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Tautological integrals on curvilinear Hilbert schemes

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We take a new look at the curvilinear Hilbert scheme of points on a smooth projective variety X as a projective completion of the nonreductive quotient of holomorphic map germs from the complex line into X by polynomial reparametrisations. Using an algebraic model of this quotient coming from global singularity theory we develop an iterated residue formula for tautological integrals over curvilinear Hilbert schemes.

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1 Introduction

Let X be a smooth projective variety of dimension n and let F be a rank-r algebraic vector bundle on X. Let $X^{[k]}$ denote the Hilbert scheme of length-k subschemes of X and let $F^{[k]}$ be the corresponding tautological rank-rk bundle on $X^{[k]}$ whose fibre at $\xi \in X^{[k]}$ is $H^0(\xi, F|_\xi)$.

Let $\mathrm{Hilb}_0^k(\mathbb{C}^n)$ be the punctual Hilbert scheme defined as the closed subset of $(\mathbb{C}^n)^{[k]} = \mathrm{Hilb}^k(\mathbb{C}^n)$ parametrising subschemes supported at the origin. Following Rennemo [33] we define punctual geometric subsets as constructible subsets $Q \subseteq \mathrm{Hilb}_0^k(\mathbb{C}^n)$ which are unions of isomorphism classes of schemes, that is, if $\xi \in Q$ and $\xi' \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ are isomorphic (ie they have isomorphic coordinate rings), then $\xi' \in Q$. Geometric subsets of $X^{[k]}$ are those generated by finite unions, intersections and complements from sets of the form

$$P(Q_1, \dots, Q_s) = \{\xi \in X^{[k]} : \xi = \xi_1 \sqcup \dots \sqcup \xi_s \text{ for some } \xi_i \in Q_i\}.$$

For a geometric subset $\mathcal{Z}$ let $\overline{\mathcal{Z}}$ denote its Zariski closure in $X^{[k]}$. Let $M(c_1, \dots, c_{rk})$ be a monomial in the Chern classes $c_i = c_i(F^{[k]})$ of weighted degree equal to $\dim \overline{\mathcal{Z}}$, where the weight of c_i is i. Let $\Omega^*(\overline{\mathcal{Z}})$ denote the algebra of differential forms supported on the smooth part of $\overline{\mathcal{Z}}$ (see eg Remark 2.3). If $\alpha_M \in \Omega^*(\overline{\mathcal{Z}})$ is a closed differential form representing the cohomology class of $M(c_1, \dots, c_{rk})$ then the Chern numbers

$$[\overline{\mathcal{Z}}] \cap M(c_1, \dots, c_{rk}) = \int_{\overline{\mathcal{Z}}} \alpha_M$$

are called tautological integrals of $F^{[k]}$. Rennemo [33] shows that these integrals can be expressed in terms of the Chern numbers of X and F.

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Theorem 1.1 (Rennemo [33]) *Let $\mathcal{M}_{r,n}$ denote the set of weighted-degree-n monomials in the Chern classes $c_1(F),\dots,c_r(F)$ and $c_1(X),\dots,c_n(X)$. For $S\in\mathcal{M}_{r,n}$ let $\alpha_S\in\Omega^{\mathrm{top}}(X)$ be a closed differential form representing the cohomology class of S and let $y_S=\int_X\alpha_S$ denote the corresponding intersection number. Let $\mathcal{Z}\subset X^{[k]}$ be a geometric subset. Then for any Chern monomial $M=M(c_1,\dots,c_{rk})$ of weighted degree $\dim\overline{\mathcal{Z}}$ there is a polynomial R_M in $|\mathcal{M}_{r,n}|$ variables depending only on M such that*

$$[\overline{\mathcal{Z}}]\cap M(c_1,\dots,c_{rk})=R_M(y_S:S\in\mathcal{M}_{r,n}).$$

The proof in [33] is nonconstructive and based on constructing homology classes supported on certain diagonals of X^n (see also Li [27]) and the fact that an element in the cohomology ring of a Grassmannian is a polynomial in the Chern classes of the universal bundle. Lacking a method of obtaining information about this polynomial, there is no apparent way of turning this proof into an algorithm. Explicit expressions for tautological integrals are not known in general. On surfaces the method of Ellingsrud, Göttsche and Lehn [15] yields a recursion which in principle computes the universal polynomial explicitly. The top Segre classes of tautological bundles over surfaces provides an example of this problem and the conjecture of Lehn [26] has been recently proved by Marian, Oprea and Pandharipande [29] for K3 surfaces using virtual localisation. However, [15] and [29] deal only with surfaces and their authors integrate over the whole Hilbert scheme rather than over geometric subsets. Our method works in any dimension for integration over a geometric subset called the curvilinear component.

Let X be a smooth projective variety of dimension n. This paper provides a closed iterated residue formula for tautological integrals over the simplest geometric subsets $P(Q)$ where $s=1$ and the punctual geometric subset Q is defined as

$$Q=\{\xi\in\mathrm{Hilb}_0^k(\mathbb{C}^n):\mathcal{O}_\xi\simeq\mathbb{C}[z]/z^k\}.$$

We will see that $\overline{Q}$ is an irreducible component of the punctual Hilbert scheme. Points of $P(\overline{Q})$ correspond to curvilinear subschemes on X, ie subschemes contained in the germ of some smooth curve on X. In other words, these are the limit points on $X^{[k]}$ where k distinct points come together along a smooth curve. We denote this curvilinear locus by CX^k and its closure by $\overline{CX}^{[k]}$, which we call the curvilinear Hilbert scheme.

The main result of the present paper is the following explicit formula for tautological integrals over curvilinear Hilbert schemes.

Theorem 1.2 *Let $k\geq 1$ and $P(\mathbf{x})=P(x_1,\dots,x_{r(k+1)})$ be a polynomial of weighted degree $\dim\overline{CX}^{[k+1]}=n+(n-1)k$ in the variables x_l of weight l for $1\leq l\leq r(k+1)$.*

Let $c_l = c_l(F^{[k+1]})$ denote the l^{th} Chern class of the tautological rank-$r(k+1)$ bundle on $X^{[k+1]}$. Then

$$\int_{\overline{CX}^{[k+1]}} P(\boldsymbol{c}) = \int_X \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{(-1)^k \prod_{1\le i<j\le k}(z_i-z_j) Q_k(\boldsymbol{z}) P(\boldsymbol{c}(\theta-\boldsymbol{z},\theta))d\boldsymbol{z}}{\prod_{i+j\le l\le k}(z_i+z_j-z_l)(z_1\cdots z_k)^n} \prod_{i=1}^k s_X\Big(\frac{1}{z_i}\Big),$$

where $\theta_1,\dots,\theta_r$ are the Chern roots of F and $c_l(\theta-\boldsymbol{z},\theta)$ denotes the l^{th} symmetric polynomial in the formal Chern roots $\{\theta_j-z_i, \theta_j : 1\le i\le k, 1\le j\le r\}$. The iterated residue is $(-1)^k$ times the coefficient of $(z_1\cdots z_k)^{-1}$ in the expansion of the rational expression in the domain $z_1 \ll \cdots \ll z_k$ and

$$s_X\Big(\frac{1}{z_i}\Big) = 1 + \frac{s_1(X)}{z_i} + \frac{s_2(X)}{z_i^2} + \cdots + \frac{s_n(X)}{z_i^n}$$

is the total Segre class of X at $1/z_i$. Finally $Q_k(\boldsymbol{z})$ is a homogeneous polynomial invariant of Morin singularities given as the equivariant Poincaré dual of a Borel orbit defined in the following Remark.

Remark (explanation and features of the residue formula) • The iterated residue gives a degree-n symmetric polynomial in Chern roots of F and Segre classes of X reproving Theorem 1.1 This shows that the dependence on Chern classes of X in fact can be expressed via the Segre classes of X. In particular, in Example 7.2 we give a formula for the top Segre classes of tautological bundles over curvilinear Hilbert schemes.

- For fixed k the rational expression

$$\mathcal{R}_k = \frac{\prod_{1\le i<j\le k}(z_i-z_j) Q_k(\boldsymbol{z}) P(c_l(\theta-\boldsymbol{z},\theta))d\boldsymbol{z}}{\prod_{i+j\le l\le k}(z_i+z_j-z_l)}$$

in the formula is independent of the dimension n and the iterated residue depends on n only through the total Segre class s_X of X. The iterated residue is then some linear combination of the coefficients of (the expansion of) $\mathcal{R}_k$ multiplied by Segre classes of X. By increasing the dimension, the iterated residue involves new terms of the expansion of $\mathcal{R}_k$, and we can think of $\mathcal{R}_k$ as a universal rational expression encoding the integrals for fixed k but varying n.

- The Chern class $c_l(\theta-\boldsymbol{z},\theta)$ is the coefficient of t^l in

$$\prod_{j=1}^r (1+\theta_j t) \prod_{i=1}^k \prod_{j=1}^r (1-z_i t+\theta_j t),$$

that is, the l^{th} Chern class of the bundle with formal Chern roots θ_j, $\theta_j - z_i$.

- The quick description of Q_k is the following; see Remark 6.2 for details. The $\mathrm{GL}(k)$–module of 3–tensors $\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k)$ has a diagonal decomposition
$$\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k) = \bigoplus_{1\le m,r,l\le k} \mathbb{C}q_l^{mr},$$
where the T_k–weight of q_l^{mr} is $(z_m + z_r - z_l)$. Define $\epsilon = \sum_{m=1}^{k}\sum_{r=1}^{k-m} q_{m+r}^{mr}$ as a point in the B_k–invariant subspace
$$W_k = \bigoplus_{1\le m+r\le l\le k} \mathbb{C}q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k).$$
Then $Q_k(\mathbf{z}) = \mathrm{eP}[\overline{B_k\epsilon}, W_k]$ is the equivariant Poincaré dual of the Borel orbit $\overline{B_k\epsilon}$ in W_k. The list of these polynomials begins as follows: $Q_1 = Q_2 = Q_3 = 1$, $Q_4 = 2z_1 + z_2 - z_4$. In principle, Q_k may be calculated for each concrete k using a computer algebra program, but at the moment, we do not have an efficient algorithm for performing such calculations for large k and Q_k is only known for $k \le 6$.

The main motivation for studying tautological integrals is their immediate applications in enumerative geometry and in particular in counting hypersurfaces in sufficiently ample linear systems on X with a prescribed set of singularities. Let X be a smooth, projective, connected variety, L a sufficiently ample line bundle on X and let $T_1, \dots, T_s$ be analytic singularity types. There are expected codimensions d_i associated with each T_i, and we let $d = \sum d_i$. Rennemo [33] shows (see also Göttsche [20] and Kleiman and Piene [24]) that there is an m and a geometric set $W = W(T_1, \dots, T_s) \subset X^{[m]}$ such that a generic hypersurface containing a $Z \in W$ has the specified singularities. Therefore in a general $\mathbb{P}^d \subseteq |L|$ the number of hypersurfaces containing a subscheme $Z \in W$ is equal to $\int_W c_{\dim(W)}(L^{[m]})$; hence this tautological integral gives the number of hypersurfaces in $\mathbb{P}^d$ with singularities $T_1, \dots, T_s$.

In the forthcoming paper Bérczi and Szenes [6], we extend the methods of the present paper to study tautological integrals over more general geometric subsets supported at more than one point on X and develop residue formulae for counts of hypersurfaces with given sets of singularities. As a special case we present a new formula for the number of δ–nodal curves on surfaces (and more generally δ–nodal hypersurfaces on projective varieties) different from the well-known Göttsche conjecture [20], which by now has several proofs; see Kazarian [23], Kool, Shende and Thomas [25], Liu [28] and Tzeng [36].

The intersection theory of the Hilbert scheme of points on surfaces has been extensively studied and it can be approached from different directions. One is the inductive recursions set up by Ellingsrud, Göttsche and Lehn [15], an other possibility is using

Nakajima calculus (see Lehn [26] and Nakajima [32]). By these methods, the integration of tautological classes is reduced to a combinatorial problem. Another strategy is to prove an equivariant version of Lehn's conjecture for the Hilbert scheme of points of $\mathbb{C}^2$ via appropriately weighted sums over partitions. More recently Marian, Oprea and Pandharipande [29] proved a conjecture of Lehn [26] on integrals of top Segre classes of tautological bundles over the Hilbert schemes of points over surfaces in the K3 case via virtual localisation on the Quot schemes of the surface.

In this paper we suggest a new approach by taking a look at Hilbert schemes of points from a different perspective. We work in arbitrary dimension, not just over surfaces. For $n \geq 3$ not much is known about the irreducible components and singularities of the punctual Hilbert scheme $\mathrm{Hilb}_0^k(\mathbb{C}^n)$ so we only focus on the curvilinear component. The crucial observation is that for $k \geq 1$ the punctual curvilinear locus $CX_p^{[k+1]}$ at $p \in X$ can be described as the nonreductive quotient of k–jets of holomorphic map germs $(\mathbb{C},0) \to (X,p)$ by polynomial reparametrisations of $\mathbb{C}$ at the origin.

Let $J_k^{\mathrm{reg}}X$ denote the regular k–jet bundle over X whose elements are equivalence classes of germs of holomorphic maps $f\colon (\mathbb{C},0) \to (X,p)$ with the equivalence relation $f \sim g$ if and only if the derivatives satisfy $f^{(j)}(0) = g^{(j)}(0)$ for $0 \leq j \leq k$ when computed in some local coordinate system of X near $p \in X$ and $f'(0) \neq 0$. The reparametrisation group $\mathrm{Diff}_k(1)$ formed by k–jets of regular reparametrisations of $\mathbb{C}$ at the origin acts fibrewise on $J_k^{\mathrm{reg}}X$ and the curvilinear locus (as a set) can be identified with the quasiprojective quotient

$$CX^{[k+1]} \simeq J_k^{\mathrm{reg}}X/\mathrm{Diff}_k(1).$$

Using an algebraic model coming from global singularity theory (we call this the test-curve model) we reinterpret the natural embedding of the punctual curvilinear locus $CX_p^{[k+1]}$ into the Grassmannian of codimension-k subspaces in the maximal ideal $\mathfrak{m} = (x_1,\dots,x_n)$ as a parametrised map $CX_p^{[k+1]} \hookrightarrow \mathrm{Grass}_k(\mathcal{D}_{X,p}^k)$, where $\mathcal{D}_X^k = \mathcal{D}_X^{\leq k}/\mathcal{O}_X$ is the bundle of order-k differential operators over X. The punctual curvilinear Hilbert scheme $\overline{CX}_p^{[k+1]}$ is the closure of the image of this map in $\mathrm{Grass}_k(\mathcal{D}_{X,p}^k)$, and moving the point p on X, this gives an embedding of the curvilinear component

$$\phi^{\mathrm{Grass}}\colon \overline{CX}^{[k+1]} \hookrightarrow \mathrm{Grass}_k(\mathcal{D}_X^k).$$

Integration on $\overline{CX}^{[k+1]}$ can be reduced to integration along the fibre $\overline{CX}_p^{[k+1]}$; see Section 7. We use equivariant localisation on $\overline{CX}_p^{[k+1]}$ following the strategy of Bérczi and Szenes [7]. However, for tautological integrals we need to modify the proof in [7] in two crucial points:

- First, the main obstacle to applying localisation directly is that we don't know which fixed points of the ambient Grassmannian sit in the image $\overline{CX}_p^{[k+1]}$. However, for $k+1 \le n$ we prove in [7] a residue vanishing theorem which tells that after transforming the localisation formula into an iterated residue only one distinguished fixed point of the torus action contributes to the sum. This mysterious property remains valid for tautological integrals but its proof needs a more detailed study of the rational differential form.

- Second, we need to extend the formula to the domain where $k+1 > n$, that is, the number of points is not smaller than the dimension of X. The trick here is to increase the dimension of the variety and study $\mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ as a subvariety of $\mathrm{Hilb}_0^{k+1}(\mathbb{C}^{k+1})$.

The developed method reflects a surprising feature of curvilinear Hilbert schemes: in order to evaluate tautological integrals and make the residue vanishing principle work we need to increase the dimension of the variety first and work in the range where the number of points does not exceed the dimension.

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2 Tautological integrals

Let X be a smooth projective variety of dimension n and let F be a rank-r bundle (locally free sheaf) on X. Let

$$X^{[k]} = \{\xi \subset X : \dim(\xi) = 0 \text{ and } \mathrm{length}(\xi) = \dim H^0(\xi, \mathcal{O}_\xi) = k\}$$

denote the Hilbert scheme of k points on X parametrising length-k subschemes of X and $F^{[k]}$ the corresponding rank-rk bundle on $X^{[k]}$ whose fibre over $\xi \in X^{[k]}$ is $F \otimes \mathcal{O}_\xi = H^0(\xi, F|_\xi)$.

Equivalently, $F^{[k]} = q_* p^*(F)$, where p and q denote the projections from the universal family of subschemes $\mathcal{U}$ to X and $X^{[k]}$ respectively:

$$\begin{array}{ccc} X^{[k]} \times X \supset \mathcal{U} & \xrightarrow{q} & X^{[k]} \\ \downarrow{\scriptstyle p} & & \\ X & & \end{array}$$

For simplicity let $\mathrm{Hilb}_0^k(\mathbb{C}^n)$ denote the punctual Hilbert scheme of k points on $\mathbb{C}^n$ defined as the closed subset of $\mathrm{Hilb}^k(\mathbb{C}^n)$ parametrising subschemes supported at the origin. Following Rennemo [33] we define punctual geometric subsets to be the constructible subsets of the punctual Hilbert scheme containing all 0–dimensional schemes of given isomorphism types.

Definition 2.1 A punctual geometric set is a constructible subset $Q \subseteq \mathrm{Hilb}_0^k(\mathbb{C}^n)$ which is the union of isomorphism classes of subschemes, that is, if $\xi \in Q$ and $\xi' \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ are isomorphic schemes then $\xi' \in Q$.

Definition 2.2 For an s–tuple $\boldsymbol{Q} = (Q_1, \dots, Q_s)$ of punctual geometric sets such that $Q_i \subseteq \mathrm{Hilb}_0^{k_i}(\mathbb{C}^n)$ and $k = \sum k_i$ define

$$P(\boldsymbol{Q}) = \{\xi \in X^{[k]} : \xi = \xi_1 \sqcup \cdots \sqcup \xi_s \text{ where } \xi_i \in X_{p_i}^{[k_i]} \cap Q_i \text{ for distinct } p_1, \dots, p_s\} \subseteq X^{[k]}.$$

A subset $\mathcal{Z} \subseteq X^{[k]}$ is geometric if it can be expressed as finite union, intersection and complement of sets of the form $P(\boldsymbol{Q})$.

A straightforward way to produce punctual geometric subsets is by taking a complex algebra A of complex dimension k and making the corresponding definition

$$Q_A = \{\xi \in X^{[k]} : \mathcal{O}_\xi \simeq A\}.$$

When $A = \mathbb{C}[z]/z^k$ then $Q_A = CX_p^{[k]}$ is the punctual curvilinear locus defined in the next section and

$$\overline{CX}^{[k]} = \bigcup_{p \in X} \overline{CX}_p^{[k]}$$

is the curvilinear Hilbert scheme, the central object of this paper.

In this paper we work with singular homology and cohomology with rational coefficients. For a smooth manifold X the degree of a class $\eta \in H_*(X)$ means its push-forward to $H_*(\mathrm{pt}) = \mathbb{Q}$. By choosing $\alpha_\eta \in \Omega^{\mathrm{top}}(X)$, a closed compactly supported differential form representing the cohomology class η, this degree is equal to the integral

$$\eta \cap [X] = \int_X \alpha_\eta.$$

Let $\mathcal{Z} \subset X^{[k]}$ be a geometric subset with closure $\overline{\mathcal{Z}}$ and $M(c_1, \dots, c_{rk})$ be a monomial in the Chern classes $c_i = c_i(F^{[k]})$ of weighted degree equal to $\dim \overline{\mathcal{Z}}$, where the weight of c_i is i. By choosing $\alpha_M \in \Omega^*(X^{[k]})$, a closed compactly supported differential form representing the cohomology class of $M(c_1, \dots, c_{rk})$, the degree

$$[\overline{\mathcal{Z}}] \cap M(c_1, \dots, c_{rk}) = \int_{\overline{\mathcal{Z}}} \alpha_M \tag{1}$$

is called a tautological integral of $F^{[k]}$.

Remark 2.3 (1) In (1) the integral of α_M on the smooth part of $\overline{\mathcal{Z}}$ is absolutely convergent and by definition we denote this by $\int_{\overline{\mathcal{Z}}} \alpha_M$.

(2) Recall (see eg Bott and Tu [10]) that if $f\colon X \to Y$ is a smooth proper map between connected oriented manifolds such that f restricted to some open subset of X is a diffeomorphism, then for a compactly supported form μ on Y, we have $\int_X f^*\mu = \int_Y \mu$. The analogous statement for singular varieties is the following. Let $f\colon M \to N$ be a smooth proper map between smooth quasiprojective varieties and assume that $X \subset M$ and $Y \subset N$ are possibly singular closed subvarieties, such that f restricted to X is a birational map from X to Y. If μ is a closed differential form on N then the integral of μ on the smooth part of Y is absolutely convergent; we denote this by $\int_Y \mu$. With this convention we again have $\int_X f^*\mu = \int_Y \mu$.

In particular this means that the integral $\int_Y \mu$ of the compactly supported form μ on N is the same as the integral $\int_{\widetilde{Y}} f^*\mu$ of the pull-back form $f^*\mu$ over any (partial) resolution $f\colon (\widetilde{Y}, \widetilde{M}) \to (Y, M)$.

3 Curvilinear Hilbert schemes

In this section we describe a geometric model for curvilinear Hilbert schemes. Let X be a smooth projective variety of dimension n and let

$$X^{[k]} = \{\xi \subset X : \dim(\xi) = 0 \text{ and } \mathrm{length}(\xi) = \dim H^0(\xi, \mathcal{O}_\xi) = k\}$$

denote the Hilbert scheme of k points on X parametrising all length-k subschemes of X. For $p \in X$ let

$$X_p^{[k]} = \{\xi \in X^{[k]} : \mathrm{supp}(\xi) = p\}$$

denote the punctual Hilbert scheme consisting of subschemes supported at p. If $\rho\colon X^{[k]} \to S^k X$ given by $\xi \mapsto \sum_{p \in X} \mathrm{length}(\mathcal{O}_{\xi,p})p$ denotes the Hilbert–Chow morphism then $X_p^{[k]} = \rho^{-1}(kp)$.

Definition 3.1 A subscheme $\xi \in X_p^{[k]}$ is called curvilinear if ξ is contained in some smooth curve $C \subset X$. Equivalently, ξ is curvilinear if $\mathcal{O}_\xi$ is isomorphic to the $\mathbb{C}$–algebra $\mathbb{C}[z]/z^k$. The punctual curvilinear locus at $p \in X$ is the set of curvilinear subschemes supported at p:

$$\begin{aligned} CX_p^{[k]} &= \{\xi \in X_p^{[k]} : \xi \subset \mathcal{C}_p \text{ for some smooth curve } \mathcal{C} \subset X\} \\ &= \{\xi \in X_p^{[k]} : \mathcal{O}_\xi \simeq \mathbb{C}[z]/z^k\}. \end{aligned}$$

If X is a surface (ie $\dim X = 2$), $CX_p^{[k]}$ is an irreducible quasiprojective variety of dimension $k-1$ which is an open dense subset in $X_p^{[k]}$ and therefore its closure is the full punctual Hilbert scheme at p, that is, $\overline{CX}_p^{[k]} = X_p^{[k]}$. When $n \geq 3$ the punctual Hilbert scheme $X_p^{[k]}$ is not necessarily irreducible or reduced, but the closure of the curvilinear locus is one of its irreducible components:

Lemma 3.2 *$\overline{CX}_p^{[k]}$ is an irreducible component of the punctual Hilbert scheme $X_p^{[k]}$ of dimension $(n-1)(k-1)$.*

Proof Note that $\xi \in \mathrm{Hilb}_0^{[k]}(\mathbb{C}^n)$ is not curvilinear if and only if $\mathcal{O}_\xi$ does not contain elements of degree $k-1$, that is, after fixing some local coordinates $x_1,\dots,x_n$ of $\mathbb{C}^n$ at the origin we have

$$\mathcal{O}_\xi \simeq \mathbb{C}[x_1,\dots,x_n]/I \quad \text{for some} \quad I \supseteq (x_1,\dots,x_n)^{k-1}.$$

This is a closed condition and hence curvilinear subschemes can't be approximated by noncurvilinear subschemes in $\mathrm{Hilb}_0^{[k]}(\mathbb{C}^n)$. The dimension of $\overline{CX}_p^{[k]}$ will come from the description of it as a nonreductive quotient in the next subsection. □

Note that any curvilinear subscheme contains only one subscheme for any given smaller length and any small deformation of a curvilinear subscheme is again locally curvilinear.

Remark 3.3 Fix coordinates $x_1,\dots,x_n$ on $\mathbb{C}^n$. Recall that the defining ideal I_ξ of any subscheme $\xi \in \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ is a codimension-k subspace in the maximal ideal $\mathfrak{m}=(x_1,\dots,x_n)$. The dual of this is a k–dimensional subspace S_ξ in $\mathfrak{m}^* \simeq \mathrm{Sym}^{\le k}\mathbb{C}^n$ giving us a natural embedding $\varphi\colon X_p^{[k+1]} \hookrightarrow \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$. In what follows, we give an explicit parametrisation of this embedding using an algebraic model coming from global singularity theory.

3.1 Test-curve model for $\overline{CX}^{[k]}$

3.1.1 Jets of holomorphic maps If u and v are positive integers let $J_k(u,v)$ denote the vector space of k–jets of holomorphic maps $(\mathbb{C}^u,0)\to(\mathbb{C}^v,0)$ at the origin, that is, the set of equivalence classes of maps $f\colon(\mathbb{C}^u,0)\to(\mathbb{C}^v,0)$, where $f\sim g$ if and only if $f^{(j)}(0)=g^{(j)}(0)$ for all $j=1,\dots,k$. This is a finite-dimensional complex vector space, which one can identify with $J_k(u,1)\otimes\mathbb{C}^v$; hence $\dim J_k(u,v)=v\binom{u+k}{k}-v$. We will call the elements of $J_k(u,v)$ *map-jets of order* k, or simply map-jets.

Eliminating the terms of degree $k+1$ results in an algebra homomorphism $J_k(u,1)\twoheadrightarrow J_{k-1}(u,1)$, and the chain $J_k(u,1)\twoheadrightarrow J_{k-1}(u,1)\twoheadrightarrow\cdots\twoheadrightarrow J_1(u,1)$ induces the following increasing filtration on $J_k(u,1)^*$:

$$J_1(u,1)^* \subset J_2(u,1)^* \subset \cdots \subset J_k(u,1)^*. \tag{2}$$

Remark 3.4 The space $J_i(u,1)^*$ may be interpreted as set of differential operators on $\mathbb{C}^u$ of degree at most i, and in particular, by taking symbols, we have

$$J_k(u,1)^* \simeq \mathrm{Sym}^{\le k}\mathbb{C}^u \stackrel{\text{def}}{=} \bigoplus_{l=1}^{k}\mathrm{Sym}^l\mathbb{C}^u, \tag{3}$$

where Sym^l stands for the symmetric tensor product and the isomorphism is that of filtered $\mathrm{GL}(n)$–modules. Given a regular k–jet $f\colon (\mathbb{C},0) \to (\mathbb{C}^n,0)$ in $J_k^{\mathrm{reg}}(1,n)$ we may push forward the differential operators of order k on $\mathbb{C}$ (with constant coefficients) to $\mathbb{C}^n$ along f which gives us a map

$$\tilde{f}\colon J_k(1,1)^* \to \mathrm{Grass}(k, J_k(n,1)^*).$$

In Section 3.1.3 we describe a parametrisation of this map, and identify the image in the Grassmannian with the punctual curvilinear locus $CX_p^{[k+1]}$ using local coordinates on X near p.

Choosing coordinates on $\mathbb{C}^u$ and $\mathbb{C}^v$ a k–jet $f \in J_k(u,v)$ can be identified with the set of derivatives at the origin, that is, the vector $(f'(0), f''(0), \dots, f^{(k)}(0))$, where $f^{(j)}(0) \in \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v)$. This way we get the identification

$$J_k(u,v) \simeq J_k(u,1) \otimes \mathbb{C}^v \simeq \bigoplus_{j=1}^{k} \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v). \tag{4}$$

One can compose map-jets via substitution and elimination of terms of degree greater than k; this leads to the composition map

$$\begin{aligned} &J_k(u,v) \times J_k(v,w) \to J_k(u,w), \\ &\qquad (\Psi_1, \Psi_2) \mapsto \Psi_2 \circ \Psi_1 \text{ modulo terms of degree} > k. \end{aligned} \tag{5}$$

When $k=1$, the map-jets in $J_1(u,v)$ may be identified with u-by-v matrices, and (5) reduces to multiplication of matrices.

The k–jet of a curve $(\mathbb{C},0) \to (\mathbb{C}^n,0)$ is simply an element of $J_k(1,n)$. We call such a curve γ *regular* if $\gamma'(0) \neq 0$; introduce the notation $J_k^{\mathrm{reg}}(1,n)$ for the set of regular curves:

$$J_k^{\mathrm{reg}}(1,n) = \{\gamma \in J_k(1,n) : \gamma'(0) \neq 0\}.$$

Note that $J_k^{\mathrm{reg}}(u,u)$ with the composition map (5) has a natural group structure and we will often use the notation

$$\mathrm{Diff}_k(u) = J_k^{\mathrm{reg}}(u,u)$$

and refer to this set as the *k–jet diffeomorphism group* to underline this property.

3.1.2 Jet bundles and differential operators Let X be a smooth projective variety. Following Green and Griffiths [21] we let $J_kX \to X$ be the bundle of k–jets of germs of parametrised curves in X; that is, J_kX is the of equivalence classes of germs of holomorphic maps $f\colon (\mathbb{C},0) \to (X,p)$, with the equivalence relation $f \sim g$ if and only if the derivatives satisfy $f^{(j)}(0) = g^{(j)}(0)$ for $0 \le j \le k$ when computed in

some local coordinate system of X near $p \in X$. The projection map $J_k X \to X$ is simply $f \mapsto f(0)$. If we choose local holomorphic coordinates on an open neighbourhood $\Omega \subset X$ around p, the elements of the fibre $J_k X_p$ can be represented by Taylor expansions
$$f(t) = p + tf'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1})$$
up to order k at $t = 0$ of $\mathbb{C}^n$–valued maps $f = (f_1, f_2, \ldots, f_n)$ on open neighbourhoods of 0 in $\mathbb{C}$. Locally in these coordinates the fibre $J_k X_p$ can be identified with the set of k–tuples of vectors $(f'(0), \ldots, f^{(k)}(0)/k!) = (\mathbb{C}^n)^k$ which further can be identified with $J_k(1,n)$. These jet bundles and the corresponding jet differential bundles play central role in the study of hyperbolic varieties and the Green–Griffiths–Lang conjecture; see Demailly [11] and Green and Griffiths [21].

Remark 3.5 Note that $J_k X$ is not a vector bundle over X since the transition functions are polynomial but not linear; see Section 5 of Demailly [11]. In fact, let Diff_X denote the principal $\mathrm{Diff}_k(n)$–bundle over X formed by all local polynomial coordinate systems on X. Then
$$J_k X = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} J_k(1,n)$$
is the associated bundle whose structure group is $\mathrm{Diff}_k(n)$.

Let $J_k^{\mathrm{reg}} X$ denote the bundle of k–jets of germs of parametrised regular curves in X, that is, where the first derivative satisfies $f' \neq 0$. After fixing local coordinates near $p \in X$ the fibre $J_k^{\mathrm{reg}} X_p$ can be identified with $J_k^{\mathrm{reg}}(1,n)$ and
$$J_k^{\mathrm{reg}} X = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} J_k^{\mathrm{reg}}(1,n).$$
Let $\mathcal{D}_X^{\le k}$ denote the bundle of k^{th}–order differential operators over X. Then we have $\mathcal{D}_X^{\le 0} = \mathcal{O}_X$, and we let $\mathcal{D}_X^k = \mathcal{D}_X^{\le k}/\mathcal{D}_X^{\le 0}$. We have a filtration

(6) $$\mathcal{O}_X = \mathcal{D}_X^{\le 0} \subset \mathcal{D}_X^{\le 1} \subset \cdots \subset \mathcal{D}_X^{\le k},$$

where the graded component $\mathcal{D}_X^{\le i}/\mathcal{D}_X^{\le i-1} \simeq \mathrm{Sym}^i T_X$ but this filtration is not split in general, so $\mathcal{D}_X^k \neq \mathrm{Sym}^{\le k} T_X$; see Section 4.1 for details. Recall from Remark 3.4 that after choosing local coordinates on X near p the fibre $\mathcal{D}_{X,p}^k$ can be identified with the space $J_k(n,1)^* \simeq \mathrm{Sym}^{\le k}\mathbb{C}^n$ of k^{th}–order differential operators on $\mathbb{C}^n$ and the filtration (6) restricted to this fibre is the one given in (2).

Remark 3.6 We have a description of $\mathcal{D}_X^k$ as an associated bundle similar to that of $J_k X$ in Remark 3.5 , namely
$$\mathcal{D}_X^k = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} \mathrm{Sym}^{\le k}\mathbb{C}^n.$$

Remark 3.7 Given a regular k–jet $(\mathbb{C},0) \to (X,p)$ we may push forward the differential operators of order k on $\mathbb{C}$ to X and obtain a k–dimensional subspace of $\mathcal{D}^{\leq k}_{X,p}$. This gives the bundle map

$$J_k^{\mathrm{reg}} X \to \mathrm{Grass}_k(\mathcal{D}^k_X), \tag{7}$$

which is the fibred version of the map in Remark 3.4. Note that $\mathrm{Diff}_k(1) = J_k^{\mathrm{reg}}(1,1)$ acts fibrewise on the jet bundle $J_k^{\mathrm{reg}} X$ via the composition map (5) and the map (7) is $\mathrm{Diff}_k(1)$–invariant, resulting in an embedding

$$J_k^{\mathrm{reg}} X/\mathrm{Diff}_k(1) \hookrightarrow \mathrm{Grass}_k(\mathcal{D}^k_X). \tag{8}$$

In Section 3.1.3 we show that the set $CX^{[k+1]}$ of curvilinear subschemes on X can be identified with the nonreductive fibrewise quotient of $J_k^{\mathrm{reg}} X$ by $\mathrm{Diff}_k(1)$:

$$CX^{[k+1]} = J_k^{\mathrm{reg}} X/\mathrm{Diff}_k(1).$$

This, together with (8) gives an embedding

$$CX^{[k+1]} \hookrightarrow \mathrm{Grass}_k(\mathcal{D}^k_X).$$

In the next subsection we describe a parametrisation of this embedding which turns out to be crucial to control equivariant localisation on $\overline{CX}_p^{[k+1]}$.

3.1.3 The test-curve model of $\overline{CX}^{[k]}$ Let $\xi \in CX_p^{[k+1]}$ be a curvilinear subscheme supported at $p \in X$. Then ξ is (scheme-theoretically) contained in a smooth curve germ $\mathcal{C}_p$ in X:

$$\xi \subset \mathcal{C}_p \subset X.$$

Let $f_\xi\colon (\mathbb{C},0) \to (X,p)$ be a k–jet of a germ parametrising $\mathcal{C}_p$. Then $f_\xi \in J_k^{\mathrm{reg}} X_p$ is determined only up to polynomial reparametrisation germs $\phi\colon (\mathbb{C},0) \to (\mathbb{C},0)$ and therefore we get the following lemma.

Lemma 3.8 *The punctual curvilinear locus $CX_p^{[k+1]}$ is equal (as a set) to the set of k–jets of regular germs at $p \in X$ modulo polynomial reparametrisations:*

$$\begin{aligned} CX_p^{[k+1]} &= \{\text{regular } k\text{–jets } (\mathbb{C},0) \to (X,p)\}/\{\text{regular } k\text{–jets } (\mathbb{C},0) \to (\mathbb{C},0)\} \\ &= J_k^{\mathrm{reg}} X_p/\mathrm{Diff}_k(1). \end{aligned}$$

Therefore the curvilinear locus $CX^{[k+1]}$ is the fibrewise quotient

$$CX^{[k+1]} = J_k^{\mathrm{reg}} X/\mathrm{Diff}_k(1).$$

Recall that after choosing local coordinates on X near p we can identify $J_k^{\mathrm{reg}} X_p$ with $J_k^{\mathrm{reg}}(1,n)$. We can explicitly write out the reparametrisation action (defined in (5)) of $\mathrm{Diff}_k(1)$ on $J_k^{\mathrm{reg}}(1,n)$ as follows:

Let

$$f_\xi(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^k}{k!}f^{(k)}(0) \in J_k^{\mathrm{reg}}(1,n)$$

be the k–jet of a germ at the origin (ie it has no constant term) in $\mathbb{C}^n$ with $f^{(i)} \in \mathbb{C}^n$ such that $f' \neq 0$ and let $\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_k z^k \in J_k^{\mathrm{reg}}(1,1)$ with $\alpha_i \in \mathbb{C}$ and $\alpha_1 \neq 0$. Then

$$\begin{aligned} &f \circ \varphi(z) \\ &= (f'(0)\alpha_1)z + \left(f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2\right)z^2 + \cdots + \left(\sum_{i_1+\cdots+i_l=k} \frac{f^{(l)}(0)}{l!}\alpha_{i_1}\cdots\alpha_{i_l}\right)z^k, \end{aligned}$$

which equals

$$(9)\qquad (f'(0), \ldots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & 2\alpha_1\alpha_{k-1} + \cdots \\ 0 & 0 & \alpha_1^3 & \cdots & 3\alpha_1^2\alpha_{k-2} + \cdots \\ 0 & 0 & 0 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix},$$

where the (i,j) entry is $p_{i,j}(\overline{\alpha}) = \sum_{a_1+a_2+\cdots+a_i=j} \alpha_{a_1}\alpha_{a_2}\cdots\alpha_{a_i}$.

Remark 3.9 The linearisation of the action of $\mathrm{Diff}_k(1)$ on $J_k^{\mathrm{reg}}(1,n)$ given as the matrix multiplication in (9) represents $\mathrm{Diff}_k(1)$ as a group of upper triangular matrices in $\mathrm{GL}(n)$. This is a nonreductive group so Mumford's reductive GIT is not applicable to study the geometry of the quotient $J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1)$; see Bérczi, Doran, Hawes and Kirwan [3; 4] for details. Note that our matrix group is parametrised along its first row with the free parameters $\alpha_1, \ldots, \alpha_k$ and the other entries are certain (weighted homogeneous) polynomials in these free parameters. It is a $\mathbb{C}^*$ extension of its maximal unipotent radical

$$\mathrm{Diff}_k(1) = U \rtimes \mathbb{C}^*,$$

where U is the subgroup we get via substituting $\alpha_1 = 1$ and the diagonal $\mathbb{C}^*$ acts with weights $0, 1, \ldots, n-1$ on the Lie algebra $\mathrm{Lie}(U)$. In Bérczi and Kirwan [5] and Bérczi, Doran, Hawes and Kirwan [3; 4] we study actions of groups of this type in a more general context.

Fix an integer $N \geq 1$ and define

$$\Theta_k = \{\Psi \in J_k(n,N) : \exists \gamma \in J_k^{\mathrm{reg}}(1,n) \text{ such that } \Psi \circ \gamma = 0\},$$

that is, Θ_k is the set of those k–jets of germs on $\mathbb{C}^n$ at the origin which vanish on some regular curve. By definition, Θ_k is the image of the closed subvariety of $J_k(n,N) \times J_k^{\mathrm{reg}}(1,n)$ defined by the algebraic equations $\Psi \circ \gamma = 0$, under the projection to the first factor. If $\Psi \circ \gamma = 0$, we call γ a *test curve* of Θ.

Remark 3.10 The subset Θ_k is the closure of an important singularity class in the jet space $J_k(n,N)$. These are called Morin singularities and the equivariant dual of Θ_k in $J_k(n,N)$ is called the Thom polynomial of Morin singularities; see Bérczi and Szenes [7] and Fehér and Rimányi [16] for details.

Test curves of germs are generally not unique. A basic but crucial observation is the following. If γ is a test curve of $\Psi \in \Theta_k$, and $\varphi \in \mathrm{Diff}_k(1)$ is a holomorphic reparametrisation of $\mathbb{C}$, then $\gamma \circ \varphi$ is, again, a test curve of Ψ:

$$\mathbb{C} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\gamma} \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^N \text{ with } \Psi \circ \gamma = 0 \quad \Longrightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0.$$

In fact, we get all test curves of Ψ in this way if the following property, open and dense in Θ_k, holds: the linear part of Ψ has 1–dimensional kernel. Before stating this in Theorem 3.12, let us write down the equation $\Psi \circ \gamma = 0$ in coordinates in an illustrative case. Let $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\mathrm{reg}}(1,n)$ and $\Psi = (\Psi', \Psi'', \dots, \Psi^{(k)}) \in J_k(n,N)$ be the k–jets of the test curve γ and the map Ψ respectively. Using the chain rule and the notation $v_i = \gamma^{(i)}/i!$, the equation $\Psi \circ \gamma = 0$ reads as follows for $k = 4$:

$$\begin{gathered}
\Psi'(v_1) = 0,\\
\Psi'(v_2) + \Psi''(v_1, v_1) = 0,\\
\Psi'(v_3) + 2\Psi''(v_1, v_2) + \Psi'''(v_1, v_1, v_1) = 0,\\
\Psi'(v_4) + 2\Psi''(v_1, v_3) + \Psi''(v_2, v_2) + 3\Psi'''(v_1, v_1, v_2) + \Psi''''(v_1, v_1, v_1, v_1) = 0.
\end{gathered} \tag{10}$$

Lemma 3.11 (Gaffney [19]; Bérczi and Szenes [7]) *Let $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\mathrm{reg}}(1,n)$ and $\Psi = (\Psi', \Psi'', \dots, \Psi^{(k)}) \in J_k(n,N)$ be k–jets. Then substituting $v_i = \gamma^{(i)}/i!$, the equation $\Psi \circ \gamma = 0$ is equivalent to the following system of k linear equations with values in $\mathbb{C}^N$:*

$$\sum_{\tau \in \mathcal{P}(m)} \Psi(\boldsymbol{v}_\tau) = 0 \quad \text{for } m = 1, 2, \dots, k. \tag{11}$$

Here $\mathcal{P}(m)$ denotes the set of partitions $\tau = 1^{\tau_1} \dots m^{\tau_m}$ of m into nonnegative integers and $\boldsymbol{v}_\tau = v_1^{\tau_1} \cdots v_m^{\tau_m}$.

For a given $\gamma \in J_k^{\mathrm{reg}}(1,n)$ and $1 \le i \le k$ let $\mathcal{S}_\gamma^{i,N}$ denote the set of solutions of the first i equations in (11), that is,

$$\mathcal{S}_\gamma^{i,N} = \{\Psi \in J_k(n,N) : \Psi \circ \gamma = 0 \text{ up to order } i\}. \tag{12}$$

The equations (11) are linear in Ψ, and hence

$$\mathcal{S}_\gamma^{i,N} \subset J_k(n,N)$$

is a linear subspace of codimension iN, ie a point of $\mathrm{Grass}_{\mathrm{codim}=iN}(J_k(n,N))$, whose orthogonal, $(\mathcal{S}_\gamma^{i,N})^\perp$, is an iN–dimensional subspace of $J_k(n,N)^*$. These subspaces are invariant under the reparametrisation of γ. In fact, $\Psi \circ \gamma$ has N vanishing coordinates and therefore

$$(\mathcal{S}_\gamma^{i,N})^\perp = (\mathcal{S}_\gamma^{i,1})^\perp \otimes \mathbb{C}^N.$$

For $\Psi \in J_k(n,N)$ let $\Psi^1 \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^N)$ denote the linear part. When $N \geq n$ then the subset

$$\widetilde{\mathcal{S}}_\gamma^{i,N} = \{\Psi \in \mathcal{S}_\gamma^{i,N} : \dim\ker \Psi^1 = 1\}$$

is an open dense subset of the subspace $\mathcal{S}_\gamma^{i,N}$. In fact it is not hard to see that the complement $\widetilde{\mathcal{S}}_\gamma^{i,N} \setminus \mathcal{S}_\gamma^{i,N}$ where the kernel of Ψ^1 has dimension at least two is a closed subvariety of codimension $N-n+2$.

Theorem 3.12 (1) *The map*

$$\phi\colon J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}_k(J_k(n,1)^*)$$

defined as $\gamma \mapsto (\mathcal{S}_\gamma^{k,1})^\perp$ *is* $\mathrm{Diff}_k(1)$*–invariant and induces an injective map on the* $\mathrm{Diff}_k(1)$*–orbits into the Grassmannian*

$$\phi^{\mathrm{Grass}}\colon J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1) \hookrightarrow \mathrm{Grass}_k(J_k(n,1)^*).$$

Moreover, ϕ *and* ϕ^{Grass} *are* $\mathrm{GL}(n)$*–equivariant with respect to the standard action of* $\mathrm{GL}(n)$ *on* $J_k^{\mathrm{reg}}(1,n) \subset \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$ *and the induced action on* $\mathrm{Grass}_k(J_k(n,1)^*)$.

(2) *Recall form Remark 3.4 that* $J_k(n,1)^* = \mathrm{Sym}^{\leq k}\mathbb{C}^n$. *The image of* ϕ *and the image of* φ *defined in Remark 3.3 coincide in* $\mathrm{Grass}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$*:*

$$\mathrm{Im}(\phi) = \mathrm{Im}(\varphi) \subset \mathrm{Grass}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n).$$

Proof For the first part it is enough to prove that for $\Psi \in \Theta_k$ with $\dim\ker\Psi^1 = 1$ and $\gamma,\delta \in J_k^{\mathrm{reg}}(1,n)$,

$$\Psi\circ\gamma = \Psi\circ\delta = 0 \iff \exists\Delta \in J_k^{\mathrm{reg}}(1,1) \text{ such that } \gamma = \delta\circ\Delta.$$

We prove this statement by induction. Let $\gamma = v_1t+\cdots+v_kt^k$ and $\delta = w_1t+\cdots+w_kt^k$. Since $\dim\ker\Psi^1 = 1$, we have $v_1 = \lambda w_1$, for some $\lambda \neq 0$. This proves the $k=1$ case.

Suppose the statement is true for $k-1$. Then, using the appropriate order-$(k-1)$ diffeomorphism, we can assume that $v_m = w_m$ for $m = 1,\dots,k-1$. It is clear then from the explicit form (11) (see (10)) of the equation $\Psi\circ\gamma=0$, that $\Psi^1(v_k)=\Psi^1(w_k)$, and hence $w_k=v_k-\lambda v_1$ for some $\lambda\in\mathbb{C}$. Then $\gamma=\Delta\circ\delta$ for $\Delta=t+\lambda t^k$, and the proof is complete.

The second part immediately follows from the definition of φ and ϕ. □

Remark 3.13 (1) For a point $\gamma\in J_k^{\mathrm{reg}}(1,n)$ let $v_i=\gamma^{(i)}/i!\in\mathbb{C}^n$ denote the normed i^{th} derivative. Then from Lemma 3.11 it immediately follows that for $1\le i\le k$ (see [7]),

$$(\mathcal{S}_\gamma^{i,1})^\perp=\mathrm{Span}_{\mathbb{C}}\Big(v_1,v_2+v_1^2,\dots,\sum_{\tau\in\mathcal{P}(i)}\boldsymbol{v}_\tau\Big)\subset\mathrm{Sym}^{\le k}\mathbb{C}^n. \tag{13}$$

This explicit parametrisation of the curvilinear component is crucial in building our localisation process in the next section.

(2) Let $\{e_1,\dots,e_n\}$ be a basis of $\mathbb{C}^n$. Since ϕ is $\mathrm{GL}(n)$–equivariant, for $k\le n$ the $\mathrm{GL}(n)$–orbit of $p_{k,n}$ satisfies

$$p_{k,n}=\phi(e_1,\dots,e_k)=\mathrm{Span}_{\mathbb{C}}\Big(e_1,e_2+e_1^2,\dots,\sum_{\tau\in\mathcal{P}(k)}e_\tau\Big)$$

and forms a dense subset of the image $J_k^{\mathrm{reg}}(1,n)$ and therefore

$$\overline{\phi(J_k^{\mathrm{reg}}(1,n))}=\overline{\mathrm{GL}(n)\cdot p_{k,n}}.$$

Recall that after choosing local coordinates on X near p we can identify the fibre $J_k^{\mathrm{reg}}X_p$ with $J_k^{\mathrm{reg}}(1,n)$ and the fibre $\mathcal{D}_{X,p}^k$ with $J_k(n,1)^*$. Lemma 3.8 and Theorem 3.12 therefore give us the following.

Corollary 3.14 *We have an embedding of the punctual curvilinear locus $CX_p^{[k+1]}$,*

$$\phi_p^{\mathrm{Grass}}\colon CX_p^{[k+1]}=J_k^{\mathrm{reg}}X_p/\mathrm{Diff}_k(1)\hookrightarrow\mathrm{Grass}_k(\mathcal{D}_{X,p}^k),$$

into the Grassmannian bundle of k–dimensional subspaces of the fibre $\mathcal{D}_{X,p}^k$. The quotient $J_k^{\mathrm{reg}}X/\mathrm{Diff}_k(1)$ has the structure of a locally trivial bundle over X which has a holomorphic embedding

$$\phi^{\mathrm{Grass}}\colon CX^{[k+1]}=J_k^{\mathrm{reg}}X/\mathrm{Diff}_k(1)\hookrightarrow\mathrm{Grass}_k(\mathcal{D}_X^k)$$

into the Grassmannian bundle of k–dimensional subspaces of $\mathcal{D}_X^k$ over X. The closure of the image

$$\overline{CX}^{[k+1]}=\overline{\phi^{\mathrm{Grass}}(J_k^{\mathrm{reg}}X)}$$

is the curvilinear component of the Hilbert scheme of $k+1$ points on X.

3.2 Tautological bundles over $\overline{CX}^{[k]}$

Let F be a rank-r vector bundle over X. The fibre of the corresponding rank-rk tautological bundle $F^{[k]}$ on $\overline{CX}^{[k]}$ at the point ξ is

$$F^{[k]}_\xi = H^0(\xi, F|_\xi) = H^0(\mathcal{O}_\xi \otimes F).$$

On the level of bundles we have the following.

Lemma 3.15 *There is an isomorphism of topological vector bundles $F^{[k]}|_{\overline{CX}^{[k]}} \simeq \mathcal{O}^{[k]}_{\overline{CX}^{[k]}} \otimes \pi^* F$ where $\pi\colon \overline{CX}^{[k]} \to X$ is the projection.*

Proof This comes from Rennemo [33, Lemma 5.2] as follows. Let us adopt the notations of [33] and denote by $X^{[\![k]\!]}$ the Hilbert scheme of k ordered points on X and let $X_0^{[\![k]\!]} \subset X^{[\![k]\!]}$ be the set of pairs $(\xi, (x_i))$ such that ξ is supported at a single point, that is, $x_1 = \cdots = x_k$. Let $\overline{TX} = \mathbb{P}(\mathcal{O}_X \oplus TX)$ denote the natural fibrewise compactification of the tangent bundle. Let $\overline{TX}_0^{[\![k]\!]} \subset \overline{TX}^{[\![k]\!]}$ denote the set of pairs $(\xi, (x_i))$ such that ξ is supported at the 0–section of $\overline{TX}$. Let $q\colon X^{[\![k]\!]} \to X$ be defined by $q(\xi, (x_i)) = x_1$ and let $r\colon \overline{TX}^{[\![k]\!]} \to X$ be defined by $r((\xi, (x_i))) = \pi(x_1)$. Let W be the set of pairs $(\xi, (x_i)) \in \overline{TX}^{[\![k]\!]}$ such that x_1 lies in the 0–section of $\overline{TX}$.

Rennemo [33] shows that there is an open neighbourhood U of $X_0^{[\![k]\!]}$ in $X^{[\![k]\!]}$, an open neighbourhood $\mathcal{U}$ of $\overline{TX}_0^{[\![k]\!]}$ in W, and a homeomorphism $f\colon U \to \mathcal{U}$ such that $q = r \circ f$ and $f|_{q^{-1}(x)}$ is holomorphic for all $x \in X$. Furthermore, there is an isomorphism of topological vector bundles $f^*(\mathcal{F}^{[\![k]\!]}) \simeq F^{[k]}$.

In particular, f is constructed using a similar but simpler statement about the neighbourhood of the diagonal in $X \times X$. Let $p_1, p_2\colon X \times X \to X$ be the projections to the first and second factors, and let $\pi\colon TX \to X$ be the tangent bundle. There is an open neighbourhood U_1 of the diagonal $\delta \subset X \times X$, an open neighbourhood $\mathcal{U}_1$ of the 0–section $X \subset TX$ and a homeomorphism $f_1\colon U_1 \to \mathcal{U}_1$, such that $\pi \circ f_1 = p_1$ and such that $f_1|_\Delta$ is the identification between Δ and the 0–section of TX. Furthermore, the restriction of f_1 to each fibre $p^{-1}(x)$ is holomorphic. There is an isomorphism of topological vector bundles $p_1^*(E)|_U \to p_2^*(E)|_U$, which is an isomorphism of holomorphic bundles on the restriction to each fibre $p^{-1}(x)$.

Then f is given by

$$f\big((\xi, (x_i))\big) = \big((f_1)_*(\{q(x)\} \times \xi), f_1(q(x), x_i)\big)$$

on a small neighbourhood U of $X_0^{[\![k]\!]}$ and over a point $(\xi, (x_i)) \in X^{[\![k]\!]}$ we have

$$\begin{aligned} f^*(\mathcal{F}^{[\![k]\!]})_{(\xi,(x_i))} &= H^0(\{x_1\} \times \xi, p_1^*(F)|_{\{x_1\}\times\xi}) \\ &\simeq H^0(\{x_1\} \times \xi, p_2^*(F)|_{\{x_1\}\times\xi}) = F^{[\![k]\!]}_{(\xi,(x_i))}. \end{aligned}$$

For $(\xi,(x_i)) \in \overline{CX}_p^{[k]} \subset X_p^{[\![k]\!]}$ we have $p = x_1 = \cdots = x_k$ and therefore

$$F_\xi^{[\![k]\!]} = H^0(\{x_1\}\times\xi, p_1^*(F)|_{\{x_1\}\times\xi}) = F_p \otimes H^0(\mathcal{O}_\xi),$$

which gives the isomorphism of the lemma. □

Our embedding $\phi^{\mathrm{Grass}}\colon \overline{CX}^{[k+1]} \hookrightarrow \mathrm{Grass}_k(\mathcal{D}^k)$ then identifies the fibres of $\mathcal{O}^{[k+1]}_{\overline{CX}^{[k+1]}}$ over $\xi \in \overline{CX}_p^{[k+1]}$ with

$$H^0(\mathcal{O}_\xi) \simeq \mathcal{O}_p \otimes \mathcal{E}_{\phi^{\mathrm{Grass}}(\xi)},$$

where $\mathcal{E}$ is the tautological rank-k bundle over $\mathrm{Grass}_k(\mathcal{D}^k)$. Hence the total Chern class of $F^{[k+1]}$ can be written as

$$c(F^{[k+1]}) = \prod_{j=1}^{r}(1+\theta_j)\prod_{i=1}^{k}\prod_{j=1}^{r}(1+\eta_i+\theta_j), \tag{14}$$

where $c(F) = \prod_{j=1}^r(1+\theta_j)$ and $c(\mathcal{E}) = \prod_{i=1}^k(1+\eta_i)$ are the Chern classes for the corresponding bundles. In particular the Chern class

$$c_i(F^{[k+1]}) = \mathcal{C}_i(c_1(\mathcal{E}),\dots c_k(\mathcal{E}), c_1(F),\dots,c_r(F)) \tag{15}$$

can be expressed as a polynomial function $\mathcal{C}_i$ in Chern classes of $\mathcal{E}$ and F.

4 Partial resolutions of $\overline{CX}^{[k+1]}$

The structure group of the bundles $\overline{CX}^{[k+1]}$ and $\mathrm{Grass}_k(\mathcal{D}_X^k)$ is the polynomial reparametrisation group $\mathrm{Diff}_k(n)$. The subgroup $\mathrm{GL}(n)$ of linear coordinate changes sit in $\mathrm{Diff}_k(n)$ and using this in Section 4.1 we define the corresponding linearised bundles $\overline{CX}_{\mathrm{GL}}^{[k+1]} \subset \mathrm{Grass}_k(\mathrm{Sym}^{\le k} T_X)$. In Section 4.2 we construct a fibrewise partial resolution of the (highly singular) curvilinear component $\overline{CX}^{[k+1]} \subset \mathrm{Grass}_k(\mathcal{D}_X^k)$ and also its linearised bundle $\overline{CX}_{\mathrm{GL}}^{[k+1]}$. This resolution is defined for any choice of parameters n,k and it uses nested Hilbert schemes. In Section 4.3 we construct a second partial resolution of $\overline{CX}_{\mathrm{GL}}^{[k+1]}$ under the very restrictive condition $k \le n$, that is, the number of points can't exceed the dimension of the variety plus one. We will see how to dispose this condition in Section 6.2.

4.1 Linearisation of $\overline{CX}^{[k+1]}$

Recall from Section 3.1.1 the notation $\mathrm{Diff}_k(n) = J_k^{\mathrm{reg}}(n,n)$ for the group of k^{th}–order diffeomorphism germs of $\mathbb{C}^n$ at the origin. Then $\mathrm{Diff}_k(n)$ is the set of local (polynomial) coordinate changes on $\mathbb{C}^n$ at the origin. The set $\mathrm{GL}(n)$ of linear coordinate changes forms a subgroup of $\mathrm{Diff}_k(n)$.

We have seen that after choosing local coordinates on X near p we can identify the fibre $\mathcal{D}^k_{X,p}$ of the bundle $\mathcal{D}^k_X$ with $J_k(n,1)^* = \mathrm{Sym}^{\le k}\mathbb{C}^n$. Then Diff_k (and therefore its subgroup $\mathrm{GL}(n)$) acts on $\mathcal{D}^k_{X,p} \simeq \mathrm{Sym}^{\le k}\mathbb{C}^n$. Let Diff_X denote the principal $\mathrm{Diff}_k(n)$–bundle over X formed by all local polynomial coordinate systems on X. Then $\mathcal{D}^k_X$ can be described as the associated bundle (see also Remark 3.4)

$$\mathcal{D}^k_X = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} \mathrm{Sym}^{\le k}\mathbb{C}^n.$$

On the other hand, if GL_X denotes the principal $\mathrm{GL}(n)$–bundle over X formed by all local linear coordinate systems on X then the vector bundle $\mathrm{Sym}^{\le k} T_X = \bigoplus_{i=1}^k \mathrm{Sym}^i T_X$ is associated to the same $\mathrm{Sym}^{\le k}\mathbb{C}^n$ considered as a $\mathrm{GL}(n)$–module:

$$\mathrm{Sym}^{\le k} T_X = \mathrm{GL}_X \times_{\mathrm{GL}(n)} \mathrm{Sym}^{\le k}\mathbb{C}^n.$$

Therefore $\mathcal{D}^k_X$ and $\mathrm{Sym}^{\le k} T_X$ are not isomorphic bundles and in particular the filtration defined in (6) does not split. Hence there is no projection map $\mathcal{D}^k_X \to T_X$ but there is a natural projection $\mathrm{Sym}^{\le k} T_X \to T_X$.

There is an induced $\mathrm{Diff}_k(n)$–action on $\mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ and the image $\mathrm{Im}(\phi^{\mathrm{Grass}})$ in Theorem 3.12 is $\mathrm{Diff}_k(n)$–invariant subvariety of $\mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$. The curvilinear locus $\overline{CX}^{[k+1]}$ is the associated bundle

$$\begin{aligned}\overline{CX}^{[k+1]} &= \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} \overline{\mathrm{Im}(\phi^{\mathrm{Grass}})}\\ &\subset \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) = \mathrm{Grass}_k(\mathcal{D}^k_X).\end{aligned}$$

We can form the corresponding linearised bundle

$$\begin{aligned}\overline{CX}^{[k+1]}_{\mathrm{GL}} &= \mathrm{GL}_X \times_{\mathrm{GL}(n)} \overline{\mathrm{Im}(\phi^{\mathrm{Grass}})}\\ &\subset \mathrm{GL}_X \times_{\mathrm{GL}(n)} \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) = \mathrm{Grass}_k(\mathrm{Sym}^{\le k} T_X),\end{aligned}$$

which is the linearised version of $\overline{CX}^{[k+1]}$ remembering the linear action on the fibres. We will explain in Section 7 that for torus localisation purposes we can replace $\overline{CX}^{[k+1]}$ with its linearised version $\overline{CX}^{[k+1]}_{\mathrm{GL}}$.

4.2 Completion in nested Hilbert schemes

Let

$$X^{[k_1,\dots,k_t]} = \{(\xi_1 \subset \xi_2 \subset \cdots \subset \xi_t) : \xi_i \in X^{[k_i]}\} \subset X^{[k_1]} \times \cdots \times X^{[k_t]}$$

denote the nested Hilbert scheme defining flags of subschemes of length vector $(k_1,\dots,k_t)$.

Curvilinear subschemes contain only one subscheme for any given smaller length. Therefore $\xi \in CX_p^{[k+1]}$ defines a unique flag

$$\mathcal{F}(\xi) = (\xi_1 \subset \xi_2 \subset \cdots \subset \xi_k) \in CX_p^{[2]} \times \cdots \times CX_p^{[k+1]} \subset X^{[2,\dots,k+1]},$$

where ξ_i is the unique subscheme of ξ satisfying

$$\mathcal{O}_{\xi_i} = \mathcal{O}_\xi / \mathcal{O}_{X,p}^{i+1} \simeq \mathbb{C}[z]/z^{i+1},$$

and therefore $\xi_i \in CX_p^{[i+1]}$. This defines an embedding

$$\widetilde{\phi}\colon CX_p^{[k+1]} \hookrightarrow X^{[2,\dots,k+1]}, \quad \xi \mapsto (\xi_1 \subset \cdots \subset \xi_k).$$

Fix local coordinates on X near p such that $J_k X_p$ is identified with $J_k(1,n)$ and $\mathcal{D}^k_{X,p}$ is identified with $J_k(n,1)^*$. Let $f_\xi \in J_k^{\mathrm{reg}}(1,n)$ denote the k–jet corresponding to $\xi \in CX_p^{[k+1]}$ and let $\mathcal{S}^i_\xi = \mathcal{S}^{i,1}_{f_\xi} \subset J_k(n,1)$ be the solution space defined in (12) where $N=1$. Then $\widetilde{\phi}$ can be equivalently written as

$$f_\xi \mapsto ((\mathcal{S}^1_\xi)^\perp \subset (\mathcal{S}^2_\xi)^\perp \subset \cdots \subset (\mathcal{S}^k_\xi)^\perp) \in \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

or, using coordinates, as

$$f_\xi \mapsto \Big(\mathrm{Span}(f') \subset \mathrm{Span}(f', f''+(f')^2) \subset \cdots \subset \mathrm{Span}\Big(f', f''+(f')^2, \dots, \sum_{\sum a_i = k} (f^{[i]})^{a_i}\Big)\Big).$$

Theorem 3.12 has the following immediate corollary:

Corollary 4.1 *The map*

$$\widetilde{\phi}\colon J_k^{\mathrm{reg}}(1,n) \to \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n), \quad \gamma \mapsto \mathcal{F}_\gamma = ((\mathcal{S}^1_\gamma)^\perp \subset \cdots \subset (\mathcal{S}^k_\gamma)^\perp)$$

is $\mathrm{Diff}_k(1)$*–invariant and induces an injective map on the* $\mathrm{Diff}_k(1)$*–orbits into the flag manifold*

$$\phi^{\mathrm{Flag}}\colon J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1) \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n).$$

Moreover, all these maps are $\mathrm{GL}(n)$*–equivariant with respect to the standard action of* $\mathrm{GL}(n)$ *on* $J_k^{\mathrm{reg}}(1,n) \subset \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ *and the induced action on* $\mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$*.*

This implies that similarly to Remark 3.13, for any basis $\{e_1,\dots,e_n\}$ *of* $\mathbb{C}^n$ *and* $k \le n$ *the* $\mathrm{GL}(n)$*–orbit of* $\mathfrak{p}_{k,n} = \widetilde{\phi}(e_1,\dots,e_k)$*, that is, in coordinates,*

$$\mathfrak{p}_{k,n} = \Big(\mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2+e_1^2) \subset \cdots \subset \mathrm{Span}\Big(e_1, e_2+e_1^2, \dots, \sum_{\tau \in \mathcal{P}(k)} e_\tau\Big)\Big),$$

forms a dense subset of the image $J_k^{\mathrm{reg}}(1,n)$ *and therefore*

$$\overline{\widetilde{\phi}(J_k^{\mathrm{reg}}(1,n))} = \overline{\mathrm{GL}(n)\cdot \mathfrak{p}_{k,n}}.$$

Definition 4.2 We define the bundle

$$\begin{aligned}\widehat{CX}^{[k+1]} &= \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} \overline{\widetilde{\phi}(J_k^{\mathrm{reg}}(1,n))} \\ &\subset \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) = \mathrm{Flag}_k(\mathcal{D}_X^k),\end{aligned}$$

which is a fibrewise partial resolution of $\overline{CX}^{[k+1]}$. The corresponding linearised bundle is defined as

$$\begin{aligned}\widehat{CX}_{\mathrm{GL}}^{[k+1]} &= \mathrm{GL}_X \times_{\mathrm{GL}(n)} \overline{\widetilde{\phi}(J_k^{\mathrm{reg}}(1,n))} \\ &\subset \mathrm{GL}_X \times_{\mathrm{GL}(n)} \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) = \mathrm{Flag}_k(\mathrm{Sym}^{\le k} T_X).\end{aligned}$$

4.3 Blowing up along the linear part

Assume $k \le n$. Let $J_k^{\mathrm{nondeg}}(1,n) \subset J_k^{\mathrm{reg}}(1,n)$ denote the Zariski open set of jets $(\gamma', \gamma'', \dots, \gamma^{[k]})$ with $\gamma', \dots, \gamma^{(k)}$ linearly independent. These correspond to the regular $n \times k$ matrices in $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, and they fibre over the set of complete flags in $\mathbb{C}^n$:

$$J_k^{\mathrm{nondeg}}(1,n) \to \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)/B_k = \mathrm{Flag}_k(\mathbb{C}^n),$$

where $B_k \subset \mathrm{GL}(k)$ is the upper Borel subgroup. Since $J_k^{\mathrm{reg}}(1,1) \subset B_k$ this induces a surjective fibration

$$\pi\colon J_k^{\mathrm{nondeg}}(1,n)/\mathrm{Diff}_k(1) \to \mathrm{Flag}_k(\mathbb{C}^n) \tag{16}$$

which factors through ϕ^{Grass}:

$$\begin{tikzcd} J_k^{\mathrm{nondeg}}(1,n)/\mathrm{Diff}_k(1) \arrow[r, "\phi^{\mathrm{Flag}}"] \arrow[dr, "\pi"'] & \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) \arrow[d, dashed] \\ & \mathrm{Flag}_k(\mathbb{C}^n) \end{tikzcd} \tag{17}$$

Here the vertical rational map is induced by the projection $\mathrm{Sym}^{\le k}\mathbb{C}^n \to \mathbb{C}^n$ and the image of ϕ^{Flag} sits in its domain.

Since $J_k^{\mathrm{nondeg}}(1,n) \subset J_k(1,n)$ is $\mathrm{GL}(n)$–invariant, we can form the associated bundle

$$J_k^{\mathrm{nondeg}} X = \mathrm{GL}_X \times_{\mathrm{GL}(n)} J_k^{\mathrm{nondeg}}(1,n).$$

Note, however, that $J_k^{\mathrm{nondeg}} X$ is not a subbundle of $J_k X = \mathrm{Diff}_X \times_{\mathrm{Diff}_k(n)} J_k(1,n)$. Similarly, we can form the bundle

$$\begin{aligned}CX_{\mathrm{nondeg}}^{[k+1]} &= \mathrm{GL}_X \times_{\mathrm{GL}(n)} (\widetilde{\phi}(J_k^{\mathrm{nondeg}}(1,n))) \subset \mathrm{GL}_X \times_{\mathrm{GL}(n)} \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) \\ &= \mathrm{Flag}_k(\mathrm{Sym}^{\le k} T_X),\end{aligned}$$

which is a dense subbundle of $\widehat{CX}_{\mathrm{GL}}^{[k+1]}$ but not a subbundle of $\widehat{CX}^{[k+1]}$. The projection $\mathrm{Sym}^{\leq k} T_X \to T_X$ then induces the following diagram whose restriction to the fibres over X was given in (17):

$$
\begin{tikzcd}
J_k^{\mathrm{nondeg}} X/\mathrm{Diff}_k(1) \arrow[r, "\phi^{\mathrm{Flag}}"] \arrow[dr, "\pi"'] & \mathrm{Flag}_k(\mathrm{Sym}^{\leq k} T_X) \arrow[d, dashed] \\
& \mathrm{Flag}_k(T_X)
\end{tikzcd}
\tag{18}
$$

The image of ϕ^{Flag} sits in the domain of the vertical rational map and therefore we have a fibration

$$\mu\colon CX_{\mathrm{nondeg}}^{[k+1]} \to \mathrm{Flag}_k(T_X)$$

of the bundles.

Definition 4.3 Let $\widetilde{CX}^{[k+1]} \to \mathrm{Flag}_k(T_X)$ denote the fibrewise compactification of the bundle $\pi\colon CX_{\mathrm{nondeg}}^{[k+1]} \to \mathrm{Flag}_k(T_X)$. In other words, if $P_{k,n} \subset \mathrm{GL}(n)$ denotes the parabolic subgroup which preserves the flag

$$\boldsymbol{f} = (\mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2) \subset \cdots \subset \mathrm{Span}(e_1, \ldots, e_k) \subset \mathbb{C}^n).$$

and $\mathfrak{p}_{k,n} = \widetilde{\phi}(e_1, \ldots, e_k)$ denotes the base point in $\mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$, then

$$\widetilde{CX}^{[k+1]} = \mathrm{GL}_X \times_{\mathrm{GL}(n)} (\mathrm{GL}(n) \times_{P_{k,n}} \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}}),$$

and we have a partial resolution map

$$\rho\colon \widetilde{CX}^{[k+1]} = \mathrm{GL}_X \times_{\mathrm{GL}(n)} (\mathrm{GL}(n) \times_{P_{k,n}} \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}}) \\ \to \mathrm{GL}_X \times_{\mathrm{GL}(n)} (\overline{\mathrm{GL}(n) \cdot \mathfrak{p}_{k,n}}) = \widehat{CX}_{\mathrm{GL}}^{[k+1]}.$$

Remark 4.4 Equivalently, let $\pi\colon J_k(n,1)^* \simeq \mathrm{Sym}^{\leq k}\mathbb{C}^n = \bigoplus_{i=1}^k \mathrm{Sym}^i\mathbb{C}^n \to \mathbb{C}^n$ denote the projection to the first (linear) factor. Then

$$\mathrm{GL}(n) \times_{P_{k,n}} \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}} \\ = \big\{\big(((\mathcal{S}_\gamma^1)^\perp \subset \cdots \subset (\mathcal{S}_\gamma^k)^\perp), (V_1 \subset \cdots \subset V_k)\big) : \pi(\mathcal{S}_\gamma^i)^\perp \subset V_i\big\} \subset \overline{\mathrm{Im}(\widetilde{\phi})} \times \mathrm{Flag}_k(\mathbb{C}^n).$$

5 Equivariant localisation on $\widetilde{CX}_p^{[k+1]}$

In this section we fix a point $p \in X$ and a holomorphic coordinate system on X near p. We identify the fibre $J_k X_p$ with $J_k(1,n)$ and $\mathcal{D}_{X,p}^k$ with $J_k(n,1)^* = \mathrm{Sym}^{\leq k}\mathbb{C}^n$.

With these identifications we can use Theorem 3.12, Remark 3.13, Definition 4.2, Definition 4.3 and the maps

$$\phi^{\mathrm{Grass}}\colon J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1) \hookrightarrow \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

and

$$\phi^{\mathrm{Flag}}\colon J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1) \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

to describe the punctual curvilinear Hilbert scheme and its partial resolutions at $p \in X$ as

$$\begin{aligned}
\overline{CX}_p^{[k+1]} &= \overline{\mathrm{Im}(\phi^{\mathrm{Grass}})} = \overline{\mathrm{GL}(n)\cdot \mathfrak{p}_{k,n}} \subset \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n),\\
\widehat{CX}_p^{[k+1]} &= \overline{\mathrm{Im}(\phi^{\mathrm{Flag}})} = \overline{\mathrm{GL}(n)\cdot \mathfrak{p}_{k,n}} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n),\\
\widetilde{CX}_p^{[k+1]} &= \mathrm{GL}(n)\times_{P_{k,n}} \overline{P_{k,n}\cdot \mathfrak{p}_{k,n}} \to \mathrm{Flag}_k(\mathbb{C}^n).
\end{aligned}$$

Let F be a rank-r vector bundle over X and let $F^{[k+1]}$ denote the corresponding rank-$(k+1)r$ tautological bundle over $X^{[k+1]}$. We use the same notation $F^{[k+1]}$ for its pull-back along the partial resolution map $\rho\colon \widetilde{CX}_p^{[k+1]} \to \overline{CX}_p^{[k+1]}$. Then $\widetilde{CX}_p^{[k+1]} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ is endowed with a natural $\mathrm{GL}(n)$–action. In this section we start developing an iterated residue formula for $\int_{\widetilde{CX}_p^{[k]}} \alpha$ for closed torus equivariant forms α. This formula is attained via equivariant localisation process using the fibration $\pi\colon \widetilde{CX}_p^{[k+1]} \to \mathrm{Flag}_k(\mathbb{C}^n)$ and it is crucially based on a vanishing theorem of residues.

5.1 Equivariant de Rham model and the Atiyah–Bott formula

This section is a short introduction to equivariant cohomology and localisation. For more details, we refer the reader to Section 2 of Bérczi and Szenes [7] and Berline, Getzler and Vergne [8].

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let M be a C^∞ manifold endowed with the action of G. The G–equivariant differential forms are defined as differential-form-valued polynomial functions on $\mathfrak{g}$:

$$\Omega_G^\bullet(M) = \{\alpha\colon \mathfrak{g} \to \Omega^\bullet(M) : \alpha(gX) = g\alpha(X) \text{ for } g\in G,\ X\in\mathfrak{g}\} = (S^\bullet\mathfrak{g}^*\otimes\Omega^\bullet(M))^G,$$

where $(g\cdot\alpha)(X) = g\cdot(\alpha(g^{-1}\cdot X))$. One can define equivariant the exterior differential d_G on $(S^\bullet\mathfrak{g}^*\otimes\Omega^\bullet(M))^G$ by the formula

$$(d_G\alpha)(X) = (d - \iota(X_M))\alpha(X),$$

where $\iota(X_M)$ denotes the contraction by the vector field X_M. This increases the degree of an equivariant form by one if the $\mathbb{Z}$–grading is given on $(S^\bullet\mathfrak{g}^*\otimes\Omega^\bullet(M))^G$ by $\deg(P\otimes\alpha) = 2\deg(P) + \deg(\alpha)$ for $P\in S^\bullet\mathfrak{g}^*$ and $\alpha\in\Omega^\bullet(M)$. The homotopy formula $\iota(X)d + d\iota(X) = \mathcal{L}(X)$ implies that $d_G^2(\alpha)(X) = -\mathcal{L}(X)\alpha(X) = 0$

for any $\alpha \in (S^\bullet \mathfrak{g}^* \otimes \Omega^\bullet(M))^G$, and therefore $(d_G, \Omega^\bullet_G(M))$ is a complex. The equivariant cohomology $H^*_G(M)$ of the G–manifold M is the cohomology of the complex $(d_G, \Omega^\bullet_G(M))$. Note that $\alpha \in \Omega^\bullet_G(M)$ is equivariantly closed if and only if

$$\alpha(X) = \alpha(X)^{[0]} + \cdots + \alpha(X)^{[n]} \quad \text{such that } \iota(X_M)\alpha(X)^{[i]} = d\alpha(X)^{[i-2]}.$$

Here $\alpha(X)^{[i]} \in \Omega^i(M)$ is the degree-i part of $\alpha(X) \in \Omega^\bullet(M)$ and $\alpha^{[i]}\colon \mathfrak{g} \to \Omega^i(M)$ is a polynomial function.

The equivariant push-forward map $\int_M\colon \Omega_G(M) \to (S^\bullet \mathfrak{g}^*)^G$ is defined by the formula

$$\left(\int_M \alpha\right)(X) = \int_M \alpha(X) = \int_M \alpha^{[n]}(X). \tag{19}$$

When the n–dimensional complex torus $T = (\mathbb{C}^*)^n$ acts on M let $K = U(1)^n$ be its maximal unipotent subgroup and $\mathfrak{t} = \mathrm{Lie}(K)$ its Lie algebra. We define the T–equivariant cohomology $H^\bullet_T(M)$ to be $H^\bullet_K(M)$, the equivariant de Rham cohomology defined by the action of K. If $M_0(X)$ is the zero locus of the vector field X_M, then the form $\alpha(X)^{[n]}$ is exact outside $M_0(X)$ (see Proposition 7.10 in [8]), and this suggests that the integral $\int_M \alpha(X)$ depends only on the restriction $\alpha(X)|_{M_0(X)}$.

Theorem 5.1 (Atiyah and Bott [1]; Berline and Vergne [9]) *Suppose that M is a compact manifold and T is a complex torus acting smoothly on M, and the fixed-point set M^T of the T–action on M is finite. Then for any cohomology class $\alpha \in H^\bullet_T(M)$,*

$$\int_M \alpha = \sum_{f \in M^T} \frac{\alpha^{[0]}(f)}{\mathrm{Euler}^T(T_f M)}.$$

Here $\mathrm{Euler}^T(T_f M)$ is the T–equivariant Euler class of the tangent space $T_f M$, and $\alpha^{[0]}$ is the differential-form-degree-0 part of α.

The right-hand side in the localisation formula sits in the fraction field of the polynomial ring $H^\bullet_T(\text{point}) = H^\bullet(BT) = S^\bullet \mathfrak{t}^*$. Part of the statement is that the denominators cancel when the sum is simplified.

5.2 Equivariant Poincaré duals and multidegrees

The Atiyah–Bott formula works for holomorphic actions of tori on nonsingular projective varieties. In our case, however, the punctual curvilinear component $\overline{CX}_p^{[k+1]}$ is highly singular at the fixed points so the AB localisation does not apply directly as the equivariant Euler class of the tangent space at a singular fixed point is not well defined. But $\overline{CX}_p^{[k+1]}$ sits in the nonsingular ambient space $\mathrm{Grass}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ and an intuitive idea would be to put $\mathrm{Euler}^T(T_f\mathrm{Grass}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ into the denominator on the right-hand side of the equation in Theorem 5.1 which we then compensate in the numerator with some sort of dual of the tangent cone of $\overline{CX}_p^{[k+1]}$ at f sitting in the tangent space of $\mathrm{Grass}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ at f. This idea indeed works and it becomes incarnate in the Rossmann formula in Section 5.3.

Let $T=(\mathbb{C}^*)^n$ be a complex torus with $K=U(1)^n$ its maximal compact subgroup and $\mathfrak{t}=\mathrm{Lie}(K)$ its Lie algebra. Let M be a manifold endowed with a T–action. The compactly supported equivariant cohomology groups $H^\bullet_{K,\mathrm{cpt}}(M)$ are obtained by restricting the equivariant de Rham complex to compactly supported (or quickly decreasing at infinity) differential forms. Clearly $H^\bullet_{K,\mathrm{cpt}}(M)$ is a module over $H^\bullet_K(M)$. When $M=W$ is an r–dimensional complex vector space, and the action is linear, one has $H^\bullet_K(W)=S^\bullet\mathfrak{t}^*$ and $H^\bullet_{K,\mathrm{cpt}}(W)$ is a free module over $H^\bullet_K(W)$ generated by a single element of degree $2r$ called the Thom class of W:

$$H^\bullet_{K,\mathrm{cpt}}(W)=H^\bullet_K(W)\cdot\mathrm{Thom}_K(W). \tag{20}$$

A T–invariant algebraic subvariety Σ of dimension d in W represents a T–equivariant $2d$–cycle in the sense that

- a compactly supported equivariant form μ is absolutely integrable over the components of maximal dimension of Σ, and $\int_\Sigma \mu\in S^\bullet\mathfrak{t}^*$,
- if $d_K\mu=0$, then $\int_\Sigma\mu$ depends only on the class of μ in $H^\bullet_{K,\mathrm{cpt}}(W)$, and
- $\int_\Sigma\mu=0$ if $\mu=d_K\nu$ for a compactly supported equivariant form ν.

Definition 5.2 Let Σ be an T–invariant algebraic subvariety of dimension d in the vector space W. Then the equivariant Poincaré dual of Σ is the polynomial on $\mathfrak{t}$ of degree $2r-2d$ defined by the integral

$$\mathrm{eP}[\Sigma,W]=\frac{1}{(2\pi)^d}\int_\Sigma\mathrm{Thom}_K(W). \tag{21}$$

An immediate consequence of this definition is that for an equivariantly closed differential form μ with compact support, we have

$$\int_\Sigma\mu=\int_W\mathrm{eP}[\Sigma,W]\cdot\mu.$$

This formula serves as the motivation for the term *equivariant Poincaré dual.* This definition naturally extends to the case of an analytic subvariety of $\mathbb{C}^n$ defined in the neighbourhood of the origin, or more generally, to any T–invariant cycle in $\mathbb{C}^n$. Note that we do not require for Σ to be smooth, and for singular Σ integration on the right-hand side means integration over the smooth part.

The fibred version of Thom classes of vector spaces are the so-called equivariant Thom classes of vector bundles. We recall the definition and basic properties from Section 2.3 of Duflo and Vergne [12] (see also Mathai and Quillen [30]). Let $\pi\colon E \to M$ be a K–equivariant rank-r complex vector bundle and assume M is compact. Then according to [12, Proposition 16], $H^\bullet_{K,\mathrm{cpt}}(E)$ is a free module over $H^\bullet_K(M)$ generated by a single element of degree $2r$ called the equivariant Thom class of E:

$$H^\bullet_{K,\mathrm{cpt}}(E) = H^\bullet_K(M)\cdot \mathrm{Thom}_K(E). \tag{22}$$

The multiplication map $\alpha \mapsto \pi^*(\alpha)\cdot \mathrm{Thom}_K(E)$ establishes an $H^\bullet_K(M)$–module isomorphism from $H^\bullet_K(M)$ to $H^\bullet_{K,\mathrm{cpt}}(E)$. In particular

$$\int_{E/M} \mathrm{Thom}_K(E) = 1$$

holds for the equivariant push-forward map $\int_{E/M}\colon H^\bullet_K(E)\to H^{\bullet-r}_K(M)$. In fact, there is an equivariantly closed form with compact support on E representing $\mathrm{Thom}_K(E)$. By an abuse of notation let $\mathrm{Thom}_K(E)\in \Omega_{K,\mathrm{cpt}}(E)\subset (S^\bullet\mathfrak{t}^*\otimes \Omega^\bullet_{K,\mathrm{cpt}}(E))^K$ denote this compactly supported form.

Note that for nonsingular Σ the definition (21) can be rewritten using the equivariant normal bundle $\mathcal{N}_\Sigma$ of Σ in W as

$$\mathrm{eP}[\Sigma, W] = \mathrm{Thom}_K(\mathcal{N}_\Sigma)\in S^\bullet\mathfrak{t}^*.$$

More generally, let $Z\subset M$ be a T–invariant complex submanifold of codimension r in the complex manifold M. Let $\mathcal{N}_Z$ denote the normal bundle of Z in M. By the equivariant tubular neighbourhood theorem there exists a K–invariant tubular neighbourhood U of Z in M and a K–invariant diffeomorphism $\gamma\colon Z\to U$ such that $\gamma\circ i_0 = i$, where $i_0\colon Z\hookrightarrow \mathcal{N}_Z$ is the embedding of Z into $\mathcal{N}_Z$ as the zero section. Let $\mathrm{Thom}_K(\mathcal{N}_Z)\in \Omega^\bullet_{K,\mathrm{cpt}}(M)$ denote the extension by zero of the equivariant Thom form of $\mathcal{N}_Z$ to M.

Definition 5.3 Let $Z\subset M$ be a T–invariant complex submanifold of codimension r in the (not necessary compact) complex manifold M. Let $\mathcal{N}_Z$ denote the normal bundle of Z in M. The equivariant Poincaré dual of Z is defined as

$$\mathrm{eP}[Z, M] = \mathrm{Thom}_K(\mathcal{N}_Z)\in \Omega^{2r}_{K,\mathrm{cpt}}(M).$$

Then for any closed (not necessarily compactly supported) form $\mu\in\Omega^\bullet_{K,\mathrm{cpt}}(M)$ we have

$$\int_Z \mu = \int_M \mathrm{eP}[Z, M]\cdot \mu.$$

More generally, for a vector bundle $\pi\colon E \to M$ over the compact variety M and $\mu \in \Omega_K^\bullet(E)$ we have

$$\int_M \mu = \int_E \mathrm{Thom}_K(E)\cdot\mu. \tag{23}$$

The following lemma is a special case of Proposition 2.8 in [7].

Lemma 5.4 *Let $\pi\colon E \to M$ be a complex vector bundle and $s\colon M \to E$ a smooth section generically transversal to the zero section $\iota\colon M \hookrightarrow E$. Then the zero locus $s^{-1}(M) \subset M$ of s defines a cycle and it is Poincaré dual to the K–equivariant Euler class* $\mathrm{Euler}_K(E) = \iota^*\,\mathrm{Thom}_K(E)$ *of E.*

Let W be again a complex N–dimensional vector space. Note that $\mathrm{eP}[\Sigma, W]$ is determined by the maximal dimensional components of Σ and in fact it can be characterised and axiomatised by some of its basic properties. These are carefully stated in Bérczi and Szenes [7, Proposition 2.3] and proofs can be found in Rossmann [34], Vergne [37] and Miller and Sturmfels [31]. The list of these properties is the following: positivity, additivity on maximal dimensional component, deformation invariance, symmetry and finally a formula for complete intersections of hypersurfaces. These properties provide an algorithm for computing $\mathrm{eP}[\Sigma, W]$ as follows (see Miller and Sturmfels [31, Chapter 8.5], Bérczi and Szenes [7] and Bérczi [2] for details). We pick any monomial order on the coordinates of W and apply Gröbner deformation on the ideal of Σ to deform it onto its initial monomial ideal (see Eisenbud [14]). The spectrum of this monomial ideal is the union of some coordinate subspaces in W with multiplicities whose equivariant dual is then given as the sum of the duals of the maximal dimensional subspaces by the additivity property. For these linear subspaces the formula for complete intersections has the following special form. Let $W = \mathrm{Spec}(\mathbb{C}[y_1, \dots, y_N])$ acted on by the n–dimensional torus T diagonally where the weight of y_i is η_i. Then for every subset $\boldsymbol{i} \subset \{1, \dots, N\}$ we have

$$\mathrm{eP}[\{y_i = 0,\ i \in \boldsymbol{i}\}, W] = \prod_{i\in\boldsymbol{i}} \eta_i. \tag{24}$$

The weights $\eta_1, \dots, \eta_N$ are linear forms of the basis elements $\lambda_1, \dots, \lambda_n$ of $\mathfrak{t}^*$. Let $\mathrm{coeff}(\eta_i, j)$ denote the coefficient of λ_j in η_i (for $1 \le i \le N$ and $1 \le j \le n$) and introduce the notation

$$\deg(\eta_1, \dots, \eta_N; m) = \#\{i : \mathrm{coeff}(\eta_i, m) \neq 0\}.$$

Let $\Sigma \subset W$ be a T–invariant subvariety. It is clear from the formula (24) that for any $1 \le m \le n$, the λ_m–degree of $\mathrm{eP}[\Sigma, W]$ satisfies

$$\deg_{\lambda_m} \mathrm{eP}[\Sigma, W] \le \deg(\eta_1, \dots, \eta_N; m). \tag{25}$$

Example 5.5 Let W be $\mathbb{C}^4$ endowed with a $T=(\mathbb{C}^*)^3$–action, whose weights η_1,η_2,η_3 and η_4 span $\mathfrak{t}^*$, and satisfy $\eta_1+\eta_3=\eta_2+\eta_4$. Choose $p=(1,1,1,1)\in W$; then the affine toric variety

$$\overline{T\cdot p}=\{(y_1,y_2,y_3,y_4)\in\mathbb{C}^4 : y_1y_3=y_2y_4\}$$

is a hypersurface and its equivariant dual is given by the weight of the equation

$$\mathrm{eP}[\overline{T\cdot p},W]=\eta_1+\eta_3=\eta_2+\eta_4.$$

Another way to see this is to fix the monomial order $>$ induced by $y_1>y_2>y_3>y_4$; then the ideal $I=(y_1y_3-y_2y_4)$ has initial ideal $\mathrm{in}_I=(y_1y_3)$ whose spectrum is the union of the hyperplanes $\{y_1=0\}$ and $\{y_3=0\}$ with duals η_1, η_3 respectively.

Remark 5.6 An alternative and slightly more general topological definition of the equivariant dual is the following; see the notes of Fulton [18], Kazarian [22] and Edidin and Graham [13] for details. For a Lie group G let $EG\to BG$ be a right principal G–bundle with EG contractible. Such a bundle is universal in the topological setting: if $E\to B$ is any principal G–bundle, then there is a map $B\to BG$, unique up to homotopy, such that E is isomorphic to the pull-back of EG. If X is a smooth algebraic G–variety then the topological definition of the G–equivariant cohomology of X is

$$H_G^*(X)=H^*(EG\times_G X).$$

If Y is a G–invariant subvariety then Y represents a G–equivariant cohomology class in the equivariant cohomology of X, namely the ordinary Poincaré dual of $EG\times_G Y$ in $EG\times_G X$. This is the equivariant dual of Y in X:

$$\mathrm{eP}[Y,X]=\mathrm{PD}(EG\times_G Y, EG\times_G X).$$

5.3 The Rossmann formula

Let Z be a complex manifold with a holomorphic T–action, and let $M\subset Z$ be a T–invariant analytic subvariety with an isolated fixed point $p\in M^T$. Then one can find local analytic coordinates near p, in which the action is linear and diagonal. Using these coordinates, one can identify a neighbourhood of the origin in T_pZ with a neighbourhood of p in Z. We denote by $\widehat{\mathrm{T}}_pM$ the part of T_pZ which corresponds to M under this identification; informally, we will call $\widehat{\mathrm{T}}_pM$ the T–invariant *tangent cone* of M at p. This tangent cone is not quite canonical: it depends on the choice of coordinates; the equivariant dual of $\Sigma=\widehat{\mathrm{T}}_pM$ in $W=\mathrm{T}_pZ$, however, does not. Rossmann named this the *equivariant multiplicity of M in Z at p*:

$$\mathrm{emult}_p[M,Z]\overset{\mathrm{def}}{=}\mathrm{eP}[\widehat{\mathrm{T}}_pM,\mathrm{T}_pZ]. \tag{26}$$

Remark 5.7 In the algebraic framework one might need to pass to the *tangent scheme* of M at p (see Fulton [17]). This is canonically defined, but we will not use this notion.

The analogue of the Atiyah–Bott formula for singular subvarieties of smooth ambient manifolds is the following statement.

Proposition 5.8 (Rossmann's localisation formula [34]) *Let $\mu \in H_T^*(Z)$ be an equivariant class represented by a holomorphic equivariant map $\mathfrak{t} \to \Omega^\bullet(Z)$. Then*

$$\int_M \mu = \sum_{p \in M^T} \frac{\mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(\mathrm{T}_p Z)} \cdot \mu^{[0]}(p), \tag{27}$$

where $\mu^{[0]}(p)$ is the differential-form-degree-0 component of μ evaluated at p.

5.4 Equivariant localisation on $\widetilde{CX}_p^{[k+1]}$ for $k \le n$.

In this subsection we start to develop a two step equivariant localisation method on $\widetilde{CX}_p^{[k+1]}$ using the Rossmann formula. As the partial resolution $\widetilde{CX}_p^{[k+1]}$ described in Section 4.3 is defined only for $k \le n$ we impose this condition in this section.

Recall from the introduction to Section 5 that we fix a holomorphic coordinate system on X near p and using this we identify the fibre $J_k X_p$ with $J_k(1,n)$ and $\mathcal{D}_{X,p}^k$ with $J_k(n,1)^* = \mathrm{Sym}^{\le k}\mathbb{C}^n$. With these identifications the partial resolution map

$$\rho\colon \widetilde{CX}_p^{[k+1]} = \mathrm{GL}(n) \times_{P_{k,n}} \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}} \to \overline{\mathrm{GL}(n) \cdot \mathfrak{p}_{k,n}} = \widehat{CX}_p^{[k+1]}$$

fits into the following diagram:

$$\begin{array}{ccc} \widetilde{CX}_p^{[k+1]} & \xrightarrow{\ \rho\ } & \widehat{CX}_p^{[k+1]} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) \\ \Big\downarrow \mu & & \\ \mathrm{Flag}_k(\mathbb{C}^n) & & \end{array}$$

The fibres of μ are isomorphic to $\overline{P_{k,n} \cdot \mathfrak{p}_{k,n}} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ and μ is $\mathrm{GL}(n)$–equivariant.

Let $e_1, \ldots, e_n \in \mathbb{C}^n$ be elements of an eigenbasis for the $T \subset \mathrm{GL}(n)$–action with weights $\lambda_1, \ldots, \lambda_n \in \mathfrak{t}^*$ and let

$$f = (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_k \rangle \subset \mathbb{C}^n)$$

denote the standard flag in $\mathbb{C}^n$ fixed by the parabolic subgroup $P_{k,n} \subset \mathrm{GL}(n)$. Since the torus action on $\widetilde{CX}_p^{[k+1]}$ is obtained by the restriction of a $\mathrm{GL}(n)$–action to its subgroup of diagonal matrices T_n, the Weyl group of permutation matrices S_n acts transitively on the fixed-point set $\mathrm{Flag}_k(\mathbb{C}^n)^{T_n}$ taking the standard flag $\boldsymbol{f}$ to $\sigma(\boldsymbol{f})$ and for any closed equivariant form $\alpha \in \Omega_T^*(\widetilde{CX}_p^{[k+1]})$ Theorem 5.1 gives us

$$\int_{\widetilde{CX}_p^{[k+1]}} \alpha = \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-k}} \frac{\alpha_{\sigma(\boldsymbol{f})}}{\prod_{1 \le m \le k} \prod_{i=m+1}^{n} (\lambda_{\sigma \cdot i} - \lambda_{\sigma \cdot m})}, \tag{28}$$

where:

- σ runs over the ordered k–element subsets of $\{1,\dots,n\}$ labelling the fixed flags $\sigma(\boldsymbol{f}) = (\langle e_{\sigma(1)}\rangle \subset \cdots \subset \langle e_{\sigma(1)},\dots,e_{\sigma(k)}\rangle \subset \mathbb{C}^n)$ in $\mathbb{C}^n$.
- $\prod_{1 \le m \le k} \prod_{i=m+1}^{n} (\lambda_{\sigma(i)} - \lambda_{\sigma(m)})$ is the equivariant Euler class of the tangent space of $\mathrm{Flag}_k(\mathbb{C}^n)$ at $\sigma(\boldsymbol{f})$. Note that $\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)} \in S^\bullet \mathfrak{t}^*$ can be identified with the Chern roots of the tautological rank-k bundle $\mathcal{E}$ at $\sigma(\boldsymbol{f})$.
- If $\widetilde{CX}_{\sigma(\boldsymbol{f})}^{[k+1]} = \mu^{-1}(\sigma(\boldsymbol{f}))$ denotes the fibre then
$$\alpha_{\sigma(\boldsymbol{f})} = \left(\int_{\widetilde{CX}_{\sigma(\boldsymbol{f})}^{[k+1]}} \alpha \right)^{[0]} (\sigma(\boldsymbol{f})) \in S^\bullet \mathfrak{t}^* \otimes H^*(X)$$
is the differential-form-degree-0 part with coefficients in $H^*(X)$ evaluated at $\sigma(\boldsymbol{f})$ and $\alpha_{\sigma(\boldsymbol{f})} = \sigma \cdot \alpha_{\boldsymbol{f}}$ with respect to the natural Weyl group action on $S^\bullet \mathfrak{t}^*$.

In particular, when $P = P(c_1,\dots,c_{r(k+1)})$ is a polynomial in the Chern classes $c_i = c_i(F^{[k+1]})$ of the tautological rank-$r(k+1)$ bundle on the curvilinear Hilbert scheme then, according to Section 3.2, P is represented by a closed form $\alpha = \alpha(\theta_1,\dots,\theta_r,\eta_1,\dots,\eta_k)$ which is a bisymmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \mathrm{Flag}_k(\mathbb{C}^n)$ and the Chern roots η_j of the tautological rank-k bundle $\mathcal{E}$. Then $\alpha_{\boldsymbol{f}}$ is a polynomial in two sets of variables: the basic weights $\lambda = (\lambda_1,\dots,\lambda_n)$ of T on $\mathbb{C}^n$ and $\theta = (\theta_1,\dots,\theta_r)$. More precisely, the Chern roots of the tautological rank-k bundle $\mathcal{E}$ over $\alpha_{\boldsymbol{f}}$ correspond to the weights $\lambda_1,\dots,\lambda_k$ and therefore

$$\alpha_{\boldsymbol{f}} = \alpha_{\boldsymbol{f}}(\theta_1,\dots,\theta_r,\lambda_1,\dots,\lambda_k) \in S^\bullet \mathfrak{t}^* \otimes H^*(X)$$

is a bisymmetric polynomial of these $r+k$ variables. Then

$$\alpha_{\sigma(\boldsymbol{f})} = \sigma \cdot \alpha_{\boldsymbol{f}} = \alpha_{\boldsymbol{f}}(\theta_1,\dots,\theta_r,\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)}) \in S^\bullet \mathfrak{t}^* \otimes H^*(X) \tag{29}$$

is the σ–shift of the polynomial $\alpha_{\boldsymbol{f}}$ corresponding to the distinguished fixed flag $\boldsymbol{f}$.

5.5 Transforming the localisation formula into iterated residue

In this section we transform the right-hand side of (28) into an iterated residue. This step turns out to be crucial in handling the combinatorial complexity of the Atiyah–Bott localisation formula and captures the symmetry of the fixed-point data in an efficient way which enables us to prove the vanishing of the contribution of all but one of the fixed points.

To describe this formula, we will need the notion of an *iterated residue* (see Szenes [35]) at infinity. Let $\omega_1,\dots,\omega_N$ be affine linear forms on $\mathbb{C}^k$; denoting the coordinates by $z_1,\dots,z_k$, this means that we can write $\omega_i=a_i^0+a_i^1z_1+\dots+a_i^kz_k$. We will use the shorthand $h(\boldsymbol{z})$ for a function $h(z_1,\dots,z_k)$, and $d\boldsymbol{z}$ for the holomorphic n–form $dz_1\wedge\dots\wedge dz_k$. Now, let $h(\boldsymbol{z})$ be an entire function, and define the *iterated residue at infinity* as follows:

$$\underset{z_1=\infty}{\operatorname{Res}}\ \underset{z_2=\infty}{\operatorname{Res}}\cdots\underset{z_k=\infty}{\operatorname{Res}}\frac{h(\boldsymbol{z})\,d\boldsymbol{z}}{\prod_{i=1}^N\omega_i}\overset{\mathrm{def}}{=}\left(\frac{1}{2\pi i}\right)^k\int_{|z_1|=R_1}\cdots\int_{|z_k|=R_k}\frac{h(\boldsymbol{z})\,d\boldsymbol{z}}{\prod_{i=1}^N\omega_i},\tag{30}$$

where $1\ll R_1\ll\dots\ll R_k$. The torus $\{|z_m|=R_m : m=1,\dots,k\}$ is oriented in such a way that $\operatorname{Res}_{z_1=\infty}\cdots\operatorname{Res}_{z_k=\infty}d\boldsymbol{z}/(z_1\cdots z_k)=(-1)^k$. We will also use the simplified notation $\operatorname{Res}_{\boldsymbol{z}=\infty}\overset{\mathrm{def}}{=}\operatorname{Res}_{z_1=\infty}\operatorname{Res}_{z_2=\infty}\cdots\operatorname{Res}_{z_k=\infty}$.

In practice, one way to compute the iterated residue (30) is the following algorithm: for each i, use the expansion

$$\frac{1}{\omega_i}=\sum_{j=0}^{\infty}(-1)^j\frac{(a_i^0+a_i^1z_1+\dots+a_i^{q(i)-1}z_{q(i)-1})^j}{(a_i^{q(i)}z_{q(i)})^{j+1}},\tag{31}$$

where $q(i)$ is the largest value of m for which $a_i^m\neq 0$, then multiply the product of these expressions with $(-1)^kh(z_1,\dots,z_k)$, and then take the coefficient of $z_1^{-1}\cdots z_k^{-1}$ in the resulting Laurent series.

Proposition 5.9 [7, Proposition 5.4] *For any homogeneous polynomial $Q(\boldsymbol{z})$ on $\mathbb{C}^k$ we have*

$$\sum_{\sigma\in\mathcal{S}_n/\mathcal{S}_{n-k}}\frac{Q(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)})}{\prod_{1\le m\le k}\prod_{i=m+1}^n(\lambda_{\sigma\cdot i}-\lambda_{\sigma\cdot m})}=\underset{\boldsymbol{z}=\infty}{\operatorname{Res}}\frac{\prod_{1\le m<l\le k}(z_m-z_l)Q(\boldsymbol{z})\,d\boldsymbol{z}}{\prod_{l=1}^k\prod_{i=1}^n(\lambda_i-z_l)}.\tag{32}$$

Remark 5.10 Changing the order of the variables in iterated residues, usually, changes the result. In this case, however, because all the poles are normal crossing, formula (32) remains true no matter in what order we take the iterated residues.

Proposition 5.9 together with (28) and (29) give us the following proposition.

Proposition 5.11 *Let $k \le n$ and $\alpha = \alpha(\theta_1, \dots, \theta_r, \eta_1, \dots, \eta_k)$ be a bisymmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ and the Chern roots η_j of the tautological rank-k bundle $\mathcal{E}$. Then*

$$\int_{\widetilde{CX}_p^{[k+1]}} \alpha = \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1 \le m < l \le k}(z_m - z_l)\, \alpha_{\boldsymbol{f}}(\theta_1, \dots, \theta_r, z_1, \dots, z_k)\, d\boldsymbol{z}}{\prod_{l=1}^k \prod_{i=1}^n (\lambda_i - z_l)}.$$

Next, we fix the T–eigenbasis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ and proceed a second localisation on the fibre

$$\widetilde{CX}_{\boldsymbol{f}}^{[k+1]} = \mu^{-1}(\boldsymbol{f}) \simeq \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

to compute $\alpha_{\boldsymbol{f}}(\boldsymbol{\theta}, \boldsymbol{z})$. Recall from Corollary 4.1 that for $k \le n$

$$\mathfrak{p}_{k,n} = \Big(\mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2 + e_1^2) \subset \cdots \subset \mathrm{Span}\Big(e_1, e_2 + e_1^2, \dots, \sum_{\tau \in \mathcal{P}(k)} e_\tau\Big)\Big)$$

and $P_{k,n} \subset \mathrm{GL}(n)$ is the parabolic subgroup which preserves the flag

$$\boldsymbol{f} = (\mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2) \subset \cdots \subset \mathrm{Span}(e_1, \dots, e_k) \subset T_p X).$$

Here for the partition $\tau = \{\tau_1 \le \tau_2 \le \cdots \le \tau_s\} \in \mathcal{P}(k)$ we use the notation

(1) k for $\mathrm{sum}(\tau) = \tau_1 + \cdots + \tau_s$,

(2) s for the length $|\tau|$,

(3) e_τ for $e_{\tau_1} e_{\tau_2} \cdots e_{\tau_s} \in \mathrm{Sym}^s(\mathbb{C}^n)$.

We define the subspaces

$$W_i = \mathrm{Span}_{\mathbb{C}}(e_\tau : \mathrm{sum}(\tau) \le i) \subset \mathrm{Sym}^{\le k}\mathbb{C}^n \quad \text{for } 1 \le i \le k.$$

These are invariant under the parabolic subgroup $P_{k,n} \subset \mathrm{GL}(n)$ which fixes the flag $\boldsymbol{f}$. Note that the fibre $\widetilde{CX}_{\boldsymbol{f}}^{[k+1]} = \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}}$ sits in the submanifold

$$\mathrm{Flag}_k^*(\mathrm{Sym}^{\le k}\mathbb{C}^n) = \{V_1 \subset \cdots \subset V_k \subset \mathrm{Sym}^{\le k}\mathbb{C}^n : \dim(V_i) = i,\ V_i \subset W_i\}$$

of $\mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$. Moreover, $\mathrm{Flag}_k^*(\mathrm{Sym}^{\le k}\mathbb{C}^n) \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ is a $P_{k,n}$–invariant subvariety.

As $\widetilde{CX}_{\boldsymbol{f}}^{[k+1]}$ is invariant under the T–action on $\mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$, we can apply Rossmann's integration formula; see Proposition 5.8. More precisely, we apply the Rossmann formula for $M = X_{\boldsymbol{f}}$, $Z = \mathrm{Flag}_k^*(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ and $\mu = \alpha_{\boldsymbol{f}}$. The fixed points on

$$Z = \mathrm{Flag}_k^*(\mathrm{Sym}^{\le k}\mathbb{C}^n) \subset \bigoplus_{i=1}^k W_1 \wedge \cdots \wedge W_i$$

are parametrised by *admissible* sequences of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$. We call a sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ admissible if

(1) $\mathrm{sum}(\pi_l) \le l$ for $1 \le l \le k$, and

(2) $\pi_l \ne \pi_m$ for $1 \le l \ne m \le k$.

We will denote the set of admissible sequences of length k by $\boldsymbol{\Pi}_k$. The corresponding fixed point is then

$$\bigoplus_{i=1}^{k} e_{\pi_1} \wedge \dots \wedge e_{\pi_i} \in \bigoplus_{i=1}^{k} W_1 \wedge \dots \wedge W_i,$$

where $e_\pi = \prod_{j \in \pi} e \in \mathrm{Sym}^{|\pi|}\mathbb{C}^n$.

Then the Rossmann formula (27) and Proposition 5.11 give us the following proposition.

Proposition 5.12 *Let $k \le n$. Let $\alpha = \alpha(\theta_1, \dots, \theta_r, \eta_1, \dots, \eta_k)$ be a bisymmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ and the Chern roots η_j of the tautological rank-k bundle $\mathcal{E}$. Then*

$$\int_{\widetilde{CX}_p^{[k+1]}} \alpha = \sum_{\boldsymbol{\pi} \in \boldsymbol{\Pi}_k \cap \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}}} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{Q_{\boldsymbol{\pi}}(\boldsymbol{z}) \prod_{m<l}(z_m - z_l)\alpha(\boldsymbol{\theta}, z_{\pi_1}, \dots, z_{\pi_k})}{\prod_{l=1}^{k} \prod_{\mathrm{sum}(\tau) \le l}^{\tau \ne \pi_1, \dots, \pi_l} (z_\tau - z_{\pi_l}) \prod_{l=1}^{k} \prod_{i=1}^{n} (\lambda_i - z_l)} \, d\boldsymbol{z}, \tag{33}$$

where

$$Q_{\boldsymbol{\pi}}(\boldsymbol{z}) = \mathrm{emult}_{\boldsymbol{\pi}}[X_{\boldsymbol{f}}, \mathrm{Flag}_k^*] \quad \textit{and} \quad z_\pi = \sum_{i \in \pi} z_i.$$

This formula reduces the computation of the tautological integrals $\int_{\widetilde{CX}_p^{[k+1]}} \alpha$ to determining the fixed-point set $\boldsymbol{\Pi}_k \cap \widetilde{CX}_{\boldsymbol{f}}^{[k+1]}$ and determining the multidegree $Q_{\boldsymbol{\pi}}(\boldsymbol{z}) = \mathrm{emult}_{\boldsymbol{\pi}}[X_{\boldsymbol{f}}, \mathrm{Flag}_k^*]$ of the tangent cone of $\widetilde{CX}_{\boldsymbol{f}}^{[k+1]}$ in $\mathrm{Flag}_k^*(\mathrm{Sym}^{\le k}\mathbb{C}^n)$.

6 The residue vanishing theorem

The first immediate problem arising with our formula (33) is our not having a complete description of the fixed-point set $\boldsymbol{\Pi}_k \cap \widetilde{CX}_{\boldsymbol{f}}^{[k+1]}$, and in fact deciding which torus fixed points on $\mathrm{Flag}_k^*(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ sit in the orbit closure $\widetilde{CX}_{\boldsymbol{f}}^{[k+1]} = \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}}$ seems to be a hard question. The second problem we face is how to compute the multidegrees $Q_{\boldsymbol{\pi}}(\boldsymbol{z}) = \mathrm{emult}_{\boldsymbol{\pi}}[X_{\boldsymbol{f}}, \mathrm{Flag}_k^*]$ for those admissible sequences representing fixed points in $\overline{P_{k,n} \cdot \mathfrak{p}_{k,n}}$. We postpone this second problem to the next section and here we focus on the first question which has a particularly nice and surprising answer. Namely, we do not need to know which fixed points sit in $\overline{P_{k,n}\cdot \mathfrak{p}_{k,n}}$ because our limited knowledge on the equations of the $P_{k,n}$–orbit is enough to show that all but one term on the right-hand side of (33) vanish. This key feature of the iterated residue has already appeared in Bérczi and Szenes [7] but here we need to prove a stronger version where the total degree of the rational forms are zero. We devote the rest of this section to the proof of the following theorem.

Theorem 6.1 (residue vanishing theorem) *Let $k+1\le n$ and let*

$$\alpha=\alpha(\theta_1,\dots,\theta_r,\eta_1,\dots,\eta_k)$$

be a bisymmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]}\subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ and the Chern roots η_j of the tautological rank-k bundle $\mathcal{E}$. Then:

(1) *All terms but the one corresponding to $\boldsymbol{\pi}_{\mathrm{dst}}=([1],[2],\dots,[k])$ vanish in (33) leaving us with*

$$\int_{\widetilde{CX}_p^{[k+1]}}\alpha=\operatorname*{Res}_{\boldsymbol{z}=\infty}\frac{Q_{([1],\dots,[k])}(\boldsymbol{z})\prod_{m<l}(z_m-z_l)\alpha(\boldsymbol{\theta},\boldsymbol{z})}{\prod_{\mathrm{sum}(\tau)\le l\le k}(z_\tau-z_l)\prod_{l=1}^k\prod_{i=1}^n(\lambda_i-z_l)}\,d\boldsymbol{z}. \tag{34}$$

(2) *If $|\tau|\ge 3$ then $Q_{([1],\dots,[k])}(\boldsymbol{z})$ is divisible by $z_\tau-z_l$ for all $l\ge \mathrm{sum}(\tau)$. Let*

$$Q_k(\boldsymbol{z})=\frac{Q_{([1],\dots,[k])}(\boldsymbol{z})}{\prod_{|\tau|\ge 3,\,\mathrm{sum}(\tau)\le l\le k}(z_\tau-z_l)}$$

denote the quotient polynomial, and then we get the simplified formula

$$\int_{\widetilde{CX}_p^{[k+1]}}\alpha=\operatorname*{Res}_{\boldsymbol{z}=\infty}\frac{Q_k(\boldsymbol{z})\prod_{m<l}(z_m-z_l)\alpha(\theta,\boldsymbol{z})}{\prod_{m+r\le l\le k}(z_m+z_r-z_l)\prod_{l=1}^k\prod_{i=1}^n(\lambda_i-z_l)}\,d\boldsymbol{z}. \tag{35}$$

Remark 6.2 (1) Here we describe the geometric meaning of $Q_k(\boldsymbol{z})$ in (35); see also [7, Theorem 6.16]. Let $T_k\subset B_k\subset \mathrm{GL}(k)$ be the subgroups of invertible diagonal and upper triangular matrices, respectively; denote the diagonal weights of T_k by $z_1,\dots,z_k$. Consider the $\mathrm{GL}(k)$–module of 3–tensors $\mathrm{Hom}(\mathbb{C}^k,\mathrm{Sym}^2\mathbb{C}^k)$; identifying the weight-$(z_m+z_r-z_l)$ symbols q_l^{mr} and q_l^{rm}, we can write this space in terms of a basis as follows:

$$\mathrm{Hom}(\mathbb{C}^k,\mathrm{Sym}^2\mathbb{C}^k)=\bigoplus_{1\le m,r,l\le k}\mathbb{C}q_l^{mr}.$$

Consider the point $\epsilon = \sum_{m=1}^{k}\sum_{r=1}^{k-m} q_{mr}^{m+r}$ in the B_k–invariant subspace

$$W_k = \bigoplus_{1\le m+r\le l\le k} \mathbb{C}q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2\mathbb{C}^k).$$

Set the notation $\mathcal{O}_k$ for the orbit closure $\overline{B_k\epsilon} \subset W_k$; then $Q_k(\boldsymbol{z})$ is the T_k–equivariant Poincaré dual $Q_k(\boldsymbol{z}) = \mathrm{eP}[\mathcal{O}_k, W_k]_{T_k}$, which is a homogeneous polynomial of degree $\dim(W_k) - \dim(\mathcal{O}_k)$. For small k these polynomials are (see [7, Section 7])

$$Q_2 = Q_3 = 1, \quad Q_4 = 2z_1 + z_2 - z_4,$$
$$Q_5 = (2z_1 + z_2 - z_5)(2z_1^2 + 3z_1z_2 - 2z_1z_5 + 2z_2z_3 - z_2z_4 - z_2z_5 - z_3z_4 + z_4z_5).$$

(2) To understand the significance of this vanishing theorem we note that while the fixed-point set $\boldsymbol{\Pi}_k$ on $\mathrm{Flag}_k^*(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ is well understood, it is not clear which of these fixed points sit in $X_{\boldsymbol{f}}$. But we have enough information to prove that none of those fixed points in $X_{\boldsymbol{f}}$ contribute to the iterated residue except for the distinguished fixed point $\boldsymbol{\pi}_{\mathrm{dst}} = ([1],[2],\dots,[k])$. This simplification is dramatic: the number of terms in (34) grows exponentially with k, and of this sum now a single term survives.

(3) The residue vanishing theorem is valid under the condition $k+1 \le n$ which is slightly stronger than the condition $k \le n$ we worked with so far and which guaranteed the existence of $\widetilde{CX}_p^{[k+1]}$. We will remedy this condition in Section 6.2.

Remark 6.3 Remark 2.3 for singular varieties and ordinary compactly supported differential forms holds for compactly supported equivariant forms as follows. Let T be a complex torus and $f\colon M \to N$ be a smooth proper T–equivariant map between smooth quasiprojective varieties. Now assume that $X \subset M$ and $Y \subset N$ are possibly singular T–invariant closed subvarieties, such that f restricted to X is a birational map from X to Y. Next, let μ be an equivariantly closed differential form on N with values in polynomials on $\mathfrak{t}$. Then the integral of μ on the smooth part of Y is absolutely convergent; we denote this by $\int_Y \mu$. With this convention we again have

$$\int_X f^*\mu = \int_Y \mu, \tag{36}$$

and we can define integrals of equivariant forms on singular quasiprojective varieties simply by passing to any partial equivariant resolution or equivalently to integration over the smooth locus. In particular, applying this for the partial resolution $\rho\colon \widetilde{CX}_p^{[k+1]} \to \overline{CX}_p^{[k+1]}$ we get

$$\int_{\overline{CX}_p^{[k+1]}} \alpha = \int_{\widetilde{CX}_p^{[k+1]}} \rho^*\alpha$$

for any closed compactly supported differential form $\alpha \in \Omega^*(\overline{CX}_p^{[k+1]})$.

6.1 The vanishing of residues

In this subsection, following Bérczi and Szenes [7, Section 6.2], we describe the conditions under which iterated residues of the type appearing in the sum in (33) vanish and we prove Theorem 6.1.

We start with the 1–dimensional case, where the residue at infinity is defined by (30) with $d=1$. By bounding the integral representation along a contour $|z|=R$ with R large, one can easily prove the following lemma.

Lemma 6.4 *Let $p(z)$ and $q(z)$ be polynomials of one variable. Then*

$$\operatorname*{Res}_{z=\infty} \frac{p(z)\,dz}{q(z)} = 0 \quad \textit{if} \;\; \deg(p(z))+1<\deg(q).$$

Consider now the multidimensional situation. Let $p(\boldsymbol{z})$ and $q(\boldsymbol{z})$ be polynomials in the k variables $z_1,\dots,z_k$, and assume that $q(\boldsymbol{z})$ is the product of linear factors $q=\prod_{i=1}^N L_i$, as in (33). We continue to use the notation $d\boldsymbol{z}=dz_1\cdots dz_k$. We would like to formulate conditions under which the iterated residue

$$\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\cdots\operatorname*{Res}_{z_k=\infty}\frac{p(\boldsymbol{z})\,d\boldsymbol{z}}{q(\boldsymbol{z})} \tag{37}$$

vanishes. Introduce the following notation:

- When $p(\boldsymbol{z})$ is the product of linear forms and $1\le m\le k$ let $\deg(p(\boldsymbol{z});m)$ denote the number of terms in $p(\boldsymbol{z})$ with nonzero coefficients in front of z_m.
- For a nonzero linear form $L=a_0+a_1z_1+\dots+a_kz_k$, denote by $\operatorname{coeff}(L,z_l)=a_i$ the coefficient in front of z_i.
- Finally, for $1\le m\le k$, set

  $$\operatorname{lead}(q(\boldsymbol{z});m)=\#\{i:\max\{l:\operatorname{coeff}(L_i,z_l)\neq 0\}=m\},$$

  which is the number of those factors L_i in which the coefficient of z_m does not vanish, but the coefficients of $z_{m+1},\dots,z_k$ are 0.

We can group the N linear factors of $q(\boldsymbol{z})$ according to the nonvanishing coefficient with the largest index; in particular, for $1\le m\le k$ we have

$$\deg(q(\boldsymbol{z});m)\ge\operatorname{lead}(q(\boldsymbol{z});m)\quad\text{and}\quad\sum_{m=1}^k\operatorname{lead}(q(\boldsymbol{z});m)=N.$$

Proposition 6.5 [7, Proposition 6.3] *Let $p(z)$ and $q(z)$ be polynomials in the variables $z_1,\dots,z_k$, and assume that $q(z)$ is a product of linear factors: $q(z)=\prod_{i=1}^N L_i$; set $dz=dz_1\cdots dz_k$. Then*

$$\operatorname*{Res}_{z_1=\infty}\operatorname*{Res}_{z_2=\infty}\cdots\operatorname*{Res}_{z_k=\infty}\frac{p(z)\,dz}{q(z)}=0$$

if for some $l\le k$, the following holds:

$$\deg(p(z);l)+1<\deg(q(z);l)=\operatorname{lead}(q(z);l).$$

Note that the equality $\deg(q(z);l)=\operatorname{lead}(q(z);l)$ means that

(38) for each $i=1,\dots,N$ and $m>l$, $\operatorname{coeff}(L_i,z_l)\neq 0$ implies $\operatorname{coeff}(L_i,z_m)=0$.

We are ready to prove the residue vanishing theorem. Recall that our goal is to show that all the terms of the sum in (33) vanish except for the one corresponding to $\boldsymbol{\pi}_{\mathrm{dst}}=([1],\dots,[k])$. The plan is to apply Proposition 6.5 in stages to show that the iterated residue vanishes unless $z_i=[i]$ holds, starting with $i=k$ and going backwards.

Fix a sequence $\boldsymbol{\pi}=(\pi_1,\dots,\pi_k)\in\boldsymbol{\Pi}_k$, and consider the iterated residue corresponding to it on the right-hand side of (33). The expression under the residue is the product of two fractions:

$$\frac{p(z)}{q(z)}=\frac{p_1(z)}{q_1(z)}\cdot\frac{p_2(z)}{q_2(z)},$$

where

$$\text{(39)}\quad \frac{p_1(z)}{q_1(z)}=\frac{Q_{\boldsymbol{\pi}}(z)\prod_{m<l}(z_m-z_l)}{\prod_{l=1}^k\prod_{\mathrm{sum}(\tau)\le l}^{\tau\neq\pi_1,\dots,\pi_l}(z_\tau-z_{\pi_l})}\quad\text{and}\quad\frac{p_2(z)}{q_2(z)}=\frac{\alpha(\theta_1,\dots,\theta_r,z_{\pi_1},\dots,z_{\pi_k})}{\prod_{l=1}^k\prod_{i=1}^n(\lambda_i-z_l)}.$$

Note that $p(z)$ is a polynomial, while $q(z)$ is a product of linear forms. As a first step we show that if $\pi_k\neq[k]$, then already the first residue in the corresponding term on the right-hand side of (33) — the one with respect to z_k — vanishes. Indeed, if $\pi_k\neq[k]$, then $\deg(q_2(z);k)=n$, while z_k does not appear in $p_2(z)$. On the other hand, $\deg(q_1(z);k)=1$, because the only term which contains z_k is the one corresponding to $l=k$ and $\tau=[k]\neq\pi_k$. This also means that the only coordinate on $T_\pi\mathrm{Flag}_k^*$ which contains the z_k coordinate of the torus is $z_k-z_{\pi_k}$, and since $Q_{\boldsymbol{\pi}}(z)=\mathrm{emult}_{\boldsymbol{\pi}}[X_{\boldsymbol{f}},\mathrm{Flag}_k^*]$, (25) tells us that $\deg(Q_{\boldsymbol{\pi}}(z);k)\le 1$ holds. Collecting this data gives

$$\text{(40)}\qquad \deg(p_1(z)p_2(z);k)=k\quad\text{and}\quad\deg(q_1(z)q_2(z);k)=n+1,$$

and $k\le n-1$, so $\deg(p(z))\le\deg(q(z))-2$ holds and we can apply Lemma 6.4.

We can thus assume that $\pi_k = [k]$, and proceed to the next step and take the residue with respect to z_{k-1}. If $\pi_{k-1} \neq [k-1]$ then

$$\deg(q_2(z); k-1) = \operatorname{lead}(q_2(z); k-1) = n \quad \text{and} \quad \deg(p_2(z); k-1) = 0. \tag{41}$$

In $q_1(z)$ the linear terms containing z_{k-1} are

$$z_{k-1} - z_k, \quad z_1 + z_{k-1} - z_k \quad \text{and} \quad z_{k-1} - z_{\pi_{k-1}}. \tag{42}$$

The first term here cancels with the identical term in the Vandermonde in p_1. The second term divides $Q_{\boldsymbol{\pi}}$, according to the following proposition from [7] applied for $l = k-1$.

Proposition 6.6 [7, Proposition 6.4] *Let $l \geq 1$, and let $\boldsymbol{\pi}$ be an admissible sequence of partitions of the form $\boldsymbol{\pi} = (\pi_1, \dots, \pi_l, [l+1], \dots, [k])$, where $\pi_l \neq [l]$. Then for $m > l$, and every partition τ such that $l \in \tau$, $\operatorname{sum}(\tau) \leq m$, and $|\tau| > 1$, we have*

$$(z_\tau - z_m) \mid Q_{\boldsymbol{\pi}}. \tag{43}$$

Therefore, after cancellation, all linear factors from $q_1(z)$ which have nonzero coefficients in front of both z_{k-1} and z_k vanish, and for the new fraction $p_1'(z)/q_1'(z)$,

$$\deg(q_1'(z); k-1) = \operatorname{lead}(q_1'(z); k-1) = 1.$$

By (42) and (25), $\deg(Q_\pi; k-1) \leq 3$ and therefore after cancellation we have

$$\deg(p_1'(z); k-1) \leq k-2+2 = k.$$

Using (41) we get

$$\deg(p_1'(z)p_2(z); k-1) \leq k$$

and

$$\deg(q_1'(z)q_2(z); k-1) = \operatorname{lead}(q_1'(z)q_2(z); k-1) = n+1,$$

so we can apply Proposition 6.5 with $l = k-1$ to deduce the vanishing of the residue with respect to $k-1$.

In general, assume that

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_l, [l+1], \dots, [k]) \quad \text{where } \pi_l \neq [l]$$

and proceed to the study of the residue with respect to z_l. For the second fraction we have again

$$\deg(q_2(z); l) = \operatorname{lead}(q_2(z); l) = n \quad \text{and} \quad \deg(p_2(z); l) = 0. \tag{44}$$

The linear terms containing z_l in $q_1(z)$ are

$$z_l - z_k, \quad z_l - z_{k-1}, \dots, \quad z_l - z_{l+1}, \tag{45}$$

$$z_\tau - z_s \qquad \text{with } l \in \tau, \quad \tau \neq l, \quad l+1 \le s \le k \quad \text{and} \quad \operatorname{sum}(\tau) \le s, \tag{46}$$

$$z_l - z_{\pi_l}. \tag{47}$$

The weights in (45) cancel out with the identical terms of the Vandermonde in $p_1(z)$, and by Proposition 6.6, $Q_{\boldsymbol{\pi}}(z)$ is divisible by the weights in (46). Hence all linear factors with nonzero coefficient in front of z_l and at least one of $z_{l+1}, \dots, z_k$ vanish from $q_1(z)$. Let again $p_1'(z)/q_1'(z)$ denote the new fraction arising from $p_1(z)/q_1(z)$ after these cancellations. Then in $q_1'(z)$ only the term (47) contains z_l, and

$$\deg(q_1'(z), l) = \operatorname{lead}(q_1'(z), l) = 1. \tag{48}$$

In $p_1'(z)$ the linear terms which are left from the Vandermonde after cancellation and contain z_l are $z_{l-1} - z_l, \dots, z_1 - z_l$. The reduced $Q'_\pi(z)$ which we get after dividing by the terms in (46) is then a polynomial of the remaining weights, and the only remaining weights which contain z_l are

$$z_l - z_{\pi_l} \qquad \text{and} \qquad z_l - z_k, \quad z_l - z_{k-1}, \dots, \quad z_l - z_{l+1}.$$

This is because Q_π is the sum of monomials of the form $\Pi_{\omega \in I}\omega$ where

$$I \subset \{z_\tau - z_{\pi_l} : \operatorname{sum}(\tau) \le l, \ \tau \neq \pi_1, \dots, \pi_l, \ 1 \le l \le k\}$$

is a subset of the weights in $q_1(z)$ and therefore Q_π does not contain repeated weights. Then (25) tells us that $\deg(Q_\pi(z); l) \le k - l + 1$. Therefore

$$\deg(p_1'(z); l) \le (l-1) + (k-l+1) = k. \tag{49}$$

Putting (48) and (49) together we get

$$\deg(p_1'(z)p_2(z), l) \le k \quad \text{and} \quad \deg(q_1'(z)q_2(z), l) = \operatorname{lead}(q_1'(z)q_2(z), l) = n+1.$$

Since $k \le n-1$, by applying Proposition 6.5 we arrive at the vanishing of the residue, forcing π_l to be $[l]$. This proves (1) of Theorem 6.1

The second part of the residue vanishing theorem is proved in Section 6.5 of [7] where we show that for $|\tau| > 1$,

$$(z_\tau - z_l) \mid Q_{([1],\dots,[k])} \quad \text{if } ([1],[2],\dots,[l-1],\tau,[l+1],\dots,[k-1],[k]) \text{ is not complete.}$$

We call an admissible sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ *complete* if for every $l \in \{1, \dots, k\}$ and every nontrivial subpartition $\tau \subset \pi_l$, there is an $m \in \{1, \dots, k\}$ such that $\pi_m = \tau$. Clearly, a sequence $([1],[2],\dots,[l-1],\tau,[l+1],\dots,[k-1],[k])$ is complete if and only if $|\tau| = 2$.

6.2 Increasing the number of points

The residue vanishing theorem provides a closed iterated residue formula for tautological integrals on $\widetilde{CX}_p^{[k+1]}$ in the case when $k+1 \le n$, that is, the number of points does not exceed the dimension of X. In this section we show how one can drop this very restrictive condition.

Recall that after fixing local coordinates on X near p, the test-curve model in Section 3.1 establishes a $\mathrm{GL}(n)$–equivariant isomorphism of quasiprojective varieties

$$J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1) \simeq CX_p^{[k+1]} \subset \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$$

between the moduli of k–jets of regular germs and the curvilinear locus of the punctual Hilbert scheme sitting in the Grassmannian of k–dimensional subspaces in $\mathrm{Sym}^{\le k}\mathbb{C}^n$.

Assume that $k+1 > \dim(X) = n$. Fix a basis $\{e_1,\dots,e_{k+1}\}$ of $\mathbb{C}^{k+1}$ and let

$$\mathbb{C}_{[n]} = \mathrm{Span}(e_1,\dots,e_n) \hookrightarrow \mathbb{C}^{k+1} \quad \text{and} \quad \mathbb{C}_{[k+1-n]} = \mathrm{Span}(e_{n+1},\dots,e_{k+1}) \hookrightarrow \mathbb{C}^{k+1}$$

denote the subspaces spanned by the first n and last $k+1-n$ basis vectors respectively. These are T_{k+1}–equivariant embeddings under the diagonal action of the maximal torus $T_{k+1} \subset \mathrm{GL}(k+1)$.

We can write $J_k^{\mathrm{reg}}(1,n) \subset J_k^{\mathrm{reg}}(1,k+1)$ as a zero locus of a smooth section of a vector bundle over $J_k^{\mathrm{reg}}(1,k+1)$. Indeed, consider the projection

$$J_k^{\mathrm{reg}}(1,k+1) \to J_k(1,k+1-n) \simeq \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]})$$

which sends a regular k–jet $f\colon \mathbb{C} \to \mathbb{C}^{k+1}$ to the composition $\mathbb{C} \to \mathbb{C}^{k+1} \to \mathbb{C}^{k+1-n}/\mathbb{C}_{[n]} = \mathbb{C}_{[k+1-n]}$. This map is $T_{k+1} \times J_k^{\mathrm{reg}}(1,1)$–equivariant and therefore it defines a T_{k+1}–equivariant section π of the bundle

$$E = J_k^{\mathrm{reg}}(1,k+1) \times_{J_k^{\mathrm{reg}}(1,1)} J_k(1,k+1-n).$$

This is a bundle over the quasiprojective base space $J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1)$ with fibres isomorphic to $J_k(1,k+1-n)$ and the zero locus of the section π is

$$\pi^{-1}(0) = J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1) \subset J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1).$$

Lemma 5.4 then suggests that for any T_{k+1}–equivariantly closed form on the quasiprojective quotient μ on $J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1)$ we have

$$\int_{J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1)} \mu = \int_{J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1)} \mu \cdot \mathrm{Euler}^{T_{k+1}}(E),$$

but our base space $J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1)$ is quasiprojective and not compact so

Lemma 5.4 does not apply directly. Note, however, that E extends T_{k+1}–equivariantly over the closure

$$\overline{J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1)} \subset \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^{k+1})$$

in the Grassmannian, namely E is the restriction of the bundle

$$\widetilde{E} = \mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathrm{Sym}^{\le k}\mathbb{C}^{k+1}) \times_{\mathrm{GL}(k)} \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]})$$

over the Grassmannian $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathrm{Sym}^{\le k}\mathbb{C}^{k+1})/\mathrm{GL}(k) = \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^{k+1})$. That is, we have a T_{k+1}–equivariant embedding

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[d, "\pi"] & & \widetilde{E} \arrow[d] \\ J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1) \arrow[rr, hook, "\phi^{\mathrm{Grass}}"] & & \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^{k+1}) \end{tikzcd}$$

of E into $\widetilde{E}$ and Lemma 5.4 gives us the following:

Proposition 6.7 *Let $\bar{E} = (\phi^{\mathrm{Grass}})^*\widetilde{E}$ denote the restriction of $\widetilde{E}$ to the closure $\overline{J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1)}$. Then for any T_{k+1}–equivariantly closed form μ on this closure we have*

$$\int_{\overline{J_k^{\mathrm{reg}}(1,n)/J_k^{\mathrm{reg}}(1,1)}} \mu = \int_{\overline{J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1)}} \mu \cdot \mathrm{Euler}^{T_{k+1}}(\bar{E}).$$

We are ready to prove the iterated residue formula on the domain $n \le k+1$.

Theorem 6.8 (extended residue vanishing theorem) *Formula (35) remains valid for any $2 \le n < k+1$.*

Proof The embedding $\phi^{\mathrm{Flag}}\colon J_k^{\mathrm{reg}}(1,k+1)/J_k^{\mathrm{reg}}(1,1) \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^{k+1})$ is T_{k+1}–equivariant, and over the flag $\boldsymbol{f}_\sigma$ the weight of $f_j^{[i]}$ is $\lambda_j - \lambda_{\sigma(i)}$. In the iterated residue formula of Proposition 5.11 we substitute $\lambda_{\sigma(i)}$ for z_i over $\boldsymbol{f}_\sigma$ and therefore $\lambda_j - z_i$ for this weight and therefore the T_{k+1}–equivariant Euler class transforms into

$$\mathrm{Euler}_{\boldsymbol{z}}^{T_{k+1}}(\bar{E}) = \prod_{i=1}^{k}\prod_{j=n+1}^{k}(\lambda_j - z_i)$$

over the flag $\boldsymbol{f}_\sigma$ corresponding to an iterated pole $\boldsymbol{z} = (z_1,\dots z_k)$. If

$$\alpha = \alpha(\theta_1,\dots,\theta_r,\eta_1,\dots,\eta_k)$$

is a bisymmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ and the Chern roots η_j of the tautological rank-k bundle $\mathcal{E}$, then the trivial extension of α to $\mathrm{Flag}_k(\mathrm{Sym}^{\le k}\mathbb{C}^{k+1})$ is closed and therefore Remark 6.3, Proposition 6.7 and Theorem 6.1 tell us

$$\int_{\overline{CX}_p^{[n]}}\alpha=\operatorname*{Res}_{z=\infty}\frac{Q_k(z)\prod_{m<l}(z_m-z_l)\alpha(\theta,z)\,dz}{\prod_{m+r\le l\le k}(z_m+z_r-z_l)\prod_{l=1}^k\prod_{i=1}^k(\lambda_i-z_l)}\cdot\prod_{l=1}^k\prod_{i=n+1}^k(\lambda_i-z_l)$$
$$=\operatorname*{Res}_{z=\infty}\frac{Q_k(z)\prod_{m<l}(z_m-z_l)\alpha(\theta,z)}{\prod_{m+r\le l\le k}(z_m+z_r-z_l)\prod_{l=1}^k\prod_{i=1}^n(\lambda_i-z_l)}\,dz.\qquad\square$$

7 Proof of Theorem 1.2 and examples

Let $P=P(c_1,\dots,c_{r(k+1)})$ be a Chern polynomial of degree $\dim\overline{CX}^{[k+1]}$, which equals $n+(n-1)k$, where the $c_i=c_i(F^{[k+1]})$ are the Chern classes of the tautological rank-$r(k+1)$ bundle on the curvilinear Hilbert scheme. To evaluate the integral $\int_{\overline{CX}^{[k+1]}}P$ we can first integrate (push forward) along the fibres of $\pi\colon\overline{CX}^{[k+1]}\to X$ followed by integration over X. By fixing local holomorphic coordinates on X near p these fibres are canonically isomorphic to $\overline{CX}_p^{[k+1]}\subset\mathrm{Grass}_k(\mathrm{Sym}^{\le k}(\mathbb{C}^n)$ endowed with a natural $\mathrm{GL}(n)$–action induced by the standard $\mathrm{GL}(n)$–action on $\mathbb{C}^n$. We can use this action to perform torus equivariant localisation on $\overline{CX}_p^{[k+1]}$ to integrate along the fibres. According to Remark 6.3, $\int_{\overline{CX}_p^{[k+1]}}P=\int_{\widetilde{CX}_p^{[k+1]}}\rho^*P$ holds, where $\rho\colon\widetilde{CX}_p^{[k+1]}\to\overline{CX}_p^{[k+1]}$ is the partial resolution constructed in Section 4.3. Applying Theorem 6.1 and its extension Theorem 6.8 and the expression in (14) for the Chern classes of $F^{[k+1]}$ we get

$$\int_{\overline{CX}_p^{[k+1]}}P=\operatorname*{Res}_{z=\infty}\frac{Q_k(z)\prod_{m<l}(z_m-z_l)P(c_l(z+\theta,\theta))}{\prod_{m+r\le l\le k}(z_m+z_r-z_l)\prod_{l=1}^k\prod_{i=1}^n(\lambda_i-z_l)}\,dz,\tag{50}$$

where $\theta_1,\dots,\theta_r$ are the Chern roots of F and $c_l(z+\theta,\theta)$ denotes the l^{th} symmetric polynomial in the formal Chern roots $\{z_i+\theta_j,\theta_j:1\le i\le k,1\le j\le r\}$.

According to Corollary 3.14 $\overline{CX}_p^{[k+1]}=J_k^{\mathrm{reg}}X_p/\mathrm{Diff}_k(1)$, which sits in $\mathrm{Grass}_k(\mathcal{D}_{X,p}^k)$. By choosing local coordinates on X near p we identify $\mathcal{D}_{X,p}^k$ with $\mathrm{Sym}^{\le k}\mathbb{C}^n$ and the weights $\lambda_1,\dots,\lambda_n$ of the $\mathrm{GL}(n)$–action on $\mathbb{C}^n$ intuitively correspond to the Chern roots of $\mathcal{D}_{X,p}^{\le 1}/\mathcal{D}_{X,p}^{\le 0}=T_pX$. To finish the proof of Theorem 1.2 we simply substitute the λ_i with the Chern roots of T_X. Indeed, this is what we have to do, but this intuitive step needs further explanation.

The crucial observation is that $\mathrm{GL}(n)$ is a (strong) deformation retract of Diff_k via the homotopy

$$\mathrm{Diff}_k\times[0,1]\to\mathrm{Diff}_k$$

which sends (ϕ, t) to the ϕ_t whose linear part is identical to the linear part of ϕ but whose quadratic and higher order terms are those of ϕ multiplied by t. This homotopy contracts the quadratic and higher order terms of ϕ to zero. This induces a retraction of the classifying spaces

$$\tau\colon B\mathrm{Diff}_k \to B\mathrm{GL}(n)$$

which is a homotopy equivalence.

Given a Diff_k–module V the embedding $\mathrm{GL}(n) \hookrightarrow \mathrm{Diff}_k$ also defines a $\mathrm{GL}(n)$–module structure on V and the corresponding universal bundles

$$E_{\mathrm{Diff}}\, V = E\mathrm{Diff}_k \times_{\mathrm{Diff}_k} V \quad \text{and} \quad E_{\mathrm{GL}(n)}\, V = E\mathrm{GL}(n) \times_{\mathrm{GL}(n)} V$$

are homotopy equivalent. In particular,

$$E_{\mathrm{Diff}}\, \overline{CX}_p^{[k+1]} = E\mathrm{Diff}_k \times_{\mathrm{Diff}_k} \overline{CX}_p^{[k+1]}$$

and

$$E_{\mathrm{GL}}\, \overline{CX}_p^{[k+1]} = E\mathrm{GL}(n) \times_{\mathrm{GL}(n)} \overline{CX}_p^{[k+1]}$$

are homotopy equivalent and therefore their pull-backs along the classifying map $\xi\colon X \to B\mathrm{Diff}_k$,

$$\overline{CX}^{[k+1]} = \xi^*\, E_{\mathrm{Diff}}\, \overline{CX}_p^{[k+1]} \quad \text{and} \quad \overline{CX}_{\mathrm{GL}}^{[k+1]} = (\tau \circ \xi)^*\, E_{\mathrm{GL}}\, \overline{CX}_p^{[k+1]},$$

are also homotopy equivalent. They sit in the corresponding Grassmannian bundles:

$$\begin{aligned}
\overline{CX}^{[k+1]} &\subset \xi^*\, E_{\mathrm{Diff}}\, \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) = \mathrm{Grass}_k(\mathcal{D}_X^k),\\
\overline{CX}_{\mathrm{GL}}^{[k+1]} &\subset (\tau \circ \xi)^*\, E_{\mathrm{GL}}\, \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n) = \mathrm{Grass}_k(\mathrm{Sym}^{\le k} T_X).
\end{aligned}$$

If α is a polynomial in the Chern classes of the tautological rank-k bundle $\mathcal{E}$ over $E_{\mathrm{GL}}\, \mathrm{Grass}_k(\mathrm{Sym}^{\le k}\mathbb{C}^n)$ then

$$\int_{\overline{CX}_{\mathrm{GL}}^{[k+1]}} (\tau \circ \xi)^* \alpha = \int_{\overline{CX}^{[k+1]}} \xi^* \alpha$$

holds, and therefore we can replace integration over $\overline{CX}^{[k+1]}$ with integration over $\overline{CX}_{\mathrm{GL}}^{[k+1]}$. The commutative diagram

$$\begin{array}{ccc}
\overline{CX}_{\mathrm{GL}}^{[k+1]} & \longrightarrow & E_{\mathrm{GL}}\, \overline{CX}_p^{[k+1]} \\
\big\downarrow & & \big\downarrow \\
X & \xrightarrow{\ \tau\circ\xi\ } & B\mathrm{GL}(n)
\end{array}$$

induces a diagram of cohomology maps

$$\begin{array}{ccc} H^*(\overline{CX}^{[k+1]}_{\mathrm{GL}}) & \longleftarrow & H^*(E_{\mathrm{GL}}\,\overline{CX}^{[k+1]}_p) \\ \Big\downarrow {\scriptstyle \int_p} & & \Big\downarrow {\scriptstyle \mathrm{Res}} \\ H^*(X) & \xleftarrow[\mathrm{Sub}]{} & H^*(B\mathrm{GL}(n)) \end{array}$$

where:

- Res is the residue operator sending a polynomial P in the Chern classes of $\mathcal{E}$ to
$$\operatorname*{Res}_{z=\infty} \frac{Q_k(z)\prod_{m<l}(z_m - z_l)P(c_l(z))}{\prod_{m+r\le l\le k}(z_m+z_r-z_l)\prod_{l=1}^k\prod_{i=1}^n(\lambda_i - z_l)}\,dz.$$
- Sub is the substitution of the Chern roots of X into the weights $\lambda_1,\dots,\lambda_n$.
- $\int_p$ is integration along the fibre.

Commutativity tells us that integration along the fibre $\overline{CX}^{[k+1]}_p$ of a class pulled back from the universal bundle $\mathcal{E}$ over $E_{\mathrm{GL}}\,\overline{CX}^{[k+1]}_p$ is given by applying the residue operation followed by the substitution of the Chern roots of X into the weights λ_i of the torus action.

To get the final version of the iterated residue formula we replace the variables z_i by $-z_i$ for $i=1,\dots,k$. This changes the sign of the iterate residue (50) with $(-1)^k$ as this substitution corresponds to changing the orientation of the contour circles. Then the terms involving the λ_i in (50) can be rewritten as

$$\frac{1}{\prod_{i=1}^n(\lambda_i+z_j)} = \frac{1}{z_j^n c(1/z_j)} = \frac{s_X(1/z_j)}{z_j^n},$$

where $s_X(1/z_i) = 1 + s_1(X)/z_i + s_2(X)/z_i^2 + \cdots + s_n(X)/z_i^n$ is the total Segre class of X. Next observe that the denominator and the numerator of the fraction

$$\frac{\prod_{i<j}(z_i - z_j)Q_k(z)}{\prod_{i+j\le l\le k}(z_i+z_j-z_l)}$$

are homogeneous polynomials of the same degree; hence this substitution will leave this rational expression unchanged and only replaces z_i by $-z_i$ in $P(c_l(z+\theta,\theta))$. So (50) can be rewritten as

$$\int_{\overline{CX}^{[k+1]}} P = \int_X \operatorname*{Res}_{z=\infty} \frac{(-1)^k\prod_{1\le i<j\le k}(z_i-z_j)Q_k(z)P(c_l(\theta - z,\theta))\,dz}{\prod_{i+j\le l\le k}(z_i+z_j-z_l)(z_1\cdots z_k)^n}\prod_{i=1}^k s_X\Big(\frac{1}{z_i}\Big),$$

and Theorem 1.2 is proved.

Remark 7.1 (1) Note that if we give the z_i and θ_j degree 1 then the total degree of the rational expression

$$\frac{(-1)^k \prod_{i<j}(z_i-z_j)Q_k(z)P(c_l(\theta-\mathbf{z},\theta))}{\prod_{i+j\le l\le k}(z_i+z_j-z_l)(z_1\cdots z_k)^n}$$

in the formula is $n-k$, so taking the iterated residue indeed gives us a bisymmetric homogeneous polynomial of degree n in the θ_j and s_i.

(2) The Chern class $c_l(\theta-\mathbf{z},\theta)$ is the coefficient of t^l in

$$c(F^{[k+1]})(t)=\prod_{j=1}^{r}(1+\theta_j t)\prod_{i=1}^{k}\prod_{j=1}^{r}(1-z_i t+\theta_j t),$$

that is, the i^{th} Chern class of the bundle with formal Chern roots θ_j, θ_j-z_i. For example,

$$c_1(\theta-\mathbf{z},\theta)=(k+1)\sum_{j=1}^{r}\theta_j-r\sum_{i=1}^{k}z_i,$$

and in general $c_l(\theta-\mathbf{z},\theta)$ is a degree-l polynomial of the form

$$c_l(\theta-\mathbf{z},\theta)=A_l c_l(\mathbf{z})+A_{l-1}c_{l-1}(\mathbf{z})+\cdots+A_0,$$

where $c_j(\mathbf{z})$ is the j^{th} elementary symmetric polynomial in $z_1,\dots,z_k$ and A_j is a degree-$(n-j)$ symmetric polynomial in $\theta_1,\dots,\theta_r$.

In certain special cases, however, we do not need this expansions of the Chern classes. We finish this paper with showing a particularly nice example of this, the Segre classes of tautological bundles over the curvilinear Hilbert schemes.

Example 7.2 (top Segre classes of tautological bundles) Top Segre classes

$$s_{\text{top}}(F^{[k+1]})=\int_{\overline{CX}^{[k+1]}} s(F^{[k+1]})$$

of tautological bundles have been long studied and they are of special interest because of their role in Donaldson–Thomas theory of counting sheaves on surfaces; see Marian, Oprea and Pandharipande [29] for details. Here $s(F^{[k+1]})=1/c(F^{[k+1]})$ is the total Segre class of $F^{[k+1]}$, that is,

$$s(F^{[k+1]})=s(\theta-\mathbf{z},\theta)=\prod_{j=1}^{r}\frac{1}{1+\theta_j}\cdot\prod_{i=1}^{k}\prod_{j=1}^{r}\frac{1}{1+\theta_j-z_i}=s_F\cdot(z_1\cdots z_k)^{-r}\prod_{i=1}^{k}\mathcal{S}\Big(\frac{1}{z_i}\Big),$$

where s_F is the total Segre class of F and

$$\mathcal{S}\Big(\frac{1}{z_i}\Big) = -\prod_{j=1}^{r}\Big(1+\frac{1+\theta_j}{z_i}+\frac{(1+\theta_j)^2}{z_i^2}+\cdots+\frac{(1+\theta_j)^n}{z_i^n}\Big)$$

is a polynomial in $1/z_i$ with coefficients polynomials in the Chern classes of F, that is, $\mathcal{S}(x) \in \mathbb{C}[c_1(F),\dots,c_r(F)][x]$.

Substituting into Theorem 1.2 we arrive at the following expression:

$$s_{\mathrm{top}}(F^{[k+1]}) = \int_X \operatorname*{Res}_{z=\infty} \frac{(-1)^k \prod_{1\le i<j\le k}(z_i-z_j)\,Q_k(\boldsymbol{z})\,s_F\,d\boldsymbol{z}}{\prod_{i+j\le l\le k}(z_i+z_j-z_l)(z_1\cdots z_k)^{r+n}} \prod_{i=1}^{k} \mathcal{S}\Big(\frac{1}{z_i}\Big)s_X\Big(\frac{1}{z_i}\Big).$$

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