P(E_3)\big) = \vec{a} + \vec{b}i$ for two real vectors $\vec{a}$ and $\vec{b}$. As $Z^\bullet_P = e^{i\pi t} Z_P$, the character $\vec{\rho} := (\vec{b}\cdot\vec{n})\vec{a} - (\vec{a}\cdot\vec{n})\vec{b}$ is the same as $\vec{\rho}^\bullet$.

At last we need to show that $\vec{\rho}$ is $\vec{\rho}_{L_{FP},\mathcal{E}}$ up to a positive scalar. Let f be the linear function

$$f(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) := (\vec{a}\cdot\vec{n})(\mathrm{ch}_1 - s\,\mathrm{ch}_0) - (\vec{b}\cdot\vec{n})(-\mathrm{ch}_2 + q\,\mathrm{ch}_0).$$

The zero locus of f contains P because $f(1,s,q) = 0$. We also have

$$f(\tilde{v}(F)) = f(n_1\tilde{v}(E_1) - n_2\tilde{v}(E_2) + n_3\tilde{v}(E_3)) = (\vec{a}\cdot\vec{n})(\vec{b}\cdot\vec{a}) - (\vec{b}\cdot\vec{n})(\vec{a}\cdot\vec{b}) = 0.$$

Therefore, the zero locus of f also contains $v(F)$.

It is easy to check that

$$(f(v(E_1)), -f(v(E_2)), f(v(E_3)))$$

is the vector $\vec{\rho}$. Since P is above the line $L_{E_1E_2}$, we have $\phi_P(E_2[1]) < \phi_P(E_1[2])$ and the determinant satisfies $\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} < 0$. Similarly, we have $\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} < 0$. Therefore, the first factor of $\vec{\rho}^\bullet$, which is

$$-\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} n_2 - \det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} n_3,$$

is always positive. So $f(v(E_1)) > 0$.

Now f satisfies the desired properties, and induces the weight character $\vec{\rho}$. Any other f satisfying the same properties induces the same character up to a positive scalar. $\square$

Remark 2.6 By construction, the character $\vec{\rho}_{L_{FP},\mathcal{E}}$ (up to a positive scalar) only depends on the wall L and not the position of P.

To conclude the construction of the moduli space of σ–stable objects via geometric invariant theory, we summarize the previous notation as follows. Let w be a character and σ_P a geometric stability condition. Suppose P is in $\mathrm{MZ}_{\mathcal{E}}$ for some exceptional triple $\mathcal{E} = \langle E_1, E_2, E_3\rangle$ such that L_{Pw} intersects $l_{E_1E_3}$. We assume that w or $-w$ can be written as $n_1\tilde{v}(E_1) - n_2\tilde{v}(E_2) + n_3\tilde{v}(E_3)$ for a positive-dimension character $\vec{n}_w = (n_1, n_2, n_3)$. Let $G_{\vec{n}}$ be the group $\mathrm{GL}(H_{n_1}) \times \mathrm{GL}(H_{n_2}) \times \mathrm{GL}(H_{n_3})$ acting naturally on the spaces $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})$ and $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)$, with stabilizer containing the scalar group $\mathbb{C}^\times$.

Proposition 2.7 *Adopting notation as above, the moduli space $\mathfrak{M}^{\mathrm{ss}}_\sigma(w)$ or $\mathfrak{M}^{\mathrm{ss}}_\sigma(-w)$ of σ_P–semistable objects in $\mathrm{Coh}_{\sharp P}$ can be constructed as the GIT quotient space*

$$\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})/\!\!/_{\det^{\vec{\rho}_{L_{w\sigma},\mathcal{E}}}}(G_{\vec{n}_w}/\mathbb{C}^\times) = \mathbf{Proj}\bigoplus_{m\geq 0}\mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})]^{G_{\vec{n}_w}/\mathbb{C}^\times,\,\det^{m\vec{\rho}_{L_{w\sigma},\mathcal{E}}}}.$$

Proof By the previous discussion and Lemma 2.5, the moduli space $\mathfrak{M}^{\mathrm{ss}}_\sigma(w)$ or $\mathfrak{M}^{\mathrm{ss}}_\sigma(-w)$ parametrizes the $\vec{\rho}_{L_{w\sigma},\mathcal{E}}$–semistable objects in $\mathcal{A}_{\mathcal{E}}$ that have dimension character $\vec{n}_w$. By King's criterion [23, Proposition 3.1], $\boldsymbol{K}$ in $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_w)$ is $\vec{\rho}_{L_{w\sigma}}$–semistable if the point of $\boldsymbol{K}$ in the space $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w)$ is $\det^{\vec{\rho}_{L_{w\sigma}}}$–semistable with respect to the $G_{\vec{n}_w}/\mathbb{C}^\times$–action. The map

$$\mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w)]^{G,\det^{m\vec{\rho}}} \to \left(\mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w)]/I_{\alpha_{\mathcal{E}}}\right)^{G,\det^{m\vec{\rho}}} = \mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_w)]^{G,\det^{m\vec{\rho}}}$$

is surjective because the group $G=G_{\vec{n}_w}/\mathbb{C}^\times$ is semisimple. Therefore, the relation $\alpha_{\mathcal{E}}$ does not affect the stability condition. In other words, a point $\boldsymbol{K}$ is $\det^{\vec{\rho}_{L_{w\sigma}}}$–semistable with respect to the $G_{\vec{n}_w}/\mathbb{C}^\times$–action on the space $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w)$ if and only if it is so on the space $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n},\alpha_{\mathcal{E}})$. As explained by Ginzburg [19, Chapter 2.2], the moduli space is constructed as the GIT quotient in the proposition. □

Now we have the following consequence on the finiteness of actual walls.

Proposition 2.8 (1) *Let w be a character in $\mathrm{K}(\mathbb{P}^2)$. Then there are only finitely many actual walls for $\mathfrak{M}^{\mathrm{ss}}_\sigma(w)$.*

(2) *Suppose $\mathrm{ch}_0(w)\geq 0$. Then for any $s<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$ and q large enough (depending on s), the moduli space $\mathfrak{M}^{\mathrm{s(ss)}}_{\sigma_{s,q}}(w)$ is the same as the moduli space $\mathfrak{M}^{\mathrm{s(ss)}}_{\mathrm{GM}}(w)$ of Simpson (semi)stable coherent sheaves.*

Proof By Corollary 1.35, we only need to consider the region from the vertical wall to the tangent line of $\overline{\Delta}_0$. We may choose finitely many quiver regions $\mathrm{MZ}_{\mathcal{E}}$ such that each ray from w contained in this region passes through at least one $\mathrm{MZ}_{\mathcal{E}}$. In each $\mathrm{MZ}_{\mathcal{E}}$, there are finitely many walls, because there are only finitely many dimension vectors of possible destabilizing subobjects.

The second statement is a consequence of the first statement and the standard fact that $\sigma_{s,q}$ tends to the Simpson stability condition when q tends to infinity. □

2.2 The Ext^2 vanishing property

In this section we prove the most important technical lemma. It is about the vanishing property of Ext^2 of σ–stable objects. This property is trivial in the slope-stable situation by Serre duality. But it is more involved in the Bridgeland stability situation since the objects may not be in the same heart.

Lemma 2.9 *Let σ_P be a geometric stability condition with P in $\overline{\Delta}_{<0}$, and let E and F be two σ_P–stable objects in $\mathrm{Coh}_{\sharp P}$. Suppose P, $v(E)$ and $v(F)$ are collinear. Then*

$$\mathrm{Hom}(E, F[2]) = \mathrm{Hom}(F, E[2]) = 0.$$

Proof We divide the proof into two cases.

Case 1 (at least one of $\widetilde{v}(E)$ and $\widetilde{v}(F)$ is an exceptional character) Assume that $\widetilde{v}(E)$ is exceptional. Suppose the dyadic number corresponding to $\widetilde{v}(E)$ is $\frac{p}{2^q}$, and let E_1 and E_3 be exceptional bundles corresponding to dyadic numbers $\frac{p-1}{2^q}$ and $\frac{p+1}{2^q}$ respectively. Then $\mathrm{Hom}(E_1, E)$ and $\mathrm{Hom}(E, E_3)$ are both nonzero. As E is σ–stable, $l_{E\sigma}$ does not intersect $l_{e_1e_1^+}$ nor $l_{e_3e_3^+}$, otherwise this contradicts Lemma 1.20. We may assume that σ is in $\mathrm{MZ}_{\mathcal{E}}$, where $\mathcal{E} = \{E_1, E, E_3\}$. As F has the same phase as E at σ, we have that $F[1]$ is in $\mathcal{A}_{\mathcal{E}}$. Since

$$\mathrm{Hom}(E[1], E_1[2+s]) = 0 \quad \text{for all } s \in \mathbb{Z},$$

$$\mathrm{Hom}(E[1], E[1+s]) = 0 \quad \text{for all } s \neq 0,$$

$$\mathrm{Hom}(E[1], E_3[s]) = 0 \quad \text{for all } s \neq 1,$$

we get that $\mathrm{Hom}(E[1], G[s]) = 0$ for any object G in $\mathcal{A}_{\mathcal{E}}$ when $s \neq 0$ or 1. Therefore, $\mathrm{Hom}(E, F[2]) = \mathrm{Hom}(E[1], F[1+2]) = 0$. Similarly, we have $\mathrm{Hom}(G, E[1+s]) = 0$ for any object G in $\mathcal{A}_{\mathcal{E}}$ when $s \neq 0$ or 1. Therefore, $\mathrm{Hom}(F, E[2]) = 0$.

Case 2 (neither $\widetilde{v}(E)$ nor $\widetilde{v}(F)$ is exceptional) By Corollary 1.33, their corresponding points are below the Le Potier curve C_{LP}.

- **Case 2.1** $\left(\text{the } \frac{\mathrm{ch}_1}{\mathrm{ch}_0}\text{–length of } L_{EF} \cap \overline{\Delta}_{\leq 0} \text{ is greater than } 3\right)$ By the construction of Bridgeland stability conditions, it is easy to see that the objects $E(-3)$ and $F(-3)$ are also stable for any geometric stability conditions on the line $L_{E(-3)F(-3)}$. Since the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{EF} \cap \overline{\Delta}_{\leq 0}$ is greater than 3, the intersection point Q of L_{EF} and $L_{E(-3)F(-3)}$ is in $\overline{\Delta}_{\leq 0}$. By Corollary 1.24, the objects E, F, $E(-3)$ and $F(-3)$ are all σ_Q–stable. By Lemma 1.20,

$$\phi_Q(E(-3)) = \phi_Q(F(-3)) < \phi_Q(E) = \phi_Q(F).$$

We have

$$\operatorname{Hom}(E, F(-3)), \operatorname{Hom}(F, E(-3)) = 0.$$

The statement then holds by Serre duality.

- **Case 2.2** $\bigl($the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{EF} \cap \overline{\Delta}_{\leq 0}$ is not greater than $3\bigr)$ Since E is σ_P–stable, by Corollary 1.35 and its proof there exists an integer k such that the points $v(\mathcal{O}(k+1)) = \bigl(1, k+1, \frac{1}{2}(k+1)^2\bigr)$ and $v(\mathcal{O}(k+2))$ are below the segment $L_{EF} \cap \overline{\Delta}_{\leq 0}$; see Figure 6, where e_1, e_2 and e_3 correspond to $\mathcal{O}(k+2)$, $\mathcal{O}(k+3)$ and $\mathcal{O}(k+4)$, respectively. Equivalently, the points $v(\mathcal{O}(k-1))$ and $v(\mathcal{O}(k-2))$ are below the segments $L_{E(-3)F(-3)} \cap \overline{\Delta}_{\leq 0}$.

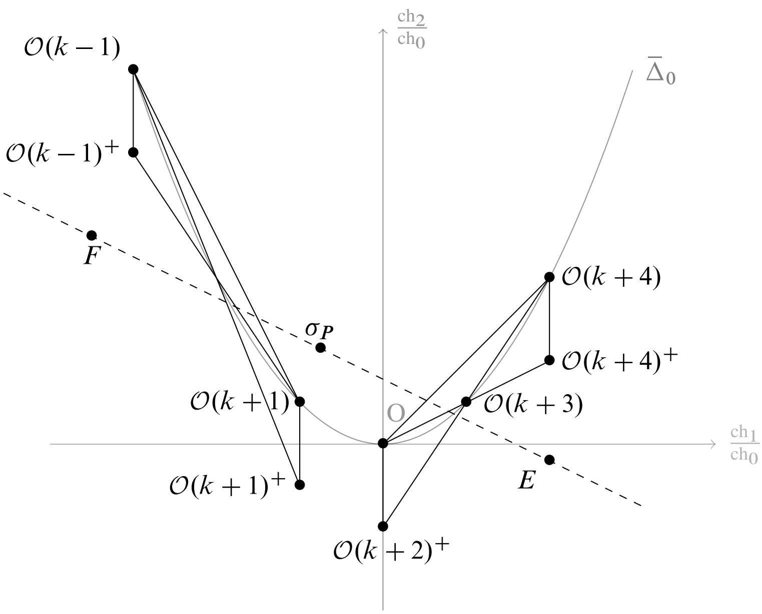

Figure 6: $L_{EF} \cap \overline{\Delta}_{\leq 0}$ is not greater than 3

Let $\mathcal{E}_k$ be the exceptional triple $\langle \mathcal{O}(k-1), \mathcal{O}(k), \mathcal{O}(k+1)\rangle$. Then by our assumption, $L_{\mathcal{O}(k-1)\mathcal{O}(k+1)}$ must intersect both L_{EF} and $L_{E(-3)F(-3)}$. By Lemma 2.1, as the point $v(\mathcal{O}(k+1))$ is below the segment $L_{EF} \cap \overline{\Delta}_{\leq 0}$, both E and F are in $\mathcal{A}_{\mathcal{E}_k}$. As the point $v(\mathcal{O}(k-1))$ is below the segment $L_{E(-3)F(-3)} \cap \overline{\Delta}_{\leq 0}$, both $E(-3)[1]$ and $F(-3)[1]$ are in $\mathcal{A}_{\mathcal{E}_k}$. Therefore, we have

$$\operatorname{Hom}(E, F(-3)) = \operatorname{Hom}(E, (F(-3)[1])[-1]) = 0.$$

By Serre duality, the statement holds. $\square$

In particular, if an object E is σ–semistable for some geometric stability condition σ, we have $\mathrm{Hom}(E, E[2]) = 0$. To see this, E admits σ–stable Jordan–Holder filtrations, and for any two stable factors we have the $\mathrm{Hom}(-,-[2])$ vanishing, hence $\mathrm{Hom}(E, E[2]) = 0$.

As an immediate application, $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ is smooth.

Corollary 2.10 *Let x be a point in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$. Then as a closed subvariety of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)$, the space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})$ is smooth at the point x.*

Proof Let $\boldsymbol{K} = (I_0, J_0)$ be the quiver representation that x stands for. The dimension of the Zariski tangent space at x is the dimension of

$$\mathrm{Hom}_{\mathbb{C}}\big(\mathbb{C}[\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w)]/(J \circ I),\ \mathbb{C}[t]/(t^2)\big)$$

at (I_0, J_0). Each tangent direction can be written in the form $(I_0, J_0) + t(I_1, J_1)$. In order to satisfy the equation $J \circ I \in (t^2)$, we need

$$J_0 \circ I_1 + J_1 \circ I_0 = 0.$$

Hence the space of (I_1, J_1) is just the kernel of $d^1\colon \mathrm{Hom}^1(\boldsymbol{K}, \boldsymbol{K}) \to \mathrm{Hom}^2(\boldsymbol{K}, \boldsymbol{K})$. By Lemma 2.9, d^1 is surjective. The Zariski tangent space has dimension

$$\hom^1(\boldsymbol{K}, \boldsymbol{K}) - \hom^2(\boldsymbol{K}, \boldsymbol{K}).$$

On the other hand, $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$ is the zero locus of

$$n_1 n_3 \cdot \hom(E_1, E_3) = \hom^2(\boldsymbol{K}, \boldsymbol{K})$$

equations, hence each irreducible component is of dimension at least $\hom^1(\boldsymbol{K}, \boldsymbol{K}) - \hom^2(\boldsymbol{K}, \boldsymbol{K})$, which is not less than the dimension of the Zariski tangent space at x. Therefore, $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})$ is smooth at the point x. □

Remark 2.11 When the dimension character $\vec{n}$ is primitive, G $(= G_{\vec{n}_w}/\mathbb{C}^\times)$ acts freely on the stable locus. By Luna's étale slice theorem,

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}} \to \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}}/G$$

is a principal G–bundle. Since $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}}$ is smooth, by Proposition IV.17.7.7 in [21] the base space is also smooth.

2.3 Generic stability

Based on Lemma 2.9, we establish some estimates on the dimension of strictly semistable objects in this section. The technical result Lemma 2.13 is useful in the proof of the irreducibility of the moduli space.

Definition 2.12 Suppose $\vec{n} = \vec{n}' + \vec{n}''$ with $\vec{n}' \cdot \vec{\rho} = \vec{n}'' \cdot \vec{\rho} = 0$. Choose $\boldsymbol{F} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}', \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ and $\boldsymbol{G} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}'', \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$. We write $\mathbf{Rep}(Q_{\mathcal{E}}, \boldsymbol{F}, \boldsymbol{G})$ for the *subspace* in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ consisting of representations $\boldsymbol{K}$ that can be written as an extension of $\boldsymbol{G}$ by $\boldsymbol{F}$,

$$0 \to \boldsymbol{F} \to \boldsymbol{K} \to \boldsymbol{G} \to 0.$$

We also write $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}', \vec{n}'')$ for the union of all $\mathbf{Rep}(Q_{\mathcal{E}}, \boldsymbol{F}, \boldsymbol{G})$ such that $\boldsymbol{F} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}', \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ and $\boldsymbol{G} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}'', \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$.

We have the following dimension estimate for $\mathbf{Rep}(Q_{\mathcal{E}}, \boldsymbol{F}, \boldsymbol{G})$:

Lemma 2.13 $\dim \mathbf{Rep}(Q_{\mathcal{E}}, \boldsymbol{F}, \boldsymbol{G}) \le -\chi(\boldsymbol{G}, \boldsymbol{F}) + \dim G_{\vec{n}} - \hom(\boldsymbol{F}, \boldsymbol{F}) - \hom(\boldsymbol{G}, \boldsymbol{G})$.

Proof Let $X(\boldsymbol{F}, \boldsymbol{G})$ be the subset of $\mathbf{Rep}(Q_{\mathcal{E}}, \boldsymbol{F}, \boldsymbol{G})$ consisting of objects of the form

$$I = \begin{pmatrix} I_{\boldsymbol{F}} & I(\boldsymbol{G}, \boldsymbol{F}) \\ 0 & I_{\boldsymbol{G}} \end{pmatrix}, \quad J = \begin{pmatrix} J_{\boldsymbol{F}} & J(\boldsymbol{G}, \boldsymbol{F}) \\ 0 & J_{\boldsymbol{G}} \end{pmatrix},$$

for a pair $(I(\boldsymbol{G}, \boldsymbol{F}), J(\boldsymbol{G}, \boldsymbol{F})) \in \mathrm{Hom}^1(\boldsymbol{G}, \boldsymbol{F})$. The morphisms are shown in the following diagram:

$$\begin{tikzcd}
\boldsymbol{F}: & \mathbb{C}^{n'_1} \arrow[r, "I_{\boldsymbol{F}}"] & \mathbb{C}^{n'_2} \arrow[r, "J_{\boldsymbol{F}}"] & \mathbb{C}^{n'_3} \\
\boldsymbol{G}: & \mathbb{C}^{n''_1} \arrow[r, "I_{\boldsymbol{G}}"'] \arrow[ur, "I(\boldsymbol{G}, \boldsymbol{F})"] & \mathbb{C}^{n''_2} \arrow[r, "J_{\boldsymbol{G}}"'] \arrow[ur, "J(\boldsymbol{G}, \boldsymbol{F})"] & \mathbb{C}^{n''_3}
\end{tikzcd}$$

Due to the condition that $J \circ I \in \ker \alpha_{\mathcal{E}} \otimes \mathrm{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_3})$, the pair $(I(\boldsymbol{G}, \boldsymbol{F}), J(\boldsymbol{G}, \boldsymbol{F}))$ is contained in the kernel of the morphism

$$d^1(\boldsymbol{G}, \boldsymbol{F})\colon \mathrm{Hom}^1(\boldsymbol{G}, \boldsymbol{F}) \to \mathrm{Hom}^2(\boldsymbol{G}, \boldsymbol{F}).$$

By Lemma 2.9, $d^1(\boldsymbol{G}, \boldsymbol{F})$ is surjective, hence

$$\dim X(\boldsymbol{F}, \boldsymbol{G}) \le \hom^1(\boldsymbol{G}, \boldsymbol{F}) - \hom^2(\boldsymbol{G}, \boldsymbol{F}).$$

Each element $g \in \mathrm{GL}_{\vec{n}}$ can be written as a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A \in \mathrm{Hom}^0(\boldsymbol{F}, \boldsymbol{F})$, $B \in \mathrm{Hom}^0(\boldsymbol{G}, \boldsymbol{F})$, $C \in \mathrm{Hom}^0(\boldsymbol{F}, \boldsymbol{G})$ and $D \in \mathrm{Hom}^0(\boldsymbol{G}, \boldsymbol{G})$. When $A \in \mathrm{Hom}(\boldsymbol{F}, \boldsymbol{F})$, $D \in \mathrm{Hom}(\boldsymbol{G}, \boldsymbol{G})$ and $C = 0$, we have $g \cdot X(\boldsymbol{F}, \boldsymbol{G}) = X(\boldsymbol{F}, \boldsymbol{G})$. Therefore,

$$\begin{aligned} \dim \mathbf{Rep}(Q_{\mathcal{E}}, \boldsymbol{F}, \boldsymbol{G}) \\ &= \dim G_{\vec{n}} \cdot X(\boldsymbol{F}, \boldsymbol{G}) \\ &\le \dim G_{\vec{n}} + \dim X(\boldsymbol{F}, \boldsymbol{G}) - \hom(\boldsymbol{F}, \boldsymbol{F}) - \hom(\boldsymbol{G}, \boldsymbol{G}) - \hom^0(\boldsymbol{G}, \boldsymbol{F}) \\ &\le -\chi(\boldsymbol{G}, \boldsymbol{F}) + \dim G_{\vec{n}} - \hom(\boldsymbol{F}, \boldsymbol{F}) - \hom(\boldsymbol{G}, \boldsymbol{G}). \qquad \square \end{aligned}$$

Definition 2.14 We set

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})_c^{\vec{\rho}\text{–ss}} := \{\boldsymbol{F} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}} \mid \hom(\boldsymbol{F}, \boldsymbol{F}) = c\},$$
$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}', \vec{n}'')_{c,d}^{\vec{\rho}-ss} := \{\boldsymbol{K} \in \mathbf{Rep}(Q_{\mathcal{E}}, \boldsymbol{F}, \boldsymbol{G}) \mid \hom(\boldsymbol{F}, \boldsymbol{F}) = c,\ \hom(\boldsymbol{G}, \boldsymbol{G}) = d\}.$$

The following proposition shows that given a Chern character w not inside $\mathrm{Cone}_{\mathrm{LP}}$ and a generic stability condition σ, stable objects are dense in the moduli space $\mathfrak{M}_\sigma^{\mathrm{ss}}(w)$. Note that this is a nontrivial statement only when w is not primitive.

Proposition 2.15 *Let $\vec{n}$ be a character for $\mathbf{Rep}(Q_{\mathcal{E}}, \alpha_{\mathcal{E}})$ such that $\chi(\vec{n}, \vec{n}) \le -1$. Let $\vec{\rho}$ be a generic weight with respect to $\vec{n}$; in other words, $\vec{\rho} \cdot \vec{n}' \ne 0$ for any $\vec{n}' < \vec{n}$ that is not proportional to $\vec{n}$. We have*

$$\dim \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}} = -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$$

for each irreducible component of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$. Moreover,

$$\dim\bigl(\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}} \setminus \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}}\bigr) \le -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} - 1.$$

In particular, there is no component whose objects are all strictly semistable objects.

Proof The first statement basically follows from the proof of Corollary 2.10. Just note that $\hom^1(\boldsymbol{K}, \boldsymbol{K}) - \hom^2(\boldsymbol{K}, \boldsymbol{K})$ in that proof is exactly $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$ here. We will repeat the proof:

$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$ is the zero locus of $n_1 n_3 \cdot \hom(E_1, E_3)$ equations, hence each irreducible component is of dimension at least $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$.

On the other hand, for any $\vec{\rho}$–semistable object $\boldsymbol{K} \in \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$, by Lemma 2.9 $d^1\colon \mathrm{Hom}^1(\boldsymbol{K}, \boldsymbol{K}) \to \mathrm{Hom}^2(\boldsymbol{K}, \boldsymbol{K})$ is surjective, and the Zariski tangent space is of dimension $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$. Since $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ is open in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})$, each irreducible component of $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ is of dimension $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$.

For the second statement, when $\vec{n}$ is primitive and $\vec{\rho}$ is generic, we have

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}} = \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}},$$

so the statement holds automatically in this case.

We may assume that $\vec{n} = m\vec{n}_0$, where $\vec{n}_0$ is primitive. Since $\vec{\rho}$ is generic, any strictly semistable object must be destabilized by an object in $\mathbf{Rep}(Q_{\mathcal{E}}, a\vec{n}_0)^{\vec{\rho}\text{–ss}}$ for some $0 < a < m$. Hence

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}} \setminus \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}} = \bigcup_{1\le a\le m-1} \mathbf{Rep}(Q_{\mathcal{E}}, a\vec{n}_0, (m-a)\vec{n}_0).$$

For each object $\boldsymbol{F} \in \mathbf{Rep}(Q_{\mathcal{E}}, a\vec{n}_0, \alpha_{\mathcal{E}})_c^{\vec{\rho}\text{–ss}}$, the orbit $G_{a\vec{n}_0}\cdot \boldsymbol{F}$ in $\mathbf{Rep}(Q_{\mathcal{E}}, a\vec{n}_0, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ is of dimension $\dim G_{a\vec{n}_0} - c$. Therefore, by Lemma 2.13, we have

$$\begin{aligned}
&\dim \mathbf{Rep}(Q_{\mathcal{E}}, a\vec{n}_0, (m-a)\vec{n}_0)_{c,d}^{\vec{\rho}-ss} \\
&\quad\le -\chi((m-a)\vec{n}_0, a\vec{n}_0) + \dim G_{\vec{n}} - c - d - (\dim G_{a\vec{n}_0} - c) - (\dim G_{(m-a)\vec{n}_0} - d) \\
&\qquad\qquad + \dim \mathbf{Rep}(Q_{\mathcal{E}}, a\vec{n}_0, \alpha_{\mathcal{E}})_c^{\vec{\rho}-ss} + \dim \mathbf{Rep}(Q_{\mathcal{E}}, (m-a)\vec{n}_0, \alpha_{\mathcal{E}})_d^{\vec{\rho}-ss} \\
&\quad\le -\chi((m-a)\vec{n}_0, a\vec{n}_0) + \dim G_{\vec{n}} - \chi((m-a)\vec{n}_0, (m-a)\vec{n}_0) - \chi(a\vec{n}_0, a\vec{n}_0) \\
&\quad= -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} + \chi(a\vec{n}_0, (m-a)\vec{n}_0) \\
&\quad\le -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} - 1.
\end{aligned}$$

The last inequality holds since $\chi(\vec{n}, \vec{n}) \le -1$. Therefore,

$$\begin{aligned}
&\dim\bigl(\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}} \setminus \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}, \alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}}\bigr) \\
&\qquad\qquad \le \max_{c,d}\bigl\{\dim \mathbf{Rep}(Q_{\mathcal{E}}, a\vec{n}_0, (m-a)\vec{n}_0)_{c,d}^{\vec{\rho}-ss}\bigr\} \\
&\qquad\qquad \le -\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}} - 1.
\end{aligned}$$

In particular, since each component is of dimension $-\chi(\vec{n}, \vec{n}) + \dim G_{\vec{n}}$, there is no component consisting of strictly semistable objects. □

2.4 The irreducibility of the moduli space

Based on the results and methods in the previous sections, we are able to estimate the dimension of the space of new stable objects after a wall-crossing. When the wall is to the left of the vertical wall, we show that the new stable objects in the next chamber have codimension at least 3. Together with Proposition 2.15, this will imply the irreducibility of the moduli space.

Let w be a character in $\mathrm{K}(\mathbb{P}^2)$ with $\mathrm{ch}_0(w) \geq 0$, and $\sigma_{s,q}$ a stability condition with $s < \frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)}$ and such that $(1, s, q)$ is contained in $\mathrm{MZ}_{\mathcal{E}}$. Let $\vec{n}$ be the dimension character for w in $Q_{\mathcal{E}}$, and $\vec{\rho}$ be the weight character corresponding to $L_{w\sigma}$. Let $\vec{\rho}_-$ be the character in the chamber below $L_{w\sigma}$ and $\vec{\rho}_+$ in the chamber above $L_{w\sigma}$. The following two lemmas will be used in the proof of Proposition 2.18.

Lemma 2.16 *Let $\boldsymbol{K}$ be an object in $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}_-\text{-s}} \setminus \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}_+\text{-s}}$. Then it can be written as a nontrivial extension*

$$0 \to \boldsymbol{K}' \to \boldsymbol{K} \to \boldsymbol{K}'' \to 0$$

of objects in $\mathbf{Rep}(Q_{\mathcal{E}}, \alpha_{\mathcal{E}})$, where the dimension character $\vec{n}'$ of $\boldsymbol{K}'$ satisfies

$$\vec{\rho}_- \cdot \vec{n}' < 0 = \vec{\rho} \cdot \vec{n}',$$

and $\mathrm{Hom}(\boldsymbol{K}'', \boldsymbol{K}') = 0$.

Proof By the assumption on $\boldsymbol{K}$, it is a strictly $\vec{\rho}$–semistable object, and is destabilized by a nonzero $\vec{\rho}$–stable proper subobject $\boldsymbol{K}'$ with $\vec{\rho} \cdot \vec{n}' = 0$. As $\boldsymbol{K}$ is $\vec{\rho}_-$–stable, we have $\vec{\rho}_- \cdot \vec{n}' < 0$. Let the quotient be $\boldsymbol{K}''$; then $\boldsymbol{K}'$ and $\boldsymbol{K}''$ are the objects we want.

In order to see that $\mathrm{Hom}(\boldsymbol{K}'', \boldsymbol{K}') = 0$, suppose there is a nonzero map in $\mathrm{Hom}(\boldsymbol{K}'', \boldsymbol{K}')$. Then its image $\widetilde{\boldsymbol{K}}$ in $\boldsymbol{K}'$ is both a subrepresentation and quotient representation of $\boldsymbol{K}$. Let $\vec{\tilde{n}}$ be the dimension vector of $\widetilde{\boldsymbol{K}}$. As $\boldsymbol{K}$ is $\vec{\rho}_-$–stable, we get $\vec{\rho}_- \cdot \vec{\tilde{n}} < 0 < \vec{\rho}_- \cdot \vec{\tilde{n}}$, which leads to a contradiction. □

For a dimension vector $\vec{n}$ of $Q_{\mathcal{E}}$, write $\mathrm{ch}_i(\vec{n})$ for $n_1 \,\mathrm{ch}_i(E_1) - n_2 \,\mathrm{ch}_i(E_2) + n_3 \,\mathrm{ch}_i(E_3)$ for $i = 0, 1, 2$.

Lemma 2.17 *Let $\vec{n}$ and $\vec{m}$ be two dimension vectors of $Q_{\mathcal{E}}$.*

(1) *The Euler character $\chi(\vec{n}, \vec{m})$ can be computed as*

$$\mathrm{ch}_2(\vec{n})\,\mathrm{ch}_0(\vec{m}) + \mathrm{ch}_2(\vec{m})\,\mathrm{ch}_0(\vec{n}) - \mathrm{ch}_1(\vec{n})\,\mathrm{ch}_1(\vec{m}) + \tfrac{3}{2}(\mathrm{ch}_1(\vec{m})\,\mathrm{ch}_0(\vec{n}) - \mathrm{ch}_0(\vec{m})\,\mathrm{ch}_1(\vec{n})) + \mathrm{ch}_0(\vec{n})\,\mathrm{ch}_0(\vec{m}).$$

(2) *Suppose $\mathrm{ch}_0(\vec{n}) \leq 0$, let w have character $-(\mathrm{ch}_0(\vec{n}), \mathrm{ch}_1(\vec{n}), \mathrm{ch}_2(\vec{n}))$ and P be a point in $\overline{\Delta}_{<0}$ to the left of the vertical wall $L_{w\pm}$ such that L_{Pw} intersects $l_{E_1E_3}$. Let $\vec{\rho}$ be $\vec{\rho}_{L_{Pw}}$, and $\vec{\rho}_-$ be the character in the chamber below L_{Pw}. Suppose $\vec{m}$ satisfies $\vec{\rho}_- \cdot \vec{m} < 0 = \vec{\rho} \cdot \vec{m}$, and $\vec{n}$ satisfies $\vec{\rho}_- \cdot \vec{n} = 0 = \vec{\rho} \cdot \vec{n}$. Then*

$$\mathrm{ch}_0(\vec{n})\,\mathrm{ch}_1(\vec{m}) - \mathrm{ch}_0(\vec{m})\,\mathrm{ch}_1(\vec{n}) > 0.$$

Proof The first statement follows from the Hirzebruch–Riemann–Roch formula for $\mathbb{P}^2$:

$$\chi(F,G)=\operatorname{ch}_2(F)\operatorname{ch}_0(G)+\operatorname{ch}_2(G)\operatorname{ch}_0(F)-\operatorname{ch}_1(F)\operatorname{ch}_1(G)\\ +\tfrac{3}{2}(\operatorname{ch}_1(G)\operatorname{ch}_0(F)-\operatorname{ch}_0(G)\operatorname{ch}_1(F))+\operatorname{ch}_0(F)\operatorname{ch}_0(G).$$

For the second statement, by definition of $\vec{\rho}$ we have that $\vec{\rho}_-$ is in the same chamber as $\vec{\rho}+\epsilon(0,n_3,-n_2)$ for small enough $\epsilon>0$. We have

$$\operatorname{ch}_0(\vec{n})\operatorname{ch}_1(\vec{m})-\operatorname{ch}_0(\vec{m})\operatorname{ch}_1(\vec{n})=(m_1,m_2,m_3)\cdot\vec{\Upsilon},$$

where $\vec{\Upsilon}$ is the vector

$$\left(\begin{vmatrix}\operatorname{ch}_0(\vec{n}) & \operatorname{ch}_1(\vec{n})\\ \operatorname{ch}_0(E_1) & \operatorname{ch}_1(E_1)\end{vmatrix}, -\begin{vmatrix}\operatorname{ch}_0(\vec{n}) & \operatorname{ch}_1(\vec{n})\\ \operatorname{ch}_0(E_2) & \operatorname{ch}_1(E_2)\end{vmatrix}, \begin{vmatrix}\operatorname{ch}_0(\vec{n}) & \operatorname{ch}_1(\vec{n})\\ \operatorname{ch}_0(E_3) & \operatorname{ch}_1(E_3)\end{vmatrix}\right).$$

The vector $\vec{\Upsilon}$ is a weight character for $\vec{n}$ since

$$\vec{n}\cdot\vec{\Upsilon}=\operatorname{ch}_0(\vec{n})\operatorname{ch}_1(\vec{n})-\operatorname{ch}_1(\vec{n})\operatorname{ch}_0(\vec{n})=0.$$

When $\mathrm{MZ}_{\mathcal{E}}$ intersects the vertical wall $L_{w\pm}$, by Lemma 2.5 $\vec{\Upsilon}$ is proportional (up to a positive scalar) to the character on the vertical wall. As $\vec{\rho}$ is to the left of the vertical wall, $\vec{\Upsilon}$ can be written as $a\vec{\rho}-b\vec{\rho}_-$ for some positive numbers a and b. Therefore, $\vec{m}\cdot\vec{\Upsilon}=-b\vec{m}\cdot\vec{\rho}_->0$.

When $\mathrm{MZ}_{\mathcal{E}}$ is to the left of the vertical wall $L_{w\pm}$, we have

$$\frac{\operatorname{ch}_1(E_i)}{\operatorname{ch}_0(E_i)}\le\frac{\operatorname{ch}_1(\vec{n})}{\operatorname{ch}_0(\vec{n})}$$

for $i=1,2,3$. As $\operatorname{ch}_0(\vec{n})\le 0$, we have

$$\begin{vmatrix}\operatorname{ch}_0(\vec{n}) & \operatorname{ch}_1(\vec{n})\\ \operatorname{ch}_0(E_i) & \operatorname{ch}_1(E_i)\end{vmatrix}>0.$$

Since the third term of $\vec{\rho}$ is negative and the character space of $\vec{n}$ is spanned by $\vec{\rho}$ and $(0,n_3,-n_2)$, the character $\vec{\Upsilon}$ can be written as $a\vec{\rho}-b(0,n_3,-n_2)$ for some positive numbers a and b. As $\vec{\rho}_-$ is in the same chamber as $\vec{\rho}+\epsilon(0,n_3,-n_2)$ and $\vec{m}\cdot\vec{\rho}_-<0$, we get $\vec{m}\cdot\vec{\Upsilon}>0$. □

Now we can give an estimate of the dimension of new stable objects after wall-crossing.

Proposition 2.18 *The dimension of* $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_- \text{-s}}\setminus\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_+ \text{-s}}$ *is less than* $-\chi(w,w)+\dim G_{\vec{n}_w}-2$.

Proof By Lemma 2.16, the space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}_- - \mathrm{s}} \setminus \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})^{\vec{\rho}_+ - \mathrm{s}}$ can be covered by

$$\mathbf{Rep}^{\vec{\rho}_- - \mathrm{s}} \setminus \mathbf{Rep}^{\vec{\rho}_+ - \mathrm{s}} = \bigcup_{\vec{m}} \mathbf{Rep}(Q_{\mathcal{E}}, \vec{m}, (\vec{n}_w - \vec{m}))^{\vec{\rho}_- - \mathrm{s}},$$

where $\vec{m}$ satisfies

- $\vec{\rho}_- \cdot \vec{m} < 0 = \vec{\rho} \cdot \vec{m}$,
- $\chi(\vec{n}_w - \vec{m}, \vec{m}) \leq 0$.

The second condition is due to Lemmas 2.9 and 2.16. Now, similar to the proof of Proposition 2.15, we have

$$\begin{aligned}
&\dim \mathbf{Rep}(Q_{\mathcal{E}}, \vec{m}, \vec{n}_w - \vec{m})^{\vec{\rho}_- - \mathrm{s}}_{c,d} \\
&\quad \leq -\chi(\vec{n}_w - \vec{m}, \vec{m}) + \dim G_{\vec{n}_w} - c - d - (\dim G_{(\vec{n}_w - \vec{m})} - c) - (\dim G_{\vec{m}} - d) \\
&\qquad\qquad + \dim \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w - \vec{m}, \alpha_{\mathcal{E}})^{\vec{\rho}_- - ss}_{c} + \dim \mathbf{Rep}(Q_{\mathcal{E}}, \vec{m})^{\vec{\rho}_- - ss}_{d} \\
&\quad \leq -\chi(\vec{n}_w - \vec{m}, \vec{m}) + \dim G_{\vec{n}_w} - \chi(\vec{n}_w - \vec{m}, \vec{n}_w - \vec{m}) - \chi(\vec{m}, \vec{m}) \\
&\quad = -\chi(\vec{n}_w, \vec{n}_w) + \dim G_{\vec{n}_w} + \chi(\vec{m}, \vec{n}_w - \vec{m}).
\end{aligned}$$

By Lemma 2.17,

$$\begin{aligned}
-\chi(\vec{m}, \vec{n}_w - \vec{m}) &\geq \chi(\vec{n}_w - \vec{m}, \vec{m}) - \chi(\vec{m}, \vec{n}_w - \vec{m}) \\
&= \chi(\vec{n}_w, \vec{m}) - \chi(\vec{m}, \vec{n}_w) \\
&= 3\big(\mathrm{ch}_0(\vec{n}_w)\,\mathrm{ch}_1(\vec{m}) - \mathrm{ch}_0(\vec{m})\,\mathrm{ch}_1(\vec{n}_w)\big) \\
&\geq 3.
\end{aligned}$$

The last inequality is due to the second statement of Lemma 2.17. □

Now we prove the irreducibility of the moduli space of stable objects. This is well known to hold for moduli of Gieseker stable sheaves. The moduli spaces are given by moduli of quiver representations, so the dimension of each component has a lower bound. The point is, by the previous results, the dimension of new stable objects is smaller than this lower bound, so an irreducible component cannot be produced after wall-crossing.

Theorem 2.19 *Let w be a primitive character in $\mathrm{K}(\mathbb{P}^2)$ such that $\mathrm{ch}_0(w) > 0$. For a generic geometric stability condition $\sigma = \sigma_{s,q}$ with $s < \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$ not on any actual wall of w, the moduli space $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$ is irreducible and smooth.*

Proof The smoothness is proved in Corollary 2.10. We only need to show the irreducibility.

For any σ, the line $L_{w\sigma}$ intersects some $\mathrm{MZ}_{\mathcal{E}}$. In fact, we may always choose $\mathcal{E}$ to be $\{\mathcal{O}(k-1),\mathcal{O}(k),\mathcal{O}(k+1)\}$. By Proposition 2.7 $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$ can always be constructed as

$$\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})/\!\!/_{\det^{\vec{\rho}_{L_{w\sigma}\mathcal{E}}}}(G_{\vec{n}_w}/\mathbb{C}^\times).$$

In the chamber near the vertical wall, the component that contains $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}}$ is irreducible since the quotient space $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}\text{–s}}/G$ is $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$, which is smooth and connected.

By Proposition 2.18, while crossing an actual wall, the new stable locus

$$\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_-\text{–s}}\setminus\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_+\text{–s}}$$

in $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_-\text{–s}}$ has codimension greater than 2. On the other hand, since $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})$ is a subspace in $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w)$ determined by $n_1n_2\hom(E_1,E_3)$ equations, each irreducible component has dimension at least

$$n_1n_2\hom(E_1,E_2)+n_2n_3\hom(E_2,E_3)-n_1n_2\hom(E_1,E_3),$$

which is the same as the dimension of $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_+\text{–s}}$ and is greater than the dimension of the new stable locus $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_-\text{–s}}\setminus\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_+\text{–s}}$. Since the stable locus is open in $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})$, the new stable locus is contained in the same irreducible component of $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_+\text{–s}}$. Also, $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_-\text{–s}}$ is still irreducible. Hence the moduli space of Bridgeland stable objects, given as the GIT quotient, is also irreducible. □

Remark 2.20 There is a natural isomorphism

$$\mathfrak{M}^{\mathrm{ss}}_{\sigma_{s,q}}(w)\simeq\mathfrak{M}^{\mathrm{ss}}_{\sigma_{-s,q}}(-\mathrm{ch}_0(w),\mathrm{ch}_1(w),-\mathrm{ch}_2(w))$$

induced by the map $\iota\colon F\mapsto\mathcal{RH}om(F,\mathcal{O})[1]$. In terms of the quiver representation, an object $\boldsymbol{K}_F\in\mathbf{Rep}(Q_{\mathcal{E}},\vec{n},\alpha_{\mathcal{E}})^{\vec{\rho}\text{–ss}}$ given by

$$\boldsymbol{K}_F\colon E_1\otimes H_{n_1}\xrightarrow{I_F}E_2\otimes H_{n_2}\xrightarrow{J_F}E_3\otimes H_{n_3}$$

is mapped to $\boldsymbol{K}_{\iota(F)}\in\mathbf{Rep}(Q_{\mathcal{E}^\vee},(n_3,n_2,n_1),\alpha_{\mathcal{E}^\vee})^{(-\rho_3,-\rho_2,-\rho_1)\text{–ss}}$ given by

$$\boldsymbol{K}_{\iota(F)}\colon E_3^\vee\otimes H^*_{n_3}\xrightarrow{J_F^T}E_2^\vee\otimes H^*_{n_2}\xrightarrow{I_F^T}E_1^\vee\otimes H^*_{n_1}.$$

The statement in Theorem 2.19 holds for $\mathfrak{M}^{\mathrm{s}}_{\sigma}(-w)$ when $\sigma=\sigma_{s,q}$ with $s>\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$.

2.5 Properties of GIT

Birational geometry via GIT has been studied by Dolgachev and Hu [17], and by Thaddeus [32]. Since the theorems in [17] and [32] are stated based on a slightly different setup, in this section we recollect some properties from these papers in the language of affine GIT.

Let X be an affine algebraic G–variety, where G is a reductive group and acts on X via a linear representation. Given a character $\rho\colon G \to \mathbb{C}^\times$, the (semi)stable locus is written as $X^{\vec{\rho}\text{–s}}$ ($X^{\vec{\rho}\text{–ss}}$). We write $\mathbb{C}[X]^{G,\chi}$ for the χ–semi-invariant functions on X; in other words, one has

$$f(g^{-1}(x)) = \chi(g)\cdot f(x) \quad \text{for all } g \in G,\ x \in X.$$

Denote the GIT quotient by $X/\!\!/_{\vec{\rho}}G := \mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X]^{G,\vec{\rho}^n}$ and the map from $X^{\vec{\rho}\text{–ss}}$ to $X/\!\!/_{\vec{\rho}}G$ by $F_{\vec{\rho}}$.

In addition, we need the following assumptions on X and G:

(1) There are only finite many walls in the space of characters on which there are strictly semistable points, and in the chamber we have $X^{\vec{\rho}\text{–s}} = X^{\vec{\rho}\text{–ss}}$.

(2) $X^{\vec{\rho}\text{–s}}$ is smooth and the action of G on $X^{\vec{\rho}\text{–s}}$ is free.

(3) $X/\!\!/_{\vec{\rho}}G$ is projective and irreducible.

(4) The closure of $X^{\vec{\rho}\text{–s}}$ (if nonempty) for any $\vec{\rho}$ is the same irreducible component.

(5) Given any point $x \in X$, the set of characters $\{\rho \mid x \in X^{\rho\text{–ss}}\}$ is closed.

Let $\vec{\rho}$ be a generic character (ie not on any walls) such that $X^{\vec{\rho}\text{–s}}$ is nonempty. By assumptions (2) and (3), we have a G–principal bundle $X^{\vec{\rho}\text{–s}} \to X/\!\!/_{\vec{\rho}}G = X^{\vec{\rho}\text{–s}}/G$.

Definition 2.21 Let $\vec{\rho}_0$ be a character of G. We define $\mathcal{L}_{\vec{\rho},\vec{\rho}_0}$ to be the *line bundle* over $X/\!\!/_{\vec{\rho}}G$ by composing the transition functions of the G–principal bundles with $\vec{\rho}_0$.

In other words, viewing $X^{\vec{\rho}\text{–s}}/G$ as a complex manifold, it has an open cover with trivialization of G fibers. The line bundle $\mathcal{L}_{\vec{\rho},\vec{\rho}_0}$ is defined by composing each transition function on the overlap of charts with $\vec{\rho}_0$.

Now we are ready to list some properties from the variation of geometric invariant theory.

Proposition 2.22 *Suppose that X is an affine algebraic G–variety that satisfies the assumptions (1)–(5), and let $\vec{\rho}$ be a generic character. The following properties hold:*

(i) $\Gamma(X/\!/_{\vec{\rho}}G, \mathcal{L}^{\otimes n}_{\vec{\rho},\vec{\rho}_1}) \simeq \mathbb{C}[X^{\vec{\rho}\text{-s}}]^{G,\vec{\rho}_1^n}$.

(ii) *Let $\vec{\rho}_+$ be a character of G in the same chamber as $\vec{\rho}$. Then*

$$\mathbb{C}[X^{\vec{\rho}\text{-s}}]^{G,\vec{\rho}_+^n} = \mathbb{C}[X]^{G,\vec{\rho}_+^n}$$

for $n \gg 1$, and $\mathcal{L}_{\vec{\rho},\vec{\rho}_+}$ is ample. Let $\vec{\rho}_0$ be a generic character on the wall of the $\vec{\rho}$–chamber. Then $\mathcal{L}_{\vec{\rho},\vec{\rho}_0}$ is nef and semiample.

(iii) *There is an inclusion $X^{\vec{\rho}_+\text{-ss}} \subset X^{\vec{\rho}_0\text{-ss}}$, which induces a canonical projective morphism $\mathrm{pr}_+\colon X/\!/_{\vec{\rho}_+}G \to X/\!/_{\vec{\rho}_0}G$.*

(iv) *A curve C (projective, smooth, connected) in $X/\!/_{\vec{\rho}_+}G$ is contracted by pr_+ if and only if it is contracted by $X/\!/_{\vec{\rho}_+}G \to \mathbf{Proj}\bigoplus_{n\geq 0}\Gamma(X/\!/_{\vec{\rho}_+}G, \mathcal{L}^{\otimes n}_{\vec{\rho}_+,\vec{\rho}_0})$.*

(v) *Let $\vec{\rho}_+$ and $\vec{\rho}_-$ be in two chambers on different sides of the wall. Assume that $X^{\vec{\rho}_+\text{-s}}$ and $X^{\vec{\rho}_-\text{-s}}$ are both nonempty. Then the morphisms $X/\!/_{\vec{\rho}_\pm}G \to X/\!/_{\vec{\rho}_0}G$ are proper and birational. If they are both small, then the rational map $X/\!/_{\vec{\rho}_-}G \dashrightarrow X/\!/_{\vec{\rho}_+}G$ is a flip with respect to $\mathcal{L}_{\vec{\rho}_+,\vec{\rho}_0}$.*

Proof (i) This is true for a general G–principal bundle by the flat descent theorem; see [16, Exposé I, Théorème 4.5].

(ii)–(iii) By assumption (5), $X^{\vec{\rho}\text{-s}} \subset X^{\vec{\rho}_*\text{-ss}}$ for $* = 0$ or $+$. By assumption (4), the natural map $\mathbb{C}[X]^{G,\vec{\rho}_*^n} \to \mathbb{C}[X^{\vec{\rho}\text{-s}}]^{G,\vec{\rho}_*^n} \simeq \Gamma(X/\!/_{\vec{\rho}}G,\, \mathcal{L}^{\otimes n}_{\vec{\rho},\vec{\rho}_*})$ is injective for $n \in \mathbb{Z}_{\geq 0}$. Hence the base locus of $\mathcal{L}_{\vec{\rho},\vec{\rho}_*}$ is empty. $\mathcal{R}(X/\!/_{\vec{\rho}}G,\, \mathcal{L}_{\vec{\rho},\vec{\rho}_*}) \simeq \bigoplus_{n\geq 0}\mathbb{C}[X^{\vec{\rho}\text{-s}}]^{G,\vec{\rho}_*^n}$ is finitely generated over $\mathbb{C}$. The canonical morphism $X/\!/_{\rho}G \to \mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X^{\vec{\rho}\text{-s}}]^{G,\vec{\rho}_*^n}$ is birational and projective when $X^{\vec{\rho}_*\text{-s}}$ is nonempty. Now we have series of morphisms

$$\mathrm{pr}_+\colon X/\!/_{\vec{\rho}}G \to \mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X^{\vec{\rho}\text{-s}}]^{G,\vec{\rho}_*^n} \to \mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X]^{G,\vec{\rho}_*^n} = X/\!/_{\vec{\rho}_*}G.$$

The morphism pr_+ maps each $\vec{\rho}_*$ S–equivariant class to itself set-theoretically. When $\vec{\rho}_+$ is in the same chamber as $\vec{\rho}$, by assumption (2) this is an isomorphism, implying that $\mathcal{L}_{\vec{\rho},\vec{\rho}_+}$ must be ample and $\mathbb{C}[X^{\vec{\rho}\text{-s}}]^{G,\vec{\rho}_*^n} = \mathbb{C}[X]^{G,\vec{\rho}_*^n}$ for n large enough. By the definition of $\mathcal{L}_{\vec{\rho},\vec{\rho}_+}$, it extends linearly to a map from the space of $\mathbb{R}$–characters of G to $\mathrm{NS}_{\mathbb{R}}(X/\!/_{\vec{\rho}}G)$. Since all elements in the $\vec{\rho}$ chamber are mapped into the ample cone, $\vec{\rho}_0$ must be nef.

(iv) $(\Leftarrow)$ The morphism

$$X/\!\!/_{\vec{\rho}_+}G \to X/\!\!/_{\vec{\rho}_0}G = \mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X]^{G,\vec{\rho}_0^n}$$

factors via the morphism

$$\mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X^{\vec{\rho}_+\text{-s}}]^{G,\vec{\rho}_0^n} \to \mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X]^{G,\vec{\rho}_0^n}.$$

If a curve C is contracted at $\mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X^{\vec{\rho}_+\text{-s}}]^{G,\vec{\rho}_0^n}$, then it is also contracted at $\mathbf{Proj}\bigoplus_{n\geq 0}\mathbb{C}[X]^{G,\vec{\rho}_0^n}$.

$(\Rightarrow)$ Suppose C is contracted to a point by pr_+. Let G' be the kernel of $\vec{\rho}_0$. We show that there is a subvariety P in $X^{\vec{\rho}_+\text{-s}}$ such that

(I) P is a G'–principal bundle, and the base space is projective and connected;

(II) $F_{\vec{\rho}_+}(P) = C$.

Suppose we find such a P. Then any function f in $\mathbb{C}[X^{\vec{\rho}_+\text{-s}}]^{G,\vec{\rho}_0^n}$ is constant on each G' fiber. Since the base space is projective and connected, it must be a constant on P. Since $F_{\vec{\rho}_+}(P) = C$, the value of f on $F_{\vec{\rho}_+}^{-1}(C)$ is determined by this constant. Hence the canonical morphism contracts C to a point.

We may assume $G' \neq G$. Choose N large enough, and finitely many f_i in $\mathbb{C}[X]^{G,\vec{\rho}_0^N}$ such that $\bigcap_i (V(f_i) \cap F_{\vec{\rho}_0}^{-1}(\mathrm{pr}_+(C)))$ is empty. All points in $F_{\vec{\rho}_0}^{-1}(\mathrm{pr}_+(C))$ are S–equivariant in $X^{\vec{\rho}_0\text{-ss}}$, so for each point x in $F_{\vec{\rho}_+}^{-1}(C)$ we have that $\overline{Gx}$ contains all minimum orbits Gy in $F_{\vec{\rho}_0}^{-1}(\mathrm{pr}_+(C))$. Choose y in $F_{\vec{\rho}_0}^{-1}(\mathrm{pr}_+(C))$ such that Gy is closed in $X^{\vec{\rho}_0\text{-ss}}$, and let

$$P_y = \bigcap_i \{x \in F_{\vec{\rho}_+}^{-1}(C) \mid f_i(x) = f_i(y)\}.$$

For any $p \in C$, since G is reductive and the G–orbit $\overline{F_{\vec{\rho}_+}^{-1}(p)}$ contains y, there is a subgroup $\beta\colon \mathbb{C}^\times \to G$ and $x_p \in F_{\vec{\rho}_+}^{-1}(p)$ such that $y \in \overline{\beta(\mathbb{C}^\times)\cdot\{x_p\}}$. Since $y \in X^{\vec{\rho}_0\text{-ss}}$, there is a $\vec{\rho}_0^N$–semi-invariant f_i such that $f_i(y) \neq 0$. Therefore $\vec{\rho}_0 \circ \beta \neq 0$, and for any $\vec{\rho}_0$–semi-invariant function f, we get that $f(x_p) = f(y)$. The point x_p is in P_y and therefore $F_{\vec{\rho}}(P_y) = C$.

Let G'' be the kernel of $\vec{\rho}_0^N$. By the choice of the f_i, another point x_q on Gx_p is in P_y if and only if they are on the same G''–orbit. Since G acts freely on all stable points, P_y becomes a G''–principal bundle over base C. As $[G'' : G']$ is finite, we may choose a connected component of P_y such that, viewing as a G'–principal bundle, the induced morphism from the base space to C is finite. This component of P_y then satisfies both conditions (I) and (II) above.

(v) This is due to Theorem 3.3 in [32]. □

Remark 2.23 When the difference between $X^{\vec{\rho}_+ -\mathrm{s}}$ and $X^{\vec{\rho}_- -\mathrm{s}}$ is of codimension two in $X^{\vec{\rho}_+ -\mathrm{s}} \cup X^{\vec{\rho}_- -\mathrm{s}}$, then since $X^{\vec{\rho}_+ -\mathrm{s}} \cup X^{\vec{\rho}_- -\mathrm{s}}$ is smooth, irreducible and quasiaffine by assumption (2), we have

$$\mathbb{C}[X^{\vec{\rho}_+ -\mathrm{s}}]^{G,\vec{\rho}_-^n} = \mathbb{C}[X^{\vec{\rho}_+ -\mathrm{s}} \cup X^{\vec{\rho}_- -\mathrm{s}}]^{G,\vec{\rho}_-^n} = \mathbb{C}[X^{\vec{\rho}_- -\mathrm{s}}]^{G,\vec{\rho}_-^n} = \mathbb{C}[X]^{G,\vec{\rho}_-^n} \quad \text{for } n \gg 0.$$

In this case, the birational morphism between $X^{s,\vec{\rho}_+}$ and $X^{s,\vec{\rho}_-}$ identifies the spaces $\mathrm{NS}_{\mathbb{R}}(X/\!/_{\vec{\rho}_+} G)$ and $\mathrm{NS}_{\mathbb{R}}(X/\!/_{\vec{\rho}_-} G)$. It maps $[\mathcal{L}_{\vec{\rho}_+,\vec{\rho}_*}]$ to $[\mathcal{L}_{\vec{\rho}_-,\vec{\rho}_*}]$ for all $\vec{\rho}_*$ in either the $\vec{\rho}_+$ or $\vec{\rho}_-$ chamber.

2.6 Wall-crossing as minimal model program

Let w be a primitive character in $\mathrm{K}(\mathbb{P}^2)$ such that $\mathrm{ch}_0(w) > 0$. We can run the minimal model program for $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$ via wall-crossing on the space of stability conditions.

Theorem 2.24 *Adopting notation as above, the actual walls $L_{w\sigma}$ (chambers) to the left of the vertical wall $L_{w\pm}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane are in one-to-one correspondence with the stable base locus decomposition walls (chambers) on one side (primitive side) of the divisor cone of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$.*

Proof Suppose $L = L_{w\sigma}$ passes through $\mathrm{MZ}_{\mathcal{E}}$ for an exceptional triple $\mathcal{E}$. By Lemma 2.5, L associates a character (up to a positive scalar) $\vec{\rho}_L$ to the group $G_{\vec{n}_w}/\mathbb{C}^\times$. By Proposition 2.7, the moduli space $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$ is constructed as the quotient space $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})/\!/_{\det\vec{\rho}_L}(G_{\vec{n}_w}/\mathbb{C}^\times)$. We first check that the G–variety $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})$ satisfies the assumptions of Proposition 2.22. Assumption (1) is due to Proposition 2.8. Assumption (2) is due to Corollary 2.10 and Remark 2.11. Assumptions (3) and (4) are due to Theorem 2.19 and its proof. Assumption (5) is automatically satisfied in our case.

By Definition 2.21, the character $\vec{\rho}_L$ induces a divisor (up to a positive scalar) $[\mathcal{L}_{\vec{\rho}_L,\vec{\rho}_L}]$ on $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})/\!/_{\det\vec{\rho}_L}(G_{\vec{n}_w}/\mathbb{C}^\times)$. We start from the chamber on the left of the vertical wall, where $\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})/\!/_{\det\vec{\rho}_L}(G_{\vec{n}_w}/\mathbb{C}^\times)$ is isomorphic to $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$, and vary the stability to the wall near the tangent line of $\overline{\Delta}_0$ across w. At an actual destabilizing wall L, let pr_+ be the morphism

$$\mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})/\!/_{\det\vec{\rho}_{L+}}(G_{\vec{n}_w}/\mathbb{C}^\times) \to \mathbf{Rep}(Q_{\mathcal{E}}, \vec{n}_w, \alpha_{\mathcal{E}})/\!/_{\det\vec{\rho}_L}(G_{\vec{n}_w}/\mathbb{C}^\times)$$

as in Proposition 2.22. One of three different cases may happen:

(a) pr_+ is a small contraction.

(b) pr_+ is birational and has an exceptional divisor.

(c) All objects in $\mathfrak{M}^{\mathrm{s}}_L(w)$ become strictly semistable.

By Proposition 2.18, in case (a), we get small contractions on both sides. By property (v) in Proposition 2.22, this is the flip with respect to the divisor $[\mathcal{L}_{\vec{\rho}_{L+},\vec{\rho}_L}]$. Since the different locus between $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_{L+}}$ and $\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})^{\vec{\rho}_{L-}}$ is of codimension at least 2, their divisor cones are identified with each other as explained in Remark 2.23. In particular, before encountering any wall of case (b) or (c), the divisor $[\mathcal{L}_{\vec{\rho}_{L+},\vec{\rho}_L}]$ is identified with a divisor $[\mathcal{L}_{\vec{\rho}_L}]$ on $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$. The flip $\mathfrak{M}^{\mathrm{s}}_{L+}(w) \dashrightarrow \mathfrak{M}^{\mathrm{s}}_{L-}(w)$ is with respect to this divisor.

In case (b), by Proposition 2.18, the morphism pr_- on the left side,

$$\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})/\!\!/_{\det\vec{\rho}_{L-}}(G_{\vec{n}_w}/\mathbb{C}^\times) \to \mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})/\!\!/_{\det\vec{\rho}_L}(G_{\vec{n}_w}/\mathbb{C}^\times),$$

does not contract any divisors. Hence the Picard number of

$$\mathbf{Rep}(Q_{\mathcal{E}},\vec{n}_w,\alpha_{\mathcal{E}})/\!\!/_{\det\vec{\rho}_{L-}}(G_{\vec{n}_w}/\mathbb{C}^\times)$$

is 1. By property (iv) in Proposition 2.22, case (b) only happens when the canonical model associated to $\mathcal{L}_{\vec{\rho}_L}$ contracts a divisor, in other words, the divisor of $\mathcal{L}_{\vec{\rho}_L}$ on $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(w)$ is on the boundary of the movable cone. The next destabilizing wall on the left corresponds to the zero divisor, it must be case (c). On the other hand, by Corollary 1.35, case (c) must happen at a wall before reaching the tangent line. This terminates the whole minimal model program.

In general, if the boundary of the movable cone is not the same as that of the nef cone, then case (b) happens. Otherwise, case (b) does not happen and the procedure ends up with a Mori fibration of case (c). □

Remark 2.25 On the vertical wall, the morphism pr_+ is the Donaldson–Uhlenbeck morphism. If it contracts a divisor, the vertical wall corresponds to the movable boundary and the minimal model program stops. If pr_+ is a small contraction, the wall-crossing behavior on the other side of the nef cone is the same as the wall-crossing behavior of $\mathfrak{M}^{\mathrm{s}}_{\mathrm{GM}}(\mathrm{ch}_0(w),-\mathrm{ch}_1(w),\mathrm{ch}_2(w))$ on the primitive side.

3 The last wall and criterion for actual walls

In this section, we describe the last wall (Section 3.2) and give a numerical criterion of actual walls (Section 3.3) for any given Chern character w.

As for the last wall, each character w is associated to an exceptional character e. We first show (Lemma 3.12) that after the wall L_{we}, for any stability condition σ, there is no σ–stable object with Chern character w. Our argument is to show that $\mathrm{Hom}(E,F)\neq 0$ for the exceptional bundle E and any σ–stable object F with Chern character w. As a consequence, such a stable object F is destabilized by the exceptional bundle E.

The subtle part is to show (Theorem 3.14) that for any stability condition σ in the chamber before the wall L_{we}, the moduli space $\mathfrak{M}^{s}_{\sigma}(w)$ is nonempty. We will show that there exist stable objects given by extensions before the wall-crossing. First, we show that on any wall before the last wall, the pair of destabilizing Chern characters w' and $w-w'$ are between their own last walls and the vertical walls. By Corollary 3.10 in [8], the discriminants of w' and $w-w'$ are less than that of w. By doing induction on the discriminant, there exist stable objects with characters w' and $w-w'$. By Lemma 3.4, the Euler pair $\chi(w',w-w')$ is negative. This implies that for any stable objects F and G with Chern characters w' and $w-w'$ respectively, the extension group $\mathrm{Ext}^1(F,G)$ is nonzero. By Lemma 3.1, the extension of such a pair of stable objects F and G is stable before the wall-crossing.

As for the criterion on actual walls, one may numerically compute all the potential destabilizing pairs of Chern characters w' and $w-w'$. Different from the situation of the last wall, in this case, not every such pair will offer pairs of stable objects that can be extended to a stable object before the wall-crossing. Either there are no stable objects with character w' and $w-w'$, or no such pair of stable objects has nontrivial extension. We show that the second scenario will not happen by Lemma 3.4 and some detailed discussions in the proof for Theorem 3.16. The first scenario indeed happens for some potential walls. In the concrete example in Section 4.3, we study the wall-crossing for the Chern character $w=(\mathrm{ch}_0,\mathrm{ch}_1,\mathrm{ch}_2)=(4,0,-15)$. The pair of Chern characters $w'=(3,-2,3)$ and $w-w'=(1,2,-18)$ indicates a potential wall. But as there is no stable object with Chern character w' on that potential wall, and there is no other pair of destabilizing Chern characters, this potential wall fails to become an actual wall. To avoid the first case, we introduce the small triangulated area TR_{wE} for each exceptional bundle E. We show that a destabilizing Chern character w' (or $w-w'$) is this area if and only if there is no stable object with Chern character w' (or $w-w'$) with respect to a stability condition on the wall $L_{ww'}$. As a consequence, a potential wall becomes an actual wall when both w and $w-w'$ avoid these triangle areas.

3.1 Stable objects by extensions

The following lemma is useful to construct new stable objects after wall-crossing.

Lemma 3.1 *Let G and F be two $\sigma_{s,q}$–stable objects of the same phase; in particular, $\sigma_{s,q}$ is on the line L_{GF}. Suppose we have*

$$\phi_{\sigma_{s,q+}}(G) > \phi_{\sigma_{s,q+}}(F) \quad \textit{and} \quad \mathrm{Hom}(G, F[1]) \neq 0.$$

Let f be a nonzero element in $\mathrm{Hom}(G, F[1])$ and C be the corresponding extension of G by F. Then C is $\sigma_{s,q+}$–stable.

Proof By Corollary 1.24 and Proposition 1.29, we may assume that $\sigma_{s,q}$ is in a quiver region $\mathrm{MZ}_{\mathcal{E}}$ so that C, F and G are in the same heart $\mathcal{A}_{\mathcal{E}}[t]$ for a homological shift $t=0$ or 1. We write σ for $\sigma_{s,q}$, and σ_+ for $\sigma_{s,q+}$.

We prove the lemma by contradiction. Suppose D is a σ_+–stable subcomplex destabilizing C in $\mathcal{A}_{\mathcal{E}}[t]$. We have the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F \arrow[r] & C \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

such that the vertical maps are all injective in $\mathcal{A}_{\mathcal{E}}[t]$. Three different cases may happen:

- If $I=0$, then $\phi_{\sigma_+}(K)=\phi_{\sigma_+}(D)\geq\phi_{\sigma_+}(C)>\phi_{\sigma_+}(F)$. But F is also σ_+–stable; this leads to a contradiction.

- If $K=0$, then either $\phi_\sigma(I)<\phi_\sigma(G)=\phi_\sigma(C)$ or $I=G$. The second case, that $I=G$, is impossible since the extension is nonsplitting. In the first case, as the phase function is continuous (by the support property), we have $\phi_{\sigma_+}(I)<\phi_{\sigma_+}(C)$. Therefore the object D does not destabilize C at σ_+, which is a contradiction.

- If both K and I are nonzero, then since F and G are σ–stable,

$$\phi_\sigma(I)\leq\phi_\sigma(G)=\phi_\sigma(C) \quad \text{and} \quad \phi_\sigma(K)\leq\phi_\sigma(F)=\phi_\sigma(C).$$

When both equalities hold, we have $I=G$ and $K=F$, and in this case, $D=C$. If at least one of the equalities does not hold, then $\phi_\sigma(D)<\phi_\sigma(C)$. Again by the continuity of the phase function, we see that $\phi_{\sigma_+}(I)<\phi_{\sigma_+}(C)$ and get a contradiction. □

In general, we also need the following direct sum version, which can be proved in a similar way.

Corollary 3.2 *Let G and F be two $\sigma_{s,q}$–stable objects of the same phase. Suppose we have*

$$\phi_{\sigma_{s,q+}}(G)>\phi_{\sigma_{s,q+}}(F) \quad \text{and} \quad \mathrm{Hom}(G,F[1])=n>0.$$

Let f be a rank m map in $\mathrm{Hom}(G,F^{\oplus m}[1])$, and let C be the object extended by G and $F^{\oplus m}$ via f. Then C is $\sigma_{s,q+}$–stable.

Now we collect some geometric properties of the Le Potier curve. For an exceptional character e, by the Hirzebruch–Riemann–Roch formula, the equation for L_{e+e^r}, ie $\chi(-,e)=0$, in the $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane is

$$\mathrm{ch}_0(e)\frac{\mathrm{ch}_2}{\mathrm{ch}_0}-\big(\mathrm{ch}_1(e)+\tfrac{3}{2}\,\mathrm{ch}_0(e)\big)\frac{\mathrm{ch}_1}{\mathrm{ch}_0}+\big(\mathrm{ch}_2(e)+\tfrac{3}{2}\,\mathrm{ch}_1(e)+\mathrm{ch}_0(e)\big)=0.$$

In particular, the slope of L_{e+e^r} is $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(e)+\frac{3}{2}$. The line L_{e+e^r} is parallel to $L_{ee(3)}$ and $L_{e^l e(3)^r}$. A similar computation shows that the slope of L_{e+e^l} is $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(e)-\frac{3}{2}$.

We first want to prove the following result, which will be used to prove Lemma 3.4.

Lemma 3.3 (1) *Let e be an exceptional character, and p be a point on the line segment l_{e+e^r} (not on the boundary). Then the line L_{ep} intersects the Le Potier curve C_{LP} at two points. In addition, the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of these two points is greater than 3.*

(2) *Let u and v be two Chern characters with $\mathrm{ch}_0(u),\mathrm{ch}_0(v)>0$ on C_{LP} whose $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length is greater than 3. Then $\chi(u,v)>0$ and $\chi(v,u)>0$.*

Proof (1) We first show that L_{ep} only intersects C_{LP} at two points. Since any point on C_{LP} to the right of e^r is above the line L_{ee^r}, we only need to consider points to the left of e.

Any e'^r to the left of e that is above L_{ep} is also strictly above L_{ee^r}. Since e, e^r and $e(-3)^r$ are collinear, e' is to the left of $e(-3)$. In other words, e' satisfies the inequality

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(e')<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(e)-3.$$

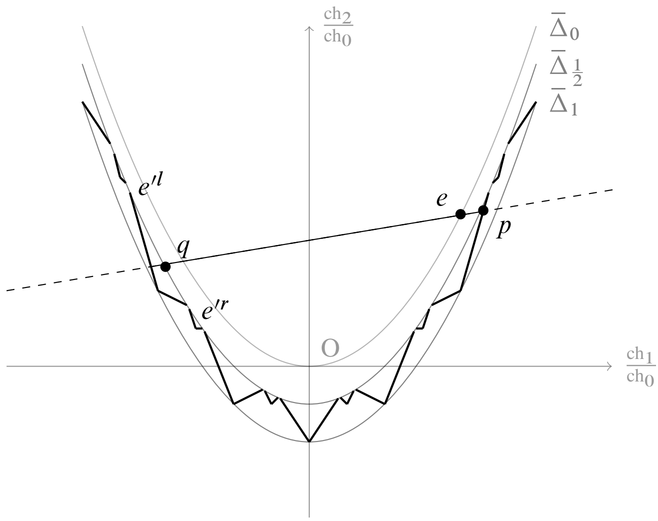

Figure 7: The intersection of L_{ep} with C_{LP}

We have that

$$\text{slope of } L_{ep} > \text{slope of } L_{ee'^l} > \text{slope of } L_{e'(3)^r e'^l} = \text{slope of } L_{e'^+ e'^r}.$$

Therefore, L_{ep} does not intersect $l_{e'^l e'^+}$ or $l_{e'^+ e'^r}$.

For any e'^l below L_{ep} to the left of e, the line segments $l_{e'^l e'^+}$ and $l_{e'^+ e'^r}$ are below $\overline{\Delta}_{\frac{1}{2}}$. The segment of $\overline{\Delta}_{\frac{1}{2}}$ between e'^l and e'^r is below L_{ep}, hence $l_{e'^l e'^+}$ and $l_{e'^+ e'^r}$ are below L_{ep}, and they do not intersect L_{ep}. Let q be the intersection point of L_{ep} and $\overline{\Delta}_{\frac{1}{2}}$ (there are two such points and we consider the one to the left of e). When q is not on any segment of $\overline{\Delta}_{\frac{1}{2}}$ between e'^l and e'^r, the intersection points of L_{ep} and C_{LP} are q and p. When q is on the segment between e'^l and e'^r for an exceptional character e', the second intersection point is either on $l_{e'^l e'^+}$ or $l_{e'^+ e'^r}$. So there is only one intersection point other than p.

The points e, e^r and $e(-3)^r$ are collinear, and the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of e^r and $e(-3)^r$ is 3. Since the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{ep} \cap \overline{\Delta}_{\frac{1}{2}}$ is increasing when p is moving from e^r to e^+, the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{ep} \cap \overline{\Delta}_{\frac{1}{2}}$ is greater than 3. Therefore, the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{ep} \cap C_{\mathrm{LP}}$ is greater than 3.

(2) Suppose u is on $\overline{\Delta}_{\frac{1}{2}}$. Then the line $\chi(u,-) = 0$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane is $L_{uu(-3)}$. Hence the point v in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane is above $L_{uu(-3)}$. As $\mathrm{ch}_0(u)$ and $\mathrm{ch}_0(v)$ are positive, $\chi(u,v) > 0$. The inequality $\chi(v,u) > 0$ is proved similarly.

Suppose u is on l_{e+e^r} for an exceptional e. We first show that $\chi(u,v)>0$. The line $\chi(u,-)=0$ passes through e, and both e^r and e^l are below the line $\chi(u,-)=0$. By the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length assumption, v is above both L_{ee^r} and L_{ee^l}. Therefore, v is above the line $\chi(u,-)=0$. Since $\mathrm{ch}_0(u)$ and $\mathrm{ch}_0(v)$ are positive, $\chi(u,v)>0$.

The line $\chi(-,u)=0$ in the $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane passes through $e(3)$, and intersects $l_{e(3)e(3)^r}$. If v is on the line segment $l_{e(3)e(3)^r}$, then by the case that u is on l_{e+e^r}, we get $\chi(v,u)>0$. If v is not on the line segment $l_{e(3)e(3)^r}$, then by the assumption on the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length, v is above the curve $\chi(-,u)=0$, so we also get $\chi(v,u)>0$.

The case that u is on l_{e+e^l} can be proved in the same way. □

Now we can state an important lemma. A similar definition also appears in [15].

Lemma 3.4 *Let u and v be Chern characters such that*

(1) *u and v are not inside the Le Potier cone;*

(2) *$\Delta(v,u)\geq 0$;*

(3) *in the $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, L_{uv} intersects C_{LP} at two points and the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length between them is greater than 3.*

Then we have

$$\chi(u,v),\ \chi(v,u)<0.$$

Remark 3.5 When both $\mathrm{ch}_0(u)$ and $\mathrm{ch}_0(v)$ are 0, the third condition does not make sense. But the statement still holds if the first two conditions hold. To see this, note that by the second condition,

$$\chi(v,w)=\chi(w,v)=-2\Delta(v,w)=-\mathrm{ch}_1(v)\,\mathrm{ch}_1(w)\leq 0.$$

Now the first condition implies that $\mathrm{ch}_1(w)$ and $\mathrm{ch}_1(v)$ are both nonzero, so

$$-\mathrm{ch}_1(v)\,\mathrm{ch}_1(w)<0.$$

Proof By the first condition, u and v are below $\overline{\Delta}_{\frac{1}{2}}$. By the third condition, let f_1 and f_2 be two characters corresponding to the intersection points of L_{uv} and C_{LP} such that $\mathrm{ch}_0(f_1)>0$, $\mathrm{ch}_0(f_2)>0$ and $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(f_1)>\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(f_2)$.

We may assume that $v=a_1f_1-a_2f_2$ and $u=b_1f_1-b_2f_2$ for some real numbers a_1, a_2, b_1 and b_2. Since u and v are not inside $\mathrm{Cone}_{\mathrm{LP}}$, we see a_1 and a_2 have the same sign (or one of them is 0) and b_1 and b_2 have the same sign (or one of them is 0). Moreover, by the second condition, we have

$$\Delta(v+au, v+au) \geq \Delta(v-au, v-au) \tag{3-1}$$

for any positive number a. Hence a_1, a_2, b_1 and b_2 all have the same sign. Without loss of generality, we may assume they are all positive.

As f_i is on C_{LP}, we have

$$\chi(f_1, f_1), \chi(f_2, f_2) \leq 0.$$

By the third condition, the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–distance of f_1 and f_2 is greater than 3. By Lemma 3.3,

$$\chi(f_1, f_2) > 0 \quad \text{and} \quad \chi(f_2, f_1) > 0.$$

Combining these results, we have

$$\begin{aligned}\chi(u, v) &\leq -b_1a_2\chi(f_1, f_2) - b_2a_1\chi(f_2, f_1) < 0,\\ \chi(v, u) &\leq -b_2a_1\chi(f_1, f_2) - b_1a_2\chi(f_2, f_1) < 0.\end{aligned} \qquad \square$$

Note that if we have stable objects A and B of characters u and v respectively, satisfying the conditions in the lemma, then the lemma implies that $\mathrm{Ext}^1(A, B) > 0$ and $\mathrm{Ext}^1(B, A) > 0$. By Lemma 3.1, this implies the existence of stable objects as extensions on both sides. This observation will be used in the proof of the last wall to show the nonemptiness of the moduli, and in the proof of the actual walls to show the existence of objects destabilized on each side of the wall.

3.2 The last wall

In this section, we describe the last wall for a given Chern character w that is not inside the Le Potier cone $\mathrm{Cone}_{\mathrm{LP}}$. By the *last wall* of w, we mean that for $P \in \overline{\Delta}_{<0}$, there is a σ_P–stable object of character w or $w[1]$ if and only if P is above the last wall. By a result from Section 3, this wall corresponds to the boundary of the effective cone. The last wall was first computed in [15] and [33] by Coskun, Huizenga and Woolf. We would like to state the result based on our setup and give a different proof. To describe the last wall for character w, we first define the exceptional bundle associated to w.

Definition 3.6 Let E be an exceptional bundle. We define $\mathfrak{R}_E$ to be *the closure of the region* bounded by $L_{e(-3)^r e e^r}$, $l_{e^r e^+}$, $l_{e^+ e^l}$ and $L_{e^l e(-3) e(-3)^l}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane; see Figure 8. Symmetrically, we define $\mathfrak{L}_E$ to be the closure of the region bounded by $L_{E(3)^l e e^l}$, $l_{e^l e^+}$, $l_{e^+ e^r}$ and $L_{e^r E(3) E(3)^r}$ in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane.

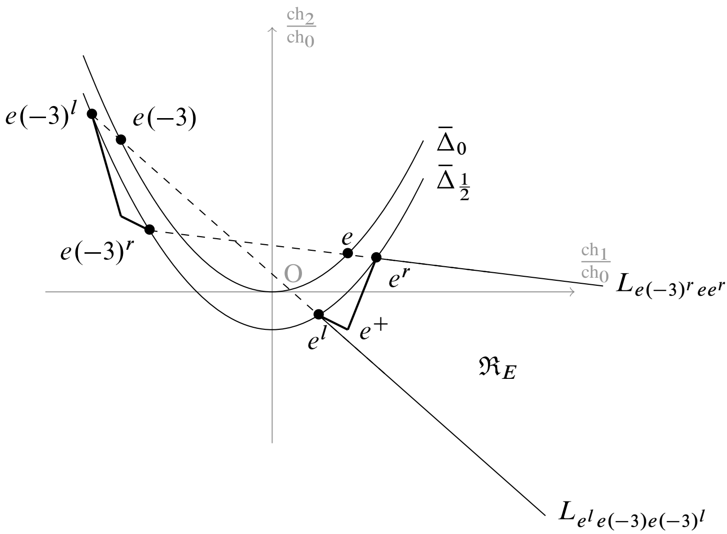

Figure 8: The region of $\mathfrak{R}_E$

The following property translates an important technical result of [15] into our setup.

Proposition 3.7 [15, Theorem 4.1] *The regions associated to the exceptional bundles cover all rational points not above the Le Potier curve:*

$$\coprod_{E \text{ exc}} \mathfrak{R}_E \supset \mathrm{P}(\mathrm{K}(\mathbb{P}^2)) \backslash \widetilde{C}_{\mathrm{LP}}.$$

A similar statement holds for $\mathfrak{L}_E$ *.*

Proof Let w be a reduced character in $\mathrm{P}(\mathrm{K}(\mathbb{P}^2))$ not above C_{LP}. There is a unique line $L_w^{\overline{\Delta}\frac{1}{2}}$ through w on the $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane which intersects $\overline{\Delta}_{\frac{1}{2}}$ at two points f_1 and f_2 , both of which are to the left of w and have a $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of 3. Let f_1 be the points with larger $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$. By Theorem 4.1 in [15], there is a unique exceptional bundle E such that on the curve $\overline{\Delta}_{\frac{1}{2}}$, the point f_1 is on the segment between e^l and e^r. For any character u on the line $L_{f_1 f_2}$, we have $\chi(f_1, u) = 0$, hence $\chi(f_1, w) = 0$. The points e^r and e^l are on the different sides of the line $\chi(-, w) = 0$, therefore

$$\chi(e^l, w) \cdot \chi(e^r, w) < 0.$$

Note that the boundary $L_{E(-3)^r E E^r}$ is the line: $\chi(e^r, -) = 0$, and the boundary $L_{E^l E(-3) E(-3)^l}$ is the line $\chi(e^l, -) = 0$. Hence, w is in $\mathfrak{R}_E$. □

Remark 3.8 It is possible to show that $L_w^{\bar{\Delta}\frac{1}{2}}$ must intersect a line segment $l_{e^l e^r}$ without using Theorem 4.1 in [15], but the argument is rather involved. The sketch of the argument is as follows:

(1) If $L_w^{\bar{\Delta}\frac{1}{2}}$ does not intersect any line segment $l_{e^l e^r}$, then for any exceptional bundle E with character below $L_w^{\bar{\Delta}\frac{1}{2}}$, by Proposition 1.30, $\mathfrak{M}^{\mathrm{s}}_\sigma(w)$ is empty for σ below L_{wE}. Hence, $\mathfrak{M}^{\mathrm{s}}_\sigma(w)$ is empty for σ below $L_w^{\bar{\Delta}\frac{1}{2}}$.

(2) By the same argument as for the last wall and Lemma 3.4, $\mathfrak{M}^{\mathrm{s}}_\sigma(w)$ is nonempty for σ on $L_w^{\bar{\Delta}\frac{1}{2}}$. This leads to the contradiction.

Thanks to this result, we can introduce the following definition, which will be related to the last wall.

Definition 3.9 Let w be a character not inside $\mathrm{Cone}_{\mathrm{LP}}$; see Definition 1.6. We define the exceptional bundle E_w associated to w to be the unique one such that $\mathfrak{R}_{E_w}$ contains w. Similarly we have the definition of $E_w^{(\mathrm{rhs})}$ according to $\mathfrak{L}_E$.

Remark 3.10 (torsion case) In the case that $\mathrm{ch}_0(w) = 0$ and $\mathrm{ch}_1(w) > 0$, we have that E_w is the unique exceptional bundle such that

$$\text{slope of } L_{e(-3)e^l} < \tfrac{\mathrm{ch}_2}{\mathrm{ch}_1}(w) < \text{ slope of } L_{ee^r}.$$

The bundle $E_w^{(\mathrm{rhs})}$ is not defined in the torsion case.

Now we can state the location of the last wall.

Definition 3.11 Let w be a character (not necessarily primitive) not inside $\mathrm{Cone}_{\mathrm{LP}}$ (it may be on the boundary but not at the origin, see Definition 1.6) and let $E = E_w$ be its associated exceptional vector bundle. We define the last wall L_w^{last} of w according to three different cases:

(1) If w is above $L_{e^+e(-3)^+}$, then $L_w^{\mathrm{last}} := L_{we}$.

(2) If w is below $L_{e^+e(-3)^+}$, then $L_w^{\mathrm{last}} := L_{we(-3)}$.

(3) If w is on $L_{e^+e(-3)^+}$, then $L_w^{\mathrm{last}} := L_{e^+e(-3)^+}$.

The last wall $L_w^{\text{right-last}}$ on the right-hand side of the vertical wall is defined similarly by using $E^{(\mathrm{rhs})}$. The torsion character does not have $E^{(\mathrm{rhs})}$ or $L^{\text{right-last}}$.

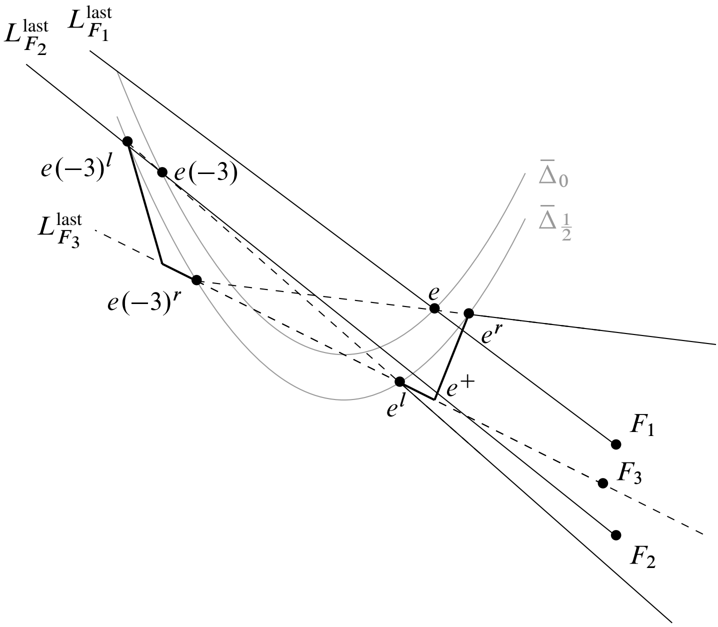

Figure 9: Three different cases of the last wall

In Figure 9, F_i shows case (i) in the definition above for $i = 1, 2, 3$.

The following lemma shows that for stability conditions below the wall L^{last}_w ($L^{\text{right-last}}_w$), there is no stable object with character w.

Lemma 3.12 *Let w be a character in $\mathrm{K}(\mathbb{P}^2)$ not inside $\mathrm{Cone}_{\mathrm{LP}}$, and σ a geometric stability condition in $\bar{\Delta}_{<0}$ below L^{last}_w or $L^{\text{right-last}}_w$. Then $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w)$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(-w)$ are both empty.*

Proof We prove the lemma in the case for L^{last}_w. The $L^{\text{right-last}}_w$ case can be proved similarly. We may assume $\mathrm{ch}_0(w) \geq 0$, since otherwise $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w)$ is empty when σ is to the left of the vertical wall $L_{w\pm}$. When w falls under case (1) or (3) in Definition 3.11, the statement follows from Proposition 1.30 directly.

When w falls under case (2) in Definition 3.11, we have $\chi(w, E_w(-3)) < 0$. For any σ–stable F with character w, the vector space $\mathrm{Hom}(F, E_w(-3)[t])$ may be nonzero only when $0 \leq t \leq 3$. Since F is in $\mathrm{Coh}_{\#s_\sigma}$, we have $\mathrm{Hom}(E_w, \mathrm{H}^{-1}(F)) = 0$. By Serre duality,

$$\hom(F, E_w(-3)[3]) = \hom(E_w, F[-1]) = \hom(E_w, \mathrm{H}^{-1}(F)) = 0.$$

On the other hand, when σ is below L^{last}_w and inside $\overline{\Delta}_{<0}$, it follows from Corollary 1.22 that $E_w(-3)[1]$ is σ–stable. By Lemma 1.20, $\phi_\sigma(E_w(-3)[1]) < \phi_\sigma(F)$. Thus, $\mathrm{Hom}(F, E_w(-3)[1]) = 0$. This contradicts the inequality $\chi(w, E_w(-3)) < 0$. □

The existence of stable objects before the last wall is more complicated. This was first proved by Coskun, Huizenga and Woolf. The authors wrote down the generic slope-stable coherent sheaves built by exceptional bundles and showed that these objects do not get destabilized before the last wall. Our approach is closer to the idea of Bayer and Macrì for K3 surfaces. We aim to show that for each wall-crossing before the last wall, new stable objects (extended by two objects) are generated on both sides, hence the moduli space is nonempty. We benefit from this approach since similar techniques can be applied in the criterion for actual walls.

Lemma 3.13 *Let w be a character in $\mathrm{K}(\mathbb{P}^2)$ with $\mathrm{ch}_0(w) > 0$ and which is not inside $\mathrm{Cone}_{\mathrm{LP}}$. Let σ be a geometric stability condition. Assume that the wall $L_{w\sigma}$ is between the vertical wall $L_{w\pm}$ and*

$$\begin{cases} L^{\mathrm{last}}_w & \text{for } w \text{ in case (1) or (2) of Definition 3.11,} \\ L_{wE_w(-3)} & \text{for } w \text{ in case (3) of Definition 3.11.} \end{cases}$$

Let $v \in \mathrm{K}(\mathbb{P}^2)$ be a character on $L_{w\sigma}$ such that $\mathrm{ch}_0(v) \geq 0$ and $\frac{\mathrm{ch}_1(v)}{\mathrm{ch}_0(v)} > \frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)}$. Then the wall $L_{v\sigma}$ is between $L_{v\pm}$ and L^{last}_v.

Proof By the definition of $\mathfrak{R}_E$ and the assumptions on $L_{w\sigma}$, the slope of $L_{w\sigma}$ is less than the slope of $L_{e_w e^r_w}$. Now $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v) > \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$ and $\mathrm{ch}_0(v) \geq 0$, so v is to the right of w in the $\left\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. Therefore, either v is in $\mathfrak{R}_{E_w}$, or $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E_v) < \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E_w)$; and L_{vE_w} is either L^{last}_v or between L^{last}_v and $L_{v\pm}$.

When w falls under case (1) of Definition 3.11, E_w is below $L_{vw\sigma}$; therefore $L_{vw\sigma}$ is between the wall L_{vE_w} and $L_{v\pm}$, and the conclusion follows.

When w falls under case (2) or (3) of Definition 3.11, v is in $\mathfrak{R}_{E_w}$ of case (3) in Definition 3.11 or E_v has slope less than E_w. In either case, $L_{vE_w(-3)}$ is either L^{last}_v or between L^{last}_v and $L_{v\pm}$. Since $E_w(-3)$ is below $L_{vw\sigma}$, it follows that $L_{vw\sigma}$ is between the wall $L_{vE_w(-3)}$ and $L_{v\pm}$, hence between the wall L^{last}_v and $L_{v\pm}$. □

Theorem 3.14 *Let w be a character in $\mathrm{K}(\mathbb{P}^2)$ not inside the Le Potier cone $\mathrm{Cone}_{\mathrm{LP}}$, and let σ be a geometric stability condition in $\overline{\Delta}_{<0}$ between L^{last}_w and $L^{\text{right-last}}_w$. When σ is not on the vertical wall $L_{w\pm}$, either $\mathfrak{M}^{\mathrm{s}}_\sigma(w)$ or $\mathfrak{M}^{\mathrm{s}}_\sigma(-w)$ is nonempty.*

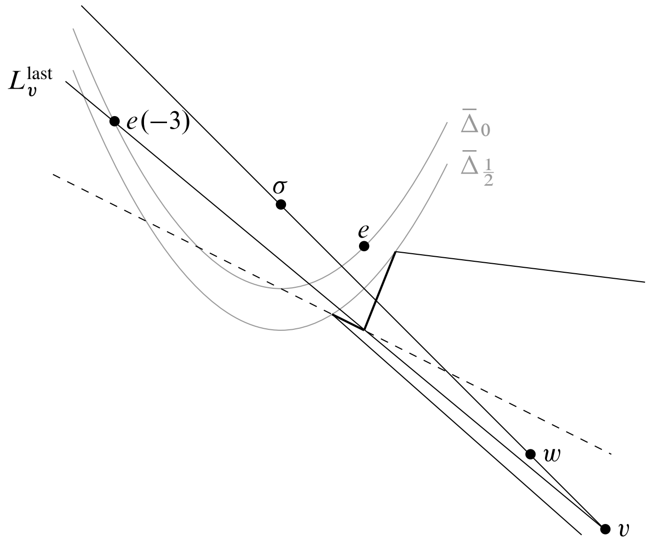

Figure 10: $L_{v\sigma}$ is between $L_{v\pm}$ and L_v^{last}

The rough idea of the proof is given at the beginning of this section. However, several different cases may happen, so the idea cannot work directly. When one of the destabilizing characters is proportional to an exceptional character, condition (1) in Lemma 3.4 fails and we need other ways to show $\chi(w', w-w') < 0$. The most complicated case is when w' is of higher rank and $L_{w-w'}^{\text{right-last}}$ is $L_{ww'}$ (Case 3.II.2 in the proof). In this case, $\mathfrak{M}_\sigma^{\text{ss}}((w'-w)[1])$ may not contain any stable objects. To deal with that, we adjust $w'-w$ to another character $\widetilde{w}$ on $L_{ww'}$ so that $\widetilde{w}$ is of positive rank and $L_{\widetilde{w}}^{\text{right-last}}$ is not $L_{ww'}$. The details of the argument are as follows.

Proof Assume the proposition does not hold. Among all the characters w not inside the Le Potier cone and such that $\mathfrak{M}_{\sigma'}^{\text{s}}(w)$ and $\mathfrak{M}_{\sigma'}^{\text{s}}(-w)$ are both empty for some σ' in $\overline{\Delta}_{<0}$ between L_w^{last} and $L_w^{\text{right-last}}$, we may choose w with the minimum discriminant Δ. We may assume that $\text{ch}_0(w) \geq 0$. When σ' is to the left of $L_{w\pm}$, the stable locus $\mathfrak{M}_{\sigma_{s,q}}^{\text{s}}(w)$ contains Gieseker–Mumford stable objects for $q \gg \frac{s^2}{2}$ and $s < \frac{\text{ch}_1(w)}{\text{ch}_0(w)}$. By Theorem 1.8, $\mathfrak{M}_{\sigma_{s,q}}^{\text{s}}(w)$ is not empty. There is a "last wall" $L_{\sigma w}$ prior to L_w^{last} such that $\mathfrak{M}_{\sigma+}^{\text{s}}(w)$ is nonempty, on the wall all objects in $\mathfrak{M}_\sigma^{\text{ss}}(w)$ are strictly semistable, and $M_{\sigma-}^{\text{ss}}(w)$ is empty. There are three main cases to treat, according to the number of exceptional characters on $L_{\sigma w}$.

Case 1 (there is no exceptional character on $L_{\sigma w}$) Let F be a σ_+–stable object of character w. Then F is destabilized by a σ–stable object G with $\widetilde{v}(G) = w'$ on the line segment $l_{\sigma w}$. Also, $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w-w')$ is not empty since it contains F/G. Since there is no exceptional character on $L_{\sigma w}$, the wall $L_{\sigma w}$ is not $L^{\text{last}}_{w-w'}$ or $L^{\text{right-last}}_{w-w'}$. By Corollary 1.33, $w-w'$ is not inside $\mathrm{Cone}_{\mathrm{LP}}$. By Lemma 3.12, $L_{\sigma w}$ is between $L^{\text{last}}_{w-w'}$ and $L^{\text{right-last}}_{w-w'}$. Corollary 3.10 in [8] implies $\Delta(w')<\Delta(w)$. By induction on Δ and the fact that $L_{\sigma(w-w')}$ is not the vertical wall, we can assume that $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w-w')$ is nonempty.

We check that the pair w' and $w-w'$ satisfies the conditions (1)–(3) in Lemma 3.4:

(1) Note that $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w-w')$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w')$ are nonempty; w' and $w-w'$ are not exceptional. By Corollary 1.33, both w' and $w-w'$ are not inside $\mathrm{Cone}_{\mathrm{LP}}$.

(2) We have that $w'+a(w-w')$ is outside the cone $\Delta_{\leq 0}$ for any $a\geq 0$. Since $L_{w\sigma}$ intersects $\overline{\Delta}_{<0}$, it follows that $w'-a(w-w')$ belongs to $\overline{\Delta}_{<0}$ for some $a>0$. Now

$$\Delta(w'-a(w-w'))=\Delta(w')+\Delta(w-w')-2a\Delta(w',w-w')<0$$

implies $\Delta(w',w-w')\geq 0$.

(3) When w is not right-orthogonal to E_w, it follows from Lemma 3.3 that the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{wE_w}\cap C_{\mathrm{LP}}$ is greater than 3. Hence, the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{w\sigma}\cap C_{\mathrm{LP}}$ is greater than 3. When w is right-orthogonal to E_w, note that w' is not in the triangle area $\mathrm{TR}_{we_we_w^+}$, otherwise $E_{w'}=E_w$ and $L_{w'\sigma}$ is to the left of $L_{w'E_w}$, by Proposition 1.30 $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w')$ is empty, and there is no σ–stable object G to destabilize F. Now since the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{wE_w}\cap C_{\mathrm{LP}}$ is greater than 3, the $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–length of $L_{w\sigma}\cap C_{\mathrm{LP}}$ is greater than 3.

Now by Lemma 3.4, we have $\chi(w',w-w')<0$. For σ–stable objects F' and F'' with characters w' and $w-w'$ respectively, and for $i\neq 0,1,2$, we have $\mathrm{Hom}(F',F''[i])=0$ since F' and F'' are in the same heart and in addition by Serre duality. These imply $\mathrm{Hom}(F',F''[1])\neq 0$. Now by Lemma 3.1, the nontrivial extension of F' by F'' is σ_-–stable, therefore $\mathfrak{M}^{\mathrm{s}}_{\sigma_-}(w)$ is nonempty, which contradicts the assumption on $L_{\sigma w}$ at the beginning.

Case 2 (there are more than two exceptional characters on $L_{\sigma w}$) This can only happen when $L_{w\sigma}$ is the line $\chi(E,-)=0$ for exceptional bundle $E=E_w$. In this case, w belongs to case (3) in Definition 3.11, and $L_{\sigma w}$ is L^{last}_{w}.

Case 3 (there are one or two exceptional characters on $L_{\sigma w}$) Similar to Case 1, we consider the character w'. We first prove the "lower-rank wall" case, ie the case $\mathrm{ch}_0(w')\leq \mathrm{ch}_0(w)$. In this case, since $\phi_{\sigma+}(w)<\phi_{\sigma+}(w-w')$, the character $w-w'$

satisfies the conditions of Lemma 3.13, therefore $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w-w')$ is nonempty by induction on Δ. We only need to show $\chi(w',w-w')<0$ so that by the same argument of the last paragraph in Case 1, $\mathfrak{M}^{\mathrm{s}}_{\sigma-}(w)$ is nonempty. If w' is not proportional to any exceptional character, then the proof for Case 1 works, and the pair w' and $w-w'$ still satisfies the conditions of Lemma 3.4. If w' is proportional to an exceptional character E, then since $L_{E(w-w')}$ is not $L^{\mathrm{last}}_{(w-w')}$, we get $\chi(E,w-w')<0$. Therefore, $\chi(E,w-E)<0$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma-}(E,w-E)$ is nonempty. This completes the argument for the lower-rank case.

Now we may assume $\mathrm{ch}_0(w')>\mathrm{ch}_0(w)$ and let $w''=w'-w$; then $\mathrm{ch}_0(w'')>0$. On the $\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, w' and w'' are in different components of $L_{w\sigma}\cap\bar{\Delta}_{\geq 0}$. If $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w''[1])$ is nonempty, then the argument for the lower-rank case still works and implies that $\mathfrak{M}^{\mathrm{s}}_{\sigma-}(w)$ is nonempty. On the other hand, by induction on Δ and Propositions 2.15 and 2.18, the semistable locus $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w''[1])$ is nonempty. So the only remaining case to consider is that $L_{w\sigma}$ is the right last wall $L^{\text{right-last}}_{w''}$ for w''.

Case 3.I (w'' is proportional to an exceptional character E, ie $w''=a\tilde{v}(E)$) Since E is to the left of E_w, we have $\chi(w,E)>0$, which implies

$$\chi(w',E)>\chi(w'',E)=a\chi(E,E)=a.$$

By Corollary 1.22, both G and $E[1]$ are σ–stable in the same heart, which implies $\mathrm{Hom}(G,E)=\mathrm{Hom}(G,(E[1])[-1])=0$. Therefore,

$$\mathrm{ext}^1(G,E[1])=\mathrm{hom}(G,E[2])\geq\chi(w',E)>a.$$

By Corollary 3.2, there exists a σ_-–stable object extended by G and $E^{\oplus a}[1]$.

Case 3.II (w'' is not proportional to any exceptional character) As $L^{\text{right-last}}_{w''}=L_{w''\sigma}$, and there are at most two exceptional characters on $L_{w''\sigma}$ by assumption, w'' does not belong to case (3) in Definition 3.11, and either $E^{(\mathrm{rhs})}_{w''}$ or $E^{(\mathrm{rhs})}_{w''}(3)$ is on the line segment $l_{ww''}$.

Case 3.II.1 (w'' belongs to (right side) case (2) in Definition 3.11, and $\tilde{v}(E^{(\mathrm{rhs})}_{w''}(3))$ is on $l_{ww''}$) The character w can be written as

$$a\tilde{v}(E^{(\mathrm{rhs})}_{w''}(3))-bw''$$

for some positive numbers a and b. Since $\chi(E^{(\mathrm{rhs})}_{w''}(3),w'')<0$, we have

$$\chi(E^{(\mathrm{rhs})}_{w''}(3),w)>0.$$

This implies $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E^{(\mathrm{rs})}_{w''}(3)) \leq \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E_w)$. As $E^{(\mathrm{rhs})}_{w''}(3)$ is above L^{last}_w, it must be E_w. On the other hand, $\chi(E_w, w) = \chi(E^{(\mathrm{rhs})}_{w''}(3), w) > 0$, so w is covered by case (1) in Definition 3.11. The wall $L_{wE_w w''}$ is just the last wall L^{last}_w of w.

Case 3.II.2 (w'' belongs to (right side) case (1) in Definition 3.11, and $E = \widetilde{v}(E^{(\mathrm{rhs})}_{w''})$ is on $l_{ww''}$) By Definition 3.11, $\chi(w'', E) > 0$. If we consider the character $\widetilde{w} := w'' - \chi(w'', E)\widetilde{v}(E)$, we have $\chi(\widetilde{w}, E) = 0$; therefore $\widetilde{w}$ is on the line $L_{e^r(e(3)^l)}$.

The character w must be above $L_{EE(3)}$, otherwise L_{wE} is the last wall L^{last}_w. The intersection $L_{wE} \cap L_{e^r(e(3)^l)}$ is outside the cone $\overline{\Delta}_{<0}$ and on a different side to w in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane. As w'', E and $\widetilde{w}$ are on the same component of $L_{wE} \cap \overline{\Delta}_{\geq 0}$, we have that $\mathrm{ch}_0(\widetilde{w})$ is greater than 0. The character $w + \widetilde{w} = w' - \chi(w'', E)\widetilde{v}(E)$ is on the line segment $l_{ww'}$.

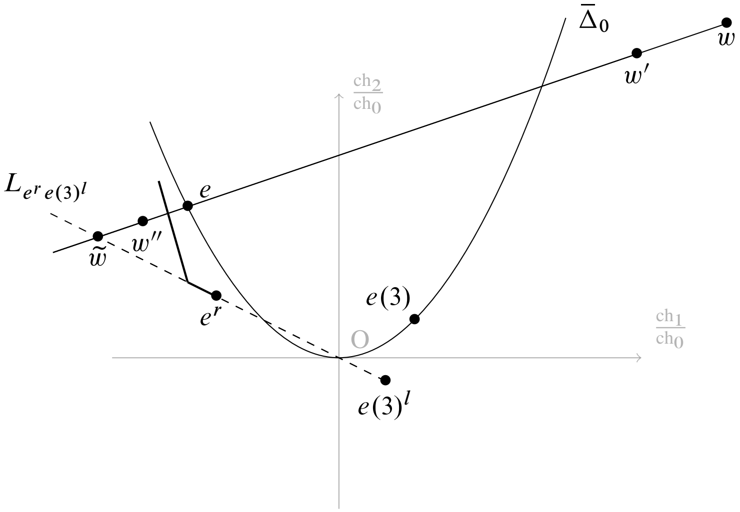

Figure 11: Definition of $\widetilde{w}$

When w' is not proportional to any exceptional character, $w + \widetilde{w}$ is outside $\mathrm{Cone}_{\mathrm{LP}}$ and on the same component of $L_{wE} \cap \overline{\Delta}_{\geq 0}$ as w. The line segment $l_{\widetilde{w}(w+\widetilde{w})}$ intersects $\overline{\Delta}_{<0}$, so $\Delta(\widetilde{w}, w + \widetilde{w}) < 0$. This implies $\Delta(\widetilde{w}) < \Delta(w)$ and $\Delta(w + \widetilde{w}) < \Delta(w)$. As $\mathfrak{M}^s_\sigma(w')$ is nonempty, by Lemma 3.13 and induction on Δ it follows $\mathfrak{M}^s_\sigma(w + \widetilde{w})$ is nonempty. $\chi(\widetilde{w}, E) = 0$ implies $L_{\widetilde{w}E}$ is not the last wall $L^{\text{right-last}}_{\widetilde{w}}$ for $\widetilde{w}$. By induction on Δ, we have that $\mathfrak{M}^s_\sigma(-\widetilde{w})$ is nonempty. The character pair $w + \widetilde{w}$ and $-\widetilde{w}$ satisfy the conditions in Lemma 3.4, hence $\chi(w + \widetilde{w}, \widetilde{w}[1]) < 0$. By Lemma 3.1, $\mathfrak{M}^s_{\sigma_-}(w)$ is nonempty.

When w' is $\widetilde{v}(E')$ for an exceptional bundle E', then since E' is to the right of $E(3)$, we have $\chi(E,E')>0$. Hence,

$$\chi(w+\widetilde{w},E')=\chi(E'-\chi(w'',E)\widetilde{v}(E),E')=1-\chi(w'',E)\chi(E,E')\leq 0.$$

This implies the characters $w+\widetilde{w}$ and $\widetilde{v}(E')$ are on two different sides of $L_{e'+e'^l}$. Therefore, $w+\widetilde{w}$ is not inside $\mathrm{Cone}_{\mathrm{LP}}$. The rest of the argument is the same as the case when w' is not proportional to an exceptional character.

We have so far finished the argument for the case that σ is on the left side of $L_{w\pm}$. When σ is on the right side of $L_{w\pm}$, the statement follows from the symmetric property $(\mathrm{ch}_0(w)>0)$

$$\mathfrak{M}^{\mathrm{s}}_{\sigma}(w)\simeq\mathfrak{M}^{\mathrm{s}}_{\sigma'}(w'[1]),\qquad F\mapsto\mathcal{RH}om(F,\mathcal{O})[1],$$

where σ' is with parameter $(-s_\sigma,q_\sigma)$ and $w'=(\mathrm{ch}_0(w),-\mathrm{ch}_1(w),\mathrm{ch}_2(w))$. □

3.3 The criterion for actual walls

In this section we give a numerical criterion for actual walls of a given Chern character. In the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane, the actual wall for w is the potential wall $L_{w\sigma}$ where new stable objects are produced on both sides and curves are contracted on at least one side. When σ is to the left of the vertical wall $L_{w\pm}$, one can always choose a destabilizing factor v with positive rank and smaller slope. As $\Delta(v)$ is less than $\Delta(w)$, there are finitely many candidates v. By checking the positions of v and $v-w$ on the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane, which are purely numerical data, Theorem 3.16 determines whether $L_{w\sigma}$ is an actual wall induced by this pair. The idea of the proof is very similar to that of the last wall: We first show there are stable objects on the wall with characters v and $w-v$ by Theorem 3.14. We then argue that the Ext^1 of the stable objects is nonzero by Lemma 3.4, and finally claim that curves must be contracted from the σ_+–side wall-crossing.

To state the criterion for actual walls, we first need to introduce the following definition.

Definition 3.15 For a Chern character w with $\mathrm{ch}_0(w)\geq 0$ and an exceptional character e, we define the *triangle* TR_{we} to be the triangle region bounded by lines L_{we}, $L_{e^le^+}$ and $L_{e^+e^r}$ in the $\left\{1,\frac{\mathrm{ch}_1}{\mathrm{ch}_0},\frac{\mathrm{ch}_2}{\mathrm{ch}_0}\right\}$–plane. See Figure 12.

Now we can state the main theorem on actual walls. The regions TR_{wE} will be used to detect the nonemptiness of moduli spaces of stable objects of any "subcharacter", as will be explained in the proof of the theorem.

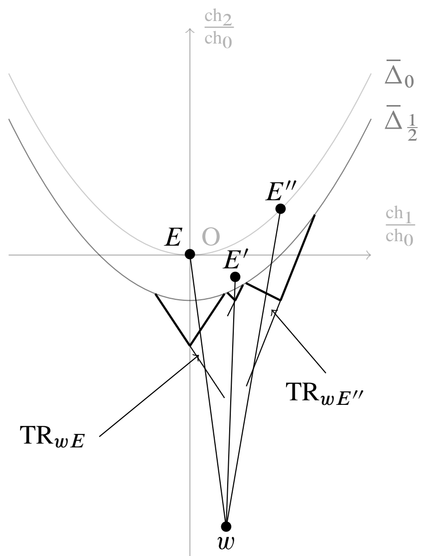

Figure 12: Definition of TR_{wE}

Theorem 3.16 *Let $w \in \mathrm{K}(\mathbb{P}^2)$ be a Chern character outside the Le Potier cone with $\mathrm{ch}_0(w) \geq 0$. For any stability condition $\sigma_{s,q}$ in $\overline{\Delta}_{<0}$ between the wall L_w^{last} and the vertical ray L_{w+}, the wall $L_{\sigma w}$ is an actual wall for w if and only if there exists a Chern character $v \in \mathrm{K}(\mathbb{P}^2)$ on the line segment $l_{\sigma w}$ such that*

- $\mathrm{ch}_0(v) > 0$ *and* $\frac{\mathrm{ch}_1(v)}{\mathrm{ch}_0(v)} < \frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)}$;
- *the characters v and $w-v$ are either proportional to exceptional characters or not inside the Le Potier cone and both of them are not in TR_{wE} for any exceptional bundle E.*

Remark 3.17 (1) For given characters w and v, one only needs to check whether v or $w-v$ are in TR_{wE} for at most two particular exceptional bundles. Suppose the intersection points $L_{\sigma w} \cap \overline{\Delta}_{\frac{1}{2}}$ fall between the segment between e_i^r and e_i^l for some exceptional character e_1 and e_2. Then one only needs to check the triangles TR_{wE_i}.

(2) By the term "in TR_{wE}", strictly speaking, we mean "in the closure of TR_{wE} but not on the line L_{e+e^l} when E is not to the right of E_w (or not on the line L_{e+e^r} when E is to the left of E_w)".

Proof The first step is to translate the second condition as nonemptiness of moduli spaces of stable objects of the characters v and $v-w$.

Lemma 3.18 *When v (or $w-v$) is not inside the Le Potier cone, the condition*

$$\text{“}v \text{ (or } w-v\text{) is not in } \mathrm{TR}_{wE} \text{ for any exceptional } E\text{”}$$

is equivalent to

$$\text{“}\mathfrak{M}^{\mathrm{s}}_{\sigma}(v) \text{ (or } \mathfrak{M}^{\mathrm{s}}_{\sigma}(w-v)\text{) is nonempty for } \sigma \text{ in } \bar{\Delta}_{<0} \text{ on the line } L_{wv}\text{”.}$$

Proof The $\Leftarrow$ direction is easy to check: Suppose v is in TR_{wE} for some E. Then E must be E_w or to the right of E_w. This implies $l_{\sigma w}$ intersects $l_{e^+e^r}$. The character v is in $\mathfrak{R}_E$ and belongs to case (1) in Definition 3.11. By Proposition 1.30, $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v)$ is empty. The $w-v$ part is proved in a similar way.

For the $\Rightarrow$ direction, let f_1 and f_2 be the intersection points of the line L_{vw} and the parabola $\bar{\Delta}_{\frac{1}{2}}$. Suppose that f_1 has larger $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$, and f_1 lies on the segment between e^l and e^r for some exceptional bundle E by Theorem 4.1 in [15]. Since v is below L_{ee^r}, it follows that E_v is either E or to the left of E.

Three different cases may happen:

(1) If v is above $L_{e^+e^l}$, then $E_v = E$. Since v is not in TR_{wE}, it follows that v is above L_{Ew}. This implies that E is below L_{vw}, and $L_{v\sigma}$ is between L^{last}_v and $L_{v\pm}$.

(2) If $E \neq E_w$ and v is not above $L_{e^+e^l}$, then w is below the line $L_{e(-3)e^l}$ and $E(-3)$ is below L_{wv}. Hence, $L_{v\sigma}$ is between $L_{vE(-3)}$ and $L_{v\pm}$.

(3) If $E = E_w$ and v is not above the line $L_{e^le^+}$, then by Remark 3.17, v is above $L^{\mathrm{last}}_w = L_{E(-3)w}$. Now w is below $L^{\mathrm{last}}_v = L_{E(-3)v}$, therefore $L_{wv\sigma}$ is between L^{last}_v and $L_{v\pm}$.

In any case, $L_{v\sigma}$ is between L^{last}_v and $L_{v\pm}$. It follows from Theorem 3.14 that $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v)$ is nonempty for any $\sigma \in L_{v\sigma} \cap \bar{\Delta}_{\leq 0}$.

Write u for $w-v$. When $\mathrm{ch}_0(u) \geq 0$, it follows from Lemma 3.13 and a similar argument as for v that $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(u)$ is nonempty for any $\sigma \in L_{vw} \cap \bar{\Delta}_{\leq 0}$. If $\mathrm{ch}_0(u) < 0$, let E be the exceptional bundle such that f_2 lies on the segment of $\bar{\Delta}_{\frac{1}{2}}$ between $x = e^l$ and $x = e^r$. By a similar argument as for v, when u is above $L_{e^+e^l}$, then L_{uw} is between $L^{\text{right-last}}_u = L_{uE}$ and $L_{i\pm}$. When u is not above $L_{e^+e^l}$, then L_{uw} is between $L_{uE(3)}$ and $L_{u\pm}$. By Theorem 3.14, $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(u)$ is nonempty for any $\sigma \in L_{vw} \cap \bar{\Delta}_{\leq 0}$. □

The second step is to prove the "only if" direction of the statement, which follows from Lemma 3.18 in the first step. If $L_{\sigma w}$ is an actual wall, an object F with character w is destabilized by a stable object with character v such that $\mathrm{ch}_0(v) > 0$ and $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v) < \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$. By Corollary 1.33, v is exceptional or outside $\mathrm{Cone}_{\mathrm{LP}}$. By the previous discussion, since $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v)$ is not empty, v is not in any TR_{wE}. For the character $w-v$, since $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w-v)$ is nonempty we only need to consider the case when all semistable objects are strictly semistable. Since we may assume that $v-w$ is not proportional to any exceptional character and $\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w-v)$ is nonempty, by Propositions 2.15 and 2.18 and Theorem 3.14 $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w-v)=\varnothing$ if and only if $L_{w\sigma}$ is $L^{\text{right-last}}_{w-v}$ or L^{last}_{w-v}. The second case is not possible by Lemma 3.13. We may assume $\mathrm{ch}_0(v) > \mathrm{ch}_0(w)$. Now $v-w$ belongs to case (1) in Definition 3.11, since otherwise $v-w$ is not in $\mathrm{TR}_{E^{(\mathrm{rhs})}_{v-w}w}$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(w-v)$ is not empty. Write $E^{(\mathrm{rhs})}_{v-w}$ as E, and let

$$v' := v-\chi(v-w,E)\cdot e.$$

Then $w-v' = w-v+\chi(v-w,E)\cdot e$. Since $\chi(w-v',E)=0$, the character $w-v'$ is the intersection point of $L_{w\sigma}$ and L_{e+e^r} and is not in TR_{wE}. By the same argument as in Case 3.II.2 of the proof of Theorem 3.14, we have inequalities $\mathrm{ch}_0(v')>0$ and $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v') < \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$, and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v')$ is nonempty. Therefore, the pair v' and $w-v'$ satisfies the requirements in the statement.

The last step is to prove the "if" direction in the statement. Similar to the proof for the last wall, objects with characters v and $u=w-v$ do not always have nontrivial extensions. We need to build Chern characters u' and v' on the line L_{vw} so that

- $w=v'+u'$;
- $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v')$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(u')$ are nonempty for $\sigma\in L_{vw}\cap\bar{\Delta}_{<0}$;
- $\mathfrak{M}^{\mathrm{s}}_{\sigma_+}(v',u')\to\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$ contracts curves.

Four cases may happen for u and v:

(i) (v and u are not proportional to any exceptional characters; in other words, they are not inside $\mathrm{Cone}_{\mathrm{LP}}$) Since they are not in the triangles TR_{wE}, it follows that $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v)$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(u)$ are nonempty. The characters v and u satisfy the conditions in Lemma 3.4 due to the same argument as for Case 1 in the proof of Theorem 3.14. This implies $\chi(v,u)<0$. By the first property of Lemma 2.17 and the same computation as in Proposition 2.18, $\chi(u,v)-\chi(v,u)\le -3$. Therefore, by Lemma 2.9, for any σ–stable objects F and G in $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v)$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(u)$, we have $\mathrm{ext}^1(G,F)\ge 3$. By Lemma 3.1, $\mathfrak{M}^{\mathrm{s}}_{\sigma_+}(F,G)\to\mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$ contracts curves.

(ii) (v is proportional to the character e of some exceptional bundle E, but u is not proportional to any exceptional characters) Write $v = ne$ for some integer $n \geq 1$. When $\mathrm{ch}_0(w) \geq \mathrm{ch}_0(e)$, the character $u' = u + (n-1)e$ is to the right of w. Therefore u' is not in TR_{wE} and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(u')$ is nonempty. Now $v' = w - u' = e$ and $\mathfrak{M}^{\mathrm{s}}_{\sigma}(e)$ is nonempty.

In the case $\mathrm{ch}_0(e) > \mathrm{ch}_0(w)$, we have

$$\frac{\mathrm{ch}_1(u)}{\mathrm{ch}_0(u)} = \frac{\mathrm{ch}_1(w-ne)}{\mathrm{ch}_0(w-ne)} > \frac{\mathrm{ch}_1(w-e)}{\mathrm{ch}_0(w-e)} = \frac{\mathrm{ch}_1(u')}{\mathrm{ch}_0(u')}.$$

Now u is outside $\mathrm{Cone}_{\mathrm{LP}}$ and on a different component of $L_{w\sigma} \cap \overline{\Delta}_{>0}$ than that of w, so u' is also outside $\mathrm{Cone}_{\mathrm{LP}}$ and not in any TR_{wE}. We may still let v' be e.

L_{ew} is not the last wall L^{last}_{w}, so $\chi(e, w) \leq 0$. We have $\chi(e, u') \leq -1$. By the same argument as in (i), we have $\mathrm{ext}^1(G, E) \geq 3$ for any object G in $\mathfrak{M}^{\mathrm{s}}_{\sigma}(u')$. Therefore, $\mathfrak{M}^{\mathrm{s}}_{\sigma+}(E, G) \to \mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$ contracts curves.

(iii) (u is proportional to the character e of some exceptional bundle E, but v is not proportional to any exceptional characters) As u is not on the line segment $l_{\sigma w}$, it has negative ch_0. Suppose $u = -ne$; we may let $v' = w + e$ and $u' = -e$ in a similar way as in (ii). The same argument on the slope of v and v' shows v' is outside any triangle TR_{wE}. As L_{we} is not the last wall L^{last}_{w}, we have $\chi(w, e) \geq 0$. Therefore $\chi(v', u') \leq -1$. By the same argument as in (i), we have $\mathrm{ext}^1(E, F) \geq 3$ for any object F in $\mathfrak{M}^{\mathrm{s}}_{\sigma}(v')$. Therefore, $\mathfrak{M}^{\mathrm{s}}_{\sigma+}(E, G) \to \mathfrak{M}^{\mathrm{ss}}_{\sigma}(w)$ contracts curves.

(iv) (u and v are proportional to the characters e_1 and e_2 of exceptional bundles E_1 and E_2, respectively) Write $u = -n_1 e_1$ and $v = n_2 e_2$. Since L_{vw} is not L^{last}_{w}, we have $\chi(e_2, w) \leq 0$. Therefore,

$$n_2 \leq n_1 \chi(e_2, e_1) = n_1 \mathrm{ext}^2(E_2, E_1) = n_1 \hom(E_1, E_2(-3)) < n_1 \hom(E_1, E_2).$$

As a consequence, we see that

$$\begin{aligned}\dim \mathfrak{M}^{\mathrm{s}}_{\sigma-}(E_1^{\oplus n_1}[1], E_2^{\oplus n_2}) &= \dim \mathrm{Kr}_{\hom(E_1,E_2)}(n_1, n_2)\\ &= n_1 n_2 \hom(E_1, E_2) - n_1^2 - n_2^2 + 1 \geq 2.\end{aligned}$$

Here $\mathrm{Kr}_{\hom(E_1,E_2)}(n_1, n_2)$ is the Kronecker model, ie the representation space

$$\mathrm{Hom}(\mathbb{C}^{n_2}, \mathbb{C}^{n_1})^{\oplus \hom(E_1,E_2)}$$

quotiented by the natural group action of $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)/\mathbb{C}^*$.

In all cases, L_{vw} is an actual wall for w. $\square$

Now the following corollary follows easily:

Corollary 3.19 (lower-rank walls) *Let w be a character with $\mathrm{ch}_0(w) \geq 0$. For any character v with $0 < \mathrm{ch}_0(v) \leq \mathrm{ch}_0(w)$, suppose that v is between the wall L_w^{last} and the vertical ray L_{w+}, outside the Le Potier cone $\mathrm{Cone}_{\mathrm{LP}}$, and not in TR_{wE} for any exceptional bundle E. Then L_{vw} is an actual wall.*

4 Applications: the ample cone and the movable cone

In this section, we work out several applications of our criterion on actual walls. We compute the boundary of the movable cone in Section 4.1 and the boundary of the nef cone in Section 4.2. In Section 4.3, we compute all the actual walls of the moduli space of stable sheaves of character $(4, 0, -15)$, as an example of how to apply the machinery of this paper in a concrete situation.

4.1 Movable cone

Let $w \in \mathrm{K}(\mathbb{P}^2)$ be a character with $\mathrm{ch}_0(w) \geq 0$ not inside $\mathrm{Cone}_{\mathrm{LP}}$. It was revealed in [15] that when σ is in the "last" chamber above L_w^{last}, the birational model $\mathfrak{M}^{\mathrm{s}}_\sigma(w)$ has Picard number 1 if and only if w is right-orthogonal to E_w. In other words, the movable cone boundary on the primary side is not the same as the effective cone boundary if and only if $\chi(E_w, w) = 0$. In this section, we determine the boundary of the movable cone in this case.

Let $(E_\alpha, E_\gamma, E_\beta)$ be a triple of exceptional bundles corresponding to dyadic numbers $\frac{p-1}{2^n}$, $\frac{p}{2^n}$ and $\frac{p+1}{2^n}$, respectively. The following property is well-known; the reader is referred to [20].

Lemma 4.1 *For the triple $(E_\alpha, E_\gamma, E_\beta)$, we have*

$$\chi(E_\alpha, E_\gamma) = \hom(E_\alpha, E_\gamma) = 3\,\mathrm{ch}_0(E_\beta),$$
$$\chi(E_\gamma, E_\beta) = \hom(E_\gamma, E_\beta) = 3\,\mathrm{ch}_0(E_\alpha),$$
$$\hom(E_\alpha, E_\gamma)\cdot\hom(E_\gamma, E_\beta) - \hom(E_\alpha, E_\beta) = 3\,\mathrm{ch}_0(E_\gamma),$$
$$\hom(E_\beta(-3), E_\alpha)\cdot\hom(E_\alpha, E_\gamma) - \hom(E_\beta(-3), E_\gamma) = 3\,\mathrm{ch}_0(E_\alpha).$$

For any exceptional $E_{(\frac{t}{2^q})}$ such that $\frac{p-1}{2^n} < \frac{t}{2^q} < \frac{p}{2^n}$, we have $\mathrm{ch}_0(E_{(\frac{t}{2^q})}) < \mathrm{ch}_0(E_\gamma)$.

The following observation is from the proof for Theorem 3.14. It will be used in the next theorem.

Lemma 4.2 *Let $w \in \mathrm{K}(\mathbb{P}^2)$ be a Chern character not inside $\mathrm{Cone}_{\mathrm{LP}}$. Let e be an exceptional character such that, in the $\{1, \frac{\mathrm{ch}_1}{\mathrm{ch}_0}, \frac{\mathrm{ch}_2}{\mathrm{ch}_0}\}$–plane, w is in the area between two parallel lines $L_{ee(3)}$ and $L_{e^r e^+}$. Then we have $|\mathrm{ch}_0(w)| > \mathrm{ch}_0(E)$.*

Let w be a primitive character outside $\mathrm{Cone}_{\mathrm{LP}}$ with $\mathrm{ch}_0(w) \geq 0$. Assume that w is right-orthogonal to the exceptional bundle $E_w = E_\gamma$, and consider the triple $(E_\alpha, E_\gamma, E_\beta)$ corresponding to dyadic numbers $\frac{p-1}{2^n}$, $\frac{p}{2^n}$ and $\frac{p+1}{2^n}$. The character w can be uniquely written as $n_2 e_\alpha - n_1 e_{\beta-3}$ for positive numbers n_1 and n_2.

Theorem 4.3 *Adopting notation as above, we may define a character P based on two different cases:*

(i) $P := e_\gamma - (3\,\mathrm{ch}_0(E_\beta) - n_2)e_\alpha$ *if* $1 \leq n_2 < 3\,\mathrm{ch}_0(E_\beta)$,

(ii) $P := e_\gamma$ *if* $n_2 \geq 3\,\mathrm{ch}_0(E_\beta)$.

On the wall L_{Pw}, a divisor of $\mathfrak{M}^{\mathrm{s}}_{L_{Pw_+}}(w)$ is contracted.

Proof In order to apply Theorem 3.16, we first show that in case (i), P is not inside $\mathrm{Cone}_{\mathrm{LP}}$, and is outside TR_{wE} for any exceptional bundle E. Now $\chi(e_\beta, P) = 0$, so P is on the line $L_{e^+_\beta e^l_\beta}$. By Lemma 4.1,

$$\begin{aligned}\chi(P, P) =& 1 + (3\,\mathrm{ch}_0(E_\beta) - n_2)^2 - (3\,\mathrm{ch}_0(E_\beta) - n_2)\chi(E_\alpha, E_\gamma) \\ =& 1 - n_2(3\,\mathrm{ch}_0(E_\beta) - n_2) \leq 0.\end{aligned}$$

Since P is on the line $L_{e^+_\beta e^l_\beta}$, it is not inside $\mathrm{Cone}_{\mathrm{LP}}$ and is outside TR_{wE} for any exceptional bundle E.

We next show that $w - P$ is outside TR_{wE} for any exceptional bundle E. By Lemma 3.13, we only need to treat the case when $\mathrm{ch}_0(w-P) < 0$. We are going to prove that for any exceptional bundle E to the left of $E_\gamma(-3)$, if E is above L_{Pw}, then $\chi(P - w, E) \leq 0$. This will imply that $w - P$ is not in TR_{wE}.

In case (i), we first treat the case of the exceptional bundle E to the left of $e_\alpha(-3)$. Note that

$$P - w = n_1 e_\beta(-3) - (3\,\mathrm{ch}_0(E_\beta) \cdot e_\alpha - e_\gamma),$$

so by Lemma 4.1 and Serre duality,

$$\chi(3\,\mathrm{ch}_0(E_\beta)e_\alpha - e_\gamma, e_\alpha(-3)) = 3\,\mathrm{ch}_0(E_\beta) - \hom(e_\alpha, e_\gamma) = 0.$$

Since $\chi(e_\beta(-3), e_\alpha(-3))$ is also 0, the point $P-w$ is on the line $L_{e_\alpha(-3)+e_\alpha(-3)^r}$. Note that $\mathrm{ch}_0(w)\geq 0$, and we have $n_2\,\mathrm{ch}_0(e_\alpha)\geq n_1\,\mathrm{ch}_0(e_\beta)$. Hence

$$n_1 \leq \frac{\mathrm{ch}_0(e_\alpha)}{\mathrm{ch}_0(e_\beta)}\cdot n_2 < 3\,\mathrm{ch}_0(e_\alpha).$$

By the equations in Lemma 4.1,

$$\begin{aligned}\chi(P-w, P-w) &= n_1^2+1-n_1\chi(e_\beta(-3), 3\,\mathrm{ch}_0(E_\beta)\cdot e_\alpha - e_\gamma)\\ &= n_1^2+1-3\,\mathrm{ch}_0(e_\alpha)\cdot n_1 < 0.\end{aligned}$$

Combining with the result that $P-w$ is on the line $L_{e_\alpha(-3)+e_\alpha(-3)^r}$, we know that $P-w$ is not above the curve $\mathrm{C_{LP}}$, and for any exceptional bundle E to the left of $e_\alpha(-3)$, we have

$$\chi(P-w, E)\leq 0.$$

Now we treat the case of the exceptional character e between $e_\alpha(-3)$ and e_γ. The line segment $l_{(P-w)P}$ is above the line $L_{(P-w)e_\gamma}$, therefore it is above the line segment $l_{e_\alpha(-3)^r e_\gamma}$. Since $l_{e_\alpha(-3)^r e_\gamma}$ is above any exceptional characters between the vertical rays $L_{e_\alpha(-3)\pm}$ and $L_{e_\gamma\pm}$, the character $P-w$ is not in the triangle TR_{wE} for any such exceptional bundle E.

In case (ii), the character $P-w$ can be rewritten as

$$\begin{aligned}P-w = e_\gamma - w &= n_1 e_{\beta-3} - (n_2-3\,\mathrm{ch}_0(E_\beta))e_\alpha - (3\,\mathrm{ch}_0(E_\beta)\cdot e_\alpha - e_\gamma)\\ &= (n_2-3\,\mathrm{ch}_0(E_\beta))\Big(\frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)}e_{\beta-3}-e_\alpha\Big)\\ &\qquad + \Big(n_1 - n_2\frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)} + 3\,\mathrm{ch}_0(E_\alpha)\Big)e_{\beta-3} - (3\,\mathrm{ch}_0(E_\beta)e_\alpha - e_\gamma).\end{aligned}$$

Note that the character $\frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)}e_{\beta-3}-e_\alpha$ in the first term is proportional to $e_\gamma(-3)-e_\gamma$ by a positive scalar, and the coefficient $n_2-3\,\mathrm{ch}_0(E_\beta)$ is nonnegative. We denote the remaining terms by

$$v' := \Big(n_1 - n_2\frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)} + 3\,\mathrm{ch}_0(E_\alpha)\Big)e_{\beta-3} - (3\,\mathrm{ch}_0(E_\beta)e_\alpha - e_\gamma).$$

By Lemma 4.1 and the assumption, $\mathrm{ch}_0(v') = \mathrm{ch}_0(P-w) > 0$. In particular,

$$n_1 - n_2\frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)} + 3\,\mathrm{ch}_0(E_\alpha) > 0.$$

Since $\mathrm{ch}_0(w) > 0$, we have the inequality

$$n_1 - n_2 \frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)} + 3\,\mathrm{ch}_0(E_\alpha) < 3\,\mathrm{ch}_0(E_\alpha).$$

By a similar computation as in case (i),

$$\begin{aligned}\chi(v',v') &< \Big(n_1 - n_2 \frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)} + 3\,\mathrm{ch}_0(E_\alpha)\Big)^2 + 1 \\ &\qquad - \Big(n_1 - n_2 \frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)} + 3\,\mathrm{ch}_0(E_\alpha)\Big)\chi(e_{\beta-3}, 3\,\mathrm{ch}_0(E_\beta)e_\alpha - e_\gamma) \\ &< (3\,\mathrm{ch}_0(E_\alpha))\Big(\Big(n_1 - n_2 \frac{\mathrm{ch}_0(E_\alpha)}{\mathrm{ch}_0(E_\beta)} + 3\,\mathrm{ch}_0(E_\alpha)\Big) - 3\,\mathrm{ch}_0(E_\alpha)\Big) + 1 \\ &< 0.\end{aligned}$$

Note that v' is on the line $L_{e_\alpha(-3)+e_\alpha(-3)^r}$, and v' is not above $\mathrm{C_{LP}}$. Since v' is to the left of $E_\beta(-3)$, after moving along the direction $e_\gamma(-3) - e_\gamma$ we have that $v' + a(e_\gamma(-3) - e_\gamma)$ is still not above $\mathrm{C_{LP}}$ or $L_{e_\alpha(-3)+e_\alpha(-3)^r}$. Therefore, $P - w$ is not above those two curves. It is not in TR_{wE} for any E to the left of $E_\alpha(-3)$.

For any exceptional e between $L_{e_\alpha(-3)\pm}$ and $L_{e_\gamma(-3)\pm}$, by the assumption we have $\mathrm{ch}_0(P - w) \le \mathrm{ch}_0(e_\gamma) < \mathrm{ch}_0(e)$. By Lemma 4.2, $P - w$ is not in the area between L_{e+e^r} and $L_{ee(3)}$. Since w is above $L_{ee(3)}$, it follows that $P - w$ is not in TR_{wE} for any exceptional E between $L_{e_\alpha(-3)\pm}$ and $L_{e_\gamma(-3)\pm}$.

The line segment $l_{(P-w)P}$ is above the character $e_\gamma(-3)$, hence it is above the line segment $l_{e_\gamma(-3)^r e_\gamma}$. Since $l_{e_\gamma(-3)^r e_\gamma}$ is above any exceptional character between the vertical rays $L_{e_\alpha(-3)\pm}$ and $L_{e_\gamma\pm}$, the character $P - w$ is not in the triangle TR_{wE} for any such exceptional bundle E.

This finishes the proof of the claim that $w - P$ is outside TR_{wE} for any exceptional bundle E. By Theorem 3.16, we know that L_{Pw} is an actual wall.

The last step is to show that a divisor of $\mathfrak{M}^{\mathrm{s}}_{L_{Pw}+}(w)$ is contracted at L_{Pw}. By Proposition 2.15 and Theorem 3.14, for $\sigma \in L_{Pw}$ we have

$$\begin{aligned}\dim \mathfrak{M}^{\mathrm{s}}_{\sigma+}(w - P) &= 1 - \chi(P - w, P - w), \\ \dim \mathfrak{M}^{\mathrm{s}}_{\sigma+}(P) &= 1 - \chi(P, P).\end{aligned}$$

By the previous argument, they are both nonnegative.

By Lemmas 2.9 and 3.1,

$$\begin{aligned}\dim \mathfrak{M}^{s}_{\sigma+}(w-P,P) &= \dim \mathfrak{M}^{s}_{\sigma+}(w-P) + \dim \mathfrak{M}^{s}_{\sigma+}(P) + \operatorname{ext}^1(w-P,P) - 1\\ &= 1-\chi(w-P,w-P)-\chi(P,P)-\chi(w-P,P)\\ &= 1-\chi(w,w)+\chi(P,w-P)\\ &= \dim \mathfrak{M}^{s}_{\sigma+}(w)+\chi(P,w-P).\end{aligned}$$

So it suffices to show that $\chi(P,w-P)=-1$. This is clear in case (ii):

$$\chi(P,w-P)=\chi(e_\gamma,w-e_\gamma)=-\chi(e_\gamma,e_\gamma)=-1.$$

In case (i),

$$\begin{aligned}\chi(P,w-P) &= -\chi(P,P)+\chi(P,w)\\ &= -\chi(P,P)-\chi((3\operatorname{ch}_0(E_\beta)-n_2)e_\alpha,w)\\ &= -\chi(P,P)-(3\operatorname{ch}_0(E_\beta)-n_2)\cdot n_2\chi(e_\alpha,e_\alpha)\\ &\qquad\qquad + (3\operatorname{ch}_0(E_\beta)-n_2)\cdot n_1\chi(e_\alpha,e_{\beta-3})\\ &= -\chi(e_\gamma,e_\gamma)-(3\operatorname{ch}_0(E_\beta)-n_2)^2\\ &\qquad + (3\operatorname{ch}_0(E_\beta)-n_2)\big(\chi(e_\alpha,e_\gamma)+\chi(e_\gamma,e_\alpha)\big)-(3\operatorname{ch}_0(E_\beta)-n_2)n_2\\ &= -1-(3\operatorname{ch}_0(E_\beta)-n_2)^2+(3\operatorname{ch}_0(E_\beta)-n_2)\cdot 3\operatorname{ch}_0(E_\beta)\\ &\qquad\qquad -(3\operatorname{ch}_0(E_\beta)-n_2)n_2\\ &= -1.\end{aligned}$$

□

4.2 Nef cone

In this section, we study the boundary of the nef cone of the moduli space $\mathfrak{M}^{ss}_{GM}(w)$. Due to Theorem 2.24, this is the first actual wall to the left of the vertical wall $L_{w\pm}$. We assume that the character w is primitive, $\operatorname{ch}_0(w)>0$ and $\frac{\operatorname{ch}_1(w)}{\operatorname{ch}_0(w)}\in(-1,0]$. The following lemma gives a first bound for the boundary of the nef cone.

Lemma 4.4 *Suppose $\overline{\Delta}(w)\geq 2$. Then $L_{\mathcal{O}(-1)w}$ is an actual wall for w.*

Proof By Corollary 3.19 and Theorem 3.14, we need to show that w is below the line $L_{\mathcal{O}(-1)\mathcal{O}(-1)^r}$.

The point $\mathcal{O}(-1)^r$ is the intersection of $\overline{\Delta}_{\frac{1}{2}}$ and $L_{\mathcal{O}\mathcal{O}(-1)}$, so in the $\{1,\frac{\operatorname{ch}_1}{\operatorname{ch}_0},\frac{\operatorname{ch}_2}{\operatorname{ch}_0}\}$–plane, its coordinates are

$$\Big(1,\frac{\operatorname{ch}_1}{\operatorname{ch}_0},\frac{\operatorname{ch}_2}{\operatorname{ch}_0}\Big)=\Big(1,\tfrac{1}{2}(1-\sqrt{5}),\tfrac{1}{4}(1-\sqrt{5})\Big).$$

Let P be the intersection point of $L_{\mathcal{O}(-1)\mathcal{O}(-1)^r}$ and $L_{\mathcal{O}\pm}$. The function $\overline{\Delta}$ on the line segment $l_{\mathcal{O}(-1)P}$ reaches its maximum at $P = \left(1, 0, -\frac{1}{2}(1+\sqrt{5})\right)$, and $\overline{\Delta}_P = \frac{1}{2}(1+\sqrt{5}) < 2$. Therefore, w is below the line $L_{\mathcal{O}(-1)\mathcal{O}(-1)^r}$. □

When $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w) \in (k-1, k]$ for some integer k and $\overline{\Delta}(w) \geq 2$, by the lemma $L_{\mathcal{O}(k-1)w}$ is an actual wall.

Lemma 4.5 *Suppose that* $\overline{\Delta}(w) \geq 10$. *Then the first lower-rank wall* L_{vw} *with* $\mathrm{ch}_0(v) \leq \mathrm{ch}_0(w)$ *is given by the character* v *satisfying the following two conditions:*

- $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)$ *is the greatest rational number less than* $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$ *with* $\mathrm{ch}_0(v) \leq \mathrm{ch}_0(w)$.
- *Given the first condition, if* $\mathrm{ch}_1(v)$ *is even (resp. odd), then* $\mathrm{ch}_2(v)$ *is the greatest integer (resp.* $2\,\mathrm{ch}_2(v)$ *is the greatest odd integer) such that the point* v *is either an exceptional character or not inside* $\mathrm{Cone}_{\mathrm{LP}}$.

Proof We may assume that $-1 < \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w) \leq 0$. Note that the slopes of $L_{e^l e^+}$ and $L_{e^r e^+}$ for any exceptional object with $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(e)$ in $[-1, 0]$ are at least $-\frac{5}{2}$.

We first show that there is no actual wall with lower rank between L_{vw} and $L_{w\pm}$. Suppose that there is a character v' with $\mathrm{ch}_0(v') \leq \mathrm{ch}_0(w)$ and $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v') < \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$, such that $L_{v'w}$ is an actual wall between L_{vw} and $L_{w\pm}$. By the previous lemma, we may assume that $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v') \geq -1$. Since v' is either an exceptional character or below C_{LP}, by the assumptions on v we have

$$\frac{\mathrm{ch}_2(v)}{\mathrm{ch}_0(v)} - \frac{\mathrm{ch}_2(v')}{\mathrm{ch}_0(v')} \geq -\frac{1}{\mathrm{ch}_0(v)} + \frac{5}{2}\left(\frac{\mathrm{ch}_1(v')}{\mathrm{ch}_0(v')} - \frac{\mathrm{ch}_1(v)}{\mathrm{ch}_0(v)}\right) - \frac{1}{\mathrm{ch}_0(v')^2}.$$

The coefficient $\frac{5}{2}$ of the second term is with respect to the minimum slope of the Le Potier curve. The last term is for the case that v' is exceptional:

$$\frac{\mathrm{ch}_2(e)}{\mathrm{ch}_0(e)} - \frac{\mathrm{ch}_2(e^+)}{\mathrm{ch}_0(e^+)} = \frac{1}{\mathrm{ch}_0(e)^2}.$$

This inequality holds as otherwise $v - (0,0,1)$ will be below C_{LP} with smaller $\mathrm{ch}_2(v)$.

Write $d_\chi := -\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v') + \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)$ for simplicity. Then

$$\frac{1}{\mathrm{ch}_0(v)\,\mathrm{ch}_0(v')} \leq d_\chi \leq 1.$$

Since $L_{v'w}$ is between L_{vw} and $L_{w\pm}$, in other words, v is below $l_{v'w}$, we have the inequality

$$\begin{aligned}\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(v')-\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(w) &\le \Big(\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(v')-\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(v)\Big)\frac{\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)-\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v')}{d_\chi}\\ &\le \Big(\frac{1}{\mathrm{ch}_0(v)}+\frac{1}{\mathrm{ch}_0(v')^2}+\frac{5}{2}d_\chi\Big)\Big(1+\frac{1}{d_\chi}\Big(\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)-\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)\Big)\Big)\\ &\le 1+1+\frac{5}{2}+\frac{1}{d_\chi}\Big(\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)-\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)\Big)\Big(\frac{1}{\mathrm{ch}_0(v)}+\frac{1}{\mathrm{ch}_0(v')^2}+\frac{5}{2}d_\chi\Big)\\ &\le \frac{9}{2}+\frac{1}{\mathrm{ch}_0(w)}\Big(\frac{1}{d_\chi}\Big(\frac{1}{\mathrm{ch}_0(v)}+\frac{1}{\mathrm{ch}_0(v')}\Big)+\frac{5}{2}\Big)\\ &\le \frac{9}{2}+1+1+\frac{5}{2}=9.\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{\Delta}_w=\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(w)+\frac{1}{2}\Big(\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(w)\Big)^2 &\le -\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(w)+\frac{1}{2}\Big(\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(v')\Big)^2\\ &=-\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(w)+\frac{\mathrm{ch}_2}{\mathrm{ch}_0}(v')+\overline{\Delta}_{v'}<9+1=10,\end{aligned}$$

which contradicts our assumption.

We next show that L_{vw} is an actual wall. By Corollary 3.19, it suffices to prove that v is not in TR_{wE} for any exceptional bundle E such that $-1\le\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E)\le 0$. Suppose that v is in TR_{wE} for such an exceptional bundle E. Now $\overline{\Delta}_w\ge 10$, so the slope of L_{wE} is less than -9. The $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–width of TR_{wE} is less than

$$\frac{\text{length of } l_{ee^+}}{9-\frac{5}{2}}<\frac{1}{6\,\mathrm{ch}_0(E)^2}.$$

Hence if v is in TR_{wE}, then

$$\frac{1}{\mathrm{ch}_0(E)\,\mathrm{ch}_0(v)}\le\frac{\mathrm{ch}_1(v)}{\mathrm{ch}_0(v)}-\frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(E)}<\frac{1}{6\,\mathrm{ch}_0(E)^2}.$$

In this way, $\mathrm{ch}_0(w)\ge\mathrm{ch}_0(v)>6\,\mathrm{ch}_0(E)$. In particular, $\mathrm{ch}_0(E)\le\mathrm{ch}_0(w)$. Note that L_{wE} becomes a lower-rank wall between L_{wv} and $L_{w\pm}$. By Corollary 3.19, L_{wE} is an actual wall. But this is not possible by the argument in the first part. Therefore, v is not in TR_{wE} for any exceptional bundle E. □

Now we can describe the boundary of the nef cone.

Theorem 4.6 *Let w be a primitive character with $\mathrm{ch}_0(w) > 0$ and $\overline{\Delta}_w \geq 10$. The first actual wall for $\mathfrak{M}^{s}_{\mathrm{GM}}(w)$ is given by L_{vw}, where v is the character defined in Lemma 4.5.*

Proof We may assume that $\frac{\mathrm{ch}_1(w)}{\mathrm{ch}_0(w)} \in (-1, 0]$, and by Lemma 4.5, we only need to show that any higher-rank actual wall is not between L_{vw} and $L_{w\pm}$. Let v' be a character satisfying the properties in Theorem 3.16 with $\mathrm{ch}_0(v') = \mathrm{ch}_0(w) + r$ for some positive integer r.

The slope of L_{vw} is less than

$$\frac{\overline{\Delta}_w - 1}{\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w) - \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)} < -9\,\mathrm{ch}_0(w).$$

So the left intersection point of $L_{vw} \cap \overline{\Delta}_0$ has $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–coordinate less than $-9\,\mathrm{ch}_0(w)$ (the slope of $\overline{\Delta}_0$ at that point is less than $-9\,\mathrm{ch}_0(w)$. Since $v' - w$ is to the left of this point, we get the inequality

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v' - w) < -9\,\mathrm{ch}_0(w),$$

hence

$$r < \frac{1}{9} \cdot \frac{\mathrm{ch}_1(w) - \mathrm{ch}_1(v')}{\mathrm{ch}_0(w)}.$$

By Section 4.2, we have $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v') > -1$, so

$$\mathrm{ch}_1(v') > -\mathrm{ch}_0(w) - r.$$

Therefore,

$$r < \frac{1}{9} \cdot \frac{\mathrm{ch}_1(w) + \mathrm{ch}_0(w) + r}{\mathrm{ch}_0(w)} \leq \frac{1}{9} \cdot \frac{\mathrm{ch}_0(w) + r}{\mathrm{ch}_0(w)} \leq \frac{1}{9} + \frac{r}{9}.$$

This leads to a contradiction since $r < 1$ and cannot be a positive integer. □

4.3 A concrete example

In this section, we apply the criterion for actual walls and compute the stable base locus/actual walls on the primitive side for the moduli space of stable objects of character $w = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2) = (4, 0, -15)$.

We first compute the last wall of w. The equation of $L_w^{\overline{\Delta}\frac{1}{2}}$ is given by

$$\frac{\mathrm{ch}_2}{\mathrm{ch}_0} + \frac{\sqrt{35}}{2}\frac{\mathrm{ch}_1}{\mathrm{ch}_0} + \frac{15}{4} = 0.$$

The $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}$–coordinates of the intersection points $L_w^{\bar{\Delta}\frac{1}{2}} \cap \bar{\Delta}_{\frac{1}{2}}$ are $-\frac{\sqrt{35}}{2} \pm \frac{3}{2}$. The larger one is approximately -1.458 and the intersection point falls in the segment between e^l and e^r, for the exceptional bundle $E_{(\frac{-3}{2})}$, which is the cotangent bundle Ω.

By the Hirzebruch–Riemann–Roch formula in the proof for Lemma 2.17,

$$\chi(\Omega, w) = \chi\big((2,-3,\tfrac{3}{2}),(4,0,-15)\big) = -30+6+18+8 = 2 > 0.$$

Therefore, w is above the line $L_{e^l e^+}$, and belongs to case (1) in Definition 3.11. The last wall of w is given by $L_{\Omega w}$ with equation

$$\frac{\mathrm{ch}_2}{\mathrm{ch}_0} + 3\frac{\mathrm{ch}_1}{\mathrm{ch}_0} + \frac{15}{4} = 0.$$

We now compute all the lower-rank walls. By Corollary 3.19, we only need determine all characters $v \in K(\mathbb{P}^2)$ such that

- $0 < \mathrm{ch}_0(v) \le \mathrm{ch}_0(w) = 4$ and $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v) < \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w) = 0$;
- v is between $L_{w\pm}$ and L_w^{last};
- v is exceptional or not inside $\mathrm{Cone}_{\mathrm{LP}}$;
- v is not in TR_{wE} for any exceptional character E.

If $\mathrm{ch}_0(v)$ is 1, then $\mathrm{ch}_1(v)$ can only be -1, and v is either

$$\big(1,-1,\tfrac{1}{2}\big) \quad \text{or} \quad \big(1,-1,-\tfrac{1}{2}\big).$$

If $\mathrm{ch}_0(v)$ is 2, then $\mathrm{ch}_1(v)$ can be -1 or -2, and v is one of

$$\big(2,-1,-\tfrac{1}{2}\big), \quad \big(2,-1,-\tfrac{3}{2}\big), \quad \big(2,-1,-\tfrac{5}{2}\big), \quad \big(2,-1,-\tfrac{7}{2}\big), \quad (2,-2,-1).$$

When $\mathrm{ch}_0(v)$ is 3, then $\mathrm{ch}_1(v)$ can be -1, -2, -3 or -4, and v is one of

- $\big(3,-1,\frac{-2n-1}{2}\big)$ for $n = 1,\dots,7$,
- $(3,-2,-n)$ for $n = 1,\dots,5$,
- $\big(3,-3,-\frac{3}{2}\big)$, $(3,-4,1)$.

When $\mathrm{ch}_0(v)$ is 4, we have $-5 \le \mathrm{ch}_1(v) \le -1$, and v is one of

- $\big(4,-1,\frac{-2n-1}{2}\big)$ for $n = 2,\dots,11$,
- $(4,-2,-n)$ for $n = 2,\dots,8$,
- $\big(4,-3,\frac{-2n-1}{2}\big)$ for $n = 1,\dots,5$,
- $(4,-4,-2)$, $\big(4,-5,\frac{1}{2}\big)$.

The nef boundary of $\mathfrak{M}^s_{\mathrm{GM}}(w)$ is the wall $L_{w(4,-1,-\frac{5}{2})}$.

We now compute the characters that are contained in TR_{wE} for some exceptional E. By Lemma 4.2, we only need consider the exceptional bundles $\mathcal{O}(-1)$ and $\Omega(1)$. The equations for the three edges of $\mathrm{TR}_{w\mathcal{O}(-1)}$ are

$$\frac{\mathrm{ch}_2}{\mathrm{ch}_0}-\frac{1}{2}\frac{\mathrm{ch}_1}{\mathrm{ch}_0}=0,\qquad \frac{\mathrm{ch}_2}{\mathrm{ch}_0}+\frac{5}{2}\frac{\mathrm{ch}_1}{\mathrm{ch}_0}+3=0,\qquad \frac{\mathrm{ch}_2}{\mathrm{ch}_0}+\frac{17}{4}\frac{\mathrm{ch}_1}{\mathrm{ch}_0}+\frac{15}{4}=0.$$

By a direct computation, the characters $(3,-2,3)$, $\left(4,-3,-\frac{5}{2}\right)$ and $\left(4,-3,-\frac{7}{2}\right)$ are in $\mathrm{TR}_{w\mathcal{O}(-1)}$. The equations for the three edges of $\mathrm{TR}_{w\Omega(1)}$ are

$$\frac{\mathrm{ch}_2}{\mathrm{ch}_0}-\frac{\mathrm{ch}_1}{\mathrm{ch}_0}=0,\qquad \frac{\mathrm{ch}_2}{\mathrm{ch}_0}+2\frac{\mathrm{ch}_1}{\mathrm{ch}_0}+\frac{3}{2}=0,\qquad \frac{\mathrm{ch}_2}{\mathrm{ch}_0}+7\frac{\mathrm{ch}_1}{\mathrm{ch}_0}+\frac{15}{4}=0.$$

The coordinates of the vertices are $\left(1,-\frac{1}{2},-\frac{1}{2}\right)$, $\left(1,-\frac{9}{20},-\frac{3}{5}\right)$ and $\left(1,-\frac{15}{32},-\frac{15}{32}\right)$. Since $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)$ is not in $\left(-\frac{1}{2},-\frac{9}{20}\right)$ for any v, there is no v in $\mathrm{TR}_{w\Omega(1)}$.

To find the higher-rank walls, we first compute a bound for $\mathrm{ch}_0(v)$.

$$L_w^{\mathrm{last}}\cap\overline{\Delta}_{\le 0}=\left\{\left(1,-3+\sqrt{\tfrac{3}{2}},\tfrac{1}{2}\left(-3+\sqrt{\tfrac{3}{2}}\right)^2\right),\left(1,-3-\sqrt{\tfrac{3}{2}},\tfrac{1}{2}\left(-3-\sqrt{\tfrac{3}{2}}\right)^2\right)\right\}.$$

Let $v\in K(\mathbb{P}^2)$ be a character such that

- $\mathrm{ch}_0(v)>\mathrm{ch}_0(w)=4$ and $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)=0$;
- v is between $L_{w\pm}$ and L_w^{last};
- v and $v-w$ are exceptional or not inside $\mathrm{Cone}_{\mathrm{LP}}$;
- v and $v-w$ are not in TR_{wE} for any exceptional character E.

Since v and $v-w$ are on different components of $L_{vw}\cap\overline{\Delta}_{\ge 0}$, we have the inequalities

$$\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)\ge -3+\sqrt{\tfrac{3}{2}},\qquad \frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v-u)\le -3-\sqrt{\tfrac{3}{2}}.$$

Therefore,

$$\left(-3+\sqrt{\tfrac{3}{2}}\right)\mathrm{ch}_0(v)\le \mathrm{ch}_1(v)\le\left(-3-\sqrt{\tfrac{3}{2}}\right)(\mathrm{ch}_0(v)-4). \tag{4-1}$$

We get a bound $\mathrm{ch}_0(v)\le 2+2\sqrt{6}<7$. When $\mathrm{ch}_0(v)$ is 6, by (4-1) we have

$$\mathrm{ch}_1(v)\le -2\left(-3-\sqrt{\tfrac{3}{2}}\right)<-8.$$

Therefore, $\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(v)\le -9=\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(E_w)$, which is not possible.

When $\mathrm{ch}_0(v)$ is 5, by (4-1), $\mathrm{ch}_1(v)$ can be -5, -6 or -7 and v is one of the characters

$$\left(5,-5,-\tfrac{7}{2}\right),\qquad (5,-6,0),\qquad \left(5,-7,\tfrac{5}{2}\right).$$

These characters v and $w-v$ are not contained in TR_{wE} for any exceptional E. Combining Theorems 2.24 and 3.16, we may draw the stable base locus decomposition walls in the divisor cone of $\mathfrak{M}^{s}_{\mathrm{GM}}(w)$ as shown in Figure 13.

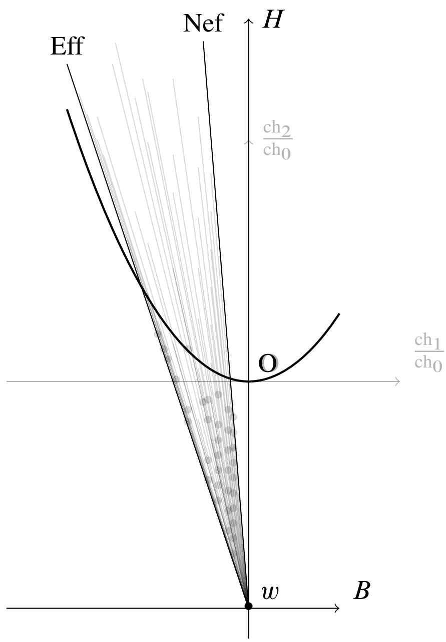

Figure 13: The stable base locus decomposition of the effective cone of $\mathfrak{M}^{s}_{\mathrm{GM}}(4,0,-15)$

Appendix Correspondence between wall-crossing and MMP

In this appendix, we sketch a different proof of Theorem B. As we are using existing results in the literature, we only outline the main idea and necessary changes. For the readers who are mainly interested in the correspondence between MMP and wall-crossing, rather than the criterion on actual walls, we believe that this proof will be conceptually easier to follow.

We start with a sketch of a different proof of Theorem 2.19, combining several existing results.

Theorem A.1 *Let w be a primitive character such that $\mathrm{ch}_0(w)>0$. For a generic geometric stability condition $\sigma=\sigma_{s,q}$ with $s<\frac{\mathrm{ch}_1}{\mathrm{ch}_0}(w)$ not on any actual wall of w, the moduli space $\mathfrak{M}^{ss}_{\sigma}(w)$ is irreducible and smooth.*

Proof The smoothness is proved in general for Poisson surfaces in [25]. Note that this proof does not make use of exceptional collections, instead it is based on a rather elementary geometric observation due to Bayer.

The irreducibility can be proved using a standard argument due to Mukai. Here we refer to Proposition A.7 in [4] and Theorem 4.1 in [22] for details. Although these papers were for K3 surfaces, the methods apply to any Poisson surfaces, as long as the moduli of stable objects has a projective, smooth component. The only necessary change is to the formula at the bottom of page 608 of [22]. The correct formula is explicitly given in Remark 2 of [30], and the rest of the argument goes through. □

It was shown in Theorem 1.1 of [9] that the wall-crossing induces a directed MMP of the moduli space. Note that the technical condition on generic points of exceptional loci being sheaves is used in Lemma 5.1 of [9] to ensure that in the case of a small contraction, the birational change on the other side is also a small contraction. This holds in general as a consequence of Proposition 2.18, so this technical condition can be removed. Also, as the moduli space is smooth and irreducible, we do not need to take the normalization of the major component as in Theorem 1.1 of [9]. This concludes the correspondence.

We close by emphasizing that the results in Section 2 cannot be replaced by the argument in this appendix. There are mainly two reasons for this. First, as we have seen, in order to remove some technical conditions, we still need the estimate on ext groups in Section 2.4, and the result here is weaker than Theorem B. The approach in Section 2 provides a more detailed structure of the wall-crossing, and relates it to variation of GIT. Second, many results in Section 2 are used in Section 3 in an essential way to prove the criterion on actual walls. In fact, such a criterion makes the wall-crossing process computable, and is crucial to most applications of this theory.

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