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Additive invariants of orbifolds

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Using the recent theory of noncommutative motives, we compute the additive invariants of orbifolds (equipped with a sheaf of Azumaya algebras) using solely "fixed-point data". As a consequence, we recover, in a unified and conceptual way, the original results of Vistoli concerning algebraic K–theory, of Baranovsky concerning cyclic homology, of the second author and Polishchuk concerning Hochschild homology, and of Baranovsky and Petrov, and Căldăraru and Arinkin (unpublished), concerning twisted Hochschild homology; in the case of topological Hochschild homology and periodic topological cyclic homology, the aforementioned computation is new in the literature. As an application, we verify Grothendieck's standard conjectures of type C^+ and D, as well as Voevodsky's smash-nilpotence conjecture, in the case of "low-dimensional" orbifolds. Finally, we establish a result of independent interest concerning nilpotency in the Grothendieck ring of an orbifold.

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1 Introduction

A *differential graded (dg) category* $\mathcal{A}$, over a base field k, is a category enriched over complexes of k–vector spaces; see Section 2.1. Let us denote by $\mathrm{dgcat}(k)$ the category of (essentially small) k–linear dg categories. Every (dg) k–algebra A gives rise naturally to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasicompact quasiseparated k–scheme X (or, more generally, suitable algebraic stack $\mathcal{X}$) admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; consult Keller [31, Section 4.6] and Lunts and Orlov [39]. Moreover, the tensor product $-\otimes_X -$ makes $\mathrm{perf}_{\mathrm{dg}}(X)$ into a commutative monoid in the category obtained from $\mathrm{dgcat}(k)$ by inverting the Morita equivalences; see Schnürer [45].

An *additive invariant* of dg categories is a functor $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an idempotent complete additive category, which inverts Morita equivalences and sends semiorthogonal decompositions to direct sums; consult Section 2.2 for further details. As explained there, examples include several variants of algebraic K–theory, of cyclic homology and of topological Hochschild homology. Given a k–scheme X (or stack) as above, let us write $E(X)$ instead of $E(\mathrm{perf}_{\mathrm{dg}}(X))$. Note that if E is lax (symmetric) monoidal, then $E(X)$ is a (commutative) monoid in D.

Now, let G be a finite group acting on a smooth separated k–scheme X. In what follows, we will write $[X/G]$ and $X/\!\!/G$ for the associated (global) orbifold and geometric quotient, respectively. The results of this article may be divided into three parts:

(i) **Decomposition of orbifolds** We establish some formulas for $E([X/G])$ in terms of fixed-point data $\{E(X^g)\}_{g\in G}$; consult Theorem 1.1 and Corollaries 1.5 and 1.8. In the particular case where E is algebraic K–theory or cyclic homology, these formulas reduce to previous results of Vistoli and Baranovsky, respectively.

(ii) **Smooth quotients** We prove that if $X/\!\!/G$ is smooth, then $E(X/\!\!/G) \simeq E(X)^G$; consult Theorem 1.22. In the particular case where E is Hochschild homology, this reduces to a previous result of the second author with Polishchuk.

(iii) **Equivariant Azumaya algebras** We extend the formulas of the above part (i) to the case where X is equipped with a G–equivariant sheaf of Azumaya algebras; consult Theorem 1.25 and Corollaries 1.28, 1.34 and 1.36. In the particular case where E is Hochschild homology, these formulas reduce to an earlier result of Baranovsky and Petrov, and Căldăraru and Arinkin (unpublished).

As an application of the formulas established in part (i), we verify Grothendieck's standard conjectures of type C^+ and D, as well as Voevodsky's smash-nilpotence conjecture, in the case of "low-dimensional" orbifolds; consult Theorem 9.2.

Statement of results

Let k be a base field of characteristic $p \geq 0$, G a finite group of order n, φ the set of all cyclic subgroups of G, and $\varphi/\sim$ a (chosen) set of representatives of the conjugacy classes in φ. Given a cyclic subgroup $\sigma \in \varphi$, we will write $N(\sigma)$ for the normalizer of σ. Throughout the article, we will assume that $1/n \in k$; we will *not* assume that k contains the n^{th} roots of unity.

Let X be a smooth separated k–scheme equipped with a G–action; we will *not* assume that X is quasiprojective. As above, we will write $[X/G]$ and $X/\!\!/G$ for the associated (global) orbifold and geometric quotient, respectively.

Decomposition of orbifolds Let $R(G)$ be the representation ring of G. As explained in Section 3, the assignment $[V] \mapsto V \otimes_k -$, where V stands for a G–representation, gives rise to an action of $R(G)$ on $E([X/G])$ for every additive invariant E.

Given a cyclic subgroup $\sigma \in \varphi$, recall from Definition 4.4 below that the $\mathbb{Z}[1/n]$–linearized representation ring $R(\sigma)_{1/n}$ comes equipped with a certain canonical idempotent e_σ. As explained in Definition 4.4, e_σ can be characterized as the maximal idempotent whose image under all the restrictions $R(\sigma)_{1/n} \to R(\sigma')_{1/n}, \sigma' \subsetneq \sigma$, is zero. Whenever an object $\mathcal{O}$ (of an idempotent complete category) is equipped with an $R(\sigma)_{1/n}$–action, we will write $\widetilde{\mathcal{O}}$ for the direct summand $e_\sigma \mathcal{O}$. In particular, $\widetilde{R}(\sigma)_{1/n}$ stands for the direct summand $e_\sigma R(\sigma)_{1/n}$.

Let $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$ be an additive invariant with values in a $\mathbb{Z}[1/n]$–linear category. On the one hand, as mentioned above, we have a canonical $R(\sigma)_{1/n}$–action on $E([X^\sigma/\sigma])$. On the other hand, $N(\sigma)$ acts naturally on $[X^\sigma/\sigma]$ and hence on $E([X^\sigma/\sigma])$. By functoriality, the former action is compatible with the latter. Consequently, we obtain an induced $N(\sigma)$–action on $\widetilde{E}([X^\sigma/\sigma])$. Under the above notations, our first main result is the following:

Theorem 1.1 *For every additive invariant $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$, with values in a $\mathbb{Z}[1/n]$–linear category, we have an isomorphism*

$$E([X/G]) \simeq \bigoplus_{\sigma \in \varphi/\sim} \widetilde{E}([X^\sigma/\sigma])^{N(\sigma)} \tag{1.2}$$

induced by pullback with respect to the morphisms $[X^\sigma/\sigma] \to [X/G]$. Moreover, if E is lax monoidal, then (1.2) is an isomorphism of monoids.

Remark 1.3 (i) Theorem 1.1 follows from a similar isomorphism in a certain category of G_0–motives over $[X/G]$; consult Theorem 5.13 below for details.

(ii) The right-hand side of (1.2) may be rewritten as $\big(\bigoplus_{\sigma\in\varphi} \widetilde{E}([X^\sigma/\sigma])\big)^G$. In this way, the isomorphism (1.2) does not depend on any choices.

(iii) Note that σ acts trivially on X^σ. Therefore, by combining Proposition 2.11 below with Tabuada and Van den Bergh [51, Lemma 4.26], we conclude that the dg category $\mathrm{perf}_{\mathrm{dg}}([X^\sigma/\sigma])$ is Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(X^\sigma \times \mathrm{Spec}(k[\sigma]))$. This implies that the above isomorphism (1.2) can be rewritten as

$$E([X/G]) \simeq \bigoplus_{\sigma\in\varphi/\sim} \widetilde{E}(X^\sigma \times \mathrm{Spec}(k[\sigma]))^{N(\sigma)}. \tag{1.4}$$

Intuitively speaking, (1.4) shows that every additive invariant of orbifolds can be computed using solely "ordinary" schemes. Unfortunately, when E is lax monoidal the isomorphism (1.4) obscures the monoid structure on $E([X/G])$.

Given a commutative ring R and an R–linear idempotent complete additive category D, let us write $-\otimes_R -$ for the canonical action of the category of finitely generated projective R–modules on D. Under some mild assumptions, Theorem 1.1 admits the following refinements:

Corollary 1.5 (i) *If k contains the n^{th} roots of unity, then (1.2) reduces to*

$$E([X/G]) \simeq \bigoplus_{\sigma\in\varphi/\sim} (E(X^\sigma) \otimes_{\mathbb{Z}[1/n]} \widetilde{R}(\sigma)_{1/n})^{N(\sigma)}. \tag{1.6}$$

(ii) *If k contains the n^{th} roots of unity and D is l–linear for a field l which contains the n^{th} roots of unity and $1/n \in l$, then (1.6) reduces to an isomorphism*

$$E([X/G]) \simeq \bigoplus_{g\in G/\sim} E(X^g)^{C(g)} \simeq \Big(\bigoplus_{g\in G} E(X^g)\Big)^G, \tag{1.7}$$

where $C(g)$ stands for the centralizer of g.

Moreover, if E is lax monoidal, then (1.6)–(1.7) are isomorphisms of monoids.

Corollary 1.8 *If E is monoidal, then (1.2) reduces to an isomorphism of monoids*

$$E([X/G]) \simeq \bigoplus_{\sigma\in\varphi/\sim} (E(X^\sigma)\otimes \widetilde{E}(B\sigma))^{N(\sigma)},$$

where $B\sigma := [\bullet/\sigma]$ stands for the classifying stack of σ.

Example 1.9 (algebraic K–theory) As mentioned in Example 2.2 below, algebraic K–theory is a lax symmetric monoidal additive invariant. Therefore, in the case where k contains the n^{th} roots of unity, Corollary 1.5(i) leads to the following isomorphism of $\mathbb{Z}$–graded commutative $\mathbb{Z}[1/n]$–algebras:[1]

$$K_*([X/G])_{1/n} \simeq \bigoplus_{\sigma\in\varphi/\sim} (K_*(X^\sigma)_{1/n}\otimes_{\mathbb{Z}[1/n]} \widetilde{R}(\sigma)_{1/n})^{N(\sigma)}. \tag{1.10}$$

Vistoli established the formula (1.10) in [54, Theorem 1] under the weaker assumptions that X is regular, Noetherian and of finite Krull dimension, but under the additional assumption that X carries an ample line bundle.[2] Moreover, since he did not assume that $1/n\in k$, he excluded from the direct sum the (conjugacy classes of) cyclic subgroups whose order is divisible by p. Note that Corollary 1.5(i) enables us to upgrade (1.10) to an homotopy equivalence of spectra.

Example 1.11 (mixed complex) As mentioned in Example 2.4 below, the mixed complex C is a symmetric monoidal additive invariant. Therefore, in the case where k contains the n^{th} roots of unity, Corollary 1.5(ii) leads to the isomorphism of commutative monoids in the derived category of mixed complexes

$$\mathsf{C}([X/G]) \simeq \Big(\bigoplus_{g\in G} \mathsf{C}(X^g)\Big)^G. \tag{1.12}$$

Note that since C is compatible with base change and the morphism (1.12), induced by pullback, is defined over k, the assumption that k contains the n^{th} roots of unity can be removed! Baranovsky established the above formula (1.12) in [4, Theorem 1.1 and Proposition 3.1] under the additional assumption that X is quasiprojective. Moreover, he used G–coinvariants instead of G–invariants. Since $1/n\in k$, G–coinvariants and G–invariants are canonically isomorphic. Hence, the difference between (1.12) and

[1] As Nick Kuhn kindly informed us, in the topological setting the above formula (1.10) goes back to the pioneering work of tom Dieck [15, Section 7.7]; see also Kuhn [37, Section 6].

[2] The assumption that X carries an ample line bundle can be removed without affecting the validity of Vistoli's results; consult Remark 5.15 below.

Baranovsky's formula is only "cosmetic"; see also Arinkin, Caldararu and Hablicsek [3, Corollary 1.17(2)]. The advantage of G–invariants over G–coinvariants is that the former construction preserves monoid structures.

Remark 1.13 (orbifold cohomology) Cyclic homology and its variants factor through the complex C. Consequently, a formula similar to (1.12) holds for all these invariants. For example, when $k=\mathbb{C}$ and E is periodic cyclic homology HP_*, the Hochschild–Kostant–Rosenberg theorem leads to an isomorphism[3] of $\mathbb{Z}/2$–graded $\mathbb{C}$–vector spaces

$$\mathrm{HP}_*([X/G]) \simeq \Big(\bigoplus_{g\in G} H^*(X^g,\mathbb{C})\Big)^G =: H^*_{\mathrm{orb}}(X/\!\!/G,\mathbb{C}), \tag{1.14}$$

where the right-hand side stands for orbifold cohomology in the sense of Chen and Ruan [12]. This is Baranovsky's [4] beautiful observation "periodic cyclic homology equals orbifold cohomology". In Corollary 1.36 below, we will extend the isomorphism (1.14) to the case where X is equipped to a G–equivariant sheaf of Azumaya algebras.

Example 1.15 (topological Hochschild homology) As explained in Section 2.7, topological Hochschild homology $\mathrm{THH}_*(-)$ is a lax symmetric monoidal additive invariant with values in the category of $\mathbb{Z}$–graded k–vector spaces. Therefore, in the case where k contains the n^{th} roots of unity, Corollary 1.5(ii) leads to the isomorphism of $\mathbb{Z}$–graded commutative k–algebras

$$\mathrm{THH}_*([X/G]) \simeq \Big(\bigoplus_{g\in G} \mathrm{THH}_*(X^g)\Big)^G. \tag{1.16}$$

To the best of the authors' knowledge, the formula (1.16) is new in the literature.

Let l be the field obtained from k by adjoining the n^{th} roots of unity. Note that by combining Proposition 2.16 below with the Morita equivalences

$$\mathrm{perf}_{\mathrm{dg}}([X_l/G]) \simeq \mathrm{perf}_{\mathrm{dg}}([X/G]) \otimes_k l \, \mathrm{perf}_{\mathrm{dg}}(X^g_l) \simeq \mathrm{perf}_{\mathrm{dg}}(X^g) \otimes_k l,$$

we can remove the assumption that k contains the n^{th} roots of unity!

Example 1.17 (periodic topological cyclic homology) Let k be a perfect field of characteristic $p>0$, $W(k)$ the associated ring of p–typical Witt vectors, and $K :=$

[3] Let $\mathcal{X} := [X/G]$ be the (global) orbifold and $I(\mathcal{X}) := \big[\big(\coprod_{g\in G} X^g\big)/G\big]$ the inertia stack of $\mathcal{X}$. Under these notations, the above isomorphism (1.14) can be rewritten as $\mathrm{HP}_*(\mathcal{X}) \simeq H^*(I(\mathcal{X}),\mathbb{C})$. Halpern-Leistner and Pomerleano [22] and Toën (unpublished), using the techniques in Toen [52], extended the latter isomorphism to all smooth Deligne–Mumford stacks $\mathcal{X}$.

$W(k)[1/p]$ the fraction field of $W(k)$. As mentioned in Example 2.5 below, periodic topological cyclic homology TP is a lax symmetric monoidal additive invariant. Since $\mathrm{TP}_0(k) \simeq W(k)$ (see Hesselholt [23, Section 4]), we observe that $\mathrm{TP}(-)_{1/p}$ can be promoted to a lax symmetric monoidal additive invariant with values in $\mathbb{Z}/2$–graded K–vector spaces. Therefore, in the case where k contains the n^{th} roots of unity[4] (eg $k = \mathbb{F}_{p^d}$ with d the multiplicative order of p modulo n), Corollary 1.5(ii) leads to the isomorphism of $\mathbb{Z}/2$–graded commutative K–algebras

$$\mathrm{TP}_*([X/G])_{1/p} \simeq \Big(\bigoplus_{g\in G} \mathrm{TP}_*(X^g)_{1/p}\Big)^G. \tag{1.18}$$

To the best of the authors' knowledge, the formula (1.18) is new in the literature.

Remark 1.19 (proof of Theorem 1.1) Our proof of Theorem 1.1, and consequently of Corollaries 1.5 and 1.8, is different from the proofs of Vistoli [54] and Baranovsky [4] (which are themselves also very different). Nevertheless, we do borrow some ingredients from Vistoli's proof. In fact, using the formalism of noncommutative motives (see Section 2.4) and G_0–motives over $[X/G]$ (see Section 5), we are able to ultimately reduce the proof of the formula (1.2) to the proof of the K_0 case of Vistoli's formula (1.10); consult Sections 5–6 for details. Note, however, that we *cannot* mimic Vistoli's arguments because they depend in an essential way on the dévissage property of G–theory (namely K–theory for smooth schemes), which does not hold for many additive invariants. For example, as explained by Keller [30, Example 1.11], Hochschild homology, and consequently the mixed complex, do *not* satisfy dévissage.

Remark 1.20 (McKay correspondence) The dg category $\mathrm{perf}_{\mathrm{dg}}([X/G])$ is known in many cases to be Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(Y)$ for a crepant resolution Y of the (singular) geometric quotient $X/\!\!/G$; see Bezrukavnikov and Kaledin [7], Bridgeland, King and Reid [11], Kapranov and Vasserot [26] and Kawamata [27]. This is generally referred to as the "McKay correspondence". Whenever it holds, we can replace $[X/G]$ by Y in all the above formulas. Here is an illustrative example (with k algebraically closed): the cyclic group $G = C_2$ acts on any abelian surface S by the involution $a \mapsto -a$ and the Kummer surface $\mathrm{Km}(S)$ is defined as the blowup of $S/\!\!/C_2$ in its 16 singular points. In this case, the dg category $\mathrm{perf}_{\mathrm{dg}}([S/C_2])$ is Morita equivalent to

[4] Recall that $W(k)$ comes equipped with a multiplicative Teichmüller map $k \to W(k)$ and that K is of characteristic zero. This implies that K contains the n^{th} roots of unity and that $1/n \in K$.

$\mathrm{perf}_{\mathrm{dg}}(\mathrm{Km}(S))$. Consequently, Corollary 1.5(i) leads to an isomorphism

$$E(\mathrm{Km}(S)) \simeq E(k)^{\oplus 16} \oplus E(S)^{C_2}. \tag{1.21}$$

Since the Kummer surface is Calabi–Yau, the category $\mathrm{perf}(\mathrm{Km}(S))$ does not admit any nontrivial semiorthogonal decompositions. Therefore, the above decomposition (1.21) is *not* induced from a semiorthogonal decomposition.

Smooth quotients Let us write $\pi\colon X \to X/\!/G$ for the quotient morphism. Our second main result is the following:

Theorem 1.22 *Let $E\colon \mathrm{dgcat}(k) \to \mathrm{D}$ be an additive invariant E with values in a $\mathbb{Z}[1/n]$–linear category. If $X/\!/G$ is k–smooth* (eg if the G–action is free), *then the induced morphism $\pi^*\colon E(X/\!/G) \to E(X)^G$ is invertible.*

Example 1.23 (Hochschild homology) As mentioned in Example 2.4, Hochschild homology HH is an additive invariant. Therefore, whenever the geometric quotient $X/\!/G$ is k–smooth, Theorem 1.22 leads to an isomorphism $\mathrm{HH}(X/\!/G) \simeq \mathrm{HH}(X)^G$. This isomorphism was established in Polishchuk and den Bergh [43, Proposition 2.1.2] using the Hochschild–Kostant–Rosenberg theorem.

Remark 1.24 Assume that we are in the situation of Corollary 1.5(ii) and that all the geometric quotients $X^g/\!/C(g)$ are k–smooth. In this case, (1.7) reduces to an isomorphism[5] $E([X/G]) \simeq \bigoplus_{g\in G/\sim} E(X^g/\!/C(g))$. This holds, for example, in the case of a symmetric group acting on a product of copies of a smooth curve.

Equivariant Azumaya algebras Let $\mathcal{F}$ be a flat quasicoherent sheaf of algebras over $[X/G]$, and $\mathrm{perf}_{\mathrm{dg}}([X/G];\mathcal{F})$ the canonical dg enhancement of the category of G–equivariant perfect $\mathcal{F}$–modules $\mathrm{perf}([X/G];\mathcal{F})$. Given an additive invariant, let us write $E([X/G];\mathcal{F})$ instead of $E(\mathrm{perf}_{\mathrm{dg}}([X/G];\mathcal{F}))$. Finally, given a cyclic subgroup $\sigma \in \varphi$, let us denote by $\mathcal{F}_\sigma$ the pullback of $\mathcal{F}$ along the morphism $[X^\sigma/\sigma] \to [X/G]$; note that $\mathcal{F}_\sigma$ is an $N(\sigma)$–equivariant sheaf of algebras over X^σ. Under the above notations, our third main result, which extends Theorem 1.1, is the following:

[5] It is natural to ask if such an isomorphism is induced from a semiorthogonal decomposition of $\mathrm{perf}([X/G])$ with components $\mathrm{perf}(X^g/\!/C(g))$. This was the motivating question for the work of Polishchuk and den Bergh [43]. A positive answer to this question was obtained therein in many cases. However, the required semiorthogonal decompositions are usually highly nontrivial; this complexity already occurs in the "simple" case of a symmetric group acting on a product of copies of a smooth curve. On the other hand, an example of a (nonfaithful) G–action where a semiorthogonal decomposition does not exist was constructed by Lunts, Bergh and Schnürer (private communication).

Theorem 1.25 *For every additive invariant $E\colon \mathrm{dgcat}(k)\to \mathrm{D}$, with values in a $\mathbb{Z}[1/n]$–linear category, we have an isomorphism*

$$E([X/G];\mathcal{F}) \simeq \bigoplus_{\sigma\in\varphi/\sim} \widetilde{E}([X^\sigma/\sigma];\mathcal{F}_\sigma)^{N(\sigma)} \tag{1.26}$$

induced by pullback with respect to the morphisms $[X^\sigma/\sigma]\to[X/G]$.

Remark 1.27 Similarly to Theorem 1.1, Theorem 1.25 also follows from an isomorphism in a certain category of G_0–motives on $[X/G]$; consult Section 8 for details.

From now on we will assume that $\mathcal{F}$ is a sheaf of Azumaya algebras over $[X/G]$, ie a G–equivariant sheaf of Azumaya algebras over X. We will write r for the product of the ranks of $\mathcal{F}$ (at each one of the connected components of X). Let us denote by $\mathcal{F}_\sigma\#\sigma$ the sheaf of skew group algebras corresponding to the σ–action on $\mathcal{F}_\sigma$. Note that since σ acts trivially on X^σ, $\mathcal{F}_\sigma\#\sigma$ is a sheaf of $\mathcal{O}_{X^\sigma}$–algebras. Finally, let us write $\mathcal{Z}_\sigma$ for the center of $\mathcal{F}_\sigma\#\sigma$ and $Y_\sigma := \underline{\mathrm{Spec}}(\mathcal{Z}_\sigma)$. By construction, $\mathcal{F}_\sigma\#\sigma$ (and hence $\mathcal{Z}_\sigma$) is σ–graded.

Let $E\colon \mathrm{dgcat}(k)\to \mathrm{D}$ be an additive invariant as in Theorem 1.25, and assume that k contains the n^{th} roots of unity. As mentioned above, $\mathcal{Z}_\sigma$ is σ–graded and therefore it comes equipped with a canonical $\sigma^\vee$–action, where $\sigma^\vee := \mathrm{Hom}(\sigma, k^\times)$ stands for the dual cyclic group. Hence, by functoriality, $E(Y_\sigma)=E(X^\sigma;\mathcal{Z}_\sigma)$ inherits a $\mathbb{Z}[\sigma^\vee]_{1/n}$–action. Since k contains the n^{th} roots of unity, we have a character isomorphism $R(\sigma)\simeq\mathbb{Z}[\sigma^\vee]$. Therefore, we can consider the direct summand $e_\sigma E(Y_\sigma)$ of $E(Y_\sigma)$ associated to the idempotent $e_\sigma\in R(\sigma)_{1/n}$. In the case where the category D is l–linear for a field l which contains the n^{th} roots of unity and $1/n\in l$, the $\sigma^\vee$–action on $E(Y_\sigma)$ may be translated back[6] into a $(\sigma^{\vee\vee}{=}\sigma)$–grading on $E(Y_\sigma)$. Whenever an object $\mathcal{O}$ is equipped with a $\langle g\rangle$–grading, we will write $\mathcal{O}_g$ for its degree g part. In particular, $E(Y_g)_g$ stands for the degree g part of $E(Y_{\langle g\rangle})$. Under the above notations, Theorem 1.25 admits the following refinements:

Corollary 1.28 *Assume that $\mathcal{F}$ is a sheaf of Azumaya algebras over $[X/G]$. Under this assumption, the following holds:*

(i) *The structural morphism $Y_\sigma\to X^\sigma$ is an $N(\sigma)$–equivariant $\sigma^\vee$–Galois cover. Moreover, for every $g\in G$, the sheaf $\mathcal{L}_g := (\mathcal{Z}_{\langle g\rangle})_g$ is a $C(g)$–equivariant*

[6] By choosing an isomorphism ϵ between the n^{th} roots of unity in k and in l, we can identify $\sigma^{\vee\vee} := \mathrm{Hom}(\mathrm{Hom}(\sigma,k^\times),l^\times)$ with σ. This identification is moreover natural on σ.

line bundle (equipped with a $C(g)$–equivariant flat connection) on $X^g := X^{\langle g\rangle}$; when $k=\mathbb{C}$, we will denote by L_g the associated $C(g)$–equivariant rank-one local system on X^g.

(ii) *If k contains the n^{th} roots of unity and the category D is $\mathbb{Z}[1/nr]$–linear, then (1.26) reduces to an isomorphism*

$$E([X/G];\mathcal{F}) \simeq \bigoplus_{\sigma\in\varphi/\sim} (e_\sigma E(Y_\sigma))^{N(\sigma)}. \tag{1.29}$$

(iii) *If k contains the n^{th} roots of unity and D is l–linear for a field l which contains the n^{th} roots of unity and $1/nr\in l$, then (1.29) reduces to an isomorphism*

$$E([X/G];\mathcal{F}) \simeq \bigoplus_{g\in G/\sim} E(Y_g)_g^{C(g)} \simeq \Big(\bigoplus_{g\in G} E(Y_g)_g\Big)^G.$$

Example 1.30 (algebraic K–theory) Algebraic K–theory is an additive invariant. Therefore, in the case where k contains the n^{th} roots of unity, Corollary 1.28(iii) leads to the isomorphism of $\mathbb{Z}$–graded $\mathbb{C}$–vector spaces

$$K_*([X/G];\mathcal{F})_{\mathbb{C}} \simeq \bigoplus_{g\in G/\sim} K_*(Y_g)_{\mathbb{C},g}^{C(g)}. \tag{1.31}$$

It may be shown that in the particular case when $k=\mathbb{C}$ and $\mathcal{F}$ is induced from a cohomology class $\alpha\in H^2(G,\mathbb{C}^\times)$ by pullback along the morphism $[X/G]\to BG$, the formula (1.31) reduces to the algebraic analogue of the formula established by Adem and Ruan [1, Section 7] in the topological setting of twisted orbifold K–theory.

Example 1.32 (mixed complex) The mixed complex C is an additive invariant. Moreover, by construction, it sends graded dg categories to graded mixed complexes. This implies that $\mathrm{C}(Y_g)=\mathrm{C}(X^g;\mathcal{Z}_g)$ comes equipped with a canonical $\langle g\rangle$–grading, which clearly agrees with the one defined above. Therefore, in the case where k contains the n^{th} roots of unity and $1/nr\in k$, Corollary 1.28(iii) leads to the isomorphism in the derived category of mixed complexes

$$\mathrm{C}([X/G];\mathcal{F}) \simeq \Big(\bigoplus_{g\in G} \mathrm{C}(Y_g)_g\Big)^G, \tag{1.33}$$

where $\mathrm{C}(Y_g)_g$ stands for the degree g part of the tautological $\langle g\rangle$–grading on $\mathrm{C}(Y_g)$. The morphism (1.33) is given by first restricting $\mathrm{C}([X/G];\mathcal{F})$ to $\mathrm{C}([X^g/\langle g\rangle];\mathcal{F}_{\langle g\rangle})\simeq \mathrm{C}(Y_g)$ and then by projecting $\mathrm{C}(Y_g)$ onto its degree g part $\mathrm{C}(Y_g)_g$. This shows, in

particular, that (1.33) is defined over k. Since C is compatible with base change, we can then remove the assumption that k contains the n^{th} roots of unity!

A formula similar to (1.33) holds for all the variants of cyclic homology. In the particular case of Hochschild homology HH and periodic cyclic homology HP, this formula admits the following (geometric) refinements:

Corollary 1.34 *Assume that $1/nr \in k$. In the case of HH_*, (1.33) reduces to an isomorphism of $\mathbb{Z}$–graded k–vector spaces*

$$\mathrm{HH}_*([X/G];\mathcal{F}) \simeq \Big(\bigoplus_{g\in G} \mathrm{HH}_*(X^g,\mathcal{L}_g)\Big)^G. \tag{1.35}$$

Proof Corollary 1.28(i) implies that $\mathrm{HH}_*(Y_g)_g \simeq \mathrm{HH}_*(X^g;\mathcal{Z}_g)_g \simeq \mathrm{HH}_*(X^g,\mathcal{L}_g)$. By combining these identifications with (1.33), we then obtain (1.35). □

In the particular case when $k=\mathbb{C}$ and $\mathcal{F}$ is induced from a cohomology class $\alpha \in H^2(G,\mathbb{C}^\times)$ by pullback along the morphism $[X/G]\to BG$, the formula (1.35) was established by Baranovsky and Petrov [5, Theorem 6]. In the case where k is of characteristic 0, the formula (1.35) was announced by Căldăraru [14]. Therein, Căldăraru and Arinkin used it in order to conceptually explain an ad hoc computation of Vafa and Witten [53] concerning elliptic curves.

Corollary 1.36 (twisted orbifold cohomology) *Assume that $k=\mathbb{C}$. In the case of HP_*, (1.33) reduces to an isomorphism[7] of $\mathbb{Z}/2$–graded $\mathbb{C}$–vector spaces*

$$\mathrm{HP}_*([X/G];\mathcal{F}) \simeq \Big(\bigoplus_{g\in G} H^*(X^g,L_g)\Big)^G. \tag{1.37}$$

Proof Let us write $f\colon Y_g\to X^g$ for the structural morphism. By the Hochschild–Kostant–Rosenberg theorem (see Feĭgin and Tsygan [16] and Keller [30; 29]), the equality $f_*(\mathbb{C})=\bigoplus_{g\in G} L_g$, and Corollary 1.28(i), we have the isomorphisms

$$\mathrm{HP}_*(Y_g)_g \simeq H^*(Y_g,\mathbb{C})_g \simeq H^*(X^g,f_*(\mathbb{C}))_g \simeq H^*(X^g,L_g).$$

By combining them with (1.33), we then obtain (1.37). □

[7] The data $L:=(L_g)_g$ defines a G–equivariant local system on $\bigsqcup_{g\in G} X^g$ or, equivalently, a local system on the inertia stack $I(\mathcal{X})$ of $\mathcal{X}:=[X/G]$. Under these notations, the above isomorphism (1.37) can be rewritten as $\mathrm{HP}_*(\mathcal{X};\mathcal{F})\simeq H^*(I(\mathcal{X}),L)$.

Notation

Throughout the article, k will be a field of characteristic $p \geq 0$ and G a finite group of order n. Except in the appendix, we will always assume that $1/n \in k$. In order to simplify the exposition, we will often write $\bullet$ instead of $\mathrm{Spec}(k)$, $(-)_{1/n}$ instead of $(-)_{\mathbb{Z}[1/n]}$, and make no notational distinction between a dg functor and its image under an additive invariant.

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2 Preliminaries

2.1 Dg categories

Let $(\mathcal{C}(k), \otimes, k)$ be the category of (cochain) complexes of k–vector spaces; we use cohomological notation. A *differential graded (dg) category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$ and a *dg functor* $F\colon \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [31]. Recall from Section 1 that $\mathrm{dgcat}(k)$ stands for the category of (essentially small) dg categories and dg functors.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and $\mathcal{A}^{\mathrm{op}}(x,y) := \mathcal{A}(y,x)$. The category $\mathrm{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms $\mathrm{H}^0(\mathcal{A})(x,y) := H^0\mathcal{A}(x,y)$, where $H^0(-)$ stands for the 0^{th} cohomology functor. A *right dg $\mathcal{A}$–module* is a dg functor $M\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category

$\mathcal{C}_{\mathrm{dg}}(k)$ of complexes of k–vector spaces. Let us write $\mathcal{C}(\mathcal{A})$ for the category of right dg $\mathcal{A}$–modules. Following [31, Section 3.2], the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the objectwise quasi-isomorphisms. Let $\mathcal{D}_c(\mathcal{A})$ be the triangulated subcategory of compact objects.

A dg functor $F\colon \mathcal{A}\to\mathcal{B}$ is called a *Morita equivalence* if the restriction functor along F induces an equivalence on derived categories $\mathcal{D}(\mathcal{B})\xrightarrow{\simeq}\mathcal{D}(\mathcal{A})$; see [31, Section 4.6]. As explained in [47, Section 1.6], the category $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by $\mathrm{Hmo}(k)$ the associated homotopy category.

The *tensor product* $\mathcal{A}\otimes\mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$ and $(\mathcal{A}\otimes\mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y)\otimes\mathcal{B}(w,z)$. As explained in [31, Section 2.3], this construction gives rise to a symmetric monoidal structure on $\mathrm{dgcat}(k)$. Moreover, since the assignment $(\mathcal{A},\mathcal{B})\mapsto\mathcal{A}\otimes\mathcal{B}$ preserves Morita equivalences (in both variables), this symmetric monoidal structure descends to the homotopy category $\mathrm{Hmo}(k)$.

A *dg $\mathcal{A}$–$\mathcal{B}$–bimodule* is a dg functor $\mathrm{B}\colon \mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathrm{dg}}(k)$ or, equivalently, a right dg $(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$–module. A standard example is the dg $\mathcal{A}$–$\mathcal{B}$–bimodule

$$ {}_F\mathrm{B}\colon \mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathrm{dg}}(k),\quad (x,z)\mapsto\mathcal{B}(z,F(x)), \tag{2.1} $$

associated to a dg functor $F\colon\mathcal{A}\to\mathcal{B}$. Let us write $\mathrm{rep}(\mathcal{A},\mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$ consisting of those dg $\mathcal{A}$–$\mathcal{B}$–modules B such that for every object $x\in\mathcal{A}$ the associated right dg $\mathcal{B}$–module $\mathrm{B}(x,-)$ belongs to $\mathcal{D}_c(\mathcal{B})$.

Following Kontsevich [34; 35; 36], a dg category $\mathcal{A}$ is called *smooth* if the dg $\mathcal{A}$–$\mathcal{A}$–bimodule ${}_{\mathrm{id}}\mathrm{B}$ belongs to the triangulated category $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A})$ and *proper* if $\sum_i \dim H^i\mathcal{A}(x,y)<\infty$ for any ordered pair of objects (x,y).

2.2 Additive invariants

Recall from Bondal and Orlov [9, Definition 2.4; 10] that a *semiorthogonal decomposition* of a triangulated category $\mathcal{T}$, denoted by $\mathcal{T}=\langle\mathcal{T}_1,\mathcal{T}_2\rangle$, consists of full triangulated subcategories $\mathcal{T}_1,\mathcal{T}_2\subseteq\mathcal{T}$ satisfying the following conditions: the inclusions $\mathcal{T}_1,\mathcal{T}_2\subseteq\mathcal{T}$ admit left and right adjoints; the triangulated category $\mathcal{T}$ is generated by the objects of $\mathcal{T}_1$ and $\mathcal{T}_2$; and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_2,\mathcal{T}_1)=0$. A functor $E\colon\mathrm{dgcat}(k)\to\mathrm{D}$, with values in an idempotent complete additive category, is called an *additive invariant* if it satisfies the following two conditions:

(i) It sends the Morita equivalences (see Section 2.1) to isomorphisms.

(ii) Given dg categories $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ such that $\mathrm{H}^0(\mathcal{B}) = \langle \mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{C}) \rangle$, the inclusions $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ induce an isomorphism $E(\mathcal{A}) \oplus E(\mathcal{C}) \simeq E(\mathcal{B})$.

Example 2.2 (algebraic K–theory) Algebraic K–theory gives rise to a (lax symmetric monoidal) additive invariant $K\colon \mathrm{dgcat}(k) \to \mathrm{Ho}(\mathrm{Spt})$ with values in the homotopy category of spectra; see [47, Section 2.2.1]. Classical variants such as mod-l^ν algebraic K–theory $K(-;\mathbb{Z}/l^\nu)$, Karoubi–Villamayor K–theory KV, nonconnective algebraic K–theory $I\!K$, homotopy K–theory KH and étale K–theory $K^{\mathrm{et}}(-;\mathbb{Z}/l^\nu)$ also give rise to additive invariants; consult [47, Sections 2.2.2–2.2.6] for details.

Example 2.3 (cyclic homology) Cyclic homology gives rise to an additive invariant $\mathrm{HC}\colon \mathrm{dgcat}(k) \to \mathcal{D}(k)$ with values in the derived category of the base field k; see [47, Section 2.2.9]. Classical variants such as Hochschild homology HH, periodic cyclic homology HP and negative cyclic homology HC^- also give rise to additive invariants; consult [47, Sections 2.2.8–2.2.11] for details.

Example 2.4 (mixed complex) A *mixed complex* is a (right) dg module over the algebra of dual numbers $\Lambda := k[\epsilon]/\epsilon^2$ with $\deg(\epsilon) = -1$ and $d(\epsilon) = 0$. Note that Λ is a cocommutative graded dg Hopf algebra with $\Delta(\epsilon) = \epsilon \otimes 1 + 1 \otimes \epsilon$. Therefore, the tensor product $- \otimes_k -$ makes the derived category $\mathcal{D}(\Lambda)$ into a symmetric monoidal category. The mixed complex gives rise to a (symmetric monoidal) additive invariant $\mathsf{C}\colon \mathrm{dgcat}(k) \to \mathcal{D}(\Lambda)$. Moreover, all the additive invariants of Example 2.3 factor through C; consult [47, Section 2.2.7] for details.

Example 2.5 (topological Hochschild homology) Topological Hochschild homology gives rise to a (lax symmetric monoidal) additive invariant $\mathrm{THH}\colon \mathrm{dgcat}(k) \to \mathrm{Ho}(\mathrm{Spt})$; see [47, Section 2.2.12]. Variants such as topological cyclic homology TC (see [47, Section 2.2.13]) and periodic topological cyclic homology TP (see [23]) also give rise to (lax symmetric monoidal) additive invariants.

2.3 Change of coefficients

Given a ring homomorphism $S \to R$ and a S–linear idempotent complete additive category D, let D_R be the idempotent completion of the category obtained by tensoring the Hom–sets of D with R. This procedure leads to a 2–functor from the 2–category of S–linear idempotent complete additive categories to the 2–category of R–linear

idempotent complete additive categories. Moreover, we have an obvious "change of coefficients" functor $(-)_R\colon \mathrm{D} \to \mathrm{D}_R$.

2.4 Noncommutative motives

Given dg categories $\mathcal{A}$ and $\mathcal{B}$, there is a natural bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A},\mathcal{B})$ and the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A},\mathcal{B})$. Under this bijection, the composition law of $\mathrm{Hmo}(k)$ corresponds to the tensor product of bimodules. Therefore, since the dg $\mathcal{A}$–$\mathcal{B}$–bimodules (2.1) belong to $\mathrm{rep}(\mathcal{A},\mathcal{B})$, we have the symmetric monoidal functor

$$\mathrm{dgcat}(k) \to \mathrm{Hmo}(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad (\mathcal{A} \xrightarrow{F} \mathcal{B}) \mapsto {}_F\mathrm{B}. \tag{2.6}$$

The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects as $\mathrm{Hmo}(k)$ and with abelian groups of morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A},\mathcal{B})$ given by the Grothendieck group $K_0\,\mathrm{rep}(\mathcal{A},\mathcal{B})$ of the triangulated category $\mathrm{rep}(\mathcal{A},\mathcal{B})$. The category of *noncommutative motives* $\mathrm{NMot}(k)$ is defined as the idempotent completion of $\mathrm{Hmo}_0(k)$ and the *universal additive invariant*

$$\mathrm{U}(-)\colon \mathrm{dgcat}(k) \to \mathrm{NMot}(k)$$

as the composition of (2.6) with the canonical functors $\mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k)$ and $\mathrm{Hmo}_0(k) \to \mathrm{NMot}(k)$. Given a commutative ring of coefficients R, the category $\mathrm{NMot}(k)_R$ is defined as in Section 2.3. As explained in [47, Section 2.3], given any R–linear idempotent complete additive ((symmetric) monoidal) category D, precomposition with the functor $\mathrm{U}(-)_R$ gives rise to induced equivalences of categories

$$\mathrm{Fun}_{R\text{–linear}}(\mathrm{NMot}(k)_R, \mathrm{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}), \tag{2.7}$$

$$\mathrm{Fun}^{\otimes}_{R\text{–linear}}(\mathrm{NMot}(k)_R, \mathrm{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\otimes}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}), \tag{2.8}$$

where the left-hand side stands for the category of R–linear ((lax) (symmetric) monoidal) functors and the right-hand side for the category of ((lax) (symmetric) monoidal) additive invariants. For further information on noncommutative motives we invite the reader to consult the recent book [47] and survey [48].

Remark 2.9 Given dg categories $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ smooth and proper, we have an equivalence $\mathrm{rep}(\mathcal{A},\mathcal{B}) \simeq \mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$. Consequently, we obtain isomorphisms

$$\mathrm{Hom}_{\mathrm{NMot}(k)}(\mathrm{U}(\mathcal{A}), \mathrm{U}(\mathcal{B})) := K_0(\mathrm{rep}(\mathcal{A},\mathcal{B})) \simeq K_0(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}).$$

2.5 Perfect complexes of trivial G–actions

Let X be a quasicompact quasiseparated k–scheme equipped with a trivial G–action. In this case, we have $[X/G] = X \times_{\bullet} BG$. This leads naturally to the dg functor

$$\mathrm{perf}_{\mathrm{dg}}(X) \otimes \mathrm{perf}_{\mathrm{dg}}(BG) \to \mathrm{perf}_{\mathrm{dg}}([X/G]), \quad (M, V) \mapsto M \boxtimes V. \tag{2.10}$$

Proposition 2.11 *The above dg functor (2.10) is a Morita equivalence.*[8]

Proof Since $1/n \in k$, $\mathrm{perf}_{\mathrm{dg}}(BG)$ is Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(k[G])$ and, consequently, to $k[G]$. Therefore, it suffices to show that the induced functor

$$\mathrm{perf}_{\mathrm{dg}}(X) \otimes k[G] \to \mathrm{perf}_{\mathrm{dg}}([X/G])$$

is a Morita equivalence. Let us denote by $\mathcal{D}_{\mathrm{Qch}}(X)$ (resp. $\mathcal{D}_{\mathrm{Qch}}([X/G])$) the full triangulated subcategory of $\mathcal{D}(X)$ (resp. $\mathcal{D}([X/G])$) consisting of those complexes of $\mathcal{O}_X$–modules (resp. G–equivariant complexes of $\mathcal{O}_X$–modules) with quasicoherent cohomology. Thanks to Neeman's celebrated result [42, Theorem 2.1], the full subcategory of compact objects of $\mathcal{D}_{\mathrm{Qch}}(X)$ agrees with $\mathrm{perf}(X)$. Under the assumption $1/n \in k$, a similar result holds for the (global) orbifold $[X/G]$; see [21, Theorem C]. As proved in [8, Theorem 3.1.1], the triangulated category $\mathcal{D}_{\mathrm{Qch}}(X)$ admits a compact generator $\mathcal{G}$. Consequently, since G acts trivially on X, $\bigoplus_{g\in G} \mathcal{G}$ is a compact generator of $\mathcal{D}_{\mathrm{Qch}}([X/G])$. By passing to compact objects, we then obtain the Morita equivalences

$$\mathrm{perf}_{\mathrm{dg}}([X/G]) \simeq \mathrm{perf}_{\mathrm{dg}}\Big(\mathbf{R}\mathrm{End}\Big(\bigoplus_{g\in G} \mathcal{G}\Big)\Big)\mathrm{perf}_{\mathrm{dg}}(X) \simeq \mathrm{perf}_{\mathrm{dg}}(\mathbf{R}\mathrm{End}(\mathcal{G})),$$

where $\mathbf{R}\mathrm{End}(-)$ stands for the (derived) dg k–algebra of endomorphisms. The proof follows now from the canonical quasi-isomorphism of dg k–algebras between $\mathbf{R}\mathrm{End}\big(\bigoplus_{g\in G} \mathcal{G}\big)$ and $\mathbf{R}\mathrm{End}(\mathcal{G}) \otimes k[G]$ □

Corollary 2.12 *We have an isomorphism of monoids* $\mathrm{U}(X) \otimes \mathrm{U}(BG) \simeq \mathrm{U}([X/G])$.

Proof Combine Proposition 2.11 with the fact that U is symmetric monoidal. □

[8] The assumption $1/n \in k$ might be unnecessary for Proposition 2.11. Nevertheless, it is necessary to guaranty that the dg categories $\mathrm{perf}_{\mathrm{dg}}(BG)$ and $k[G]$ are Morita equivalent.

2.6 Galois descent for representation rings

Let l/k be a finite Galois field extension with Galois group Γ. Consider the induced homomorphism

$$-\otimes_k l\colon R(G)_{1/n} \to R_l(G)^{\Gamma}_{1/n}, \tag{2.13}$$

where $R_l(G)$ stands for the representation ring of G over l.

Proposition 2.14 *The above homomorphism (2.13) is invertible.*

Proof Since $1/n \in k$, the group algebra $k[G]$ is semisimple. Hence, we have $k[G] \simeq \bigoplus_{i=1}^{m} A_i$ with A_i a central simple algebra over its center l_i. As explained in [13, Theorem (33.7)],[9] the index of A_i divides n. Consequently, the functor $A_i \otimes_{l_i} -$ gives rise to isomorphisms $K_0(l_i)_{1/n} \simeq K_0(A_i)_{1/n}$ and $K_0(l_i \otimes_k l)_{1/n} \simeq K_0(A_i \otimes_k l)_{1/n}$. Therefore, it suffices to show that the homomorphism $-\otimes_k l\colon K_0(l_i) \to K_0(l_i \otimes_k l)^{\Gamma}$ is invertible. Note that $l_i \otimes_k l$ decomposes into a direct sum of copies of a field. Moreover, Γ permutes these copies. This implies that $K_0(l_i \otimes_k l) \simeq \mathbb{Z}\{S\}$, where S is a set on which Γ acts transitively. The proof follows now from the fact that the diagonal inclusion $\mathbb{Z} \to \mathbb{Z}\{S\}$ induces an isomorphism $\mathbb{Z} \simeq (\mathbb{Z}\{S\})^{\Gamma}$. □

2.7 Galois descent for topological Hochschild homology

As mentioned in Example 2.5, topological Hochschild homology THH is a lax symmetric monoidal additive invariant. Since the "inclusion of the 0^{th} skeleton" yields a ring isomorphism $k \xrightarrow{\simeq} \mathrm{THH}_0(k)$, we observe that $\mathrm{THH}_*(-)$ can be promoted to a lax symmetric monoidal additive invariant with values in $\mathbb{Z}$–graded k–vector spaces.

Let l/k be a finite Galois field extension with Galois group Γ. Given a dg category $\mathcal{A}$, consider the induced homomorphism of $\mathbb{Z}$–graded k–vector spaces

$$-\otimes_k l\colon \mathrm{THH}_*(\mathcal{A}) \to \mathrm{THH}_*(\mathcal{A}\otimes_k l)^{\Gamma}. \tag{2.15}$$

Proposition 2.16 *The above homomorphism (2.15) is invertible.*

Proof Note first that by definition of topological Hochschild homology, the following two procedures lead to the same $\mathbb{Z}$–graded k–vector spaces:

[9] The result in loc. cit. is stated in characteristic 0. However, it is well known that if $p \nmid |G|$, then the blocks in characteristic p are obtained by reduction from those in characteristic 0.

(i) First apply the functor $\mathrm{THH}_*(-)$ to the dg l–linear category $\mathcal{A}\otimes_k l$, and then consider $\mathrm{THH}_*(\mathcal{A}\otimes_k l)$ as a $\mathbb{Z}$–graded k–vector space.

(ii) First consider $\mathcal{A}\otimes_k l$ as a dg k–linear category, and then apply $\mathrm{THH}_*(-)$.

Consider $\mathcal{A}\otimes_k l$ as a dg k–linear category. Consider also the dg $(\mathcal{A}\otimes_k l)$–$\mathcal{A}$–bimodule

$$(\mathcal{A}\otimes_k l)\otimes\mathcal{A}^{\mathrm{op}}\to\mathcal{C}_{\mathrm{dg}}(k),\quad (z,x)\mapsto\mathcal{A}\otimes_k l((-\otimes_k l)(x),z). \tag{2.17}$$

Since the field extension l/k is finite, (2.17) belongs to the category $\mathrm{rep}(\mathcal{A}\otimes_k l,\mathcal{A})$. Consequently, (2.17) corresponds to a morphism $\mathcal{A}\otimes_k l\to\mathcal{A}$ in the homotopy category $\mathrm{Hmo}(k)$. Using the equivalence of categories (2.7), we then obtain an induced homomorphism of $\mathbb{Z}$–graded k–vector spaces res: $\mathrm{THH}_*(\mathcal{A}\otimes_k l)\to\mathrm{THH}_*(\mathcal{A})$. In the particular case where $\mathcal{A}=k$, the homomorphism res: $\mathrm{THH}_0(l)\to\mathrm{THH}_0(k)$ agrees with the field trace homomorphism tr: $l\to k$. Note that tr is surjective.

We now have all the ingredients necessary to conclude the proof. Choose an element $\lambda\in l$ such that $\mathrm{tr}(\lambda)=1$, and consider the composition

$$\mathrm{THH}_*(\mathcal{A}\otimes_k l)^\Gamma\subset\mathrm{THH}_*(\mathcal{A}\otimes_k l)\xrightarrow{\lambda\cdot -}\mathrm{THH}_*(\mathcal{A}\otimes_k l)\xrightarrow{\mathrm{res}}\mathrm{THH}_*(\mathcal{A}). \tag{2.18}$$

We claim that the homomorphisms (2.15) and (2.18) are inverse to each other. On the one hand, given $\alpha\in\mathrm{THH}_*(\mathcal{A})$, we have the equalities

$$\mathrm{res}(\lambda\cdot(\alpha\otimes_k l))=\mathrm{tr}(\lambda)\cdot\alpha=1\cdot\alpha=\alpha,$$

where the first equality follows from the projection formula. On the other hand, given $\beta\in\mathrm{THH}_*(\mathcal{A}\otimes_k l)^\Gamma$, we have the equalities

$$\begin{aligned}\mathrm{res}(\lambda\cdot\beta)\otimes_k l&=\sum_{\gamma\in\Gamma}\gamma(\lambda\cdot\beta)\\&=\sum_{\gamma\in\Gamma}\gamma^{-1}(\lambda)\cdot\gamma(\beta)\\&=\Big(\sum_{\gamma\in\Gamma}\gamma^{-1}(\lambda)\Big)\cdot\beta\\&=(\mathrm{tr}(\lambda)\otimes_k l)\cdot\beta=\beta,\end{aligned}$$

where the first equality follows from [46, Proposition 5.5] and the fourth from the fact that l/k is a finite Galois field extension. This finishes the proof. □

3 Action of the representation ring

In this section we describe the action of the representation ring $R(G)$ on additive invariants. We start with some generalities. Let $(\mathrm{D}, \otimes, \mathbf{1})$ be a $\mathbb{Z}$–linear monoidal category, ie a monoidal category which is enriched over abelian groups in a compatible way. Given an object $\mathcal{O} \in \mathrm{D}$, consider the abelian group $\mathrm{Hom}(\mathbf{1}, \mathcal{O})$.

(i) If $\mathcal{O}$ is a monoid in D, then $\mathrm{Hom}(\mathbf{1}, \mathcal{O})$ is a ring. Similarly, if D is a symmetric monoidal category and $\mathcal{O}$ is a commutative monoid, then $\mathrm{Hom}(\mathbf{1}, \mathcal{O})$ is also a commutative monoid.

(ii) If $\mathcal{O}$ is a monoid in D, then we have an induced ring homomorphism

$$\mathrm{Hom}(\mathbf{1}, \mathcal{O}) \to \mathrm{Hom}(\mathcal{O}, \mathcal{O}), \quad f \mapsto \phi(f) := m \circ (f \otimes \mathrm{id}_{\mathcal{O}}),$$

where $m\colon \mathcal{O} \otimes \mathcal{O} \to \mathcal{O}$ stands for the multiplication map. In other words, the ring $\mathrm{Hom}(\mathbf{1}, \mathcal{O})$ acts on the object $\mathcal{O}$.

(iii) If D is a symmetric monoidal category and $\mathcal{O}$ is a commutative monoid, then the monoid structure of $\mathcal{O}$ is $\mathrm{Hom}(\mathbf{1}, \mathcal{O})$–linear. In other words, the equality $\phi(fg) \circ m = m \circ (\phi(g) \otimes \phi(g))$ holds for every $f, g \in \mathrm{Hom}(\mathbf{1}, \mathcal{O})$.

Let us now consider the case where $\mathrm{D} = \mathrm{NMot}(k)$. Given a quasicompact, quasi-separated k–scheme X (or, more generally, a suitable algebraic stack $\mathcal{X}$), the tensor product $- \otimes_X -$ makes $\mathrm{perf}_{\mathrm{dg}}(X)$ into a commutative monoid in $\mathrm{dgcat}(k)$. Since the universal additive invariant is symmetric monoidal, we obtain a commutative monoid $\mathrm{U}(X)$ in $\mathrm{NMot}(k)$. The above general considerations, combined with the isomorphism (see Remark 2.9)

$$\mathrm{Hom}_{\mathrm{NMot}(k)}(\mathrm{U}(k), \mathrm{U}(X)) \simeq K_0(X),$$

allow us then to conclude that the Grothendieck ring $K_0(X)$ acts on $\mathrm{U}(X)$. Moreover, the monoid structure of $\mathrm{U}(X)$ is $K_0(X)$–linear. Note that under the notations of Section 2.1, the $K_0(X)$–action on $\mathrm{U}(X)$ may be explicitly described by

$$K_0(\mathrm{perf}(X)) \to K_0\big(\mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(X), \mathrm{perf}_{\mathrm{dg}}(X))\big), \quad [M] \mapsto [{}_{(M \otimes_X -)}\mathrm{B}].$$

Given an additive invariant E, the equivalence of categories (2.7) implies, by functoriality, that the Grothendieck ring $K_0(X)$ acts on $E(X)$. If E is moreover symmetric monoidal, then the monoid structure of $E(X)$ is $K_0(X)$–linear.

Finally, let us consider the case where $\mathcal{X} = [X/G]$. In this case, the morphism $[X/G] \to BG$ induces a ring homomorphism $R(G) \to K_0([X/G])$. By combining it

with the above considerations, we obtain a canonical $R(G)$–action on $E([X/G])$ for every additive invariant E, which may be explicitly described by

$$R(G) \to \mathrm{End}_{\mathrm{D}}(E([X/G])), \quad [V] \mapsto \big(E([X/G]) \xrightarrow{V\otimes_k -} E([X/G])\big).$$

4 Decomposition of the representation ring

In this section, we describe a decomposition of the representation ring $R(G)$. Given a cyclic subgroup $\sigma \in \varphi$, we start by introducing the primitive part of $R(\sigma)$.

4.0.1 Cyclic group rings Let Σ be a commutative ring and ρ a cyclic group of order m. Assume that $1/m \in \Sigma$. The choice of a generator $t \in \rho$ leads to a decomposition

$$\Sigma[\rho] \simeq \Sigma[t]/(t^m - 1) \simeq \bigoplus_{j\,|\,m} \Sigma[t]/(\Phi_j(t)),$$

where $\Phi_j(t)$ stands for the cyclotomic polynomial corresponding to the primitive j^{th} roots of unity. Consider the idempotent $e_{\mathrm{prim}} := \prod_{\rho' \subsetneq \rho}(1 - e_{\rho'})$, with $e_{\rho'} := \big(\sum_{h\in\rho'} h\big)/|\rho'|$, where the product runs over all minimal nontrivial subgroups ρ' of ρ. Note that e_{prim}, being a product of idempotents is, indeed, an idempotent. Moreover, it is mapped to $\delta_{j,m}$ under the projection $\Sigma[\rho] \to \Sigma[t]/(\Phi_j(t))$. The *primitive part* of $\Sigma[\rho]$ is defined by

$$\Sigma[\rho]_{\mathrm{prim}} := e_{\mathrm{prim}}\Sigma[\rho] \simeq \Sigma[t]/(\Phi_m(t)). \tag{4.1}$$

Remark 4.2 If $u\colon \rho \to \varrho$ is a noninjective group homomorphism, then $u(e_{\mathrm{prim}}) = 0$. Indeed, since $\mathrm{Ker}(u)$ contains a minimal subgroup ρ', we have $u(1 - e_{\rho'}) = 0$. The idempotent e_{prim} is maximal with respect to this property.

The following result follows easily from the above definitions/considerations:

Lemma 4.3 *Every automorphism of $\Sigma[\rho]$ that permutes the elements of ρ leaves e_{prim}, and consequently $\Sigma[\rho]_{\mathrm{prim}}$, invariant.*

4.0.2 Representation rings of cyclic groups Let l be the field obtained from k by adjoining the n^{th} roots of unity, and $\Gamma := \mathrm{Gal}(l/k)$ the associated Galois group. Given a cyclic subgroup $\sigma \in \varphi$, character theory provides an isomorphism $R_l(\sigma) \simeq \mathbb{Z}[\sigma^\vee]$, where $\sigma^\vee := \mathrm{Hom}(\sigma, l^\times)$ stands for the dual cyclic group. Moreover, the Γ–action on $\mathbb{Z}[\sigma^\vee]$ permutes the elements of $\sigma^\vee$. Therefore, thanks to Lemma 4.3 (with $\rho = \sigma$

and $\Sigma = \mathbb{Z}[1/n]$), the canonical idempotent $e_{\mathrm{prim}} \in \mathbb{Z}[1/n][\sigma^\vee]$ is Γ–invariant. Using Proposition 2.14, we can then consider e_{prim} as an element of the representation ring $R(\sigma)_{1/n} \simeq R_l(\sigma)^\Gamma_{1/n}$.

Definition 4.4 (primitive part) Let $e_\sigma \in R(\sigma)_{1/n}$ be the idempotent corresponding to $e_{\mathrm{prim}} \in \mathbb{Z}[1/n][\sigma^\vee]^\Gamma$ under the character isomorphism $R(\sigma)_{1/n} \simeq \mathbb{Z}[1/n][\sigma^\vee]^\Gamma$. The *primitive part*[10] $\widetilde{R}(\sigma)_{1/n}$ of $R(\sigma)_{1/n}$ is defined as the direct summand $e_\sigma R(\sigma)_{1/n}$.

Remark 4.5 Thanks to Lemma 4.3, the group $\mathrm{Aut}(\sigma)$ of automorphisms of σ acts on the primitive part $\widetilde{R}(\sigma)_{1/n}$.

Lemma 4.6 *The idempotent e_σ belongs to the kernel of the restriction homomorphism $R(\sigma)_{1/n} \to R(\sigma')_{1/n}$ for every proper subgroup $\sigma' \subsetneq \sigma$.*

Proof We may assume without loss of generality that k contains the n^{th} roots of unity. Therefore, the proof follows from Remark 4.2. □

Proposition 4.7 (computation) *Assume that k contains the n^{th} roots of unity. Let l be another field (not necessarily of characteristic p) which contains the n^{th} roots of unity and $1/n \in l$. Choose an isomorphism ϵ between the n^{th} roots of unity in k and in l (eg the identity if $k = l$). Under these assumptions, we have the following commutative diagram of l–algebras (compatible with the $\mathrm{Aut}(\sigma)$–action):*

(4.8)
$$\begin{array}{ccc} \widetilde{R}(\sigma)_l & \xrightarrow{\simeq} & \mathrm{Map}(\mathrm{gen}(\sigma), l) \\ \downarrow & & \downarrow \\ R(\sigma)_l & \xrightarrow[\simeq]{} & \mathrm{Map}(\sigma, l). \end{array}$$

Some explanations are in order: $\mathrm{gen}(\sigma)$ is the set of generators of σ; $\mathrm{Map}(\mathrm{gen}(\sigma), l)$ is the set of functions from $\mathrm{gen}(\sigma)$ to l; the lower arrow in (4.8) sends an irreducible σ–representation V to the composition of its character χ_V with ϵ; and the right vertical arrow in (4.8) identifies a function on $\mathrm{gen}(\sigma)$ to the function on σ that is zero on the complement of $\mathrm{gen}(\sigma)$.

Proof Note first that we have the identifications

(4.9)
$$R(\sigma)_l \simeq R(\sigma)_{1/n} \otimes_{\mathbb{Z}[1/n]} l \simeq \mathbb{Z}[\sigma^\vee]_{1/n} \otimes_{\mathbb{Z}[1/n]} l \simeq l[\sigma^\vee].$$

[10] There is no obvious relation between the rings $\widetilde{R}(\sigma)_{1/n}$ and $K_0(k[\sigma]_{\mathrm{prim}})$. Indeed, the ring $K_0(k[\sigma]_{\mathrm{prim}})_{1/n}$ is *not* a direct summand of the ring $R(\sigma)_{1/n}$.

By composing them with the l–algebra isomorphism

$$l[\sigma^\vee] \xrightarrow{\sim} l[\sigma]^\vee \simeq \mathrm{Map}(\sigma, l), \quad \chi \mapsto (g \mapsto (\epsilon \circ \chi)(g)),$$

we obtain the lower arrow in diagram (4.8). Note also that (4.9) implies that $\widetilde{R}(\sigma)_l \simeq l[\sigma^\vee]_{\mathrm{prim}}$. Thanks to the right-hand side of (4.1) (with $\Sigma = l$), the dimension of the l–vector space $\widetilde{R}(\sigma)_l$ is equal to $\phi(|\sigma|)$, where ϕ stands for Euler's totient function. Now, Lemma 4.6 leads to the inclusion of l–algebras

$$\widetilde{R}(\sigma)_l \subset \bigcap_{\sigma' \subsetneq \sigma} \ker\big(\mathrm{Map}(\sigma, l) \to \mathrm{Map}(\sigma', l)\big).$$

This shows that an element of $\widetilde{R}(\sigma)_l$ is a function on σ which is zero on all proper subgroups σ' of σ. Consequently, we have $\widetilde{R}(\sigma)_l \subseteq \mathrm{Map}(\mathrm{gen}(\sigma), l)$. Finally, since both sides of this inclusion have the same dimension $\phi(|\sigma|)$, the proof is finished. □

4.1 Decomposition

Given a cyclic subgroup $\sigma \in \varphi$, consider the restriction homomorphism

$$\mathrm{Res}^G_\sigma \colon R(G)_{1/n} \to R(\sigma)_{1/n}.$$

This homomorphism factors through the direct summand $R(\sigma)^{N(\sigma)}_{1/n}$. Moreover, since $N(\sigma)$ maps naturally into the group $\mathrm{Aut}(\sigma)$ of automorphisms of σ, it follows from Remark 4.5 that $N(\sigma)$ acts on the primitive part $\widetilde{R}(\sigma)_{1/n}$. The next proposition refines a previous result of Vistoli:

Proposition 4.10 *The following homomorphism of $\mathbb{Z}[1/n]$–algebras is invertible:*

$$R(G)_{1/n} \xrightarrow{\oplus_\sigma e_\sigma \circ \mathrm{Res}^G_\sigma} \bigoplus_{\sigma \in \varphi/\sim} \widetilde{R}(\sigma)^{N(\sigma)}_{1/n}. \tag{4.11}$$

Proof In the case where k contains the n^{th} roots of unity, the isomorphism (4.11) was proved in [54, Proposition (1.5)]; see also [15, Corollary 7.7.10]. Using Proposition 2.14, the general case follows now from Galois descent. □

Remark 4.12 Let l be the field obtained from k by adjoining the n^{th} roots of unity, and r the degree of the (finite) field extension l/k. Proposition 4.10, with $1/n$ replaced by $1/nr$, was proved by Vistoli in [54, Theorem (5.2)].

Notation 4.13 Let $\widetilde{e}_\sigma \in R(G)_{1/n}$ be the idempotent corresponding to the direct summand $\widetilde{R}(\sigma)^{N(\sigma)}_{1/n}$ under the above isomorphism (4.11).

5 G_0–motives over an orbifold

This section is the (technical) heart of the article. Given a smooth separated k–scheme X equipped with a G–action, we start by constructing a certain category of G_0–motives $\mathbb{M}\mathrm{ot}(\mathcal{X})$ over the (global) orbifold $\mathcal{X} := [X/G]$, as well as a functor $\Psi\colon \mathbb{M}\mathrm{ot}(\mathcal{X}) \to \mathrm{NMot}(k)$; in the case where $\mathcal{X}$ is equipped with a sheaf of $\mathcal{O}_{\mathcal{X}}$–algebras $\mathcal{F}$, we construct also a functor $\Psi_{\mathcal{F}}\colon \mathbb{M}\mathrm{ot}(\mathcal{X}) \to \mathrm{NMot}(k)$. Then, we establish a key decomposition result in the category $\mathbb{M}\mathrm{ot}(\mathcal{X})$; consult Theorem 5.13. By applying the functor Ψ and $\Psi_{\mathcal{F}}$ to the latter key decomposition result, we then obtain immediately proofs of Theorems 1.1 and 1.25, respectively.

5.1 Construction

Let $\mathrm{DM}(\mathcal{X})$ be the category whose objects are the finite morphisms of smooth separated Deligne–Mumford stacks $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ with the property that the stabilizer orders in $\mathcal{Y}$ are invertible in k. Note that $\mathrm{DM}(\mathcal{X})$ is a 2–category. Concretely, a morphism from $\mathcal{Y}_1 \xrightarrow{f_1} \mathcal{X}$ to $\mathcal{Y}_2 \xrightarrow{f_2} \mathcal{X}$ consists of a pair (α, η), where $\alpha\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ is a 1–morphism and $\eta\colon f_2 \circ \alpha \Rightarrow f_1$ a 2–isomorphism. Whenever f_1, f_2 and η are clear from the context, we will omit them from the notation and write simply $\alpha\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ for a morphism in $\mathrm{DM}(\mathcal{X})$.

Let $\mathbb{H}\mathrm{mo}(\mathcal{X})$ be the category with the same objects as $\mathrm{DM}(\mathcal{X})$. Given any two objects $\mathcal{Y}$ and $\mathcal{Y}'$ of $\mathbb{H}\mathrm{mo}(\mathcal{X})$, let $\mathrm{Hom}_{\mathbb{H}\mathrm{mo}(\mathcal{X})}(\mathcal{Y}, \mathcal{Y}')$ be the set of isomorphism classes of the bounded derived category

$$\mathcal{D}^b\mathrm{coh}_{\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}'}(\mathcal{Y} \times_{BG} \mathcal{Y}') \subset \mathrm{perf}(\mathcal{Y} \times_{BG} \mathcal{Y}') \tag{5.1}$$

of those coherent $\mathcal{O}_{\mathcal{Y}\times_{BG}\mathcal{Y}'}$–modules that are (topologically) supported on the closed substack $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}'$ of $\mathcal{Y} \times_{BG} \mathcal{Y}'$. Note that since $\mathcal{Y} \times_{BG} \mathcal{Y}'$ is smooth, every bounded coherent complex on it is perfect. Note also that the above definition (5.1) depends on the fact that $\mathcal{X}$ is a (global) quotient stack. The composition law

$$\mathrm{Hom}_{\mathbb{H}\mathrm{mo}(\mathcal{X})}(\mathcal{Y}', \mathcal{Y}'') \times \mathrm{Hom}_{\mathbb{H}\mathrm{mo}(\mathcal{X})}(\mathcal{Y}, \mathcal{Y}') \to \mathrm{Hom}_{\mathbb{H}\mathrm{mo}(\mathcal{X})}(\mathcal{Y}, \mathcal{Y}'')$$

is induced by the classical (derived) "pullback/pushforward" formula

$$(\mathcal{E}', \mathcal{E}) \mapsto (p_{13})_*((p_{23})^*(\mathcal{E}') \otimes^{\mathbf{L}} (p_{12})^*(\mathcal{E})), \tag{5.2}$$

where p_{ij} stands for the projection from the triple fiber product onto its ij–factor. Finally, the identity of an object $\mathcal{Y}$ of $\mathbb{H}\mathrm{mo}(\mathcal{X})$ is the (isomorphism class of the) structure sheaf $\mathcal{O}_\Delta$ of the diagonal Δ in $\mathcal{Y} \times_{BG} \mathcal{Y}$.

The *additivization* of $\mathbb{H}\mathrm{mo}(\mathcal{X})$ is the category $\mathbb{H}\mathrm{mo}_0(\mathcal{X})$ with the same objects as $\mathbb{H}\mathrm{mo}(\mathcal{X})$ and with abelian groups of morphisms $\mathrm{Hom}_{\mathbb{H}\mathrm{mo}_0(\mathcal{X})}(\mathcal{Y},\mathcal{Y}')$ given by the Grothendieck group[11] of the triangulated category (5.1). Thanks to Quillen's dévissage theorem [44, Section 5] and to the definition of G–theory, we have isomorphisms

$$\mathrm{Hom}_{\mathbb{H}\mathrm{mo}_0(\mathcal{X})}(\mathcal{Y},\mathcal{Y}') \simeq G_0(\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}'). \tag{5.3}$$

In particular, we have a ring isomorphism

$$\mathrm{End}_{\mathbb{H}\mathrm{mo}_0(\mathcal{X})}(\mathcal{X}) \simeq K_0(\mathcal{X}). \tag{5.4}$$

Note also that, since the assignment (5.2) is exact in each one of the variables, the composition law of $\mathbb{H}\mathrm{mo}(\mathcal{X})$ extends naturally to $\mathbb{H}\mathrm{mo}_0(\mathcal{X})$.

Definition 5.5 (G_0–motives) The *category of G_0–motives* $\mathbb{M}\mathrm{ot}(\mathcal{X})$ is defined by formally adding to $\mathbb{H}\mathrm{mo}_0(\mathcal{X})$ all finite direct sums and direct summands. Let us write $\mathbb{U}(-)$ for the canonical functor from $\mathbb{H}\mathrm{mo}(\mathcal{X})$ to $\mathbb{M}\mathrm{ot}(\mathcal{X})$.

Given objects $\mathcal{Y}$ and $\mathcal{Y}'$ of $\mathbb{H}\mathrm{mo}(\mathcal{X})$, let us write i for the finite morphism $\mathcal{Y}\times_{BG}\mathcal{Y}'\to\mathcal{Y}\times\mathcal{Y}'$. Under this notation, we have the exact functor

$$\mathcal{D}^b\mathrm{coh}_{\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}'}(\mathcal{Y}\times_{BG}\mathcal{Y}')\to \mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}),\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}')),\quad \mathcal{E}\mapsto {}_{\Phi_{i_*(\mathcal{E})}}\mathrm{B}, \tag{5.6}$$

where $\Phi_{i_*(\mathcal{E})}$ stands for the Fourier–Mukai dg functor

$$\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y})\to\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}'),\quad \mathcal{G}\mapsto (p_2)_*((p_1)^*(\mathcal{G})\otimes^{\mathbf{L}} i_*(\mathcal{E})), \tag{5.7}$$

and ${}_{\Phi_{i_*(\mathcal{E})}}\mathrm{B}$ for the associated bimodule; see Section 2.1. By construction of the categories $\mathbb{H}\mathrm{mo}(\mathcal{X})$ and $\mathrm{Hmo}(k)$, these considerations lead to a functor

$$\mathbb{H}\mathrm{mo}(\mathcal{X})\to\mathrm{Hmo}(k),\quad \mathcal{Y}\mapsto\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}),\quad \mathcal{E}\mapsto {}_{\Phi_{i_*(\mathcal{E})}}\mathrm{B},$$

which naturally extends to the motivic categories:

$$\Psi\colon \mathbb{M}\mathrm{ot}(\mathcal{X})\to\mathrm{NMot}(k),\quad \mathbb{U}(\mathcal{Y})\mapsto \mathrm{U}(\mathcal{Y}).$$

Remark 5.8 (sheaves of algebras) Suppose that $\mathcal{X}$ is equipped with a *flat* quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}}$–algebras $\mathcal{F}$. In this case, as we now explain, it is possible to

[11] The idea of using the Grothendieck group in the construction of categories of "motivic nature" goes back to the work of Manin [40] and Gillet and Soulé [19, Section 5.2].

construct a variant $\Psi_{\mathcal{F}}$ of the functor Ψ. Given objects $\mathcal{Y}_1 \xrightarrow{f_1} \mathcal{X}$ and $\mathcal{Y}_2 \xrightarrow{f_2} \mathcal{X}$ of $\mathbb{H}\mathrm{mo}(\mathcal{X})$, we may view (5.6) as an exact functor

$$\mathcal{D}^b\mathrm{coh}_{\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}'}(\mathcal{Y}\times_{BG}\mathcal{Y}') \to \mathrm{rep}\big(\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}_1; f_1^*(\mathcal{F})), \mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}_2; f_2^*(\mathcal{F}))\big),$$

where $\Phi_{i_*(\mathcal{E})}$ is defined by the above formula (5.7) but considered as a Fourier–Mukai dg functor from $\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}_1; f_1^*(\mathcal{F}))$ to $\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}_2; f_2^*(\mathcal{F}))$. By construction of the categories $\mathbb{H}\mathrm{mo}(\mathcal{X})$ and $\mathrm{Hmo}(k)$, these considerations lead to a functor

$$\mathbb{H}\mathrm{mo}(\mathcal{X}) \to \mathrm{Hmo}(k), \quad (\mathcal{Y} \xrightarrow{f} \mathcal{X}) \mapsto \mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}; f^*(\mathcal{F})), \quad \mathcal{E} \mapsto {}_{\Phi_{i_*(\mathcal{E})}}\mathrm{B},$$

which naturally extends to the motivic categories:

$$\Psi_{\mathcal{F}}\colon \mathbb{M}\mathrm{ot}(\mathcal{X}) \to \mathrm{NMot}(k), \quad \mathbb{U}(\mathcal{Y} \xrightarrow{f} \mathcal{X}) \mapsto \mathrm{U}(\mathcal{Y}; f^*(\mathcal{F})).$$

5.2 Properties

In what follows, we establish some (structural) properties of the category of G_0–motives. These will be used in the next sections.

5.2.1 Pushforward and pullback Let $\alpha\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism in $\mathrm{DM}(\mathcal{X})$. Its *pushforward* $\alpha_*\colon \mathbb{U}(\mathcal{Y}_1) \to \mathbb{U}(\mathcal{Y}_2)$ and *pullback* $\alpha^*\colon \mathbb{U}(\mathcal{Y}_2) \to \mathbb{U}(\mathcal{Y}_1)$ are defined as the Grothendieck classes

$$[(\mathrm{id}, \alpha)_*(\mathcal{O}_{\mathcal{Y}})] \in G_0(\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}') \quad \text{and} \quad [(\alpha, \mathrm{id})_*(\mathcal{O}_{\mathcal{Y}})] \in G_0(\mathcal{Y}'\times_{\mathcal{X}}\mathcal{Y}),$$

respectively; see (5.3). Note that $\Psi(\alpha_*) = \alpha_*$ and $\Psi(\alpha^*) = \alpha^*$. Note also that if $\alpha, \beta\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ are isomorphic 1–morphisms in $\mathrm{DM}(\mathcal{X})$, then $\alpha_* = \beta_*$ and $\alpha^* = \beta^*$.

5.2.2 K_0–action Let $\mathbb{U}(\mathcal{Y})$ be an object of $\mathbb{M}\mathrm{ot}(\mathcal{X})$. The pushforward along the diagonal map $i_\Delta\colon \mathcal{Y} \to \mathcal{Y}\times_{BG}\mathcal{Y}$ leads to an exact functor

$$i_{\Delta,*}\colon \mathrm{perf}(\mathcal{Y}) \to \mathcal{D}^b\mathrm{coh}_{\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}}(\mathcal{Y}\times_{BG}\mathcal{Y}) \tag{5.9}$$

that sends the tensor product $-\otimes_{\mathcal{Y}}-$ on the left-hand side to the "pullback/pushforward" formula (5.2) on the right-hand side. Therefore, by applying $K_0(-)$ to (5.9), we obtain an induced ring morphism $K_0(\mathcal{Y}) \to \mathrm{End}_{\mathbb{M}\mathrm{ot}(\mathcal{X})}(\mathbb{U}(\mathcal{Y}))$. In other words, we obtain an action of the Grothendieck ring $K_0(\mathcal{Y})$ on the G_0–motive $\mathbb{U}(\mathcal{Y})$.

Lemma 5.10 *The functor Ψ is compatible with the $K_0(\mathcal{Y})$–action on $\mathbb{U}(\mathcal{Y})$* (defined above) *and on* $\mathrm{U}(\mathcal{Y})$ (defined in Section 3).

Proof Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{perf}(\mathcal{Y}) & \xrightarrow{i_{\Delta,*}} & \mathcal{D}^b\mathrm{coh}_{\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y}}(\mathcal{Y}\times_{BG}\mathcal{Y}) \\ \| & & \downarrow{\scriptstyle \mathcal{E}\mapsto\Phi_{i_*(\mathcal{E})}\mathrm{B}} \\ \mathrm{perf}(\mathcal{Y}) & \xrightarrow[M\mapsto(M\otimes_{\mathcal{Y}}-)\mathrm{B}]{} & \mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y}),\mathrm{perf}_{\mathrm{dg}}(\mathcal{Y})) \end{array} \tag{5.11}$$

By applying $K_0(-)$ to (5.11), we obtain the claimed compatibility. □

5.2.3 $K_0(\mathcal{X})$–linearity Let $\mathcal{Y}\xrightarrow{f}\mathcal{X}$ be an object of $\mathrm{DM}(\mathcal{X})$. By composing the induced ring homomorphism $f^*\colon K_0(\mathcal{X})\to K_0(\mathcal{Y})$ with the $K_0(\mathcal{Y})$–action on $\mathbb{U}(\mathcal{Y})$ described in Section 5.2.2, we obtain a $K_0(\mathcal{X})$–action on $\mathbb{U}(\mathcal{Y})$. A simple verification shows that this $K_0(\mathcal{X})$–action is compatible with the morphisms of $\mathbb{M}\mathrm{ot}(\mathcal{X})$. In other words, $\mathbb{M}\mathrm{ot}(\mathcal{X})$ is a $K_0(\mathcal{X})$–linear category. The map $\mathcal{X}\to BG$ induces a ring homomorphism $R(G)\to K_0(\mathcal{X})$. Therefore, $\mathbb{M}\mathrm{ot}(\mathcal{X})$ is also a $R(G)$–linear category.

5.2.4 G–action Let $Y\hookrightarrow X$ be a smooth closed k–subscheme of X which is preserved by a subgroup $H\subseteq G$, and $[Y/H]\xrightarrow{f}\mathcal{X}$ the corresponding object of $\mathrm{DM}(\mathcal{X})$. Consider also the objects $[gY/gHg^{-1}]\xrightarrow{f_g}\mathcal{X}$ of $\mathrm{DM}(\mathcal{X})$ with $g\in G$. Note that we have a commutative 2–diagram of (global) orbifolds

$$\begin{array}{ccc} [Y/H] & \xrightarrow{\alpha_g} & [gY/gHg^{-1}] \\ & {\scriptstyle f}\searrow \quad \overset{\eta_g}{\Longleftarrow} \quad \swarrow{\scriptstyle f_g} & \\ & \mathcal{X} & \end{array}$$

where the 1–isomorphism α_g is given by $(y\mapsto gy, h\mapsto ghg^{-1})$ and the evaluation of the 2–isomorphism $\eta_g\colon f_g\circ\alpha_g\Rightarrow f$ at $y\in Y$ is given by $gy\xrightarrow{g^{-1}}y$. Therefore, α_g may be considered as an isomorphism in the category $\mathrm{DM}(\mathcal{X})$; in the sequel, we will write g instead of α_g (note that α_g acts as g on Y). By functoriality (see Section 5.2.1), we obtain inverse isomorphisms g^* and g_* between $\mathbb{U}([Y/H])$ and $\mathbb{U}([gY/gHg^{-1}])$.

Now, assume that Y is stabilized by the normalizer $N(H)$ of H. In this case, the above considerations lead to a $N(H)$–action on $[Y/H]$ and, consequently, on the G_0–motive $\mathbb{U}([Y/H])$. Moreover, the induced morphism $f^*\colon\mathbb{U}(\mathcal{X})_{1/n}\to\mathbb{U}([Y/H])_{1/n}$ factors through the direct summand $\mathbb{U}([Y/H])_{1/n}^{N(H)}$.

Example 5.12 Let $\sigma\in\varphi$ be a cyclic subgroup. By choosing $Y=X^\sigma$, (resp. $Y=X$) and $H=\sigma$, we obtain an induced $N(\sigma)$–action on $[X^\sigma/\sigma]$ (resp. $[X/\sigma]$) and, consequently, on the G_0–motive $\mathbb{U}([X^\sigma/\sigma])$ (resp. $\mathbb{U}([X/\sigma])$).

5.2.5 Base-change functoriality Given a field extension l/k, base-change $-\times_k l$ leads to a (2–)functor $\mathrm{DM}(\mathcal{X}) \to \mathrm{DM}(\mathcal{X}_l)$, where $\mathcal{X}_l = [X_l/G]$. By construction, this functor extends to G_0–motives $(-)\times_k l\colon \mathbb{M}\mathrm{ot}(\mathcal{X}) \to \mathbb{M}\mathrm{ot}(\mathcal{X}_l)$, $\mathbb{U}(\mathcal{Y}) \mapsto \mathbb{U}(\mathcal{Y}_l)$.

5.2.6 Pushforward functoriality Let $[Y/H] \to [X/G]$ be a morphism of (global) orbifolds induced by a finite map $Y \to X$ and by a group homomorphism $H \to G$. The associated pushforward functor $\mathrm{DM}([Y/H]) \to \mathrm{DM}([X/G])$ sends $\mathcal{Z} \to [Y/H]$ to the composition $\mathcal{Z} \to [Y/H] \to [X/G]$. By construction, this functor extends naturally to G_0–motives $\mathbb{M}\mathrm{ot}([Y/H]) \to \mathbb{M}\mathrm{ot}([X/G])$, $\mathbb{U}(\mathcal{Z}) \mapsto \mathbb{U}(\mathcal{Z})$.

5.2.7 Pullback functoriality The morphism of (global) orbifolds $[X/G] \to BG$ leads to a pullback functor $\mathbb{M}\mathrm{ot}(BG) \to \mathbb{M}\mathrm{ot}(\mathcal{X})$, $\mathbb{U}(\mathcal{Y}) \mapsto \mathbb{U}(\mathcal{X}\times_{BG}\mathcal{Y})$.

5.3 Decomposition

Following Section 5.2.3, the category $\mathbb{M}\mathrm{ot}([X/G])$ is $R(G)$–linear. Given a cyclic subgroup $\sigma \in \varphi$, let us write γ_σ for the morphism of (global) orbifolds $[X^\sigma/\sigma] \to [X/G]$. The normalizer $N(\sigma)$ acts on the G_0–motive $\mathbb{U}([X^\sigma/\sigma])$; see Example 5.12. Moreover, the pullback morphism $\gamma_\sigma^*\colon \mathbb{U}([X/G])_{1/n} \to \mathbb{U}([X^\sigma/\sigma])_{1/n}$ factors through the direct summand $\mathbb{U}([X^\sigma/\sigma])_{1/n}^{N(\sigma)}$. Let $e_\sigma \in R(\sigma)_{1/n}$ be the canonical idempotent introduced in Definition 4.4. As explained in Section 4.0.2, since $N(\sigma)$ maps naturally into $\mathrm{Aut}(\sigma)$, the idempotent e_σ is invariant under the $N(\sigma)$–action on $R(\sigma)_{1/n}$. Hence, we can (and will) identify $e_\sigma(\mathbb{U}([X^\sigma/\sigma])_{1/n}^{N(\sigma)})$ with $(e_\sigma(\mathbb{U}[X^\sigma/\sigma])_{1/n})^{N(\sigma)}$. Under the above notations and identifications, we have the morphism in the category $\mathbb{M}\mathrm{ot}([X/G])_{1/n}$

$$\overline{\theta}\colon \mathbb{U}([X/G])_{1/n} \xrightarrow{\oplus_\sigma e_\sigma\circ\gamma_\sigma^*} \bigoplus_{\sigma\in\varphi/\sim} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)}.$$

Theorem 5.13 *The above morphism $\overline{\theta}$ is invertible.*

Proof Consider the inclusion ι and projection π morphisms

$$\widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)} \xrightarrow{\iota_\sigma} \mathbb{U}([X^\sigma/\sigma])_{1/n}\mathbb{U}([X^\sigma/\sigma])_{1/n} \xrightarrow{\pi_\sigma} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)}.$$

Under these notations, we have $\overline{\theta} = \bigoplus_\sigma \pi_\sigma \circ \gamma_\sigma^*$. Thanks to Lemmas 5.17, 5.22 and 5.25 below (see Remark 5.16), the morphism $\overline{\theta}$ admits a right inverse $\overline{\psi}$. This implies that $e := \mathrm{id} - \overline{\psi}\circ\overline{\theta}$ is an idempotent in $\mathrm{End}(\mathbb{U}([X/G])_{1/n}) \simeq K_0([X/G])_{1/n}$. We need to prove that $e = 0$. Let l be the field obtained from k by adjoining the

n^{th} roots of unity. Thanks to Corollary A.3, it is sufficient to prove that $e \times_k l = 0$ (consult Section 5.2.5 for the notation $(-) \times_k l$). Hence, we may assume without loss of generality that $l = k$ and, consequently, that k contains the n^{th} roots of unity.

In order to prove that $e = 0$, let us consider the additive invariant $K_0(-)_{1/n}$ with values in the category of $\mathbb{Z}[1/n]$–modules. The composition

$$\textbf{(5.14)} \qquad \mathrm{End}(\mathbb{U}([X/G])_{1/n}) \to \mathrm{End}(\mathrm{U}([X/G])_{1/n}) \to \mathrm{End}(K_0([X/G])_{1/n}),$$

where the first map is induced by the functor Ψ and the second by the functor corresponding to $K_0(-)_{1/n}$ under the equivalence of categories (2.7), sends $[\mathcal{E}]_{1/n} \in K_0([X/G])_{1/n}$ to left multiplication by $[\mathcal{E}]_{1/n}$. Since $K_0([X/G])_{1/n}$ is a unital $\mathbb{Z}[1/n]$–algebra, this implies that the morphism (5.14) is injective. By an abuse of notation, let us still denote by $K_0(-)_{1/n}$ the composition (5.14). Under this notation, $e = 0$ if and only if $K_0(e)_{1/n} = 0$. By construction, we have $K_0(e)_{1/n} = \mathrm{id} - K_0(\bar{\psi})_{1/n} \circ K_0(\bar{\theta})_{1/n}$. Therefore, it suffices to prove that $K_0(\bar{\theta})_{1/n}$ is an isomorphism. Modulo the caveat in Remark 5.15 below, this was proved in [54, Theorem (5.2)]. □

Remark 5.15 (ample line bundle) Vistoli's results [54] were proved under the assumption that X carries an ample line bundle. However, as explained by him in [54, page 402], this assumption is only used in the proof of [54, Lemma 1.1]. By invoking [50, Theorem 2.3 and Corollary 2.4], we observe that the proof of [54, Lemma 1.1] remains valid under the much weaker assumption that X is quasicompact and quasi-separated. Therefore, we can freely use the results from [54] is our current setting.

Remark 5.16 (right inverse) Let $\bar{\psi}' := \bigoplus_\sigma \gamma_{\sigma,*} \circ \iota_\sigma$. In what follows, we will prove that the composition $\bar{\theta} \circ \bar{\psi}'$ is invertible. This implies that $\bar{\theta}$ admits a right inverse $\bar{\psi}$.

Lemma 5.17 *For every $\sigma \neq \sigma' \in \varphi/\sim$, we have $\pi_{\sigma'} \circ \gamma_{\sigma'}^* \circ \gamma_{\sigma,*} \circ \iota_\sigma = 0$.*

Proof Since the category $\mathbb{M}\mathrm{ot}([X/G])$ is $R(G)$–linear, the morphism

$$\textbf{(5.18)} \qquad \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)} \xrightarrow{\pi_{\sigma'} \circ \gamma_{\sigma'}^* \circ \gamma_{\sigma,*} \circ \iota_\sigma} \widetilde{\mathbb{U}}([X^{\sigma'}/\sigma'])_{1/n}^{N(\sigma')}$$

is $R(G)_{1/n}$–linear. The idempotent $\widetilde{e}_\sigma \in R(G)_{1/n}$ (see Notation 4.13) acts as the identity on the source of (5.18), whereas the idempotent $\widetilde{e}_{\sigma'}$ acts as the identity on the target. Therefore, since $\widetilde{e}_\sigma \widetilde{e}_{\sigma'} = 0$, the homomorphism (5.18) is zero. □

We now claim that the following composition is invertible:

$$\widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)} \xrightarrow{\pi_\sigma \circ \gamma_\sigma^* \circ \gamma_{\sigma,*} \circ \iota_\sigma} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)}. \tag{5.19}$$

Note that this implies that the composition $\overline{\theta} \circ \overline{\psi}'$ is invertible, and hence concludes the proof of Theorem 5.13. In order to prove this claim, consider the factorization

$$\gamma_\sigma \colon [X^\sigma/\sigma] \xrightarrow{\alpha_\sigma} [X/\sigma] \xrightarrow{\beta_\sigma} [X/G],$$

as well as the inclusion ι' and projection π' morphisms

$$\widetilde{\mathbb{U}}([X/\sigma])_{1/n}^{N(\sigma)} \xrightarrow{\iota'_\sigma} \mathbb{U}([X/\sigma])_{1/n}\mathbb{U}([X/\sigma])_{1/n} \xrightarrow{\pi'_\sigma} \widetilde{\mathbb{U}}([X/\sigma])_{1/n}^{N(\sigma)}.$$

This data leads to the commutative diagrams

$$\begin{array}{ccccc} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)} & \xrightarrow{\iota_\sigma} & \mathbb{U}([X^\sigma/\sigma])_{1/n} & \xrightarrow{\gamma_{\sigma,*}} & \mathbb{U}([X/G])_{1/n} \\ \downarrow{\scriptstyle \alpha_{\sigma,*}} & & \downarrow{\scriptstyle \alpha_{\sigma,*}} & & \| \\ \widetilde{\mathbb{U}}([X/\sigma])_{1/n}^{N(\sigma)} & \xrightarrow[\iota'_\sigma]{} & \mathbb{U}([X/\sigma])_{1/n} & \xrightarrow[\beta_{\sigma,*}]{} & \mathbb{U}([X/G])_{1/n} \end{array} \tag{5.20}$$

$$\begin{array}{ccccc} \mathbb{U}([X/G])_{1/n} & \xrightarrow{\gamma_\sigma^*} & \mathbb{U}([X^\sigma/\sigma])_{1/n} & \xrightarrow{\pi_\sigma} & \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)} \\ \| & & \uparrow{\scriptstyle \alpha_\sigma^*} & & \uparrow{\scriptstyle \alpha_\sigma^*} \\ \mathbb{U}([X/G])_{1/n} & \xrightarrow[\beta_\sigma^*]{} & \mathbb{U}([X/\sigma])_{1/n} & \xrightarrow[\pi'_\sigma]{} & \widetilde{\mathbb{U}}([X/\sigma])_{1/n}^{N(\sigma)} \end{array} \tag{5.21}$$

in the category of G_0–motives $\mathbb{M}\mathrm{ot}([X/G])_{1/n}$; the existence of the left-hand square in (5.20) and the right-hand square in (5.21) follows from the $R(G)$–linearity of $\mathbb{M}\mathrm{ot}([X/G])$ and from the fact that α_σ commutes with the $N(\sigma)$–actions on $[X^\sigma/\sigma]$ and $[X/\sigma]$ (as objects of $\mathrm{DM}([X/G])$) introduced in Example 5.12.

Lemma 5.22 *For every $\sigma \in \varphi$, the composition $\pi'_\sigma \circ \beta_\sigma^* \circ \beta_{\sigma,*} \circ \iota'_\sigma$ is equal to $[N(\sigma):\sigma] \cdot \mathrm{id}$. In particular, it is invertible.*

Proof Thanks to the pullback functor $\mathbb{M}\mathrm{ot}(BG) \to \mathbb{M}\mathrm{ot}([X/G])$ (see Section 5.2.7), it suffices to prove Lemma 5.22 in the particular case where $X = \bullet$. Let us write $\mathcal{H}$ for a (chosen) set of representatives of the double cosets $\sigma\backslash G/\sigma$. The fiber product $[\bullet/\sigma] \times_{[\bullet/G]} [\bullet/\sigma]$ decomposes into the disjoint union $\bigsqcup_{\tau\in\mathcal{H}}[\bullet/(\sigma \cap \tau\sigma\tau^{-1})]$. By

unpacking the definitions, we observe that the composition

$$\mathbb{U}([\bullet/\sigma]) \xrightarrow{\beta_{\sigma,*}} \mathbb{U}([\bullet/G]) \xrightarrow{\beta_\sigma^*} \mathbb{U}([\bullet/\sigma]) \tag{5.23}$$

is the sum over $\tau \in \mathcal{H}$ of the compositions

$$\mathbb{U}([\bullet/\sigma]) \xrightarrow{\mu^*} \mathbb{U}([\bullet/(\sigma \cap \tau\sigma\tau^{-1})]) \xrightarrow{\nu_*} \mathbb{U}([\bullet/\sigma]), \tag{5.24}$$

where $\mu\colon [\bullet/(\sigma \cap \tau\sigma\tau^{-1})] \to [\bullet/\sigma]$ corresponds to the inclusion $\sigma \cap \tau\sigma\tau^{-1} \subseteq \sigma$ and $\nu\colon [\bullet/(\sigma \cap \tau\sigma\tau^{-1})] \to [\bullet/\sigma]$ to the inclusion $\sigma \cap \tau\sigma\tau^{-1} \xrightarrow{\tau^{-1}\cdot\tau} \sigma$; note that, as explained in Section 5.2.4, ν is a morphism in $\mathrm{DM}([\bullet/G])$.

If $\tau \notin N(\sigma)$, then $e_\sigma \in R(\sigma)_{1/n}$ acts on $\mathbb{U}([\bullet/(\sigma \cap \tau\sigma\tau^{-1})])_{1/n}$ by it its image in $R(\sigma \cap \tau\sigma\tau^{-1})_{1/n}$. Hence, thanks to Lemma 4.6, e_σ acts as zero. This implies that the precomposition (of the $\mathbb{Z}[1/n]$–linearization) of (5.24) with the inclusion $\widetilde{\mathbb{U}}([\bullet/\sigma])_{1/n} \hookrightarrow \mathbb{U}([\bullet/\sigma])_{1/n}$ is zero. Therefore, in the evaluation of $\pi'_\sigma \circ \beta_\sigma^* \circ \beta_{\sigma,*} \circ \iota'_\sigma$, we only have to consider the terms in (5.24) where $\tau \in N(\sigma)$.

If $\tau \in N(\sigma)$, then $\mu = \mathrm{id}$ and $\nu = \alpha_{\tau^{-1}}$ as defined in Section 5.2.4. Therefore, (5.24) agrees with the action of $\tau^{-1} \in N(\sigma)$ on $\mathbb{U}([\bullet/\sigma])_{1/n}$. In particular, it induces the identity on the direct summand $\mathbb{U}([\bullet/\sigma])_{1/n}^{N(\sigma)}$. Now, the number of terms (5.24) where $\tau \in N(\sigma)$ is equal to $[N(\sigma):\sigma]$. This concludes the proof. □

Using Lemma 5.22 and the commutative diagrams (5.20)–(5.21), we observe that in order to prove our claim it suffices to prove that the composition

$$\widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)} \xrightarrow{\alpha_{\sigma,*}} \widetilde{\mathbb{U}}([X/\sigma])_{1/n}^{N(\sigma)} \xrightarrow{\alpha_\sigma^*} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)}$$

is invertible. The preceding composition can be rewritten as $\pi_\sigma \circ \alpha_\sigma^* \circ \alpha_{\sigma,*} \circ \iota_\sigma$. Therefore, the proof of our claim follows now automatically from the next result:

Lemma 5.25 *For every $\sigma \in \varphi$, the composition $\pi_\sigma \circ \alpha_\sigma^* \circ \alpha_{\sigma,*} \circ \iota_\sigma$ is invertible.*

Proof Consider the factorizations

$$\iota_\sigma\colon \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)} \xrightarrow{\iota_{\sigma,1}} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n} \xrightarrow{\iota_{\sigma,2}} \mathbb{U}([X^\sigma/\sigma])_{1/n},$$
$$\pi_\sigma\colon \mathbb{U}([X^\sigma/\sigma])_{1/n} \xrightarrow{\pi_{\sigma,2}} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n} \xrightarrow{\pi_{\sigma,1}} \widetilde{\mathbb{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)}.$$

Since $\pi_\sigma \circ \alpha_\sigma^* \circ \alpha_{\sigma,*} \circ \iota_\sigma = (\pi_{\sigma,2} \circ \alpha_\sigma^* \circ \alpha_{\sigma,*} \circ \iota_{\sigma,2})^{N(\sigma)}$, it suffices to prove that the composition $\pi_{\sigma,2} \circ \alpha_\sigma^* \circ \alpha_{\sigma,*} \circ \iota_{\sigma,2}$ is invertible. Moreover, thanks to the pushforward functor $\mathbb{M}\mathrm{ot}([X/\sigma]) \to \mathbb{M}\mathrm{ot}([X/G])$ (see Section 5.2.6), it is enough to consider the

particular case where $G=\sigma$. In this case, we have $\pi_{\sigma,2}=\pi_\sigma$ and $\iota_{\sigma,2}=\iota_\sigma$. Moreover, we have ring isomorphisms

$$\textbf{(5.26)}\quad \mathrm{End}_{\mathbb{M}\mathrm{ot}([X/\sigma])}(\mathbb{U}([X^\sigma/\sigma]))\simeq G_0([X^\sigma/\sigma]\times_{[X/\sigma]}[X^\sigma/\sigma])\simeq K_0([X^\sigma/\sigma]).$$

Furthermore, under the ring isomorphism (5.26), the composition

$$\textbf{(5.27)}\qquad \mathbb{U}([X^\sigma/\sigma])\xrightarrow{\alpha_{\sigma,*}}\mathbb{U}([X/\sigma])\xrightarrow{\alpha_\sigma^*}\mathbb{U}([X^\sigma/\sigma])$$

corresponds to the Grothendieck class

$$[\mathcal{O}_{[X^\sigma/\sigma]}\otimes^{\mathbf{L}}_{X/\sigma}\mathcal{O}_{[X^\sigma/\sigma]}]=\sum_i(-1)^i[H^i(\mathcal{O}_{X^\sigma}\otimes^{\mathbf{L}}_{X/\sigma}\mathcal{O}_{X^\sigma})]=\sum_i(-1)^i\big[\textstyle\bigwedge^i(I/I^2)\big],$$

where I stands for the sheaf of ideals associated to the closed immersion $X^\sigma\hookrightarrow X$. By composing and precomposing (the $\mathbb{Z}[1/n]$–linearization of) (5.27) with π_σ and ι_σ, respectively, we obtain the image ξ of the above Grothendieck class in the direct summand $e_\sigma K_0([X^\sigma/\sigma])_{1/n}$. In order to prove that ξ is invertible, consider the field l obtained from k by adjoining the n^{th} roots of unity, and the homomorphism

$$\textbf{(5.28)}\qquad e_\sigma K_0([X^\sigma/\sigma])_{1/n}\to e_\sigma K_0([X_l^\sigma/\sigma])_{1/n}.$$

Thank to Corollary 2.12 (with $X=X_l^\sigma$ and $G=\sigma$) and to the fact that l contains the n^{th} roots of unity, we have $K_0([X_l^\sigma/\sigma])\simeq K_0(X_l^\sigma)\otimes_{\mathbb{Z}[1/n]}R_l(\sigma)$. Under this ring isomorphism, the canonical $R_l(\sigma)$–action on $K_0([X_l^\sigma/\sigma])$ corresponds to the tautological $R_l(\sigma)$–action on $R_l(\sigma)$. Therefore, by first applying $(-)_{1/n}$ and then taking the direct summands corresponding to $e_\sigma\in\widetilde{R}_l(\sigma)_{1/n}$, we conclude that the right-hand side of (5.28) is isomorphic to $K_0(X_l^\sigma)_{1/n}\otimes_{\mathbb{Z}[1/n]}\widetilde{R}_l(\sigma)_{1/n}$. Let us write $\{X_{l,i}^\sigma\}_{i\in I}$ for the connected components of X_l^σ. Under these notations, we have also the rank map

$$\textbf{(5.29)}\quad e_\sigma K_0([X_l^\sigma/\sigma])_{1/n}\simeq\bigoplus_{i\in I}K_0(X_{l,i}^\sigma)_{1/n}\otimes_{\mathbb{Z}[1/n]}\widetilde{R}_l(\sigma)_{1/n}\xrightarrow{\text{rank}}\bigoplus_{i\in I}\widetilde{R}_l(\sigma)_{1/n}.$$

Let us denote by $\overline{\xi}$ the image of ξ under the composition (5.29)$\circ$(5.28). As proved in [54, Lemma (1.8)] (with the caveat of Remark 5.15), the image of ξ under the homomorphism (5.28) is invertible. Consequently, $\overline{\xi}$ is also invertible. Thanks to Corollary A.3 (with $X=X^\sigma$ and $G=\sigma$) and [50, Theorem 2.3], respectively, the elements in the kernel of the homomorphism (5.28) and (5.29) are nilpotent. Therefore, in order to prove that ξ is invertible, it suffices to show that the inverse ζ of $\overline{\xi}$ belongs to the image of the composition (5.29)$\circ$(5.28). Since $\bigoplus_{i\in I}\widetilde{R}_l(\sigma)_{1/n}$ is a Noetherian

$\mathbb{Z}[1/n]$–module, there exists an integer $N \geq 1$ such that $\zeta^N + b_1\zeta^{N-1} + \cdots + b_N = 0$ with $b_i \in \mathbb{Z}[1/n]$. By multiplying this equality with $\overline{\xi}^N$, we hence conclude that

$$1 + b_1\overline{\xi} + \cdots + b_N\overline{\xi}^N = 1 + (b_1 + \cdots + b_N\overline{\xi}^{N-1})\overline{\xi} = 0.$$

This shows that the inverse $\zeta = -(b_1 + \cdots + b_N\overline{\xi}^{N-1})$ of $\overline{\xi}$ belongs, indeed, to the image of the composition (5.29) $\circ$ (5.28). □

6 Proofs: decomposition of orbifolds

In this section we prove Theorem 1.1 as well as Corollaries 1.5 and 1.8.

Proof of Theorem 1.1

As explained in Section 5.2.1 (resp. Lemma 5.10), the functor Ψ is compatible with pullbacks (resp. with K_0–actions). Therefore, by applying it to the isomorphism $\overline{\theta}$ of Theorem 5.13 we obtain the isomorphism

$$\theta\colon \mathrm{U}([X/G])_{1/n} \xrightarrow{\oplus_\sigma e_\sigma \circ \gamma_\sigma^*} \bigoplus_{\sigma\in\varphi/\sim} \widetilde{\mathrm{U}}([X^\sigma/\sigma])_{1/n}^{N(\sigma)}$$

in the category $\mathrm{NMot}(k)_{1/n}$. Since θ is an isomorphism of monoids, the proof of Theorem 1.1 follows now from the equivalences of categories (2.7)–(2.8).

Proof of Corollary 1.5

We start with some computations:

Proposition 6.1 *Let $\sigma \in \varphi$ be a cyclic subgroup. If k contains the n^{th} roots of unity, then we have the isomorphism of (commutative) monoids*

$$\widetilde{\mathrm{U}}([X^\sigma/\sigma])_{1/n} \simeq \mathrm{U}(X^\sigma)_{1/n} \otimes_{\mathbb{Z}[1/n]} \widetilde{R}(\sigma)_{1/n}. \tag{6.2}$$

Proof Recall from Remark 2.9 that we have the isomorphism

$$\mathrm{Hom}_{\mathrm{NMot}(k)}(\mathrm{U}(k), \mathrm{U}(B\sigma)) \simeq K_0(B\sigma) = R(\sigma).$$

This leads naturally to a $R(\sigma)$–action on $\mathrm{U}(B\sigma)$, ie to a morphism of monoids

$$\mathrm{U}(k) \otimes_{\mathbb{Z}} R(\sigma) \to \mathrm{U}(B\sigma). \tag{6.3}$$

Note that by applying $\mathrm{Hom}_{\mathrm{NMot}(k)}(\mathrm{U}(k),-)$ to (6.3), we obtain an isomorphism. Therefore, thanks to the enriched Yoneda lemma, in order to show that (6.3) is an isomorphism, it suffices to show that $\mathrm{U}(B\sigma)$ is isomorphic to a (finite) direct sum of copies of $\mathrm{U}(k)$. This is, indeed, the fact, because $\mathrm{U}(B\sigma)\simeq\mathrm{U}(k[\sigma])$ and $k[\sigma]$ is isomorphic to a (finite) direct sum of copies of k (this uses the assumption that k contains the n^{th} roots of unity). Now, note that the cyclic subgroup $\sigma\in\varphi$ acts trivially on X^σ. By combining Corollary 2.12 (with $X=X^\sigma$ and $G=\sigma$) with (6.3), we then obtain an isomorphism of (commutative) monoids

$$\mathrm{U}([X^\sigma/\sigma])\simeq\mathrm{U}(X^\sigma)\otimes\mathrm{U}(k)\otimes_{\mathbb{Z}[1/n]}R(\sigma)\simeq\mathrm{U}(X^\sigma)\otimes_{\mathbb{Z}[1/n]}R(\sigma). \tag{6.4}$$

Under the isomorphism (6.4), the canonical $R(\sigma)$–action on $\mathrm{U}([X^\sigma/\sigma])$ (described in Section 3) corresponds to the tautological $R(\sigma)$–action on $R(\sigma)$. Therefore, the above isomorphism of (commutative) monoids (6.2) is obtained from (6.4) by applying $(-)_{1/n}$ and then by taking the direct summands corresponding to $e_\sigma\in\widetilde{R}(\sigma)_{1/n}$. □

Item (i) of Corollary 1.5 follows automatically from the combination of Proposition 6.1 with the equivalences of categories (2.7)–(2.8). Let us now prove item (ii). Thanks to the assumption on the additive invariant E and to Proposition 4.7, we have the isomorphisms

$$E(X^\sigma)\otimes_{\mathbb{Z}[1/n]}\widetilde{R}(\sigma)_{1/n}\simeq E(X^\sigma)\otimes_l\widetilde{R}(\sigma)_l\simeq E(X^\sigma)\otimes_l\mathrm{Map}(\mathrm{gen}(\sigma),l).$$

Therefore, the right-hand side of (1.6) reduces to

$$\begin{aligned}\bigoplus_{\sigma\in\varphi/\sim}\big(E(X^\sigma)\otimes_l\mathrm{Map}(\mathrm{gen}(\sigma),l)\big)^{N(\sigma)}&\simeq\Big(\bigoplus_{\sigma\in\varphi}E(X^\sigma)\otimes_l\mathrm{Map}(\mathrm{gen}(\sigma),l)\Big)^G\\&\simeq\Big(\bigoplus_{g\in G}E(X^g)\Big)^G,\end{aligned} \tag{6.5}$$

where (6.5) follows from the fact that $G=\coprod_{\sigma\in\varphi}\mathrm{gen}(\sigma)$ and $X^{\langle g\rangle}=X^g$. This concludes the proof of item (ii) and, consequently, of Corollary 1.5.

Proof of Corollary 1.8

Note that σ acts trivially on X^σ. Therefore, thanks to Corollary 2.12 (with $X=X^\sigma$ and $G=\sigma$), we have an isomorphism of (commutative) monoids between $\mathrm{U}([X^\sigma/\sigma])$ and $\mathrm{U}(X^\sigma)\otimes\mathrm{U}(B\sigma)$. Under this isomorphism, the canonical $R(\sigma)$–action on $\mathrm{U}([X^\sigma/\sigma])$

(described in Section 3) corresponds to the tautological $R(\sigma)$–action on $\mathrm{U}(B\sigma)$. Consequently, by applying $(-)_{1/n}$ and then taking the direct summands corresponding to $e_\sigma \in \widetilde{R}(\sigma)_{1/n}$, we obtain an induced isomorphism of (commutative) monoids between $\widetilde{\mathrm{U}}([X^\sigma/\sigma])_{1/n}$ and $\mathrm{U}(X)_{1/n} \otimes \widetilde{\mathrm{U}}(B\sigma)_{1/n}$. Finally, since the additive invariant E is monoidal, the equivalence of categories (2.8) leads to an isomorphism of (commutative) monoids between $\widetilde{E}([X^\sigma/\sigma])$ and $E(X^\sigma) \otimes \widetilde{E}(B\sigma)$. This finishes the proof.

7 Proofs: smooth quotients

In this section we prove Theorem 1.22. In order to simplify the exposition, let us write Y for the coarse moduli space $X/\!/G$; see [28]. We start with some reductions. Firstly, we may (and will) assume that X is connected. Secondly, we can assume without loss of generality that G acts generically free; otherwise, simply replace G by G/N, where N stands for the generic stabilizer of X. Following Section 5.2.1, consider the induced pullback $\pi^*\colon \mathbb{U}(Y)_{1/n} \to \mathbb{U}(X)_{1/n}$ and pushforward $\pi_*\colon \mathbb{U}(X)_{1/n} \to \mathbb{U}(Y)_{1/n}$ morphisms in the category[12] $\mathbb{M}\mathrm{ot}(Y)_{1/n}$. The proof will consist in showing that π^* induces an isomorphism $\mathbb{U}(Y)_{1/n} \simeq \mathbb{U}(X)^G_{1/n}$. Similarly to the proof of Theorem 1.1, by applying the functor Ψ to the latter isomorphism, we then conclude that $\mathrm{U}(Y)_{1/n} \simeq \mathrm{U}(X)^G_{1/n}$.

By construction of the category $\mathbb{M}\mathrm{ot}(Y)$, the composition $\pi_*\pi^*$ is equal to the class $\xi := [\pi_*(\mathcal{O}_X)]_{1/n} \in K_0(Y)_{1/n} \simeq \mathrm{End}(\mathbb{U}(Y)_{1/n})$. Therefore, since $\pi_*(\mathcal{O}_X)$ has rank n, it follows from [50, Corollary 2.4][13] that ξ is invertible in $\mathrm{End}(\mathbb{U}(Y)_{1/n})$.

Consider the endomorphisms $e := (1/\xi)(\pi^*\pi_*)$ (recall from Section 5.2.3 that the category $\mathbb{M}\mathrm{ot}(Y)_{1/n}$ is $K_0(Y)_{1/n}$–linear) and $e' := (1/n)\big(\sum_{g\in G} g_*\big)$ of the G_0–motive $\mathbb{U}(X)_{1/n}$. Both e and e' are idempotents. Moreover, since π is G–equivariant, we have $e'e = ee' = e$. We claim that $e = e'$. Note that this claim implies that π^* and $(1/\xi)\pi_*$ define inverse isomorphisms between $\mathbb{U}(Y)_{1/n}$ and $\mathbb{U}(X)^G_{1/n}$.

In order to prove the preceding claim, since $e' - e$ is also an idempotent, it suffices to show that $e - e'$ is nilpotent. Let $\Gamma := \coprod_{g\in G} \Gamma_g$, where $\Gamma_g \subset X \times_k X$ stands for the graph of g, and $\gamma\colon \Gamma \to X \times_Y X$ the map whose restriction to Γ_g is given by the inclusion $\Gamma_g \subset X \times X$. Under these notations, the endomorphisms e and e'

[12] Note that since Y may be an algebraic space, we are not necessarily in the setting of Section 5 (with trivial G). However, the generalization of Section 5 to algebraic spaces is purely formal.

[13] The proof of [50, Corollary 2.4] holds similarly for algebraic spaces: simply replace the Zariski topology by the Nisnevich topology and [9, Proposition 3.3.1] by [24, Tag 08GL, Lemma 62.8.3].

are represented in $\mathrm{End}(\mathrm{U}(X)_{1/n}) \simeq G_0(X \times_Y X)_{1/n}$ by the Grothendieck classes $(1/\xi)[\mathcal{O}_{X\times_Y X}]$ and $(1/n)[\gamma_*(\mathcal{O}_\Gamma)]$, respectively. Since G acts generically free, the map γ is birational. Moreover, using the fact that Y is the orbit space for X, we have $(X \times_Y X)_{\mathrm{red}} = \bigcup_{g\in G} \Gamma_g \subset X \times X$ (in this case $X \times_Y X$ is already reduced!). Since generically the class ξ corresponds to multiplication by n, the difference $e - e'$ is represented by a class in $F^1 G_0(X \times_Y X)_{1/n}$, where $\{F^j G_0(X \times_Y X)_{1/n}\}_{j\geq 0}$ stands for the (decreasing) codimension filtration of $G_0(X \times_Y X)_{1/n}$. This implies that in order to prove that $e - e'$ is nilpotent, it suffices to show that the composition law of $\mathrm{End}(\mathbb{U}(X)_{1/n})$ is compatible with the codimension filtration. Clearly, we have $F^j G_0(X \times_Y X)_{1/n} = \sum_{g\in G} F^j G_0(\Gamma_g)_{1/n}$. Therefore, the homomorphism

$$\gamma_*\colon K_0(\Gamma)_{1/n} \to G_0(X \times_Y X)_{1/n} \simeq \mathrm{End}_{\mathbb{M}\mathrm{ot}(Y)_{1/n}}(\mathbb{U}(X)_{1/n}) \tag{7.1}$$

induces a surjection $F^j K_0(\Gamma)_{1/n} \twoheadrightarrow F^j G_0(X \times_Y X)_{1/n}$ for every $j \geq 0$. Note that we may view

$$\Gamma \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X$$

as a groupoid in the category of k–schemes. Under this viewpoint, the homomorphism (7.1) sends the convolution product on $K_0(\Gamma)_{1/n}$ to the composition law of $\mathrm{End}(\mathbb{U}(X)_{1/n})$. Therefore, the proof of Theorem 1.22 follows now automatically from Lemma 7.2:

Lemma 7.2 *Let X be a smooth k–scheme and*

$$\Gamma \underset{t}{\overset{s}{\rightrightarrows}} X$$

a groupoid in the category of k–schemes with s and t finite étale maps. Under these assumptions, the convolution product $\cup\colon K_0(\Gamma) \times K_0(\Gamma) \to K_0(\Gamma)$ preserves the codimension filtration.

Proof The convolution product $\cup$ may be written as the composition

$$K_0(\Gamma) \otimes K_0(\Gamma) \xrightarrow{-\boxtimes-} K_0(\Gamma \times_k \Gamma) \xrightarrow{i^*} K_0(\Gamma \times_{s,X,t} \Gamma) \xrightarrow{\mu_*} K_0(\Gamma),$$

where μ stands for the multiplication map $\Gamma \times_{s,X,t} \Gamma \to \Gamma$ and i for the inclusion $\Gamma \times_{s,X,t} \Gamma \to \Gamma \times_k \Gamma$. Thanks to [18, Lemma 82] and [18, Theorem 83], respectively, $-\boxtimes-$ and i^* preserve the codimension filtration. Since μ_* is a finite map, it also preserves the codimension filtration. Hence, the proof is finished. □

8 Proofs: equivariant Azumaya algebras

In this section we prove Theorem 1.25 and Corollary 1.28. Recall from Section 1 that $\mathcal{F}$ is a flat quasicoherent sheaf of algebras over $[X/G]$, that $\mathcal{F}_\sigma$ stands for the pullback of $\mathcal{F}$ along the morphism $[X^\sigma/\sigma]\to[X/G]$, that $\mathcal{Z}_\sigma$ stands for the center of the σ–graded sheaf of $\mathcal{O}_{X^\sigma}$–algebras $\mathcal{F}_\sigma\#\sigma$, and that $Y_\sigma := \underline{\mathrm{Spec}}(\mathcal{Z}_\sigma)$. We start with a (geometric) result concerning sheaves of Azumaya algebras:

Proposition 8.1 *Assume that $\mathcal{F}$ is a sheaf of Azumaya algebra over $[X/G]$. Under this assumption, the following holds:*

(i) *The σ–graded sheaf of $\mathcal{O}_{X^\sigma}$–algebra $\mathcal{Z}_\sigma$ is* strongly graded *in the sense of [41]. Concretely, $(\mathcal{Z}_\sigma)_g$ is a line bundle on X^σ for every $g\in\sigma$ and the multiplication on $\mathcal{Z}_\sigma$ induces an isomorphism $(\mathcal{Z}_\sigma)_g\otimes_{X^\sigma}(\mathcal{Z}_\sigma)_h\simeq(\mathcal{Z}_\sigma)_{gh}$.*

(ii) *The multiplication map induces an isomorphism $\mathcal{F}_\sigma\otimes_{X^\sigma}\mathcal{Z}_\sigma\simeq\mathcal{F}_\sigma\#\sigma$ of σ–graded $\mathcal{O}_{X^\sigma}$–algebras. Hence, $\mathcal{F}_\sigma\#\sigma$ is a sheaf of Azumaya algebras over $\mathcal{Z}_\sigma$.*

(iii) *The sheaf $\mathcal{Z}_\sigma$ is equipped with a unique flat connection which is compatible with its algebra structure and which extends the tautological connection on $\mathcal{O}_{X^\sigma}$. This connection is, moreover, compatible with the σ–grading.*

Proof Since $\mathcal{F}_\sigma$ is a sheaf of Azumaya algebras over X^σ, the functors

$$\mathrm{coh}(X^\sigma)\xrightarrow{\mathcal{F}_\sigma\otimes_{X^\sigma}-}\mathrm{coh}(X^\sigma;\mathcal{F}_\sigma^{\mathrm{op}}\otimes_{X^\sigma}\mathcal{F}_\sigma),\quad \mathrm{coh}(X^\sigma;\mathcal{F}_\sigma^{\mathrm{op}}\otimes_{X^\sigma}\mathcal{F}_\sigma)\xrightarrow{(-)^{\mathcal{F}_\sigma}}\mathrm{coh}(X^\sigma)$$

are inverse monoidal equivalences of categories. Let $\mathcal{Z}'_\sigma := (\mathcal{F}_\sigma\#\sigma)^{\mathcal{F}_\sigma}$ be the centralizer of $\mathcal{F}_\sigma$ in $\mathcal{F}_\sigma\#\sigma$. Thanks to the preceding equivalences, the multiplication map $\mathcal{F}_\sigma\otimes_{X^\sigma}\mathcal{Z}'_\sigma\to\mathcal{F}_\sigma\#\sigma$ is an isomorphism of $\mathcal{O}_{X^\sigma}$–algebras. Moreover, the σ–grading on $\mathcal{F}_\sigma\#\sigma$ induces a σ–grading on $\mathcal{Z}'_\sigma$ with $(\mathcal{Z}'_\sigma)_g=(\mathcal{F}_\sigma g)^{\mathcal{F}_\sigma}$. Since the $\mathcal{F}_\sigma$–bimodule $\mathcal{F}_\sigma g$ is invertible and $\mathcal{F}_\sigma g\otimes_{\mathcal{F}_\sigma}\mathcal{F}_\sigma h\simeq\mathcal{F}_\sigma gh$, we then conclude that $\mathcal{Z}'_\sigma$ is strongly graded. We now claim that $\mathcal{Z}'_\sigma$ is commutative. Note that thanks to the isomorphism $\mathcal{F}_\sigma\otimes_{X^\sigma}\mathcal{Z}'_\sigma\simeq\mathcal{F}_\sigma\#\sigma$, this claim implies that $\mathcal{Z}'_\sigma=\mathcal{Z}_\sigma$, and consequently proves items (i)–(ii). Let $g\in\mathrm{gen}(\sigma)$. Since σ is a cyclic group and $(\mathcal{Z}'_\sigma)_g$ is locally principal, the sections of $(\mathcal{Z}'_\sigma)_g$ commute between themselves. Consequently, we have $(\mathcal{Z}'_\sigma)_{g^a}=((\mathcal{Z}'_\sigma)_g)^a$. This shows that $\mathcal{Z}'_\sigma$ is commutative.

Let us now prove item (iii). Since $\mathcal{Z}_\sigma$ is étale over $\mathcal{O}_{X^\sigma}$ (this follows from the fact that $\mathcal{Z}_\sigma$ is strongly graded), every local derivation of $\mathcal{O}_{X^\sigma}$ extends uniquely to a local derivation of $\mathcal{Z}_\sigma$. This leads to the unique flat connection on $\mathcal{Z}_\sigma$ extending the one

on $\mathcal{O}_{X^\sigma}$. Via a local computation, it can be checked that this connection respects the grading; alternatively, first base-change to a field which contains the n^{th} roots of unity and then use the fact that the grading corresponds to a $\sigma^\vee$–action which is necessarily compatible with the unique connection. □

Proof of Theorem 1.25

By applying the functor $\Psi_{\mathcal{F}}$ of Remark 5.8 to the isomorphism $\overline{\theta}$ of Theorem 5.13, we obtain an induced isomorphism

$$\mathrm{U}([X/G];\mathcal{F})_{1/n} \simeq \bigoplus_{\sigma\in\varphi/\sim} \widetilde{\mathrm{U}}([X^\sigma/\sigma];\mathcal{F}_\sigma)_{1/n}^{N(\sigma)} \tag{8.2}$$

in $\mathrm{NMot}(k)_{1/n}$; we are implicitly using the obvious analogues for $\Psi_{\mathcal{F}}$ of Section 5.2.1 and Lemma 5.10. The proof follows now from the equivalence of categories (2.7).

Proof of Corollary 1.28(i)

The structure map $Y_\sigma \to X^\sigma$ is finite, étale, and $\mathcal{Z}_\sigma^{\sigma^\vee} = \mathcal{O}_{X^\sigma}$. Hence, it is a $\sigma^\vee$–Galois cover. The fact that $\mathcal{L}_g := (\mathcal{Z}_{\langle g\rangle})_g$ is a line bundle (equipped with a flat connection) on X^σ follows from Proposition 8.1.

Proof of Corollary 1.28(ii)

Note first that $\mathrm{U}([X^\sigma/\sigma];\mathcal{F}_\sigma)$ is canonically isomorphic to $\mathrm{U}(X^\sigma;\mathcal{F}_\sigma\#\sigma)$. Recall from Section 1 that $\mathrm{U}(X^\sigma;\mathcal{F}_\sigma\#\sigma)_{1/n}$ carries a $\mathbb{Z}[\sigma^\vee]_{1/n}$–action, where $\sigma^\vee$ stands for the dual cyclic group. Moreover, in the case where k contains the n^{th} roots of unity, we have a character isomorphism $R(\sigma)\simeq\mathbb{Z}[\sigma^\vee]$. Therefore, in this case, we can consider the direct summand $e_\sigma\mathrm{U}(X^\sigma;\mathcal{F}_\sigma\#\sigma)_{1/n}$ associated to the canonical idempotent $e_\sigma\in R(\sigma)_{1/n}$.

Proposition 8.3 *If k contains the n^{th} roots of unity, then (8.2) reduces to*

$$\mathrm{U}([X/G];\mathcal{F})_{1/n} \simeq \bigoplus_{\sigma\in\varphi/\sim} (e_\sigma\mathrm{U}(X^\sigma;\mathcal{F}_\sigma\#\sigma)_{1/n})^{N(\sigma)}. \tag{8.4}$$

Proof Given a character $\chi\in\sigma^\vee$, let us write V_χ for the associated 1–dimensional σ–representation and τ_χ for the automorphism of $\mathcal{F}_\sigma\#\sigma$ corresponding to the σ–grading.

Note that τ_χ acts trivially on $\mathcal{F}_\sigma$ and that $\tau_\chi(g) = \chi(g)g$ for every $g \in \sigma$. A simple verification shows that we have the commutative diagram

$$\begin{array}{ccc} \mathrm{perf}_{\mathrm{dg}}([X^\sigma/\sigma]; \mathcal{F}_\sigma) & \xrightarrow{V_\chi \otimes_k -} & \mathrm{perf}_{\mathrm{dg}}([X^\sigma/\sigma]; \mathcal{F}_\sigma) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{perf}_{\mathrm{dg}}(X^\sigma; \mathcal{F}_\sigma \# \sigma) & \xrightarrow[\tau_\chi^*]{} & \mathrm{perf}_{\mathrm{dg}}(X^\sigma; \mathcal{F}_\sigma \# \sigma) \end{array} \tag{8.5}$$

Consequently, by applying the universal additive invariant to (8.5), we conclude that the $R(\sigma)$–action on $\mathrm{U}([X^\sigma/\sigma]; \mathcal{F}_\sigma)$ corresponds to the $\mathbb{Z}[\sigma^\vee]$–action on $\mathrm{U}(X^\sigma; \mathcal{F}_\sigma \# \sigma)$ associated to the σ–grading. The proof follows now from isomorphism (8.2). □

Recall that $\mathcal{F}$ is a sheaf of Azumaya algebras over $[X/G]$. Thanks to Proposition 8.1(ii), $\mathcal{F} \# \sigma$ becomes a sheaf of Azumaya algebras over $\mathcal{Z}_\sigma$. Consequently, using [50, Theorem 2.1], we conclude that

$$\mathrm{U}(X^\sigma; \mathcal{F}_\sigma \# \sigma)_{1/r} \simeq \mathrm{U}(X^\sigma; \mathcal{Z}_\sigma)_{1/r} = \mathrm{U}(Y_\sigma)_{1/r}, \tag{8.6}$$

where r stands for the product of the ranks of $\mathcal{F}$ (at each one of the connected components of X). Note that the isomorphism (8.6) is preserved by the $\sigma^\vee$–action because it is induced by the (inverse) of the Morita equivalence

$$- \otimes_{\mathcal{Z}_\sigma} (\mathcal{F}_\sigma \# \sigma) \colon \mathrm{perf}_{\mathrm{dg}}(X^\sigma; \mathcal{Z}_\sigma) \to \mathrm{perf}_{\mathrm{dg}}(X^\sigma; \mathcal{F}_\sigma \# \sigma).$$

Therefore, the proof of Corollary 1.28(ii) follows now from the combination of the above isomorphisms (8.4) and (8.6) with the equivalence of categories (2.7).

Proof of Corollary 1.28(iii)

Let l be a field which contains the n^{th} roots of unity and $1/nr \in l$. Recall from Section 1 that the noncommutative motive $\mathrm{U}(X^\sigma; \mathcal{F}_\sigma \# \sigma)_l$ may be considered as a σ–graded object in $\mathrm{NMot}(k)_l$ as soon as we choose an isomorphism ϵ between the n^{th} roots of unity in k and in l (eg the identity if $k = l$). In particular, $\mathrm{U}(X^\sigma; \mathcal{F}_\sigma \# \sigma)_{l,g}$ stands for the degree g part of $\mathrm{U}(X^\sigma; \mathcal{F}_\sigma \# \sigma)_l$.

Proposition 8.7 *If k contains the n^{th} roots of unity and l is a field which contains the n^{th} roots of unity and $1/n \in l$, then (8.4) reduces to an isomorphism*

$$\mathrm{U}([X/G]; \mathcal{F})_l \simeq \Big(\bigoplus_{g \in G} \mathrm{U}(X^g; \mathcal{F}_g \# \langle g \rangle)_{l,g} \Big)^G. \tag{8.8}$$

Proof Recall first, from the proof of Proposition 4.7, that we have the identification

$$l[\sigma^\vee] \xrightarrow{\simeq} l[\sigma]^\vee \simeq \mathrm{Map}(\sigma, l), \quad \chi \mapsto \big(g \mapsto (\epsilon \circ \chi)(g)\big). \tag{8.9}$$

Let $\sigma \in \varphi$ be a (fixed) cyclic subgroup. Note that for every $g \in \sigma$ we have

$$\mathrm{U}(X^g; \mathcal{F}_g \# \langle g \rangle)_{l,g} = e_g \mathrm{U}(X^g; \mathcal{F}_g \# \langle g \rangle)_l,$$

where $e_g \in l[\sigma^\vee]$ stands for the idempotent $(1/|\sigma|)\big(\sum_{\chi \in \sigma^\vee} (\epsilon \circ \chi)(g)^{-1}\chi\big)$. Under the above identification (8.9), the idempotent e_σ corresponds to the characteristic function of $\mathrm{gen}(\sigma)$ (since this is the unit element of $\mathrm{Map}(\mathrm{gen}(\sigma), l) \subset \mathrm{Map}(\sigma, l)$) and the idempotent e_g to the characteristic function of g. Therefore, the equality $e_\sigma = \sum_{g \in \mathrm{gen}(\sigma)} e_g$ holds. Consequently, the right-hand side of (8.4) reduces to

$$\begin{aligned}\Big(\bigoplus_{\sigma \in \varphi} e_\sigma \mathrm{U}(X^\sigma; \mathcal{F}_\sigma \# \sigma)_l\Big)^G &\simeq \Big(\bigoplus_{\sigma, g} e_g \mathrm{U}(X^\sigma; \mathcal{F}_\sigma \# \sigma)_l\Big)^G \\ &\simeq \Big(\bigoplus_{g \in G} \mathrm{U}(X^g; \mathcal{F}_g \# \langle g \rangle)_{l,g}\Big)^G,\end{aligned}$$

where the middle direct sum runs over the pairs $\sigma \in \varphi$ and $g \in \mathrm{gen}(\sigma)$. □

Recall that $\mathcal{F}$ is a sheaf of Azumaya algebras over $[X/G]$. Similarly to the proof of Corollary 1.28(ii), the proof of Corollary 1.28(iii) follows from the combination of the above isomorphisms (8.6) and (8.8) with the equivalence (2.7).

9 Grothendieck and Voevodsky's conjectures for orbifolds

Assume that k is of characteristic 0. Given a smooth projective k–scheme X and a Weil cohomology theory H^*, let us denote by $\pi_X^i \colon H^*(X) \to H^*(X)$ the i^{th} Künneth projector, by $Z^*(X)_\mathbb{Q}$ the $\mathbb{Q}$–vector space of algebraic cycles on X (up to rational equivalence), and by $Z^*(X)_\mathbb{Q}/_{\sim\otimes\text{nil}}$, $Z^*(X)_\mathbb{Q}/_{\sim\text{hom}}$ and $Z^*(X)_\mathbb{Q}/_{\sim\text{num}}$ the quotients with respect to the smash-nilpotence, homological and numerical equivalence relations; consult [2, Section 3.2] for details. Following Grothendieck [20] (see also Kleiman [33; 32]), the standard conjecture[14] of type C^+, denoted by $C^+(X)$, asserts that the even Künneth projector $\pi_X^+ := \sum_i \pi_X^{2i}$ is algebraic, and the standard conjecture of type D, denoted by $D(X)$, asserts that $Z^*(X)_\mathbb{Q}/_{\sim\text{hom}} = Z^*(X)_\mathbb{Q}/_{\sim\text{num}}$.

[14] The standard conjecture of type C^+ is also usually known as the *sign conjecture*. Note that if π_X^+ is algebraic, then the odd Künneth projector $\pi_X^- := \sum_i \pi_X^{2i+1}$ is also algebraic.

Following Voevodsky [55], the smash-nilpotence conjecture, denoted by $V(X)$, asserts that $Z^*(X)_{\mathbb{Q}}/_{\sim\otimes\mathrm{nil}} = Z^*(X)_{\mathbb{Q}}/_{\sim\mathrm{num}}$.

Remark 9.1 (status) (i) Thanks to the work of Grothendieck and Kleiman (see [20; 33; 32]), the conjecture $C^+(X)$ holds when X is of dimension ≤ 2, and also for abelian varieties. Moreover, this conjecture is stable under products.

(ii) Thanks to the work of Lieberman [38], the conjecture $D(X)$ holds when X is of dimension ≤ 4, and also for abelian varieties.

(iii) Thanks to the work Voevodsky [55] and Voisin [56], the conjecture $V(X)$ holds when X is of dimension ≤ 2. Moreover, thanks to the work of Kahn and Sebastian [25], the conjecture $V(X)$ also holds for abelian 3–folds.

In [6; 49], the aforementioned conjectures of Grothendieck and Voevodsky were proved in some new cases (eg quadric fibrations, intersections of quadrics, intersections of bilinear divisors, linear sections of Grassmannians, linear sections of determinantal varieties, Moishezon varieties) and extended from smooth projective k–schemes X to smooth proper algebraic stacks $\mathcal{X}$. Using Theorem 1.1, we are now able to verify these conjectures in the case of "low-dimensional" orbifolds:

Theorem 9.2 *Let X be a smooth projective k–scheme equipped with a G–action.*

(i) *If $\dim(X) \leq 2$ or X is an abelian variety and G acts by group homomorphisms, then the conjecture $C^+([X/G])$ holds.*

(ii) *If $\dim(X) \leq 4$, then the conjecture $D([X/G])$ holds.*

(iii) *If $\dim(X) \leq 2$, then the conjecture $V([X/G])$ holds.*

Proof Recall from [47, Section 4.1] that the category of *noncommutative Chow motives* $\mathrm{NChow}(k)_{\mathbb{Q}}$ is defined as the smallest full, additive, idempotent complete, symmetric monoidal subcategory of $\mathrm{NMot}(k)_{\mathbb{Q}}$ containing the objects $\mathrm{U}(\mathcal{A})_{\mathbb{Q}}$, with $\mathcal{A}$ a smooth proper dg category. Let $\mathrm{NM} \in \mathrm{NChow}(k)_{\mathbb{Q}}$ be a noncommutative Chow motive. As explained in [6; 49] (see also [48, Section 3]), the conjectures C^+, D and V admit noncommutative analogues $C^+_{\mathrm{nc}}(\mathrm{NM})$, $D_{\mathrm{nc}}(\mathrm{NM})$ and $V_{\mathrm{nc}}(\mathrm{NM})$, respectively. Moreover, given a smooth projective k–scheme X, we have the equivalences

$$C^+(X) \iff C^+_{\mathrm{nc}}(\mathrm{U}(X)_{\mathbb{Q}})D(X) \iff D_{\mathrm{nc}}(\mathrm{U}(X)_{\mathbb{Q}})V(X) \iff V_{\mathrm{nc}}(\mathrm{U}(X)_{\mathbb{Q}}).$$

This motivated the extension of Grothendieck and Voevodsky's conjectures from smooth projective schemes X to smooth proper algebraic stacks $\mathcal{X}$ by setting

$$C^+(\mathcal{X}) := C^+_{\mathrm{nc}}(\mathrm{U}(\mathcal{X})_{\mathbb{Q}}), \quad D(\mathcal{X}) := D_{\mathrm{nc}}(\mathrm{U}(\mathcal{X})_{\mathbb{Q}}), \quad V(\mathcal{X}) := V_{\mathrm{nc}}(\mathrm{U}(\mathcal{X})_{\mathbb{Q}}).$$

Now, note that in the case where $\mathcal{X} = [X/G]$, the formula (1.4) implies that the noncommutative motive $\mathrm{U}([X/G])_{\mathbb{Q}}$ is a direct summand of $\bigoplus_{\sigma\in\varphi} \mathrm{U}(X^\sigma \times \mathrm{Spec}(k[\sigma]))$. Therefore, since by construction the conjectures C^+_{nc}, D_{nc} and V_{nc} are stable under direct sums and direct summands, the above considerations show that in order to prove Theorem 9.2 it suffices to prove the conjectures

$$C^+(X^\sigma \times \mathrm{Spec}(k[\sigma])), \quad D(X^\sigma \times \mathrm{Spec}(k[\sigma])), \quad V(X^\sigma \times \mathrm{Spec}(k[\sigma])).$$

(i) Using the fact that the conjecture C^+ is moreover stable under products, it is enough to prove the conjectures $C^+(X^\sigma)$ and $C^+(\mathrm{Spec}(k[\sigma]))$. On the one hand, since $\mathrm{Spec}(k[\sigma])$ is 0–dimensional, the conjecture $C^+(\mathrm{Spec}(k[\sigma]))$ holds. On the other hand, since the assumptions imply that $\dim(X^\sigma) \leq 2$ or that X^σ is an abelian variety, the conjecture $C^+(X^\sigma)$ also holds.

(ii) The assumptions imply that $\dim(X^\sigma \times \mathrm{Spec}\, k[\sigma]) \leq 4$.

(iii) The assumptions imply that $\dim(X^\sigma \times \mathrm{Spec}\, k[\sigma]) \leq 2$.

The above considerations (i)–(iii) conclude the proof. □

Appendix: Nilpotency in the Grothendieck ring of an orbifold

In what follows, we don't assume that $1/n \in k$. Given a connected k–scheme X, it is natural to ask if the elements in the kernel of the rank map $K_0(X) \to \mathbb{Z}$ are nilpotent. This is well known in the case where X is Noetherian and admits an ample line bundle;[15] consult [17, Chapter V, Section 3] for example. The more general case where X is quasicompact and quasiseparated was proved in [50, Theorem 2.3]. In this appendix, we further extend the latter result to the case of (global) orbifolds.

Theorem A.1 *Let X be quasicompact quasiseparated scheme equipped with a G–action. Assume that $1/n \in \mathcal{O}_X$ and that $\mathcal{X} := [X/G]$ is equipped with a finite morphism towards a quasicompact quasiseparated algebraic space Y. Under these assumptions,*

[15] Under these strong assumptions, the kernel of the rank map is itself nilpotent.

the elements in the kernel of the induced map

$$K_0(\mathcal{X}) \to \prod_{\overline{y}\to Y} K_0(\mathcal{X}_{\overline{y}}),$$

where the product runs over the geometric points of Y, are nilpotent.

Proof The inductive argument used in the proof of [50, Theorem 2.3] for quasicompact, quasiseparated schemes holds similarly for quasicompact, quasiseparated algebraic spaces; simply replace the Zariski topology by the Nisnevich topology[16] and [9, Proposition 3.3.1] by [24, Tag 08GL, Lemma 62.8.3]. Therefore, by applying this inductive argument to the algebraic space Y, it suffices to prove the following local statement: *Let $(R,\mathfrak{m})$ be a local ring with residue field $\kappa := R/\mathfrak{m}$, $Y := \operatorname{Spec}(R)$, $y := \operatorname{Spec}(\kappa)$ the closed point of Y and $A := \mathcal{O}_{X,y} \# G$ the skew group algebra. Under these notations, for every field extension l/κ, the induced homomorphism $K_0(A) \to K_0(A_l)$ is injective.* This local statement is a particular case of Lemma A.2 below. □

Lemma A.2 *Let $(R,\mathfrak{m})$ be a local ring with residue field $\kappa := R/\mathfrak{m}$, A an R–algebra (which we assume finitely generated as an R–module), and l/κ a field extension. Under these assumptions, the induced homomorphism $K_0(A) \to K_0(A_l)$ is injective.*

Proof Note first that every simple A–module V is finitely generated as an R–module. If $r \in \mathfrak{m}$, then rV is an A–submodule of V. Therefore, by Nakayama's lemma applied to R, we have $rV = 0$. This implies that $\mathfrak{m}A \subset \operatorname{rad}(A)$, where $\operatorname{rad}(A)$ stands for the Jacobson's radical of A. Using Nakayama's lemma once again, we conclude that the induced homomorphism $K_0(A) \to K_0(A_\kappa)$ is injective. Thanks to the above considerations, it suffices then to prove the particular case where $R = \kappa$ is a field and A a finite-dimensional κ–algebra. By replacing A with $A/\operatorname{rad}(A)$, we can moreover assume that A is semisimple, and, using Morita theory, we can furthermore assume that A is a division κ–algebra. In this latter case, we have an isomorphism $K_0(A) \simeq \mathbb{Z}$ induced by the rank map. The proof follows now from the fact that the rank map is preserved under base change $- \otimes_\kappa l$. □

Corollary A.3 *Let X be a quasicompact separated k–scheme equipped with a G–action. Assume that $1/n \in \mathcal{O}_X$. For every finite field extension l/k, the elements in the kernel of the induced homomorphism $K_0([X/G]) \to K_0([X_l/G])$ are nilpotent.*

[16] It is well known that (nonconnective) algebraic K–theory satisfies not only Zariski descent but also Nisnevich descent.

Proof Let $\pi\colon [X/G] \to Y$ be the coarse moduli space of $[X/G]$; see [28]. Note that the morphism π is finite and that $[X_l/G] = [X/G] \times_Y Y_l$. Therefore, the proof follows from the combination of Theorem A.1 with the fact that every geometric point of Y factors through a geometric point of Y_l. □

Remark A.4 The homomorphism $K_0([X/G]) \to K_0([X_l/G])$ is not necessarily injective. However, if r stands for the degree of the (finite) field extension l/k, then the induced homomorphism $K_0([X/G])_{1/r} \to K_0([X_l/G])_{1/r}$ is injective.

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