

Boundaries and automorphisms of hierarchically hyperbolic spaces

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Hierarchically hyperbolic spaces provide a common framework for studying mapping class groups of finite-type surfaces, Teichmüller space, right-angled Artin groups, and many other cubical groups. Given such a space \mathcal{X} , we build a bordification of \mathcal{X} compatible with its hierarchically hyperbolic structure.

If \mathcal{X} is proper, eg a hierarchically hyperbolic group such as the mapping class group, we get a compactification of \mathcal{X} ; we also prove that our construction generalizes the Gromov boundary of a hyperbolic space.

In our first main set of applications, we introduce a notion of geometrical finiteness for hierarchically hyperbolic subgroups of hierarchically hyperbolic groups in terms of boundary embeddings.

As primary examples of geometrical finiteness, we prove that the natural inclusions of finitely generated Veech groups and the Leininger–Reid combination subgroups extend to continuous embeddings of their Gromov boundaries into the boundary of the mapping class group, both of which fail to happen with the Thurston compactification of Teichmüller space.

Our second main set of applications are dynamical and structural, built upon our classification of automorphisms of hierarchically hyperbolic spaces and analysis of how the various types of automorphisms act on the boundary.

We prove a generalization of the Handel–Mosher “omnibus subgroup theorem” for mapping class groups to all hierarchically hyperbolic groups, obtain a new proof of the Caprace–Sageev rank-rigidity theorem for many CAT(0) cube complexes, and identify the boundary of a hierarchically hyperbolic group as its Poisson boundary; these results rely on a theorem detecting *irreducible axial* elements of a group acting on a hierarchically hyperbolic space (which generalize pseudo-Anosov elements of the mapping class group and rank-one isometries of a cube complex not virtually stabilizing a hyperplane).

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Introduction

The class of hierarchically hyperbolic spaces (HHSs) was introduced by Behrstock, Hagen and Sisto [5], and they gave a streamlined definition in [6], to provide a common framework for studying cubical groups and mapping class groups of surfaces. The definition was motivated by the observation that, under natural hypotheses, a $CAT(0)$ cube complex is equipped with a collection of projections to hyperbolic spaces obeying rules reminiscent of the hierarchical structure of mapping class groups and projections to curve graphs introduced by Masur and Minsky [59; 60]. The class of HHSs includes the aforementioned spaces (mapping class groups and many $CAT(0)$ cube complexes, including all universal covers of compact special cube complexes), along with Gromov-hyperbolic spaces, Teichmüller space with any of the usual metrics, and many others; see Behrstock, Hagen and Sisto [5; 6; 7] for an account of the current scope of the theory.

Much of the utility of HHSs comes from the fact that many features of Gromov-hyperbolic spaces have natural generalizations in the HHS world. Since one of the most useful objects associated to a hyperbolic space is its Gromov boundary, we provide here a generalization of the Gromov boundary to hierarchically hyperbolic spaces. The boundary of a hierarchically hyperbolic space is inspired by various boundaries associated to the salient examples of HHSs, eg the simplicial boundary of a $CAT(0)$ cube complex and the Thurston compactification of Teichmüller space, projective measured lamination space $\mathbb{PML}(S)$.

Just as the Gromov boundary does for hyperbolic spaces and groups, the HHS boundary provides considerable information about the geometry of an HHS and the dynamics of its automorphisms; our aim in this paper is to explore some of these properties.

Introduction to HHSs

We first briefly and softly recall the HHS theory. A *hierarchically hyperbolic space* is a pair $(\mathcal{X}, \mathfrak{S})$ equipped with some additional data: \mathcal{X} is a quasigeodesic metric space and \mathfrak{S} is an index set equipped with a partial order \sqsubseteq , called *nesting*, with a unique maximal element S . There is also an *orthogonality* relation on \mathfrak{S} ; when \mathfrak{S} is the set of essential subsurfaces of a surface S , up to isotopy, orthogonality is just disjointness. We often call elements of \mathfrak{S} *domains*.

Each $U \in \mathfrak{S}$ is equipped with a uniformly hyperbolic space $\mathcal{C}U$ and a coarse map $\pi_U: \mathcal{X} \rightarrow \mathcal{C}U$. There are also *relative projections* ρ_V^U , which are coarse maps $\mathcal{C}U \rightarrow \mathcal{C}V$ defined unless U and V are orthogonal. In the case where \mathcal{X} is the marking complex of the surface S and \mathfrak{S} is the set of subsurfaces of S , the associated hyperbolic spaces are the curve graphs of these subsurfaces and the projections are subsurface projections. We impose other rules reminiscent of the hierarchical structure of the mapping class group; see [Definition 1.1](#).

The *distance formula* is crucial: for any $x, y \in \mathcal{X}$, the distance $d_{\mathcal{X}}(x, y)$ differs, up to bounded multiplicative and additive error, from the sum of the distances

$$d_{\mathcal{C}U}(\pi_U(x), \pi_U(y))$$

as $U \in \mathfrak{S}$ varies over those domains where that distance exceeds some predefined threshold; see Behrstock, Hagen and Sisto [\[6\]](#).

Just as quasiconvexity is vital to the study of hyperbolic spaces, *hierarchical quasiconvexity* is central in the study of HHSs. Roughly, $\mathcal{Y} \subseteq \mathcal{X}$ is hierarchically quasiconvex if $\pi_U(\mathcal{Y})$ is uniformly quasiconvex for each $U \in \mathfrak{S}$, and any point in \mathcal{X} projecting under π_U close to $\pi_U(\mathcal{Y})$ for each U must lie close (in \mathcal{X}) to \mathcal{Y} . The fundamental example of a hierarchically quasiconvex subspace is the *standard product region* P_U associated to each $U \in \mathfrak{S}$. Roughly, the subspace P_U consists of those points $x \in \mathcal{X}$ where $\pi_V(x)$ is close to ρ_V^U for any $V \in \mathfrak{S}$ that is not orthogonal to, or nested in, U . The factor of P_U obtained by fixing, in addition, the projections to domains orthogonal to U (and allowing movement in domains nested in U) is denoted by F_U , and the other factor is E_U . A familiar example here is the region of Teichmüller space with the Teichmüller metric where the boundary curves of some subsurface U are short: Minsky [\[61\]](#) proved that these so-called thin parts are quasiisometric to products of the Teichmüller spaces of the complementary subsurfaces, one of which is U .

What’s needed from [5; 6] Behrstock, Hagen and Sisto [6] is the main foundational paper in the theory of HHSs. In the current paper, we use most of the background material developed in [6], with the notable exception of the combination theorems. In particular, we use the main definition of HHSs (which is equivalent to, but much simpler than, the original definition from [5]), the realization theorem, the distance formula, and the existence of hierarchy paths. The fact that mapping class groups are hierarchically hyperbolic groups, which is crucial for our applications to Veech and Leininger–Reid subgroups in Section 5, could be deduced from Behrstock [3], Behrstock, Kleiner, Minsky and Mosher [8] and Masur and Minsky [59; 60], but is also given a streamlined proof by Behrstock, Hagen and Sisto [6, Section 11].

From Behrstock, Hagen and Sisto [5], we need the acylindricity result (Theorem 14.3) and, for the purposes of Section 10, the HHS structure on CAT(0) cube complexes. We note that the acylindricity result from [5] is independent of the other HHS results in that paper.

Finally, the recent paper Behrstock, Hagen and Sisto [7] is completely independent of this one.¹

The boundary

Consider an HHS $(\mathcal{X}, \mathfrak{S})$. Since any two points of \mathcal{X} are joined by a *hierarchy path* — a uniform quasigeodesic projecting to a uniform unparametrized quasigeodesic in \mathcal{CU} for each $U \in \mathfrak{S}$ (see [6]) — a natural approach to constructing a boundary is to imitate the construction of the Gromov boundary, or the visual boundary of a CAT(0) space: boundary points would be asymptotic classes of “hierarchy rays” emanating from a fixed basepoint, and one might imagine topologizing this set by defining two boundary points to be close if the corresponding rays stay close “for a long time”.

The boundary construction is motivated by this intuition. Given a hierarchy ray $\gamma: \mathbb{N} \rightarrow \mathcal{X}$, one first observes that the set of $U \in \mathfrak{S}$ for which $\pi_U \circ \gamma$ is unbounded is a pairwise-orthogonal collection — γ either spends a bounded amount of time in each standard product region, or γ wanders permanently into the (coarse) intersection of several standard product regions. Accordingly, the underlying set of the boundary $\partial(\mathcal{X}, \mathfrak{S})$ is the set of formal linear combinations $p = \sum_{U \in \mathfrak{U}} a_U p_U$, where $\mathfrak{U} \subset \mathfrak{S}$ (the *support* of p) is a pairwise-orthogonal set, each p_U is a point in the Gromov boundary of \mathcal{CU} , each $a_U \in (0, 1]$, and $\sum_U a_U = 1$.

Regarding each $\partial\mathcal{CU}$ as a discrete set, the above construction yields a (highly disconnected, locally infinite) simplicial complex. The “rank-one hierarchy rays” — ie the

¹The picture on [Hagen’s website](#) shows the current state of the theory, indicating the main concepts and results and their interdependencies.

points of $\partial\mathcal{C}S$ — correspond to isolated 0–simplices, while the standard product regions contribute boundary subcomplexes isomorphic to simplicial joins. This complex is a kind of “Tits boundary” for $(\mathcal{X}, \mathfrak{S})$. The actual boundary we define is related to this complex in much the same way that the visual boundary of a CAT(0) space is related to the Tits boundary; we define the boundary $\partial(\mathcal{X}, \mathfrak{S})$ by imposing a coarser topology, described in [Section 2](#). (When the context is clear, we denote $\partial(\mathcal{X}, \mathfrak{S})$ by $\partial\mathcal{X}$, being mindful that this space depends, as far as we know, on the particular HHS structure \mathfrak{S} .)

The resulting space $\bar{\mathcal{X}} = \mathcal{X} \cup \partial\mathcal{X}$ is Hausdorff and separable; $\partial\mathcal{X}$ is a closed subset and \mathcal{X} is dense ([Proposition 2.17](#)). Moreover, the Gromov boundary $\partial\mathcal{C}U$ embeds in $\partial(\mathcal{X}, \mathfrak{S})$, in the obvious way, for each $U \in \mathfrak{S}$, by [Theorem 4.3](#). Crucially:

Theorem 3.4 (compactness) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with \mathcal{X} proper. Then $\bar{\mathcal{X}}$ is compact.*

The definition of $\partial(\mathcal{X}, \mathfrak{S})$ is given strictly in terms of \mathfrak{S} and the accompanying hyperbolic spaces and projections; the standing assumption that $(\mathcal{X}, \mathfrak{S})$ is *normalized* — each π_U is coarsely surjective — connects the boundary to the space \mathcal{X} by ensuring that \mathcal{X} is dense in $\bar{\mathcal{X}}$. Even so, it is not clear whether the homeomorphism type of $\partial(\mathcal{X}, \mathfrak{S})$ depends on the particular choice of HHS structure:

Question 1 *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and let $(\mathcal{X}, \mathfrak{S}')$ be a different hierarchically hyperbolic structure on the same space. Does the identity map $\mathcal{X} \rightarrow \mathcal{X}$ extend to a map $\mathcal{X} \cup \partial(\mathcal{X}, \mathfrak{S}) \rightarrow \mathcal{X} \cup \partial(\mathcal{X}, \mathfrak{S}')$ which restricts to a homeomorphism of boundaries?*

A positive answer to [Question 1](#) would stand in contrast to the situation for CAT(0) spaces. For example, the right-angled Artin group A , presented by a path of length 3, famously has the property that the universal cover \tilde{X} of the Salvetti complex can be endowed with different CAT(0) metrics (obtained by perturbing angles in the 2–cells) with nonhomeomorphic visual boundaries; see Croke and Kleiner [\[22\]](#). On the other hand, \tilde{X} admits a hierarchically hyperbolic structure $(\tilde{X}, \mathfrak{S})$ coming from the cubical structure of \tilde{X} (with no dependence on the CAT(0) metric). Perturbing the CAT(0) metric within its quasiisometry type does not change the HHS structure (and hence the HHS boundary), so the HHS boundary is in a sense more “canonical” than the visual boundary in this example (and indeed for all CAT(0) cube complexes with *factor systems*, which we discuss in more detail below).

Automorphisms and their actions on the boundary

An *automorphism* of $(\mathcal{X}, \mathfrak{S})$ is a bijection $g: \mathfrak{S} \rightarrow \mathfrak{S}$ and an isometry $\mathcal{C}U \rightarrow \mathcal{C}g(U)$ for each $U \in \mathfrak{S}$ which satisfy certain compatibility conditions. The distance for-

mula ensures that automorphisms induce uniform quasiisometries of \mathcal{X} , so the group $\text{Aut}(\mathfrak{S})$ of automorphisms uniformly quasiacts by (uniform) quasiisometries on \mathcal{X} . The (quasi)action of $\text{Aut}(\mathfrak{S})$ on \mathcal{X} extends to an action on $\bar{\mathcal{X}}$ that restricts to an action by homeomorphisms on $\partial\mathcal{X}$ (Corollary 6.1).

In one of the main cases of interest, \mathcal{X} is a Cayley graph of a finitely generated group G , and the action of G on itself by left multiplication corresponds to an action on (G, \mathfrak{S}) by HHS automorphisms. In this situation, if the action on \mathfrak{S} is cofinite, then (G, \mathfrak{S}) is a *hierarchically hyperbolic group structure*; if a group G admits a hierarchically hyperbolic group structure, then G is a *hierarchically hyperbolic group* (HHG). The archetypal hierarchically hyperbolic group is the mapping class group of a connected, oriented surface of finite type [6, Section 11]. Other examples include many cubical groups [5], many graphs of hierarchically hyperbolic groups [6], and certain quotients of hierarchically hyperbolic groups [7]. If (G, \mathfrak{S}) is a hierarchically hyperbolic group, then the isometric action of G on itself by left multiplication extends to an action by homeomorphisms on \bar{G} (Corollary 6.2). We describe in detail below our results regarding the dynamics and structure of groups of automorphisms.

Embeddings of subspace boundaries and geometrical finiteness

A desirable property of a boundary is that inclusions of subspaces that are “convex” in an appropriate sense induce embeddings of boundaries with closed images. In Section 5, we show that hierarchically quasiconvex subspaces of \mathcal{X} , which admit their own natural HHS structures [6], have this property: if $\mathcal{Y} \subset \mathcal{X}$ is hierarchically quasiconvex, then \mathcal{Y} has a limit set in $\partial\mathcal{X}$ which is homeomorphic to $\partial\mathcal{Y}$ with the HHS structure inherited from \mathcal{X} . In fact, Theorem 5.6 provides more, by giving natural conditions on maps between HHSs ensuring that they extend continuously to the HHS boundary. This motivates the following definition:

Definition 2 (geometrical finiteness) We say a hierarchically hyperbolic subgroup H of a hierarchically hyperbolic group G is *geometrically finite* if the natural inclusion $\iota: H \hookrightarrow G$ extends continuously to an H -equivariant embedding $\partial\iota: \partial H \hookrightarrow \partial G$.

In what follows, we will be interested in developing this notion and establishing examples in the context of the mapping class group of a finite-type surface.

Comparison of the mapping class group boundary with $\mathbb{P}\mathcal{ML}(S)$

The archetypal hierarchically hyperbolic group is the mapping class group $\mathcal{MCG}(S)$ of a connected, oriented surface S of finite type. The hierarchically hyperbolic structure is provided by results of Aougab [1], Behrstock [3], Behrstock, Kleiner, Minsky and

Mosher [8], Bowditch [12], Clay, Rafi and Schleimer [21], Hensel, Przytycki and Webb [43], Mangahas [55], Masur and Minsky [59; 60], Przytycki and Sisto [66] and Webb [74] and is discussed in detail in Section 11 of Behrstock, Hagen and Sisto [6]. Roughly, \mathfrak{S} is the set of essential subsurfaces of S , up to isotopy, \mathcal{CU} is the curve graph of U for each $U \in \mathfrak{S}$, and projections are usual subsurface projections.

Traditionally, $\mathcal{MCG}(S)$ has been studied via its action on Teichmüller space $\mathcal{T}(S)$ with its Thurston compactification by $\mathbb{P}\mathcal{ML}(S)$. This approach has been fruitful especially when considering subgroups of $\mathcal{MCG}(S)$ defined via flat or hyperbolic geometry. Nonetheless, the $\mathcal{MCG}(S)$ action on $\mathcal{T}(S)$ is not cocompact and the orbits of many subgroups (in fact, any with Dehn twists) are distorted in $\mathcal{T}(S)$, which make $\mathcal{T}(S)$ imperfect for studying the coarse geometry of $\mathcal{MCG}(S)$ and its subgroups.

The situation is further complicated when one attempts to extend the $\mathcal{MCG}(S)$ action on $\mathcal{T}(S)$ to its various boundaries. Teichmüller geodesics are unique and thus geodesic rays based at a point form a natural visual compactification of $\mathcal{T}(S)$, but Kerckhoff [49] proved that it is basepoint dependent and thus the $\mathcal{MCG}(S)$ action fails to extend continuously. While Thurston [72] defined a compactification via $\mathbb{P}\mathcal{ML}(S)$ to which the $\mathcal{MCG}(S)$ action does extend continuously, Thurston's compactification is defined via hyperbolic geometry and the Teichmüller metric is defined via flat geometry, which leads to an incoherence between the internal geometry and its asymptotics in $\mathbb{P}\mathcal{ML}(S)$; see Brock, Leininger, Modami and Rafi [16], Chaika, Masur and Wolf [19], Leininger, Lenzhen and Rafi [51], Lenzhen [53] and Masur [58].

The boundary $\partial(\mathcal{MCG}(S), \mathfrak{S})$ provides the first compactification of $\mathcal{MCG}(S)$, so the action of $\mathcal{MCG}(S)$ on itself by left multiplication extends to a continuous action on the boundary with the dynamical properties we discuss below (see also Section 6). While many of these dynamical properties were originally proven via the $\mathcal{MCG}(S)$ -action on $\mathcal{T}(S)$ with its Thurston compactification, many of the pathologies described above vanish in our construction, as we discuss presently.

On geometrically finite subgroups of $\mathcal{MCG}(S)$

Problem 5 of Hamenstädt [41] and Section 6 of Mosher [62] regard the development of a notion of geometrical finiteness for subgroups of $\mathcal{MCG}(S)$. Mosher suggests a definition that requires an external proper hyperbolic space X on which the candidate subgroup acts with a collection of cusp subgroups in some appropriate sense; geometric finiteness would then require that X and ∂X embed quasiisometrically in $\mathcal{T}(S)$ and continuously in $\mathbb{P}\mathcal{ML}(S)$, respectively. Masur's theorem makes it unreasonable to expect a simultaneous continuous embedding $X \cup \partial X \rightarrow \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$.

We will argue that replacing $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$ with $\mathcal{MCG}(S) \cup \partial\mathcal{MCG}(S)$ as in [Definition 2](#) generates a robust theory of geometrical finiteness. In particular, we prove:

Theorem 3 *Suppose that $H < \mathcal{MCG}(S)$ is one of the following:*

- (1) *The standard embedding of $\mathcal{MCG}(Y)$ for some proper subsurface $Y \subset S$.*
- (2) *Convex cocompact in the sense of Farb and Mosher [31].*
- (3) *A finitely generated Veech group.*
- (4) *A Leininger–Reid combination subgroup [52].*

Then H is a geometrically finite subgroup of $\mathcal{MCG}(S)$.

Hence geometrical finiteness generalizes convex cocompactness for subgroups of $\mathcal{MCG}(S)$ to a broader class of groups. [Theorem 3\(a\)](#) is proven in [Theorem 5.11](#) and [Theorem 3\(b\)](#) is [Theorem 5.12](#). We discuss presently the Veech and Leininger–Reid examples in more detail.

Veech and Leininger–Reid combinations subgroups For Mosher (see [Problem 6.1](#) of [\[62\]](#)), the main test cases for a definition of geometrical finiteness for subgroups of mapping class groups are finitely generated Veech groups and the Leininger–Reid subgroups. It is worth noting that while the former are explicitly defined via flat geometry and the latter somewhat less so, the aforementioned coherence pathologies between the Teichmüller geometry and the Thurston compactification give an obstruction to considering embeddings of natural boundaries associated to them into $\mathbb{P}\mathcal{ML}(S)$. We prove that this obstruction disappears with $\partial\mathcal{MCG}(S)$. We now briefly give some background.

Given a holomorphic quadratic differential q on S , there is an associated copy of \mathbb{H}^2 called a Teichmüller disk, $\text{TD}(q)$, which is a convex subset of $\mathcal{T}(S)$. The stabilizer of $\text{TD}(q)$ in $\mathcal{MCG}(S)$ is $\text{Aff}(q)$, those elements with a representative which act by affine homeomorphisms with respect to the flat metric determined by q . A Veech group V is a subgroup of $\text{Aff}(q)$ which acts properly on $\text{TD}(q)$; we consider only finitely generated Veech groups. The visual boundary of $\text{TD}(q)$ is naturally identified by $\mathbb{P}\mathcal{ML}(q)$, which admits a natural embedding in $\mathbb{P}\mathcal{ML}(S)$ that parametrizes the limit set of V in $\mathbb{P}\mathcal{ML}(S)$ — see Kent and Leininger [\[47\]](#) — but a theorem of Masur [\[58\]](#) implies that this embedding does not give an everywhere continuous extension $\text{TD}(q) \cup \mathbb{P}\mathcal{ML}(q) \hookrightarrow \mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$.

Leininger and Reid [\[52\]](#) construct subgroups of $\mathcal{MCG}(S)$ which are combinations of Veech groups; some are surface groups in which all but one conjugacy class is pseudo-Anosov. The *boundary* of such a surface subgroup is its limit set in $\partial\mathbb{H}^2$. [Problem 3.3](#) of Reid [\[68\]](#) asks if there is a continuous, equivariant embedding of this boundary into $\mathbb{P}\mathcal{ML}(S)$.

While we do not answer this question directly, we do prove something strictly stronger for $\partial\mathcal{MCG}(S)$:

Theorem 5.20 *Let $H < \mathcal{MCG}(S)$ be either a finitely generated Veech or Leininger–Reid subgroup as above. Then the inclusion $H \hookrightarrow \mathcal{MCG}(S)$ extends to a continuous H –equivariant embedding $\partial H \hookrightarrow \partial\mathcal{MCG}(S)$ with closed image. In particular, H is a geometrically finite subgroup of $\mathcal{MCG}(S)$.*

Other candidates for geometrical finiteness Perhaps the next best candidates for geometrically finite subgroups of $\mathcal{MCG}(S)$ are the various right-angled Artin groups constructed by Clay, Leininger and Mangahas [20] and Koberda [50]. These subgroups are HHGs and the former are even known to be quasiisometrically embedded in $\mathcal{MCG}(S)$.

Question 4 *Are the Clay–Leininger–Mangahas and Koberda right-angled Artin subgroups of $\mathcal{MCG}(S)$ geometrically finite? Hierarchically quasiconvex?²*

The HHS boundary of Teichmüller space and $\mathbb{P}\mathcal{ML}(S)$ Slight modifications of the above hierarchical structures endow the Teichmüller space, $\mathcal{T}(S)$, with either the Teichmüller or Weil–Peterson metrics, with an HHS structure, as explained in [5; 6] using results of Brock [15], Durham [26] and Eskin, Masur and Rafi [29]; see also Bowditch [14; 13] for closely related results.

Question 5 *How is the HHS boundary $\partial\mathcal{T}(S)$ of $\mathcal{T}(S)$, with the Teichmüller metric and the above HHS structure, related to the projective measured lamination space $\mathbb{P}\mathcal{ML}(S)$?*

In fact, there is a natural map $\mathbb{P}\mathcal{ML}(S) \rightarrow \partial\mathcal{T}(S)$ which collapses certain simplices of measures on given laminations to points, while being injective on the set of uniquely ergodic laminations, whose image in $\partial\mathcal{T}(S)$ can be identified with a subset of $\partial\mathcal{CS} \subset \partial\mathcal{T}(S)$. A promising strategy is to attempt to use this map, along with a result of Edwards — see Daverman [24] and Edwards [28] — to prove that $\partial\mathcal{T}(S)$ is homeomorphic to $\mathbb{P}\mathcal{ML}(S)$, ie to $\mathbb{S}^{2\xi(S)-1}$. The missing ingredient is a positive answer to:

Question 6 *Does $\partial\mathcal{T}(S)$ have the **disjoint discs property**?*

A metric space M has the disjoint disks property if any two maps $D^2 \rightarrow M$ admit arbitrarily small perturbations with disjoint image; the above question makes sense since it is not hard to see, using Proposition 2.17, that $\partial\mathcal{T}(S)$ is metrizable. The difficulty

²Since we initially posted this paper, Mousley [63] answered this question negatively.

here involves nonuniquely ergodic laminations, which cause a similar problem to the extensions discussed above related to the Leininger–Reid subgroups.

Another question, subject to much recent study, is about the limit sets of Teichmüller geodesics in Thurston’s compactification. The analogous question in our setting is:

Question 7 *What are the limit sets of Teichmüller geodesics in $\partial\mathcal{T}(S)$?*

There are now several constructions of geodesics with limits sets that are bigger than a point—see Brock, Leininger, Modami and Rafi [16], Chaika, Masur and Wolf [19], Leininger, Lenzhen and Rafi [51] and Lenzhen [53]—but these constructions fundamentally depend on the fact that filling minimal laminations can admit simplices of measures, which collapse in $\partial\mathcal{T}(S)$. The geodesics constructed in [16; 19; 51] will have unique limits $\partial\mathcal{T}(S)$ as their asymptotics with respect to $\partial\mathcal{T}(S)$ are determined by their asymptotics in the curve graph \mathcal{CS} . On the other hand, the situation becomes more opaque for Teichmüller geodesics with vertical laminations with multiple components. Using work of Rafi [67], one can determine that the coefficients a_Y of the components $Y \subset S$ supporting the potential limits in $\partial\mathcal{T}(S)$ are determined by limits of ratios of the rates of divergence in the various subsurface curve graphs \mathcal{CY} . However, it seems unlikely that these limits of ratios always exist, suggesting that such geodesics need not have unique limits in $\partial\mathcal{T}(S)$.

Dynamical and structural results

Our second main collection of applications of the boundary are about the dynamics of the action on the boundary and the structure of subgroups. In Section 6, we study automorphisms of hierarchically hyperbolic spaces:

Classification of automorphisms Given $f \in \text{Aut}(\mathfrak{S})$, the set $\text{Big}(f)$ of $U \in \mathfrak{S}$ for which $\langle f \rangle \cdot x$ (for some basepoint $x \in \mathcal{X}$) projects to an unbounded set in \mathcal{CU} is a possibly empty finite set of pairwise-orthogonal domains preserved by the action of $\langle f \rangle$ on \mathfrak{S} . We classify f according to the nature of $\text{Big}(f)$. First, if $\text{Big}(f) = \emptyset$, then f has bounded orbits in each \mathcal{CU} and hence has bounded orbits in \mathcal{X} , by Proposition 6.4; in this case, f is *elliptic*. Second, if $\langle f \rangle \cdot x$ projects to a quasiline in \mathcal{CU} for some $U \in \text{Big}(f)$, then $\langle f \rangle \cdot x$ is a quasiline in \mathcal{X} , by Proposition 6.12, and f is *axial*. Otherwise, f is *distorted*.

If $\text{Big}(f) = \{S\}$, then f is *irreducible*, and f is *reducible* otherwise. Perhaps the most important class of HHS automorphisms are irreducible axial automorphisms. In the mapping class group, these are the pseudo-Anosov elements; in a hierarchically hyperbolic cube complex, these are the rank-one elements that do not virtually preserve hyperplanes; see [5] and Hagen [36]. In the case where (G, \mathfrak{S}) is a hierarchically

hyperbolic group, each irreducible axial element is Morse — this follows from [Theorem 6.15](#) — but the converse does not hold. The question of when irreducible axial elements exist is of major interest later.

Dynamics and fixed points In [Section 6.2](#), we study the dynamics of $f \in \text{Aut}(\mathfrak{S})$ on $\partial\mathcal{X}$. First, we show that irreducible axial automorphisms act as expected:

Proposition 6.18 (north–south dynamics) *If $g \in \text{Aut}(\mathfrak{S})$ is irreducible axial, then g has exactly two fixed points $\lambda_+, \lambda_- \in \partial\mathcal{X}$. Moreover, for any boundary neighborhoods $\lambda_+ \in U_+$ and $\lambda_- \in U_-$, there exists an $N > 0$ such that $g^N(\partial\mathcal{X} - U_-) \subset U_+$.*

In [Propositions 6.19](#) and [6.20](#), we show that if f is irreducible distorted, then f fixes a unique point $p \in \partial\mathcal{X}$, which is an “attracting fixed point”. We also prove analogues of these results for reducible automorphisms ([Propositions 6.22](#) and [6.25](#)).

We then study hierarchically hyperbolic groups. First, we rule out distortion:

Theorem 7.1 (coarse semisimplicity) *If (G, \mathfrak{S}) is a hierarchically hyperbolic group, then each $g \in G$ is either elliptic or axial; in fact g is undistorted in each element of $\text{Big}(g)$.*

In the event that G contains irreducible axial elements, we have:

Theorem 6.29 (topological transitivity) *Let (G, \mathfrak{S}) be hierarchically hyperbolic with an irreducible axial element and let G be nonelementary. Then any G –orbit in ∂G is dense.*

Below, we will describe when (G, \mathfrak{S}) has an irreducible axial element.

Uses of the boundary

We use the boundary, and actions thereon, in numerous ways.

Finding and exploiting irreducible axials In [Section 9](#), we study irreducible axial elements of groups of automorphisms of hierarchically hyperbolic spaces. The setting is an HHS $(\mathcal{X}, \mathfrak{S})$ with \mathcal{X} proper and \mathfrak{S} countable, and we consider a countable subgroup $G \leq \text{Aut}(\mathfrak{S})$. This holds, for example, when $\mathcal{X} = G$ is an HHG. The main technical statement is:

Propositions 9.4 and 9.2 (finding irreducible axials) *Suppose that either G acts properly and coboundedly on \mathcal{X} and cofinitely on \mathfrak{S} , or G acts with unbounded orbits in \mathcal{X} and no fixed point in $\partial\mathcal{CS}$. Then either G contains an irreducible axial element, or there exists $U \in \mathfrak{S} - \{U\}$ which is fixed by a finite-index subgroup of G .*

These two propositions are proved in tandem. The strategy is to consider probability measures on G and corresponding G -stationary measures on $\partial\mathcal{X}$; the main lemma, [Lemma 9.8](#), shows that, unless G has a finite orbit in $\partial\mathcal{CS}$ or $\mathfrak{S} - \{S\}$, such a measure must be supported on $\partial\mathcal{CS} \subset \partial\mathcal{X}$. In particular, if \mathcal{CS} is bounded, then there must be a finite orbit in $\mathfrak{S} - \{S\}$. We emphasize that, for the above proposition and all of its applications, compactness of the HHS boundary (ie [Theorem 3.4](#)) is absolutely vital.

Using the above propositions, we prove:

Theorem 9.15 (HHG Tits alternative) *Let (G, \mathfrak{S}) be an HHG and let $H \leq G$. Then H either contains a nonabelian free group or is virtually abelian.*

By analyzing supports of global fixed points in the boundary of an HHS, we then prove:

Theorem 9.20 (omnibus subgroup theorem) *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group and let $H \leq G$. Then there exists an element $g \in H$ with $\mathfrak{A}(H) = \text{Big}(g)$. Moreover, for any $g' \in H$ and each $U \in \text{Big}(g')$, there exists $V \in \text{Big}(g)$ with $U \sqsubseteq V$.*

Here, $\mathfrak{A}(H)$ is the set of domains U on which H has unbounded projection. The theorem we actually prove is more general than the above, but the version stated here is sufficient to imply the omnibus subgroup theorem for mapping class groups, due to Handel and Mosher [\[42\]](#), which they proved as an umbrella theorem for several subgroup structure theorems, including the Tits alternative; see also Mangahas [\[56\]](#) for further discussion.

We also obtain a coarse/HHS version of the rank-rigidity conjecture for CAT(0) spaces:

Theorems 9.13 and 9.14 (coarse rank-rigidity) *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with \mathcal{X} unbounded and proper and \mathfrak{S} countable. Let $G \leq \text{Aut}(\mathfrak{S})$ be a countable subgroup and suppose that one of the following holds:*

- (1) G acts essentially on \mathcal{X} with no fixed point in $\partial\mathcal{X}$.
- (2) G acts properly and coboundedly on \mathcal{X} and cofinitely on \mathfrak{S} .

*Then either $(\mathcal{X}, \mathfrak{S})$ is a **product HHS with unbounded factors** or there exists an axial element $g \in G$ such that $\text{Big}(g)$ consists of a single domain W , for which \mathcal{CU} is bounded if $U \perp W$.*

Such an element g is a *rank-one automorphism*; all of its quasigeodesic axes of any fixed quality lie in some neighborhood of one another (of radius depending on the quality). The HHS is a *product with unbounded factors* if there exists $U \in \mathfrak{S}$ such that \mathcal{X} coarsely coincides with the standard product region P_U , and each of E_U and F_U is unbounded.

In particular, if \mathcal{X} is any of the cube complexes shown in [5] to be hierarchically hyperbolic (ie those admitting “factor systems”), then our methods allow us to recover the Caprace–Sageev rank-rigidity theorem [18] for \mathcal{X} :

Corollary 9.24 (rank-rigidity for many cube complexes) *Let \mathcal{X} be a CAT(0) cube complex with a factor system. Let G act on \mathcal{X} and suppose that one of the following holds:*

- (1) \mathcal{X} is unbounded and G acts on \mathcal{X} properly and cocompactly.
- (2) G acts on \mathcal{X} with no fixed point in $\mathcal{X} \cup \partial_{\Delta}\mathcal{X}$.

Then \mathcal{X} contains a G -invariant convex subcomplex \mathcal{Y} such that either G contains a rank-one isometry of \mathcal{Y} or $\mathcal{Y} = \mathcal{A} \times \mathcal{B}$, where \mathcal{A} and \mathcal{B} are unbounded convex subcomplexes.

It is difficult to construct cube complexes without factor systems that satisfy the remaining hypotheses of this theorem. At least in the cocompact case, we believe that our proof works without explicitly hypothesizing the existence of a factor system—see Question A of [6], which asks whether the presence of a geometric group action on a cube complex guarantees that a factor system exists (see Remark 9.25).³

Other applications, examples, and questions

The HHS boundary in the cubical case If \mathcal{X} is a CAT(0) cube complex with a factor system \mathfrak{F} (here \mathfrak{F} more properly denotes the set of parallelism classes of elements of the factor system), then the resulting hierarchically hyperbolic structure (which is fundamentally derived from the hyperplanes of \mathcal{X} and how they interact) has a boundary which is, perhaps unsurprisingly, closely related to the *simplicial boundary* $\partial_{\Delta}\mathcal{X}$ introduced by Hagen [36] (which is derived from how certain infinite families of hyperplanes interact). Specifically:

Theorem 10.1 (simplicial and HHS boundaries) *Let \mathcal{X} be a CAT(0) cube complex with a factor system \mathfrak{F} , and let $(\mathcal{X}, \mathfrak{F})$ be the associated hierarchically hyperbolic structure. There is a topology \mathcal{T} on the simplicial boundary $\partial_{\Delta}\mathcal{X}$ such that:*

- (1) *There is a homeomorphism $b: (\partial_{\Delta}\mathcal{X}, \mathcal{T}) \rightarrow \partial(\mathcal{X}, \mathfrak{F})$.*
- (2) *For each component C of the simplicial complex $\partial_{\Delta}\mathcal{X}$, the inclusion $C \hookrightarrow (\partial_{\Delta}\mathcal{X}, \mathcal{T})$ is an embedding.*

In particular, if \mathfrak{F} and \mathfrak{F}' are factor systems on \mathcal{X} , then $\partial(\mathcal{X}, \mathfrak{F})$ is homeomorphic to $\partial(\mathcal{X}, \mathfrak{F}')$.

³After we initially posted this paper, Hagen and Susse [38] showed that every CAT(0) cube complex with a geometric group action admits a factor system and is thus hierarchically hyperbolic.

This theorem highlights the relationship between the question of when factor systems exist, and when \mathcal{X} is *visible* in the sense that every simplex of the simplicial boundary corresponds to a geodesic ray in \mathcal{X} ; this is discussed in [Remark 10.9](#).

Detecting splittings and cubulations from the boundary It is not difficult to show, from the definitions and Stallings' theorem [71] on ends of groups, that if (G, \mathfrak{S}) is a hierarchically hyperbolic group, then $\partial(G, \mathfrak{S})$ is disconnected if and only if G splits over a finite subgroup.

Question 8 *Can the JSJ splitting of G over slender subgroups (see Dunwoody and Sageev [25], Fujiwara and Papasoglu [33] and Rips and Sela [69]) be detected by examining separating spheres in $\partial(G, \mathfrak{S})$, as is the case for hyperbolic groups and splittings over two-ended subgroups (see Bowditch [10])?*

One can also consider producing actions of hierarchically hyperbolic groups on CAT(0) cube complexes other than trees. As usual, this divides into two separate issues, namely detecting a profusion of codimension-1 subgroups and then choosing a finite collection sufficient to produce an action on a cube complex with good finiteness properties. It appears as though $\partial(G, \mathfrak{S})$ can be used to produce a proper action on a cube complex from a sufficiently rich collection of hierarchically quasiconvex codimension-1 subgroups by a method exactly analogous to that used to cubulate various hyperbolic groups by Bergeron and Wise [9]. The main difference is that G does not act as a uniform convergence group on $\partial(G, \mathfrak{S})$; one must replace the space of triples of distinct boundary points by the space of triples $(p, q, r) \in \partial G$ such that any two of p , q and r are *antipodal*, ie joined by a biinfinite hierarchy path.

Question 9 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group. Give conditions on G ensuring that for any antipodal $p, q \in \partial G$, there exists a hierarchically quasiconvex codimension-1 subgroup H such that p and q are in distinct components of ∂gH for some $g \in G$.*

We have not included a detailed discussion of the above “boundary cubulation for HHG” technique in the present paper since there are not yet any applications; these could be provided by an answer to [Question 9](#).

Poisson boundaries and C^* -simplicity In [Section 9.8.1](#), we show that the boundary of an HHG is a topological model for the Poisson boundary:

Theorem 9.26 (Poisson boundary) *Let (G, \mathfrak{S}) be an HHG with $\text{diam } \mathcal{CS} = \infty$, μ be a nonelementary probability measure on G with finite entropy and finite first logarithmic moment, and ν the resulting μ -stationary measure on ∂G . Then $(\partial G, \nu)$ is the Poisson boundary for (G, μ) .*

In fact, $\partial\mathcal{CS}$ is a model for the Poisson boundary [5], but $\partial(G, \mathfrak{S})$ has the advantage of being compact, while in general $\partial\mathcal{CS}$ is not compact. The space ∂G is a G -boundary, ie a compactum on which G acts minimally and proximally. Moreover:

Proposition 10 *The action of G on ∂G is topologically free, ie for each $g \in G - \{1\}$, the set of $p \in \partial\mathcal{X}$ with $gp \neq p$ is dense in $\partial\mathcal{X}$.*

Proof Let $g \in G - \{1\}$, let $q \in \partial G$, and let U be a neighborhood of q . Suppose for a contradiction that g fixes U pointwise. By Proposition 9.4, G contains an irreducible axial element, so by Proposition 6.28, $\partial\mathcal{CS}$ is dense in ∂G , whence, since G is nonelementary, g fixes infinitely many distinct points of $\partial\mathcal{CS}$. If g is reducible axial, then Lemma 6.24 yields a contradiction, since g cannot fix any point in $\partial\mathcal{CS}$ by the lemma. If g is irreducible axial, then g fixes exactly two points in $\partial\mathcal{CS}$, again a contradiction. Otherwise, g is elliptic and hence has finite order and we are done by hypothesis. \square

By a result of Kalantar and Kennedy [46, Theorem 1.5], the above proposition gives a new proof that a nonelementary HHG G with $\partial\mathcal{CS}$ unbounded is C^* -simple (ie the reduced C^* -algebra of G is simple) provided finite-order elements have finite fixed-point sets in $\partial\mathcal{CS}$. However, G is known to be C^* -simple under these circumstances, since G is acylindrically hyperbolic [5] and has no finite normal subgroup; see Dahmani, Guirardel and Osin [23].

In light of the HHG structure on cubulated groups discussed above, Theorem 9.26 should be compared to the results of [64], in which Nevo and Sageev construct the Poisson boundary for a cubical group using the Roller boundary of the cube complex.

Outline of this paper

In Section 1, we review hierarchically hyperbolic spaces. In Section 2, we define the HHS boundary. Section 3 is devoted to the proof that proper HHSs have compact boundaries, and in Section 4, we show that the HHS boundary of a hyperbolic HHS is homeomorphic to the Gromov boundary. In Section 5, we discuss continuous extensions of maps between HHSs to the boundary, and consider this phenomenon in the context of Veech and Leininger–Reid subgroups of the mapping class group. Automorphisms of hierarchically hyperbolic structures induce homeomorphisms of the boundary; in Section 6, we classify their automorphisms and study fixed sets and dynamics of the actions of automorphisms on the boundary. In particular, in Section 7, we show that cyclic subgroups of hierarchically hyperbolic groups are undistorted. Section 8 is a brief technical discussion of essential HHSs and actions, supporting Section 9, in which

we prove the coarse rank-rigidity theorem and some of its consequences. In [Section 10](#), we consider CAT(0) cube complexes with HHS structures coming from [\[5\]](#), relating the HHS boundary to the *simplicial boundary* from Hagen [\[36\]](#).

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1 Background

1.1 Hierarchically hyperbolic spaces

We begin by recalling the definition of a hierarchically hyperbolic space, introduced in [\[5\]](#) and axiomatized in a more efficient fashion in [\[6\]](#), as follows. We begin by defining a hierarchically hyperbolic space. We will work in the context of a *quasigeodesic space*, \mathcal{X} , ie a metric space where any two points can be connected by a uniform-quality quasigeodesic.

Definition 1.1 (hierarchically hyperbolic space) The q -quasigeodesic space $(\mathcal{X}, d_{\mathcal{X}})$ is a *hierarchically hyperbolic space* if there exists $\delta \geq 0$, an index set \mathfrak{S} , whose elements we call *domains*, and a set $\{CW : W \in \mathfrak{S}\}$ of δ -hyperbolic spaces (CU, d_U) , such that the following conditions are satisfied:

(1) **Projections** There is a set $\{\pi_W : \mathcal{X} \rightarrow 2^{CW} \mid W \in \mathfrak{S}\}$ of *projections* sending points in \mathcal{X} to sets of diameter bounded by some $\xi \geq 0$ in the various $CW \in \mathfrak{S}$. Moreover, there exists K such that each π_W is (K, K) -coarsely Lipschitz.

(2) **Nesting** \mathfrak{S} is equipped with a partial order \sqsubseteq , and either $\mathfrak{S} = \emptyset$ or \mathfrak{S} contains a unique \sqsubseteq -maximal element; when $V \sqsubseteq W$, we say V is *nested* in W . We require that $W \sqsubseteq W$ for all $W \in \mathfrak{S}$. For each $W \in \mathfrak{S}$, we denote by \mathfrak{S}_W the set of $V \in \mathfrak{S}$ such that $V \sqsubseteq W$. Moreover, for all $V, W \in \mathfrak{S}$ with $V \not\sqsubseteq W$ there is a specified subset $\rho_W^V \subset CW$ with $\text{diam}_{CW}(\rho_W^V) \leq \xi$. There is also a *projection* $\rho_W^V: CW \rightarrow 2^{CV}$. (The notation is justified by viewing ρ_W^V as a coarsely constant map $CW \rightarrow 2^{CV}$.)

(3) **Orthogonality** \mathfrak{S} has a symmetric and antireflexive relation called *orthogonality*: we write $V \perp W$ when V and W are orthogonal. Also, whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$. We require that for each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which $\{V \in \mathfrak{S}_T : V \perp U\} \neq \emptyset$, there exists $W \in \mathfrak{S}_T - \{T\}$ such that whenever $V \perp U$ and $V \sqsubseteq T$, we have $V \sqsubseteq W$. Finally, if $V \perp W$, then V and W are not \sqsubseteq -comparable.

(4) **Transversality and consistency** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V and W are *transverse*, denoted by $V \pitchfork W$. There exists $\kappa_0 \geq 0$ such that if $V \pitchfork W$, then there are sets $\rho_W^V \subseteq CW$ and $\rho_V^W \subseteq CV$ each of diameter at most ξ and satisfying:

$$\min\{d_W(\pi_W(x), \rho_W^V), d_V(\pi_V(x), \rho_V^W)\} \leq \kappa_0$$

for all $x \in \mathcal{X}$.

For $V, W \in \mathfrak{S}$ satisfying $V \sqsubseteq W$ and for all $x \in \mathcal{X}$,

$$\min\{d_W(\pi_W(x), \rho_W^V), \text{diam}_{CV}(\pi_V(x) \cup \rho_V^W(\pi_W(x)))\} \leq \kappa_0.$$

The preceding two inequalities are the *consistency inequalities* for points in \mathcal{X} . Finally, if $U \sqsubseteq V$, then $d_W(\rho_W^U, \rho_W^V) \leq \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies $W \not\sqsubseteq U$ and either $V \sqsubseteq W$ or $V \pitchfork W$.

(5) **Finite complexity** There exists $n \geq 0$, the *complexity* of \mathcal{X} (with respect to \mathfrak{S}), such that any set of pairwise- \sqsubseteq -comparable elements has cardinality at most n .

(6) **Large links** There exist $\lambda \geq 1$ and $E \geq \max\{\xi, \kappa_0\}$ such that the following holds: Let $W \in \mathfrak{S}$ and let $x, x' \in \mathcal{X}$. Let $N = \lambda d_W(\pi_W(x), \pi_W(x')) + \lambda$. Then there exists $\{T_i\}_{i=1, \dots, [N]} \subseteq \mathfrak{S}_W - \{W\}$ such that for all $T \in \mathfrak{S}_W - \{W\}$, either $T \in \mathfrak{S}_{T_i}$ for some i or $d_T(\pi_T(x), \pi_T(x')) < E$. Also, $d_W(\pi_W(x), \rho_W^{T_i}) \leq N$ for each i .

(7) **Bounded geodesic image** For all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W - \{W\}$, and all geodesics γ of CW , either $\text{diam}_{CV}(\rho_V^W(\gamma)) \leq E$ or $\gamma \cap \mathcal{N}_E(\rho_V^W) \neq \emptyset$.

(8) **Partial realization** There exists a constant α with the following property: Let $\{V_j\}$ be a family of pairwise-orthogonal elements of \mathfrak{S} and let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq CV_j$. Then there exists $x \in \mathcal{X}$ such that

- $d_{V_j}(x, p_j) \leq \alpha$ for all j ,

- for each j and each $V \in \mathfrak{S}$ with $V_j \sqsubseteq V$, we have $d_V(x, \rho_V^{V_j}) \leq \alpha$, and
- if $W \pitchfork V_j$ for some j , then $d_W(x, \rho_W^{V_j}) \leq \alpha$.

(9) **Uniqueness** For each $\kappa \geq 0$, there exists $\theta_u = \theta_u(\kappa)$ such that if $x, y \in \mathcal{X}$ and $d(x, y) \geq \theta_u$, then there exists $V \in \mathfrak{S}$ such that $d_V(x, y) \geq \kappa$.

We often refer to \mathfrak{S} , together with the nesting and orthogonality relations, the projections, and the hierarchy paths, as a *hierarchically hyperbolic structure* for the space \mathcal{X} .

Notation 1.2 Given $U \in \mathfrak{S}$, we often suppress the projection map π_U when writing distances in \mathcal{CU} : given $x, y \in \mathcal{X}$ and $p \in \mathcal{CU}$ we write $d_U(x, y)$ for $d_U(\pi_U(x), \pi_U(y))$ and $d_U(x, p)$ for $d_U(\pi_U(x), p)$. To measure distance between a pair of sets, we take the infimal distance between the two sets. Given $A \subset \mathcal{X}$ and $U \in \mathfrak{S}$ we let $\pi_U(A)$ denote $\bigcup_{a \in A} \pi_U(a)$.

Remark 1.3 (summary of constants) Each hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ is associated with a collection of constants often, as above, denoted by $\delta, \xi, n, \kappa_0, E, \theta_u$ and K , where

- (1) \mathcal{CU} is δ -hyperbolic for each $U \in \mathfrak{S}$,
- (2) each π_U has image of diameter at most ξ and is (K, K) -coarsely Lipschitz, and each ρ_V^U has (image of) diameter at most ξ ,
- (3) for each $x \in \mathcal{X}$, the tuple $(\pi_U(x))_{U \in \mathfrak{S}}$ is κ_0 -consistent,
- (4) E is the constant from the bounded geodesic image axiom.

Whenever working in a fixed hierarchically hyperbolic space, we use the above notation freely. We can, and shall, assume that $E \geq q, E \geq \delta, E \geq \xi, E \geq \kappa_0, E \geq K$ and $E \geq \alpha$.

Lemma 1.4 (“finite dimension”) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space of complexity n and let $U_1, \dots, U_k \in \mathfrak{S}$ be pairwise orthogonal. Then $k \leq n$.*

Proof Definition 1.1(3) provides $W_1 \in \mathfrak{S}$, not \sqsubseteq -maximal, such that $U_2, \dots, U_k \sqsubseteq W_1$. Using Definition 1.1 inductively yields a sequence $W_{k-1} \sqsubset W_{k-2} \sqsubset \dots \sqsubset W_1 \sqsubseteq S$, with $S \sqsubseteq$ -maximal, such that $U_{i-1}, \dots, U_k \sqsubseteq W_i$ for $1 \leq i \leq k-1$. Hence $k \leq n$ by Definition 1.1(5). □

The next lemma is a simple consequence of the axioms and also appears in [7]:

Lemma 1.5 *Let $U, V, W \in \mathfrak{S}$ satisfy $U \perp V$, and $U, V \not\perp W$, and $W \not\sqsubseteq U, V$. Then $d_W(\rho_W^U, \rho_W^V) \leq 2E$.*

Proof Our assumptions imply that $U \sqsubseteq W$ or $U \pitchfork W$, and the same is true for V . Applying partial realization yields a point $x \in \mathcal{X}$ such that $d_T(x, \rho_T^U), d_T(x, \rho_T^V) \leq E$ whenever $T \not\sqsubseteq U, V$ and $T \not\pitchfork U, V$. The claim follows from the triangle inequality. \square

Definition 1.6 For $D \geq 1$, a path γ in \mathcal{X} is a D -*hierarchy path* if

- (1) γ is a (D, D) -quasigeodesic,
- (2) $\pi_W \circ \gamma$ is an unparametrized (D, D) -quasigeodesic for each $W \in \mathfrak{S}$.

An unbounded hierarchy path $[0, \infty) \rightarrow \mathcal{X}$ is a *hierarchy ray*.

The following theorems are proved in [6]:

Theorem 1.7 (realization theorem) *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic. Then for each κ there exist θ_e and θ_u such that the following holds. Let $\vec{b} \in \prod_{W \in \mathfrak{S}} 2^{CW}$ have each coordinate correspond to a subset of CW of diameter at most κ ; for each W , let b_W denote the CW -coordinate of \vec{b} . Suppose that whenever $V \pitchfork W$ we have*

$$\min\{d_W(b_W, \rho_W^V), d_V(b_V, \rho_V^W)\} \leq \kappa$$

and whenever $V \sqsubseteq W$ we have

$$\min\{d_W(b_W, \rho_W^V), \text{diam}_{CW}(b_V \cup \rho_V^W(b_W))\} \leq \kappa.$$

Then the set of all $x \in \mathcal{X}$ such that $d_W(b_W, \pi_W(x)) \leq \theta_e$ for all $CW \in \mathfrak{S}$ is nonempty and has diameter at most θ_u .

Theorem 1.8 (existence of hierarchy paths) *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic. Then there exists D_0 such that any $x, y \in \mathcal{X}$ are joined by a D_0 -hierarchy path.*

Theorem 1.9 (distance formula) *Let (X, \mathfrak{S}) be hierarchically hyperbolic. Then there exists $s_0 \geq \xi$ such that for all $s \geq s_0$ there exist constants K and C such that, for all $x, y \in \mathcal{X}$,*

$$d_X(x, y) \asymp_{(K, C)} \sum_{W \in \mathfrak{S}} \{\{d_W(\pi_W(x), \pi_W(y))\}\}_s.$$

The notation $\{\{A\}\}_B$ denotes the quantity which is A if $A \geq B$ and 0 otherwise.

1.2 Hieromorphisms, automorphisms and hierarchically hyperbolic groups

Morphisms in the category of hierarchically hyperbolic spaces were defined in [6], along with the related notion of a hierarchically hyperbolic group; we recall these definitions here.

Definition 1.10 (hieromorphism) Let $(\mathcal{X}, \mathfrak{S})$ and $(\mathcal{X}', \mathfrak{S}')$ be hierarchically hyperbolic structures on the spaces \mathcal{X} and \mathcal{X}' , respectively. A *hieromorphism*

$$(f, \pi(f), \{\rho(f, U): U \rightarrow \pi(f)(U) \mid U \in \mathfrak{S}\}): (\mathcal{X}, \mathfrak{S}) \longrightarrow (\mathcal{X}', \mathfrak{S}')$$

consists of a map $f: \mathcal{X} \rightarrow \mathcal{X}'$, a map $\pi(f): \mathfrak{S} \rightarrow \mathfrak{S}'$ preserving nesting, transversality and orthogonality, and a set $\{\rho(f, U): U \rightarrow \pi(f)(U) \mid U \in \mathfrak{S}\}$ of quasiisometric embeddings with uniform constants such that the following two diagrams coarsely commute for all nonorthogonal $U, V \in \mathfrak{S}$:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{X}' \\ \downarrow \pi_U & & \downarrow \pi_{\pi(f)(U)} \\ \mathcal{C}U & \xrightarrow{\rho(f,U)} & \mathcal{C}\pi(f)(U) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{C}U & \xrightarrow{\rho(f,U)} & \mathcal{C}\pi(f)(U) \\ \downarrow \rho_V^U & & \downarrow \rho_{\pi(f)(V)}^{\pi(f)(U)} \\ \mathcal{C}V & \xrightarrow{\rho(f,V)} & \mathcal{C}\pi(f)(V) \end{array}$$

where $\rho_V^U: \mathcal{C}U \rightarrow \mathcal{C}V$ is the map from Definition 1.1.

Definition 1.11 (automorphism of an HHS, automorphism group) A hieromorphism $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}, \mathfrak{S})$ is an *automorphism* if $\pi(f): \mathfrak{S} \rightarrow \mathfrak{S}$ is a bijection and $\rho(f, U): \mathcal{C}U \rightarrow \mathcal{C}\pi(f)(U)$ is an isometry for each $U \in \mathfrak{S}$. When the context is clear, we will continue to use f to denote $f, \pi(f)$ and $\rho(f, U)$.

Observe that if f and f' are automorphisms of $(\mathcal{X}, \mathfrak{S})$, then $f \circ f': \mathcal{X} \rightarrow \mathcal{X}$ is also an automorphism: compose the maps $\mathfrak{S} \rightarrow \mathfrak{S}$, and compose isometries of the hyperbolic spaces in the obvious way. Declare automorphisms f and f' *equivalent* if $\pi(f) = \pi(f')$ and $\rho(f, U) = \rho(f', U)$ for all $U \in \mathfrak{S}$. Note that $f, f': \mathcal{X} \rightarrow \mathcal{X}$ uniformly coarsely coincide in this case.

Denote by $\text{Aut}(\mathfrak{S})$ the set of equivalence classes of automorphisms, so $\text{Aut}(\mathfrak{S})$ is a group with the obvious multiplication. If $[f] \in \text{Aut}(\mathfrak{S})$, then $[f]^{-1}$ is represented by the quasiinverse of f associated to $\pi(f)^{-1}$ and $\{\rho(f, U)^{-1} \mid U \in \mathfrak{S}\}$.

Observe that $\text{Aut}(\mathfrak{S})$ quasiacts on \mathcal{X} by uniform quasiisometries. We will sometimes abuse language and refer to individual automorphisms as elements of $\text{Aut}(\mathfrak{S})$, and refer to the “action” of $\text{Aut}(\mathfrak{S})$ on \mathcal{X} . By an *action* of a group G on $(\mathcal{X}, \mathfrak{S})$, we mean a homomorphism $G \rightarrow \text{Aut}(\mathfrak{S})$. “Coarse” properties of an action, like properness and coboundedness, make sense in this context.

Definition 1.12 (equivariant) Let $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}', \mathfrak{S}')$ be a hieromorphism, $G \leq \text{Aut}(\mathfrak{S})$ and $G' \leq \text{Aut}(\mathfrak{S}')$, and $\phi: G \rightarrow G'$ a homomorphism. Then f is ϕ -*equivariant* if

$$\begin{array}{ccc}
 \mathfrak{S} & \xrightarrow{f} & \mathfrak{S}' \\
 g \downarrow & & \downarrow \phi(g) \\
 \mathfrak{S} & \xrightarrow{f} & \mathfrak{S}'
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \mathcal{C}U & \xrightarrow{f} & \mathcal{C}f(U) \\
 g \downarrow & & \downarrow \phi(g) \\
 \mathcal{C}gU & \xrightarrow{f} & \mathcal{C}\phi(g)f(U)
 \end{array}$$

(coarsely) commute for all $g \in G$ and $U \in \mathfrak{S}$. This implies that $\phi(g)f(x) \asymp f(gx)$ for all $x \in \mathcal{X}$ and $g \in G$. If ϕ is an isomorphism and f is ϕ -equivariant, then f is G -equivariant.

Definition 1.13 (hierarchically hyperbolic group) A finitely generated group G is *hierarchically hyperbolic* if there exists a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ such that $G \leq \text{Aut}(\mathfrak{S})$, the action on \mathcal{X} is proper and cobounded, and G acts on \mathfrak{S} with finitely many orbits. In this case we can assume $\mathcal{X} = G$ (with any fixed word-metric) and that the action $G \rightarrow \text{Aut}(\mathfrak{S})$ sends each $g \in G$ to an automorphism whose underlying map $G \rightarrow G$ is left multiplication by g . In this case, we say that (G, \mathfrak{S}) is *hierarchically hyperbolic*.

1.3 Standard product regions

The notion of a standard product region in a hierarchically hyperbolic space, introduced in [6], plays an important role in several places, so we recall the definition here. Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and let $U \in \mathfrak{S}$. Let \mathfrak{S}_U be the set of $V \in \mathfrak{S}$ with $V \sqsubseteq U$ (in particular, $U \in \mathfrak{S}_U$ is the unique \sqsubseteq -maximal element). Let \mathfrak{S}_U^\perp be the set of $V \in \mathfrak{S}$ such that $V \perp U$, together with some \sqsubseteq -minimal $A \in \mathfrak{S}$ such that $V \sqsubseteq A$ for all such V .

Fix $\kappa \geq \kappa_0$ and let F_U be the space of κ -consistent tuples in $\prod_{V \in \mathfrak{S}_U} 2^{cV}$ whose coordinates are sets of diameter $\leq \xi$. Similarly, let E_U be the set of κ -consistent tuples in $\prod_{V \in \mathfrak{S}_U^\perp - \{A\}} 2^{cV}$ whose coordinates are sets of diameter $\leq \xi$. In fact, (F_U, \mathfrak{S}_U) and $(E_U, \mathfrak{S}_U^\perp)$ are HHSs (the hyperbolic space associated to A is $\text{im}_A(E_U)$), and there are hieromorphisms (see [6] or Definition 1.10), inducing quasiisometric embeddings, $F_U \rightarrow \mathcal{X}$ and $E_U \rightarrow \mathcal{X}$, extending to a coarsely defined map $F_U \times E_U \rightarrow \mathcal{X}$ whose image is hierarchically quasiconvex in the sense of [6] (or see below). Specifically, each tuple $\vec{b} \in F_U$ is sent to the tuple that coincides with \vec{b} on \mathfrak{S}_U and has coordinate ρ_V^U for all $V \in \mathfrak{S} - \{U\}$ such that $V \pitchfork U$ or $U \sqsubseteq V$, and is fixed at some base element of E_U on $\mathfrak{S}_U^\perp - \{A\}$. The map $E_U \rightarrow \mathcal{X}$ is defined analogously. The spaces F_U and E_U are the *standard nesting factor* and the *standard orthogonality factor*, respectively, associated to U . The maps are the *standard hieromorphisms* associated to U , and the image P_U of $F_U \times E_U$ is a *standard product region*. Where it will not cause

confusion, we sometimes denote by E_U and F_U the images of the corresponding standard hieromorphisms.

Remark 1.14 (automorphisms of product regions) Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and let $U \in \mathfrak{S}$. Recall that (F_U, \mathfrak{S}_U) is a hierarchically hyperbolic space, where the hyperbolic spaces and projections implicit in the hierarchically hyperbolic structure are exactly those inherited from \mathfrak{S} . Recall that $(E_U, \mathfrak{S}_U^\perp)$ is a hierarchically hyperbolic space, where $\mathcal{C}V$ is as in $(\mathcal{X}, \mathfrak{S})$ except when $V = A$ is the \sqsubseteq -maximal element. The hieromorphism $(E_U, \mathfrak{S}_U^\perp) \rightarrow (\mathcal{X}, \mathfrak{S})$ is determined by the choice of $A \in \mathfrak{S}$ that is \sqsubseteq -minimal among all those containing each V with $V \perp U$, which we take as the \sqsubseteq -maximal element of \mathfrak{S}_U^\perp .

Let \mathcal{A}_U be the group of automorphisms g of \mathfrak{S} such that $g \cdot U = U$. Then there are restriction homomorphisms $\theta_U, \theta_U^\perp: \mathcal{A}_U \rightarrow \text{Aut}(\mathfrak{S}_U), \text{Aut}(\mathfrak{S}_U^\perp)$, defined as follows. Given $g \in \mathcal{A}_U$, let $\theta_U(g)$ act like g on \mathfrak{S}_U and like g on each $\mathcal{C}V$ with $V \sqsubseteq U$.

Define θ^\perp analogously, to give an automorphism of $\mathfrak{S}_U^\perp - \{A\}$ restricting the action of g on \mathfrak{S} , and fixing A . When defining $g: \text{im}_A(E_U) \rightarrow \text{im}_A(E_U)$, we draw attention to two cases, which it will be important to distinguish in [Section 9](#):

- There exist infinitely many $A_i \in \mathfrak{S}$ that are \sqsubseteq -minimal with the property that $V \sqsubseteq A_i$ whenever $V \perp U$. The minimality assumption implies that these A_i are pairwise nonnested, so, using [Lemma 1.4](#) and the consistency axiom, we see that $\pi_{A_i}(E_U)$ has diameter bounded independently of A_i (in fact, just in terms of E); thus, when building the HHS $(E_U, \mathfrak{S}_U^\perp)$, we can take the hyperbolic space $\text{im}_A(E_U)$ associated to the maximal element A to be a single point, and define $g: \text{im}_A(E_U) \rightarrow \text{im}_A(E_U)$ in the obvious way. This conclusion holds, more generally, if there are two transverse \sqsubseteq -minimal “containers” A_i and A_j for the domains orthogonal to U .
- The set $\{A_i\}$ of domains that are \sqsubseteq -minimal with the property that $V \sqsubseteq A_i$ whenever $V \perp U$ is a pairwise-orthogonal set. In this case, there are at most n such A_i , where n is the complexity, by [Lemma 1.4](#). Again, we choose $A \in \{A_i\}$ arbitrarily and define the HHS structure on $(E_U, \mathfrak{S}_U^\perp)$ using A as the \sqsubseteq -maximal element, with associated hyperbolic space $\text{im}_A(E_U)$. Now, if there exists $h \in \text{Aut}(\mathfrak{S})$ such that $hA = A_i$ for some i , then $\text{im}_{A_i}(E_U)$ is uniformly quasiisometric to $\text{im}_A(E_U)$. In particular, $g: \text{im}_A(E_U) \rightarrow \text{im}_A(E_U)$ can be defined so that the restriction homomorphism θ_U^\perp makes sense.

Note that, if $f \in \mathcal{A}_U$ and $x \in P_U \subset \mathcal{X}$, then $d_{F_U \times E_U}(\theta_U(f)(r_U(x)), r_U(f(x)))$ is uniformly bounded, where $r_U: P_U \cong_{\text{qi}} F_U \times E_U \rightarrow F_U$ is coarse projection to the first factor, and a similar statement holds for θ_U^\perp and projection to E_U .

Finally, recall that the standard product region P_U is defined to be the image of $F_U \times E_U$ under the product of the hieromorphisms $(F_U, \mathfrak{S}_U), (E_U, \mathfrak{S}_U^{-1}) \rightarrow (\mathcal{X}, \mathfrak{S})$. This map is coarsely defined, but it is convenient to fix maps $F_U \times E_U \rightarrow \mathcal{X}$ (realizing those hieromorphisms) such that $P_{gU} = gP_U$ for all $U \in \mathfrak{S}$ and $g \in \text{Aut}(\mathfrak{S})$. Similarly, the image of F_{gU} coincides with gF_U , etc. The set $\{P_U : U \in \mathfrak{S}\}$ is $\text{Aut}(\mathfrak{S})$ -invariant.

1.4 Normalized hierarchically hyperbolic spaces and hierarchical quasiconvexity

Hierarchically hyperbolic spaces, in the sense of Definition 1.1, need not coarsely surject to the associated hyperbolic spaces, but in almost all cases of interest, they do. Accordingly:

Definition 1.15 (normalized HHS) The HHS $(\mathcal{X}, \mathfrak{S})$ is *normalized* if there exists C such that for all $U \in \mathfrak{S}$, we have $\mathcal{C}U = \mathcal{N}_{\mathcal{C}U}(\pi_U(\mathcal{X}))$.

Proposition 1.16 Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then \mathcal{X} admits a normalized hierarchically hyperbolic structure $(\mathcal{X}, \mathfrak{S}')$ with a hieromorphism $f: (\mathcal{X}, \mathfrak{S}') \rightarrow (\mathcal{X}, \mathfrak{S})$, where $f: \mathcal{X} \rightarrow \mathcal{X}$ is the identity and $f: \mathfrak{S}' \rightarrow \mathfrak{S}$ is a bijection. Moreover, if $G \leq \text{Aut}(\mathfrak{S})$, then there is a monomorphism $G \rightarrow \text{Aut}(\mathfrak{S}')$ making f equivariant.

Proof Let $\mathfrak{S}' = \mathfrak{S}$, and retain the same nesting, orthogonality, and transversality relations. For each $U \in \mathfrak{S}'$, the associated hyperbolic space $\mathcal{C}_{\text{norm}}U$ is chosen to be uniformly quasiisometric to the uniformly quasiconvex subset $\pi_U(\mathcal{X})$ of $\mathcal{C}U$. The projection $\pi_U: \mathcal{X} \rightarrow \mathcal{C}_{\text{norm}}U$ is, up to composition with a uniform quasiisometry, unchanged (and therefore continues to be coarsely Lipschitz). Let $p_U: \mathcal{C}U \rightarrow \mathcal{C}_{\text{norm}}U$ be the composition of the coarse closest-point projection $\mathcal{C}U \rightarrow \pi_U(\mathcal{X})$, composed with the uniform quasiisometry $\pi_U(\mathcal{X}) \rightarrow \mathcal{C}_{\text{norm}}U$. Then, for all U and V with $U \pitchfork V$ or $U \sqsubseteq V$, define the relative projection $\mathcal{C}_{\text{norm}}U \rightarrow \mathcal{C}_{\text{norm}}V$ to be the composition of $p_U \circ \rho_V^U: \pi_U(\mathcal{X}) \rightarrow \mathcal{C}_{\text{norm}}V$ with the quasiisometry $\mathcal{C}_{\text{norm}}U \rightarrow \pi_U(\mathcal{X})$. The remaining assertions are a matter of checking definitions. □

Recall from [6] that the subspace \mathcal{Y} of $(\mathcal{X}, \mathfrak{S})$ is *hierarchically quasiconvex* if there exists $k_0 \geq 0$ such that $\pi_U(\mathcal{Y})$ is k_0 -quasiconvex in $\mathcal{C}U$ for all $U \in \mathfrak{S}$ and if, for all $\kappa \geq \kappa_0$, each κ -consistent tuple $\bar{b} \in \prod_{U \in \mathfrak{S}} \mathcal{C}U$ with U -coordinate in $\pi_U(\mathcal{Y})$ for all U has the property that any associated realization point $x \in \mathcal{X}$ lies at distance from \mathcal{Y} , depending only on κ .

In the interest of staying in the class of normalized hierarchically hyperbolic spaces, we will always work with a normalized hierarchically hyperbolic structure on \mathcal{Y} , namely

the one provided by Proposition 1.16. Moreover, we will (abusively) eschew the notation $\mathcal{C}_{\text{norm}}U$ and use the same notation for $\pi_U(\mathcal{Y})$ and its thickening; in other words, we will regard $\pi_U(\mathcal{Y})$ as a genuine (uniformly) hyperbolic geodesic space.

Finally, we recall the following notion from [6, Definition 5.3, Lemma 5.4]. Let $\mathcal{Y} \subset \mathcal{X}$ be a hierarchically quasiconvex subspace. Then there is a coarsely Lipschitz map $g_{\mathcal{Y}}: \mathcal{X} \rightarrow \mathcal{Y}$ (the coarse Lipschitz constants depend only on the constants from Definition 1.1 and the constants implicit in the definition of hierarchical quasiconvexity) with the following property: for each $U \in \mathfrak{S}$ and $x \in \mathcal{X}$, the projection $\pi_U(g_{\mathcal{Y}}(x))$ uniformly coarsely coincides with the coarse closest-point projection of $\pi_U(x)$ to the quasiconvex subspace $\pi_U(\mathcal{Y})$. The map $g_{\mathcal{Y}}$ is the *gate map* associated to \mathcal{Y} .

2 Definition of the boundary

Fix a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$. For each $S \in \mathfrak{S}$, denote by $\partial\mathcal{C}S$ the Gromov boundary, ie the space of equivalence classes of sequences $(x_n \in \mathcal{C}S)$, where (x_n) and (y_n) are equivalent if for some (hence any) fixed basepoint $x \in \mathcal{C}S$, we have $(x_n, y_n)_x \rightarrow \infty$. In particular, $\partial\mathcal{C}S$ need not be compact if $\mathcal{C}S$ is not proper. The topology is as usual.

Remark 2.1 (extending the Gromov product) For $U \in \mathfrak{S}$, any $p, q \in \mathcal{C}U \cup \partial\mathcal{C}U$ are joined to $u \in \mathcal{C}U$ by $(1, 20\delta)$ -quasigeodesics, enabling extension of the Gromov product to $\partial\mathcal{C}U$.

2.1 Supports and boundary points

We first define $\partial\mathcal{X} = \partial(\mathcal{X}, \mathfrak{S})$ as a set.

Definition 2.2 (support set, boundary point) A *support set* $\bar{S} \subset \mathfrak{S}$ is a set with $S_i \perp S_j$ for all $S_i, S_j \in \bar{S}$. Given a support set \bar{S} , a *boundary point* with *support* \bar{S} is a formal sum $p = \sum_{S \in \bar{S}} a_S^p p_S$, where each $p_S \in \partial\mathcal{C}S$, and $a_S^p > 0$, and $\sum_{S \in \bar{S}} a_S^p = 1$. Such sums are necessarily finite, by Lemma 1.4. We denote the support \bar{S} of p by $\text{Supp}(p)$.

Definition 2.3 (boundary) The *boundary* $\partial(\mathcal{X}, \mathfrak{S})$ of $(\mathcal{X}, \mathfrak{S})$ is the set of boundary points.

Notation 2.4 When the specific HHS structure is clear, we write $\partial\mathcal{X}$ to mean $\partial(\mathcal{X}, \mathfrak{S})$.

2.2 Topologizing $\partial\mathcal{X}$

We topologize $\partial\mathcal{X}$ using the visual topologies on the Gromov boundaries of elements of $\{\mathcal{C}S : S \in \mathfrak{S}\}$. The main challenge is to incorporate these topologies into a coherent

topology on the whole boundary, allowing boundary points supported on nonorthogonal domains to interact. This requires some preliminary definitions.

Definition 2.5 (remote point) Let $\bar{S} \subset \mathfrak{S}$ be a support set. A point $p \in \partial\mathcal{X}$ is *remote* (with respect to \bar{S} , or with respect to some $q \in \partial\mathcal{X}$ with support \bar{S}) if

- (1) $\text{Supp}(p) \cap \bar{S} = \emptyset$, and
- (2) for all $S \in \bar{S}$, there exists $T \in \text{Supp}(p)$ such that S and T are *not* orthogonal.

Denote by $\partial_{\bar{S}}^{\text{rem}}\mathcal{X}$ the set of all remote points with respect to \bar{S} .

For each $S \in \mathfrak{S}$, let $\mathcal{B}(CS)$ be the set of all bounded sets in CS . If $\bar{S} \subset \mathfrak{S}$ is a support set, we denote by \bar{S}^\perp the set of all $U \in \mathfrak{S}$ such that $U \perp S$ for all $S \in \bar{S}$.

Definition 2.6 (boundary projection) Let $\bar{S} \subset \mathfrak{S}$ be a support set. For each $q \in \partial_{\bar{S}}^{\text{rem}}\mathcal{X}$, let \bar{S}_q be the union of \bar{S} and the set of domains $T \in \bar{S}^\perp$ such that T is not orthogonal to W_T for some $W_T \in \text{Supp}(q)$. Define a *boundary projection* $\partial\pi_{\bar{S}}(q) \in \prod_{S \in \bar{S}_q} CS$ as follows. Let $q = \sum_{T \in \bar{T}} a_T^p q_T$ be a remote point with respect to \bar{S} . For each $S \in \bar{S}_q$, let $T_S \in \text{Supp}(q)$ be chosen so that S and T_S are not orthogonal. Define the S -coordinate $(\partial\pi_{\bar{S}}(q))_S$ of $\partial\pi_{\bar{S}}(q)$ as follows:

- (1) If $T_S \sqsubseteq S$ or $T_S \pitchfork S$, then $(\partial\pi_{\bar{S}}(q))_S = \rho_S^{T_S}$.
- (2) Otherwise, $S \sqsubseteq T_S$. Choose a $(1, 20\delta)$ -quasigeodesic ray γ in CT_S joining $\rho_S^{T_S}$ to q_{T_S} . By the bounded geodesic image axiom, there exists $x \in \gamma$ such that $\rho_S^{T_S}$ is coarsely constant on the subray of γ beginning at x . Let $(\partial\pi_{\bar{S}}(q))_S = \rho_S^{T_S}(x)$.

Lemma 2.7 *The map $\partial\pi_{\bar{S}}$ is coarsely independent of the choice of $\{T_S\}_{S \in \bar{S}}$.*

Proof Suppose that $T_S, T'_S \in \bar{T}$ are chosen so that T_S and T'_S are not orthogonal to S and suppose that $S \not\sqsubseteq T_S, T'_S$. In other words, either $T_S \pitchfork S$ or $T_S \pitchfork S$ and the same is true for T'_S . By partial realization (Definition 1.1(8)), there therefore exists $y \in \mathcal{X}$ such that $d_S(\rho_S^{T_S}, y), d_S(\rho_S^{T'_S}, y) \leq E$, whence $\rho_S^{T_S}$ and $\rho_S^{T'_S}$ coarsely coincide. If $S \sqsubseteq T_S$, then $S \perp T'_S$ since $T_S \perp T'_S$; this contradicts the defining property of T'_S . Hence, in all allowable situations, $\rho_S^{T_S}$ coarsely coincides with $\rho_S^{T'_S}$; the claim follows. □

Fix a basepoint $x_0 \in \mathcal{X}$. We are now ready to define a neighborhood basis for each $p = \sum_{S \in \bar{S}} a_S^p p_S$, where $p_S \in CS$ for all $S \in \text{Supp}(p) = \bar{S}$. For each $S \in \mathfrak{S}$, choose a cone-topology neighborhood U_S of p_S in $CS \cup \partial CS$, and choose $\epsilon > 0$. For convenience, given $q \in \partial\mathcal{X}$, we let $a_T^q = 0$ when $T \in \mathfrak{S} - \text{Supp}(q)$.

We define the basic set $\mathcal{N}_{\{U_S\}, \epsilon}(p)$ as the union of a *remote part*, a *nonremote part*, and an *interior part*, as follows:

Definition 2.8 (remote part) The *remote part* is

$$\mathcal{N}_{\{U_S\},\epsilon}^{\text{rem}}(p) = \left\{ q \in \partial_{\bar{S}}^{\text{rem}} \mathcal{X} \mid \forall S \in \bar{S}, (\partial\pi_{\bar{S}}(q))_S \in U_S \text{ and } \sum_{T \in \bar{S}^\perp} a_T^q < \epsilon \right. \\ \left. \text{and } \forall S \in \bar{S}_q, S' \in \bar{S}, \left| \frac{d_S(x_0, (\partial\pi_{\bar{S}}(q))_S)}{d_{S'}(x_0, (\partial\pi_{\bar{S}}(q))_{S'})} - \frac{a_S^p}{a_{S'}^p} \right| < \epsilon \right\}.$$

Definition 2.9 (nonremote part) Given $p, q \in \partial\mathcal{X}$, let $A = \text{Supp}(p) \cap \text{Supp}(q)$. The *nonremote part* is

$$\mathcal{N}_{\{U_S\},\epsilon}^{\text{non}}(p) = \left\{ q = \sum_T a_T^q q_T \in \partial\mathcal{X} - \partial_{\bar{S}}^{\text{rem}} \mathcal{X} \mid \forall T \in A, |a_T^q - a_T^p| < \epsilon \text{ and } q_T \in U_T, \right. \\ \left. \text{and } \sum_{V \in \text{Supp}(q) - A} a_V^q < \epsilon \right\}.$$

Definition 2.10 (interior part) The *interior part* is

$$\mathcal{N}_{\{U_S\},\epsilon}^{\text{int}}(p) = \left\{ x \in \mathcal{X} \mid \forall S, S' \in \bar{S}, \forall T \in \bar{S}^\perp, \pi_S(x) \in U_S \text{ and } \left| \frac{a_S}{a_{S'}} - \frac{d_S(x_0, x)}{d_{S'}(x_0, x)} \right| < \epsilon \right. \\ \left. \text{and } \frac{d_T(x_0, x)}{d_S(x_0, x)} < \epsilon \right\}.$$

Definition 2.11 (topology on $\mathcal{X} \cup \partial\mathcal{X}$) For each $p \in \partial\mathcal{X}$ with $\text{Supp}(p) = \bar{S}$, and $\{U_S : S \in \bar{S}\}$ and $\epsilon > 0$ as above, let

$$\mathcal{N}_{\{U_S\},\epsilon}(p) = \mathcal{N}_{\{U_S\},\epsilon}^{\text{rem}}(p) \cup \mathcal{N}_{\{U_S\},\epsilon}^{\text{non}}(p) \cup \mathcal{N}_{\{U_S\},\epsilon}^{\text{int}}(p).$$

We declare the set of all such $\mathcal{N}_{\{U_S\},\epsilon}(p)$ to form a neighborhood basis at p . Also, we include in the topology on $\mathcal{X} \cup \partial\mathcal{X}$ the open sets in \mathcal{X} . This topology does not depend on x_0 .

Remark 2.12 The $\mathcal{N}_{\{U_S\},\epsilon}(p)$ need not be open; a priori, they may have empty interior!

The following is an obvious consequence of the definitions:

Proposition 2.13 For all $U \in \mathfrak{S}$, the inclusion $\partial\mathcal{C}U \hookrightarrow \partial\mathcal{X}$ is an embedding.

Proposition 2.17 gives basic properties of $\partial\mathcal{X}$; first we need a definition and some lemmas.

Definition 2.14 (basically Hausdorff) Let \mathcal{H} be a topological space and let \mathcal{B} be a neighborhood basis. Then $(\mathcal{H}, \mathcal{B})$ is *basically Hausdorff* if for all distinct $h, h' \in \mathcal{H}$, there exist disjoint $B, B' \in \mathcal{B}$ with $h \in B$ and $h' \in B'$.

Lemma 2.15 *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic and let $\bar{\mathcal{X}} = \mathcal{X} \cup \partial\mathcal{X}$. Then, equipped with the neighborhood basis declared above, $\bar{\mathcal{X}}$ is basically Hausdorff.*

Proof Let $p, q \in \bar{\mathcal{X}}$ be distinct. The statement is obvious when p or q is in \mathcal{X} , so assume that $p, q \in \partial\mathcal{X}$. Fix a basepoint $x_0 \in \mathcal{X}$.

For each $U \in \text{Supp}(p)$, choose a neighborhood Y_U^p of p in $\mathcal{C}U \cup \partial\mathcal{C}U$ that does not contain $(\partial\pi_{\text{Supp}(p)}(q))_U$, provided it is defined. For each $T \in \text{Supp}(q)$, choose a neighborhood Y_T^q of q in $\mathcal{C}T \cup \partial\mathcal{C}T$ that does not intersect $\mathcal{N}_{1000E+\omega}(\{\pi_T(x_0)\})$ and, when it is defined, $\mathcal{N}_{1000E+\omega}((\partial\pi_{\text{Supp}(q)}(p))_T)$, where $\omega \geq 0$ is to be determined; also choose Y_T^q so that $Y_T^p \cap Y_T^q = \emptyset$ when $T \in \text{Supp}(p) \cap \text{Supp}(q)$, unless $p_T = q_T$, in which case we choose $Y_T^p = Y_T^q$. Fix $\epsilon > 0$, to be determined. Let $\mathcal{N}(p) = \mathcal{N}_{\{Y_U^p\}, \epsilon}(p)$ and $\mathcal{N}(q) = \mathcal{N}_{\{Y_T^q\}, \epsilon}(q)$.

Finally, for any $w, v \in \partial\mathcal{X}$, let $\text{Supp}(w)_v = \text{Supp}(w) \cup (\text{Supp}(w)^\perp - \text{Supp}(v)^\perp)$.

We need an auxiliary claim:

Claim 1 Let $x, p, q \in \partial\mathcal{X}$. Suppose there exist $W_p, W_q \in \text{Supp}(x)$, and $U \in \text{Supp}(p)_x$ and $V \in \text{Supp}(q)_x$, such that $W_p \not\perp U$ and $W_p \neq U$, and $W_q \not\perp V$ and $W_q \neq V$. Then there exists $y \in P_{W_p} \cap P_{W_q} \subset \mathcal{X}$ such that $(\partial\pi_{\text{Supp}(p)}(x))_U$ $100E$ -coarsely coincides with $\pi_U(y)$, and $(\partial\pi_{\text{Supp}(q)}(x))_V$ $100E$ -coarsely coincides with $\pi_V(y)$.

(P_{W_p} is the standard product region associated to W_p , defined in Section 1.3.)

Proof of Claim 1 If $W_p \pitchfork U$ or $W_p \sqsubseteq U$, and $W_q \pitchfork V$ or $W_q \sqsubseteq V$, then any $y \in P_{W_p} \cap P_{W_q}$ suffices. If $U \sqsubsetneq W_p$, use partial realization to see that, given a $(1, 20\delta)$ -quasigeodesic ray γ in $\mathcal{C}W_p$ with endpoint x_{W_p} , we can choose a sequence (y_n) in $P_{W_p} \cap P_{W_q}$ projecting uniformly close to an unbounded sequence in γ . This provides the desired y . \triangleleft

Suppose that $x \in \mathcal{N}(p) \cap \mathcal{N}(q)$. We consider the following cases:

- (1) **$x \in \partial\mathcal{X}$ is p -remote and q -remote** First of all, notice that by definition of remote, for any $U \in \text{Supp}(p)$ there exists W_p as in Claim 1, and similarly for $V \in \text{Supp}(q)$. We now consider the following subcases:
 - (a) There exists $U \in \text{Supp}(p) \cap \text{Supp}(q)$ with $p_U \neq q_U$.
 - (b) There exists $U \in \text{Supp}(p) \cap \text{Supp}(q)$ with $p_U = q_U$ but $a_U^p \neq a_U^q$.
 - (c) Up to swapping p and q , there exists $V \in \text{Supp}(q) - \text{Supp}(p)$, and there exists $U \in \text{Supp}(p)$ not orthogonal to V .
 - (d) The previous case does not apply and, up to swapping p and q , there exists $V \in (\text{Supp}(q) - \text{Supp}(p)) \cap \text{Supp}(p)^\perp$.

(a) Then we would have that $(\partial\pi_{\text{Supp}(p)}(x))_U$ is contained in both Y_U^p and Y_U^q , which are disjoint, a contradiction.

(b) Let $\mathcal{U} = \text{Supp}(p) \cap \text{Supp}(q)$. For each $V \in \mathcal{U}$ we have that the ratio

$$\frac{d_V(x_0, (\partial\pi_{\text{Supp}(p)}(x))_V)}{d_U(x_0, (\partial\pi_{\text{Supp}(p)}(x))_U)}$$

is ϵ -close to both a_V^p/a_U^p and a_V^q/a_U^q . Hence, if there exists $V \in \mathcal{U}$ such that $a_V^p/a_U^p \neq a_V^q/a_U^q$, we can choose ϵ small enough to give a contradiction. Otherwise, since the coefficients sum to 1, the supports of p and q do not coincide, and we deal with this in the next subcases.

(c) If $U \pitchfork V$, then by our choice of $\mathcal{N}(p)$ and $\mathcal{N}(q)$, we have $d_U(y, \rho_V^U) > E$ and $d_V(y, \rho_V^U) > E$ for y as in Claim 1, contradicting consistency. If $U \sqsubset V$ or $V \sqsubset U$, then we reach a similar contradiction of consistency.

(d) Suppose also that $\text{Supp}(p) \subseteq \text{Supp}(q) \cup \text{Supp}(q)^\perp$ but $\text{Supp}(p) \cap \text{Supp}(q)^\perp \neq \emptyset$, since otherwise either (a) or (b) holds. Let $U \in \text{Supp}(p) - \text{Supp}(q)$. By remoteness of x , we have $U \in \text{Supp}(q)^\perp - \text{Supp}(x)^\perp$, so $U \in \text{Supp}(q)_x$. Hence the definition of q -remoteness gives

$$\left| \frac{d_U(x_0, (\partial\pi_{\text{Supp}(q)}(x))_U)}{d_V(x_0, (\partial\pi_{\text{Supp}(q)}(x))_V)} - \frac{a_U^q}{a_V^q} \right| < \epsilon.$$

Similarly, we have $V \in \text{Supp}(p)_x$, so the definition of p -remoteness gives

$$\left| \frac{d_V(x_0, (\partial\pi_{\text{Supp}(p)}(x))_V)}{d_U(x_0, (\partial\pi_{\text{Supp}(p)}(x))_U)} - \frac{a_V^p}{a_U^p} \right| < \epsilon.$$

Now, since $V \notin \text{Supp}(p)$, $U \notin \text{Supp}(q)$, we have $a_V^p = a_U^q = 0$, so, we may take y to be the point in \mathcal{X} provided by Claim 1, and hence we have $d_V(y, x_0)/d_U(y, x_0) < 2\epsilon$ and $d_U(y, x_0)/d_V(y, x_0) < 2\epsilon$ provided ω in Claim 1 was chosen sufficiently large in terms of ϵ and E . This is a contradiction.

(2) $x \in \mathcal{X}$ In this case, x can play the role of y in the above arguments.

(3) $x \in \partial\mathcal{X}$ is p -nonremote and q -nonremote In this case, first choose $\epsilon \in (0, \frac{1}{2})$ smaller than $\frac{1}{10}|a_W^p - a_W^q|$ for each $W \in \text{Supp}(p) \cap \text{Supp}(q)$. The definition of the nonremote part now ensures that x cannot exist.

(4) $x \in \partial\mathcal{X}$ is p -remote and q -nonremote In this case, there exists $U \in \text{Supp}(p)$ and $V \in \text{Supp}(q)$, and $W_p, W_q \in \text{Supp}(x)$, such that W_p is distinct from and not orthogonal to U while $W_p = V$ or $W_p \perp V$. If for each such W_q we have $W_q \in \text{Supp}(q)^\perp$,

then by choosing $\epsilon < 1$, we have that $\sum_{T \in \text{Supp}(x)} a_T^x < 1$, a contradiction. Thus we may take $W_q = V \in \text{Supp}(q)$.

Now, choose $y \in P_{W_p}$ so that $(\partial\pi_{\text{Supp}(p)}(x))_U \ 100E$ -coarsely coincides with $\pi_U(y)$. If $U = W_q$, then our choice of $\mathcal{N}(p)$ and $\mathcal{N}(q)$ ensures that x cannot lie in both. Suppose that $U \pitchfork W_q$. Then $\pi_U(y)$, $\rho_U^{W_q}$ and $\rho_U^{W_p}$ all $10E$ -coarsely coincide and lie at distance $50E$ from the required neighborhood of p_U , so $x \notin \mathcal{N}(p)$. When $U \sqsubset W_q$ or $W_q \sqsubset U$, a similar argument shows that $x \notin \mathcal{N}(p) \cap \mathcal{N}(q)$.

Hence it remains to consider the case where $W_q \perp U$. By definition, $|a_{W_q}^x - a_{W_q}^q| < \epsilon$. On the other hand, we can assume $W_q \in \text{Supp}(p)^\perp$, for otherwise we could rechoose U and W_q to be in one of the above cases. Thus, by definition, $a_{W_q}^x < \epsilon$. This yields a contradiction provided we choose, say, $\epsilon \in \bigcap_{T \in \text{Supp}(q)} (0, \frac{1}{10}a_T^q)$.

Hence our choice of $\mathcal{N}(p)$ and $\mathcal{N}(q)$ ensures $\mathcal{N}(p) \cap \mathcal{N}(q) = \emptyset$, as required. \square

Lemma 2.16 $\bar{\mathcal{X}}$ is Hausdorff.

Proof In light of Lemma 2.15, it suffices to show that for all $p \in \partial\mathcal{X}$, with $p = \sum_{T \in \text{Supp}(p)} a_T p_T$, all $\epsilon > 0$, and all collections $\{U_T : T \in \text{Supp}(p)\}$ with each U_T a neighborhood of p_T in $\mathcal{C}T \cup \partial\mathcal{C}T$, the corresponding basic set $\mathcal{N}_{\{U_T\}, \epsilon}(p)$ has nonempty interior.

The topology of basic convergence Given a sequence $\{p_n\}$ with each $p_n \in \bar{\mathcal{X}}$, we say that p_n *basically converges* to $p \in \partial\mathcal{X}$ if for all $\epsilon > 0$ and all choices of $\{U_T\}$ as above, we have $p_n \in \mathcal{N}_{\{U_T\}, \epsilon}(p)$ for all but finitely many $n \in \mathbb{N}$. Similarly, $\{p_n\}$ *basically converges* to $p \in \mathcal{X}$ if, for all $\epsilon > 0$, we have $p_n \in \mathcal{N}_\epsilon(p)$ for all sufficiently large n .

Define a topology on $\bar{\mathcal{X}}$ as follows: the set $A \subset \bar{\mathcal{X}}$ is declared to be closed if $a \in A$ whenever there is a sequence $\{a_n\}$ such that $a_n \in A$ for all n and a_n basically converges to a . Denote by \mathfrak{M} the space $\bar{\mathcal{X}}$ endowed with this topology.

Nonempty interior of basic sets Let $\mathcal{N} = \mathcal{N}_{\{U_T\}, \epsilon}(p)$ be a basic set as above. We claim that $p \in \text{Int}(\mathcal{N})$. Otherwise, there exists a sequence $\{p_n\}$ in $\bar{\mathcal{X}} - \mathcal{N}$ that basically converges to p . This is a contradiction since basic convergence to p needs $\{p_n\}$ to enter \mathcal{N} .

Equivalence of the topologies To complete the proof that basic sets in $\bar{\mathcal{X}}$ have nonempty interior (with respect to the original topology), and thereby complete the proof of the lemma, it suffices to show that $\bar{\mathcal{X}}$ is homeomorphic to \mathfrak{M} .

Now, a set $A \subseteq \bar{\mathcal{X}}$ is closed in $\bar{\mathcal{X}}$ (ie has open complement) if and only if, for each $p \in \bar{\mathcal{X}} - A$, we can choose $\epsilon > 0$ and neighborhoods $\{U_T : T \in \text{Supp}(p)\}$ so that

$\mathcal{N}_{\{U_T\},\epsilon}(p)$ is disjoint from A . But this is equivalent to the following: for all basically convergent $\{a_n\}$ with each $a_n \in A$, the (basic) limit a lies in A . This is in turn equivalent to the assertion that A is closed in \mathfrak{M} . \square

Proposition 2.17 *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic, and let $\bar{\mathcal{X}} = \mathcal{X} \cup \partial(\mathcal{X}, \mathfrak{S})$.*

- (1) $\bar{\mathcal{X}}$ is Hausdorff and, if \mathcal{X} is separable (eg if it is proper), then $\bar{\mathcal{X}}$ is separable.
- (2) $\partial\mathcal{X}$ is closed in $\bar{\mathcal{X}}$.
- (3) \mathcal{X} is dense in $\bar{\mathcal{X}}$.

Proof The ‘‘Hausdorff’’ part of assertion (1) follows from Lemma 2.16. Separability of $\bar{\mathcal{X}}$ follows from density of the metric space \mathcal{X} in $\bar{\mathcal{X}}$, ie part (3). Assertion (2) is obvious: no bounded neighborhood of an interior point contains a boundary point, so no sequence of boundary points converges to an interior point.

It remains to prove (3). Pick a neighborhood $\mathcal{N}_{\{U_S\},\epsilon}(p)$ of $p = \sum_{S \in \text{Supp}(p)} a_S^p p_S \in \partial\mathcal{X}$ with $p_S \in \partial\mathcal{C}S$ for $S \in \text{Supp}(p)$. For each $S_i \in \text{Supp}(p) = \{S_1, \dots, S_d\}$, fix a uniform quasigeodesic ray γ_i in $\mathcal{C}S$ from $\pi_S(x_0)$ to p_S .

First, suppose that $d = 1$. Then for each t , there exists x_1^t such that $\pi_{S_1}(x_1^t)$ coarsely coincides with $\gamma_1(a_{S_1}^p \cdot t)$ and, in view of the quasiisometric embedding $F_{S_1} \times E_{S_1} \rightarrow \mathcal{X}$ described in Section 1.3, the point x_1^t can be chosen so that $\pi_T(x_1^t)$ coarsely equals $\pi_T(x_0)$ for each $T \perp S_1$. (Here we have used that $(\mathcal{X}, \mathfrak{S})$ is normalized.)

Now suppose $d \geq 2$. By induction, for all t , there exists $x_{d-1}^t \in E_{S_d}$ such that for all $i \leq d - 1$, the projection $\pi_{S_i}(x_{d-1}^t)$ coarsely coincides with $\gamma_i(a_{S_i}^p \cdot t)$, and also $\pi_T(x_{d-1}^t)$ coarsely coincides with $\pi_T(x_0)$ for each T orthogonal to each S_i . In view of the quasiisometric embedding $F_{S_d} \times E_{S_d} \rightarrow \mathcal{X}$, there exists a point x_d^t such that $\mathfrak{g}_{E_{S_d}}(x_d^t)$ coarsely coincides with x_{d-1}^t and $\pi_{S_d}(x_d^t)$ coarsely coincides with $\gamma_d(a_{S_d}^p \cdot t)$. (Here, $\mathfrak{g}_{E_{S_d}}$ is the gate map defined at the end of Section 1.) For each sufficiently large t , the point x_d^t lies in $\mathcal{N}_{\{U_{S_i}\},\epsilon}(p)$, as required. \square

Remark 2.18 By regarding each $\partial\mathcal{C}U$, with $U \in \mathfrak{S}$, as a discrete set, we can endow $\partial(\mathcal{X}, \mathfrak{S})$ with an alternate topology as a simplicial complex, as follows. For each $U \in \mathfrak{S}$ and each $p \in \partial\mathcal{C}U$, we have a 0–simplex, and the 0–simplices $p_i \in \partial\mathcal{C}U_i$ for $i = 0, \dots, k$ span a k –simplex if $U_i \perp U_j$ for $0 \leq i < j \leq k$. There is an obvious bijection from the resulting simplicial complex to $\partial(\mathcal{X}, \mathfrak{S})$, which is an embedding on each simplex.

3 Compactness for proper HHS

In this section, we will prove that proper HHSs have compact HHS boundaries.

3.1 Preliminary lemmas

Definition 3.1 Let (X, \mathfrak{S}) be hierarchically hyperbolic. The level ℓ_U of $U \in \mathfrak{S}$ is defined inductively as follows. If U is \sqsubseteq -minimal, then $\ell_U = 1$. We inductively define $\ell_U = k + 1$ if k is the maximal integer such that there exists $V \sqsubseteq U$ with $\ell_V = k$ and $V \neq U$.

The following is a slightly modified version of Lemma 2.5 in [6].

Lemma 3.2 Let (X, \mathfrak{S}) be hierarchically hyperbolic. Then there exists N with the following property: Let $x, y \in \mathcal{X}$ and let $\{S_i\}_{i=1, \dots, N} \subseteq \mathfrak{S}$ be such that $d_{CS_i}(x, y) \geq 50E$ for each $i = 1, \dots, N$. Then there exist $S \in \mathfrak{S}$ and i such that $S_i \sqsubset S$ and $d_{CS}(x, y) \geq 100E$. Moreover, for each $T \in \mathfrak{S}$ such that each $S_i \sqsubseteq T$, we can choose $S \sqsubseteq T$.

Proof The proof is by induction on the level k of a \sqsubseteq -minimal $S \in \mathfrak{S}$ into which each S_i is nested. The base case $k = 1$ is empty.

Suppose that the statement holds for a given $N = N(k)$ when the level of S as above is at most k . Suppose instead that $|\{S_i\}| \geq N(k + 1)$ (where $N(k + 1)$ is a constant much larger than $N(k)$ that will be determined shortly) and there exists a \sqsubseteq -minimal $S \in \mathfrak{S}$ of level $k + 1$ into which each S_i is nested. There are two cases.

If $d_{CS}(x, y) \geq 100E$, then we are done. If not, then the large link axiom (Definition 1.1(6)) implies that there exists $K = K(100E)$ and T_1, \dots, T_K , each properly nested into S (and hence of level less than $k + 1$), so that any S_i is nested into some T_j . In particular, if $N(k + 1) \geq KN(k)$, there exists j such that at least $N(k)$ elements of $\{S_i\}$ are nested into T_j . By the induction hypothesis, we are done.

Note that the proof still works replacing \mathfrak{S} with \mathfrak{S}_T when each $S_i \sqsubseteq T$. In this case, we can take $S \sqsubseteq T$ and the T_i produced by the large link axiom will also have $T_i \sqsubseteq S \sqsubseteq T$ for each i , as required for the second statement. □

Lemma 3.3 Let (X, \mathfrak{S}) be hierarchically hyperbolic. Then for every hierarchy ray γ there exists $S \in \mathfrak{S}$ such that $\pi_S(\gamma)$ is unbounded. Moreover, if $T \in \mathfrak{S}$ has the property that $\{\text{diam}_{CT'}(\gamma) : T' \sqsubseteq T\}$ is unbounded, then there exists $S \sqsubseteq T$ such that $\pi_S(\gamma)$ is unbounded.

Proof The proof of the “moreover” part is a minor variation; we prove the first assertion and indicate parenthetically how to adapt the proof.

By the distance formula (Theorem 1.9) and the fact that γ is a quasigeodesic, there exists an increasing sequence $\{n_i\}$ of natural numbers such that for each positive integer i ,

there exists $S'_i \in \mathfrak{S}$ such that $d_{CS'_i}(\gamma(n_i), \gamma(n_{i+1})) \geq 100E$. (For the purposes of the “moreover” part, we choose S'_i nested into T .) Since γ is a hierarchy path, it makes coarsely monotonic progress in \mathcal{CU} for each $U \in \mathfrak{S}$, and thus for each $t \geq 0$ we have

$$d_{CU}(\gamma(0), \gamma(t)) \geq 50E \cdot |\{i : n_i \leq t, S'_i = U\}|.$$

Let $\mathcal{S} \subset \mathfrak{S}$ be the collection of domains in which γ makes significant progress; that is, \mathcal{S} is the set of all $S \in \mathfrak{S}$ for which there exists $t_S \geq 0$ such that for any $t \geq t_S$ we have $d_{CS}(\gamma(0), \gamma(t)) \geq 50E$. (In the proof of the “moreover” part, we further require that S is nested into T .) If $|\mathcal{S}| < \infty$, then we are done by the above inequality, so assume $|\mathcal{S}| = \infty$.

Let $S \in \mathfrak{S}$ be \sqsubseteq -minimal with the property that there are infinitely many $S' \in \mathcal{S}$ nested into S . (In the proof of the “moreover” part, S is nested into T .) Suppose for a contradiction that $\text{diam}_{\mathcal{S}}(\pi_{\mathcal{S}}(\gamma)) = D < \infty$.

Denote by \mathcal{S}^j the set of all level- j elements of \mathcal{S} nested into S , and let k be maximal with the property that \mathcal{S}^k is infinite. Note that this assumption and finite complexity imply that $\bigcup_{k' > k} \mathcal{S}^{k'}$ is finite. To derive a contradiction, we will use the large link axiom and Lemma 3.2 to construct an infinite sequence of distinct $S_i \in \bigcup_{k' > k} \mathcal{S}^{k'}$.

By the large link axiom (Definition 1.1(6)), there exists $K = K(D)$ such that, for any t , there exist T_1^t, \dots, T_K^t properly nested into S such that if $X \in \mathcal{S}$ has $X \sqsubseteq S$ and $t_X \leq t$, then $X \sqsubseteq T_j^t$ for some j . If we take t_0 large enough, we can apply Lemma 3.2 to a sufficiently large subset of \mathcal{S}^k , all of whose elements are nested into some $T_j^{t_0}$, and we get some S_0 of level $k_0 > k$ such that $d_{CS_0}(\gamma(0), \gamma(t)) \geq 100E$ for $t \geq t_0$. Note that Lemma 3.2 allows us to take $S_0 \sqsubseteq T_j^{t_0}$, so that $S_0 \sqsubseteq S$ and thus $S_0 \in \mathcal{S}^{k_0}$. By minimality of S , there are finitely many elements of \mathcal{S}^k nested into S_0 . We can now choose $t_1 > t_0$ and apply Lemma 3.2 to a sufficiently large subset of \mathcal{S}^k all of whose elements are nested into some $T_j^{t_1}$ but not nested into S_0 , and get another element $S_1 \in \mathcal{S}^{k_1}$ for some $k_1 > k$ which is properly nested into S . We can then proceed inductively and construct infinitely many distinct elements $S_i \sqsubseteq S$ of level greater than k , giving us our contradiction. \square

3.2 Compactness

We are ready to prove:

Theorem 3.4 *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic, and let $\bar{\mathcal{X}} = \mathcal{X} \cup \partial(\mathcal{X}, \mathfrak{S})$. If \mathcal{X} is proper, then $\bar{\mathcal{X}}$ is compact.*

Proof It suffices to show that $\bar{\mathcal{X}}$ is sequentially compact since it is separable by Proposition 2.17. We will first show that any internal sequence $\{x_n\} \subset \mathcal{X}$ subconverges to some point in $\bar{\mathcal{X}}$. Then we will show this suffices for the theorem.

Internal sequences subconverge Let $\{x_n\} \subset \mathcal{X}$ be a sequence of interior points. For each n , let γ_n be a uniformly Lipschitz hierarchy path between x_0 and x_n , whose existence is guaranteed by [Theorem 1.8](#). Since \mathcal{X} is proper, either the sequence x_n subconverges to an interior point and we are done, or we can assume that the sequence of hierarchy paths γ_n converges to a hierarchy ray, γ_∞ .

[Lemma 3.3](#) implies there exists $T \in \mathfrak{S}$ such that $\pi_T \circ \gamma_\infty$ is unbounded. The collection $\{T_i\}_{i=1}^k$ for which this is true must be a collection of pairwise-orthogonal elements by the consistency inequalities ([Definition 1.1\(4\)](#)). For each T_i , the quasigeodesic ray $\pi_{T_i} \circ \gamma_\infty \subset CT_i$ represents a point $p_{T_i} \in \partial CT_i$. Set $\bar{T} = \{T_i\}_{i=1}^k$.

We now consider two cases, depending on the behavior of the sequence $\{x_n\}$ in \bar{T}^\perp . First, suppose $\liminf_n \sup\{d_{CT}(x_0, x_n) : T \in \bar{T}^\perp\} < \infty$. Up to passing to a further subsequence of $\{x_n\}$, we have well-defined limits for $1 \leq i, j \leq k$,

$$r_{i,j} = \lim_n \frac{d_{CT_i}(x_0, x_n)}{d_{CT_j}(x_0, x_n)} \in [0, \infty],$$

which determine coefficients $\{a_i^p \in [0, 1]\}$ such that $a_i^p/a_j^p = r_{i,j}$ and $\sum a_i^p = 1$. It is straightforward to check that $\{x_n\}$ eventually lies in the interior part of any $\mathcal{N}_{\{U_{T_i}\}, \epsilon}(p)$, implying that $\{x_n\}$ subconverges to $p = \sum_{T \in \bar{T}} a_T^p p_T$.

Now suppose that, up to passing to a subsequence,

$$\liminf_n \sup\{d_{CT}(x_0, x_n) : T \in \bar{T}^\perp\} = \infty.$$

Consider the sequence $\{y_n\} = \{g_{E_{\bar{T}}}(x_n)\}$ of gates in the orthogonal complement of \bar{T} .

Since $(E_{\bar{T}}, \mathfrak{S}_{\bar{T}^\perp})$ is an HHS with complexity strictly less than that of $(\mathcal{X}, \mathfrak{S})$, by induction on the complexity of $(\mathcal{X}, \mathfrak{S})$, the sequence $\{y_n\}$ subconverges to $q \in \partial \mathcal{X}$, where $\text{Supp}(q) = \{T_i\}_{i=k+1}^{k'}$ and $T_i \perp T_j$ whenever $i \leq k < j$. Since $(E_{\bar{T}}, \mathfrak{S}_{\bar{T}^\perp}) \subset (\mathcal{X}, \mathfrak{S})$ is hierarchically quasiconvex, we can take $q \in \partial E_{\bar{T}}$. For each $j > k$, let $q_{T_j} \in \partial CT_j$, so that q is a linear combination of the q_{T_j} . As before, up to passing to a further subsequence, for any $1 \leq i, j \leq k'$, we can define

$$r_{i,j} = \lim_n \frac{d_{CT_i}(x_0, x_n)}{d_{CT_j}(x_0, x_n)} \in [0, \infty],$$

which determine coefficients $\{a_{T_i}^p\}_{i=1}^k \cup \{a_{T_j}^q\}_{j=k+1}^{k'}$ such that

- $a_{T_i}^r/a_{T_j}^{r'}$ = $r_{i,j}$ when $r, r' \in \{p, q\}$ and $a_{T_i}^r$ and $a_{T_j}^{r'}$ are defined, and
- $\sum_{i=1}^k a_{T_i}^p + \sum_{j=k+1}^{k'} a_{T_j}^q = 1$.

If some $a_{T_i}^r = 0$ for $r \in \{p, q\}$, we disregard T_i . We now claim that $\{x_n\}$ (sub)converges to

$$p = \sum_{i=1}^k a_{T_i}^p p_{T_i} + \sum_{i=k+1}^{k'} a_{T_i}^q q_{T_i}.$$

Pick a neighborhood $\mathcal{N}_{\{U_{T_i}\}, \epsilon}(p)$ of p . For large enough n , we have $x_n \in \mathcal{N}_{\{U_{T_i}\}, \epsilon}(p)$ because

- $\pi_{T_i}(x_n) \in U_{T_i}$ for $i \leq k$ since $(\pi_{T_i}(x_n) | p_{T_i})_{\pi_{T_i}(x_0)} \rightarrow \infty$,
- $\pi_{T_i}(x_n) \in U_{T_i}$ for $i > k$ since $\pi_{T_i}(x_n)$ coarsely equals $\pi_{T_i}(y_n)$ and $y_n \rightarrow q$,
- $|a_{T_j}^r/a_{T_j}^{r'} - d_{T_j}(x_0, x_n)/d_{T_j}(x_0, x_n)| < \epsilon$ by definition, when $r, r' \in \{p, q\}$ and $a_{T_i}^r$ and $a_{T_j}^{r'}$ are defined, and
- $d_T(x_0, x_n)/d_{T_i}(x_0, x_n) < \epsilon$ for $T \in (\{T_i\}_{i=1}^{k'})^\perp$ and any $1 \leq i \leq k'$, as we now show.

Let $T \in (\{T_i\}_{i=1}^{k'})^\perp$ and choose i so that $a_{T_i}^r \neq 0$ for $r \in \{p, q\}$. Observe that

$$\frac{d_T(x_0, x_n)}{d_{T_i}(x_0, x_n)} = \frac{d_T(x_0, x_n)}{d_{T_{k+1}}(x_0, x_n)} \cdot \frac{d_{T_{k+1}}(x_0, x_n)}{d_{T_i}(x_0, x_n)}.$$

The first term on the right-hand side can be made arbitrarily small by increasing n since $d_T(x_0, x_n)$ and $d_{T_{k+1}}(x_0, x_n)$ coarsely coincide with $d_T(x_0, y_n)$ and $d_{T_{k+1}}(x_0, y_n)$, respectively, and $\{y_n\}$ converges to q . Since the second term converges to $r_{k+1, i} < \infty$, this proves the claim and completes the internal sequence case.

Reduction to the internal sequence case Recall the definition of the boundary projection, Definition 2.8. By passing to a subsequence if necessary, it suffices to consider any boundary sequence $\{z_n\} \subset \partial\mathcal{X}$, where $z_n = \sum_{S \in \text{Supp}(z_n)} a_S^{z_n} p_S^n$ for each n .

We first find $\{x_n\} \subset \mathcal{X}$ with the properties (1)–(7) below, and then verify that $\{z_n\}$ subconverges to the limit of $\{x_n\}$:

- (1) $d_{\mathcal{X}}(x_0, x_n) \geq n$.
- (2) $(\pi_S(x_n) | p_S^n)_{\pi_S(x_0)} \geq n$ for each $S \in \text{Supp}(z_n)$ (we remind the reader that the notation $(\bullet | \bullet)_\bullet$ denotes the Gromov product with respect to the subscripted basepoint).
- (3) $|a_S^n/a_{S'}^n - d_S(x_0, x_n)/d_{S'}(x_0, x_n)| < 1/n$ for any distinct $S, S' \in \text{Supp}(z_n)$.
- (4) $d_T(x_0, x_n)/d_S(x_0, x_n) < 1/n$ for any $T \in (\text{Supp}(z_n))^\perp$ and $S \in \text{Supp}(z_n)$.
- (5) For all n and $S^n \in \text{Supp}(z_n)$, if $T \pitchfork S^n$ or $S^n \sqsubseteq T$, then $d_T(\rho_T^{S^n}, x_n) < K$ for some uniform $K > 0$. Moreover, $d_T(x_0, x_n) \leq d_{S^n}(x_0, x_n)$ for all such T .

- (6) $\{x_n\}$ converges to $p = \sum_{T \in \text{Supp}(p)} a_T^p p_T \in \partial \mathcal{X}$ with the following property: if there are infinitely many n for which $z_n \in \partial^{\text{rem}} \mathcal{X}$ (with respect to $\text{Supp}(p)$), then there are infinitely many remote z_n such that the following holds for some fixed $T \in \text{Supp}(p)$: there exists $S_T^n \in \text{Supp}(z_n)$ such that $S_T^n \pitchfork T$ or $S_T^n \sqsubset T$, or $T \sqsubset S_T^n$ but $d_{S_T^n}(\rho_{S_T^n}^T, x_0) \leq 100K'E$ for some constant $K' \geq 1$ depending on $\{z_n\}$ and p but not on n . Moreover, $d_{cT}(x_0, x_n) \leq d_{cS^n}(x_0, x_n)$ for all such T .
- (7) $\{x_n\}$ converges to $p = \sum_{T \in \text{Supp}(p)} a_T^p p_T \in \partial \mathcal{X}$ with the following property: if there are infinitely many n for which $z_n \in \partial^{\text{rem}} \mathcal{X}$ (with respect to $\text{Supp}(p)$), then there are infinitely many remote z_n such that $d_T((\partial \pi_{\text{Supp}(p)}(z_n))_T, x_n) \leq K''$ for some K'' independent of n and all $T \in \text{Supp}(p)_{z_n}$. Moreover, $d_{cT}(x_0, x_n) \leq d_{cS^n}(x_0, x_n)$ for all such T .

To see that such an internal sequence exists, choose a sequence $\{x_n\}$ so that $x_n \in P$ for all n , where

$$P = \text{im} \left(\prod_{S \in \text{Supp}(z_n)} F_S \rightarrow \mathcal{X} \right);$$

the sequence $\{x_n\}$ satisfies (1)–(4) (which can be done since they are componentwise conditions); and

$$\min_{S \in \text{Supp}(z_n)} \frac{d_{\mathcal{X}}(\mathfrak{g}_{F_S}(x_n), x_0)}{d_{\mathcal{X}}(\mathfrak{g}_{F_S}(x_0), x_0)} \rightarrow \infty$$

as $n \rightarrow \infty$. Here we fix, for each n , a basepoint $(p_S)_{S \in \text{Supp}(z_n)}$ and let $F_S = F_S \times \{(P_{S'})_{S' \neq S}\}$.

(Recall from [6, Remark 5.12] that, whenever $U_1, \dots, U_k \in \mathfrak{S}$ are pairwise orthogonal, we have a standard quasiisometric embedding $\prod_{i=1}^k F_{U_i} \rightarrow \mathcal{X}$ whose image is hierarchically quasiconvex and which is, for each $i \leq k$, the restriction of the usual map $F_{U_i} \times E_{U_i} \rightarrow \mathcal{X}$.)

We can verify condition (5) by examining the product regions $\prod_{S \in \text{Supp}(z_n)} F_S \rightarrow \mathcal{X}$. Let $T \pitchfork S^n$ or $S^n \sqsubset T$ for $S^n \in \text{Supp}(z_n)$. Since x_n coarsely lies in $\prod_{S \in \text{Supp}(z_n)} F_S$, it follows that $\text{diam}_T(\rho_T^{S^n} \cup \pi_T(F_{S^n})) \asymp 1$ and $d_T(\pi_T(F_{S^n}), x_n) \asymp 1$. We thus have, for some uniform C ,

$$d_T(x_0, x_n) \leq C d_{\mathcal{X}} \left(x_0, \prod_{S \in \text{Supp}(z_n)} F_S \right) + C.$$

For sufficiently large n , our choice of $\{x_n\}$ ensures that

$$d_{S_n}(x_0, x_n) \geq C d \left(x_0, \prod_{S \in \text{Supp}(z_n)} F_S \right) + C,$$

verifying the “moreover” part of assertion (5).

Let $\{x_n\}$ satisfy (1)–(5). We now prove there is a subsequence of $\{x_n\}$ satisfying (6).

By replacing $\{x_n\}$ with a subsequence (and replacing $\{z_n\}$ with the corresponding subsequence of $\{z_n\}$), we can apply the proof that internal sequences subsequentially converge to conclude $\{x_n\}$ converges to $p = \sum_{T \in \text{Supp}(p)} a_T^p p_T \in \partial \mathcal{X}$.

Consider the set \mathbb{G} of $n \in \mathbb{N}$ such that z_n is remote with respect to p . If \mathbb{G} is finite, then (6) holds vacuously. Otherwise, by replacing \mathbb{G} with an infinite subset, we find $T \in \text{Supp}(p)$ such that for all $n \in \mathbb{G}$, there exists $S^n \in \text{Supp}(z_n)$ with either $T \pitchfork S^n$ or $S^n \sqsubset T$ or $T \sqsubset S^n$.

First consider the case where $\{S^n : n \in \mathbb{G}\}$ is infinite. By passing to a subsequence if necessary, and then applying finite complexity, Lemma 1.4, and Ramsey’s theorem, we can assume that $S^n \pitchfork S^m$ when $n \neq m$. Let $\mathbb{G}_T \subseteq \mathbb{N}$ be the set of $n \in \mathbb{G}$ such that $T \sqsubset S^n$. Then for all $m, n \in \mathbb{G}_T$, we have $d_{S^m}(\rho_{S^m}^T, \rho_{S^m}^{S^n}) \leq E$ by the consistency inequalities. Hence, again by the consistency inequalities and the triangle inequality, we have $d_{S^n}(\rho_{S^n}^T, x_0) \leq 2E$ for all but at most one element of \mathbb{G}_T . Indeed, if $d_{S^n}(\rho_{S^n}^T, x_0) > 2E$, then $d_{S^n}(\rho_{S^n}^{S^m}, x_0) > E$ for any $m \in \mathbb{G}_T - \{n\}$, so by consistency $d_{S^m}(\rho_{S^m}^{S^n}, x_0) \leq E$; the claim follows from the triangle inequality since $d_{S^m}(\rho_{S^m}^T, \rho_{S^m}^{S^n}) \leq E$. Hence, by replacing $\{z_n\}$ with a subsequence, for all $T \in \text{Supp}(p)$ with $T \sqsubset S^n$, we have $d_{S^n}(\rho_{S^n}^T, x_0) \leq 100K'E$. Letting $S_T^n = S^n$ for $n \in \mathbb{G}$, this establishes assertion (6) when $\{S^n : n \in \mathbb{G}\}$ is infinite.

When $\{S^n : n \in \mathbb{G}\}$ is finite, we can assume that $S^n = S^m$ for all m, n by passing to a subsequence. Hence, there exists $S \in \mathfrak{S}$ such that for all $n \in \mathbb{G}$, and all $U \in \text{Supp}(z_n)$, either $U = S$ or $U \perp T$. Fix T and S as above, and replace (z_n) with a subsequence so that for each $n \in \mathbb{G}$, we have $S \in \text{Supp}(z_n)$. Then, for each $n \in \mathbb{G}$, set $S_T^n = S$ and observe that either $S \sqsubset T$, $S \pitchfork T$ or $T \sqsubset S$. In the latter case, take $K' = d_S(\rho_S^T, x_0)$, which depends on p and $\{z_n\}$ but not on n . This completes the proof of (6).

We now deduce condition (7) from (1)–(6). Assume \mathbb{G} is infinite, so that, by (6), there exists $T' \in \text{Supp}(p)$ such that, after replacing \mathbb{G} with an infinite subset if necessary, we have, for each $n \in \mathbb{G}$, some $S_{T'}^n \in \text{Supp}(z_n)$ such that $d_{S_{T'}^n}(\rho_{S_{T'}^n}^T, x_0) \leq 100K'E$. Let $T \in \text{Supp}(p)_{z_n}$. First suppose that $T \sqsubset S_{T'}^n$. Then, since $T \perp T'$ or $T = T'$, Lemma 1.5 implies that $d_{S_{T'}^n}(\rho_{S_{T'}^n}^T, x_0) \leq 200K'E$. It follows from (2) that

$$(\pi_{S_{T'}^n}(x_n) | p_{S_{T'}^n}^n, \rho_{S_{T'}^n}^T) \rightarrow \infty$$

as $n \rightarrow \infty$, so that, by discarding finitely many n and applying the bounded geodesic image axiom, we have $d_T((\partial \pi_{\text{Supp}(p)}(z_n))_T, x_n) \leq E$ for all $n \in \mathbb{G}$. In the remaining cases, where $T \pitchfork S_{T'}^n$, or $S_{T'}^n \sqsubset T$, we reach the same conclusion, using (5) instead of (6). This completes the proof of condition (7).

Subconvergence of $\{z_n\}$ Fix a neighborhood $\mathcal{N} = \mathcal{N}_{\{U_S\}, \epsilon}(p)$ of p ; we must check that for infinitely many values of n , we have $z_n \in \mathcal{N}$. For each n , either $z_n \in \partial^{\text{rem}} \mathcal{X}$ (recall that this means that $\text{Supp}(z_n) \cap \text{Supp}(p) = \emptyset$ and for all $T \in \text{Supp}(p)$, there exists $S \in \text{Supp}(z_n)$ with $T \not\perp S$) or $z_n \in \partial \mathcal{X} - \partial^{\text{rem}} \mathcal{X}$ (so that either $\text{Supp}(z_n) \cap \text{Supp}(p) \neq \emptyset$ or there exists $T \in \text{Supp}(p)$ with $T \perp S$ for all $S \in \text{Supp}(z_n)$).

The nonremote case We will consider the nonremote case first. Recall that $z_n = \sum_{S \in \text{Supp}(z_n)} a_S^{z_n} p_S^n$. We must check the following conditions:

- (a) For each $S \in \text{Supp}(p) \cap \text{Supp}(z_n)$, and infinitely many n , we have $p_S^n \in U_S$.
- (b) For each $S \in \text{Supp}(p) \cap \text{Supp}(z_n)$ and infinitely many n , we have $a_S^n \rightarrow a_S^p$.
- (c) $\sum_{T \in \text{Supp}(p) - \text{Supp}(z_n)} a_T^p < K'\epsilon$ for infinitely many n and some uniform K' .

Up to passing to a subsequence, (a) follows from (2) and the fact that $x_n \rightarrow p$.

For (b), we have three cases. If $\text{Supp}(p) \cap \text{Supp}(z_n) = \emptyset$, then this holds vacuously. If $\text{Supp}(p) \cap \text{Supp}(z_n)$ has multiple elements, then this follows from (3) and the fact that $x_n \rightarrow p$. If $\text{Supp}(p) \cap \text{Supp}(z_n) = \{S\}$, then this follows from (3) and (c), proved momentarily.

To see (c), first observe that $\text{Supp}(p) - \text{Supp}(z_n) \subset (\text{Supp}(z_n))^\perp$ by nonremoteness. Let $T \in \text{Supp}(p) - \text{Supp}(z_n)$ and $S \in \text{Supp}(p) \cap \text{Supp}(z_n)$; note that such an $S \in \text{Supp}(p) \cap \text{Supp}(z_n)$ exists, otherwise one of $x_n \rightarrow p$ or (4) is contradicted. By definition of $x_n \rightarrow p$,

$$\left| \frac{a_T^p}{a_S^p} - \frac{d_T(x_0, x_n)}{d_S(x_0, x_n)} \right| < \epsilon.$$

It follows from (4) that $d_T(x_0, x_n)/d_S(x_0, x_n) < 1/n$. Since each $a_S^p \leq 1$, it follows that

$$\sum_{T \in \text{Supp}(p) - \text{Supp}(z_n)} a_T^p < \xi(\mathcal{X}) \left(\epsilon + \frac{1}{n} \right) \leq 2\xi(\mathcal{X})\epsilon,$$

completing the proof of (c) and thus the nonremote case.

The remote case We must check the following conditions:

- (i) For any $T \in \text{Supp}(p)$, and infinitely many n , we have $(\partial\pi_{\text{Supp}(p)}(z_n))_T \in U_T$.
- (ii) For infinitely many n and any $T \in \text{Supp}(p)_{z_n}$, $T' \in \text{Supp}(p)$, we have

$$\left| \frac{d_T(x_0, (\partial\pi_{\text{Supp}(p)}(z_n))_T)}{d_{T'}(x_0, (\partial\pi_{\text{Supp}(p)}(z_n))_{T'})} - \frac{a_T^p}{a_{T'}^p} \right| < \epsilon.$$

- (iii) We have $\sum_{T \in \text{Supp}(p)^\perp \cap \text{Supp}(z_n)} a_T^{z_n} < K\epsilon$ for some uniform K .

For any $T \in \text{Supp}(p)$ and each n , choose $S_T^n \in \text{Supp}(z_n)$ so that T and S_T^n are not orthogonal. If \mathbb{G} is infinite, then we may pass to a subsequence such that S_T^n and T are always nonorthogonal: that is, $T \not\subseteq S_T^n$, or $T \pitchfork S_T^n$, or $S_T^n \not\subseteq T$.

We now show that assertion (i) holds for infinitely many n ; the proof divides into three cases according to the above possibilities, which influence the definition of $(\partial\pi_{\text{Supp}(p)}(z_n))_T$.

First, if $S_T^n \pitchfork T$, then $(\partial\pi_{\text{Supp}(p)}(z_n))_T = \rho_T^{S_T^n}$. In this case, (i) follows immediately from conditions (2) and (5) in the definition of $\{x_n\}$. The same is true if $S_T^n \not\subseteq T$. If $T \subseteq S_T^n$, then (i) follows from (2), (7) and the triangle inequality.

Assertion (ii), in the case when $T, T' \in \text{Supp}(p)$, follows from (7). In fact, since $\{x_n\}$ converges to p , we have

$$(*) \quad \left| \frac{d_T(x_0, x_n)}{d_{T'}(x_0, x_n)} - \frac{a_T^p}{a_{T'}^p} \right| \rightarrow 0,$$

and $d_T(x_0, x_n) \rightarrow \infty, d_{T'}(x_0, x_n) \rightarrow \infty$. By (7), we have that $d_T(x_0, x_n)$ coarsely coincides with $d_T(x_0, (\partial\pi_{\text{Supp}(p)}(z_n))_T)$, and similarly for T' . Hence, (*) implies that the ratio in assertion (ii) satisfies the required inequality. If $T \in \text{Supp}(p)_{z_n} - \text{Supp}(p)$, then we have to verify $|d_T(x_0, (\partial\pi_{\text{Supp}(p)}(z_n))_T) / d_{T'}(x_0, (\partial\pi_{\text{Supp}(p)}(z_n))_{T'})| \rightarrow 0$. We still know (*) (with $a_T^p/a_{T'}^p$ replaced by 0) and $d_{T'}(x_0, x_n) \rightarrow \infty$. If $d_T(x_0, x_n)$ does not diverge, we are done. If it does, we can approximate $d_T(x_0, (\partial\pi_{\text{Supp}(p)}(z_n))_T)$ by $d_T(x_0, x_n)$ and we can conclude as above.

It remains to verify assertion (iii). For each n , let $T^n \in (\text{Supp}(p))^\perp \cap \text{Supp}(z_n)$ and choose $S^n \in \text{Supp}(z_n) - (\text{Supp}(p))^\perp$. Fix $P \in \text{Supp}(p)$ so that, after passing to a subsequence, P is not orthogonal to any of the S^n . By either (5) or (7), we have $d_{CS^n}(x_0, x_n) / d_{CP}(x_0, x_n) \leq 1$, while $d_{CP}(x_0, x_n) / d_{CT^n}(x_0, x_n) < \epsilon$ since $x_n \rightarrow p$. Hence $a_{T^n}^{z_n} / a_{S^n}^{z_n} \leq \epsilon + 1/n$, by (3), and the desired inequality follows since the number of terms in the sum is bounded by $\xi(\mathcal{X})$, as in the nonremote case. This completes the proof that $\{z_n\}$ subconverges to p , and thus completes the proof that $\partial\mathcal{X}$ is compact. \square

4 The HHS boundary of a Gromov-hyperbolic space

In this section, we prove that the HHS boundary of a hyperbolic space is its Gromov boundary, regardless of the chosen HHS structure.

Lemma 4.1 *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic. If \mathcal{X} is hyperbolic, then there exists $C > 0$ such that if $U, V \in \mathfrak{S}$ and $U \perp V$, then either $\text{diam } CU < C$ or $\text{diam } CV < C$.*

Proof Recall from [6] that if $U \perp V$, then there exists a quasiisometric embedding $F_U \times F_V \hookrightarrow \mathcal{X}$. Hyperbolicity uniformly bounds the diameter of one of the factors. \square

Lemma 4.2 *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic and let \mathcal{X} be hyperbolic. If $\gamma: [0, \infty) \rightarrow \mathcal{X}$ is a hierarchy ray with $\gamma(0) = x_0$, then there exists a unique $U \in \mathfrak{S}$ with $\pi_U \circ \gamma: [0, \infty) \rightarrow \mathcal{C}U$ a parametrized quasigeodesic ray. In particular, $\text{diam}_{\mathcal{C}V}(\gamma) < \infty$ for all $V \in \mathfrak{S}$ with $V \neq U$.*

Proof By Lemma 3.3, there exists $U \in \mathcal{CS}$ such that $\text{diam}_{\mathcal{C}U}(\gamma)$ is unbounded. Let $V \in \mathfrak{S}$ be such that $V \neq U$; by Lemma 4.1, there are three cases: $V \sqsubseteq U$, $U \sqsubseteq V$ and $V \pitchfork U$.

Let $t_M \in [0, \infty)$ be such that $d_{\mathcal{C}U}(\gamma(0), \gamma(t)) > E^2$ for $t \geq t_M$. If $U \sqsubseteq V$, then by the consistency inequality, $d_V(\gamma(t), \rho_U^V(\gamma(0))) < E$ for all $t > t_M$. If $V \sqsubseteq U$, then $d_{\mathcal{C}V}(\gamma(t), \rho_V^U) < E$ for all $t > t_M$. Similarly, if $U \pitchfork V$, then $d_{\mathcal{C}V}(\gamma(t), \rho_V^U) < E$ for all $t > t_M$ by the transverse case of the consistency inequality. Thus, in each case, $\text{diam}_{\mathcal{C}V}(\gamma) < \infty$. \square

Theorem 4.3 *Let $(\mathcal{X}, \mathfrak{S})$ be hierarchically hyperbolic and suppose that \mathcal{X} is hyperbolic. Let $\bar{\mathcal{X}}^{\text{Gr}} = \mathcal{X} \cup \partial^{\text{Gr}} \mathcal{X}$, where $\partial^{\text{Gr}} \mathcal{X}$ is the Gromov boundary of \mathcal{X} , and let $\bar{\mathcal{X}} = \mathcal{X} \cup \partial \mathcal{X}$. Then the identity map $\mathcal{X} \rightarrow \mathcal{X}$ extends uniquely to a homeomorphism $\bar{\mathcal{X}}^{\text{Gr}} \rightarrow \bar{\mathcal{X}}$.*

Proof Lemma 4.1 gives $\partial \mathcal{X} = \bigsqcup_{U \in \mathfrak{S}} \partial \mathcal{C}U$ and Lemma 4.2 gives $|\text{Supp}(p)| = 1$ for all $p \in \partial \mathcal{X}$.

Fix $x_0 \in \mathcal{X}$ and let $p \in \partial^{\text{Gr}} \mathcal{X}$. Let $\gamma_p: [0, 1) \rightarrow \bar{\mathcal{X}}^{\text{Gr}}$ be a geodesic from x_0 to p . For any $n \in \mathbb{N}$, let $\gamma_n: [0, n) \rightarrow \mathcal{X}$ be a hierarchy path between x_0 and $\gamma_p(n)$. Since \mathcal{X} is hyperbolic, each γ_n uniformly fellow-travels γ_p and thus $\gamma = \lim_n \gamma_n$ is a hierarchy ray from x_0 to p . The ray γ is independent of the choice of (γ_n) and is thus uniquely determined by p . By Lemma 4.2, there exists a unique $U \in \mathfrak{S}$ such that $\text{diam}_{\mathcal{C}U}(\gamma)$ is an unbounded quasigeodesic ray. By hyperbolicity of $\mathcal{C}U$, there exists $q \in \partial \mathcal{C}U$ such that $\pi_{\mathcal{C}U}(\gamma)$ limits to q .

The above discussion yields a well-defined map $\phi^{\text{Gr}}: \partial^{\text{Gr}} \mathcal{X} \rightarrow \partial \mathcal{X}$ given by $\phi^{\text{Gr}}(p) = q$. Define $\phi: \bar{\mathcal{X}}^{\text{Gr}} \rightarrow \bar{\mathcal{X}}$ by $\phi|_{\mathcal{X}} = \text{id}_{\mathcal{X}}$ and $\phi|_{\bar{\mathcal{X}}^{\text{Gr}}} = \phi^{\text{Gr}}$. We claim that ϕ is a homeomorphism.

Bijectivity The map ϕ is clearly bijective on \mathcal{X} . Let $p, q \in \partial^{\text{Gr}} \mathcal{X}$ and suppose that $p \neq q$. Then there exist geodesic rays $\gamma_p, \gamma_q: [0, \infty) \rightarrow \mathcal{X}$ with $[\gamma_p] = p$, $[\gamma_q] = q$ and $\gamma_p(0) = \gamma_q(0) = x_0$. Since $p \neq q$, hyperbolicity of \mathcal{X} implies that $d_{\mathcal{X}}(\gamma_p(t), \gamma_q(t)) \rightarrow \infty$.

By Lemma 4.2, γ_p and γ_q have unique domains U_p and U_q , respectively, to which they have unbounded projections. If $U_p \neq U_q$, we are done. Otherwise, $U_p = U_q = U$, and Lemma 4.2, the distance formula, and the triangle inequality imply that $d_U(\gamma_p(t), \gamma_q(t)) \rightarrow \infty$, whence $\phi(p) \neq \phi(q)$, by definition. Thus ϕ is injective; surjectivity of ϕ follows from Theorem 1.7.

Basic sets in $\bar{\mathcal{X}}$ For convenience, we describe basic sets $\mathcal{N}(p)$ for $p \in \partial(\mathcal{X}, \mathfrak{S})$, in our current simple situation. Observe that $\text{Supp}(p)$ consists of a single $S \in \mathfrak{S}$, while $\partial_{\text{Supp}(p)}^{\text{rem}} \mathcal{X}$ consists of those $q \in \partial(\mathcal{X}, \mathfrak{S})$ with $\text{Supp}(q) = \{T\}$ with $T \neq S$. It is automatic that T is not orthogonal to S if $T \perp S$, then Lemma 4.1 implies only one of $\mathcal{C}S$ or $\mathcal{C}T$ can be unbounded and thus have nonempty Gromov boundary. It follows that $\text{Supp}(q) \cap (\text{Supp}(p))^\perp = \emptyset$.

Choosing $\epsilon > 0$ and $p \in \mathcal{U}_S \subset \mathcal{C}S \cup \partial\mathcal{C}S$, a remote neighborhood of p in $\bar{\mathcal{X}}$ is

$$\mathcal{N}_{\mathcal{U}_S, \epsilon}^{\text{rem}}(p) = \left\{ q \in \bigsqcup_{S \neq T} \partial\mathcal{C}T \mid \rho_S^T \in \mathcal{U}_S \right\}.$$

Meanwhile, the nonremote part of the boundary is just $\partial\mathcal{C}S$, so

$$\mathcal{N}_{\mathcal{U}_S, \epsilon}^{\text{non}}(p) = \mathcal{U}_S.$$

Finally, the interior part is

$$\mathcal{N}_{\mathcal{U}_S, \epsilon}^{\text{int}}(p) = \left\{ x \in \mathcal{X} \mid \pi_S(x) \in \mathcal{U}_S, \frac{d_T(x_0, x)}{d_S(x_0, x)} < \epsilon \text{ for all } T \perp S \right\}.$$

The above descriptions will be useful in proving that ϕ is a homeomorphism.

Continuity of ϕ and ϕ^{-1} Choose $p \in \partial(\mathcal{X}, \mathfrak{S})$, supported on $S \in \mathfrak{S}$, a neighborhood \mathcal{U}_S of $p \in \partial\mathcal{C}S$, and $\epsilon > 0$. We may assume that

$$\mathcal{U}_S = \{y \in \mathcal{C}S \cup \partial\mathcal{C}S \mid p_n \rightarrow p \text{ and } \liminf_n (y \mid \pi_S(p_n))_{\pi_S(x_0)} > r \text{ for some } (p_n)\}$$

for some $r \geq 0$. Choose $q \in \partial^{\text{Gr}} \mathcal{X}$ so that $\phi(q) = p$. For each $r' \geq 0$, let

$$U(q, r') = \{y \in \mathcal{X} \cup \partial^{\text{Gr}} \mathcal{X} \mid (y \mid q)_{x_0} \geq r'\}.$$

Recall that sets of this type yield a neighborhood basis in $\bar{\mathcal{X}}^{\text{Gr}}$.

We exhibit $r' \geq 0$, depending on p, r, ϵ and the distance formula constants, such that

$$\phi(U(q, r')) \subseteq \mathcal{N}_{\mathcal{U}_S, \epsilon}(p).$$

Indeed, if $y \in U(q, r') \cap \partial^{\text{Gr}} \mathcal{X}$, and r' is sufficiently large, then any geodesic ray or segment representing $[\pi_S \circ \gamma_y]$ has an initial segment of length at least r lying 2δ -close to the corresponding segment for p . This implies that $\phi(y) \in \mathcal{U}_S$, which is

exactly the nonremote part of $\mathcal{N}_{\mathcal{U}_S, \epsilon}(p)$ (regardless of the choice of ϵ). If $y \in U(q, r')$ is an interior point, and r' is sufficiently large, then similarly $\pi_S(x) \in \mathcal{U}_S$.

If $T \perp S$, then, by Lemma 4.1, there exists a uniform $C > 0$ such that $d_T(x_0, y) \leq C$. Moreover, choosing r' sufficiently large compared to r , C and the constants in the distance formula, we have $d_S(x_0, y) \geq C/\epsilon$. Hence either y is interior or $y \in \partial \mathcal{C}S$, and so

$$\phi(U(q, r')) \subseteq \mathcal{N}_{\mathcal{U}_S, \epsilon}^{\text{non}}(p) \cup \mathcal{N}_{\mathcal{U}_S, \epsilon}^{\text{int}}(p).$$

Continuity follows easily: Given an open set $\mathcal{O} \subseteq \bar{\mathcal{X}}$, let $q \in \phi^{-1}(\mathcal{O})$. Then, since \mathcal{O} is open, it contains a neighborhood \mathcal{N} of $\phi(q)$. The preceding discussion shows that q lies in some neighborhood U which in turn lies in $\phi^{-1}(\mathcal{N}) \subset \phi^{-1}(\mathcal{O})$, so $\phi^{-1}(\mathcal{O})$ is open. Continuity of ϕ^{-1} is proved similarly. \square

5 Extending hieromorphisms to the boundary

Hieromorphisms need not extend continuously to the boundary, but under additional hypotheses on the quasiisometries implicit in the hieromorphism, such extensions do exist. However, the class of hieromorphisms that extend continuously to the boundary is contained in a larger class of maps with this property, and, given the examples we study later in this section, it is in our interest to focus on this larger class of maps.

Definition 5.1 (slanted hieromorphism) Let $(\mathcal{X}, \mathfrak{S})$ and $(\mathcal{X}', \mathfrak{S}')$ be hierarchically hyperbolic spaces. A *slanted hieromorphism* $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}', \mathfrak{S}')$ consists of

- (1) a map $f: \mathcal{X} \rightarrow \mathcal{X}'$;
- (2) a map $\pi(f): \mathfrak{S} \rightarrow 2^{\mathfrak{S}'}$ such that $\pi(f)(U)$ is a collection of pairwise-orthogonal elements of \mathfrak{S}' for each $U \in \mathfrak{S}$;
- (3) for each $U \in \mathfrak{S}$, a map $\rho(f, U): \mathcal{C}U \rightarrow \prod_{V \in \pi(f)(U)} \mathcal{C}V$

such that:

- (I) If $U, V \in \mathfrak{S}$ satisfy $U \sqsubset V$, then for each $W' \in \pi(f)(V)$, there exists $W \in \pi(f)(U)$ with $W \sqsubset W'$, and for every $W \in \pi(f)(U)$ there exists (a unique) $W' \in \pi(f)(V)$ with $W \sqsubset W'$.
- (II) If $U, V \in \mathfrak{S}$ satisfy $U \perp V$, then $W \perp W'$ for all distinct $W \in \pi(f)(U)$ and $W' \in \pi(f)(V)$.
- (III) If $U, V \in \mathfrak{S}$ satisfy $U \pitchfork V$, then for all $W \in \pi(f)(U)$ there exists $W' \in \pi(f)(V)$ with $W \pitchfork W'$ and vice versa.
- (IV) Each $\rho(f, U)$ is a (uniform) quasiisometric embedding.

(V) For all $U \in \mathfrak{S}$, the following diagram (uniformly) coarsely commutes:

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{f} & \mathcal{X}' \\
 \downarrow \pi_U & & \downarrow \prod_{W \in \pi(f)(U)} \pi_W \\
 \mathcal{C}U & \xrightarrow{\rho(f,U)} & \prod_{W \in \pi(f)(U)} \mathcal{C}W
 \end{array}$$

(VI) If $U, V \in \mathfrak{S}$ satisfy $U \sqsubset V$ or $U \pitchfork V$, then

$$\begin{array}{ccc}
 \mathcal{C}U & \xrightarrow{\rho(f,U)} & \prod_{W \in \pi(f)(U)} \mathcal{C}W \\
 \downarrow \rho_V^U & & \downarrow g \\
 \mathcal{C}V & \xrightarrow{\rho(f,V)} & \prod_{W' \in \pi(f)(V)} \mathcal{C}W'
 \end{array}$$

uniformly coarsely commutes, where g is a coarsely constant map such that: if $U \sqsubset V$, then for each $W' \in \pi(f)(V)$, the W' -coordinate of g is $\rho_{W'}^W$, for some (hence any, by Lemma 1.5) $W \in \pi(f)(U)$ with $W \sqsubset W'$, and if $U \pitchfork V$, then for each $W' \in \pi(f)(V)$, the W' -coordinate of g is $\rho_{W'}^W$, for some (hence any) $W \in \pi(f)(U)$ with $W \pitchfork W'$.

(VII) If $V \sqsubset U$, then

$$\begin{array}{ccc}
 \mathcal{C}U & \xrightarrow{\rho(f,U)} & \prod_{W \in \pi(f)(U)} \mathcal{C}W \\
 \downarrow \rho_V^U & & \downarrow h \\
 \mathcal{C}V & \xrightarrow{\rho(f,V)} & \prod_{W' \in \pi(f)(V)} \mathcal{C}W'
 \end{array}$$

uniformly coarsely commutes, where the map h is defined as follows: given $(x_{W'})_{W' \in \pi(f)(U)}$, for each $W \in \pi(f)(V)$, the W -coordinate of $h((x_{W'}))$ is $\rho_{W'}^{W''}(x_{W''})$, where W'' is the unique element of $\pi(f)(U)$ with $W \sqsubset W''$.

Remark 5.2 (hieromorphisms are slanted hieromorphisms) Any hieromorphism f is a slanted hieromorphism in which $|\pi(f)(U)| = 1$ for all $U \in \mathfrak{S}$.

Remark 5.3 There is presumably a still more general version of Definition 5.1 encompassing morphisms $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}', \mathfrak{S}')$ where $f: \mathcal{X} \rightarrow \mathcal{X}'$ is a map, $f: 2^{\mathfrak{S}} \rightarrow 2^{\mathfrak{S}'}$

sends pairwise-orthogonal sets to pairwise-orthogonal sets, and f sends appropriate products of hyperbolic spaces to products of hyperbolic spaces. Simple examples like rotation in \mathbb{E}^2 require such a definition in order to be regarded as maps of hierarchically hyperbolic spaces.

Definition 5.4 (coarse similarity) Let M and M' be metric spaces. Then $f: M \rightarrow M'$ is a (λ, ϵ) -coarse similarity if there exist $\lambda > 0$ and $\epsilon \geq 0$ such that for all $p, q \in M$,

$$\lambda d_M(p, q) - \epsilon \leq d_{M'}(f(p), f(q)) \leq \lambda d_M(p, q) + \epsilon.$$

Definition 5.5 (extensible slanted hieromorphism) Let $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}', \mathfrak{S}')$ be a slanted hieromorphism. Then f is *extensible* if there exist $0 < \lambda_1 \leq \lambda_2$ and $K < \infty$ such that:

- (1) $\pi(f): \mathfrak{S} \rightarrow 2^{\mathfrak{S}'}$ is injective.
- (2) For all $V \in \mathfrak{S}'$, either there is $U \in \mathfrak{S}$ with $V \in \pi(f)(U)$ or

$$\text{diam}_{\mathcal{C}V}(\pi_V(f(\mathcal{X}))) \leq K.$$

- (3) For all $U \in \mathfrak{S}$ and $W \in \pi(f)(U)$, the composition

$$\mathcal{C}U \xrightarrow{\rho(f,U)} \prod_{V \in \pi(f)(U)} \mathcal{C}V \rightarrow \mathcal{C}W$$

is a (λ, λ') -coarse similarity, where the second map is the canonical projection and $\lambda \in [\lambda_1, \lambda_2]$ (λ can depend on U and V) and $\lambda' \geq 0$.

Theorem 5.6 (extending slanted hieromorphisms to the boundary) Let $(\mathcal{X}, \mathfrak{S})$ and $(\mathcal{X}', \mathfrak{S}')$ be hierarchically hyperbolic structures on the spaces \mathcal{X} and \mathcal{X}' , respectively. Suppose that $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}', \mathfrak{S}')$ is an extensible slanted hieromorphism. Then there is a map $\bar{f}: \bar{\mathcal{X}} \rightarrow \bar{\mathcal{X}}'$ such that

- (1) $\bar{f}|_{\mathcal{X}} = f$;
- (2) $\bar{f}|_{\partial \mathcal{X}}$ is injective;
- (3) for all $f(p) \in \partial \mathcal{X}'$ and basic neighborhoods \mathcal{N} of $f(p)$ in \mathcal{X}' , the set $\bar{f}^{-1}(\mathcal{N})$ contains a basic neighborhood of $p \in \bar{\mathcal{X}}$, ie \bar{f} is continuous at each point in $\partial \mathcal{X}$.

In particular, if \mathcal{X} is proper, then $\bar{f}|_{\partial \mathcal{X}}$ is an embedding with closed image and, if f is an embedding, then $\bar{f}: \bar{\mathcal{X}} \rightarrow \bar{\mathcal{X}}'$ is an embedding whose image is closed.

Proof For convenience, when the domains of the various maps are understood, we shall denote each map $f: \mathcal{X} \rightarrow \mathcal{X}'$, $\pi(f): \mathfrak{S} \rightarrow 2^{\mathfrak{S}'}$, and $\rho(f, U): \mathcal{C}U \rightarrow \prod_{W \in \pi(f)(U)} \mathcal{C}W$ by f .

Boundary maps on hyperbolic domains Let $U \in \mathfrak{S}$. To each sequence (x_n) in \mathcal{CU} , associate the sequence $(f(x_n))_n$ in $\prod_{W \in \mathcal{C}\pi(f)(U)} \mathcal{CW}$. For each $W \in \pi(f)(U)$, let $w_n(W) \in \mathcal{CW}$ be the W -coordinate of $f(x_n)$. Fix a basepoint $x \in \mathcal{CU}$ and $p_W = \pi_W(\rho(f, U)(x)) \in \mathcal{CW}$ for each $W \in \pi(f)(U)$.

Suppose that $(x_n)_n$ represents a point in $\partial\mathcal{CU}$, ie $(x_i | x_j)_x \rightarrow \infty$ as $i, j \rightarrow \infty$. Since $\rho(f, U)$ is a uniform quasiisometric embedding, we have for each $W \in \pi(f)(U)$ that $(w_i(W) | w_j(W))_{p_W} \rightarrow \infty$ as $i, j \rightarrow \infty$. Hence $w_i(W)$ converges to a point $p(W) \in \partial\mathcal{CW}$.

For each $W \in \pi(f)(U)$, choose $\alpha_W \in (0, 1]$ so that

$$\frac{\alpha_W}{\alpha_{W'}} = \lim_n \frac{d_W(p_W, w_n(W))}{d_{W'}(p_{W'}, w_n(W'))}$$

for all $W, W' \in \pi(f)(U)$, which exists because of the coarse similarity assumption. Then define $p \in \star_{W \in \mathcal{C}\pi(f)(U)} \partial\mathcal{CW}$ to be the linear combination $\sum_{W \in \pi(f)(U)} \alpha_W p_W$. The assignment $\bar{f}_U((x_n)) = p$ thus provides a map

$$\bar{f}_U: \mathcal{CU} \cup \partial\mathcal{CU} \rightarrow \prod_{W \in \mathcal{C}\pi(f)(U)} \mathcal{CW} \cup \star_{W \in \mathcal{C}\pi(f)(U)} \mathcal{CW}$$

extending the map $\rho(f, U)$.

For any $U \in \mathfrak{S}$, the map \bar{f}_U defined above is injective since the composition of f with any of the canonical projections $\prod_{W \in \pi(f)(U)} \mathcal{CW} \rightarrow \mathcal{CW}$ is a uniform quasiisometric embedding, and quasiisometric embeddings coarsely preserve Gromov products.

Definition of \bar{f} Let $p \in \partial\mathcal{X}$, so that $p = \sum_{U \in \text{Supp}(p)} \beta_U p_U$, where $p_U \in \partial\mathcal{CU}$ for each U , each $\beta_U \in (0, 1]$, and $\sum_U \beta_U = 1$. For each $U \in \text{Supp}(p)$, we defined $\bar{f}_U(p_U) = \sum_{W \in \pi(f)(U)} \alpha_W^U q_W$ above, where $q_W \in \partial\mathcal{CW}$ and $\sum_W \alpha_W^U = 1$. Let

$$\bar{f}(p) = \sum_{U \in \text{Supp}(p)} \sum_{W \in \pi(f)(U)} \beta_U \alpha_W^U \cdot q_W,$$

which is a point in $\partial\mathcal{X}'$ since $\sum_U \sum_W \beta_U \alpha_W^U = 1$ and since $\bigcup_{U \in \text{Supp}(p)} \pi(f)(U)$ is a pairwise-orthogonal set by [Definition 5.1](#) since f is a slanted hieromorphism.

Injectivity of $\bar{f}|_{\partial\mathcal{X}}$ Injectivity of $\bar{f}|_{\partial\mathcal{X}}$ follows from injectivity of \bar{f}_U on each $\partial\mathcal{CU}$ for $U \in \mathfrak{S}$ together with injectivity of $\pi(f)$ and the fact that each $\bar{f}_U: \mathcal{CU} \rightarrow \prod_{W \in \pi(f)(U)} \mathcal{CW}$ is “fully supported” in the sense that each $\alpha_W^U > 0$.

Continuity at boundary points First consider $p \in \partial\mathcal{X}$. By [Proposition 2.17](#), there exists (x_n) in \mathcal{X} such that $x_n \rightarrow p$ as $n \rightarrow \infty$. We check that $f(x_n)$ converges to $\bar{f}(p)$.

Fix a basepoint $x \in \mathcal{X}$, so that $p = \sum_{U \in \text{Supp}(p)} a_U p_U$ with $\sum_U a_U = 1$, each $a_U > 0$, and for all $U, U' \in \text{Supp}(p)$,

$$\left| \frac{d_U(x, x_n)}{d_{U'}(x, x_n)} - \frac{a_U}{a_{U'}} \right| \rightarrow 0 \quad \text{and} \quad \frac{d_V(x, x_n)}{d_U(x, x_n)} \rightarrow 0$$

whenever $U \in \text{Supp}(p)$ and $V \in \text{Supp}(p)^\perp$, and finally $\pi_U(x_n) \rightarrow p_U$ for all $U \in \text{Supp}(p)$.

Consider the sequence $(w_n) = (f(x_n))$. For each $U \in \text{Supp}(p)$ and $W \in \pi(f)(U)$, let $c_W: \prod_{V \in \pi(f)(U)} \overline{C^V} \rightarrow \overline{C^W}$ be the canonical projection. By hypothesis, for each such W we have $|d_W(f(x), w_n) - \lambda'_W d_U(x, x_n)| \leq \lambda'_W$, where $\lambda_W \in [\lambda_1, \lambda_2]$ and $\lambda'_W \geq 0$. Hence for each $U \in \text{Supp}(p)$ and $W \in \pi(f)(U)$, we have that $\pi_W(w_n) = c_W \circ \bar{f}(\pi_U(x_n)) \rightarrow c_W \circ \bar{f}(p_U)$ and $\bar{f}(\pi_U(x_n)) \rightarrow \sum_{W \in \pi(f)(U)} \beta_U \alpha_W c_W \cdot \bar{f}(p_U)$ as required. Moreover, if $V \in \mathfrak{S}'$ does not belong to $\pi(f)$, then $d_V(f(x), w_n)$ is uniformly bounded by Definition 5.5(2).

Finally, if $V \in \mathfrak{S} - \text{Supp}(p)$, then $d_V(x, x_n)$ is dominated by $d_U(x, x_n)$ for any $U \in \text{Supp}(p)$. Hence, for such V , we have that $d_W(f(x), f(x_n))$ is dominated by $d_Z(f(x), f(x_n))$ whenever $W \in \pi(f)(V)$ and $Z \in \pi(f)(U)$ for some $U \in \text{Supp}(p)$, since each $\rho(f, U)$ is a uniform quasiisometric embedding. Thus $f(x_n)$ converges to $\bar{f}(p)$.

More generally, given any sequence (z_k) in $\bar{\mathcal{X}}$ converging to $p \in \partial X$, we can use the ideas in the proof of Theorem 3.4 to build a sequence of internal sequences $(x_{k,i})$ such that $\lim_i x_{k,i} = z_k$ for each k . Namely, for each k , we can take a sequence $(x_{k,i}) \rightarrow z_k$ (if $z_k \in \mathcal{X}$, then we choose $x_{k,i} = z_k$ to be constant), and then we choose $N_k > 0$ large enough that if $n > N_k$, then the sequence $(x_{k,n})$ will satisfy conditions (1)–(7) from the proof of Theorem 3.4. This will force that $\lim_i x_{k,i} = z_k$, and then since $\lim_k z_k = p$, the above conditions will force $\lim_k x_{k,n} = p$.

Now, since $\lim_n x_{k,n} = p$ and $\lim_i x_{k,i} = z_k$, the internal case above implies

$$\lim_n \bar{f}(x_{k,n}) = \bar{f}(p) \quad \text{and} \quad \lim_i \bar{f}(x_{k,i}) = z_k.$$

Together, these imply that $\lim_k \bar{f}(z_k) = \bar{f}(p)$. Thus \bar{f} is continuous at boundary points.

When \mathcal{X} is proper Assertion (3) combines with Theorem 3.4 and Proposition 2.17(1) to imply that \bar{f} is an embedding; compactness of $\partial \mathcal{X}$ implies that its image is closed. If in addition, f is an embedding, then $\bar{f}: \bar{\mathcal{X}} \rightarrow \bar{\mathcal{X}'}$ is an embedding, since assertion (3) again combines with Proposition 2.17(1) and Theorem 3.4 to imply that \bar{f} is a continuous injection from a compact space to a Hausdorff space. \square

Remark 5.7 *Theorem 5.6* holds under slightly more general conditions: condition (3) of *Definition 5.5* need only be imposed on $U \in \mathfrak{G}$ in cases where either there exists $V \in \mathfrak{G}$ with $U \perp V$ or $|\pi(f)(U)| > 1$ or both. For any U with empty orthogonal complement and for which $\pi(f)(U) = \{V\}$ for some $V \in \mathfrak{G}'$, it suffices to require that $\rho(f, U): \mathcal{C}U \rightarrow \mathcal{C}V$ is a uniform quasiisometric embedding.

5.1 Limit sets of hierarchically quasiconvex sets

Let $(\mathcal{X}, \mathfrak{G})$ be a proper hierarchically hyperbolic space and let $\mathcal{Y} \subseteq \mathcal{X}$ be hierarchically quasiconvex. Let $\Lambda\mathcal{Y}$ be the set of boundary points $p = \sum_{U \in \text{Supp}(p)} a_U p_U \in \partial\mathcal{X}$ such that for all $U \in \text{Supp}(p)$, there is a sequence $p_U^n \in \pi_U(\mathcal{Y})$ converging to p_U .

Proposition 5.8 (hierarchically quasiconvex subspaces have limit sets) *The set $\mathcal{Y} \cup \Lambda\mathcal{Y}$ is a closed subset of $\bar{\mathcal{X}}$, and \mathcal{Y} is dense in $\mathcal{Y} \cup \Lambda\mathcal{Y}$. Hence \mathcal{Y} has an HHS structure such that $\mathcal{Y} \cup \Lambda\mathcal{Y} = \bar{\mathcal{Y}}$.*

Proof This is a definition chase and an application of *Proposition 2.17*. □

Remark 5.9 When $\pi_U|_{\mathcal{Y}}$ is either surjective or uniformly bounded for each U , *Theorem 5.6*, together with the HHS structure on \mathcal{Y} inherited from \mathcal{X} , implies that $\Lambda\mathcal{Y}$ is homeomorphic to the HHS boundary $\partial\mathcal{Y}$. This holds in particular for the main examples of hierarchically quasiconvex subspaces that we use, namely product regions:

Remark 5.10 (boundaries of standard product regions) Let $U \in \mathfrak{G}$, and recall from *Section 1.3* that there is a quasiisometric embedding $F_U \times E_U \rightarrow \mathcal{X}$ coming from the standard hieromorphisms. By definition, ∂F_U consists of exactly those $\sum_V a_V p_V \in \partial\mathcal{X}$ where the support set $\{V\}$ consists entirely of elements of \mathfrak{G}_U , while ∂E_U consists of linear combinations of the same form, but with each $V \in \mathfrak{G}_U^\perp$. In particular, under the map $F_U \times E_U \rightarrow \mathcal{X}$, we see that the images of

$$\partial(F_U \times \{e_1\}), \partial(F_U \times \{e_2\}) \rightarrow \partial\mathcal{X}$$

are identical. Moreover, the subspace $\partial F_U \subset \partial\mathcal{X}$ is closed. Finally, $\partial P_U \subset \mathcal{X}$ is a closed subset homeomorphic to $\partial F_U \star \partial E_U$, where \star denotes the spherical join.

5.2 Geometrically finite subgroups of mapping class groups

In this subsection, we will show that certain interesting subgroups of mapping class groups have a well-defined limit set in the boundary. Before doing so, we give a quick sketch of relevant facts about mapping class groups and Teichmüller spaces. For more details about the HHG structure of the mapping class group, the reader is referred to [6, Section 11].

Fix a finite-type surface S . The mapping class group $\mathcal{MCG}(S)$ of S acts properly and cocompactly on the marking graph $\mathcal{M}(S)$ of S [60]. The vertices of the marking graph, called markings, are isotopy classes of certain collections of curves on S (pants decomposition together with certain transverse curves). $\mathcal{MCG}(S)$ and $\mathcal{M}(S)$ are quasiisometric via the orbit map, and we will identify $\mathcal{MCG}(S)$ with an orbit in $\mathcal{M}(S)$ from now on. The mapping class group can be given a hierarchically hyperbolic structure by considering the collection \mathfrak{S} of all its (isotopy classes of essential) subsurfaces and associating to each $Y \in \mathfrak{S}$ its curve graph $\mathcal{C}Y$, a graph whose vertices are isotopy classes of essential simple closed curves on Y , except when Y is an annulus (a case that will be more subtle to deal with later, and which we will hence explain in more detail here). When Y is an annulus, $\mathcal{C}Y$ has vertices the isotopy classes of arcs connecting the two boundary components, and two such vertices are adjacent if they can be represented by disjoint arcs. The maps $\pi_Y: \mathcal{MCG}(S) \rightarrow 2^{\mathcal{C}Y}$ are called subsurface projections and, when Y is not an annulus, they are defined more or less by intersecting the curves in the marking with Y . When Y is an annulus π_Y is defined in the following way. Let \hat{Y} be the annular cover of S where the core of the annulus lifts to a simple closed curve. There is a natural compactification \bar{Y} of \hat{Y} which is a closed annulus, and that can be identified with Y . Given a marking m , lift to \hat{Y} all the curves in m , except possibly the (only) one which is isotopic to the core of Y . Each such lift can be compactified to an arc in \bar{Y} , and we can finally define $\pi_Y(m)$ to be the collection of all such arcs that connect distinct boundary components of \bar{Y} .

We now comment briefly on Teichmüller space $\mathcal{T}(S)$ endowed with the Teichmüller metric. A point on Teichmüller space corresponds to a hyperbolic metric on S , and we can hence consider the systole map $\text{Sys}: \mathcal{T}(S) \rightarrow 2^{\mathcal{C}S}$ that maps points in Teichmüller space to the shortest curves in the corresponding hyperbolic metric. The set of systoles is nonempty and pairwise disjoint, thus giving a bounded subset of $\mathcal{C}S$.

5.2.1 Subsurface mapping class groups For any nonpants subsurface $Y \subset S$ there is a natural embedding $\iota_Y: \mathcal{MCG}(Y) \hookrightarrow \mathcal{MCG}(S)$ which takes any mapping class $f_Y \in \mathcal{MCG}(Y)$ to a mapping class $f \in \mathcal{MCG}(S)$ such that $f|_Y \equiv f_Y$ and $f|_{S \setminus Y} \equiv \text{id}_{S \setminus Y}$; if Y is an annulus, we take $\mathcal{MCG}(Y)$ to be the cyclic subgroup generated by the Dehn (half) twist about the core of Y .

We can also see this map in terms of markings: For each component $X \subset S \setminus Y$ (including annuli with core curves in ∂Y), fix a marking $\mu_X \in \mathcal{M}(X)$; if X is an annulus, then $\mu_X \in \mathcal{C}X$. Define a map $\iota_Y: \mathcal{M}(Y) \rightarrow \mathcal{M}(S)$ by

$$\iota_Y(\mu_Y) = \mu_Y \sqcup \bigsqcup_{\alpha \in \partial Y} \alpha \sqcup \bigsqcup_{X \in S \setminus Y} \mu_X$$

for any marking $\mu_Y \in \mathcal{M}(Y)$.

The map ι_Y extends to a hieromorphism in the obvious way and it follows from the distance formula that it is a quasiisometric embedding. Since $\text{diam}_Z(\iota_Y(\mathcal{M}(Y)))$ is uniformly bounded for each $Z \in \mathfrak{S} \setminus \mathfrak{S}_Y$ and ι_Y is surjective for each $W \in \mathfrak{S}_Y$, it is, moreover, easy to see that $\iota_Y(\mathcal{M}(Y))$ is a hierarchically quasiconvex subspace of $\mathcal{M}(S)$. Hence we have, by [Proposition 5.8](#):

Theorem 5.11 *The natural inclusion $\iota_Y: \mathcal{MCG}(Y) \hookrightarrow \mathcal{MCG}(S)$ equivariantly extends to a continuous embedding $\partial\iota_Y: \partial\mathcal{MCG}(Y) \hookrightarrow \partial\mathcal{MCG}(S)$ for any nonpants subsurface $Y \subset S$.*

5.3 Convex cocompactness subgroups

Convex cocompact subgroups of mapping class groups are a much-studied class of hyperbolic subgroups of mapping class groups, mainly because they are precisely the class of subgroups of $\mathcal{MCG}(S)$ whose corresponding surface subgroup extensions are hyperbolic. Importantly, they satisfy several strong equivalent characterizations, which we state in the following theorem-definition with parts due variously to Farb and Mosher [\[31\]](#), Hamenstädt [\[40\]](#), Kent and Leininger [\[48\]](#), and the first author with Taylor [\[27\]](#):

Theorem 5.12 *A subgroup $H < \mathcal{MCG}(S)$ is convex cocompact if it satisfies any of the following equivalent conditions:*

- (1) *Any orbit of H in $\mathcal{T}(S)$ is quasiconvex.*
- (2) *Any orbit of H in \mathcal{CS} is quasiisometrically embedded.*
- (3) *Any orbit of H in $\mathcal{M}(S)$ is quasiisometrically embedded and has uniformly bounded subsurface projections.*
- (4) *H is a stable subgroup of $\mathcal{MCG}(S)$.*
- (5) *The corresponding extension Γ_H of $\pi_1(S)$ is Gromov-hyperbolic.*

The following is a corollary of [Proposition 5.8](#) and [Theorems 4.3](#) and [5.12](#):

Corollary 5.13 *If $H < \mathcal{MCG}(S)$ is a convex cocompact subgroup of $\mathcal{MCG}(S)$, then the inclusion map $H \hookrightarrow \mathcal{MCG}(S)$ H -equivariantly extends to a continuous embedding $\partial_{\text{Gr}}H \hookrightarrow \partial\mathcal{MCG}(S)$.*

Proof It follows immediately from properties (2) and (3) of [Theorem 5.12](#) that H is a hierarchically quasiconvex subgroup of $\mathcal{MCG}(S)$. Since H is hyperbolic, [Theorem 4.3](#) implies that the boundary of the induced HHS structure on H inside of $\mathcal{MCG}(S)$ is homeomorphic to $\partial_{\text{Gr}}H$. The result then follows from [Proposition 5.8](#). \square

In the rest of the section, we will consider finitely generated Veech subgroups and the Leininger–Reid combination subgroups of $\mathcal{MCG}(S)$, which are generally not hierarchically quasiconvex. Recall that for both classes of groups, their actions on $\mathcal{T}(S)$ do not extend continuously everywhere to embeddings of their boundaries into $\mathbb{P}\mathcal{ML}(S)$. The main goal of the remainder of this section is to prove that such an extension does exist for both classes of groups into $\partial\mathcal{MCG}(S)$.

5.3.1 Veech subgroups The construction of Veech and Leininger–Reid subgroups involves holomorphic quadratic differentials. We will not work with them directly, so we do not need to define them, but we will rather work with the q -metric associated to a holomorphic quadratic differential q on the surface S . This is a singular flat metric on S which is locally isometric to \mathbb{R}^2 except at finitely many points called *singularities*.

Given a holomorphic quadratic differential q on S , there exists a convex subset $\text{TD}(q) \subset \mathcal{T}(S)$ with $\text{TD}(q) \cong \mathbb{H}^2$ called a *Teichmüller disk*. Let $\text{Aff}^+(q)$ denote the affine group of q . Following [52], we call any subgroup $G(q) \leq \text{Aff}^+(q) \leq \mathcal{MCG}(S)$, with $G(q)$ acting properly on $\text{TD}(q)$, a *Veech subgroup*, except that we will also ask that $G(q)$ be finitely generated. Veech subgroups have the property that every element of $G(q)$ is either pseudo-Anosov or a multitwist about some annular decomposition A of q [73], where this annular decomposition comes from a finite measured foliation with only closed leaves naturally associated to q .

Consider the Veech subgroup $G = G(q) \leq \mathcal{MCG}(S)$. Let \mathcal{X}_G be the orbit of G of a fixed marking μ in the marking graph $\mathcal{M}(S)$. Given a multitwist $g \in G$ with annular decomposition $A_g = \{\alpha_1, \dots, \alpha_{n_g}\}$, let

$$\pi_g: \mathcal{X}_G \rightarrow \prod_{1 \leq i \leq n_g} C\alpha_i$$

be given by $\pi_g(v) = (\pi_{\alpha_1}(v), \dots, \pi_{\alpha_{n_g}}(v))$ for $v \in \mathcal{X}_G$. If $g = T_{\alpha_1}^{k_1} \dots T_{\alpha_{n_g}}^{k_{n_g}}$, let

$$L_g = \langle g \rangle \cdot \pi_g(\mu) \subset \prod_{1 \leq i \leq n_g} C\alpha_i.$$

Note that $L_g \cong \mathbb{R}$, and in fact L_g is the projection of the g -orbit of μ and thus coarsely the line in \mathbb{R}^{n_g} with slope (k_1, \dots, k_{n_g}) , where we identify the origin of \mathbb{R}^{n_g} with the projection of μ . For each L_g , let $\pi_{L_g}: \prod_{1 \leq i \leq n_g} C\alpha_i \rightarrow L_g$ be the standard projection onto L_g , considered as a subspace of \mathbb{R}^{n_g} identified as above.

We now define an HHS structure (G, \mathfrak{S}_G) on G as follows:

Domains S is the unique nest-maximal domain in \mathfrak{S}_G , and for every primitive multitwist $g \in G$ with corresponding annular decomposition $A_g = \{\alpha_{g,1}, \dots, \alpha_{g,n_g}\}$, we include a domain $U_g \in \mathfrak{S}_G$.

The spaces To S , we associate $\pi_S(G \cdot \mu) \subset \mathcal{CS}$ and to each U_g , we set $\mathcal{CU}_g = L_g$ and declare $U_g \sqsubseteq S$ for each g ; moreover, we specify that $U_g \pitchfork U_{g'}$ for each primitive $g \neq g'$.

Projections $\pi_S: \mathcal{X}_G \rightarrow \mathcal{CS}$ is the standard projection; for each U_g , we define $\pi_{U_g}: \mathcal{X}_G \rightarrow L_g$ by $\pi_{U_g}(v) = \pi_{L_g}(\pi_g(v))$ for each $v \in \mathcal{X}_G$.

Relative projections Given $U, V \in \mathfrak{S}_G$, we define $\rho_V^U: \mathcal{CU} \rightarrow \mathcal{CV}$ by:

$(U \sqsubseteq V)$ In this case, $V = S$ and $U = U_g$ for some primitive g ; then $\rho_U^V = \pi_{L_g} \circ \pi_g$.

$(U \pitchfork V)$ If $U = U_g$ and $V = U_{g'}$, then

$$\rho_{U_{g'}}^{U_g} = \pi_{U_{g'}}(\langle g \rangle \cdot \mu).$$

Lemma 5.14 *If G is finitely generated, then (G, \mathfrak{S}_G) is an HHS structure on G , and $G < \text{Aut}(\mathfrak{S}_G)$.*

Proof We need to prove that (G, \mathfrak{S}_G) satisfies the axioms; since it clearly satisfies projections, nesting, orthogonality, and finite complexity, it suffices to prove it satisfies the consistency, large link, bounded geodesic image, partial realization, and uniqueness axioms. Hyperbolicity of the associated spaces uses [Lemma 5.15](#) (the only part for which we need finite generation of G).

There is no nontrivial orthogonality, so partial realization holds by construction. Bounded geodesic image holds by the bounded geodesic image axiom in $(\mathcal{MCG}(S), \mathfrak{S})$ and the definition of $\rho_{U_g}^S$. The consistency and large link axioms hold for a similar reason. Uniqueness follows from uniqueness in $(\mathcal{MCG}(S), \mathfrak{S})$ together with [Lemma 5.16](#). □

Lemma 5.15 *The projection $\pi_S(G \cdot \mu)$ is quasiconvex in \mathcal{CS} .*

Proof Consider the action of G on the corresponding Teichmüller disk $\text{TD}(q)$. Since the action is proper, this makes G a finitely generated Fuchsian group. Hence, G is geometrically finite [\[57\]](#), so that it acts with cofinite volume on a convex subspace $C_G \subseteq \text{TD}(q)$. Consider now the image of C_G and $\text{TD}(q)$ in \mathcal{CS} . Since geodesics in $\mathcal{T}(S)$ map to quasigeodesics in \mathcal{CS} [\[59\]](#) and C_G is a convex subspace of $\mathcal{T}(S)$, it follows that $\pi_S(C_G)$ is quasiconvex in \mathcal{CS} .

Now, it is not hard to see that $\pi_S(C_G)$ coarsely coincides with $\pi_S(G \cdot \mu)$. In fact, C_G contains a G -equivariant collection of horodisks such that the action on the complement C'_G is cocompact, and cocompactness implies that $\pi_S(G \cdot \mu)$ coarsely coincides with the image in \mathcal{CS} of C'_G . Moreover, each horodisk is stabilized by a multitwist, and the corresponding curves are short in all hyperbolic metrics corresponding to points in the horodisk. This implies that the whole horodisk maps to a uniformly bounded subset of \mathcal{CS} under the systole map, namely a neighborhood of the aforementioned curves. To sum up, the projection of the Teichmüller disk to \mathcal{CS} is quasiconvex and coarsely coincides with the projection of C'_G , which in turn coarsely coincides with the projection of $G \cdot \mu$, and we are done. \square

Lemma 5.16 *There exists $V > 0$ such that for any $U \in \mathfrak{S} - \{S\}$, either*

$$\text{diam}_U(\pi_U(G \cdot \mu)) \leq V$$

or $U = \alpha_i \in A_g$ for some annular decomposition A_g . In the latter case, π_U is (uniformly) coarsely surjective.

Proof Let $U \sqsupseteq S$ be a subsurface and let $\Delta \subset U$ be its spine, which is obtained by pulling tight ∂U with respect to the q -metric, so that vertices of Δ are singular points and edges are saddle connections (ie geodesics connecting singularities and intersecting the singular set only at the endpoints). There exists a natural retraction $r: U \rightarrow \Delta$ and for each edge e of Δ , let $\delta_e = r^{-1}(m_e)$, where m_e is the midpoint of e . Each δ_e is either a curve or an arc in $(U, \partial U)$. We now divide into three cases.

U is nonannular In this case, Δ has a degree-3 vertex v . Suppose that μ has a base curve α that traverses each saddle connection in Δ at most once. Then v has some incident edge e such that δ_e is disjoint from α . Now, for any $g \in \text{Aff}^+(q)$, we have that $g \cdot \Delta$ is the spine of $g \cdot U$, with vertices that are singular points and edges saddle connections. In particular, $g \cdot \alpha$ is a curve using each saddle connection of Δ at most once, so $d_{\mathcal{AC}U}(\alpha, g \cdot \alpha) \leq 3$, where $\mathcal{AC}U$ denotes the arc-and-curve graph of U . Since there is a 2-Lipschitz retraction $\mathcal{AC}(U) \rightarrow \mathcal{CU}$ [60, Lemma 2.2], it follows that $\text{diam}_U(G \cdot \mu)$ is uniformly bounded.

Since $G(q)$ preserves the set of all singularities, saddle connections, and geodesic representatives of curves, we are done provided we choose the marking μ in such a way that each of its base curves traverses each saddle connection at most once.

$U \in A_g$ for some g Let $g \in G(q)$ be a multitwist about curves $\alpha_1, \dots, \alpha_n$, with $g = \prod_{i=1}^n T_{\alpha_i}^{k_i}$, where $k_i \in \mathbb{Z} - \{0\}$. Hence π_U is k_i -surjective (where $U = \alpha_i$). Indeed, $\pi_U(g \cdot \mu) = \pi_U(T_{\alpha_i}^{k_i} \cdot \mu)$, and the k_i are uniformly bounded since the action of $G(q)$ on the corresponding Teichmüller disc is geometrically finite, and thus there are finitely many conjugacy classes of multitwists in $G(q)$; see the proof of Lemma 5.15.

U an annulus and $U \notin A_g$ for any g The spine Δ of U contains at least one singularity, and the angle at the singularity is greater than π on both sides. Let \hat{U} be the annular cover of S corresponding to U . The lift $\hat{\Delta}$ of Δ disconnects \hat{U} into two connected components, and we will refer to the closure of each such connected component as a *side* of $\hat{\Delta}$. Consider a singularity along $\hat{\Delta}$ and a saddle connection entering the singularity. Then, for any side of $\hat{\Delta}$ there exists a unique geodesic ray emanating from the given singularity, forming an angle of π with the given saddle connection and contained in the given side of $\hat{\Delta}$. We let $\{\alpha_i\}$ be the open arcs in \hat{U} that can be formed by concatenating two such rays lying in opposite sides of $\hat{\Delta}$. It is readily seen that any two α_i have intersection number at most 1. The bound on the diameter of the projection onto \mathcal{CU} now follows from the fact that any arc in the subsurface projection onto \mathcal{CU} of some curve in S can be represented either by a geodesic transverse to a saddle connection in $\hat{\Delta}$, which is easily seen to be disjoint from some α_i , or a geodesic containing one of the singularities, which is easily seen to intersect an appropriate α_i containing that singularity at most once. \square

Lemma 5.17 *There exists a G -equivariant extensible slanted hieromorphism*

$$(G, \mathfrak{S}_G) \rightarrow (\mathcal{MCG}(S), \mathfrak{S}).$$

Proof At the level of spaces, the map $G \rightarrow \mathcal{MCG}(S)$ is the inclusion. Define $\pi(f): \mathfrak{S}_G \rightarrow 2^{\mathfrak{S}}$ as follows: let $\pi(f)(S) = \{S\}$, and for each primitive multitwist g , let $\pi(f)(U_g) = A_g$, where A_g is the set of pairwise-disjoint annuli corresponding to the multicurve supporting g . This is G -equivariant since $hA_g = A_{hg h^{-1}}$ for each multitwist g and each $h \in G$.

The map $\rho(f, S): \mathcal{CS} \rightarrow \mathcal{CS}$ is the identity. For each primitive multitwist $g = T_{\alpha_1}^{k_1} \dots T_{\alpha_{n_g}}^{k_{n_g}}$, the map $\rho(f, U): L_g \rightarrow \prod_i \mathcal{C}\alpha_i$ was specified above. Observe that the composition of this map with any of the canonical projections to $\mathcal{C}\alpha_i$ is a coarse similarity with multiplicative constants determined by $\{k_1, \dots, k_{n_g}\}$. These constants are uniformly bounded since there are finitely many conjugacy classes of multitwists in $G(q)$. \square

Combining [Lemma 5.17](#) and [Theorem 5.6](#), [Remark 5.7](#), and [Theorem 4.3](#) yields:

Corollary 5.18 *For any Veech subgroup $G < \mathcal{MCG}(S)$, the inclusion $G \rightarrow \mathcal{MCG}(S)$ extends continuously to an equivariant embedding $\partial_{\text{Gr}}G \rightarrow \partial\mathcal{MCG}(S)$ with closed image.*

Remark 5.19 [Corollary 5.18](#) does not follow from [Proposition 5.8](#) because the Veech subgroup G is not hierarchically quasiconvex in $\mathcal{MCG}(S)$ whenever it contains a

multitwist supported on a multicurve with more than one component; indeed, in this case there are realization points in $\mathcal{MCG}(S)$ whose images in each curve graph lie in the image of G , but which are arbitrarily far from G .

5.3.2 Leininger–Reid surface subgroups We now turn to the Leininger–Reid surface subgroups constructed in [52, Theorem 6.1]. Again, we show that these are nonhierarchically quasiconvex subgroups of $\mathcal{MCG}(S)$ that nonetheless have well-defined limit sets in $\partial\mathcal{MCG}(S)$. The setup is as follows:

- (1) Let q_1, \dots, q_n be holomorphic quadratic differentials, with $A_0 \in \mathcal{CS}$ the core of the annular decomposition of each q_i such that each complementary component has negative Euler characteristic.
- (2) Suppose $G_0 = G_0(q_i)$ for all $i \leq n$.
- (3) Suppose $h \in \mathcal{MCG}(S)$ centralizes G_0 and is pure and pseudo-Anosov on all components of $S - A_0$.

Then, for

$$H = G(q_1) *_{G_0} h^{k_2} G(q_2) h^{-k_2} *_{G_0} \dots *_{G_0} h^{k_n} G(q_n) h^{-k_n},$$

the map $H \rightarrow \mathcal{MCG}(S)$ is an embedding whenever

$$N = \min\{|k_i - k_j| : i, j \in \{1, \dots, n\}, i \neq j\}$$

(where we set $k_1 = 0$) is large enough. Moreover, every element of $\text{im}(H \rightarrow \mathcal{MCG}(S))$ (which we denote by H) is either pseudo-Anosov or conjugate into an elliptic or parabolic subgroup of some $h^{k_i} G(q_i) h^{-k_i}$. In particular, the $G(q_i)$ can be chosen so that H fails to be hierarchically quasiconvex for the reason explained in Remark 5.19.

In the remainder of this section, we prove:

Theorem 5.20 *The inclusion $H \rightarrow \mathcal{MCG}(S)$ extends continuously to an equivariant embedding $\partial H \rightarrow \partial\mathcal{MCG}(S)$ with closed image.*

Proof This follows from Theorem 5.6, Remark 5.7, and Proposition 5.25 below. \square

Our goal is now to state and prove Proposition 5.25, which says that the inclusion of H into $\mathcal{MCG}(S)$ is a slanted hieromorphism. We need control over various projections, which we achieve in the following preliminary lemmas.

Lemma 5.21 *There exists a constant Q such that $\pi_S(h^k G(q_i) h^{-k})$ is Q -quasiconvex for any i and any k .*

Proof Apply quasiconvexity of the $\pi_S(G(q_i))$ and boundedness of $\{\pi_S(1, h^k)\}_{k \in \mathbb{Z}}$. \square

Denote by \mathcal{Y} the set of connected components in S of the complement of the annuli in the annular decomposition of the multitwists in G_0 .

Lemma 5.22 *There exists K such that for any Y transverse to some $Y_0 \in \mathcal{Y}$ we have $d_Y(\rho_{Y_0}^Y, 1) \leq K$.*

Proof This is because $\rho_{Y_0}^Y$ coarsely coincides with $\pi_Y(P_{Y_0})$, and the fact that π_Y is coarsely Lipschitz (note that there are finitely many Y_0). □

Lemma 5.23 *For each $g \in G(q_i) - G_0$ for some i and each $Y \in \mathcal{Y}$, there exists $Y' \in \mathcal{Y}$ such that $g \cdot Y'$ is transverse to Y .*

Proof This is a restatement of [52, Lemma 4.1]. □

Lemma 5.24 *There exist C and M with the following property. For any $g = g_1 h^{m_1} \dots g_k h^{m_k}$ with $g_i \in G(q_{j(i)}) - G_0$ and $|m_i| \geq M$ for each $i \leq k$, we have $d_{Y_0}(1, g) \leq C$ for each $Y_0 \in \mathcal{Y}$.*

Proof Let K be as in Lemma 5.22. Proceed by induction on k , with C to be determined. If $k = 0$, there is nothing to prove.

Suppose $k \geq 1$. Fix $Y_0 \in \mathcal{Y}$ and let $Y = g_1 Y'$ with $Y' \in \mathcal{Y}$ chosen via Lemma 5.23, so that $Y' \pitchfork Y_0$. By induction, $d_Y(g_1 h^{m_1}, g) = d_{Y'}(1, g_2 h^{m_2} \dots g_k h^{m_k}) \leq C$, since $hY = Y$ for any $Y \in \mathcal{Y}$ by hypothesis, so that $g_1 h^{m_1} \cdot Y' = g_1 \cdot Y' = Y$.

By Lemma 5.16, $d_Y(1, g_1)$ is uniformly bounded by some V . Hence $d_Y(1, g) \geq d_Y(g_1, g_1 h^{m_1}) - C - V = d_{Y'}(1, h^{m_1}) - C - V$. If $|m_1|$ is large enough, then this quantity is larger than $K + 10E$. Since $Y_0 \pitchfork Y$, consistency implies that we have $d_{Y_0}(\rho_{Y_0}^Y, g) \leq E$. Also,

$$d_{Y_0}(\rho_{Y_0}^Y, 1) \leq d_{Y_0}(\rho_{Y_0}^Y, g_1) + V = d_{g_1^{-1}Y_0}(\rho_{g_1^{-1}Y_0}^{Y'}, 1) + V \leq V + K,$$

hence $d_{Y_0}(1, g) \leq 2E + V + K$. Thus we set $C = 2E + V + K$, which determines M . □

Proposition 5.25 *The subgroup $H \leq \mathcal{MCG}(S)$ admits a hierarchically hyperbolic space structure (H, \mathfrak{S}_H) such that there is an extensible slanted hieromorphism $(H, \mathfrak{S}_H) \rightarrow (\mathcal{MCG}(S), \mathfrak{S})$ induced by the inclusion $H \hookrightarrow \mathcal{MCG}(S)$.*

Proof We follow a very similar procedure to that used for individual Veech subgroups. In particular, \mathfrak{S}_H is defined exactly as \mathfrak{S}_G was, except that there is now a domain U_g for each primitive multitwist in H . To verify that this yields an HHS structure, we must check that:

- (1) $\pi_S(H)$ is quasiconvex.
- (2) $\pi_U(H)$ is uniformly bounded unless $U \in A_g$ for some $g \in H$.

Once the properties above are proven, arguing exactly as in the proof of [Lemma 5.14](#) and [Lemma 5.17](#) yields the desired slanted hieromorphism and completes the proof.

We now set conventions and notations that we use throughout the proof. When some $g = g_1 \cdots g_k \in H$ with $g_i \in h^{k_{j(i)}}G(q_{j(i)})h^{-k_{j(i)}} - G_0$ is any fixed element of H , we write $p_l = \pi_S(g_1 \cdots g_l)$ (with $p_0 = \pi_S(1)$), and let γ_l be a geodesic in \mathcal{CS} from p_{l-1} to p_l , so that the concatenation of the γ_l is a path from $\pi_S(1)$ to $\pi_S(g)$. Furthermore, notice that we can write $g = h^{m_0}g'_1h^{m_1} \cdots g'_kh^{m_k}$ for some $g'_i \in G(q_{j(i)}) - G_0$ (more specifically, $g'_i = h^{-k_{j(i)}}g_ih^{k_{j(i)}}$), and that $|m_l|$ for $l < k$ is bounded below by N (recall that this is the minimal value of $|k_i - k_j|$ for $i \neq j$). We set $h_l = h^{m_0}g'_1h^{m_1} \cdots g'_l$.

In the following claim, we study geodesics connecting $\pi_S(1)$ to $\pi_S(g)$ for arbitrary $g \in G$. The claim easily implies that geodesics from $\pi_S(1)$ to $\pi_S(g)$ stay close to $\pi_S(H)$ for any $g \in H$ because each γ_l is contained in a coset of some $h^{k_{j(i)}}G(q_{j(i)})h^{-k_{j(i)}}$ and such cosets are uniformly quasiconvex by [Lemma 5.21](#). Hence, the claim proves that $\pi_S(H)$ is quasiconvex, which is item (1) above.

Claim 2 There exists a constant R with the following property. For any $g \in H$, the Hausdorff distance between $\bigcup_l \gamma_l$ and $[\pi_S(1), \pi_S(g)]$ is bounded by R , where $[\pi_S(1), \pi_S(g)]$ is any geodesic in \mathcal{CS} from $\pi_S(1)$ to $\pi_S(g)$. Moreover, for any $Y \in \mathcal{Y}$ we have that $d_{h_l Y}(1, h_l), d_{h_l Y}(g, h_l h^{m_l}) \leq C$.

Proof We first show $\bigcup_l \gamma_l$ is uniformly close to $[\pi_S(1), \pi_S(g)]$.

It suffices to show that the endpoints of all γ_l lie within controlled distance of $[\pi_S(1), \pi_S(g)]$. Any such endpoint x coarsely coincides with both $\pi_S(h_l)$ and $\pi_S(h_l h^{m_l})$, for some l (since $\{\pi_S(h^m)\}_{m \in \mathbb{Z}}$ is a bounded set). Pick any $Y \in \mathcal{Y}$, and set $Z = h_l \cdot Y$. By [Lemma 5.24](#) we have $d_Z(h_l h^{m_l}, g) \leq C$ and $d_Z(1, h_l) \leq C$. Hence, if m_l is large enough, we get $d_Z(1, g) \geq d_Y(1, h^{m_l}) - 2C \geq 100E$. Notice that by bounded geodesic image ρ_S^Z needs to be within $10E$ of geodesics from $\pi_S(h_l)$ and $\pi_S(h_l h^{m_l})$, which both coarsely coincide with the endpoint x we are interested in. If geodesics from $\pi_S(1)$ to $\pi_S(g)$ did not pass close to x we could then conclude that they do not pass close to ρ_S^Z , which would imply by bounded geodesic image that $d_Z(1, g) \leq 5E$. But this is not the case, and hence we get a bound on the distance from x to $[\pi_S(1), \pi_S(g)]$, as required.

Let us now prove that points on $[\pi_S(1), \pi_S(g)]$ are close to $\bigcup_l \gamma_l$. Suppose by contradiction that there exists $x \in [\pi_S(1), \pi_S(g)]$ with $d_S(x, \bigcup_l \gamma_l) \geq 2C + 1$. Let

$x_1, x_2 \in [\pi_S(1), \pi_S(g)]$ lie on distinct sides of x (in the natural order of $[\pi_S(1), \pi_S(g)]$) with x_1 closer to $\pi_S(1)$ than x , and satisfy $d_S(x_i, x) = C + 1$. Then any $y \in \bigcup \gamma_l$ lies in $\mathcal{N}_C([\pi_S(1), x_1]) \cup \mathcal{N}_C([x_2, \pi_S(g)])$. However, the two neighborhoods are disjoint and the connected set $\bigcup \gamma_l$ contains points in both, a contradiction. \triangleleft

Let us now take $U \in \mathfrak{S} - \{S\}$ and $g \in H$ with $d_U(1, g) \geq 100E$. We need to show that either U belongs to some $A_{g'}$ or $d_U(1, g)$ is bounded independently of U and g .

We proved in the claim that, for any $Y \in \mathcal{Y}$, the projections of 1 and g on $h_l \cdot Y$ coarsely coincide with the projections of h_l and $h_l h^{m_l}$, respectively, and hence that $d_{h_l \cdot Y}(1, g) > 100E$ if $|m_l| \geq N$ is large enough. Since m_l can take finitely many values, we therefore get the desired bound whenever U is of the form $h_l \cdot Y$. We now assume that U is neither belongs to some $A_{g'}$ nor it is of the form $h_l \cdot Y$. Hence, for any l there exists Y such that $h_l \cdot Y \pitchfork U$ overlap, and hence are comparable in the partial order \preceq ; see Proposition 2.8 of [6].

Another fact about \preceq is that whenever $Y, Y' \in \mathcal{Y}$ and l are such that $h_l \cdot Y \pitchfork h_{l+1} \cdot Y'$, we have $h_l \cdot Y \preceq h_{l+1} \cdot Y'$, again provided $|m_l| \geq N$ is large enough. In fact,

$$\rho_{h_l Y}^{h_{l+1} Y'} = h_{l+1} \rho_{h_{l+1}^{-1} h_l Y'}^Y$$

coarsely coincides with $\pi_{h_l \cdot Y}(h_{l+1})$ (Lemma 5.22), which in turn coarsely coincides with $\pi_{h_l \cdot Y}(h_l h^{m_l})$ by Lemma 5.16 since $h_{l+1} = h_l h^{m_l} g'_{l+1}$. Finally, $\pi_{h_l \cdot Y}(h_l h^{m_l})$ coarsely coincides with $\pi_{h_l \cdot Y}(g)$ by what we said above.

By looking at a predecessor and a successor of U , we then see that the projections of $1, g$ onto U coarsely coincide with those of $h_l \cdot Y, h_{l+1} \cdot Y'$ for some l and Y and Y' . But these latter projections coarsely coincide with those of h_l and $h_l h^{m_l} g'_{l+1}$. The projections of h_l and $h_l h^{m_l}$ are uniformly close by boundedness of m_l , while the projections of $h_l h^{m_l}$ and $h_l h^{m_l} g'_{l+1}$ are uniformly close by Lemma 5.16. This concludes the proof. \square

6 Automorphisms of HHSs and their actions on the boundary

The most important special case of an extensible hieromorphism is an automorphism of $(\mathcal{X}, \mathfrak{S})$. For any automorphism $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}, \mathfrak{S})$, each isometry $f: \mathcal{CU} \rightarrow \mathcal{C}(f(U))$ extends to a homeomorphism $\hat{f}: \partial \mathcal{CU} \rightarrow \partial \mathcal{C}(f(U))$, yielding an application of Theorem 5.6:

Corollary 6.1 (extensions of automorphisms to the boundary) *Any $f \in \text{Aut}(\mathfrak{S})$ extends to a bijection $\bar{\mathcal{X}} \rightarrow \bar{\mathcal{X}}$ which restricts to a homeomorphism on $\partial \mathcal{X}$.*

Proof Let $f: (\mathcal{X}, \mathfrak{S}) \rightarrow (\mathcal{X}, \mathfrak{S})$ be an automorphism. Let $p \in \partial\mathcal{X}$, with $p = \sum_{i=1}^n a_{T_i}^p p_{T_i}$, where the T_i are pairwise orthogonal and $p_{T_i} \in \partial CT_i$. Define a map $\hat{f}: \partial\mathcal{X} \rightarrow \partial\mathcal{X}$ by

$$\hat{f}(p) = \sum_{i=1}^n a_{T_i}^p \hat{f}(p_{T_i}),$$

where $\hat{f}: \partial CT_i \rightarrow \partial\mathcal{C}(f(T_i))$ is induced by $f: CT_i \rightarrow CT_i$. Let $\bar{f}: \bar{\mathcal{X}} \rightarrow \bar{\mathcal{X}}$ be the extension of f that is \hat{f} on $\partial\mathcal{X}$; extend f^{-1} similarly. Since f is an automorphism, \bar{f} is clearly a bijection. Continuity of \bar{f} and \bar{f}^{-1} on the boundary follows from Theorem 5.6. \square

When (G, \mathfrak{S}) is a hierarchically hyperbolic group, ∂G is defined. In general, if \mathcal{X} and \mathcal{X}' are hierarchically hyperbolic with respect to the same collection \mathfrak{S} , then there is a quasiisometry $\mathcal{X} \rightarrow \mathcal{X}'$ extending to the identity on the boundary. Indeed, the definition of $\partial\mathcal{X}$ depends only on \mathfrak{S} and the attendant hyperbolic spaces.

Corollary 6.2 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group. Then the action of G on itself by left multiplication extends to an action of G on \bar{G} by homeomorphisms.*

Section 6.1 is devoted to automorphisms, whose fixed points in $\partial\mathcal{X}$ we study in Section 6.2.

6.1 Classification of HHS automorphisms

In this subsection, we will classify HHS automorphisms by their actions on \mathfrak{S} . Let $g \in \text{Aut}(\mathfrak{S})$ and fix a basepoint $X \in \mathcal{X}$. Set

$$\text{Big}(g) = \{U \in \mathfrak{S} \mid \text{diam}_{\mathcal{C}U}(\langle g \rangle \cdot X) \text{ is unbounded}\}.$$

Observe that $g \cdot U \in \text{Big}(g)$ if $U \in \text{Big}(g)$, since $g: \mathcal{C}U \rightarrow \mathcal{C}(gU)$ is an isometry.

Lemma 6.3 *There exists $M = M(\mathfrak{S}) > 0$ such that for all $g \in \text{Aut}(\mathfrak{S})$ and $U \in \text{Big}(g)$, we have $g^M \cdot U = U$.*

Proof Consider the orbit $\langle g \rangle \cdot U$ in \mathfrak{S} .

If there exists $n \geq 1$ such that $g^n \cdot U \sqsubset U$, then $g^{kn} \cdot U \sqsubset g^{(k-1)n} \cdot U \sqsubset \dots \sqsubset g^n \cdot U \sqsubset U$ for all $k \geq 1$, so we either contradict finite complexity (if $\langle g \rangle \cdot U$ is infinite) or the fact that \sqsubset is a partial order (if $\langle g \rangle \cdot U$ is finite). Hence $g^n \cdot U \not\sqsubset U$ unless $n = 0$. Similarly, $U \not\sqsupseteq g^n U$ unless $n = 0$.

Next, consider the case where $U \in \text{Big}(g)$ and $g^n \cdot U \pitchfork U$ for some $n \geq 1$. Then, since $U \in \text{Big}(g)$, we can choose arbitrarily large $m \in \mathbb{N}$ such that $d_U(X, g^m \cdot X) >$

$T = 100E + d_U(g^{-1} \cdot X, X) + f(m)$, where $f: \mathbb{N} \rightarrow \mathbb{N}$ is increasing. Hence $d_{g^n U}(g^{m+1} \cdot X, g \cdot X) > T$, since $g: \mathcal{C}U \rightarrow \mathcal{C}gU$ is an isometry. The triangle inequality shows that $d_{g^n U}(g^M \cdot X, X) > T - 2d_{gU}(X, g^n \cdot X) = 100E + f(m)$. By considering at least two such values of m , we see that consistency is contradicted (specifically, we contradict Lemma 2.3 of [6]).

It follows that if $U \in \text{Big}(g)$, then, for all $n \in \mathbb{Z}$, either $g^n \cdot U = U$ or $g^n \cdot U \perp U$. Hence $\langle g \rangle \cdot U$ is a pairwise-orthogonal collection. Hence there exists a global M , depending only on the complexity and Lemma 1.4, such that $g^M \cdot U = U$ for each $U \in \text{Big}(g)$, establishing the first assertion. \square

Proposition 6.4 *The automorphism $g \in \text{Aut}(\mathfrak{S})$ is elliptic if and only if $\text{Big}(g) = \emptyset$.*

Proof If $\langle g \rangle \cdot X$ is bounded, then $\text{Big}(g) = \emptyset$ since projections are coarsely Lipschitz. Conversely, suppose that $\text{Big}(g) = \emptyset$. We will show that there exists $D = D(g)$ such that $\text{diam}_V(\pi_V(\langle g \rangle \cdot X)) \leq D$ for all $V \in \mathfrak{S}$. From this and the distance formula (Theorem 1.9), it follows that g is elliptic. Hence suppose that no such D exists.

We need two facts:

- (a) For each $N \geq 0$, there exists $P = P(N, \mathfrak{S})$ such that for all $U \in \mathfrak{S}$ and $h \in \text{Aut}(\mathfrak{S})$, either some positive power of h fixes U or $\{U, g \cdot U, \dots, g^P \cdot U\}$ contains a set of N pairwise-transverse elements. Indeed, as in the proof of Lemma 6.3, for any p , the elements of $\{U, g \cdot U, \dots, g^{p-1} \cdot U\}$ are pairwise \sqsubseteq -incomparable, and any pairwise-orthogonal subset has cardinality bounded by the complexity χ of \mathfrak{S} . Hence, if p exceeds the Ramsey number $\text{Ram}(\chi + 1, N)$, we have by Ramsey’s theorem that $\{U, g \cdot U, \dots, g^{p-1} \cdot U\}$ contains a set of N pairwise-transverse elements, so we can take $P = \text{Ram}(\chi + 1, N) - 1$.
- (b) For each $C \geq 0$ there exists $Q \in \mathbb{N}$ with the following property. Let $x, y \in \mathcal{X}$ and suppose $\{V_i\}_{i \in I}$ satisfies $d_{V_i}(x, y) > E$ for all i , and that $|I| \geq Q$. Then there exists $V \in \mathfrak{S}$ such that $V_i \sqsubset V$ for some $i \in I$, and $d_V(x, y) > C$. This is a slight strengthening of Lemma 3.2; this exact statement is [7, Lemma 1.8].

Recall that χ denotes the complexity — ie the maximum level — in \mathfrak{S} , so that S is the unique element of level χ . Since $\text{Big}(g) = \emptyset$ but there are arbitrarily large projections, by assumption, there exists a level $\ell < \chi$ and a constant $R < \infty$ such that:

- $\text{diam}_U(\pi_U(\langle g \rangle \cdot X)) \leq R$ when U has level greater than ℓ .
- For each $D < \infty$, there exists $U \in \mathfrak{S}$, of level ℓ , with $\text{diam}_U(\pi_U(\langle g \rangle \cdot X)) > D$.

Let $U \in \mathfrak{S}$ be chosen so that $d_U(X, g^n \cdot U) > \ddot{R}$, where \ddot{R} is a constant to be determined. We can and shall assume that our U has been chosen at level ℓ , and we emphasize that such a U can be chosen so as to make \ddot{R} arbitrarily large.

Let $Q = Q(R)$ be the constant provided by setting $C = R$ in fact (b) and let $P = \text{Ram}(\chi + 1, Q)$. Fact (a) provides $U_1, \dots, U_Q \in \{U, g \cdot U, \dots, g^P \cdot U\}$ such that $U_i \pitchfork U_j$ when $i \neq j$. Now, for $1 \leq j \leq Q$, we have $d_{U_j}(X, g^n \cdot X) \geq \ddot{R} - 100KEQ$. So, provided \ddot{R} — which can be chosen *independently* of R and hence of Q — satisfies $\ddot{R} > 100KEQ + 10E$, fact (b) provides $T \in \mathfrak{S}$ such that $U_j \not\sqsubseteq T$ for some j and such that $d_T(X, g^n \cdot X) > R$. Now, since U_j is a translate of U and $\text{Aut}(\mathfrak{S})$ preserves the levels, the level of U_j is ℓ , and hence T has level strictly greater than ℓ , which is a contradiction since $d_T(X, g^n \cdot X) > R$. \square

Remark 6.5 In the case where \mathcal{X} is proper, there is a quick proof of Proposition 6.4 relying on the more powerful tools from Section 9.

Lemma 6.6 *Let $g \in \text{Aut}(\mathfrak{S})$. Then there exists $D = D(g, E)$ such that*

$$\text{diam}_U(\pi_U(\langle g \rangle \cdot X)) \leq D$$

for all $U \in \mathfrak{S} - \text{Big}(g)$.

Proof Let $\text{Big}(g) = \{U_i\}_{i \in I}$. Note that it suffices to prove the lemma for some positive power of g , so by Lemma 6.3, we may assume that $g \cdot U_i = U_i$ for all $i \in I$.

If $\text{Big}(g) = \emptyset$, then g is elliptic by Proposition 6.4, from which the lemma follows immediately: for each $V \in \mathfrak{S}$, we have $\text{diam}_V(\pi_V(\langle g \rangle \cdot X)) \leq K \text{diam}_{\mathcal{X}}(\langle g \rangle \cdot X)$, which is bounded independently of V .

Next, suppose that $\text{Big}(g) \neq \emptyset$ and $S \notin \text{Big}(g)$ (as usual, $S \in \mathfrak{S}$ is the unique \sqsubseteq -maximal element). Then, for each $i \in I$, the element U_i is maximal in an HHS $(F_{U_i}, \mathfrak{S}_{U_i})$ admitting a g -equivariant hieromorphism to $(\mathcal{X}, \mathfrak{S})$. Since $U_i \neq S$, the complexity of $(F_{U_i}, \mathfrak{S}_{U_i})$ is strictly lower than that of $(\mathcal{X}, \mathfrak{S})$, so it follows by induction that $\text{diam}_V(\pi_V(\langle g \rangle \cdot X))$ is bounded independently of V when $V \sqsubseteq U_i$. Indeed, in the base case, when the complexity is 1, \mathcal{X} is itself a hyperbolic space and the lemma follows from the usual elliptic/parabolic/loxodromic classification of isometries of hyperbolic spaces [35].

Now, let \mathfrak{T} be the set of all $U \in \mathfrak{S}$ such that $U \sqsubseteq U_i$ for some $i \in I$. Observe that \mathfrak{T} is g -invariant and downward-closed under nesting. Then Proposition 2.4 of [7] provides an HHS $(\hat{\mathcal{X}}_{\mathfrak{T}}, \mathfrak{S} - \mathfrak{T})$ with the same associated nesting and orthogonality relations, hyperbolic spaces, and projections. Since \mathfrak{T} was g -invariant, g descends to an automorphism of $(\hat{\mathcal{X}}_{\mathfrak{T}}, \mathfrak{S} - \mathfrak{T})$ such that the action of g on $\mathfrak{S} - \mathfrak{T}$ is the restriction of the original action on \mathfrak{S} and, for each $V \in \mathfrak{S} - \mathfrak{T}$, the isometry $\mathcal{C}V \rightarrow \mathcal{C}gV$ is the original one. Now g has $\text{Big}(g) = \emptyset$ with respect to $(\hat{\mathcal{X}}_{\mathfrak{T}}, \mathfrak{S} - \mathfrak{T})$ and hence we are done by the proof of Proposition 6.4.

The preceding two analyses prove the lemma except in the case where $S \in \text{Big}(g)$. Hence, suppose $S \in \text{Big}(g)$, so that g acts either loxodromically or parabolically on \mathcal{CS} . In this case, we cannot induct on complexity, so we argue directly using consistency, bounded geodesic image, and simple properties of isometries of hyperbolic spaces.

If $U \in \mathfrak{S} - \{S\}$, then $U \sqsubset S$, and $\rho_S^U \subset \mathcal{CS}$ is a well-defined subset of diameter $\leq E$.

First suppose that g acts loxodromically on \mathcal{CS} . Then there exists $N = N(g)$ such that $\leq N$ elements of $\pi_S(\langle g \rangle \cdot X)$ lie in the $100E$ -neighborhood of ρ_S^U . Let $\{g^i \cdot X\}_{i=n}^{n'}$ be the points in $\langle g \rangle \cdot X \subset \mathcal{X}$ projecting into $\mathcal{N}_{100E}^S(\rho_S^U) \subset \mathcal{CS}$, so that $n' - n \leq N$. Then for all $i, j \in \mathbb{Z}$, consistency and bounded geodesic image imply that

$$\begin{aligned} d_U(g^i \cdot X, g^j \cdot X) &\leq E + \max_{n \leq k, k' \leq n'} d_S(g^k \cdot X, g^{k'} \cdot X) \\ &\leq E + \max_{0 \leq k, k' \leq N} \text{Kd}_{\mathcal{X}}(g^k \cdot X, g^{k'} \cdot X) + K, \end{aligned}$$

which is independent of U (here K is the coarse Lipschitz constant from Definition 1.1).

Next, suppose that g acts parabolically on \mathcal{CS} . By definition, $\langle g \rangle \cdot X$ has a unique limit point in the Gromov boundary of \mathcal{CS} , so there is an increasing function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $(g^n \cdot \pi_S(X) | g^m \cdot \pi_S(X))_{\pi_S(X)} > f(k)$ whenever $\min\{|m|, |n|\} \geq k$. In particular, there exists k , independent of U , such that no \mathcal{CS} -geodesic from $\pi_S(g^n \cdot X)$ to $\pi_S(g^m \cdot X)$ passes $100E$ -close to ρ_S^U provided $|m| \geq k$ and $|n| \geq k$. We now argue exactly as in the loxodromic case to bound $\text{diam}_U(\pi_U(\langle g \rangle \cdot X))$ independently of U . This completes the proof. □

Lemma 6.7 For any distinct $U, V \in \text{Big}(g)$, we have $U \perp V$.

Proof Lemma 6.3 shows that by passing to a uniformly bounded power, if necessary (which does not affect the big-set), we can assume that $gU = U$ and $gV = V$. Hence g acts as an isometry of both of the (not necessarily proper) hyperbolic spaces $\mathcal{CU}, \mathcal{CV}$. Since $U, V \in \text{Big}(g)$, the isometry g cannot be elliptic on either \mathcal{CU} or \mathcal{CV} . Hence, by eg [35, Section 8.1], g is either parabolic or loxodromic on \mathcal{CU} and \mathcal{CV} .

If $U \sqsubset V$ or $U \pitchfork V$, then ρ_V^U is a uniformly bounded subset of \mathcal{CV} , and, since $g^n \cdot \rho_V^U \simeq \rho_{g^n V}^{g^n U} = \rho_V^U$ for all $n \in \mathbb{Z}$, we have that $\langle g \rangle$ -orbits in \mathcal{CV} are bounded, contradicting that $U \in \text{Big}(g)$. □

Definition 6.8 (elliptic) An automorphism $g \in \text{Aut}(\mathfrak{S})$ is *elliptic* if some (hence any) orbit of $\langle g \rangle$ in \mathcal{X} is bounded.

Definition 6.9 (axial) An automorphism $g \in \text{Aut}(\mathfrak{S})$ is *axial* if some (hence any) orbit of $\langle g \rangle$ in \mathcal{X} is quasiisometrically embedded.

Definition 6.10 (distorted) An element $g \in \text{Aut}(\mathfrak{S})$ is *distorted* if it is not elliptic or axial.

Example 6.11 (distorted automorphisms in familiar examples) Let S be a surface of finite type and α a simple closed curve. In $\mathcal{MCG}(S)$, the subgroup $\langle \tau_\alpha \rangle$ generated by the Dehn twist about α is quasiisometrically embedded [30], but in $(\mathcal{T}(S), d_T)$, the orbit of τ_α is distorted. In fact, $\mathcal{MCG}(S)$ has no distorted automorphisms, as is the case for cube complexes with factor systems, since cubical automorphisms are combinatorially semisimple [39]. In Theorem 7.1 below, we prove that HHGs have no distorted elements. A simple example of an HHS with a distorted automorphism is obtained by gluing a combinatorial horoball to \mathbb{Z} ; this encapsulates the difference between the HHS structures of $\mathcal{MCG}(S)$ and $(\mathcal{T}(S), d_T)$, where annular curve graphs are replaced by horoballs over annular curve graphs.

Proposition 6.12 The automorphism $g \in \text{Aut}(\mathfrak{S})$ is axial if and only if there exists $U \in \text{Big}(g)$ such that $n \rightarrow g^n \cdot \pi_U(X)$ is a quasiisometric embedding $\mathbb{Z} \rightarrow \mathcal{CU}$ for any $X \in \mathcal{X}$.

Proof Suppose that there exists $U \in \text{Big}(g)$ such that $n \rightarrow g^n \cdot \pi_U(X)$ is a quasiisometric embedding. Then the distance formula (Theorem 1.9) yields a lower bound on $d_{\mathcal{X}}(g^m \cdot X, g^n \cdot X)$ which is (at least) linear in $|m - n|$, ie g is axial.

Conversely, suppose that g is axial. Lemma 6.7 bounds the number of $U \in \text{Big}(g)$ by the complexity of \mathfrak{S} . Lemma 6.6 ensures that $\text{diam}_V(\pi_V(\langle g \rangle \cdot X))$ is bounded independently of V for $V \notin \text{Big}(g)$. Since g acts axially on \mathcal{X} , the distance formula (Theorem 1.9) now implies that there exists at least one $U \in \text{Big}(g)$ such that g acts axially on \mathcal{CU} . □

The next proposition is an immediate consequence of Propositions 6.4 and 6.12:

Proposition 6.13 The automorphism $g \in \text{Aut}(\mathfrak{S})$ is distorted if and only if there exists $U \in \text{Big}(g)$ such that $\langle g \rangle \cdot \pi_U(X)$ is unbounded but, for all $U \in \text{Big}(g)$, we have

$$d_{\mathcal{CU}}(X, g^n \cdot X) = o(n).$$

Definition 6.14 (reducible) The automorphism $g \in \text{Aut}(\mathfrak{S})$ is *irreducible* if $\text{Big}(g) = \{S\}$, where $S \in \mathfrak{S}$ is the unique \sqsubseteq -maximal element. Otherwise, $S \notin \text{Big}(g)$ and g is *reducible*.

Finally, we have the following strong characterization of irreducible axials:

Theorem 6.15 Let $G \leq \text{Aut}(\mathfrak{S})$ act properly and coboundedly on the hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$. Suppose that $g \in G$ is irreducible axial. Then g is Morse.

Proof By [5, Corollary 14.4], G acts acylindrically on \mathcal{CS} , where S is \sqsubseteq -maximal in \mathfrak{S} , while g acts hyperbolically on \mathcal{CS} . By [70, Proposition 3.8], g is *weakly contracting* for the *path system* consisting of all geodesics in \mathcal{CS} , so g is Morse, by [70, Lemma 2.9]. □

Remark 6.16 (reducible Morse elements) The converse of [Theorem 6.15](#) does not hold, as can be seen by examining a Morse element of an appropriately chosen right-angled Artin group whose support does not include all generators.

6.2 Dynamics of action on the boundary

In the remainder of this section, we impose the standing assumption that \mathcal{X} is proper. We will analyze the action of an infinite-order automorphism g on $\partial(\mathcal{X}, \mathfrak{S})$, according to whether g is irreducible or reducible and according to whether g is axial or distorted.

6.2.1 Irreducible automorphisms

Lemma 6.17 *Let the irreducible $g \in \text{Aut}(\mathfrak{S})$ fix some $\lambda \in \partial\mathcal{X}$. Then $\text{Supp}(\lambda) = \{S\}$.*

Proof Suppose $U \in \text{Supp}(\lambda) - \{S\}$. Since g is irreducible, its orbit in \mathcal{CS} is unbounded. In particular, this means that the orbit of ρ_S^U is unbounded. By definition, $g \cdot \rho_S^U \asymp \rho_S^{g \cdot U}$ and thus U could not be fixed by g , completing the proof. □

Proposition 6.18 (irreducible axials act with north–south dynamics) *If $g \in \text{Aut}(\mathfrak{S})$ is irreducible axial, then g has exactly two fixed points $\lambda_+, \lambda_- \in \partial\mathcal{X}$. Moreover, for any boundary neighborhoods $\lambda_+ \in U_+$ and $\lambda_- \in U_-$, there exists an $N > 0$ such that $g^N(\partial\mathcal{X} - U_-) \subset U_+$.*

Proof Let $g \in \text{Aut}(\mathfrak{S})$ be irreducible axial. For the rest of the proof, fix a basepoint $X \in \mathcal{X}$.

Existence of $\lambda_+, \lambda_- \in \partial\mathcal{X}$ For any n , let $X_n = g^n \cdot X$. We will show that (X_n) converges to some point in $\partial\mathcal{X}$; a similar argument will show that (X_{-n}) converges to some other point, and then we will prove they are distinct. By compactness ([Theorem 3.4](#)), there exists a subsequence $(X_{n_k}) \subset (X_n)$ which converges to some point $\lambda_+ \in \partial\mathcal{X}$. By irreducibility of g , we must have that $\lambda_+ \in \partial\mathcal{CS} \subset \partial\mathcal{X}$. By irreducibility and the definition of convergence, we have that $\pi_{\mathcal{CS}}(X_{n_k}) \rightarrow \lambda_+ \in \partial\mathcal{CS}$. Axiality of g then implies that, for any other subsequence $(X_{n_l}) \subset (X_n)$, the Gromov product $(X_{n_k}, X_{n_l})_X \rightarrow \infty$ in \mathcal{CS} as $k, l \rightarrow \infty$. This implies that $\pi_{\mathcal{CS}}(X_n) \rightarrow \lambda_+ \in \partial\mathcal{CS}$, which implies that $X_n \rightarrow \lambda_+ \in \partial\mathcal{X}$.

Similarly, we define $X_{-n} \rightarrow \lambda_- \in \partial\mathcal{X}$. Observe that $(\pi_{\mathcal{CS}}(X_n), \pi_{\mathcal{CS}}(X_{-n}))_{\pi_{\mathcal{CS}}(X)}$ is uniformly bounded by Proposition 6.12, implying that $\lambda_+ \neq \lambda_-$. Since g stabilizes the orbit, it obviously fixes λ_+ and λ_- . Note that λ_+ and λ_- are independent of our choice of $X \in \mathcal{X}$.

Uniqueness of $\lambda_+, \lambda_- \in \partial\mathcal{X}$ By Lemma 6.17, any point $\lambda \in \partial\mathcal{X}$ fixed by g has $\text{Supp}(\lambda) = S$. If g fixes three points in $\partial\mathcal{X}$, then it fixes three points in $\partial\mathcal{CS}$. As such, g coarsely fixes the coarse median of those points, producing a bounded orbit, a contradiction.

North–south dynamics on $\partial\mathcal{X}$ Fix boundary neighborhoods $\lambda_+ \in U_+$ and $\lambda_- \in U_-$ with $U_+ \cap U_- = \emptyset$.

Claim 1 For any $p \in \partial\mathcal{X} - \{\lambda_-\}$, the sequence $(g^n(p))$ does not converge to λ_- .

Proof of Claim 1 If $\text{Supp}(p) \neq \{S\}$, then $(g^n(p))$ cannot converge to a point in $\partial\mathcal{X}$ supported on S , as g does not alter the coefficients of the pieces of p supported on proper subdomains. In particular, since $\text{Supp}(\lambda_-) = \{S\}$, as shown above, $(g^n(p))$ cannot converge to λ_- . Thus we may assume that $\text{Supp}(p) = \{S\}$.

Let $[X, p]$ be a hierarchy ray in $\bar{\mathcal{X}}$. Since $\text{Supp}(p) = \{S\}$, $[X, p]$ projects to a D -quasigeodesic, $[X, p]_S \subset \bar{\mathcal{CS}}$. Let $[X, \lambda_-]$ be the orbit $(g^{-n}(X))$, which is a quasigeodesic with quality depending on g .

Consider $m \in \mathcal{CS}$, the coarse median of (λ_-, p, X) . By hyperbolicity, there exist points $Y \in [X, p]_S$ and $Z \in [X, \lambda_-]$ sufficiently far out along $[X, p]_S$ and $[X, \lambda_-]$ such that any geodesic $[Y, Z]$ between Y and Z comes uniformly close to m , independent of Y and Z ; in particular, the coarse median of (X, Y, Z) is uniformly close to m . Moreover, there is a uniform constant $\delta' > 0$ (depending on D, g , and the hyperbolicity constant, $\delta > 0$) such that each of $[Y, Z], [X, Y]$, and $[X, Z]$ is δ' -close to m .

Let $m_{Y,Z} \in [Y, Z]$ and $m_{X,Z} \in [X, Z]$ be points δ' -close to m . Then there exists a uniform $\delta'' > 0$ such that $[m_{Y,Z}, Z]$ and $[m_{X,Z}, Z]$ must δ'' -fellow-travel. By axiomaticity, there exists $N > 0$ such that, for all $n > N$, $g^n(m_{X,Z})$ is between X and $g^n(X)$ along the quasigeodesic axis of g in \mathcal{CS} . This implies that the coarse median of $(X, g^n(Y), g^n(Z))$ is uniformly close to X . Thus $(g^n(p), \lambda_-)_X$ is uniformly bounded and $(g^n(p))$ cannot converge to λ_- in $\partial\mathcal{CS}$ and thus not in $\partial\mathcal{X}$ as well. \triangleleft

Since the limit of $(g^n(p))$ is a fixed point, uniqueness of λ_-, λ_+ and Claim 1 imply that $g^n(p) \rightarrow \lambda_+$ for any $p \in \partial\mathcal{X} - \{\lambda_-\}$.

Now consider the function $f: \partial\mathcal{X} - U_- \rightarrow \mathbb{N}$, where $f(p)$ is the least power N_p such that $g^{N_p}(p) \in U_+$. Since λ_+ and λ_- are the unique fixed points of g , such a power

exists, otherwise the sequence $(g^n(p)) \subset \partial\mathcal{X}$ would subconverge to another fixed point. Since $\partial\mathcal{X}$ is compact (Theorem 3.4) the function f attains a maximum, N_f . By definition, $g^{N_f}(\partial\mathcal{X} - U_-) \subset U_+$, completing the proof. \square

We now treat the irreducible distorted case:

Proposition 6.19 (irreducible distorteds act parabolically) *If $g \in \text{Aut}(\mathfrak{S})$ is irreducible distorted, then g has exactly one fixed point $\lambda_g \in \partial\mathcal{X}$, and $g^n \cdot X, g^{-n} \cdot X \rightarrow \lambda_g$ for any $X \in \bar{\mathcal{X}}$.*

Proof Let $S \in \mathfrak{S}$ be the unique \sqsubseteq -maximal element, so that $gS = S$ and $g: CS \rightarrow CS$ is an isometry. By the definition of irreducibility, $\text{Big}(g) = \{S\}$, so g has unbounded orbits in the δ -hyperbolic space CS . We now apply the classification of isometries of hyperbolic spaces, as summarized in [17, Section 3], emphasizing that these results do not rely on properness of the space in question.

First, by Proposition 3.2 of [17] and the fact that $\langle g \rangle \cdot \pi_X(X)$ (which coarsely coincides with $\pi_S(\langle g \rangle \cdot X)$) is distorted — ie not quasiconvex — in CS , we have that the action of $\langle g \rangle$ on CS is not linear or focal. By Lemma 3.3, the action of $\langle g \rangle$ on CS is not of general type. Hence the action is horocyclic, ie the limit set of $\langle g \rangle$ on ∂CS consists of exactly one point λ_g with $g\lambda_g = \lambda_g$. Moreover, Proposition 3.1 of [17] implies that every $\lambda \neq \lambda_g$ in ∂CS has infinite $\langle g \rangle$ -orbit. We also denote by λ_g the image of this limit point under the usual $(\text{Aut}(\mathfrak{S})$ -equivariant) embedding $\partial CS \rightarrow \partial\mathcal{X}$. We thus have a fixed point $\lambda_g \in \partial\mathcal{X}$ for g . Now, suppose that $\lambda \in \partial\mathcal{X}$ is fixed by g . By Lemma 6.17, $\lambda \in \partial CS \cap \partial\mathcal{X}$. If $\lambda \neq \lambda_g$, then (as a point of ∂CS), λ cannot be fixed by g , so λ_g is the unique fixed point in $\partial\mathcal{X}$.

Finally, if $p \in \partial\mathcal{X} - \lambda_g$, then $g^n \cdot p \rightarrow \lambda_g$, for it subconverges to some point by compactness of $\bar{\mathcal{X}}$ (Theorem 3.4), which is fixed by g and thus must be λ_g by uniqueness. \square

Proposition 6.20 *Let $g \in \text{Aut}(\mathfrak{S})$ be irreducible distorted and fix $\lambda_g \in \partial\mathcal{X}$. For any neighborhood $U \subset \partial\mathcal{X}$ of λ_g , there exists $N > 0$ such that if $p \in \partial\mathcal{X} - U$, then $g^N \cdot p \in U$.*

Proof Fix a neighborhood $\lambda_g \in U \subset \partial\mathcal{X}$ and let $p \in \partial\mathcal{X} - U$. Let $F: \bar{\mathcal{X}} \rightarrow \mathbb{N}$ be the map which takes each $p \in \bar{\mathcal{X}}$ to the minimal $n \in \mathbb{N}$ such that $g^n \cdot p \in U$; note that F is defined by Proposition 6.19. We prove that F is bounded.

Assume not; then there exists a sequence $(p_i) \subset \partial\mathcal{X}$ such that $F(p_i) = n_i \rightarrow \infty$ as $i \rightarrow \infty$. By compactness of $\bar{\mathcal{X}}$, the sequence (p_i) accumulates on some point $\mu \in \partial\mathcal{X}$. If $N_\mu = F(\mu)$, then $g^{N_\mu} \cdot \mu \in U$. Choose an open neighborhood $g^{N_\mu} \cdot \mu \in V \subset U$.

By passing to a subsequence if necessary, we may assume $p_i \rightarrow \mu$ and continuity of the action of g on $\partial\mathcal{X}$ implies that $g^{N\mu} \cdot p_i \rightarrow g^{N\mu} \cdot \mu$. In particular, this implies that the sequence $(g^{N\mu} \cdot p_i)$ eventually lies in $V \subset U$, a contradiction. \square

6.2.2 Reducible automorphisms We now turn to nonelliptic reducible automorphisms. As before, we assume \mathcal{X} is proper, $g \in \text{Aut}(\mathfrak{S})$ has infinite order and is thus axial or distorted, and $\text{Big}(g) \neq \emptyset$ denotes the set of (pairwise orthogonal) $U \in \mathfrak{S}$ where $\text{diam}_{\mathcal{C}U}(\langle g \rangle \cdot X) = \infty$.

If g is reducible, then $\text{Big}(g) = \{A_i\} \sqcup \{B_j\}$, where g acts axially on $\mathcal{C}A_i$ and distortedly on $\mathcal{C}B_j$ for all i and j , and $A_i, B_j \neq S$ for all i and j . Proposition 6.12 implies that g is axial if and only if $\{A_i\} \neq \emptyset$; otherwise g is distorted.

We must be careful with nontrivial finite orbits in \mathfrak{S} . To that end, recall that by Lemma 6.3 there exists $M = M(\mathfrak{S}) > 0$ such that g^M fixes $\text{Big}(g)$ pointwise. The proof of that lemma shows that g^M in fact fixes $\{A_i\}$ and $\{B_j\}$ pointwise, since we cannot have $g \cdot A_i = B_j$ for any i and j . Let $h = g^M$, and note that $\text{Big}(h) = \text{Big}(g)$. Note that we can choose M so that any pairwise-orthogonal subset of \mathfrak{S} stabilized by h is fixed by h pointwise.

Lemma 6.21 *Let $V \in \mathfrak{S}$ and suppose that $V \sqsubseteq U$ or $V \pitchfork U$ for some $U \in \text{Big}(g)$. Suppose also that $p \in \partial\mathcal{X}$ is fixed by g . Then $V \notin \text{Supp}(p)$.*

Proof By hypothesis, $h \cdot p = p$. Observe that $\langle h \rangle \cdot \rho_U^V$ is unbounded. Since $U \in \text{Big}(g)$, we have that $h \cdot \rho_U^V = \rho_U^{h \cdot V}$ and $h \cdot U$ is infinite, implying $U \notin \text{Supp}(p)$, as required. \square

We denote by \mathbb{S}^k a k -sphere and by \mathbb{D}^k a k -ball. Given spaces X and Y , we denote by $X \star Y$ their join. For each i and j , let $F_i = F_{A_i}$ and $F'_j = F_{B_j}$ be the standard factors associated to A_i and B_j , so that there is a quasiconvex hieromorphism $\prod_i F_i \times \prod_j F'_j \rightarrow \mathcal{X}$, inducing an embedding $\star_i \partial F_i \rightarrow \star_j \partial F'_j \rightarrow \partial\mathcal{X}$ whose image is a closed g -invariant subset which we denote by $\mathfrak{E}(g)$. (Note: The image of $\prod_i F_i \times \prod_j F'_j$ need not be g -invariant, but since g stabilizes each standard product region $F'_j \times E_{B_j}$, the subspaces gF_i, F_i are parallel, and thus have the same boundary.)

For each i , the action of $h = g^M$ on $P_{F_i} \cong F_i \times E_{A_i}$ induces an action of h on F_i by applying the restriction homomorphism $\theta_{A_i}: \text{Stab}_{\text{Aut}(\mathfrak{S})}(A_i) \rightarrow \text{Aut}(\mathfrak{S}_{A_i})$. For each A_i , let h_i be the image of h under this homomorphism, and let h_j be the image of h under the corresponding restriction homomorphism for B_j .

The following proposition says that, up to taking a power, a reducible automorphism can be decomposed into irreducible automorphisms on subdomains:

Proposition 6.22 *If g is nonelliptic reducible and $h = g^M$, then the following hold:*

- (1) *For each i , h_i is an irreducible axial automorphism of F_i which fixes a unique pair of points $\lambda_{i,+}, \lambda_{i,-} \in \partial\mathcal{C}A_i$ and acts with north–south dynamics on $\partial\mathcal{C}A_i$.*
- (2) *For each j , h_j is an irreducible distorted automorphism of F'_j and fixes a unique point $\lambda_{h_j} \in \partial\mathcal{C}B_j$.*

Hence, g stabilizes (and h fixes pointwise) a nonempty subspace $S(g) \star C(g) \subseteq \partial\mathcal{X}$, where $S(g) = \emptyset$ or $S(g) \cong \mathbb{S}^{|\{A_i\}|-1}$ and $C(g) = \emptyset$ or $C(g) \cong \mathbb{D}^{|\{B_j\}|}$. Moreover, for each $n > 0$, g^n does not fix any point in $\mathfrak{E}(g) - S(g) \star C(g)$.

Proof For each i , h_i acts on $\mathcal{C}A_i$ axially by the assumption on g and irreducibly by construction. Hence, Proposition 6.18 implies that h_i fixes two points $\lambda_{i,+}, \lambda_{i,-} \in \partial\mathcal{C}A_i$ and acts with north–south dynamics on $\partial\mathcal{C}A_i$. Similarly, for each j , h_j acts on $\mathcal{C}B_j$ distortedly by assumption and irreducibly by construction. Proposition 6.19 then implies that h_j fixes a unique point $\lambda_{h_j} \in \partial\mathcal{C}B_j$.

If $\{A_i\} \neq \emptyset$, then each A_i contributes a pair of points $\lambda_{i,+}, \lambda_{i,-} \in \partial\mathcal{C}A_i$ fixed by h , which we can think of as a copy of \mathbb{S}^0 , namely \mathbb{S}_i^0 . Moreover, h clearly fixes the join of these spheres, $\star_i \mathbb{S}_i^0 \cong \mathbb{S}^{|\{A_i\}|-1} = S(g)$, as required.

Similarly, if $\{B_j\} \neq \emptyset$, then each B_j contributes a point $\lambda_{h_j} \in \partial\mathcal{C}B_j$ fixed by h , and h fixes the join of these points, $\star_j \lambda_{h_j} \cong \mathbb{D}^{|\{B_j\}|} = C(g)$, as required.

Since h fixes these $S(g)$ and $C(g)$, h clearly fixes $S(g) \star C(g)$. Now, if g^n fixes a point $\lambda \in \mathfrak{E}(g)$, then $h^n = (g^n)^M$ fixes λ . If $\lambda = \sum_i a_i p_i + \sum_j b_j q_j$, where $p_i \in \partial F_i$ and $q_j \in \partial F'_j$, then the uniqueness of the $\lambda_{i,+}, \lambda_{i,-}$ and λ_{h_j} implies that, for $a_i \neq 0$ and $b_j \neq 0$, we must have $q_j = \lambda_{h_j}$ and either $p_i = \lambda_{i,+}$ or $p_i = \lambda_{i,-}$. □

Remark 6.23 Set $\text{Comp}(g) = \{p \in \partial\mathcal{X} \mid \text{Supp}(p) \subset \{A_i, B_j\}_{i,j}^\perp\}$ and let $\text{Fix}(h) \subset \partial\mathcal{X}$ be the set of fixed points of h . It is not difficult to show that

$$\text{Fix}(h) \subseteq S(g) \star C(g) \star \text{Comp}(g),$$

but proper containment can happen.

Lemma 6.24 *Let $U \in \text{Big}(g)$ and $U \sqsubseteq V$. For all $p \in \partial\mathcal{X}$ such that $g^n(p) = p$ for some $n > 0$, we have $V \notin \text{Supp}(p)$.*

Proof It suffices to prove the lemma for $h = g^M$. Suppose for a contradiction that $V \in \text{Supp}(p)$. Since $U \in \text{Big}(h)$, $\text{diam}_V(\langle h \rangle \cdot \rho_V^U)$ is uniformly bounded. Take any sequence $X_k \rightarrow p$ in $\bar{\mathcal{X}}$; note that this implies $X_k \rightarrow p_V$ in $\bar{\mathcal{C}V}$. Thus, there exists $K > 0$ such that $d_V(X_k, \rho_V^U) > 100E$ if $k \geq K$.

Since h is unbounded on \mathcal{CU} , there exists $N > 0$ depending only on K such that $d_U(X_k, h^n(X_k)) > 100E$ if $n \geq N$ and $k \geq K$. If γ is a hierarchy path between X_k and $h^n(X_k)$ in \mathcal{X} , then the bounded geodesic image axiom (Definition 1.1(7)) implies that $\pi_V(\gamma) \cap N_E(\rho_V^U) \neq \emptyset$. In particular, this implies that $d_V(X_k, h^n(X_k)) > 100E$. Thus, for any $n > N$, we have that $(X_k, h^n(X_k))_{\rho_V^U}$ is uniformly bounded as $k \rightarrow \infty$, which implies that no power of h could fix p , a contradiction. \square

Proposition 6.25 *Let $p \in \partial\mathcal{X}$ be such that $g^M(p) = p$ for some $M > 0$. Then*

$$p \in S(g) \star C(g) \star \left(\bigcap_i \partial E_{A_i} \cap \bigcap_j \partial E_{B_j} \right).$$

Proof Lemmas 6.21 and 6.24 imply

$$\text{Supp}(p) \subset \bigcup_{i,j} (\mathfrak{S}_{A_i} \cup \mathfrak{S}_{B_j} \cup (\{A_i\}^\perp \cap \{B_j\}^\perp)),$$

which, together with Proposition 6.22 and g -invariance of $\text{Big}(g)$, gives the claim. \square

6.3 Dynamics on boundaries of HHGs

Fix a hierarchically hyperbolic group (G, \mathfrak{S}) .

Definition 6.26 (stable boundary points) A point $p \in \partial G$ is a *stable boundary point* if p is a fixed point of some irreducible axial element of $\text{Aut}(\mathfrak{S})$.

The next lemma states that irreducible axials have cobounded orbits.

Lemma 6.27 *Let $g \in G$ be an irreducible axial. Then, given any $X \in \mathcal{X}$, there exists $N > 0$ such that $\text{diam}_{\mathcal{CU}}(\langle g \rangle \cdot X) < N$ for any $U \in \mathfrak{S} - \{S\}$.*

Proof If not, then there is a sequence of domains $U_n \in \mathfrak{S}$ such that $\text{diam}_{\mathcal{CU}_n}(\langle g \rangle \cdot x) \geq n$ for each n . Since g is irreducible axial, $\langle g \rangle \cdot X$ projects to a uniform quasigeodesic in \mathcal{CS} .

By the bounded geodesic image axiom and hyperbolicity of \mathcal{CS} , for each $n > 100E$, there exists a sequence $(k_n) \subset \mathbb{Z}$ such that $\rho_S^{U_n} \in \mathcal{N}_E([g^{k_n} \cdot X, g^{k_n+1} \cdot X]) \subset \mathcal{CS}$, where $[g^{k_n} \cdot X, g^{k_n+1} \cdot X]$ is any geodesic between $g^{k_n} \cdot X$ and $g^{k_n+1} \cdot X$ in \mathcal{CS} . Moreover, since $\langle g \rangle \cdot X$ is a uniform quasigeodesic in \mathcal{CS} , it follows that $d_{U_n}(g^{k_n} \cdot X, g^{k_n+1} \cdot X) \asymp \text{diam}_{U_n}(\langle g \rangle \cdot X) \geq n$.

It follows that there exists a sequence of domains $U'_n = g^{-kn} \cdot U_n \in \mathfrak{S}$ with $\rho_S^{U'_n} \in \mathcal{N}_E([X, g \cdot X])$ and $d_{U'_n}(X, g \cdot X) \asymp \text{diam}_{U'_n}(\langle g \rangle \cdot X) \geq n$, which is impossible by the distance formula. This completes the proof. \square

Proposition 6.28 *If G has an irreducible axial element, then the set of stable boundary points is dense in ∂G .*

Proof Let $p \in \partial G$ be any point and let $\lambda \in \partial G$ be a stable boundary point for some irreducible axial $g \in G$. Choose $X \in \mathcal{X}$ and let $\gamma_n = [X, g^n \cdot X]$ be a D -hierarchy path between X and $g^n \cdot X$. Let $\gamma = [X, \lambda]$ be the limiting D -hierarchy ray as $n \rightarrow \infty$. Since $\gamma_n \rightarrow \gamma$ uniformly on compact sets and $\langle g \rangle \cdot X$ is uniformly cobounded by Lemma 6.27, it follows that γ is uniformly cobounded.

By coboundedness of the action of G and density of the interior (Proposition 2.17), there exists a sequence $(g_n) \subset G$ and $N > 0$ such that $g_n(X) \rightarrow p$ and thus $g_n \cdot \lambda \rightarrow p$. Since G acts on itself by automorphisms, we have that $g_n \cdot [X, \lambda]$ projects to an infinite quasigeodesic in \mathcal{CS} , implying that $g_n \cdot \lambda \in \partial \mathcal{CS} \subset \partial G$, which completes the proof. \square

Theorem 6.29 (topological transitivity of the G -action on ∂G) *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group with G not virtually cyclic and containing an irreducible axial element. For any $p \in \partial G$, the orbit $G \cdot p$ is dense in ∂G .*

Proof Let $U \subseteq \partial G$ be an open set. By Proposition 6.28, there exists an irreducible axial $g \in G$ with stable boundary points $\lambda_{g,+}, \lambda_{g,-} \in \partial G$, one of which is contained in U . Suppose that $\lambda_{g,+} \in U$ and $\lambda_{g,-} \neq p$. Then, since ∂G is Hausdorff, it follows from Proposition 6.18 that some power of g moves p into U , as required. Hence either we are done, or for every irreducible axial g with $\lambda_{g,+} \in U$, we have $\lambda_{g,-} = p$. Now, suppose that there exists $q \in \partial G - U \cup \{p\}$. Then, by Proposition 6.28, and the fact that ∂G is Hausdorff, we may argue as above, using Proposition 6.18, that some irreducible axial element takes p arbitrarily close to q , and thus that some power of g takes a translate of p into U , as required, unless p is a stable point for every irreducible axial element of G . But then G does not contain two independent irreducible axial elements whence, since G acts acylindrically on \mathcal{CS} by [5, Theorem 14.3], a theorem of Osin (see Theorem 9.3 below) implies that G is virtually cyclic. \square

Corollary 6.30 *If (G, \mathfrak{S}) is an HHG with an irreducible axial, then $\partial \mathcal{CS}$ is dense in ∂G .*

Remark 6.31 In Section 9, we investigate the question of when groups of HHS automorphisms contain irreducible axial elements. In that section, we consider a more general class, so-called “rank-one” elements, of which irreducible axial elements are the main examples.

7 Coarse semisimplicity in hierarchically hyperbolic groups

Theorem 7.1 *If (G, \mathfrak{S}) is a hierarchically hyperbolic group, then each $g \in G$ is either elliptic or axial, and $\pi_U(\langle g \rangle)$ is a quasiisometrically embedded copy of \mathbb{Z} for each $U \in \text{Big}(g)$.*

Proof of Theorem 7.1 This follows from Lemmas 7.3 and 7.4 below. □

Our main tool here is the following result of Bowditch:

Lemma 7.2 [11, Lemma 2.2] *If G acts acylindrically by isometries on a hyperbolic space M , then each element of G acts either elliptically or loxodromically on M .*

Lemma 7.2 and [5, Theorem 14.3] combine to yield:

Lemma 7.3 *If $g \in G$ is irreducible, then g is either elliptic or axial.*

Recall that for any reducible $g \in G$, we have $\text{Big}(g) = \{A_i\} \cup \{B_j\}$, where g acts axially on each $\mathcal{C}A_i$ and distortedly on each $\mathcal{C}B_j$. It remains to prove:

Lemma 7.4 *If $g \in G$ is reducible, then $\{B_j\} = \emptyset$.*

For each $U \in \mathfrak{S}$, let $G_U = \mathcal{A}_U \cap G$ be the subgroup of G fixing $U \in \mathfrak{S}$ and let $\bar{G}_U = \theta_U(G_U)$, where $\mathcal{A}_U = \text{Stab}_{\text{Aut}(\mathfrak{S})}(U)$ and $\theta_U: \mathcal{A}_U \rightarrow \text{Aut}(\mathfrak{S}_U)$ is the restriction homomorphism.

Lemma 7.5 *Let $U \in \mathfrak{S}$. Then \bar{G}_U acts acylindrically on $\mathcal{C}U$.*

Proof of Lemma 7.5 By definition, \bar{G}_U acts by automorphisms on the hierarchically hyperbolic space (F_U, \mathfrak{S}_U) . We first establish:

Claim 1 For each $R \geq 0$, there exists $K = K(R)$ such that any R -ball $B \subseteq F_U$ intersects gB for at most K elements $g \in \bar{G}_U$.

Proof of Claim 1 Since the inclusion homomorphism $(F_U, \mathfrak{S}_U) \rightarrow (G, \mathfrak{S})$ is a quasiisometric embedding (with constants independent of U), it suffices to bound the number of cosets $g(\ker \theta_U)$ in G_U for which $g(\ker \theta_U) \cdot (B' \times E_U) = (\bar{g}B') \times E_U$ intersects $B' \times E_U$, where B' is a ball in $F_U \subset P_U \subset \mathcal{X}$ of radius depending on R and the quasiisometry constants. Such a bound exists because G acts on itself geometrically. ◁

We now follow the proof of Theorem 14.3 of [5]. Let $\epsilon > 0$ be given and let $R \geq 1000\epsilon$. Consider the set \mathfrak{H} of $g \in \bar{G}_U$ such that $d_U(x, gx), d_U(y, gy) < \epsilon$, where $x, y \in F_U$. Choose s_0 as in the distance formula for (F_U, \mathfrak{S}_U) and, for each $r \geq 0$, consider the set $\mathfrak{L}(r)$ of \sqsubseteq -maximal $V \in \mathfrak{S}_U - \{U\}$ such that $d_V(x, y) > s_0$ and $|d_U(x, \rho_V^U) - \frac{1}{2}R| < r\epsilon$. Arguing exactly as in the proof of Theorem 14.3 of [5] yields a uniform bound on $|\mathfrak{L}(11)|$. We then divide into two cases.

First, if $\mathfrak{L}(10) \neq \emptyset$, then we again argue as in the proof of [5, Theorem 14.3], reaching the conclusion that, if $V \in \mathfrak{L}(10)$ and $g \in \mathfrak{H}$, then $g_{P_V}(x)$ coarsely coincides with $g \cdot g_{P_V}(x)$, from which it follows from Claim 1 that \mathfrak{H} has uniformly bounded cardinality. The argument in [5] uses only the \bar{G}_U -equivariance of the gate construction and Definition 1.1 and thus goes through.

Similarly, if $\mathfrak{L}(10) = \emptyset$, then the argument in [5] uses only the existence of hierarchy paths, large links, bounded geodesic image, the distance formula, and a bound on the cardinalities of stabilizers of balls in F_U . The latter comes from Claim 1, and thus the argument works verbatim in the present context. □

Proof of Lemma 7.4 Let $U \in \text{Big}(g)$. Let $M > 0$ be as in Lemma 6.3 and set $h = g^M$; note that $h \cdot U = U$, ie $h \in \mathcal{A}_U$. Let $h_U = \theta_U(h) \in \bar{G}_U$. By Lemma 7.5, \bar{G}_U acts acylindrically on $\mathcal{C}U$, so by Lemma 7.2, h_U is either elliptic or loxodromic on $\mathcal{C}U$. Since $U \in \text{Big}(h)$, it must be the case that h_U is loxodromic on $\mathcal{C}U$. Since h acts like h_U on $\mathcal{C}U$, the claim follows. □

8 Essential structures, essential actions and product HHSs

8.1 Product HHSs

It is shown in [6] that, if $\mathcal{X}_0, \mathcal{X}_1$ admit hierarchically hyperbolic structures, then $\mathcal{X}_0 \times \mathcal{X}_1$ admits a hierarchically hyperbolic structure making the inclusions $\mathcal{X}_i \rightarrow \mathcal{X}_0 \times \mathcal{X}_1$ into hieromorphisms with hierarchically quasiconvex image. Rather than recall the construction, we now give a more streamlined (equivalent) definition.

Definition 8.1 Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then $(\mathcal{X}, \mathfrak{S})$ is a *product HHS* if there exists $K < \infty$ and $U \in \mathfrak{S}$ such that for all $V \in \mathfrak{S}$, either $V \sqsubseteq U$, or $V \perp U$, or $\text{diam}(\mathcal{C}V) \leq K$. If, in addition, for each $n \in \mathbb{N}$ there exist $V, W \in \mathfrak{S}$ with $V \sqsubseteq U$, $W \perp U$ and $\text{diam}(\pi_V(\mathcal{X})), \text{diam}(\pi_W(\mathcal{X})) > n$, then $(\mathcal{X}, \mathfrak{S})$ is a *product region with unbounded factors*. Observe that $(\mathcal{X}, \mathfrak{S})$ is a product HHS if and only if there exists $U \in \mathfrak{S}$ such that $P_U \rightarrow \mathcal{X}$ is coarsely surjective, and that $(\mathcal{X}, \mathfrak{S})$ is a product region with unbounded factors if in addition F_U and E_U are both unbounded.

8.2 Essential structures and cores

Definition 8.2 (essential HH structures) Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and let $G \leq \text{Aut}(\mathfrak{S})$. Then $(\mathcal{X}, \mathfrak{S})$ is G -essential if, for any G -invariant hierarchically quasiconvex $\mathcal{Y} \subset \mathcal{X}$, all of \mathcal{X} is contained in some regular neighborhood of \mathcal{Y} .

Remark 8.3 Compare Definition 8.2 to the definition of a G -essential cube complex from [18], which requires that the cube complex be the cubical convex hull of a G -orbit (but actually requires something stronger).

Proposition 8.4 (essential core) Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and let $G \leq \text{Aut}(\mathfrak{S})$ be a subgroup. Suppose that one of the following holds:

- (1) G acts properly and cocompactly on \mathcal{X} and with finitely many orbits on \mathfrak{S} , ie (G, \mathfrak{S}) is an HHG.
- (2) G acts on \mathcal{X} with unbounded orbits and with no fixed point in $\partial\mathcal{X}$.

Then there exists a G -invariant, G -essential, hierarchically quasiconvex subspace $\mathcal{Y} \subset \mathcal{X}$ such that whichever of (1) or (2) held for $G \curvearrowright \mathcal{X}$ holds for the action of G on \mathcal{Y} .

Proof If $(\mathcal{X}, \mathfrak{S})$ is an HHG, the claim follows immediately with $\mathcal{Y} = \mathcal{X}$. In the second case, we will build $\mathcal{Y} \subset \mathcal{X}$ so that \mathcal{Y} is hierarchically quasiconvex and G -invariant, with the property that if $\mathcal{Y}' \subset \mathcal{X}$ is hierarchically quasiconvex and G -invariant, then there exists an $R > 0$ such that $\mathcal{Y} \subset \mathcal{N}_R(\mathcal{Y}')$. Given such a \mathcal{Y} , the fact that G does not fix a point in $\partial\mathcal{Y}$ follows from Proposition 5.8 and the hypothesis that G does not fix a point in $\partial\mathcal{X}$.

To construct \mathcal{Y} , for each $U \in \mathfrak{S}$, let $H_U \subseteq \mathcal{C}U$ be the union of all geodesics starting and ending in $\pi_U(G \cdot x)$ for some fixed basepoint $x \in \mathcal{X}$. A thin quadrilateral argument shows that H_U is uniformly quasiconvex. Let \mathcal{Y} consist of all realization points y with $\pi_U(y) \in H_U$ for all $U \in \mathfrak{S}$; this subspace is easily seen to have the required properties. □

Recall that, by hierarchical quasiconvexity, $(\mathcal{Y}, \mathfrak{S})$ is normalized: for each $U \in \mathfrak{S}$, the associated hyperbolic space is uniformly quasiisometric to $\pi_U(\mathcal{Y}) \subseteq \mathcal{C}U$.

9 Coarse rank-rigidity and its consequences

Throughout this section, $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space with \mathcal{X} proper and \mathfrak{S} countable; we always let S denote the \sqsubseteq -maximal element of \mathfrak{S} . In this section,

we consider countable subgroups $G \leq \text{Aut}(\mathfrak{S})$ (so that, by the distance formula, G acts discretely on \mathcal{X}). These standing hypotheses cover the case where (G, \mathfrak{S}) is an HHG. We emphasize our standing assumption that all HHSs are normalized.

Definition 9.1 (rank-one automorphism) The automorphism $g \in \text{Aut}(\mathfrak{S})$ is *rank-one* (on $(\mathcal{X}, \mathfrak{S})$) if

- g is axial;
- $|\text{Big}(g)| = 1$;
- if $U \in \mathfrak{S}$ is orthogonal to the domain in $\text{Big}(g)$, then $\text{diam}(\pi_U(\mathcal{X})) < \infty$.

Irreducible axial elements are rank-one.

Our first goal is to show that, under the above hypotheses, either G contains an irreducible axial element or the G -essential core of \mathcal{X} is a product HHS (not necessarily with unbounded factors). This is done in Section 9.1, using tools from Sections 9.2, 9.3 and 9.4. In Section 9.5, we apply results of Section 9.1.

9.1 Irreducible axials or fixed domains

We now prove the following two parallel propositions (one covering the nonparabolic case, and one covering the HHG case):

Proposition 9.2 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with \mathcal{X} proper and \mathfrak{S} countable. Let the countable group $G \leq \text{Aut}(\mathfrak{S})$ act with unbounded orbits in \mathcal{X} and without a global fixed point in $\partial\mathcal{CS}$. Then either G contains an irreducible axial element, or there exists $U \in \mathfrak{S} - \{S\}$ such that $|G \cdot U| < \infty$. Moreover, any G -essential hierarchically quasi-convex subspace $\mathcal{Y} \subset \mathcal{X}$ coarsely coincides with the standard product region $P_U \cap \mathcal{Y}$.*

Proof By Proposition 8.4, there exists a G -invariant hierarchically quasiconvex subspace \mathcal{Y} with a hierarchically hyperbolic structure $(\mathcal{Y}, \mathfrak{S})$ admitting a G -equivariant hieromorphism $(\mathcal{Y}, \mathfrak{S}) \rightarrow (\mathcal{X}, \mathfrak{S})$ that is the inclusion on \mathcal{Y} and the identity on \mathfrak{S} , and such that $(\mathcal{Y}, \mathfrak{S})$ is G -essential. Moreover, G continues to act without a global fixed point in $\partial\mathcal{CS}$. Hence, since \mathcal{Y} is proper and \mathfrak{S} is countable, Proposition 9.11 provides an irreducible axial isometry of $(\mathcal{Y}, \mathfrak{S})$ (hence of $(\mathcal{X}, \mathfrak{S})$) unless $\text{diam}(\pi_S(\mathcal{Y})) < \infty$. If $\text{diam}(\pi_S(\mathcal{Y})) < \infty$, then Proposition 9.10 completes the proof. □

The HHG version requires the following theorem of Osin, which we also use elsewhere:

Theorem 9.3 [65, Theorem 1.1] *Let G be a group acting acylindrically on a hyperbolic space. Then exactly one of the following holds:*

- (1) G has bounded orbits.
- (2) G is virtually infinite cyclic and contains a loxodromic element.
- (3) G contains infinitely many independent loxodromic elements.

Proposition 9.4 *Let (G, \mathfrak{S}) be an HHG. Then either G contains an irreducible axial element or there exists $U \in \mathfrak{S}$ such that $|G \cdot U| < \infty$ and G coarsely coincides with P_U .*

Proof The G -action on (G, \mathfrak{S}) is essential. If $\text{diam}(\mathcal{CS}) = \infty$, then, since G acts acylindrically on \mathcal{CS} , as proved in [5, Section 14], Theorem 9.3 implies that G contains an irreducible axial element. Hence we can assume that $\text{diam}(\mathcal{CS}) < \infty$, and in particular that G has no fixed point in $\partial\mathcal{CS} = \emptyset$. The claim now follows from Proposition 9.10. □

9.2 Finding finite orbits in \mathfrak{S}

Let μ be a probability measure on G , whose support generates G . All spaces are equipped with their Borel σ -algebra, so every subset of G is measurable, while the measurable subsets of $\bar{\mathcal{X}}$ are determined by Definition 2.11.

Lemma 9.5 (stationary measure on $\bar{\mathcal{X}}$) *There exists a μ -stationary probability measure ν on $\bar{\mathcal{X}}$, ie for all ν -measurable $E \subseteq \bar{\mathcal{X}}$,*

$$\nu(E) = \sum_{g \in G} \mu(g)\nu(g^{-1}E) = \mu * \nu(E).$$

Proof This is a standard fact, relying on compactness of $\bar{\mathcal{X}}$, ie Theorem 3.4. See [34, Lemma 1.2], for example. □

Remark 9.6 (sampling \mathcal{X}) Since our aim in this section is to establish that, after passing if necessary to a G -essential core, G contains an irreducible axial element or \mathcal{X} is a product HHS, and these properties are insensitive to modifications of \mathcal{X} within its quasiisometry type, we now “discretize” \mathcal{X} , for convenience in the proof of Lemma 9.8.

Let $\mathcal{D} = G \setminus \mathcal{X}$, and let \bar{d} be the quotient pseudometric, so (\mathcal{D}, \bar{d}) is proper since \mathcal{X} is proper. Hence there exists $\epsilon > 0$ and a countable set $\{\bar{x}_n\}_{n \geq 0}$ in \mathcal{D} such that $\mathcal{N}_\epsilon^{\mathcal{D}}(\{\bar{x}_n\}) = \mathcal{D}$. Thus \mathcal{X} contains a countable, G -invariant set $\{x_n\}_{n \geq 0}$ for which the inclusion $\{x_n\} \hookrightarrow \mathcal{X}$ is a quasiisometry, and we replace \mathcal{X} with $\{x_n\}$. We can thus assume that \mathcal{X} is countable.

Lemma 9.7 *For each $U \subset \mathfrak{S}$, the set $\{p \in \partial\mathcal{X} : \text{Supp}(p) = U\}$ is ν -measurable.*

Proof Either $\{p \in \partial\mathcal{X} : \text{Supp}(p) = \mathcal{U}\} = \emptyset$, in which case we're done, or $\mathcal{U} = \{U_i\}$ is a set of pairwise-orthogonal domains. Let \mathcal{X}_0 be the set of points $q \in \partial\mathcal{X}$ such that, for all $V \in \text{Supp}(q)$, there exists $U \in \mathcal{U}$ with $V \sqsubseteq U$. Note that

$$\mathcal{Y} = \{p \in \partial\mathcal{X} : \text{Supp}(p) = \mathcal{U}\} \subseteq \mathcal{X}_0.$$

Let \mathcal{X}_1 be the subset of \mathcal{X}_0 consisting of those $q \in \mathcal{X}_0$ such that for some $V \in \text{Supp}(q)$, we have $V \not\subseteq \mathcal{U}$ (so V is properly nested in some $U \in \mathcal{U}$ and orthogonal to the remaining elements).

\mathcal{X}_0 is closed in $\bar{\mathcal{X}}$ We will check that for any sequence $\{q_n\}$ with each $q_n \in \mathcal{X}_0$, if $q_n \rightarrow q$, then $q \in \mathcal{X}_0$. Suppose not, ie suppose that there exists $V \in \text{Supp}(q)$ such that $V \not\subseteq U$ for all $U \in \mathcal{U}$. Consider a basic neighborhood $\mathcal{N} = \mathcal{N}_{\epsilon, \{N_T\}}(q)$ of q . There are two cases.

First case This is the case where there exists $U \in \mathcal{U}$ such that $U \pitchfork V$ or $U \sqsubset V$ and, for infinitely many n , there exists $W \in \text{Supp}(q_n)$ such that $W \sqsubseteq U$ and $W \not\perp V$. Let \mathcal{I} be the set of such n .

First, suppose that q_n is remote with respect to q . Suppose that the basic neighborhood \mathcal{N} has been chosen so that N_V does not meet the $10^9 E$ -neighborhood of ρ_V^U . Then for arbitrarily large $n \in \mathcal{I}$, the subsets ρ_V^U and ρ_V^W coarsely coincide, and hence $(\partial\pi_{\text{Supp}(q)}(q_n))_V = \rho_V^W$ does not lie in N_V . It follows that for arbitrarily large $n \in \mathcal{I}$, we have $q_n \notin \mathcal{N}$, by the definition of the remote part of a basic set. This is a contradiction.

Second, suppose that q_n is nonremote with respect to q , where $n \in \mathcal{I}$. Exactly as before, suppose that N_V does not meet the $10^9 E$ -neighborhood of ρ_V^U (which is still defined by assumption). We still have that ρ_V^W is defined and coarsely coincides with ρ_V^U for some $W \in \text{Supp}(q_n)$, by assumption. Hence, again, we have that $(\partial\pi_{\text{Supp}(q)}(q_n))_V = \rho_V^W$ does not lie in N_V . From the final condition in the definition of the nonremote part of a basic set, it follows that $q_n \notin \mathcal{N}$, which is again a contradiction.

Second case In this case, for all but finitely many n , we have $V \perp W$ for all $W \in \text{Supp}(q_n)$. The point q_n is nonremote with respect to q . Indeed, there exists $V \in \text{Supp}(q)$ which is orthogonal to every element of $\text{Supp}(q_n)$. In particular, $V \in \text{Supp}(q) - \text{Supp}(q_n) \cap \text{Supp}(q)$. Now, $\sum_{T \in \text{Supp}(q_n) - \text{Supp}(q)} a_T^{q_n} < \epsilon$, so

$$\sum_{T \in \text{Supp}(q_n) \cap \text{Supp}(q)} a_T^{q_n} > 1 - \epsilon,$$

while $|a_T^q - a_T^{q_n}| < \epsilon$ whenever $T \in \text{Supp}(q_n) \cap \text{Supp}(q)$. Hence

$$\sum_{T \in \text{Supp}(q) \cap \text{Supp}(q_n)} a_T^q > 1 - \epsilon (|\text{Supp}(q_n) \cap \text{Supp}(q)|),$$

which is impossible when ϵ is sufficiently small compared to a_V^q , since $V \notin \text{Supp}(q_n)$. Hence $q_n \notin \mathcal{N}$, a contradiction.

Conclusion Let \mathfrak{T} be the set of support sets $\mathcal{V} \neq \mathcal{U}$ such that for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ with $V \sqsubseteq U$. Then \mathfrak{T} is countable, being a set of finite subsets of the countable set \mathfrak{S} . Now, \mathcal{X}_1 is the union over all $\mathcal{V} \in \mathfrak{T}$ of the set $\mathcal{X}_0(\mathcal{V})$ of $q \in \partial\mathcal{X}$ such that for each $W \in \text{Supp}(q)$, there exists $V \in \mathcal{V}$ with $W \sqsubseteq V$. Hence, by the previous part of the proof, \mathcal{X}_1 is a countable union of closed sets. Thus $\mathcal{Y} = \mathcal{X}_0 - \mathcal{X}_1$ is Borel, and hence ν -measurable. \square

Lemma 9.8 *If G has no finite orbit in $(\mathfrak{S} - \{S\}) \cup \partial\mathcal{C}S$, then ν is supported on $\partial\mathcal{C}S \subset \bar{\mathcal{X}}$.*

Proof Let D be the set of finite subsets of \mathfrak{S} , so that D is countable and G acts on D in the obvious way. By construction, $\{S\}$ and \emptyset are the only elements of D whose G -orbits are finite. We first define a map $\mathcal{O}: \bar{\mathcal{X}} \rightarrow D$. Note that if $\mathfrak{S} = \{S\}$, then $\partial\mathcal{X} = \partial\mathcal{C}S$, and the claim follows, so we assume that there exists $U \sqsubset S$.

Defining \mathcal{O} on boundary points For each $p \in \partial\mathcal{X}$, let $\mathcal{O}(p) = \text{Supp}(p)$. Observe that this assignment is G -equivariant and that $\mathcal{O}(p) = \{S\}$ if and only if $p \in \partial\mathcal{C}S$.

Defining \mathcal{O} on interior points Let $\mathcal{B} \subset \mathcal{X}$ contain exactly one point from each G -orbit, and choose $F \in D - \{\{S\}, \emptyset\}$. For each $x \in \mathcal{B}$, let $\mathcal{O}(x) = F$. Then, for any $x \in \mathcal{B}$ and $g \in G$, let $\mathcal{O}(gx) = gF$. Then \mathcal{O} is G -equivariant and, for all $x \in \mathcal{X}$, the nonempty finite set $\mathcal{O}(x)$ differs from $\{S\}$. For any $F' \in D$, either $\mathcal{O}^{-1}(F') = \emptyset$ or $F' = gF$ for some $g \in G$. Hence, for any subset D' of D , we can write $\mathcal{O}^{-1}(D') = \bigcup_{gF \in D'} g\mathcal{B}$. It follows that $\mathcal{O}^{-1}(D')$ is a countable union of translates of \mathcal{B} , which is a countable union of closed sets (singletons) by [Remark 9.6](#), and thus $\mathcal{O}^{-1}(D')$ is Borel.

Measurability of $\bar{\mathcal{X}} - \partial\mathcal{C}S$ Since $\partial\mathcal{C}S = \{p \in \partial\mathcal{X} : \text{Supp}(p) = \{S\}\}$, it follows from [Lemma 9.7](#) that $\bar{\mathcal{X}} - \partial\mathcal{C}S$ is measurable.

Measurability of \mathcal{O} There is a probability measure $\tilde{\nu}$ on D given by $\tilde{\nu}(A) = \nu(\mathcal{O}^{-1}(A))$, for each $A \subseteq D$. A set $\mathcal{O}^{-1}(A)$ decomposes as

$$\{x \in \mathcal{X} : \mathcal{O}(x) \in A\} \cup \{p \in \partial\mathcal{X} : \text{Supp}(p) \in A\}.$$

The set $\{p \in \partial\mathcal{X} : \text{Supp}(p) \in A\} = \bigcup_{\mathcal{U} \in A} \{p : \text{Supp}(p) = \mathcal{U}\}$, which is ν -measurable by [Lemma 9.7](#). Since $A \subseteq D$ is countable, it suffices to show that $\mathcal{O}^{-1}(F) \cap \mathcal{X}$ is Borel for each $F \in D$, but this was established above.

Conclusion We have that $\mathcal{O}: \bar{\mathcal{X}} \rightarrow D$ is a measurable G -equivariant map. Since G preserves $\partial\mathcal{C}S$, it follows that $\bar{\mathcal{X}} - \partial\mathcal{C}S$ is a G -invariant ν -measurable set.

Suppose that $F' \in D$ has the property that $G \cdot F'$ is finite. Then $G \cdot U$ is a finite G -invariant subset of \mathfrak{S} for each $U \in F'$ and, by our hypothesis that there is no finite G -orbit in $\mathfrak{S} - \{S\}$, we have that $F' = \{S\}$. Since $\mathcal{O}(e) \neq \{S\}$ for all $e \in \bar{\mathcal{X}} - \partial\mathcal{CS}$, it follows that $\mathcal{O}(\bar{\mathcal{X}} - \partial\mathcal{CS})$ does not contain a finite G -orbit. As shown in eg [2; 45, Lemma 2.2.2; 75, Lemma 3.4; 44, Lemma 3.3], we must have $\nu(\bar{\mathcal{X}} - \partial\mathcal{CS}) = 0$. \square

Corollary 9.9 *If $\text{diam}(\mathcal{CS}) < \infty$, then G stabilizes a finite subset of $\mathfrak{S} - \{S\}$.*

Proof By hypothesis, $\partial\mathcal{CS} = \emptyset$, so ν cannot be supported on $\partial\mathcal{CS}$. Hence G has a finite orbit in $\mathfrak{S} \cup \partial\mathcal{CS}$ by Lemma 9.8 and thus G must have a finite orbit in $\mathfrak{S} - \{S\}$. \square

9.3 Finding product structures when $\text{diam}(\mathcal{CS}) < \infty$

Proposition 9.10 *Suppose $G \leq \text{Aut}(\mathfrak{S})$ is a countable subgroup with $\text{diam}(\mathcal{CS}) < \infty$. Then there exists $U \in \mathfrak{S} - \{S\}$ and a finite-index subgroup G' such that $G' \cdot U = U$ and \mathcal{X} coarsely coincides with P_U . Hence either $(\mathcal{X}, \mathfrak{S})$ is a product HHS with unbounded factors or \mathcal{X} coarsely coincides with F_U or E_U .*

Proof By Corollary 9.9, there exists $U \in \mathfrak{S} - \{S\}$ and a finite-index subgroup $G' \leq G$ such that $G' \cdot U = U$. Note that G' continues to act essentially on $(\mathcal{X}, \mathfrak{S})$, coarsely stabilizing P_U . Since P_U is hierarchically quasiconvex, \mathcal{X} coarsely equals P_U by essentiality. The last assertion is immediate. \square

9.4 Finding irreducible axial elements when $\text{diam}(\mathcal{CS}) = \infty$

Proposition 9.11 *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Let $G \leq \text{Aut}(\mathfrak{S})$ act essentially and suppose that G acts on \mathcal{X} with no global fixed point in $\partial\mathcal{CS}$ and that \mathcal{CS} is unbounded. Then G contains an irreducible axial automorphism of $(\mathcal{X}, \mathfrak{S})$.*

Proof Suppose that every orbit of G in \mathcal{CS} is bounded, so that, fixing $x_0 \in \mathcal{X}$, there exist $Q, R < \infty$ such that $\text{diam}_S(G \cdot \pi_S(x_0)) \leq R$ and $G \cdot \pi_S(x_0)$ is Q -quasiconvex. Consider the set of all E -consistent tuples $(b_U)_{U \in \mathfrak{S}}$ such that $b_S \in G \cdot \pi_S(x_0)$. Let \mathcal{Y} be the set of realization points in \mathcal{X} corresponding to such tuples, provided by Theorem 1.7, and note that G acts on \mathcal{Y} . By definition, \mathcal{Y} is hierarchically quasiconvex in \mathcal{X} provided $\pi_U(\mathcal{Y})$ is uniformly quasiconvex in \mathcal{CU} for each $U \in \mathfrak{S}$, which we now verify.

If \vec{b} is such a tuple, with $d_S(b_S, \rho_S^U) \leq E$, then consistency puts no constraint on the U -coordinate of \vec{b} , ie for any such U , the map $\pi_U: \mathcal{Y} \rightarrow \mathcal{CU}$ is uniformly coarsely surjective, and in particular $\pi_U(\mathcal{Y})$ is uniformly quasiconvex in \mathcal{CU} . On the other

hand, if $d_S(\rho_S^U, G \cdot \pi_S(x_0)) > E$, then consistency and bounded geodesic image imply that $\pi_U(\mathcal{Y})$ is uniformly bounded, and hence uniformly quasiconvex.

The existence of \mathcal{Y} contradicts G -essentiality of \mathcal{X} . Hence G has an unbounded orbit in \mathcal{CS} , so either there exists $g \in G$ acting loxodromically on \mathcal{CS} , so g is irreducible axial, or there exists a unique fixed point $p \in \partial\mathcal{CS}$, which is impossible. \square

9.5 Coarse rank-rigidity

Recall that a metric space \mathcal{X} is *wide* if no asymptotic cone of \mathcal{X} has a cut-point. The following lemma is well-known and elementary:

Lemma 9.12 *Let \mathcal{X} be a metric space quasiisometric to the product $\mathcal{X}_0 \times \mathcal{X}_1$, where each \mathcal{X}_i is unbounded. Then \mathcal{X} is wide, ie no asymptotic cone of \mathcal{X} has a cut-point.*

We now prove the main theorems of this section. Much of the work was done in proving Propositions 9.2 and 9.4; the remaining work is largely in sorting out technical issues that arise when attempting to induct on complexity; these issues mainly stem from the fact that, given $U \in \mathfrak{S}$, the induced HHS structure on E_U does not have a uniquely determined \sqsubseteq -maximal element.

Theorem 9.13 (coarse rank-rigidity for nonparabolic actions) *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with \mathcal{X} proper and \mathfrak{S} countable. Let the countable group $G \leq \text{Aut}(\mathfrak{S})$ act essentially with unbounded orbits in \mathcal{X} and without a fixed point in $\partial(\mathcal{X}, \mathfrak{S})$. Then one of the following holds:*

- (1) \mathcal{X} is a product HHS with unbounded factors; specifically, \mathcal{X} is coarsely equal to P_U for some $U \in \mathfrak{S}$ with $|GU| < \infty$.
- (2) There exists $g \in G$ such that g is rank-one.

If conclusion (1) holds, then \mathcal{X} is wide.

Proof By Proposition 9.2, either G contains an irreducible axial element, which is rank-one by definition, so conclusion (2) holds, or there is a finite-index subgroup $G' \leq G$ fixing some $U \in \mathfrak{S} - \{S\}$, so that by essentiality, \mathcal{X} coarsely coincides with the standard product region P_U . This implies that \mathcal{X} is a product HHS. Choose U of minimal level with this property, ie no domain of lower level has a finite G -orbit in \mathfrak{S} .

Since G has unbounded orbits in \mathcal{X} , at least one of E_U and F_U is unbounded. If F_U and E_U are both unbounded, then conclusion (1) holds, and we are done. The statement about wideness follows from Lemma 9.12.

If F_U is unbounded and E_U is bounded, then (F_U, \mathfrak{S}_U) is an HHS with F_U proper and \mathfrak{S}_U countable, on which G' acts by HHS automorphisms with no fixed point

in $\partial\mathcal{C}U$ (for otherwise G would have a fixed point in $\partial\mathcal{X}$). By minimality, G' has no finite orbit in $\mathfrak{S}_U - \{U\}$, so Proposition 9.11 provides $g \in G'$ acting as an irreducible axial element of $\text{Aut}(\mathfrak{S}_U)$. As an element of $\text{Aut}(\mathfrak{S})$, we see that g is rank-one, for otherwise there would be some $V \perp U$ with $\text{diam}(\mathcal{C}V) = \infty$, contradicting that E_U is bounded.

Finally, suppose that E_U is unbounded and F_U is bounded. Let \mathfrak{C} be a minimal G' -invariant set of \sqsubseteq -minimal elements C of $\mathfrak{S} - \{S\}$ such that $W \sqsubseteq C$ whenever $W \perp U$.

Suppose that there exists $C \in \mathfrak{C}$ with $C \perp U$. Then $g \cdot C \perp g \cdot U = U$ for all $g \in G'$, so $g \cdot C \sqsubseteq C$, from which it follows that (passing if necessary to a further finite-index subgroup if necessary) $G' \cdot C = C$. Then (E_U, \mathfrak{S}_C) is an HHS satisfying the hypotheses of the theorem, and $G' \leq \text{Aut}(\mathfrak{S}_C)$ acts without a fixed point in ∂E_U (since it stabilizes $\partial E_U \subset \partial\mathcal{X}$). In this case, the claim follows by induction on complexity. Indeed, in the base case, $|\mathfrak{S}| = 1$ and the theorem is obvious. Otherwise, induction shows that either conclusion (1) holds, or there exists $g \in G$ that acts as a rank-one element of $\text{Aut}(\mathfrak{S}_C)$. Since G' preserves P_U and P_U coarsely equals \mathcal{X} , this implies that g is rank-one on $(\mathcal{X}, \mathfrak{S})$, as required.

The definition of \mathfrak{C} and Definition 1.1(3) imply that $C \not\sqsubseteq U$ and $U \not\sqsubseteq C$ for all $C \in \mathfrak{C}$. Hence it remains to consider the case where each $C \in \mathfrak{C}$ satisfies $C \pitchfork U$; fix such a C . Since G' stabilizes U , it coarsely stabilizes the image \bar{P}_U of $P_U = F_U \times E_U \rightarrow \mathcal{X}$. In other words, for any basepoint $x \in \mathcal{X}$, the orbit $G' \cdot x$ lies in a neighborhood of \bar{P}_U . Now, since $C \pitchfork U$, the definition of P_U implies that $\pi_C(gx)$ uniformly coarsely coincides with ρ_C^U for all $g \in G'$, whence $\text{diam}(\pi_C(G' \cdot x)) < \infty$, so, by essentiality, $\text{diam}(\pi_C(\mathcal{X})) < \infty$.

In this case, form a new index set \mathfrak{S}_U^\perp by appending to the set of domains orthogonal to U a new domain C . In $\mathfrak{S}_U^\perp \cap \mathfrak{S}$, the associated hyperbolic spaces, projections from E_U , and relative projections are defined as in \mathfrak{S} . The hyperbolic space $\mathcal{C}C$ is a single point, so the projections $\pi_C: \mathcal{X} \rightarrow \mathcal{C}C$ and ρ_C^V for $V \perp U$ are defined in an obvious way. We thus have an HHS structure $(E_U, \mathfrak{S}_U^\perp)$ with $G' \leq \text{Aut}(\mathfrak{S}_U^\perp)$, of complexity less than that of \mathfrak{S} , and we can argue as above by induction. Observe that, if $g \in \text{Aut}(\mathfrak{S}_U^\perp)$ is rank-one on E_U , then $\text{Big}(g)$ consists of some element of $\mathfrak{S}_U^\perp \cap \mathfrak{S}$, and since $\pi_C(\mathcal{X})$ is bounded for all $C \in \mathfrak{C}$, and we can argue as above that g is rank-one on $(\mathcal{X}, \mathfrak{S})$. □

Theorem 9.14 (coarse rank-rigidity for HHG) *Let (G, \mathfrak{S}) be an infinite hierarchically hyperbolic group. Then exactly one of the following holds:*

- (1) (G, \mathfrak{S}) is a product HHS with unbounded factors, and G is wide.

(2) G contains a rank-one element, and is thus not wide.

Moreover, conclusion (1) holds if and only if $\text{diam}(\mathcal{CS}) < \infty$.

Proof By Proposition 9.4, either G contains an irreducible axial element, which is rank-one, or there exists $U \in \mathfrak{S} - \{S\}$ with $G' \cdot U = U$ for some finite-index $G' \leq G$, and G coarsely coincides with P_U . In the latter case, we argue as in the proof of Theorem 9.13, by induction on complexity, using the following observation: if $V \in \mathfrak{S} - \{S\}$ and a finite-index subgroup G' fixes V , then the action of G' on F_V is proper and cobounded. Moreover, G' acts with finitely many orbits on \mathfrak{S}_V , so (G', \mathfrak{S}_V) is an HHG structure on G' , enabling induction. \square

9.6 Tits alternative for HHGs

The goal of this subsection is the following theorem:

Theorem 9.15 (Tits alternative for HHGs) *Let (G, \mathfrak{S}) be an HHG and let $H \leq G$. Then H either contains a nonabelian free group or is virtually abelian.*

Before we proceed with the proof, we need some supporting results:

Proposition 9.16 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group. Then any $H \leq G$ containing an irreducible axial element is virtually \mathbb{Z} or contains a nonabelian free group.*

Proof Since G acts on \mathcal{CS} acylindrically [5], and hence $H \leq G$ does, Theorem 9.3 implies that either H is virtually cyclic or H contains irreducible axial elements g and h such that $\{h^\pm\} \cap \{g^\pm\} = \emptyset$. Propositions 6.18 and 2.17(1) enable an application of the ping-pong lemma, showing that g^N and h^N freely generate a free subgroup F for some $N > 0$. Or, one can apply [5, Corollary 14.6], which uses [32, Proposition 2.4]. \square

Lemma 9.17 *Let (G, \mathfrak{S}) be an HHG with $S \in \mathfrak{S}$ \sqsubseteq -maximal. Suppose that $H \leq G$ has bounded orbits in \mathcal{CS} and fixes some $p \in \partial\mathcal{CS}$. Then $|H| < \infty$.*

Proof By Theorem 14.3 of [5], G acts acylindrically on \mathcal{CS} , ie for each $\epsilon > 0$, there exists $R \geq 0$ and $N \in \mathbb{N}$ such that whenever $s, s' \in \mathcal{CS}$ satisfy $d_S(s, s') \geq R$, there are at most N elements $g \in G$ for which $d_S(s, g \cdot s), d_S(s', g \cdot s') \leq \epsilon$.

Fix $s \in \mathcal{CS}$ and let ϵ_1 bound the diameter of the orbit $H \cdot s$. Let γ be a $(1, 20\delta)$ -quasigeodesic ray with endpoint p and initial point s , where \mathcal{CS} is δ -hyperbolic. Then, for all $h \in H$, the ray $h \cdot \gamma$ emanates from $h \cdot s$ and has endpoint $h \cdot p = p$. This fact, together with a thin quadrilateral argument, shows that there exists $k = k(\delta)$

and R_0 such that for all $h \in H$, we have $d_S(t, h \cdot t) \leq k\delta$ whenever $t \in \gamma$ satisfies $d_S(s, t) \geq R_0$. Let $\epsilon = \max\{\epsilon_1, k\delta\}$ and let R and N be the associated constants coming from acylindricity. Then we can choose $t \in \gamma$ so that $d_S(s, t) > R$ while $d_S(s, h \cdot s), d_S(t, h \cdot t) \leq \epsilon$ for all $h \in H$, and hence $|H| \leq N$. \square

Proof of Theorem 9.15 Note that H is a countable subgroup of $\text{Aut}(\mathfrak{S})$, since G is finitely generated. We divide into cases, according to whether H fixes some $p \in \partial G$.

H fixes $p \in \partial \mathcal{CS}$ In this case, by Proposition 9.16, H is either virtually cyclic, contains a nonabelian free group, or, by Theorem 9.3, H has a bounded orbit in \mathcal{CS} . Lemma 9.17 implies that H is finite in the latter case.

H has no fixed boundary point Suppose there is an irreducible axial $g \in H$. Then either H contains a nonabelian free group or H is virtually \mathbb{Z} , by Proposition 9.16.

Otherwise, Proposition 9.2 provides $U \in \mathfrak{S} - \{S\}$ such that $H \cdot U$ is finite and the H -essential core \mathcal{Y} of in G coarsely coincides with $P_U \cap \mathcal{Y}$. By replacing H with a finite-index subgroup if necessary, we can assume that $H \cdot U = U$.

Thus we have an H -essential product HHS $(\mathcal{X}_0 \times \mathcal{X}_1, \mathfrak{S}^\times)$ with $H \leq \text{Aut}(\mathfrak{S}^\times)$ acting on $\mathcal{X}_0 \times \mathcal{X}_1$. Here \mathfrak{S}^\times consists of two disjoint subsets \mathfrak{S}_0 and \mathfrak{S}_1 , together with various domains whose associated spaces are uniformly bounded, with the property that $U_0 \perp U_1$ for all $U_0 \in \mathfrak{S}_0$ and $U_1 \in \mathfrak{S}_1$, and each \mathfrak{S}_i gives \mathcal{X}_i an HHS structure (for more on product decompositions, see [6]). Let $H_i \leq H$ be the stabilizer of some (hence any) parallel copy of \mathcal{X}_i .

Observe that $H_i \leq \text{Aut}(\mathfrak{S}_i)$ is an action on an HHS of strictly lower complexity for $i \in \{0, 1\}$, namely $(\mathcal{X}_i, \mathfrak{S}_i)$. If H_i contains no irreducible axial element, then \mathcal{X}_i decomposes as a product HHS, by Theorem 9.13. Otherwise, applying Lemma 7.5 and Theorem 9.3, we see that either H_0 or H_1 (hence H) contains a nonabelian free group, or H_i is virtually \mathbb{Z} for $i \in \{0, 1\}$. Hence, either H contains a nonabelian free subgroup, or by induction on complexity, we have a product HHS $(\prod_j L_j^i, \mathfrak{S}_i)$ such that $H_i \leq \text{Aut}(\mathfrak{S}_i)$ and each $L_j^i \cong_{\text{qi}} \mathbb{R}$. In the latter case, we conclude that H virtually acts geometrically by HHS automorphisms on $(\prod_{ij} L_j^i, \mathfrak{S}^\times)$. Hence, for some n , a finite-index subgroup of H acts by uniform quasiisometries on \mathbb{R}^n , so H is virtually abelian.

H fixes $p \in \partial G - \partial \mathcal{CS}$ In this case, H has a finite-index subgroup fixing some $U \in \text{Supp}(p)$ (so $U \not\sqsubset S$). We now argue by induction on complexity as above. \square

9.7 The “omnibus subgroup theorem”

Our next result generalizes the Handel–Mosher “omnibus subgroup theorem” from [42]. Theorem 9.20 below implies the omnibus subgroup theorem in the case where \mathcal{X} is the

mapping class group of a connected, oriented surface of finite type. In order to state the theorem, we need to restrict the class of HHSs we consider, and give some definitions.

Definition 9.18 (hierarchical acylindricity) Given an HHS $(\mathcal{X}, \mathfrak{S})$, we say that $G \leq \text{Aut}(\mathfrak{S})$ is *hierarchically acylindrical* if, for each $U \in \mathfrak{S}$, the image of $G \cap \mathcal{A}_U$ under the restriction homomorphism $\theta_U: \mathcal{A}_U \rightarrow \text{Aut}(\mathfrak{S}_U)$ acts acylindrically on $\mathcal{C}U$.

Lemma 7.5 implies that every group of automorphisms of an HHG is hierarchically acylindrical. Moreover, hierarchical acylindricity passes to subgroups. For the rest of this subsection, fix $G \leq \text{Aut}(\mathfrak{S})$ to be hierarchically acylindrical.

Definition 9.19 (active domains) Let $G \leq \text{Aut}(\mathfrak{S})$ be a group of HHS automorphisms. We say $U \in \mathfrak{S}$ is an *active domain* for G if $\text{diam}_U(\pi_U(G \cdot x))$ is unbounded for some (hence any) $x \in \mathcal{X}$. Let $\mathfrak{A}(G)$ be the set of \sqsubseteq -maximal active domains for G . Note that if $G = \langle g \rangle$, then $\mathfrak{A}(G) = \text{Big}(g)$.

Theorem 9.20 (omnibus subgroup theorem) Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with \mathfrak{S} countable and \mathcal{X} proper. Let $G \leq \text{Aut}(\mathfrak{S})$ be a countable hierarchically acylindrical subgroup. Then there exists an element $g \in G$ with $\mathfrak{A}(G) = \text{Big}(g)$. Moreover, for any $g' \in G$ and each $U \in \text{Big}(g')$, there exists $V \in \text{Big}(g)$ with $U \sqsubseteq V$.

Before we prove Theorem 9.20, we prove a lemma related to fixed boundary points of G . Throughout, $\xi(\mathfrak{S})$ denotes the complexity of $(\mathcal{X}, \mathfrak{S})$, ie the length of a longest \sqsubseteq -chain.

Definition 9.21 (fixed-point set) Given an arbitrary HHS $(\mathcal{X}, \mathfrak{S})$ and $G \leq \text{Aut}(\mathfrak{S})$, let $\text{Fix}(G) = \{p \in \partial(\mathcal{X}, \mathfrak{S}) \mid G \cdot p = p\}$.

Given $p \in \text{Fix}(G)$, let $G' \leq_{\text{f.i.}} G$ be a finite-index subgroup of G which fixes each $U \in \text{Supp}(p)$. Let $U \in \text{Supp}(p)$ and suppose that G is hierarchically acylindrical. Since G' fixes U , the restriction homomorphism θ_U gives a group G'_U which (coarsely) acts on F_U and acts acylindrically on $\mathcal{C}U$. The next lemma relates supports of fixed points to active domains.

Lemma 9.22 If $p \in \text{Fix}(G)$, $U \in \text{Supp}(p)$, and $V \in \mathfrak{A}(G)$, then either $U \perp V$ or $U = V$. Moreover, in the latter case, there exists $g'_U \in G'_U$ such that $U \in \text{Big}(g'_U)$ and $\langle g'_U \rangle \leq_{\text{f.i.}} G'_U$.

Proof We separately analyze two cases.

The case $U \pitchfork V$ or $U \sqsubset V$ Suppose that $U \pitchfork V$ or $U \sqsubset V$, ie ρ_V^U is a well-defined coarse point. Since $G' \cdot U = U$, we have that G' coarsely stabilizes the image of $P_U = F_U \times E_U \rightarrow \mathcal{X}$, which we denote by \bar{P}_U . In other words, $G' \cdot x_0$ is uniformly close to \bar{P}_U for all $x_0 \in \bar{P}_U$.

By definition of the standard embedding, if $V \pitchfork U$ or $U \sqsubset V$, then $\pi_V(\bar{P}_U) \asymp \rho_V^U \in \mathcal{C}V$ (see Section 1.3). Thus for any $x_0 \in \bar{P}_U$ and $V \in \mathfrak{S}$ with $U \pitchfork V$ or $U \sqsubset V$, we have

$$\text{diam}_V(G' \cdot x_0) \asymp 1,$$

which implies that any orbit of G' projects to a bounded subset of $\mathcal{C}V$. Hence $V \notin \mathfrak{A}(G)$, a contradiction. Thus either $V \sqsubseteq U$ or $V \perp U$.

The case $V \sqsubseteq U$ Now suppose $V \sqsubseteq U$. Since $U \in \text{Supp}(p)$, it follows that G'_U fixes a point $p_U \in \partial F_U$, where $p_U \in \partial \mathcal{C}U$. Since G is hierarchically acylindrical, G'_U acts acylindrically on $\mathcal{C}U$. By Theorem 9.3 and the fact that G'_U fixes a point in $\partial \mathcal{C}U$, one of the following holds:

- (1) G'_U has bounded orbits in $\mathcal{C}U$.
- (2) G'_U contains an element g'_U which acts axially on $\mathcal{C}U$, and $\langle g'_U \rangle \leq_{\text{f.i.}} G'_U$.

If (1) holds, then, since G'_U fixes a point of $\partial \mathcal{C}U$, Lemma 9.17 implies that $|G'_U| < \infty$. In this case, since $V \sqsubseteq U$, we have $\pi_V(G' \cdot x) = \pi_V(G'_U \cdot x)$ is finite, so $V \notin \mathfrak{A}(G)$, a contradiction.

If (2) holds, then we have found the desired element g'_U . Moreover, the existence of this element shows that U is nested into some element of $\mathfrak{A}(G)$. On the other hand, $V \sqsubseteq U$ and $V \in \mathfrak{A}(G)$, so $U = V$ by maximality of V .

Thus the only possibilities are that either $V \perp U$ or $U = V$ and the desired g'_U exists. □

We are now ready for the proof of Theorem 9.20:

Proof of Theorem 9.20 The “moreover” part of the statement follows automatically from the first assertion and the definition of $\mathfrak{A}(G)$, for if $g' \in G$ and $U \in \text{Big}(g')$, then U is an active domain for G and thus U must nest into some domain in $\mathfrak{A}(G) = \text{Big}(g)$.

We now prove the main part of the statement. By Proposition 8.4, we can assume that G acts essentially on \mathcal{X} . Let $S \in \mathfrak{S}$ to be the unique \sqsubseteq -maximal domain in \mathfrak{S} . Note that if G contains an irreducible axial element or has finite order, then we are done. Moreover, by acylindricity of the action of G on $\mathcal{C}S$, either G contains an irreducible axial or has bounded orbits in $\mathcal{C}S$ (so $S \notin \mathfrak{A}(G)$).

In particular, if G fixes a point of $\partial \mathcal{C}S$, then Lemma 9.17 implies that $|G| < \infty$, and we are done. We may therefore assume that G does not fix a point in $\partial \mathcal{C}S$ and $S \notin \mathfrak{A}(G)$.

We now argue by induction on complexity of \mathfrak{S} . Suppose that $\xi(\mathfrak{S}) = 1$. Then either there is an irreducible axial element, and we are done, or G acts with bounded orbits on \mathcal{CS} , in which case $\mathfrak{A}(G) = \emptyset$ since $\mathfrak{S} = \{S\}$, and we are done.

Now assume that the statement holds for any group of automorphisms of an HHS that satisfies the hypotheses of the theorem and has complexity less than $\xi(\mathfrak{S})$.

There are two main cases, depending on whether or not G has a fixed point in $\partial\mathcal{X}$.

First consider the case where G fixes no point of $\partial\mathcal{X}$. Proposition 9.2 implies that either G contains an irreducible axial, in which case we are done, or there exists $U \in \mathfrak{S} - \{S\}$ such that $|G \cdot U| < \infty$ and \mathcal{X} is coarsely equal to $P_U \subset \mathcal{X}$. In the latter case, after passing to a finite-index subgroup if necessary, we have G acting by automorphisms on the HHS (P_U, \mathfrak{S}) (with complexity $\xi(\mathfrak{S})$).

The remaining possibility is that G fixes some $p \in \partial\mathcal{X} - \partial\mathcal{CS}$. In this case, after passing if necessary to a finite-index subgroup, we again find $U \in \mathfrak{S} - \{S\}$ with $GU = U$ and G acting by automorphisms on the HHS (P_U, \mathfrak{S}) (with complexity $\xi(\mathfrak{S})$).

In either case, let $P_U = F_U \times E_U$, so that \mathfrak{S} contains orthogonal subsets \mathfrak{S}_U and \mathfrak{S}_U^\perp such that (F_U, \mathfrak{S}_U) and $(E_U, \mathfrak{S}_U^\perp)$ are HHSs of complexity at most $\xi(\mathfrak{S}) - 1$. By replacing G with an index-2 subgroup if necessary, we can assume that G stabilizes \mathfrak{S}_U . Moreover, G stabilizes $\mathfrak{S}_U^{\perp o} := \{V \in \mathfrak{S} : V \perp U\}$, ie $\mathfrak{S}_U^{\perp o}$ is obtained from \mathfrak{S}_U^\perp by removing W if $W \not\perp U$, where $W \sqsubset S$ is the (arbitrarily chosen) \sqsubseteq -minimal “container” domain containing everything orthogonal to U .

Recall that \mathfrak{S}_U^\perp consists of all domains $V \in \mathfrak{S}$ with $V \perp U$ along with a \sqsubseteq -minimal domain $W \in \mathfrak{S}$ such that $V \sqsubseteq W$ for all $V \perp U$. If W is the unique such domain, then $G \cdot W = W$, and thus G admits a natural restriction homomorphism to $\text{Aut}(\mathfrak{S}_U^\perp)$.

Otherwise, $W \notin \mathfrak{A}(G)$. Since $\text{diam}_W(\pi_W(P_U)) \asymp 1$, we may replace W with a single point W^* such that $\mathcal{CW}^* = \{*\}$. From this we obtain a new HHS structure on $(E_U, \mathfrak{S}_U^{\perp o})$, where $\mathfrak{S}_U^{\perp o} = \mathfrak{S}_U^\perp - W \cup \{W^*\}$, by making the obvious alterations to the projection and domain maps associated to W .

In either case, let G_U be the image of G under the usual restriction homomorphism $\mathcal{A}_U \rightarrow \text{Aut}(\mathfrak{S}_U)$. Let G_U^\perp be the image of G under the restriction map $\psi: \mathcal{A}_U \rightarrow \text{Aut}(\mathfrak{S}_U^\perp)$ or, if W is not unique, we take G_U^\perp be the image of $\psi: \mathcal{A}_U \rightarrow \text{Aut}(\mathfrak{S}_U^{\perp o})$ defined as follows: for all $g \in \mathcal{A}_U$, the map $\psi(g)$ acts like g on $\mathfrak{S}_U^{\perp o}$ and acts as the identity on \mathcal{CW}^* .

Hence we have HHSs (F_U, \mathfrak{S}_U) and $(E_U, \mathfrak{S}_U^\perp)$, of complexity at most $\xi(\mathfrak{S}) - 1$, and groups $G_U \leq \text{Aut}(\mathfrak{S}_U)$ and $G_U^\perp \leq \text{Aut}(\mathfrak{S}_U^\perp)$ or $\text{Aut}(\mathfrak{S}_U^{\perp o})$ that satisfy the hypotheses of the theorem.

We now show that $\mathfrak{A}(G) = \mathfrak{A}(G_U) \sqcup \mathfrak{A}(G_U^\perp)$. The inclusions $\mathfrak{A}(G_U), \mathfrak{A}(G_U^\perp) \rightarrow \mathfrak{A}(G)$ are obvious. Conversely, suppose that $V \in \mathfrak{A}(G)$. If $U \in \text{Supp}(p)$ for some $p \in \text{Fix}(G)$ (as we can assume is the case whenever $\text{Fix}(G) \neq \emptyset$), then Lemma 9.22 implies that $V = U$ or $V \perp U$, ie $V \in \mathfrak{S}_U \sqcup \mathfrak{S}_U^\perp$ (and, if $V = W$, then W is the unique container and hence G -invariant). Otherwise, the proof of Lemma 9.22 shows that $V \perp U$ or $V \sqsubseteq U$. Hence $V \in \mathfrak{A}(G_U) \sqcup \mathfrak{A}(G_U^\perp)$.

By induction on complexity, either $\mathfrak{A}(G_U) = \emptyset$, or there exists $\bar{h} \in G_U$ with $\text{Big}(\bar{h}) = \mathfrak{A}(G_U)$. Likewise, either $\mathfrak{A}(G_U^\perp) = \emptyset$, or there exists $\bar{h}^\perp \in G_U^\perp$ with $\text{Big}(\bar{h}^\perp) = \mathfrak{A}(G_U^\perp)$. If $\mathfrak{A}(G_U) = \emptyset$ (resp. $\mathfrak{A}(G_U^\perp) = \emptyset$), we take $\bar{h} = 1$ (resp. $\bar{h}^\perp = 1$). Since $\mathfrak{A}(G) = \mathfrak{A}(G_U) \sqcup \mathfrak{A}(G_U^\perp)$, we must use \bar{h} and \bar{h}^\perp to find $g \in G$ with $\text{Big}(g) = \mathfrak{A}(G_U) \sqcup \mathfrak{A}(G_U^\perp)$.

Choose $h, h^\perp \in G$ stabilizing \mathfrak{S}_U and \mathfrak{S}_U^\perp and mapping to $\bar{h} \in G_U$ and $\bar{h}^\perp \in G_U^\perp$, respectively, under the above restriction maps. Let k be the image of h in G_U^\perp and let k^\perp be the image of h^\perp in G_U , so we are considering the action of h and k^\perp on \mathfrak{S}_U and h^\perp and k on \mathfrak{S}_U^\perp .

Let $\{U_1, \dots, U_\ell\} = \text{Big}(\bar{h}) \subset \mathfrak{S}_U$ and let $\{V_1, \dots, V_k\} = \text{Big}(\bar{h}^\perp) \subset \mathfrak{S}_U^\perp$. By passing to powers, we can assume that $hU_i = U_i$ and $h^\perp V_j = V_j$ for all i and j . Since the action of G_U on \mathfrak{S}_U preserves $\mathfrak{A}(G_U)$, and the action of G_U^\perp on \mathfrak{S}_U^\perp preserves $\mathfrak{A}(G_U^\perp)$, we can, by passing to powers, assume that k^\perp preserves each U_i and k preserves each V_j .

Let $N \gg 0$ and consider $F = \langle h^N, (h^\perp)^{10N} \rangle \leq G$. The image of F in G_U is $\bar{F} = \langle \bar{h}^N, (k^\perp)^{10N} \rangle$, and the image of F in G_U^\perp is $\bar{F}^\perp = \langle k^N, (\bar{h}^\perp)^{10N} \rangle$. The above discussion shows that \bar{F} acts acylindrically on each $\mathcal{C}U_i$ and \bar{F}^\perp acts acylindrically on each $\mathcal{C}V_j$. Examining the various cases that arise according to how k acts on the $\mathcal{C}V_i$ and how k^\perp acts on the $\mathcal{C}U_i$ shows that, in each case, there exists $g \in F$ whose image in \bar{F} is loxodromic on each $\mathcal{C}U_i$ and whose image in \bar{F}^\perp is loxodromic on each V_j . Hence $\text{Big}(g) = \mathfrak{A}(G_U) \sqcup \mathfrak{A}(G_U^\perp)$, as required. \square

The following is an immediate but useful corollary of Theorem 9.20:

Corollary 9.23 *If $G \leq \text{Aut}(\mathfrak{S})$ is hierarchically acylindrical, then $\mathfrak{A}(G)$ is pairwise orthogonal.*

9.8 Rank-rigidity for some CAT(0) cube complexes

We now use Theorems 9.14 and 9.13 to reprove the rank-rigidity theorem of Caprace and Sageev [18], in the case where the cube complex in question contains a factor system. See Section 10 for a discussion of the definition, and the definition of the simplicial boundary $\partial_\Delta \mathcal{X}$ of the cube complex \mathcal{X} .

Corollary 9.24 (rank-rigidity for cube complexes with factor systems) *Let \mathcal{X} be an unbounded CAT(0) cube complex with a factor system \mathfrak{F} . Let G act on \mathcal{X} and suppose that one of the following holds:*

- (1) G acts on \mathcal{X} properly and cocompactly.
- (2) G acts on \mathcal{X} with no fixed point in $\mathcal{X} \cup \partial_{\Delta}\mathcal{X}$.

Then \mathcal{X} contains a G -invariant convex subcomplex \mathcal{Y} such that either G contains a rank-one isometry of \mathcal{Y} or $\mathcal{Y} = \mathcal{A} \times \mathcal{B}$, where \mathcal{A} and \mathcal{B} are unbounded convex subcomplexes.

We remark that in view of [36, Remark 5.3], we could have stated the corollary in terms of fixed points in the CAT(0) boundary rather than the simplicial boundary, but we have opted for the latter because of the close relationship between the simplicial and HHS boundaries discussed in Section 10.

Proof of Corollary 9.24 First suppose that G acts on \mathcal{X} essentially, in the sense that every halfspace contains points of some G -orbit arbitrarily far from the associated hyperplane (in particular, \mathcal{X} does not contain a G -invariant proper convex subcomplex). Recall from [5] that \mathcal{X} is equipped with a hierarchically hyperbolic structure $(\mathcal{X}, \mathfrak{S})$, where \mathfrak{S} is the set of *factored contact graphs* of elements of \mathfrak{F} , and that $G \leq \text{Aut}(\mathfrak{S})$. If G acts on \mathcal{X} properly and cocompactly, then (G, \mathfrak{S}) is an HHG; if G acts on \mathcal{X} with no fixed point in $\partial_{\Delta}\mathcal{X}$, then G does not fix a point in $\partial(\mathcal{X}, \mathfrak{S})$, by Theorem 10.1 below.

Depending on which hypothesis we invoke, one of Theorem 9.14 or Theorem 9.13 implies that either there exists $g \in G$ which is rank-one (in the HHS sense) or there exists $U \in \mathfrak{S}$ such that \mathcal{X} coarsely coincides with P_U , which has unbounded factors, and $G'U = U$ for some finite-index $G' \leq G$. In the former case, elements that are rank-one in the HHS sense (with respect to this particular HHS structure on \mathcal{X}) are rank-one isometries of \mathcal{X} in the usual sense, by [36, Proposition 5.1] and the definition of a factor system [5, Section 8].

In the latter case, $P_U = F_U \times E_U$ is a genuine convex product subcomplex with unbounded factors (see [5]). Let $g \in G$ and suppose that H is a hyperplane intersecting P_U but not gP_U . Since P_U is coarsely equal to \mathcal{X} and \mathcal{X} is essential, the halfspace of P_U separated from gP_U by H contains points arbitrarily far from H , whence P_U and gP_U cannot lie at finite Hausdorff distance. This contradicts that P_U is invariant under a finite-index subgroup of G . Hence P_U and gP_U are *parallel* for all $g \in G$, ie they are crossed by exactly the same hyperplanes. Thus $\mathcal{X} = P_U \times Y$ for some compact cube complex Y , whence Y is a single point, by essentiality. It follows that

P_U is G -invariant, so $\mathcal{X} = P_U$ by essentiality. Hence \mathcal{X} decomposes as a product with unbounded factors. In general, we first replace \mathcal{X} by its G -essential core in either preceding argument, using Proposition 3.5 of [18]. □

Remark 9.25 Question A of [6] asks whether the existence of a proper cocompact action of G on the CAT(0) cube complex \mathcal{X} ensures that \mathcal{X} contains a factor system. By a result in [5], the answer is affirmative provided \mathcal{X} embeds as a convex subcomplex in the universal cover of the Salvetti complex of some right-angled Artin group. Although it is a strong condition, we believe that such embeddings always exist (although there is in general no algebraic relationship between G and the RAAG).

9.8.1 The Poisson boundary of an HHG Results in [5] show that, if G is an HHG with $\text{diam } \mathcal{CS} = \infty$, then, given a nonelementary probability measure μ on G , the boundary $\partial \mathcal{CS}$ admits a μ -stationary measure making it the Poisson boundary. As a topological model of the Poisson boundary, $\partial \mathcal{CS}$ is unsatisfactory since it need not be compact. However:

Theorem 9.26 (the HHS boundary is the Poisson boundary) *Let (G, \mathfrak{S}) be an HHG with $\text{diam } \mathcal{CS} = \infty$, μ be a nonelementary probability measure on G with finite entropy and finite first logarithmic moment, and ν the resulting μ -stationary measure on ∂G . Then $(\partial G, \nu)$ is the Poisson boundary for (G, μ) .*

We use acylindricity of the action of G on \mathcal{CS} and a result of Maher and Tiozzo [54]:

Theorem 9.27 [54, Theorem 1.5] *Let G be a countable group which acts acylindrically on a separable Gromov-hyperbolic space X . If μ is a nonelementary probability measure on G with finite entropy and finite first logarithmic moment with corresponding stationary measure ν , then $(\partial X, \nu)$ is the Poisson boundary for (G, μ) .*

Proof of Theorem 9.26 Let μ be a nonelementary probability measure on G with finite entropy and finite first logarithmic moment. Since G acts on \mathcal{CS} acylindrically [5, Theorem 14.3], Theorem 9.27 implies that there exists a μ -stationary measure ν' on $\partial \mathcal{CS}$ such that $(\partial \mathcal{CS}, \nu')$ is the Poisson boundary for (G, μ) .

Let $f: \partial \mathcal{CS} \hookrightarrow \partial G$ be the embedding from Proposition 2.13. By Lemma 9.7, $f(\partial \mathcal{CS})$ is Borel, so for any Borel subset $V \subset \partial G$, the set $V \cap f(\partial \mathcal{CS})$ is Borel. Define a new measure ν on ∂G by $\nu(V) = \nu'(f^{-1}(V \cap f(\partial \mathcal{CS})))$.

Since f is G -equivariant, it follows that ν is μ -stationary. By definition, $f(\partial \mathcal{CS})$ has full ν -measure. Moreover, $(\partial G, \nu)$ is a μ -boundary by measurability of f and it is maximal since $(\partial \mathcal{CS}, \nu')$ is maximal. Thus $(\partial G, \nu)$ models the Poisson boundary for (G, μ) . □

10 Case study: CAT(0) cube complexes

Throughout this section, \mathcal{X} is a locally finite CAT(0) cube complex in which each collection of pairwise-intersecting hyperplanes is (not necessarily uniformly) finite. In [5], it is shown that CAT(0) cube complexes can often be given HH structures using certain collections of convex subcomplexes called *factor systems*. We recall the definition in Section 10.2. When \mathfrak{F} is a factor system for \mathcal{X} , denote the resulting HH structure by $(\mathcal{X}, \overline{\mathfrak{F}})$.

The *simplicial boundary* of \mathcal{X} was introduced in [36]; we recall the definition below. The simplicial boundary and the HH structure are closely related by the following theorem:

Theorem 10.1 (simplicial and HHS boundaries) *Let \mathcal{X} be a CAT(0) cube complex with a factor system \mathfrak{F} . There is a topology \mathcal{T} on the simplicial boundary $\partial_{\Delta}\mathcal{X}$ such that:*

- (1) *There is a homeomorphism $b: (\partial_{\Delta}\mathcal{X}, \mathcal{T}) \rightarrow \partial(\mathcal{X}, \overline{\mathfrak{F}})$.*
- (2) *For each component C of the simplicial complex $\partial_{\Delta}\mathcal{X}$, the inclusion $C \hookrightarrow (\partial_{\Delta}\mathcal{X}, \mathcal{T})$ is an embedding.*

In particular, if \mathfrak{F} and \mathfrak{F}' are factor systems on \mathcal{X} , then $\partial(\mathcal{X}, \overline{\mathfrak{F}})$ is homeomorphic to $\partial(\mathcal{X}, \overline{\mathfrak{F}'})$.

We prove Theorem 10.1 in Section 10.3.

Remark 10.2 Proposition 3.37 of [36] relates $\partial_{\Delta}\mathcal{X}$ to its Tits boundary $\partial_{\mathcal{T}}\mathcal{X}$. There is an analogous relationship between the HHS boundary and the visual boundary when the former is defined (ie when \mathcal{X} has a factor system). Specifically, one can show that there is a commutative diagram

$$\begin{array}{ccc}
 \partial_{\Delta}\mathcal{X} & \xrightarrow{I} & \partial_{\mathcal{T}}\mathcal{X} \\
 \downarrow b & & \downarrow \text{id} \\
 \partial(\mathcal{X}, \overline{\mathfrak{F}}) & \xrightarrow{J} & \partial_{\text{vis}}\mathcal{X}
 \end{array}$$

where b is the bijection from Theorem 10.1, I and J are embeddings, J is $\frac{\pi}{2}$ -quasisurjective, and $\partial(\mathcal{X}, \overline{\mathfrak{F}})$ is a deformation retract of $\partial_{\text{vis}}\mathcal{X}$. The CAT(0) metric on \mathcal{X} is far afield from our present discussion, since the HHS structure depends only on the combinatorics of \mathcal{X} and is insensitive to changes in the CAT(0) metric (unlike the visual boundary [22]), so we will not give a detailed proof of the above. The top part

of the diagram comes from [36, Proposition 3.37]; the missing ingredient is to show that J is an embedding, which is a tedious exercise in the definition of the topology on $\partial(\mathcal{X}, \bar{\mathfrak{F}})$.

10.1 The simplicial boundary

We first recall the necessary definitions from [36].

Definition 10.3 (UBS, boundary equivalence, minimal UBS) A set \mathcal{U} of hyperplanes in \mathcal{X} is a *unidirectional boundary set (UBS)* if each of the following holds:

- \mathcal{U} is infinite.
- If $U, U' \in \mathcal{U}$ and a hyperplane V separates U and U' , then $V \in \mathcal{U}$.
- If $U, U', U'' \in \mathcal{U}$ are pairwise disjoint, then one of them separates the other two.
- For all hyperplanes W , at least one component of $\mathcal{X} - W$ contains at most finitely many elements of \mathcal{U} .

Given UBSs \mathcal{U} and \mathcal{V} , let $\mathcal{U} \preceq \mathcal{V}$ if all but finitely many elements of \mathcal{U} lie in \mathcal{V} . The UBSs \mathcal{U} and \mathcal{V} are *boundary equivalent* if $\mathcal{U} \preceq \mathcal{V}$ and $\mathcal{V} \preceq \mathcal{U}$, and \mathcal{U} is *minimal* if \mathcal{U} and \mathcal{V} are boundary equivalent for all UBSs \mathcal{V} with $\mathcal{V} \preceq \mathcal{U}$.

Remark 10.4 Any infinite set of hyperplanes which is closed under separation contains a minimal UBS [36, Lemma 3.7].

Proposition 3.10 of [36] shows that each UBS \mathcal{U} is boundary equivalent to a UBS of the form $\bigsqcup_{i=0}^k \mathcal{U}_i$, where each \mathcal{U}_i is a minimal UBS, and this decomposition is unique up to boundary equivalence. Up to reordering, for $0 \leq i < j \leq k$, for all but finitely many $U \in \mathcal{U}_j$, the hyperplane U intersects all but finitely many elements of \mathcal{U}_i . In this situation, \mathcal{U}_j *dominates* \mathcal{U}_i . The number k is the *dimension* of \mathcal{U} .

Definition 10.5 (simplicial boundary) A *k -simplex at infinity* is a boundary equivalence class of k -dimensional UBSs. If v and v' are simplices at infinity, represented by boundary sets \mathcal{V} and \mathcal{V}' , then $\mathcal{V} \cap \mathcal{V}'$ is, if infinite, a boundary set representing the simplex $v \cap v'$. The *simplicial boundary* $\partial_{\Delta} \mathcal{X}$ of \mathcal{X} is the simplicial complex with a closed k -simplex for each k -dimensional simplex at infinity; the simplex u represented by the UBS \mathcal{U} is a face of the simplex v , represented by \mathcal{V} , if $\mathcal{U} \preceq \mathcal{V}$.

Remark 10.6 (boundaries of convex subcomplexes) It is shown in [36] that if $\mathcal{Y} \subseteq \mathcal{X}$ is a convex subcomplex, then $\partial_{\Delta} \mathcal{Y} \subset \partial_{\Delta} \mathcal{X}$ in a natural way: each simplex at infinity in $\partial_{\Delta} \mathcal{Y}$ corresponds to a UBS in \mathcal{X} consisting of hyperplanes that intersect \mathcal{Y} , and these hyperplanes intersect in \mathcal{X} exactly when they intersect in \mathcal{Y} , by convexity.

10.1.1 Visibility

Definition 10.7 (visible simplex) The simplex u at infinity is *visible* if there exists a combinatorial geodesic ray γ in $\mathcal{X}^{(1)}$ such that the set \mathcal{U} of hyperplanes intersecting γ represents the boundary–equivalence class u . Otherwise, the simplex u at infinity is *invisible*. If every simplex at infinity is visible, then \mathcal{X} is *fully visible*.

Theorem 3.19 of [36] states that every maximal simplex of $\partial_\Delta \mathcal{X}$ is visible. Visibility is also related to a subtlety in the definition of $\partial_\Delta \mathcal{X}$:

Remark 10.8 (visibility and proper faces) Let $\bigsqcup_{i=0}^k \mathcal{U}_i$ be a UBS, with each \mathcal{U}_i a minimal UBS, numbered so that for $0 \leq i < j \leq k$ and all $U \in \mathcal{U}_j$, we have that $U \cap V \neq \emptyset$ for all but finitely many $V \in \mathcal{U}_i$. If, up to modifying each \mathcal{U}_i in its boundary equivalence class, $U \cap V \neq \emptyset$ whenever $U \in \mathcal{U}_i$, $V \in \mathcal{U}_j$, and $i \neq j$, then the simplex u represented by $\bigsqcup_{i=0}^k \mathcal{U}_i$ is visible. In this case, \mathcal{X} contains an isometrically embedded (on the 1–skeleton) cubical orthant, the boundary of whose convex hull is u . Conversely, if we know that each \mathcal{U}_i represents a visible 0–simplex, then $\bigsqcup_{i \in K} \mathcal{U}_i$ represents a visible simplex at infinity for any $K \subset \{0, \dots, k\}$, as is proved in [36]. If this does not occur, then there may be subsets $K \subset \{0, \dots, k\}$ such that $\bigsqcup_{i \in K} \mathcal{U}_i$ represents an invisible simplex at infinity, or is not even a UBS (by virtue of failing to satisfy the condition on separation). In other words, when \mathcal{X} is not fully visible, simplices at infinity may have *proper faces* that are not genuine simplices at infinity represented by UBSs.

A visible simplex $v \subseteq \partial_\Delta \mathcal{X}$ is *represented* by the combinatorial geodesic ray $\gamma \subseteq \mathcal{X}^{(1)}$ if the UBS of hyperplanes intersecting γ represents the boundary equivalence class v .

Remark 10.9 (factor systems and visibility) Conjecture 2.8 of [4] states that if \mathcal{X} is a CAT(0) cube complex on which some group acts geometrically, then \mathcal{X} is fully visible. Also, the proof of Theorem 10.1 shows that, if \mathcal{X} contains a factor system (see Definition 10.10), then every simplex of $\partial_\Delta \mathcal{X}$ is visible. This is another reason for interest in Question A of [6], which asks whether every CAT(0) cube complex on which some group acts geometrically contains a factor system.

10.2 Factor systems: hierarchical hyperbolicity of cube complexes

We now summarize results from [5] yielding hierarchically hyperbolic structures on \mathcal{X} . We refer the reader to Section 2 of [5] for discussion of convex subcomplexes and the gate map $\mathfrak{g}_F: \mathcal{X} \rightarrow F$ from \mathcal{X} to any convex subcomplex F .

Recall that each hyperplane H of \mathcal{X} lies in a *carrier*, $\mathcal{N}(H)$, which is the union of closed cubes intersecting H . For all H , there is a cubical isomorphism $\mathcal{N}(H) \cong H \times [-\frac{1}{2}, \frac{1}{2}]$; a subcomplex of \mathcal{X} which is the image under the inclusion $\mathcal{N}(H) \rightarrow \mathcal{X}$

of either of the subcomplexes $H \times \{\frac{1}{2}\}$ or $H \times \{-\frac{1}{2}\}$ is a *combinatorial hyperplane*. We say that two convex subcomplexes F and F' of \mathcal{X} are *parallel* if for any hyperplane H of \mathcal{X} , we have $H \cap F \neq \emptyset$ if and only if $H \cap F' \neq \emptyset$. We let \mathfrak{F} denote a choice of representatives for each parallelism class of elements of \mathfrak{F} .

Definition 10.10 A *factor system* \mathfrak{F} is a set of convex subcomplexes such that:

- (1) Each nontrivial combinatorial hyperplane of \mathcal{X} belongs to \mathfrak{F} , as does each convex subcomplex parallel to a nontrivial combinatorial hyperplane.
- (2) $\mathcal{X} \in \mathfrak{F}$.
- (3) There exists $\xi > 0$ such that, for all $F, F' \in \mathfrak{F}$, either

$$g_F(F') \in \mathfrak{F} \quad \text{or} \quad \text{diam}(g_F(F')) \leq \xi.$$

- (4) There exists $\Delta \geq 1$ such that each point in \mathcal{X} belongs to at most Δ elements of \mathfrak{F} .

We require that elements of \mathfrak{F} are not single points. (This condition is only imposed to ensure that nesting and orthogonality are mutually exclusive: if F is a single point and $F' \in \mathfrak{F}$, then $F \perp F'$ and $F \sqsubseteq F'$, so we exclude this situation.)

The *contact graph* $\mathcal{C}\mathcal{X}$ of \mathcal{X} (see [37]) has a vertex for each hyperplane, with two hyperplanes joined by an edge if no third hyperplane separates them. If $F \sqsubseteq \mathcal{X}$ is a convex subcomplex, then F is a CAT(0) cube complex whose hyperplanes have the form $H \cap F$, where H is a hyperplane of \mathcal{X} , and, by convexity of F , this yields an embedding $\mathcal{C}F \hookrightarrow \mathcal{C}\mathcal{X}$ of \mathcal{F} as a full subgraph.

Given a factor system \mathfrak{F} on \mathcal{X} , we define the *factored contact graph* $\widehat{\mathcal{C}}F$ of each $F \in \mathfrak{F}$ as follows. Begin with $\mathcal{C}F$. For each parallelism class of subcomplexes $F' \in \mathfrak{F}$, parallel to a proper subcomplex of F that is not a single 0-cube, we have $\mathcal{C}F' \subsetneq \mathcal{C}F$, and we cone off $\mathcal{C}F'$ by adding a vertex $v_{F'}$ to $\mathcal{C}F$ and joining each vertex of $\mathcal{C}F' \subset \mathcal{C}F$ to $v_{F'}$. The resulting factored contact graph $\widehat{\mathcal{C}}F$ is uniformly quasiisometric to a tree [5, Proposition 8.24].

Let us now define the maps $\pi_F: \mathcal{X} \rightarrow 2^{\widehat{\mathcal{C}}F}$. For each $F \in \mathfrak{F}$, given $x \in \mathcal{X}$, let $g_F(x) \in F$ be its gate. There is a nonempty finite set of hyperplanes H of F that are not separated from x by any other hyperplane; these form a nonempty clique in $\mathcal{C}F$, to which we send x . We then compose with $2^{\mathcal{C}F} \hookrightarrow 2^{\widehat{\mathcal{C}}F}$ to yield $\pi_F: \mathcal{X} \rightarrow 2^{\widehat{\mathcal{C}}F}$ sending each point to a clique.

Let $F \sqsubseteq F'$ if F is parallel to a subcomplex of F' , and $F \perp F'$ if there is a cubical isometric embedding $F \times F' \rightarrow \mathcal{X}$ (after possibly varying F and F' in their parallelism classes). Otherwise, F and F' are transverse. With these definitions, it is shown in [5; 6] that $(\mathcal{X}, \widetilde{\mathfrak{F}})$ is a hierarchically hyperbolic space.

10.3 Relating the simplicial and HHS boundaries

Fix \mathcal{X} with a factor system \mathfrak{F} ; necessarily, \mathcal{X} is uniformly locally finite.

Proof of Theorem 10.1 We will first exhibit a bijection $b: \partial_{\Delta}\mathcal{X} \rightarrow \partial(\mathcal{X}, \overline{\mathfrak{F}})$. We then define $\mathcal{T} = \{b^{-1}(\mathcal{O})\}$, where \mathcal{O} varies over all open sets in $\partial(\mathcal{X}, \overline{\mathfrak{F}})$, so as to make b a homeomorphism. It then suffices to verify that this topology agrees with the simplicial topology on each component of $\partial_{\Delta}\mathcal{X}$; the “in particular” statement then follows immediately.

Reduction to the single-simplex case Let m be a maximal simplex of $\partial_{\Delta}\mathcal{X}$. By the definition of the simplicial boundary, m is a simplex at infinity, ie it is represented by some UBS \mathcal{M} . Moreover, by [36, Theorem 3.19], we can take \mathcal{M} to be the set of hyperplanes intersecting some combinatorial geodesic ray γ_m emanating from the (fixed) basepoint x_0 . Let \mathcal{Y}_m be the convex hull of γ_m .

By [5, Lemma 8.4], $\mathfrak{F}_m = \{F \cap \mathcal{Y}_m : F \in \mathfrak{F}\}$ is a factor system. (We emphasize that \mathfrak{F}_m is a set, not a multiset: if $F, F' \in \mathfrak{F}$ satisfy $F \cap \mathcal{Y}_m = F' \cap \mathcal{Y}_m$, we count this subcomplex once.) We adopt the following convention: for each $F \cap \mathcal{Y}_m \in \mathfrak{F}_m$, we assume that F has been chosen so that F is \sqsubseteq -minimal among all $F' \in \mathfrak{F}$ with $F' \cap \mathcal{Y}_m = F \cap \mathcal{Y}_m$. (Note that there is a unique such minimal F : if $F \cap \mathcal{Y}_m = F' \cap \mathcal{Y}_m$, then $F \cap \mathcal{Y}_m = F \cap F' \cap \mathcal{Y}_m$, and $F \cap F' \sqsubseteq F, F'$.)

Also, if $F \sqsubseteq F'$, then $F \cap \mathcal{Y}_m \sqsubseteq F' \cap \mathcal{Y}_m$, obviously. Conversely, suppose that $F \cap \mathcal{Y}_m \sqsubseteq F' \cap \mathcal{Y}_m$. Let $F'' = \mathfrak{g}_F(F')$, so $F'' \sqsubseteq F'$ and $F'' \sqsubseteq F$. Then $F'' \cap \mathcal{Y}_m = F \cap \mathcal{Y}_m$, so $F'' = F$ by minimality, whence $F \sqsubseteq F'$.

If $F \perp F'$, then convexity of \mathcal{Y}_m implies $(F \times \perp F') \cap \mathcal{Y}_m = (F \cap \mathcal{Y}_m) \times (F' \cap \mathcal{Y}_m)$, so $(F \cap \mathcal{Y}_m) \perp (F' \cap \mathcal{Y}_m)$. Conversely, suppose that $(F \cap \mathcal{Y}_m) \perp (F' \cap \mathcal{Y}_m)$. For brevity, let $A = F \cap \mathcal{Y}_m$ and $B = F' \cap \mathcal{Y}_m$, so that \mathcal{X} contains $A \times B$. By Lemma 10.13, there exist $F_A, F_B \in \mathfrak{F}$ such that $A \subset F_A, B \subset F_B$ and $F_A \perp F_B$. Let $F'_A = F \cap F_A$ and $F'_B = F' \cap F_B$. Then $F'_A \cap \mathcal{Y}_m = F \cap \mathcal{Y}_m$ and $F'_A \sqsubseteq F$, so minimality of F implies $F'_A = F$; similarly $F'_B = F'$. But since $F_A \perp F_B$ and $F'_A \sqsubseteq F_A$ and $F'_B \sqsubseteq F_B$, we have $F \perp F'$.

It follows that there is a hieromorphism $(\mathcal{Y}_m, \overline{\mathfrak{F}}_m) \rightarrow (\mathcal{X}, \overline{\mathfrak{F}})$ defined as follows: the map $\mathcal{Y}_m \rightarrow \mathcal{X}$ is the inclusion; the map $\overline{\mathfrak{F}}_m \rightarrow \overline{\mathfrak{F}}$ is given by $F \cap \mathcal{Y}_m \mapsto F$ for each $F \cap \mathcal{Y}_m \in \overline{\mathfrak{F}}_m$ (where F is \sqsubseteq -minimal in \mathfrak{F} with the given intersection with \mathcal{Y}), and for each $F \cap \mathcal{Y}_m$, the map $\widehat{\mathcal{C}}(F \cap \mathcal{Y}_m) \rightarrow \widehat{\mathcal{C}}F$ is the inclusion on contact graphs and sends cone vertices to cone vertices in the obvious way.

We will see below that $\mathcal{Y}_m = \prod_{i=0}^k \mathcal{Y}_{m_i}$, where each \mathcal{Y}_{m_i} has the property that $\partial\widehat{\mathcal{C}}(F \cap \mathcal{Y}_{m_i}) = \emptyset$ for all $F \in \mathfrak{F}$ except for a unique $\tilde{F}_i \in \mathfrak{F}$ for which $\partial\widehat{\mathcal{C}}(\tilde{F}_i \cap \mathcal{Y}_{m_i})$

consists of a single point p_i . Moreover, $\ddot{F}_i \perp \ddot{F}_j$ for $i \neq j$. Lemma 10.11 shows that for each $F \cap \mathcal{Y}_m$, the map $\widehat{C}(F \cap \mathcal{Y}_m) \hookrightarrow \widehat{C}F$ is a uniform quasisisometric embedding, inducing a boundary map, ie p_i may be regarded as a point in $\partial \widehat{C}\ddot{F}_i$ for each i . We thus obtain an injective map $b_m: \partial(\mathcal{Y}_m, \overline{\mathfrak{F}}_m) \rightarrow \partial(\mathcal{X}, \overline{\mathfrak{F}})$ given by

$$b_m \left(\sum_{i=0}^k a_i m_i \right) = \sum_{i=0}^k a_i p_i.$$

Constructing b We will observe below that if m, m' are maximal simplices, then the associated collections $\{p_i\}_{i=0}^k$ and $\{p'_i\}_{i=0}^{k'}$ intersect in a set corresponding precisely to the set of 0–simplices of $m \cap m'$. It follows that the maps constructed above are compatible, ie $b_m|_{\mathcal{Y}_m \cap m'} = b_{m'}|_{\mathcal{Y}_m \cap m'}$ and that, if m and m' are disjoint maximal simplices of $\partial_\Delta \mathcal{X}$, then b_m and $b_{m'}$ have disjoint images. Pasting together the b_m thus yields an injection $b: \partial_\Delta \mathcal{X} \rightarrow \partial(\mathcal{X}, \overline{\mathfrak{F}})$.

Surjectivity of b Let $\{\ddot{F}_i\}_{i=1}^k$ be a support set in $\overline{\mathfrak{F}}$, choose for each i a point $p_i \in \partial \widehat{C}\ddot{F}_i$ and let $p = \sum_{i=1}^k a_i p_i$. For each i , let σ_i be a geodesic ray in the quasitree $\widehat{C}\ddot{F}_i$ joining $\pi_{\widehat{C}\ddot{F}_i}(x_0)$ to p_i . Let $\{H_n^i\}$ be a sequence of hyperplanes of \mathcal{X} , each crossing \ddot{F}_i , corresponding to vertices of σ_i , ordered so that H_n^i separates H_{n+1}^i from x_0 . Any $P \in \overline{\mathfrak{F}}$ that crosses infinitely many of these hyperplanes satisfies $\ddot{F}_i \sqsubseteq P$, or else some element of $\overline{\mathfrak{F}}$ nested into \ddot{F}_i would “kill” the p_i direction in $\partial \widehat{C}\ddot{F}_i$. Every simplex of $\partial_\Delta(\prod_{j=0}^k \ddot{F}_j) \subset \partial_\Delta \mathcal{X}$ is visible, from which it is easy to check that there is a unique (up to boundary–equivalence) minimal UBS \mathcal{M}_i containing $\{H_n^i\}$ and representing a 0–simplex m_i of $\partial_\Delta \mathcal{X}$ such that $\{m_0, \dots, m_k\}$ span a simplex m . By definition, $b_m(\sum_i a_i m_i) = p$.

Analysis of components To prove that each component C of $\partial_\Delta \mathcal{X}$, with the simplicial topology, is embedded in $(\partial_\Delta \mathcal{X}, \mathcal{T})$, we must show that $b \circ \text{id}: \partial_\Delta \mathcal{X} \rightarrow \partial(\mathcal{X}, \overline{\mathfrak{F}})$ restricts to an embedding on C , where $\text{id}: \partial_\Delta \mathcal{X} \rightarrow (\partial_\Delta \mathcal{X}, \mathcal{T})$ is the identity. Let m be a maximal simplex of $\partial_\Delta \mathcal{X}$. Let $p = \sum_i a_i p_i \in b \circ \text{id}(M)$ and let $\mathcal{N} = \mathcal{N}_{\{U_i\}, \epsilon}(p) \cap \partial(\mathcal{Y}_m, \overline{\mathfrak{F}}_m)$ be a basic neighborhood of p , as defined in Section 1.1. Observe that \mathcal{N} is completely nonremote, whence it is clear from the definition that $b_m^{-1}(\mathcal{N})$ is basic in the simplicial topology on $\partial_\Delta \mathcal{Y}_m = m$, so b_m is continuous. It follows that $b \circ \text{id}$ is continuous. A similar argument shows that the restriction of $b \circ \text{id}$ to C is an open map. To complete the proof, it now suffices to produce the F_i and analyze their factored contact graphs, which we do in the next several steps.

Visibility of faces of m Let m be a maximal simplex of $\partial_\Delta \mathcal{X}$ and observe that $\partial_\Delta \mathcal{Y}_m$ is exactly the simplex m . We now verify that each face of m is a visible simplex at infinity. Let m_0, \dots, m_k be the 0–simplices of m ; represent m_i by a minimal UBS \mathcal{M}_i such that \mathcal{M}_j dominates \mathcal{M}_i when $i < j$ and $\mathcal{M} = \bigsqcup_{i=0}^k \mathcal{M}_i$. Recall from Remark 10.8 that if \mathcal{M}_i dominates \mathcal{M}_j for all i and j , then each subsimplex of m is visible.

By projecting γ_m to a combinatorial hyperplane on the carrier of some element of \mathcal{M}_k , we see that $\mathcal{M} - \mathcal{M}_k$ represents a visible codimension-1 face m' of m , represented by a ray $\gamma_{m'}$. The convex hull $\mathcal{Y}_{m'} \subset \mathcal{Y}_m$ of $\gamma_{m'}$ inherits a factor system from \mathcal{Y}_m as above. Hence, by induction, for $i < k$, the 0-simplex represented by \mathcal{M}_i is visible. Thus it suffices to show that the 0-simplex m_k represented by \mathcal{M}_k is visible. (In the base case, m is a maximal 0-simplex, and is visible by maximality.) Suppose, for a contradiction, that m_k is not visible, so there exists $i < k$ such that \mathcal{M}_i fails to dominate \mathcal{M}_k . In particular, $k \geq 1$.

The UBS \mathcal{M}_k contains a sequence $\{M_n\}_{n \geq 0}$ of pairwise-disjoint hyperplanes such that M_n separates $M_{n \pm 1}$ for all $n \geq 1$. For each n , let M_n^+ be the combinatorial hyperplane in $\mathcal{N}(M_n)$ in the same component of $\mathcal{X} - M_n$ as M_{n+1} . For each n , let $P_n = \mathfrak{g}_{M_0^+}(M_n^+)$ be the projection of M_n^+ on M_0^+ . The set of hyperplanes crossed by both M_0 and M_n contains all but finitely many elements of \mathcal{M}_i ; hence each P_n is unbounded and thus belongs to the factor system \mathfrak{F}_m . Moreover, for all $N \geq 0$, the intersection $\bigcap_{n=0}^N P_n \neq \emptyset$. Hence, since \mathfrak{F}_m has multiplicity $\Delta < \infty$, it must be the case that there exists N such that $P_n = P_N$ for all $N \geq n$. Thus, when $n, n' \geq N$, the set of elements of \mathcal{M}_j crossed by M_n coincides with the set crossed by $M_{n'}$ for all $j \leq k - 1$. Hence each \mathcal{M}_j dominates \mathcal{M}_k , whence m_k is visible.

Structure of \mathcal{Y}_m By [36, Theorem 3.23] and visibility of the m_i established above, after moving x_0 if necessary, $\mathcal{Y}_m = \prod_{i=0}^k \mathcal{Y}_{m_i}$, where \mathcal{Y}_{m_i} is the convex hull in \mathcal{X} of a combinatorial geodesic ray γ^i at the basepoint x_0 representing a 0-simplex m_i of m . Each point of $m = \partial_\Delta \mathcal{Y}_m$ can be uniquely written as $\sum_{i=0}^k a_i m_i$, where $a_i \geq 0$ and $\sum_{i=1}^k a_i = 1$.

For each i , let $\{H_n^i\}_{n \geq 0}$ be the set of hyperplanes crossing γ^i ; this is a minimal UBS and is numbered according to the order in which γ^i crosses the H_n^i . Thus, if $n > m$, the hyperplane H_n^i does not separate H_m^i from x_0 (in fact, either $H_n^i \cap H_m^i \neq \emptyset$ or H_m^i separates H_n^i from x_0). Choose $F_i \in \overline{\mathfrak{F}}_m$ to be \sqsubseteq -minimal such that all but finitely many H_n^i cross F_i . Observe that $F_i \perp F_j$ for all $i \neq j$, and that $F_i \subseteq \mathcal{Y}_{m_i}$.

Suppose that m' is some other maximal simplex and $\mathcal{Y}_{m'} = \prod_{i=0}^{k'} \mathcal{Y}_{m'_i}$. For each i , let $F'_i \in \overline{\mathfrak{F}}_{m'}$ be \sqsubseteq -minimal among those factors crossing all but finitely many of the elements crossing $\mathcal{Y}_{m'_i}$. Suppose that $\partial \hat{C}F_i = \partial \hat{C}F'_j$ for some $i \leq k$ and $j \leq k'$. Then the set of hyperplanes crossing \mathcal{Y}_{m_i} , which is boundary-equivalent to that crossing F_i , is boundary-equivalent to that crossing F'_j and hence that crossing $\mathcal{Y}_{m'_j}$, ie $m_i = m'_j$.

Orthogonality Each F_i has the form $F_i = \hat{F}_i \cap \mathcal{Y}_m$, where $\hat{F}_i \in \overline{\mathfrak{F}}$. While orthogonality of elements of $\overline{\mathfrak{F}}$ implies orthogonality of the corresponding elements of $\overline{\mathfrak{F}}_m$, the converse need not hold, but we will require that $\hat{F}_i \perp \hat{F}_j$ for all $i \neq j$, in order

to construct points of $\partial(\mathcal{X}, \overline{\mathfrak{F}})$. However, finitely many applications of Lemma 10.13 below show that for each i , there exists $\ddot{F}_i \in \overline{\mathfrak{F}}$ such that $F_i \subseteq \ddot{F}_i \subseteq \hat{F}_i$ and such that $\ddot{F}_i \perp \ddot{F}_j$ for all $i \neq j$.

Factored contact graphs in $\overline{\mathfrak{F}}_m$ For any $F \in \overline{\mathfrak{F}}_m$, we have, by convexity and [18, Proposition 2.5], that $F = \prod_{i=0}^k \mathfrak{g}_{\mathcal{Y}_{m_i}}(F)$, whence $\mathcal{C}F$ decomposes as a join, so $\hat{\mathcal{C}}F$ is obtained from a join by coning off certain subgraphs. Thus $\hat{\mathcal{C}}F$ is bounded (and $\partial\hat{\mathcal{C}}F = \emptyset$) unless F is parallel to a subcomplex of some \mathcal{Y}_{m_i} . We claim that $\partial\hat{\mathcal{C}}F_i$ consists of exactly one point p_i for each i , and that, for all other $F \in \overline{\mathfrak{F}}_m$, we have $\partial\hat{\mathcal{C}}F = \emptyset$.

Observe that $\mathcal{C}F_i$ coarsely coincides with $\mathcal{C}\mathcal{Y}_i$, the $\{H_n^i\}$ are partially ordered by the order in which γ_i crosses them, and that $\mathcal{C}F_i$ is coarsely equal to a maximal chain in this partial order (ie a combinatorial ray σ in $\mathcal{C}F_i$). By Theorem 2.4 of [36], σ is unbounded in $\mathcal{C}F_i$, since F_i is \sqsubseteq -minimal, and thus determines a point $p_i \in \partial\mathcal{C}F_i$. Moreover, p_i is unique, since $\hat{\mathcal{C}}F_i$ lies in the 1-neighborhood of σ ($\hat{\mathcal{C}}F_i$ is obtained from σ by adding edges reflecting intersections of elements of the $\{H_n^i\}$).

Hence, if $\sigma \subset \hat{\mathcal{C}}F_i$ is unbounded, then $\partial\hat{\mathcal{C}}F_i = \{p_i\}$. By \sqsubseteq -minimality of F_i , no hyperplane of F_i crosses infinitely many $\{H_n^i\}$, so hyperplanes of F_i are compact. By minimality of the UBS $\{H_n^i\}$, any element of $\overline{\mathfrak{F}}_m$ corresponding to a cone-vertex in $\hat{\mathcal{C}}F_i$ crosses finitely many hyperplanes. It follows that for all $n \geq 0$, there exists $N \geq n$ such that H_n^i and H_m^i cannot be adjacent to the same cone-vertex of $\hat{\mathcal{C}}F_i$ when $m \geq N$. Hence $\partial\hat{\mathcal{C}}F_i = \{p_i\}$.

We have shown that if $F \in \overline{\mathfrak{F}}_m$ has unbounded factored contact graph, then F is (up to parallelism) contained in some \mathcal{Y}_{m_i} . If F intersects only finitely many elements of $\{H_i\}$, then F is compact and thus $\hat{\mathcal{C}}F$ is bounded. If F intersects infinitely many, then it intersects all but finitely many, whence either F is parallel to F_i or $\hat{\mathcal{C}}F$ contains a subgraph, containing all but finitely many hyperplane-vertices, whose vertices are all adjacent to the cone-point corresponding to $\mathfrak{g}_F(F_i)$; thus $\hat{\mathcal{C}}F$ is bounded. This completes the description of the boundaries of the factored contact graphs of the elements of $\overline{\mathfrak{F}}_m$. \square

Lemma 10.11 *Let \mathfrak{F} be a factor system in \mathcal{X} , let $\mathcal{Y} \subseteq \mathcal{X}$ be a convex subcomplex, and let \mathfrak{F}' be the factor system in \mathcal{Y} consisting of all subcomplexes of the form $F' \cap \mathcal{Y}$, where $F' \in \mathfrak{F}$. Let $F \cap \mathcal{Y} \in \mathfrak{F}'$, and suppose that if $F' \in \mathfrak{F}$ satisfies $F' \cap \mathcal{Y} = F \cap \mathcal{Y}$, then $F \sqsubseteq F'$.*

Then the following map $\phi: \hat{\mathcal{C}}(F \cap \mathcal{Y}) \rightarrow \hat{\mathcal{C}}F$ is a $(3, 0)$ -quasiisometric embedding: ϕ is the inclusion on contact graphs; for each $F' \cap \mathcal{Y} \in \mathfrak{F}'$ properly nested in $F \cap \mathcal{Y}$ (with F' minimal with this intersection with \mathcal{Y}), the cone-point in $\hat{\mathcal{C}}(F \cap \mathcal{Y})$ corresponding to $F' \cap \mathcal{Y}$ is sent to the cone-point of $\hat{\mathcal{C}}\mathcal{X}$ corresponding to F' .

Remark 10.12 Recall from the discussion in the proof of [Theorem 10.1](#) of the homeomorphism $(\mathcal{Y}_m, \overline{\mathfrak{F}}_m) \rightarrow (\mathcal{X}, \overline{\mathfrak{F}})$ that if $F' \cap \mathcal{Y} \sqsubseteq F \cap \mathcal{Y}$ and F and F' are each \sqsubseteq -minimal with the given intersections with \mathcal{Y} , then $F \sqsubseteq F'$.

Proof of Lemma 10.11 Let v and v' be vertices of $\widehat{C}(F \cap \mathcal{Y}_m)$. Let $v = v_0, v_1, \dots, v_n = v'$ be a geodesic sequence in $\widehat{C}F$ from v to v' . If v_i is a hyperplane vertex, let H_i be the corresponding hyperplane of F (so H crosses $F \cap \mathcal{Y}$). If v_i is a cone-vertex, let H_i be a subcomplex in \mathfrak{F} , properly contained in F , that represents the parallelism class corresponding to the cone-vertex v_i . (For $i \in \{0, n\}$, if H_i is a hyperplane, then it crosses \mathcal{Y} . Otherwise, $H_i \in \mathfrak{F}$ is \sqsubseteq -minimal among all $U \in \mathfrak{F}_F$ with $U \cap \mathcal{Y} = H_i \cap \mathcal{Y}$.)

If H_i is a cone-vertex, then $H_{i\pm 1}$ are hyperplanes crossing H_i . This gives a sequence H_0, H_1, \dots, H_n of hyperplanes or factor-system elements in F such that $\mathcal{N}(H_i) \cap \mathcal{N}(H_{i+1}) \neq \emptyset$ when H_i and H_{i+1} are hyperplanes, and $H_i \cap H_{i+1} \neq \emptyset$ when H_{i+1} is a subcomplex in \mathfrak{F} .

For each i such that $H_i \in \mathfrak{F}$, we have $H_i \not\sqsubseteq F$. In particular, our minimality assumption on F ensures that if $H_i \cap \mathcal{Y} \neq \emptyset$, then $H_i \cap \mathcal{Y} \not\sqsubseteq F \cap \mathcal{Y}$. Otherwise, we would have $H_i \cap \mathcal{Y} = F \cap \mathcal{Y}$ while $H_i \not\sqsubseteq F$, contradicting minimality of F .

For each i with H_i a hyperplane, choose a combinatorial geodesic $\gamma_i \rightarrow \mathcal{N}(H_i)$ joining the terminal point of γ_{i-1} to a closest point on H_{i+1} (or $\mathcal{N}(H_{i+1})$ if v_{i+1} is a hyperplane vertex). Similarly, choose $\gamma_i \rightarrow H_i$ when v_i is a cone-vertex. The geodesic $\gamma_1 \rightarrow H_1$ joins $H_1 \cap \mathcal{Y}$ (or $\mathcal{N}(H_1) \cap \mathcal{Y}$ to $H_1 \cap H_2$, or $\mathcal{N}(H_1) \cap H_2$ etc), and $\gamma_n \rightarrow H_n$ (or $\mathcal{N}(H_n)$) is similarly chosen to end in \mathcal{Y} . Let $D \rightarrow F$ be a minimal-area disc diagram bounded by $\gamma_1 \cdot \gamma_2 \cdots \gamma_n$ and a geodesic of \mathcal{Y} joining its endpoints. Moreover, suppose that each of the geodesics, and indeed the sequence v_0, \dots, v_n and the representative subspaces, are chosen so as to minimize the area of D among all possible such choices.

Then, arguing exactly as in the proof of Proposition 3.1 of [\[5\]](#), we see that $\gamma_1 \cdots \gamma_n$ can be chosen to be a geodesic since a minimal D cannot contain a dual curve traveling from γ_i to γ_j for any i and j . It follows that $\gamma_1 \cdots \gamma_n$ lies in \mathcal{Y} , so each H_i that is a hyperplane either crosses \mathcal{Y} or contributes a combinatorial hyperplane to \mathfrak{F}' , while each H_i that is a subcomplex contributes an element to \mathfrak{F}' ; as explained above, for each such H_i , we have $H_i \cap \mathcal{Y} \not\sqsubseteq F \cap \mathcal{Y}$, so $H_i \cap \mathcal{Y}$ corresponds to a cone-point in $\widehat{C}(F \cap \mathcal{Y})$. We thus have a sequence H_1, \dots, H_n of (non- \sqsubseteq -maximal) elements of \mathfrak{F}' and hyperplanes crossing \mathcal{Y} , which determines a path of length between $n - 1$ and $3(n - 1)$ in $\widehat{C}(F \cap \mathcal{Y})$. □

Lemma 10.13 *Let \mathcal{X} be a CAT(0) cube complex with a factor system \mathfrak{F} . Suppose that A and B are unbounded convex subcomplexes of \mathcal{X} such that there is a cubical*

isometric embedding $A \times B \rightarrow \mathcal{X}$ extending $A, B \hookrightarrow \mathcal{X}$. Then there exist $P_A, P_B \in \mathfrak{F}$ with $P_A \perp P_B$ and $A \subseteq P_A$ and $B \subseteq P_B$.

Proof Let $x = A \cap B$. Then A and B are contained in combinatorial hyperplanes H_A and H_B , respectively. Indeed, every hyperplane crossing A (including the one whose carrier contains H_B) crosses every hyperplane crossing B (including the one whose carrier contains H_A). For each hyperplane V' crossing H_B , let V be one of the two associated combinatorial hyperplanes and consider $\mathfrak{g}_{H_A}(V)$. Observe that $\mathfrak{g}_{H_A}(V) \in \mathfrak{F}$ since it contains A and is thus unbounded. Since \mathfrak{F} has finite multiplicity, there are only finitely many distinct subcomplexes $\mathfrak{g}_{H_A}(V)$, as V varies over all hyperplanes whose projection to H_A contains A ; let $P_A \in \mathfrak{F}$ be their intersection. Define P_B analogously. Then P_A and P_B have the desired properties. (Indeed, a hyperplane H crosses P_A if and only if H crosses every hyperplane V whose projection to H_A contains A ; the projection of H to H_B thus contains B , so every hyperplane crossing P_B crosses H , whence $P_A \times P_B \subset \mathcal{X}$.) \square

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