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Mirror theorem for elliptic quasimap invariants

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We propose and prove a mirror theorem for the elliptic quasimap invariants of smooth Calabi–Yau complete intersections in projective spaces. This theorem, combined with the wall-crossing formula of Ciocan-Fontanine and Kim, implies mirror theorems of Zinger and Popa for the elliptic Gromov–Witten invariants of those varieties. This paper and the wall-crossing formula provide a unified framework for the mirror theory of rational and elliptic Gromov–Witten invariants.

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1 Introduction

Let W be a codimension-r affine subvariety in $\mathbb{C}^n$ defined by homogeneous degree $l_1, \dots, l_r$ polynomials such that the origin is the only singular point of W. Assume

$$\sum_{a=1}^{r} l_a = n,$$

and let $\boldsymbol{G} := \mathbb{C}^*$ act on $\mathbb{C}^n$ by the standard diagonal action so that its associated GIT quotient

$$X := W /\!\!/ \boldsymbol{G}$$

is a codimension-r, nonsingular Calabi–Yau complete intersection in $\mathbb{P}^{n-1}$.

With this setup, for each positive rational number ε, there is a so-called ε–stable, genus-g, k–pointed, degree-d quasimap moduli space

$$Q^{\varepsilon}_{g,k}(X, d)$$

with the canonical virtual fundamental class $[Q^{\varepsilon}_{g,k}(X, d)]^{\mathrm{vir}}$; see Ciocan-Fontanine, Kim and Maulik [6]. We are mainly interested in the space $Q^{\varepsilon}_{g,k}(X, d)$ with small enough ε with respect to degree d, which will be denoted by $Q^{0+}_{g,k}(X, d)$ and also simply by $Q_{g,k}(X, d)$. When $\varepsilon > 2$, the space $Q^{\varepsilon}_{g,k}(X, d)$ coincides with the moduli space $\overline{M}_{g,k}(X, d)$ of stable maps, which will be denoted also by $Q^{\infty}_{g,k}(X, d)$.

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When $g=1$, the virtual dimension of $Q^{\varepsilon}_{g,0}(X,d)$ is always zero. The main goal of this paper is to discover an explicit description of $\deg[Q_{1,0}(X,d)]^{\mathrm{vir}}$ in terms of Givental's I–function for X. Let

$$\langle\,\rangle^{\varepsilon}_{1,0}:=\sum_{d=1}^{\infty}q^d\deg[Q^{\varepsilon}_{1,0}(X,d)]^{\mathrm{vir}},$$

where q is a formal Novikov variable. We express the generating function $\langle\,\rangle^{\varepsilon}_{1,0}$ in terms of Givental's $\boldsymbol{T}$–equivariant I–function for X, where $\boldsymbol{T}:=(\mathbb{C}^*)^n$ is the complex torus group acting on $\mathbb{P}^{n-1}$; see Givental [7].

The equivariant I–function is the $H^*_{\boldsymbol{T}}(\mathbb{P}^{n-1})\otimes\mathbb{Q}(\lambda)$–valued formal function in formal variables q,z,t_H:

$$I_{\boldsymbol{T}}(t,q):=e^{t_HH/z}\sum_{d=0}^{\infty}q^de^{t_Hd}\frac{\prod_{a=1}^{r}\prod_{k=1}^{l_ad}(l_aH+kz)}{\prod_{k=1}^{d}\prod_{j=1}^{n}(H-\lambda_j+kz)},\tag{1-1}$$

where $\lambda_1,\dots,\lambda_n$ are the $\boldsymbol{T}$–equivariant parameters, $\mathbb{Q}(\lambda)$ denotes the quotient field of the polynomial ring in $\lambda_1,\dots,\lambda_n$, H is the $\boldsymbol{T}$–equivariant hyperplane class, and $t:=t_HH$.

Let λ_0 be another formal parameter. Consider the restriction $I_{\boldsymbol{T}}(0,q)|_{p_i}$ of $I_{\boldsymbol{T}}(0,q)$ to the i^{th} $\boldsymbol{T}$–fixed point

$$p_i:=[\underbrace{0,\dots,1}_{i},\dots,0]\in\mathbb{P}^{n-1}.\tag{1-2}$$

Define the q–series $\mu(q)$, $R_0(q)\in\mathbb{Q}[\![q]\!]$ by the asymptotic expansion

$$I(0,q)|_{p_i}\equiv e^{\mu(q)\lambda_i/z}(R_0(q)+O(z)),$$

where $\equiv$ means the equality after the specialization

$$\lambda_i=\lambda_0\exp(2\pi i\sqrt{-1}/n),\quad i=1,\dots,n.\tag{1-3}$$

For the existence of the asymptotic expansion, see (2-7). Denote by

$$\underline{I}_{\boldsymbol{T}}$$

the specialization of $I_{\boldsymbol{T}}$ with (1-3).

For $k=0,1,\dots,n-1$, define the initial constants $C_k(q)\in\mathbb{Q}[\![q]\!]$ of the form $1+O(q)$ inductively by the requirements

$$C_k(q)H^k=B_k+O(1/z)$$

in the following Birkhoff factorization procedure:

$$B_0:=\underline{I}_{\boldsymbol{T}}(0,q),\quad B_k:=\Big(H+zq\frac{d}{dq}\Big)\frac{B_{k-1}}{C_{k-1}(q)}\quad\text{for } 1\le k\le n-1.$$

There is an interpretation of C_k as a $\boldsymbol{T}$–equivariant quasimap invariant; see Remark 3.3.

Theorem 1.1 *The following holds:*

$$\langle\,\rangle_{1,0}^{0+} = -\frac{3(n-1-r)^2+n-r-3}{48}\log\Big(1-q\prod_{a=1}^{r} l_a^{l_a}\Big) - \frac{1}{2}\sum_{k=0}^{n-2-r}\binom{n-r-k}{2}\log C_k(q).$$

In *all genera*, there is a wall-crossing formula, which relates $[Q_{g,k}^{\infty}(X,d)]^{\mathrm{vir}}$ with $[Q_{g,k}^{0+}(X,d)]^{\mathrm{vir}}$, conjectured by Ciocan-Fontanine and Kim in [5] and proven in [4]. Its consequence for $g=1$ is as follows. Define $I_0 := C_0$ and I_1 by the $1/z$–expansion of

$$I_{\boldsymbol{T}}|_{\lambda=t=0} = I_0 + I_1/z + O(1/z^2).$$

Theorem 1.2 [4] *Let $\chi_{\mathrm{top}}(X)$ be the topological Euler characteristic of X and $c_{\dim X-1}(TX)$ the $(\dim X-1)^{st}$ Chern class of the tangent bundle TX. Then*

$$\langle\,\rangle_{1,0}^{\infty}|_{q^d\mapsto q^d\exp(\int_{d[\mathrm{line}]} I_1/I_0)} - \langle\,\rangle_{1,0}^{0+} = \frac{1}{24}\chi_{\mathrm{top}}(X)\log I_0 + \frac{1}{24}\int_X \frac{I_1}{I_0}c_{\dim X-1}(TX). \tag{1-4}$$

Without any usage of the reduced Gromov–Witten invariants, Theorem 1.1 combined with the wall-crossing formula (1-4) reproves the following mirror theorem of Popa and Zinger for Calabi–Yau complete intersections in projective spaces.

Theorem 1.3 [11; 12] *The following holds:*

$$\begin{aligned}\langle\,\rangle_{1,0}^{\infty}|_{q^d\mapsto q^d\exp(\int_{d[\mathrm{line}]} I_1/I_0)} = \frac{1}{24}\chi_{\mathrm{top}}(X)\log I_0 &+ \frac{1}{24}\int_X \frac{I_1}{I_0}c_{\dim X-1}(TX)\\ &- \frac{3(n-1-r)^2+n-r-3}{48}\log\Big(1-q\prod_{a=1}^{r} l_a^{l_a}\Big)\\ &- \frac{1}{2}\sum_{k=0}^{n-2-r}\binom{n-r-k}{2}\log C_k(q).\end{aligned}$$

The above three theorems are logically independently proven, and any pair of them implies the remaining theorem. Theorem 1.1 combined with Theorem 1.2 answers the question raised by Marian, Oprea, and Pandharipande in Section 10.2 of [10].

The strategy to prove Theorem 1.1 consists of two steps. The first step is quite general and conceptual. We obtain Theorem 2.6, one of two main results of this paper. The theorem is a quasimap version of Givental's expression [8] of the elliptic

Gromov–Witten generating function for a smooth projective toric variety twisted by a vector bundle. The latter expression is given in terms of the equivariant Frobenius structure of the equivariant quantum cohomology. One may regard Theorem 2.6 as a mirror theorem for elliptic quasimap invariants for Calabi–Yau complete intersections in toric varieties (as well as in partial flag varieties; see Remark 2.7), in the following sense.

Whenever one computes the right-hand side in Conjecture 2.5 in closed form, one obtains a mirror theorem in closed form. Inspired by [11; 12], we accomplish the computation for Calabi–Yau complete intersections in projective spaces. This second step is completely algebraic. We will, however, see that the geometric natures of various generating functions of quasimap invariants make the step crucially simple.

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2 Localized elliptic expression

Let $\boldsymbol{G}$ be a complex reductive group and let V be a finite-dimensional representation space of $\boldsymbol{G}$. Let θ be a character of $\boldsymbol{G}$ such that the semistable locus $V^{\mathrm{ss}}(\theta)$ with respect to θ has no nontrivial isotropy subgroup of $\boldsymbol{G}$. Following the twisted theory as in [2, Section 7], we assume that a complex torus $\boldsymbol{T}$ acts on a vector space V, and this action commutes with the $\boldsymbol{G}$–action on V. Assume furthermore that the induced action on $Y := V /\!\!/_\theta \boldsymbol{G}$ allows only finitely many 0–dimensional and 1–dimensional $\boldsymbol{T}$–orbits.

Let E be a $\boldsymbol{G}\times\boldsymbol{T}$–representation space. Let s be a $\boldsymbol{G}$–equivariant map from V to E whose zero-locus W has only locally complete intersection singularities. Assume that the semistable locus $W^{\mathrm{ss}}(\theta)$ is nonsingular.

Let $X = W /\!\!/_\theta \boldsymbol{G}$. For $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{\boldsymbol{G}} V, \mathbb{Z})$, let

$$Q_{g,k}(X,\beta) \quad (\text{resp. } Q_{g,k}(Y,\beta))$$

be the moduli space of k–pointed genus-g stable quasimaps to X (resp. Y) of degree class β. Denote by f the universal map from the universal curve $\mathcal{C}$ to the stack quotient $[V/\boldsymbol{G}]$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & {[V/\boldsymbol{G}]} \\ Q_{g,k}(Y,\beta) & \end{tikzcd}$$

Let

$$\widetilde{E} := [(E \times V)/\boldsymbol{G}],$$

which is a vector bundle on $[V/\boldsymbol{G}]$. Note that s induces a section of a coherent sheaf $\pi_* f^* \widetilde{E}$. Assume that for $g = 0$ and also for $g = 1$, $k = 0$ and $\beta \neq 0$, we have

$$R^1 \pi_* f^* \widetilde{E} = 0.$$

For example, this is the case when E is a sum $\bigoplus_a E_a$ of 1–dimensional $\boldsymbol{G} \times \boldsymbol{T}$–representations E_a with $\boldsymbol{G}$–weight $m_a \theta$ for some positive integers m_a.

Let ι denote the closed immersion of $Q_{g,k}(X, \beta)$ into $Q_{g,k}(Y, \beta)$. By the functoriality in [9], we have

$$\iota_*[Q_{g,k}(X, \beta)]^{\mathrm{vir}} = \mathrm{e}(\pi_* f^* \widetilde{E}) \cap [Q_{g,k}(Y, \beta)]^{\mathrm{vir}} \tag{2-1}$$

for $g = 0$, $k = 2, 3, \ldots$ and also for $g = 1$, $k = 0$ and $\beta \neq 0$. In this paper, we study $[Q_{1,0}(X, \beta)]^{\mathrm{vir}}$ using (the obvious $\boldsymbol{T}$–equivariant version of) the right-hand side of (2-1).

2.1 Genus-0 theory

We introduce the definitions of various generating functions of rational quasimap invariants with the ordinary markings. We prove the relation in Corollary 2.3, which will be needed later.

First we set the notation for the cohomology basis and its dual basis. Let $\{p_i\}_i$ be the set of $\boldsymbol{T}$–fixed points of Y, and let ϕ_i be the "delta" basis of $H^*_{\boldsymbol{T}}(Y) \otimes \mathbb{Q}(\lambda)$; that is,

$$\phi_i|_{p_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let ϕ^i be the dual basis with respect to the *E–twisted $\boldsymbol{T}$*–equivariant Poincaré pairing; ie

$$\int_Y \phi_i \phi^j \,\mathrm{e}^{\boldsymbol{T}}(\widetilde{E}|_Y) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\mathrm{e}^{\boldsymbol{T}}(\widetilde{E}|_Y)$ is the $\boldsymbol{T}$–equivariant Euler class of $\widetilde{E}|_Y$.

We assume, for every i, that $\mathrm{e}^{\boldsymbol{T}}(\widetilde{E}|_{p_i})$ is *invertible* in $\mathbb{Q}(\lambda)$, so the twisted Poincaré pairing is a perfect pairing on $H^*_{\boldsymbol{T}}(Y) \otimes \mathbb{Q}(\lambda)$. Note that

$$\phi^i = e_i \phi_i, \quad \text{where } e_i := \frac{1}{\int_Y \phi_i \phi_i \mathrm{e}^{\boldsymbol{T}}(\widetilde{E}|_Y)} = \frac{\mathrm{e}^{\boldsymbol{T}}(T_{p_i} Y)}{\mathrm{e}^{\boldsymbol{T}}(\widetilde{E}|_{p_i})}.$$

Integrating along the twisted virtual fundamental class

$$\mathrm{e}^{\boldsymbol{T}}(\pi_* f^* \widetilde{E}) \cap [Q_{0,k}(Y,\beta)]^{\mathrm{vir}},$$

we define correlators $\langle \cdots \rangle_{0,k,\beta}^{0+}$ as follows. For $\gamma_i \in H_{\boldsymbol{T}}^*(Y) \otimes \mathbb{Q}(\lambda)$,

$$\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_k \psi^{a_k} \rangle_{0,k,\beta}^{0+} := \int_{\mathrm{e}^{\boldsymbol{T}}(\pi_* f^* \widetilde{E}) \cap [Q_{0,k}(Y,\beta)]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^*(\gamma_i) \psi_i^{a_i},$$

where ψ_i is the psi-class associated to the i^{th} marking and ev_i is the i^{th} evaluation map. Let

$$Q_{0,k}(Y,\beta)^{\boldsymbol{T},p_i}$$

be the $\boldsymbol{T}$–fixed part of $Q_{0,k}(Y,\beta)$ whose elements have domain components only over p_i. Integrating along the localized cycle class

$$\mathrm{LCC} := \frac{\mathrm{e}^{\boldsymbol{T}}(\pi_* f^* \widetilde{E}) \cap [Q_{0,k}(Y,\beta)^{\boldsymbol{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\boldsymbol{T}}(N^{\mathrm{vir}}_{Q_{0,k}(Y,\beta)^{\boldsymbol{T},p_i}/Q_{0,k}(Y,\beta)})},$$

we define $\langle \cdots \rangle_{0,k,\beta}^{0+,p_i}$ and $\langle\!\langle \cdots \rangle\!\rangle_{0,k}^{0+,p_i}$ as follows:

$$\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_k \psi^{a_k} \rangle_{0,k,\beta}^{0+,p_i} := \int_{\mathrm{LCC}} \prod_i \mathrm{ev}_i^*(\gamma_i) \psi_i^{a_i},$$

$$\langle\!\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_k \psi^{a_k} \rangle\!\rangle_{0,k}^{0+,p_i} := \sum_{m,\beta} \frac{q^\beta}{m!} \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_k \psi^{a_k}, t, \ldots, t \rangle_{0,k+m,\beta}^{0+,p_i}$$

for $t \in H_{\boldsymbol{T}}^*(Y) \otimes \mathbb{Q}(\lambda)$, and where q is a formal Novikov variable.

In what follows, let z be a formal variable. We will need the following $\boldsymbol{T}$–localized generating functions:

$$\begin{aligned}
D_i &:= e_i \langle\!\langle 1,1,1 \rangle\!\rangle_{0,3}^{0+,p_i} = 1 + O(q), \\
u_i &:= e_i \langle\!\langle 1,1 \rangle\!\rangle_{0,2}^{0+,p_i} = t|_{p_i} + O(q), \\
S_t^{0+,p_i}(\gamma) &:= e_i \left\langle\!\!\left\langle \frac{1}{z-\psi}, \gamma \right\rangle\!\!\right\rangle_{0,2}^{0+,p_i} = e^t \gamma|_{p_i} + O(q), \\
J^{0+,p_i} &:= e_i \left\langle\!\!\left\langle \frac{1}{z(z-\psi)} \right\rangle\!\!\right\rangle_{0,1}^{0+,p_i} = e^t|_{p_i} + O(q)
\end{aligned}$$

for $\gamma \in H^*_{\boldsymbol{T}}(Y) \otimes \mathbb{Q}(\lambda)[\![q]\!]$, and where the unstable terms of J^{0+,p_i} are defined by the quasimap graph spaces $\mathrm{QG}^{0+}_{0,0,\beta}(Y)$ as in [1; 2] so that

$$J^{0+,p_i}|_{t=0} = J^{0+}|_{t=0,\,p_i};$$

see Section 5 of [2] for the definition of J^{0+}. Here the front terms e_i are inserted as the class E–Poincaré dual to $\phi_i|_{p_i} = 1$. The parameter z naturally appears as the $\mathbb{C}^*$–equivariant parameter in the graph construction; see Section 4 of [2]. It is originated from the $\mathbb{C}^*$–action on $\mathbb{P}^1$.

Denote by $\mathrm{QG}^{0+}_{0,k,\beta}(Y)$ the quasimap graph spaces (see [2]), and by

$$\mathrm{QG}^{0+}_{0,k,\beta}(Y)^{\boldsymbol{T},p_i}$$

the $\boldsymbol{T}$–fixed part of $\mathrm{QG}_{0,k,\beta}(Y)$ whose elements have domain components only over p_i. Furthermore, we define invariants and generating functions on the graph spaces: for $\gamma_i \in H^*_{\boldsymbol{T}}(Y) \otimes H^*_{\mathbb{C}^*}(\mathbb{P}^1) \otimes \mathbb{Q}(\lambda)$ and letting

$$\mathrm{LCCG} := \frac{\mathrm{e}^{\boldsymbol{T}}(\pi_* f^* \widetilde{E}) \cap [\mathrm{QG}_{0,k,\beta}(Y)^{\boldsymbol{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\boldsymbol{T}}(N^{\mathrm{vir}}_{\mathrm{QG}_{0,k,\beta}(Y)^{\boldsymbol{T},p_i}/\mathrm{QG}_{0,k,\beta}(Y)})},$$

we define

$$\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k}\rangle^{\mathrm{QG}^{0+},p_i}_{k,\beta} := \int_{\mathrm{LCCG}} \prod_i \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i},$$

$$\langle\!\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k}\rangle\!\rangle^{\mathrm{QG}^{0+},p_i}_{k} := \sum_{m,\beta} \frac{q^\beta}{m!}\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k},t,\dots,t\rangle^{\mathrm{QG}^{0+},p_i}_{k+m,\beta},$$

for $t \in H^*_{\boldsymbol{T}}(Y) \otimes \mathbb{Q}(\lambda)$. Here we denote by ev_i the i^{th} evaluation map to $Y \times \mathbb{P}^1$ from the quasimap graph spaces and regard t also as the element $t \otimes 1$ in $H^*_{\boldsymbol{T}}(Y) \otimes H^*_{\mathbb{C}^*}(\mathbb{P}^1) \otimes \mathbb{Q}(\lambda)$.

In what follows, let $\boldsymbol{p}_\infty$ be the equivariant cohomology class $H^*_{\mathbb{C}^*}(\mathbb{P}^1)$ defined by the requirements

$$\boldsymbol{p}_\infty|_0 = 0, \quad \boldsymbol{p}_\infty|_\infty = -z.$$

Proposition 2.1 *The following holds:*

$$J^{0+,p_i} = S^{0+,p_i}_t(P^{0+,p_i}), \quad \text{where } P^{0+,p_i} := e_i\langle\!\langle 1 \otimes \boldsymbol{p}_\infty\rangle\!\rangle^{\mathrm{QG}^{0+},p_i}_1. \tag{2-2}$$

Proof The proof is completely parallel to the proof of Theorem 5.4.1 of [2]. Fix the number of markings and the degree class β and then apply the $\mathbb{C}^*$–localization to the definition of P^{0+,p_i}. □

By the uniqueness lemma in Section 7.7 of [2],

$$S_t^{0+,p_i}(\gamma) = e^{u_i/z}\gamma|_{p_i}. \tag{2-3}$$

Hence Proposition 2.1 gives the expression

$$J^{0+,p_i} = e^{u_i/z}(r_{i,0} + O(z)),$$

where $r_{i,0} \in \mathbb{Q}(\lambda)[\![t,q]\!]$ is the constant term of P^{0+,p_i} in z.

Corollary 2.2 *The equality*

$$\log J^{0+,p_i} = u_i/z + \log r_{i,0} + O(z) \in \mathbb{Q}(\lambda)((z))[\![t,q]\!] \tag{2-4}$$

holds as Laurent series in z over the coefficient ring $\mathbb{Q}(\lambda)$ in each power expansion of t and q, after regarding t as a formal element.

Proof It is clear that both sides belong to $\mathbb{Q}(\lambda)((z))[\![t,q]\!]$. □

Corollary 2.3 $$D_i|_{t=0} = \frac{1}{r_{i,0}|_{t=0}}.$$

Proof By (2-2) at $t=0$, (2-3) with $\gamma = 1$, and the definition of J^{0+,p_i}, we see that

$$J^{0+,p_i} = e^{u_i|_{t=0}/z}P^{0+,p_i}|_{t=0} + \frac{t}{z}S_{t=0}^{0+,p_i}(1) + O(t^2). \tag{2-5}$$

Also by (2-3) with $\gamma = 1$ and the definition of S_t^{0+,p_i}, we see that

$$S_t^{0+,p_i} = e^{u_i|_{t=0}/z}\Big(1 + \frac{t}{z}(D_i|_{t=0})\Big) + O(t^2). \tag{2-6}$$

The multiplication of $e^{-u_i|_{t=0}/z}$ and (2-2) after the replacements of (2-5) and (2-6) gives

$$P^{0+,p_i}|_{t=0} + \frac{t}{z} + O(t^2) = P^{0+,p_i}|_{t=0}\Big(1 + D_i|_{t=0}\frac{t}{z}\Big) + O(t^2).$$

Now the comparison of the (t/z)–coefficient yields the result. □

2.2 Insertions of 0+ weighted markings

To break the symmetry of the localization computation for the virtual fundamental classes of the elliptic quasimap moduli spaces, we will need to introduce a marking. However, to keep the relation (2-1) even with markings for $g=1$, we will use the infinitesimally (ie $0+$) weighted markings.

Denote by

$$Q^{0+,0+}_{g,k|m}(Y,\beta) \quad (\text{resp. } \mathrm{QG}^{0+,0+}_{0,k|m,\beta}(Y))$$

the moduli space (resp. graph moduli space) of genus-g (resp. genus-0), degree class β stable quasimaps to Y with ordinary k–pointed markings and infinitesimally weighted m–pointed markings; see Sections 2 and 5 of [3]. They are isomorphic to the universal curve $\mathcal{C}$ of $Q^{0+}_{g,k|m-1}(Y,\beta)$ (resp. $\mathrm{QG}^{0+}_{0,k|m-1,\beta}(Y)$). Denote by

$$Q^{0+,0+}_{g,k|m}(Y,\beta)^{\boldsymbol{T},p_i} \quad (\text{resp. } \mathrm{QG}^{0+,0+}_{0,k|m,\beta}(Y)^{\boldsymbol{T},p_i})$$

the $\boldsymbol{T}$–fixed part of $Q^{0+,0+}_{g,k|m}(Y,\beta)$ (resp. $\mathrm{QG}^{0+,0+}_{0,k|m,\beta}(Y)$) whose domain components are only over p_i.

For $\gamma_i \in H^*_{\boldsymbol{T}}(Y)\otimes\mathbb{Q}(\lambda)$, $\tilde{t},\delta_j \in H^*_{\boldsymbol{T}}([V/\boldsymbol{G}],\mathbb{Q})$, let

$$\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k};\delta_1,\dots,\delta_m\rangle^{0+,0+}_{0,k|m,\beta} := \int_{\mathrm{e}^{\boldsymbol{T}}(\pi_* f^*\widetilde{E})\cap[Q^{0+,0+}_{0,k|m}(Y,\beta)]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i}\prod_j \widehat{\mathrm{ev}}_j^*(\delta_j),$$

$$\langle\!\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k};\delta_1,\dots,\delta_m\rangle\!\rangle^{0+,0+}_{0,k} := \sum_{m',\beta}\frac{q^\beta}{m'!}\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k};\delta_1,\dots,\delta_m,\tilde{t},\dots,\tilde{t}\rangle^{0+,0+}_{0,k|m+m',\beta},$$

$$\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k};\delta_1,\dots,\delta_m\rangle^{0+,0+,p_i}_{0,k|m,\beta} := \int_{\mathrm{LCCI}} \prod_i \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i}\prod_j \widehat{\mathrm{ev}}_j^*(\delta_j),$$

$$\langle\!\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k};\delta_1,\dots,\delta_m\rangle\!\rangle^{0+,0+,p_i}_{0,k|m} := \sum_{m',\beta}\frac{q^\beta}{m'!}\langle \gamma_1\psi^{a_1},\dots,\gamma_k\psi^{a_k};\delta_1,\dots,\delta_m,\tilde{t},\dots,\tilde{t}\rangle^{0+,0+,p_i}_{0,k|m+m',\beta},$$

where $\widehat{\mathrm{ev}}_j$ is the evaluation map to $[V/\boldsymbol{G}]$ at the j^{th} infinitesimally weighted marking, and

$$\mathrm{LCCI} := \frac{\mathrm{e}^{\boldsymbol{T}}(\pi_* f^* \widetilde{E}) \cap [Q^{0+,0+}_{0,k|m}(Y,\beta)^{\boldsymbol{T},p_i}]^{\mathrm{vir}}}{\mathrm{e}^{\boldsymbol{T}}(N^{\mathrm{vir}}_{Q^{0+,0+}_{0,k|m}(Y,\beta)^{\boldsymbol{T},p_i}/Q^{0+,0+}_{0,k|m}(Y,\beta)})}.$$

Here and below, *double brackets with superscript* $0+,0+$ will indicate the sum over all degree classes β and all possible $\tilde{t}$ insertions *only at the infinitesimally weighted markings*. Similarly, we define

$$\langle \dots ; \dots \rangle^{\mathrm{QG}^{0+,0+}}_{k|m,\beta} \quad \text{and} \quad \langle\!\langle \dots ; \dots \rangle\!\rangle^{\mathrm{QG}^{0+,0+},p_i}_{k|m}.$$

Consider

$$\begin{aligned}
\mathbb{S}(\gamma) &:= \sum_i \phi^i \left\langle\!\!\left\langle \frac{\phi_i}{z-\psi}, \gamma \right\rangle\!\!\right\rangle^{0+,0+}_{0,2}, \\
\mathbb{V}_{ii}(x,y) &:= \left\langle\!\!\left\langle \frac{\phi_i}{x-\psi}, \frac{\phi_i}{y-\psi} \right\rangle\!\!\right\rangle^{0+,0+}_{0,2} = \frac{1}{e_i(x+y)} + O(q), \\
\mathbb{U}_i &:= e_i \langle\!\langle 1, 1 \rangle\!\rangle^{0+,0+,p_i}_{0,2} = \tilde{t}|_{p_i} + O(q), \\
\mathbb{S}^{0+,p_i}_i(\gamma) &:= e_i \left\langle\!\!\left\langle \frac{1}{z-\psi}, \gamma \right\rangle\!\!\right\rangle^{0+,0+,p_i}_{0,2} = e^{\tilde{t}/z}\gamma|_{p_i} + O(q), \\
\mathbb{J}^{0+,p_i} &:= e_i \left\langle\!\!\left\langle \frac{1}{z(z-\psi)} \right\rangle\!\!\right\rangle^{0+,0+,p_i}_{0,1} = e^{\tilde{t}}|_{p_i} + O(q) = J^{0+,p_i}|_{t=0} + O(\tilde{t}).
\end{aligned}$$

(Here $e_i^2 \mathbb{V}_{ii}$ at $\tilde{t}=0$ coincides with $V^{0+}_{t=0}|_{p_i}$ of [5].) As before,

$$\mathbb{S}^{0+,p_i}_i(\gamma) = e^{\mathbb{U}_i/z}\gamma|_{p_i},$$

$$\mathbb{J}^{0+,p_i} = e^{\mathbb{U}_i/z}\left(\sum_{k=0}^{m} \mathbb{R}_{i,k} z^k + O(z^{m+1})\right) \tag{2-7}$$

for some unique $\mathbb{R}_{i,k} \in \mathbb{Q}(\lambda)[\![\tilde{t}, q]\!]$ (after regarding $\tilde{t}$ as a formal element).

2.3 Birkhoff factorization

In this subsection, we do not need to assume that the $\boldsymbol{T}$–action on Y has isolated fixed points. Therefore, in this subsection, $\{\phi_i\}_i$ will denote any chosen basis of $H^*_{\boldsymbol{T}}(Y) \otimes \mathbb{Q}(\lambda)$ with its E–Poincaré dual basis $\{\phi^i\}_i$.

Denote by $\mathbb{I}$ the infinitesimal I–function $\mathbb{J}^{0+,0+}$ defined in [3]. The $\mathbb{S}$ introduced in the previous section is, by the very definition, the infinitesimal S–operator $\mathbb{S}^{0+,0+}$

defined in [3]. Hence

$$\mathbb{I} := \mathbb{J}^{0+,0+} \quad \text{and} \quad \mathbb{S} := \mathbb{S}^{0+,0+}.$$

For $\gamma \in H^*_{\boldsymbol{T}}(Y)$, we denote by $\tilde{\gamma} \in H^*_{\boldsymbol{T}}([V/\boldsymbol{G}])$ a lift of γ; ie $\tilde{\gamma}|_Y = \gamma$.

Let $\boldsymbol{p}_0$ be the equivariant cohomology class $H^*_{\mathbb{C}^*}(\mathbb{P}^1)$ defined by

$$\boldsymbol{p}_0|_0 = z, \quad \boldsymbol{p}_0|_\infty = 0.$$

Consider

$$P_{\tilde{\gamma}} := \sum_i \phi^i \langle\langle \phi_i \otimes \boldsymbol{p}_\infty ; \tilde{\gamma} \otimes \boldsymbol{p}_0 \rangle\rangle^{\mathrm{QG}^{0+,0+}}_{0,1|1} \in H^*_{\boldsymbol{T}}(Y) \otimes \mathbb{Q}(\lambda)[z][\![\tilde{t}, q]\!]$$

and its virtual $\mathbb{C}^*$ localization factorization. As in Proposition 4.3 of [3], there is a Birkhoff factorization

$$z\partial_\gamma \mathbb{I} := z \frac{d}{ds}\Big|_{s=0} \mathbb{I}(\tilde{t} + s\tilde{\gamma}) = \mathbb{S}(P_{\tilde{\gamma}}). \tag{2-8}$$

Since $P_{\tilde{\gamma}} = \gamma + O(q)$, the factorization (2-8) implies that for each $\tilde{t}$, there is a unique expression of $\mathbb{S}(\gamma)$ as a linear combination of $\partial_{\phi_i}\mathbb{I}$ with coefficients in $\mathbb{Q}(\lambda)[z][\![q]\!]$. Hence we conclude the following proposition.

Proposition 2.4 *For each $\tilde{t}$, there are unique coefficients $a_i(z,q) \in \mathbb{Q}(\lambda)[z][\![q]\!]$ making*

$$\sum_i a_i(z,q) z \partial_{\phi_i} \mathbb{I} = \gamma + O(1/z).$$

Furthermore the left-hand side coincides with $\mathbb{S}(\gamma)$.

2.4 Genus-1 theory

From now on, we assume that the Calabi–Yau condition holds; ie

$$c_1(Y) - c_1(\tilde{E}|_Y) = 0 \quad \text{in } H^2(Y, \mathbb{Q}).$$

We apply Givental's localization method [8] to express a genus-1 generating function in terms of the genus-0 generating functions.

Consider the genus-1 generating function with one insertion at an infinitesimally (ie $0+$) weighted marking:

$$\langle \cdot ; \tilde{\gamma} \rangle^{0+,0+}_{1,0|1} := \sum_{d=1}^{\infty} q^d \langle \cdot ; \tilde{\gamma} \rangle^{0+,0+}_{1,0|1,d},$$

where $\tilde{\gamma} \in H^2_{\boldsymbol{T}}([V/\boldsymbol{G}], \mathbb{Q})$. We will study the generating function using the virtual $\boldsymbol{T}$–localization.

In the following conjecture, $c_i(\lambda)$ denotes the element in $\mathbb{Q}(\lambda)$ uniquely determined by

$$1 + c_i(\lambda)\mathrm{e}(\mathbb{E}) = \frac{\mathrm{e}^{\boldsymbol{T}}(\mathbb{E}^\vee \otimes T_{p_i} Y)\mathrm{e}^{\boldsymbol{T}}(\widetilde{E}|_{p_i})}{\mathrm{e}^{\boldsymbol{T}}(T_{p_i} Y)\mathrm{e}^{\boldsymbol{T}}(\mathbb{E}^\vee \otimes \widetilde{E}|_{p_i})}, \tag{2-9}$$

where $\mathbb{E}$ is the Hodge bundle on the moduli stack $\overline{M}_{1,1}$ of stable 1–pointed genus-1 curves.

Conjecture 2.5 *For $\widetilde{\gamma} \in H^2_{\boldsymbol{T}}([V/\boldsymbol{G}], \mathbb{Q})$,*

$$\begin{aligned} \langle \cdot ; \widetilde{\gamma} \rangle^{0+,0+}_{1,0|1} &= \sum_i q_{\widetilde{\gamma}} \frac{\partial}{\partial q_{\widetilde{\gamma}}} \left(-\frac{\log \mathbb{R}_{i,0}|_{\tilde{t}=0}}{24} + c_i(\lambda) \frac{\mathbb{U}_i|_{\tilde{t}=0}}{24} \right) \\ &\quad + \frac{1}{2} \sum_i (\partial_{\widetilde{\gamma}} \mathbb{U}_i|_{\tilde{t}=0}) \lim_{(x,y)\to(0,0)} \left(\left(e^{-\mathbb{U}_i\left(\frac{1}{x}+\frac{1}{y}\right)} e_i \mathbb{V}_{ii}(x,y) - \frac{1}{x+y} \right) \Big|_{\tilde{t}=0} \right), \end{aligned} \tag{2-10}$$

where $q_{\widetilde{\gamma}}(\partial/\partial q_{\widetilde{\gamma}})$ acts on q^β by $q_{\widetilde{\gamma}}(\partial/\partial q_{\widetilde{\gamma}}) q^\beta = q^\beta \int_\beta \widetilde{\gamma}$.

We prove Conjecture 2.5 in the following toric setting. Let Y be a projective smooth toric variety defined by a fan Σ. Let $\Sigma(1)$ be the collection of all 1–dimensional cones ρ in Σ and let $V = \mathbb{C}^{\Sigma(1)}$. Then Y is also given by a GIT quotient $\mathbb{C}^{\Sigma(1)} /\!\!/_\theta \boldsymbol{G}$ for the complex torus $\boldsymbol{G} = (\mathbb{C}^*)^{|\Sigma(1)| - \dim Y}$ and some character θ of $\boldsymbol{G}$. Denote by $\boldsymbol{T}$ the big torus $(\mathbb{C}^*)^{\Sigma(1)}$. Let E and W be as in the beginning of Section 2.

Theorem 2.6 *Conjecture 2.5 holds true for the toric setting.*

2.5 The proof of Theorem 2.6

There is a natural one-to-one correspondence between the $\boldsymbol{T}$–fixed points of Y and the maximal cones of Σ. For a maximal cone σ, denote by p_σ the corresponding $\boldsymbol{T}$–fixed point. The $\boldsymbol{T}$–fixed loci of $Q^{0+,0+}_{1,0|1}(Y,\beta)$ are divided into two types according to their images. We will call a quasimap in $Q^{0+,0+}_{1,0|1}(Y,\beta)^{\boldsymbol{T}}$ a *vertex type* over p_σ if all domain components of the quasimap are all over p_σ. Otherwise, the quasimap will be called a *loop type*. The loop type quasimap is called a loop type over p_σ if the marking of the quasimap is over p_σ.

Let $Q^{\boldsymbol{T}}_{\mathrm{vert},\sigma}$ and $Q^{\boldsymbol{T}}_{\mathrm{loop},\sigma}$ be the substacks of $Q^{0+,0+}_{1,0|1}(Y,\beta)^{\boldsymbol{T}}$ consisting of the vertex and loop types over p_σ, respectively.

By the virtual localization theorem, $\langle\,\cdot\,;\widetilde{\gamma}\rangle^{0+,0+}_{1,0|1}$ is the sum of the localization contribution $\mathbf{Vert}^{\widetilde{\gamma}}_{\sigma}$ from all the vertex types over $p_\sigma\in Y^{\boldsymbol{T}}$ and the localization contribution $\mathbf{Loop}^{\widetilde{\gamma}}_{\sigma}$ from all the loop types over $p_\sigma\in Y^{\boldsymbol{T}}$. That is,

$$\langle\,\cdot\,;\gamma\rangle^{0+,0+}_{1,0|1}:=\sum_{\sigma}\mathbf{Vert}^{\widetilde{\gamma}}_{\sigma}+\sum_{\sigma}\mathbf{Loop}^{\widetilde{\gamma}}_{\sigma},$$

where

$$\mathbf{Vert}^{\widetilde{\gamma}}_{\sigma}:=\sum_{\beta\neq 0}q^{\beta}\int_{[Q^{\boldsymbol{T}}_{\mathrm{vert},\sigma}]^{\mathrm{vir}}}\frac{\mathrm{e}^{\boldsymbol{T}}(\pi_*f^*\widetilde{E})|_{Q^{\boldsymbol{T}}_{\mathrm{vert},\sigma}}\widehat{\mathrm{ev}}^*_1(\widetilde{\gamma})}{\mathrm{e}^{\boldsymbol{T}}(N^{\mathrm{vir}}_{Q^{\boldsymbol{T}}_{\mathrm{vert},\sigma}/Q^{0+,0+}_{1,0|1}(Y,\beta)})},$$

$$\mathbf{Loop}^{\widetilde{\gamma}}_{\sigma}:=\sum_{\beta\neq 0}q^{\beta}\int_{[Q^{\boldsymbol{T}}_{\mathrm{loop},\sigma}]^{\mathrm{vir}}}\frac{\mathrm{e}^{\boldsymbol{T}}(\pi_*f^*\widetilde{E})|_{Q^{\boldsymbol{T}}_{\mathrm{loop},\sigma}}\widehat{\mathrm{ev}}^*_1(\widetilde{\gamma})}{\mathrm{e}^{\boldsymbol{T}}(N^{\mathrm{vir}}_{Q^{\boldsymbol{T}}_{\mathrm{loop},\sigma}/Q^{0+,0+}_{1,0|1}(Y,\beta)})}.$$

The above loop term can be identified with the last term in (2-10) by an argument completely parallel to the corresponding procedure in the proof of Theorem 2.1 of [8].

The analysis of vertex terms needs a nontrivial modification to the corresponding procedure of [8] due to the appearance of diagonal classes Δ_J of $\overline{M}_{g,m|d}$, where $J\subset[d]:=\{1,2,\dots,d\}$. Here Δ_J is the codimension-$(|J|-1)$ cycle class represented by the locus where $0+$ weighted markings of J coincide to each other.

Let $\mathbf{Vert}_{\sigma}$ be the p_σ–vertex part of $\langle\,\rangle^{0+}_{1,0}$. Then by the divisor axiom for the infinitesimally weighted marking, $\mathbf{Vert}^{\widetilde{\gamma}}_{\sigma}=q_{\widetilde{\gamma}}(\partial/\partial q_{\widetilde{\gamma}})\mathbf{Vert}_{\sigma}$. Therefore, it is enough to show

$$\mathbf{Vert}_{\sigma}=-\tfrac{1}{24}\log\mathbb{R}_{\sigma,0}|_{\tilde{t}=0}+\tfrac{1}{24}c_{\sigma}(\lambda)\mathbb{U}_{\sigma}|_{\tilde{t}=0}. \tag{2-11}$$

For $\rho\in\Sigma(1)$, let ξ_ρ be the character of the $\boldsymbol{G}$–action on the corresponding coordinate of $\mathbb{C}^{\Sigma(1)}$. Recall that $\xi_{\rho'},\rho'\not\subset\sigma$ form a basis of the character group of $\boldsymbol{G}$. Hence we may let $\xi_\rho=\sum_{\rho'\not\subset\sigma}a_{\rho,\rho'}\xi_{\rho'}$ for some unique integers $a_{\rho,\rho'}$. For a curve class $\beta\in\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{\boldsymbol{G}}V,\mathbb{Z})$, denote by $\beta(\rho)$ the integer value of β at the line bundle associated to ξ_ρ.

Let β_σ be the set of all pairs (ρ,j) with $\rho\not\subset\sigma$ and $j\in[\beta(\rho)]:=\{1,2,\dots,\beta(\rho)\}$. Then the $\boldsymbol{T}$–fixed p_σ–vertex part of $Q_{1,0}(Y,\beta)$ is the quotient of $\overline{M}_{1,0|\beta_\sigma}$ by a finite group of order $\prod_{\rho\not\subset\sigma}\beta(\rho)!$.

For $C\in\overline{M}_{1,0|\beta_\sigma}$, denote the marked point by $x_{(\rho',j)}$ attached to the index $(\rho',j)\in\beta_\sigma$. Let $\hat{\boldsymbol{x}}_{\rho'}$ for $\rho'\not\subset\sigma$ denote the effective divisor $\sum_{j\in[\beta(\rho')]}x_{(\rho',j)}$, and let $\hat{\boldsymbol{x}}_{\rho}$ for $\rho\in\Sigma(1)$ denote the divisor $\sum_{\rho'}a_{\rho,\rho'}\hat{\boldsymbol{x}}_{\rho'}$ of C. Here for $\rho\not\subset\sigma$ with $\beta(\rho')=0$, we set $\hat{\boldsymbol{x}}_{\rho'} = 0$. Then the corresponding quasimap in $Q_{1,0}(Y,\beta)$ is a pair

$$\big(C, \{\mathcal{O}_C(\hat{\boldsymbol{x}}_\rho), u_\rho\}_{\rho\in\Sigma(1)}\big),$$

where u_ρ is the canonical section of $\mathcal{O}_C(\hat{\boldsymbol{x}}_\rho)$ if $\rho \not\subset \sigma$; otherwise, u_ρ is zero.

Let r be the dimension of E. Decompose $E = \bigoplus_{i=1}^r E_i$ by 1–dimensional $\boldsymbol{T}\times\boldsymbol{G}$–representations E_i. Denote by ξ_i the character of $\boldsymbol{G}$ associated to the $\boldsymbol{G}$–action on E_i. Then $\xi_i = \sum_{\rho'\not\subset\sigma} b_{i,\rho'}\xi_{\rho'}$ for some unique integers $b_{i,\rho'}$. Let $\hat{\boldsymbol{x}}_i = \sum_{\rho'\not\subset\sigma} b_{i,\rho'}\hat{\boldsymbol{x}}_{\rho'}$.

In the following, for a divisor $D = \sum_i a_i p_i$ of C with $p_i \in C$ and $a_i \in \mathbb{Z}$, we define $D^+ := \sum_{a_i>0} a_i p_i$ and $D^- := \sum_{a_i<0} a_i p_i$. By the localization formula (see Section 5.4 of [5]), note that

$$\mathbf{Vert}_\sigma = \sum_{d\neq 0} \frac{q^\beta}{\prod_{\rho\not\subset\sigma}\beta(\rho)!}\int_{\overline{M}_{1,0|\beta_\sigma}} (1 + c_\sigma(\lambda)\mathrm{e}(\mathbb{E}))F^{(1,0)}_{\sigma,\beta},$$

where

$$F^{(1,0)}_{\sigma,\beta} = \prod_{\rho\subset\sigma} \frac{\mathrm{e}^{\boldsymbol{T}}(\pi_*(\mathcal{O}_{\hat{\boldsymbol{x}}_\rho^-}(\hat{\boldsymbol{x}}_\rho^+))\otimes\mathbb{C}_{\sigma,\rho})}{\mathrm{e}^{\boldsymbol{T}}(\pi_*(\mathcal{O}_{\hat{\boldsymbol{x}}_\rho^+}(\hat{\boldsymbol{x}}_\rho^+))\otimes\mathbb{C}_{\sigma,\rho})} \prod_{i=1}^r \frac{\mathrm{e}^{\boldsymbol{T}}(\pi_*(\mathcal{O}_{\hat{\boldsymbol{x}}_i^+}(\hat{\boldsymbol{x}}_i^+))\otimes\widetilde{E}|_{p_\sigma})}{\mathrm{e}^{\boldsymbol{T}}(\pi_*(\mathcal{O}_{\hat{\boldsymbol{x}}_i^-}(\hat{\boldsymbol{x}}_i^+))\otimes\widetilde{E}|_{p_\sigma})};$$

here $\mathbb{C}_{\sigma,\rho}$ denotes the 1–dimensional $\boldsymbol{T}$–subspace of $T_{p_\sigma}Y$ corresponding to the facet of σ complementary to ρ, and π denotes the projection from the universal curve to $\overline{M}_{1,0|\beta_\sigma}$.

By the decomposition sequence $0\to\mathcal{O}_A(-B)\to\mathcal{O}_{A+B}\to\mathcal{O}_B\to 0$ of effective Cartier divisors A, B, the computation of $\mathrm{e}^{\boldsymbol{T}}(\pi_*(\mathcal{O}_{A+B}(D)))$ is reduced to that of $\mathrm{e}^{\boldsymbol{T}}(\pi_*(\mathcal{O}_{x_i}(x_j)))$, which is $-\hat{\psi}_i$ if $i=j$ or $\Delta_{\{i,j\}}$ if $i\neq j$. Here $\hat{\psi}_i$ is the psi-class associated to the $0+$ weighted k^{th} marking. Hence the above expression for $F^{(1,0)}_{\sigma,\beta}$ can be considered as a formal element in $\mathbb{Q}(\lambda)[\![\hat{\psi}_k, \Delta_{\{i,j\}}]\!]$ by expanding denominator as geometric series. For dimensional reasons, $F^{(1,0)}_{\sigma,\beta}$ is expressed as a polynomial in psi-classes and diagonal classes $\Delta_{\{i,j\}}$.

For nonnegative integers g, m, the above expression for $F^{(1,0)}_{\sigma,\beta}$ also defines $F^{(g,m)}_{\sigma,\beta}$ as an element in $H^*(\overline{M}_{g,m|\beta_\sigma}, \mathbb{Q}(\lambda))$ by replacing π_* with the π_* for the universal curve of $\overline{M}_{g,m|\beta_\sigma}$. We will consider only cases where $(g,m) = (1,0)$, $(0,2)$ or $(0,3)$.

We may simplify the expression of $F^{(g,m)}_{\sigma,\beta}$ by universal calculus not depending on g, m, as follows. Let d be a positive integer. For $J\subset[d]$, let $\hat{\psi}_J$ denote $\hat{\psi}_j|_{\Delta_J}$ for any $j\in J$. Note that for $J_1\cap J_2\neq\varnothing$,

$$\Delta_{J_1}\Delta_{J_2} = (-\hat{\psi}_{J_1\cup J_2})^{|J_1\cap J_2|-1}\Delta_{J_1\cup J_2}$$

in $H^*(\overline{M}_{g,m|d},\mathbb{Q})$; see Section 4.4 of [10]. For $j\in[d]$, define Δ_j to be the fundamental class. For a partition $J=\{J_1,\dots,J_k\}$ of $[d]$ (ie $\varnothing\neq J_i\subset[d]$ and $\bigsqcup_{i=1}^k J_i=[d]$), define

$$\Delta_J:=\Delta_{J_1}\cdots\Delta_{J_k}.$$

Then $F^{(g,m)}_{\sigma,\beta}$ can be written

$$F^{(g,m)}_{\sigma,\beta}=\sum_{\substack{J=\{J_1,\dots,J_k\}\\ \text{a partition of }\beta_\sigma}} a_J\Delta_J \tag{2-12}$$

as a linear sum of Δ_J over the coefficient ring $\mathbb{Q}(\lambda)[\widehat{\psi}_\bullet|\bullet\in\beta_\sigma]$ such that the right-hand side does not depend on g, m as long as g, m are bounded.

We claim that $\mathrm{e}(\mathbb{E})\widehat{\psi}_\bullet=0$ in $H^*(\overline{M}_{1,0|k})$ for all $k\geq 1$. The claim is trivial for $k=1$ by dimension. Consider the morphism π which forgets the last marking:

$$\pi\colon\thinspace \overline{M}_{1,0|k+1}\to\overline{M}_{1,0|k}.$$

Since $\pi^*(\mathrm{e}(\mathbb{E}))=\mathrm{e}(\mathbb{E})$ and $\pi^*(\widehat{\psi}_1)=\widehat{\psi}_1$, we have $\mathrm{e}(\mathbb{E})\widehat{\psi}_1=0$ by induction on k. Similarly, $\widehat{\psi}_\bullet=0$ in $H^*(\overline{M}_{0,2|\beta_\sigma})$.

Since $\mathrm{e}(\mathbb{E})\widehat{\psi}_\bullet=0$ in $H^*(\overline{M}_{1,0|\beta_\sigma})$ and $\widehat{\psi}_\bullet=0$ in $H^*(\overline{M}_{0,2|\beta_\sigma})$, we see that

$$\int_{\overline{M}_{1,0|\beta_\sigma}}\mathrm{e}(\mathbb{E})F^{(1,0)}_{\sigma,\beta}=\frac{1}{24}(\text{coeff. of the const. term in } a_{\{\beta_\sigma\}})=\frac{1}{24}\int_{\overline{M}_{0,2|\beta_\sigma}}F^{(0,2)}_{\sigma,\beta}.$$

The second equality follows by the expression (2-12) independent of g,m. This explains the last term of (2-11).

The verification of the first term in the right-hand side of (2-11) requires a further analysis of $F^{(g,m)}_{\sigma,\beta}$. First, observe again that the j^{th} cotangent line on $\overline{M}_{g,m|d}$ for $j\in[d]$ is naturally isomorphic to the j^{th} cotangent line on $\overline{M}_{g,m|d-1}$ under the pullback of the forgetting map of the last $0+$ weighted point. Therefore, $\widehat{\psi}_\bullet^2=0$ in $H^*(\overline{M}_{g,m|\beta_\sigma})$. By $\widehat{\psi}_\bullet^2=0$ and dimensional reasons, it is easy to check that for any partition $J=\{J_1,\dots,J_k\}$ of β_σ, the following equations hold:

$$\int_{\overline{M}_{1,0|\beta_\sigma}}\Delta_J\widehat{\psi}^{a_1}_{J_1}\cdots\widehat{\psi}^{a_k}_{J_k}=\begin{cases}\frac{1}{24} & \text{if } k=1 \text{ and } a_1=1,\\ 0 & \text{otherwise,}\end{cases} \tag{2-13}$$

$$\int_{\overline{M}_{0,3|\beta_\sigma}}\Delta_J\widehat{\psi}^{a_1}_{J_1}\cdots\widehat{\psi}^{a_k}_{J_k}=\begin{cases}1 & \text{if } a_1=\cdots=a_k=1,\\ 0 & \text{otherwise.}\end{cases} \tag{2-14}$$

Here we use the well-known fact that $\int_{\overline{M}_{1,1}} \psi = \frac{1}{24}$ and inductive equalities

$$\begin{aligned}\int_{\overline{M}_{0,3|d}} \widehat{\psi}_1 \cdots \widehat{\psi}_d &= \int_{\overline{M}_{0,2|d-1}} \widehat{\psi}_1 \cdots \widehat{\psi}_{d-1} \\ &\;\vdots \\ &= \int_{\overline{M}_{0,3|1} \cong \overline{M}_{0,4}} \widehat{\psi}_1 \\ &= 1\end{aligned}$$

by the dilaton equation. Let

$$A^{\beta}_{J_1,\dots,J_k} := \text{coeff. of } \prod_{i=1}^{k} \widehat{\psi}_{J_i} \text{ in } a_{\{J_1,\dots,J_k\}}.$$

When $k=1$, we denote $A^{\beta}_{J_1,\dots,J_k}$ simply by A^{β}. Denote by β_{J_i} the set of all pairs (ρ, j) such that $j \in [|J_i(\rho)|]$, where $J_i(\rho) := \{(\rho, j) \in J_i\}$. Then note that

$$A^{\beta}_{J_1,\dots,J_k} = \prod_{i=1}^{k} A^{\beta_{J_i}}, \tag{2-15}$$

which follows from two properties:

(1) $F_{\sigma,\beta}$ is a product of the $\boldsymbol{T}$–equivariant Euler classes of vector bundles with fibers $H^0(C, \mathcal{O}_D(B))$, where D is an effective divisor and B is a divisor of C. Here, supports of D and B are contained in β_σ.

(2) Let $D = D_1 + D_2$, where D_1, D_2 are effective, and let $B = B_1 + B_2$. Then in the K–group element,

$$\mathcal{O}_{D_1+D_2}(B_1+B_2) = \mathcal{O}_{D_1}(B_1) \otimes \mathcal{O}_{D_1}(B_2) + \mathcal{O}_{D_2}(B_2) \otimes \mathcal{O}_{D_2}(-D_1+B_1).$$

Suppose that β_σ is a disjoint union of S_1, S_2 such that supports of D_i, B_i are in S_i for each $i=1,2$. Then

$$\mathrm{e}^{\boldsymbol{T}}\big(H^0(C, \mathcal{O}_{D_1+D_2}(B_1+B_2))\big)|_{S_1,S_2} = \mathrm{e}^{\boldsymbol{T}}\big(H^0(C, \mathcal{O}_{D_1}(B_1))\big)\, \mathrm{e}^{\boldsymbol{T}}\big(H^0(C, \mathcal{O}_{D_2}(B_2))\big),$$

where the restriction to S_1, S_2 is defined to be letting $\Delta_J = 0$ whenever there is J_i in the partition J such that J_i intersects with S_1 and S_2 simultaneously.

By (2-13), (2-14) and (2-15), we note that

$$\begin{aligned}\int_{\overline{M}_{0,3|\beta_\sigma}} F^{(0,3)}_{\sigma,\beta} &= \sum_{k=1}^{\infty} \sum_{\substack{\{J_1,\dots,J_k\}\\ \text{a partition of } \beta_\sigma}} \prod_{i=1}^{k} A^{\beta_{J_i}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{\text{ordered partition}\\ (J_1,\dots,J_k) \text{ of } \beta_\sigma}} \prod_{i=1}^{k} A^{\beta_{J_i}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{\text{ordered}\\ (\beta_{J_1},\dots,\beta_{J_k})}} \prod_{\rho\not\subset\sigma} \binom{\beta(\rho)}{|J_1(\rho)|,\dots,|J_k(\rho)|} \prod_{i=1}^{k} A^{\beta_{J_i}} \\ &= \prod_{\rho\not\subset\sigma} \beta(\rho)! \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{\text{ordered}\\ (\beta_{J_1},\dots,\beta_{J_k})}} \prod_{i=1}^{k} \left(\frac{24}{\prod_{\rho\not\subset\sigma} |J_i(\rho)|!} \int_{\overline{M}_{1,0|\beta_{J_i}}} F^{(1,0)}_{\sigma,\beta_{J_i}} \right).\end{aligned}$$

Hence

$$\sum_{\beta\neq 0} \frac{q^\beta}{\prod_{\rho\not\subset\sigma} \beta(\rho)!} \int_{\overline{M}_{1,0|\beta_\sigma}} F^{(1,0)}_{\sigma,\beta} = \frac{1}{24} \log D_\sigma|_{t=0}.$$

This combined with Corollary 2.3 verifies the first term in the right-hand side of (2-11).

Remark 2.7 By Section 5.9.2 of [5], it is clear that the above proof works also for Calabi–Yau zero loci of homogeneous vector bundles on partial flag varieties Y, local toric varieties, local Grassmannians, and the total spaces of the cotangent bundles of partial flag varieties.

3 Explicit computations

In this section, we prove Theorem 1.1. From now on unless stated otherwise, let $\boldsymbol{G} = \mathbb{C}^*$ and $\boldsymbol{T} = (\mathbb{C}^*)^n$, and let $\mathbb{C}_{l_a}$ be the 1–dimensional representation space of $\boldsymbol{G}$ with positive weight l_a. Let $E = \bigoplus_{a=1}^{r} \mathbb{C}_{l_a}$ with $\sum_{a=1}^{r} l_a = n$. We take the standard $\boldsymbol{T}$–action on V and the $\boldsymbol{T}$–trivial action on E. This gives rise to a $\boldsymbol{T}$–equivariant vector bundle $\widetilde{E}$ on $[V/\boldsymbol{G}]$. Choose a character θ such that $Y := V /\!\!/_\theta \boldsymbol{G}$ becomes $\mathbb{P}^{n-1}$. Under the natural isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^{\boldsymbol{G}} V, \mathbb{Z}) \cong \mathbb{Z}$, we use a nonnegative integer d instead of β. Let p_i be the i^{th} $\boldsymbol{T}$–fixed point of Y as in (1-2).

3.1 Birkhoff factorization revisited

By [7] (see also (5.3.1) of [3]),

$$\mathbb{I}_{\tilde{t}=0} = I_{\boldsymbol{T}}|_{t=0},$$

where the left- and right-hand sides are defined in Section 2.3 and (1-1), respectively. We define the degrees of λ, H and q as

$$\deg \lambda_j = 1 = \deg H, \quad \deg q = 0.$$

Then it is easy to check that, for $k = 0, 1, \dots, n-1$, the $1/z^k$–coefficient I_k of

$$I_{\boldsymbol{T}}|_{t=0}$$

is a homogeneous degree-k element in $\mathbb{Q}[\sigma_1, \dots, \sigma_{n-1}, H][\![q]\!]$ satisfying

$$I_k \in \mathbb{Q}[\![q]\!]H^k \text{ modulo (1-3).} \tag{3-1}$$

On the other hand,

$$\mathbb{S}_{\tilde{t}=0}(H^k) = H^k + O(1/z) \text{ for } 0 \le k \le n-1.$$

Throughout Section 3, *we impose the condition* (1-3). After [12], we define an operation as follows. For

$$F \in \left(\frac{\mathbb{Q}[H]}{(H^n - \lambda_0^n)}\right)[\![1/z]\!][\![q]\!] \quad \text{with} \quad \left(\left(zq\frac{d}{dq} + H\right)F\right)\Big|_{H=1, z=\infty, q=0} \neq 0,$$

let

$$\mathfrak{B}(F) := \frac{\left(zq\frac{d}{dq} + H\right)F(z, H, q)}{\left(\left(zq\frac{d}{dq} + H\right)F(z, H, q)\right)\big|_{H=1, z=\infty}}.$$

Consider $\mathfrak{B}^k(I_{\boldsymbol{T}}|_{t=0}/I_0)$, and note that it is of the form $H^k + O(1/z)$ and homogenous of degree k if we put $\deg z = 1$.

Corollary 3.1 *Recall that* $\equiv$ *denotes the equality modulo relations* (1-3). *Then*

$$\mathbb{S}_{\tilde{t}=0}(H^k) \equiv \mathfrak{B}^k\left(\frac{I_{\boldsymbol{T}}|_{t=0}}{I_0}\right), \quad k = 0, 1, \dots, n-1. \tag{3-2}$$

Proof Let $\widetilde{H} \in H^*_{\boldsymbol{T}}([\mathbb{C}^n/\mathbb{C}^*])$ be the natural lift of H, and let

$$\tilde{t} = \sum_{i=0}^{n-1} t_i \widetilde{H}^i$$

with formal variables t_i. Then there is (the $\boldsymbol{T}$–equivariant version of) the derivative form formula

$$\mathbb{I}(\tilde{t}) = \left(\exp\left(\sum_{i=0}^{n-1} \frac{t_i}{z}\Big(zq\frac{d}{dq} + H\Big)^i\right)\right) I_{\boldsymbol{T}}\bigg|_{t=0}$$

of the big I–function as in Section 5.3 of [3], which shows that

$$(z\partial_{H^i}\mathbb{I})|_{\tilde{t}=0} = \Big(zq\frac{d}{dq} + H\Big)^i (\mathbb{I}|_{\tilde{t}=0}). \tag{3-3}$$

By (3-3) and Proposition 2.4, in order to verify (3-2), it is enough to recall that both sides of (3-2) are of the form $H^k + O(1/z)$. □

Now consider an equivariant cohomology basis

$$\{1, H, \dots, H^{n-1}\}, \quad \text{where } H := c_1^{\boldsymbol{T}}(\mathcal{O}(1))$$

of the $\boldsymbol{T}$–equivariant cohomology ring

$$H^*_{\boldsymbol{T}}(\mathbb{P}^{n-1}) \cong \mathbb{Q}[\lambda_1, \dots, \lambda_n, h]\Big/\left(\prod_{i=1}^{n}(h-\lambda_i)\right), \quad H \mapsto h.$$

Its E–twisted Poincaré metric modulo relations (1-3) becomes

$$g_{ij} := \begin{cases} \prod_a l_a & \text{if } i+j = n-1-r, \\ \lambda_0^n \prod_a l_a & \text{if } i+j = 2n-1-r, \end{cases}$$

for $0 \le i, j \le n-1$. Here we use the relation $H^n = -\prod_{j=1}^n(-\lambda_j) = \lambda_0^n$.

There is an expression of V–correlators in terms of S–correlators by [5, Theorem 3.2.1]:

$$e_i \mathbb{V}_{ii}(x,y)|_{\tilde{t}=0} = \frac{1}{e_i}\frac{\sum_j \mathbb{S}_{z=x,\tilde{t}=0}(\phi_j)|_{p_i}\mathbb{S}_{z=y,\tilde{t}=0}(\phi^j)|_{p_i}}{x+y}.$$

Hence

$$\begin{aligned} e_i \mathbb{V}_{ii}|_{\tilde{t}=0} \equiv{}& \frac{1}{(\prod l_a)e_i(x+y)} \\ &\times \left(\sum_{k=0}^{n-1-r} \mathbb{S}_{z=x,\tilde{t}=0}(H^k)|_{p_i}\mathbb{S}_{z=y,\tilde{t}=0}(H^{n-1-r-k})|_{p_i} \right. \\ &\qquad \left. + \frac{1}{\lambda_0^n}\sum_{b=0}^{r-1} \mathbb{S}_{z=x,\tilde{t}=0}(H^{n-r+b})|_{p_i}\mathbb{S}_{z=y,\tilde{t}=0}(H^{n-1-b})|_{p_i}\right). \end{aligned} \tag{3-4}$$

3.2 Vertex terms

Applying (2-7) to

$$\mathbb{I}|_{\tilde{t}=0,p_i} = I_{\boldsymbol{T}}|_{t=0,p_i} \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{a=1}^{r}\prod_{k=1}^{l_a d}(l_a + kz/\lambda_i)}{\prod_{k=1}^{d}((1+kz/\lambda_i)^n - 1)} \equiv I_{\boldsymbol{T}}|_{t=0,p_n,z\mapsto z/\lambda_i},$$

we obtain

$$\mathbb{I}|_{\tilde{t}=0,p_i} \equiv e^{\mu(q)\lambda_i/z}\left(\sum_{k=0}^{\infty} R_k(q)(z/\lambda_i)^k\right)$$

for some $\mu(q) \in q\mathbb{Q}[\![q]\!]$ and $R_k(q) \in \mathbb{Q}[\![q]\!]$. Hence

$$\begin{aligned}\mathbb{I}|_{\tilde{t}=t_H\tilde{H},p_i} &\equiv e^{\lambda_i t_H/z}(I_{\boldsymbol{T}}|_{t=0,q\mapsto qe^{t_H}}) \\ &\equiv e^{\lambda_i t_H/z} e^{\mu(qe^{t_H})\lambda_i/z}\left(\sum_{k=0}^{\infty} R_k(qe^{t_H})(z/\lambda_i)^k\right).\end{aligned}$$

Thus

$$\text{(3-5)} \qquad \mathbb{U}_i|_{\tilde{t}=t_H\tilde{H}} \equiv \lambda_i(t_H + \mu(qe^{t_H})) \quad \text{and} \quad \mathbb{R}_{i,k} \equiv R_k(qe^{t_H})/(\lambda_i)^k.$$

Since

$$r_{i,0}|_{t=0} = \mathbb{R}_{i,0}|_{\tilde{t}=0}, \quad u_i|_{t=0} = \mathbb{U}_i|_{\tilde{t}=0} \quad \text{and} \quad c_i(\lambda) = \left(\sum_{j\neq i}\frac{1}{\lambda_j - \lambda_i}\right) + \sum_a \frac{1}{l_a\lambda_i},$$

we conclude that

$$\text{(3-6)} \qquad \sum_i \frac{1}{24}\log D_i|_{t=0} \equiv \frac{1}{24}(-n\log R_0(q)),$$

$$\text{(3-7)} \qquad \sum_i \frac{1}{24}c_i(\lambda)u_i|_{t=0} \equiv \frac{1}{24}\left(\sum_a \frac{n}{l_a} - \binom{n}{2}\right)\mu(q).$$

3.3 Loop terms

If we let

$$\mathrm{W}_{p,p'} := \left((\mathfrak{B})^p \frac{I_{\boldsymbol{T}}|_{t=0}}{I_0}\right)\bigg|_{H=1,z=x}\left((\mathfrak{B})^{p'} \frac{I_{\boldsymbol{T}}|_{t=0}}{I_0}\right)\bigg|_{H=1,z=y},$$

$$\mathrm{V}(x,y,q) := \sum_{p+p'=n-1-r} \mathrm{W}_{p,p'} + \sum_{\substack{p+p'=2n-1-r \\ n-r\leq p\leq n-1}} \mathrm{W}_{p,p'},$$

then by (3-2) and (3-4),

$$e_i\mathbb{V}_{ii}(x,y)|_{\tilde{t}=0} \equiv \frac{\lambda_i^{n-1-r}}{(\prod l_a)e_i(x+y)}\mathrm{V}\left(\frac{x}{\lambda_i}, \frac{y}{\lambda_i}, q\right).$$

Here we use also the degree property that $\deg(\mathbb{S}(\phi_j)\mathbb{S}(\phi^j)) = n-1-r$. Therefore,

$$e^{-\mathbb{U}_i|_{\tilde{t}=0}(1/x+1/y)}e_i\mathbb{V}_{ii}(x,y)|_{\tilde{t}=0} \equiv \frac{\lambda_i^{n-1}}{\mathrm{e}^{\boldsymbol{T}}(T_{p_i}Y)(x+y)}e^{-\mu(q)(\lambda_i/x+\lambda_i/y)}\mathrm{V}\Big(\frac{x}{\lambda_i},\frac{y}{\lambda_i},q\Big).$$

Now the limit as $x, y \to 0$ of (2-10) (or equivalently the residue at $x=0,\ y=0$ of (2-10)$/(xy)$) as computed in [11, Lemma 5.4] becomes

$$\lim_{(x,y)\to(0,0)}\Big(e^{-\mathbb{U}_i(1/x+1/y)}e_i\mathbb{V}_{ii}(x,y)-\frac{1}{x+y}\Big)\Big|_{\tilde{t}=0} \equiv \frac{\lambda_i^{n-2}}{\mathrm{e}^{\boldsymbol{T}}(T_{p_i}Y)L(q)}q\frac{d}{dq}\mathrm{Loop}(q),$$

where

$$L(q) := (1-q\prod_a l_a^{l_a})^{-1/n},$$

$$\mathrm{Loop}(q) := \frac{n}{24}\Big(n-1-2\sum_{a=1}^{r}\frac{1}{l_a}\Big)\mu(q) - \frac{3(n-1-r)^2+(n-2)}{24}\log\Big(1-q\prod l_a^{l_a}\Big) - \sum_{k=0}^{n-2-r}\binom{n-r-k}{2}\log C_k(q).$$

Since

$$\frac{\partial\mathbb{U}_i}{\partial t_\gamma}|_{\tilde{t}=0} = L(q)\lambda_i$$

by (3-5) and Proposition 3.2 below, we conclude that

$$\sum_i \mathbf{Loop}_i = \frac{1}{2}q\frac{d}{dq}\mathrm{Loop}(q). \tag{3-8}$$

3.4 Proof of Theorem 1.1

Now the sum

$$\text{(3-6)} + \text{(3-7)} + \tfrac{1}{2}\mathrm{Loop}(q)$$

can be explicitly obtained by Proposition 3.2, and hence we complete the proof of Theorem 1.1.

3.5 Explicit computations of μ, R_0, R_1 and loop terms

Recall we assume (1-3). Let $\lambda_0 = 1$. Note that $I_{\boldsymbol{T}}$ satisfies the differential equation

$$\mathrm{PF}\, I_{\boldsymbol{T}}|_{t=t_H\cdot H} = 0, \quad \text{where } \mathrm{PF} := \Big(z\frac{d}{dt}\Big)^n - 1 - q\prod_a\prod_{m=1}^{l_a}\Big(l_a z\frac{d}{dt}+mz\Big);$$

see [7, Corollary 11.7]. Applying the differential operator PF to the asymptotic form of

$$I_{\boldsymbol{T}}|_{t=t_H H, p_n},$$

one obtains μ, R_0, R_1 and the loop limit; see Sections 4.2 and 4.3 of [11] for details. For the reader's convenience, we state the following proposition due to Popa [11].

Proposition 3.2 [11, Propositions 4.3 & 4.4] *Consider C_b for $b = 0, 1, \ldots, n-1$.*

(1) $\displaystyle\prod_{i=0}^{n-r} C_i = \Big(1 - q \prod_a l_a^{l_a}\Big)^{-1}.$

(2) $C_b = C_{n-r-b}$ *for* $b = 0, 1, \ldots, n-r$.

(3) $C_b = 1$ *for* $b = n-r+1, \ldots, n-1$.

(4) $\displaystyle\mu(q) = \int_0^q \frac{(1 - x\prod_a l_a^{l_a})^{-1/n} - 1}{x} dx$ *and* $R_0 = L^{(r+1)/2}$.

Remark 3.3 Let $\{(H^b)^\vee\}_b$ be the E–twisted Poincaré dual basis of $\{H^b\}_b$. Note that

$$\mathbb{S}(H^{b-1}) \equiv H^{b-1} + \frac{1}{z}\big(H^b \langle (H^b)^\vee, H^{b-1}; H\rangle_{0,2|1}^{0+,0+} \tilde{t}\big) + O\Big(\tilde{t}^2, \frac{1}{z^2}\Big).$$

By (3-2) and the definition of C_b, we have

$$C_b \equiv \langle (H^b)^\vee, H^{b-1}; H\rangle_{0,2|1}^{0+,0+} \quad \text{for } b = 1, \ldots, n-1,$$

and hence Proposition 3.2(2) naturally follows except for the claim that $C_0 = C_{n-r}$.

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