A higher chromatic analogue of the image of J

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We prove a higher chromatic analogue of Snaith's theorem which identifies the K–theory spectrum as the localisation of the suspension spectrum of $\mathbb{CP}^\infty$ away from the Bott class; in this result, higher Eilenberg–MacLane spaces play the role of $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$. Using this, we obtain a partial computation of the part of the Picard-graded homotopy of the $K(n)$–local sphere indexed by powers of a spectrum which for large primes is a shift of the Gross–Hopkins dual of the sphere. Our main technical tool is a $K(n)$–local notion generalising complex orientation to higher Eilenberg–MacLane spaces. As for complex-oriented theories, such an orientation produces a one-dimensional formal group law as an invariant of the cohomology theory. As an application, we prove a theorem that gives evidence for the chromatic redshift conjecture.

1 Introduction

The J–homomorphism $J\colon \pi_j(O) \to \pi_j^S(S^0)$ gives a map from the homotopy groups of the infinite orthogonal group $O = \varinjlim O(m)$ to the stable homotopy groups of S^0. It is induced in homotopy by the limit over m of maps

$$J_m\colon O(m) \to \Omega^m S^m,$$

where for a matrix $M \in O(m)$ regarded as a linear transformation $M\colon \mathbb{R}^m \to \mathbb{R}^m$, $J_m(M) = M \cup \{\infty\}\colon S^m \to S^m$. There is an analogous function from the infinite unitary group U; namely, $J_U\colon U \to QS^0$ is given by composition with the forgetful map $U \to O$. The homotopy groups of the domains are computable via Bott periodicity, and are:

$j \bmod 8$	0	1	2	3	4	5	6	7
$\pi_j(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_j(U)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

The work of Adams [2] shows that for an odd prime p, in dimensions 3 mod 4, the p–torsion of the image in $\pi_j^S(S^0)$ of the cyclic group $\pi_j(O)$ is isomorphic to $\mathbb{Z}/p^{k+1}$

when we can write $j + 1 = 2(p-1)p^k m$ with m coprime to p. When j cannot be written in this form, the p–torsion in the image of J is zero. If we are working away from $p = 2$, this computation may be done using U in place of O. A computation of the 2–torsion in the image of J follows from Quillen's proof of the Adams conjecture [38].

As this result indicates, the burden of computation in stable homotopy theory often encourages one to work locally at a prime p. That is, one computes the p–power torsion in $\pi_*^S(X)$, and then assembles the results together into an integral statement. One of the deeper insights of homotopy theorists in the last half century is that one may, following Bousfield [8], do such computations local to any *cohomology theory* E^* to get more refined computations. Morava's extraordinary K–theories $K(n)$ and E–theories E_n [34] have proven particularly suited to this purpose; see eg Miller, Ravenel and Wilson [32] and Hopkins and Smith [25].

When $n = 1$, $K(1)$ is identified with (a split summand of) mod p K–theory. The fact that $\pi_*(U) = \pi_{*+1}(K)$ for $* > 0$ suggests that Adams' computation of the p–torsion in the image of J is related to $K(1)$–local homotopy theory. This is in fact the case; the localisation map

$$\pi_*^S(S^0) \to \pi_*(L_{K(1)}S^0)$$

carries im J isomorphically onto the codomain in positive degrees.

One substantial difference between the stable homotopy category and its $K(n)$–local variant is the existence of exotic invertible elements. In the stable homotopy category, the only spectra which admit inverses with respect to the smash product are spheres; thus the *Picard group* of equivalence classes of such spectra is isomorphic to $\mathbb{Z}$. In contrast, the Picard group of the $K(n)$–local category, Pic_n, includes p–complete factors as well as torsion; see eg Hopkins, Mahowald and Sadofsky [24] and Goerss, Henn, Mahowald and Rezk [19].

Our main result is a computation of part of the *Picard graded* homotopy of the $K(n)$–local sphere analogous to the image of J computation. Throughout this paper p will denote an odd prime; when localising with respect to $K(n)$, the prime p is implicitly used. We will use the notation $S := L_{K(n)}S^0$ for the $K(n)$–local sphere and $A \otimes B := L_{K(n)}(A \wedge B)$ for the $K(n)$–local smash product.

Theorem 1.1 Let $\ell \in \mathbb{Z}$, and write $\ell = p^k m$, where m is coprime to p. Then the group $[S\langle\det\rangle^{\otimes\ell(p-1)}, L_{K(n)}S^1]$ contains a subgroup isomorphic to $\mathbb{Z}/p^{k+1}$. Furthermore, if $n^2 < 2p - 3$, there is an exact sequence

$$0 \to \mathbb{Z}/p^{k+1} \to [S\langle\det\rangle^{\otimes\ell(p-1)}, L_{K(n)}S^1] \to N_{k+1} \to 0,$$

where $N_{k+1} \le \pi_{-1}(S)$ is the subgroup of p^{k+1}–torsion elements.

This is proved as Corollary 5.10. Here, $S\langle\det\rangle \in \mathrm{Pic}_n$ was introduced by Goerss et al in [19]; it is defined below. When $n = 1$ and $p > 2$, $S\langle\det\rangle$ may be identified as $L_{K(1)}S^2$, and so this result recovers the constructive part of the classical image of J computation. More generally, $S\langle\det\rangle$ may be identified as a shift of the Brown–Comenetz dual of the nth monochromatic layer of the sphere spectrum if $\max\{2n+2, n^2\} < 2(p-1)$ (see Gross and Hopkins [23]). This identification fails for small primes; see eg Goerss and Henn [16] for the case $n = 2$ and $p = 3$.

1.1 The invertible spectrum $S\langle\det\rangle$

Morava's E–theories are Landweber exact cohomology theories E_n associated to the universal deformation of the Honda formal group Γ_n from $\mathbb{F}_{p^n}$ to

$$\mathbb{W}(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}].$$

When $n = 1$, E_1 is precisely p–adic K–theory. The Goerss–Hopkins–Miller theorem [20; 21] equips the spectrum E_n with an action of the *Morava stabiliser group*

$$\mathbb{G}_n := \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathrm{Aut}(\Gamma_n)$$

which lifts the defining action in homotopy. The work of Devinatz and Hopkins [13], Davis [11; 12], and Behrens and Davis [7] then allows one to define continuous[1] homotopy fixed point spectra with respect to closed subgroups of $\mathbb{G}_n$ in a consistent way. The homotopy fixed point spectrum of the full group is the $K(n)$–local sphere: $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$.

The automorphism group $\mathrm{Aut}(\Gamma_n)$ is known to be the group of units of an order of a rank n^2 division algebra over $\mathbb{Q}_p$; the determinant of the action by left multiplication defines a homomorphism $\det: \mathrm{Aut}(\Gamma_n) \to \mathbb{Z}_p^\times$. Extend this to a homomorphism

$$\det_\pm: \mathbb{G}_n \to \mathbb{Z}_p^\times$$

by sending the Frobenius generator of $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ to $(-1)^{n-1}$. We will write $S\mathbb{G}_n^\pm$ for the kernel of this map.

We may define the homotopy fixed point spectrum $E_n^{hS\mathbb{G}_n^\pm}$ for the restricted action of this subgroup. We note: in many references, such as [23; 19], a different extension $\det: \mathbb{G}_n \to \mathbb{Z}_p^\times$ is used; there $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is in the kernel of $\det$. In an earlier version of this article, we incorrectly used this older form of $\det$; we thank Charles Rezk for clarifying this point. Nonetheless, the homotopy fixed point spectra for the kernels of $\det$ and $\det_\pm$ are equivalent (though not as ring spectra); see Section 3.10.

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[1] All homotopy fixed point spectra considered in this article will be of the continuous sort.

The spectrum $E_n^{hS\mathbb{G}_n^{\pm}}$ retains a residual action of $\mathbb{Z}_p^{\times} = \mathbb{G}_n/S\mathbb{G}_n^{\pm}$; for an element $k \in \mathbb{Z}_p^{\times}$, we will write the associated map as

$$\psi^k \colon E_n^{hS\mathbb{G}_n^{\pm}} \to E_n^{hS\mathbb{G}_n^{\pm}}.$$

The reader is encouraged to think of these automorphisms as analogues of Adams operations. Noting that $\mathbb{Z}_p^{\times} = \mu_{p-1} \times (1 + p\mathbb{Z}_p)^{\times}$ is topologically cyclic with generator $g = \zeta_{p-1}(1 + p)$, we define F_{γ} as the homotopy fibre of

$$\psi^g - \gamma \colon E_n^{hS\mathbb{G}_n^{\pm}} \to E_n^{hS\mathbb{G}_n^{\pm}}$$

for any $\gamma \in \mathbb{Z}_p^{\times}$. These spectra are always invertible, and in fact the construction $\gamma \mapsto F_{\gamma}$ defines a homomorphism $\mathbb{Z}_p^{\times} \to \mathrm{Pic}_n$. When $\gamma = 1$, the associated homotopy fibre defines the homotopy fixed point spectrum for the action of $\mathbb{Z}_p^{\times}$, and so

$$F_1 = (E_n^{hS\mathbb{G}_n^{\pm}})^{h\mathbb{Z}_p^{\times}} \simeq E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0.$$

In contrast, one defines $S\langle\det\rangle := F_g$.

1.2 A Snaith theorem

The K–theory spectrum admits a remarkable description due to Snaith [43]. He shows that the natural inclusion of $\mathbb{CP}^{\infty}$ into $BU \times \mathbb{Z}$ localises to an equivalence

$$\Sigma^{\infty}\mathbb{CP}_+^{\infty}[\beta^{-1}] \simeq K.$$

Here the *Bott map* $\beta \colon S^2 \to \Sigma^{\infty}\mathbb{CP}_+^{\infty}$ is a reduced, stable form of the inclusion $\mathbb{CP}^1 \to \mathbb{CP}^{\infty}$.

Theorem 1.2 *There is a map* $\rho_n \colon S\langle\det\rangle \to L_{K(n)}\Sigma^{\infty}K(\mathbb{Z}_p, n+1)_+$ *and an equivalence of* E_{∞}*–ring spectra*

$$L_{K(n)}\Sigma^{\infty}K(\mathbb{Z}_p, n+1)_+[\rho_n^{-1}] \to E_n^{hS\mathbb{G}_n^{\pm}}.$$

We will refer to the equivalent ring spectra of the theorem as R_n. One inverts the map ρ_n in the same fashion as for normal homotopy elements of a ring spectrum, via a telescope construction. A consequence of this theorem is that the homotopy of R_n is $S\langle\det\rangle$–periodic. When $n = 1$, R_1 is simply p–adic K–theory, and this result is familiar as Bott periodicity. Indeed, the map ρ_1 is the $K(1)$–localisation of β, and so we refer to ρ_n as a *higher Bott map*.

The proof of this theorem uses the computations of Ravenel and Wilson [40] of the Morava K–theories of Eilenberg–MacLane spaces, as well as the E_{∞} obstruction theory developed by Goerss and Hopkins in [20]. It is proven as Corollary 3.25, below.

1.3 Higher orientation for $K(n)$–local cohomology theories

The map β is traditionally used to define the notion of complex orientation for a cohomology theory; the associated formal group law is an important invariant of the cohomology theory, and central to the chromatic approach to homotopy theory. We use the map ρ_n to give a similar notion:

Definition 1.3 An n–*orientation* of a $K(n)$–local ring spectrum E is a class $x \in E^{S\langle \det \rangle}(K(\mathbb{Z}_p, n+1))$ with the property that $\rho_n^*(x) \in E^{S\langle \det \rangle}(S\langle \det \rangle) = \pi_0 E$ is a unit.

Here, for an element $A \in \mathrm{Pic}_n$ and a space X, the notation $E^A(X)$ indicates the group
$$E^A(X) = [\Sigma^\infty X_+, A \otimes E];$$
this yields $E^m(X)$ when $A = S^m$. Grading the associated groups over all of Pic_n, we define
$$E^\star(X) := \bigoplus_{A \in \mathrm{Pic}_n} E^A(X).$$

As for complex orientation, we show in Theorem 5.3 that an n–orientation of E gives a ring isomorphism $E^\star(K(\mathbb{Z}_p, n+1)) \cong E^\star[\![x]\!]$. The multiplication in $K(\mathbb{Z}_p, n+1)$ then yields a formal group law $F(x, y) \in E^\star[\![x, y]\!]$. An n–oriented cohomology theory is said to be *multiplicative* if its associated formal group law is of the form $F(x, y) = x + y + txy$ for a unit $t \in E^\star$.

Theorem 1.4 R_n *is the universal (ie initial) multiplicative* n–*oriented cohomology theory.*

This is Theorem 5.5 below. Our proof is modelled on an argument of Spitzweck and Østvær [44] which gives a version of Snaith's theorem in the motivic setting.

This theorem allows us to identify $\mathbb{Z}_p$ as a summand of $\pi_0(R_n)$, and hence of $[S\langle \det \rangle^{\otimes j}, R_n]$ for every j by Theorem 1.2. Furthermore, it implies that the action of the Adams operation ψ^g on this summand is via the jth power of the identity character. Theorem 1.1 follows from these facts by the long exact sequence in $[S\langle \det \rangle^{\otimes j}, -]$–groups associated to a fibre sequence
$$S \xrightarrow{\ \eta\ } R_n \xrightarrow{\ \psi^g - 1\ } R_n$$
which presents $S \simeq E_n^{h\mathbb{G}_n} = R_n^{h\mathbb{Z}_p^\times}$ as the homotopy equaliser of ψ^g and 1; see Section 3.6 for details.

1.4 Algebraic K–theory and the chromatic redshift conjecture

For an A_∞ ring spectrum A, the space $\mathrm{GL}_1(A)$ of units (the invertible components in $\Omega^\infty A$) admits a delooping $B\,\mathrm{GL}_1(A)$. If A is in fact an E_∞ ring spectrum, its multiplication equips $B\,\mathrm{GL}_1(A)$ with the structure of an infinite loop space. We will write $K(A)$ for the algebraic K–theory spectrum of A. There is a map of E_∞ ring spectra

$$i\colon \Sigma^\infty B\,\mathrm{GL}_1(A)_+ \to K(A)$$

whose adjoint is given by the inclusion of A–lines into all cell A–modules (see Ando, Blumberg, Gepner, Hopkins and Rezk [3] for the construction on the level of ∞–categories; then Elmendorf and Mandell [15] gives the map of E_∞ ring spectra).

In [41], with Hisham Sati, we constructed an E_∞ map $\varphi_n\colon K(\mathbb{Z}, n+1) \to \mathrm{GL}_1 E_n$. Delooping once and composing with i, we obtain a map of E_∞ ring spectra

$$i \circ B\varphi_n\colon \Sigma^\infty K(\mathbb{Z}, n+2)_+ \to K(E_n).$$

We may localise both the domain and range of this map with respect to $K(n+1)$; it is natural to ask about the behaviour of the composite map

$$\beta_{n+1} = i \circ B\varphi_n \circ \rho_{n+1}\colon S\langle\det\rangle \to L_{K(n+1)}K(E_n)$$

(we note that the domain is the $K(n+1)$–local $S\langle\det\rangle$).

Our methods are not suitable to construct such a map for the case $n = 0$ but it is instructive to consider this setting nonetheless. We interpret E_0 as singular cohomology with $\mathbb{Q}_p$ coefficients; the resulting K–theory spectrum is the algebraic K–theory of $\mathbb{Q}_p$. The map β_1, were it to exist, would be of the form $S^2 \to L_{K(1)}K(\mathbb{Q}_p)$; it should be considered as the image in the $K(1)$–local category of the *Bott element*[2] (considered in eg Thomason [46] and Mitchell [33]) in the p–adic algebraic K–theory of $\mathbb{Q}_p$.

Theorem 1.5 *For $p > 3$, multiplication by $\beta_{n+1} \in \pi_{S\langle\det\rangle}L_{K(n+1)}K(E_n)$ is an equivalence. Therefore*

$$L_{K(n+1)}K(E_n) \simeq L_{K(n+1)}K(E_n)[\beta_{n+1}^{-1}],$$

and so the map $i \circ B\varphi_n$ makes $L_{K(n+1)}K(E_n)$ an R_{n+1}–algebra spectrum.

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[2]This class doesn't exist unless μ_{p^∞} is adjoined to $\mathbb{Q}_p$; however, its $p - 1^{\mathrm{st}}$ power does exist in $K(\mathbb{Q}_p)$. Alternatively, β_1 exists after smashing with $M\mathbb{Z}/p$. See Section 6 for a similar phenomenon in our setting.

This is proven as Corollary 6.3 below. It provides some evidence for the chromatic redshift conjecture, as enunciated in Ausoni and Rognes [6, Conjecture 4.4]. The spectrum of the theorem is the algebraic K–theory spectrum of a prominent $K(n)$–local spectrum. After localisation at $K(n+1)$, we have shown it to support an algebra structure one chromatic level higher.

It is worth pointing out that this result is entirely $K(n+1)$–local. The corresponding statement for $n = 0$ — that multiplication by the Bott element as considered by Thomason is a $K(1)$–local equivalence — follows naturally from the fact that it descends to the familiar Bott class in the complex K–theory spectrum. A deeper statement in fact holds for $n = 0$: the Bott element exists prior to localisation, and multiplication by it is not nilpotent. Furthermore, inversion of this element *defines* the $K(1)$–localisation after smashing with an appropriate Moore spectrum; see Mitchell [33].

For $n = 1$, Ausoni has constructed in [5] a class that he calls the *higher Bott element*, $b \in V(1)_{2p+2}(K(E_1))$, and has verified that it, too, is not nilpotent using topological cyclic homology techniques. Again, this construction occurs at a stage prior to $K(2)$–localisation. It follows from his Theorem 1.1 that for $p > 5$, after localisation, b^{p-1} is a unit multiple of a $V(1)$–Hurewicz image of the element β_2^{p-1} in

$$\left[S^{2p^2-2}, L_{K(n)}(V(1) \wedge K(E_1)) \right] \cong \left[S\langle \det \rangle^{p-1}, L_{K(n)}(V(1) \wedge K(E_1)) \right]$$

(this isomorphism follows from the same arguments as Proposition 6.1). In contrast to the intricate computational methods of [5], our proof of the periodicity of β_{n+1} is achieved by detecting it modulo p as multiplication by an invertible element of the Picard-graded homotopy of the $K(n+1)$–local Moore spectrum.

However, an important caveat to Theorem 1.5 is that it does not actually show that the localisation $L_{K(n+1)}K(E_n)$ is nonzero. When $n = 0$, the fact that $L_{K(1)}K(\mathbb{Q}_p) \neq 0$ is visible in Hesselholt and Madsen [22], where it is shown that $K_*(\mathbb{Q}_p)$ contains the image of J as a factor. When $n = 1$, the nonvanishing of $L_{K(2)}K(E_1)$ follows from Ausoni's explicit computations of $V(1)_* K(E_1)$.

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2 Generalities on the $K(n)$–local category

2.1 Morava K and E–theories

A central object of the chromatic program is the Honda formal group law Γ_n of height n over the finite field $\mathbb{F}_{p^n}$; the p–series of Γ_n is $[p](x) = x^{p^n}$. We will write K or K_n for the 2–periodic Landweber-exact cohomology theory that supports this formal group law; thus the coefficients of K are $\pi_* K = \mathbb{F}_{p^n}[u^{\pm 1}]$. This is closely related to the "standard" Morava K–theory, $K(n)$, whose homotopy groups are $\mathbb{F}_p[v_n^{\pm 1}]$, with $|v_n| = 2p^n - 2$.

The (Lubin–Tate) universal deformation of Γ_n is defined over the ring

$$\mathbb{W}(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}],$$

where $\mathbb{W}(\mathbb{F}_{p^n})$ is the ring of Witt vectors over $\mathbb{F}_{p^n}$. There is a cohomology theory E_n — *Morava E–theory* — whose homotopy groups are given by that ring, and support this formal group law. The theorem of Goerss, Hopkins and Miller [20] equips E_n with the structure of an E_∞ ring spectrum.

If we write $\mathfrak{m}$ for the maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1}) \subseteq \pi_*(E_n)$, these cohomology theories are related via

$$K^*(X) \cong (E_n/\mathfrak{m})^*(X) \cong \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} K(n)^*(X)[u]/(u^{p^n-1} - v_n),$$

and similarly in homology. Notice that the representing spectrum K is equivalent to a bouquet of copies of $K(n)$. Thus its Bousfield class is the same as that of $K(n)$.

Recall that $\mathbb{F}_{p^n}^\times$ consists of $p^n - 1^{\text{st}}$ roots of unity. Factoring

$$p^n - 1 = (p-1)(1 + p + \cdots + p^{n-1}),$$

we note that $2(p-1)$ divides $p^n - 1$ when p is odd and n is even. Choose a $p - 1^{\text{st}}$ root of $(-1)^{n-1}$, $\xi \in \mathbb{F}_{p^n}$, to be

$$\xi := \begin{cases} 1 & \text{for } n \text{ odd,} \\ \text{a primitive } (2p-2)^{\text{nd}} \text{ root of } 1 & \text{for } n \text{ even.} \end{cases}$$

2.2 The Morava stabiliser group

We will briefly summarise some preliminaries regarding the Morava stabiliser group. We refer the reader to [34] and [18] for more complete discussions of this material.

We will use the notation $\mathbb{S}_n := \mathrm{Aut}_{\mathbb{F}_{p^n}}(\Gamma_n)$ to denote the group of automorphisms of Γ_n. Then $\mathbb{G}_n$ is defined as the semidirect product

$$\mathbb{G}_n := \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

The Goerss–Hopkins–Miller theorem lifts the defining action $\mathbb{G}_n$ on Γ_n to an action on the spectrum E_n through E_∞ maps. The work of Devinatz and Hopkins [13], Davis [11], and Behrens and Davis [7] defines homotopy fixed point spectra with respect to closed subgroups of $\mathbb{G}_n$ and constructs associated descent spectral sequences.

We write $\mathcal{O}_n$ for the noncommutative ring

$$\mathcal{O}_n = \mathbb{W}(\mathbb{F}_{p^n})\langle S \rangle / (S^n - p, Sa = a^\sigma S),$$

where σ denotes a lift of the Frobenius on $\mathbb{F}_{p^n}$ to the ring $\mathbb{W}(\mathbb{F}_{p^n})$ of Witt vectors. Then $\mathcal{O}_n \cong \mathrm{End}_{\mathbb{F}_{p^n}}(\Gamma_n)$ is the ring of endomorphisms of Γ_n, and so $\mathbb{S}_n \cong \mathcal{O}_n^\times$.

Now, $\mathbb{S}_n$ naturally acts on $\mathcal{O}_n$ (by right multiplication) through left $\mathbb{W}(\mathbb{F}_{p^n})$–module homomorphisms, and so defines a map $\mathbb{S}_n \to \mathrm{GL}_n(\mathbb{W}(\mathbb{F}_{p^n}))$, since $\mathcal{O}_n$ is free of rank n over $\mathbb{W}(\mathbb{F}_{p^n})$, with basis $\{1, S, \dots, S^{n-1}\}$. It turns out that the determinant of elements coming from $\mathbb{S}_n$ actually lie in $\mathbb{Z}_p^\times$, instead of $\mathbb{W}(\mathbb{F}_{p^n})^\times$; this gives a homomorphism

$$\det \colon \mathbb{S}_n \to \mathbb{Z}_p^\times.$$

Note that the Frobenius generator F of $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on the division algebra $\mathcal{O}_n \otimes \mathbb{Q}$ by conjugation by S. Therefore, we may extend det to $\det_\pm \colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ by sending $F \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ to $(-1)^{n-1}$; the action of F on $\mathbb{S}_n$ is such that this is indeed a homomorphism. We will write $S\mathbb{G}_n^\pm$ for the kernel of this homomorphism.

The sign character is somewhat unusual (eg it does not appear in [23] or [18] and further work of these authors), but is necessary in order to deal with a troublesome sign in the Ravenel–Wilson computations of the Verschiebung in the Morava K–theory of Eilenberg–MacLane spaces. This is what requires the introduction of the root of unity ξ in Section 3.1 and is further reflected in the proof of Proposition 3.21.

We recall that there is an isomorphism $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)^\times \cong \mu_{p-1} \times \mathbb{Z}_p$. Composing the determinant with the projection onto the second factor yields the

reduced determinant[3] $\overline{\det_{\pm}}\colon \mathbb{G}_n \to \mathbb{Z}_p$. If we write $\mathbb{G}_n^1$ for the kernel of $\overline{\det_{\pm}}$, then there is a split short exact sequence

$$(2.2.1) \qquad 1 \to S\mathbb{G}_n^{\pm} \to \mathbb{G}_n^1 \to \mu_{p-1} \to 1.$$

Furthermore, one may define a homomorphism $z\colon \mathbb{Z}_p^{\times} \to \mathbb{S}_n$ by

$$a \mapsto a + 0 \cdot S + \cdots + 0 \cdot S^{n-1}.$$

The image is central, and the composite $\det_{\pm} \circ z\colon \mathbb{Z}_p^{\times} \to \mathbb{Z}_p^{\times}$ is $a \mapsto a^n$. In particular, if n is coprime to both p and $p-1$, this map is an isomorphism, yielding a splitting $\mathbb{G}_n \cong S\mathbb{G}_n^{\pm} \times \mathbb{Z}_p^{\times}$. Similarly, we get a splitting $\mathbb{G}_n \cong \mathbb{G}_n^1 \times \mathbb{Z}_p$.

2.3 The Picard group

We recall that the category of $K(n)$–local spectra is symmetric monoidal; the tensor product

$$A \otimes B := L_{K(n)}(A \wedge B)$$

is the $K(n)$–localisation of the smash product of A and B. The unit is the $K(n)$–local sphere $S := L_{K(n)}(S^0)$.

An *invertible spectrum* A is invertible with respect to this tensor product; that is, there exists another spectrum B with $A \otimes B \simeq S$. The set of weak equivalence classes of invertible spectra forms an abelian group under tensor product, the *Picard group* of the $K(n)$–local category, denoted Pic_n.

It was shown in [24] that A is invertible if and only if $K(n)_*(A)$ is free of rank one over $K(n)_*$. Consequently there is a well-defined *dimension* $\dim(A) \in \mathbb{Z}/2(p^n - 1)$ which records the degree of a generator of $K(n)_*(A)$.

For spectra A, B, we will denote by $F(A, B)$ the function spectrum of maps from A to B. If A is an element of Pic_n, it is easily seen that the Spanier–Whitehead dual $F(A, S) = A^{-1}$ is the inverse to A: the evaluation map $A \otimes F(A, S) \to S$ is an equivalence.

2.4 Pic_n–graded invariants

In [24] it was suggested that one should take a larger view of homotopy groups in the $K(n)$–local category, and grade homotopy by the Picard group. To invent some

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[3]Since $\pm 1 \in \mu_{p-1}$, this is the same as the reduced determinant of eg [18], so our $\mathbb{G}_n^1$ is the same as in that article. However, the homomorphism to μ_{p-1} in Equation (2.2.1) differs.

notation, we will write $\pi_A(X) := [A, X]$, for $K(n)$–local spectra X and invertible A; then $\pi_{S^m}(X) = \pi_m(X)$.

We propose that the same approach be used to index[4] the coefficients of more general (co)homology theories.

Definition 2.1 Let X and E be $K(n)$–local spectra. For an invertible spectrum A, define

$$E_A(X) := [A, X \wedge E], \quad E_A^\vee(X) := [A, X \otimes E], \quad \text{and} \quad E^A(X) := [X, A \otimes E].$$

These are the Pic_n–graded *E–homology, completed E–homology,* and *E–cohomology.*

We note that $E_A(X)$ need not be isomorphic to the $E_A^\vee(X)$, even for $A = S^m$. Indeed, when $E = E_n$ is Morava E–theory, the completed homology theory is the *Morava module* of X, which may differ from $(E_n)_m(X)$ when X is an infinite complex. In this paper, we will largely be concerned with $E_A^\vee(X)$, rather than $E_A(X)$.

In order to be completely balanced, we should also consider the cohomological functor $[X, A \wedge E]$ (and perhaps decorate the functor $E^A(X)$ above with a $\vee$). However, $[X, A \wedge E] = [X, A \otimes E]$ when A is a sphere (since $S^m \wedge E = \Sigma^m E$ is already $K(n)$–local). Furthermore, as in the homological setting, we will largely concern ourselves with $K(n)$–local constructions, making the nonlocal $A \wedge E$ awkward.

To distinguish from the standard indexing (by elements $m \in \mathbb{Z}$, corresponding to S^m), we will write

$$E_\star^\vee(X) := \bigoplus_{A \in \mathrm{Pic}_n} E_A^\vee(X), \quad \text{but} \quad E_*^\vee(X) := \bigoplus_{n \in \mathbb{Z}} E_n^\vee(X),$$

and similarly for cohomology.

Example 2.2 Since K satisfies a strong Künneth theorem,

$$K^A(X) = [X, A \otimes K] = [X \otimes A^{-1}, K] = K^0(X \otimes A^{-1}) = (K^*(X) \otimes_{K_*} K^*(A^{-1}))_0.$$

If A has dimension d, then A^{-1} has dimension $-d$, so we conclude that $K^A(X) \cong K^d(X)$. So grading $K^\star$ by the full Picard group does not recover any more information than grading by the integers.

---

[4]The point is raised in [27] that grading homotopy (and hence other such invariants) by the Picard group does not necessarily yield a well-behaved ring structure on the resulting direct sum. In practice, however, we will only ever consider indexing on integral powers of spheres and $S\langle \det \rangle$, for which their Theorem 14.11 is more than sufficient.

The same argument works if one replaces K with E_n, since $E_n^*(A^{-1})$ is free of rank 1 over E_n^*, giving a collapsing Künneth spectral sequence. However, the result is far from true for all (co)homology theories, most notably stable homotopy.

Additionally, we note the following, whose proof is immediate:

Proposition 2.3 If A is invertible and E a $K(n)$–local ring spectrum, then $E_\star^\vee(A)$ and $E^\star(A)$ are free $E_\star := E_\star(S)$–modules of rank one.

Note that, by the (collapsing) Künneth spectral sequence, this implies that for any spectrum X, the natural map

$$E_\star(A) \otimes_{E_\star} E_\star(X) \to E_\star(A \otimes X)$$

is an isomorphism (and similarly for the completed homology and cohomology).

2.5 Localising ring spectra

If X is a $K(n)$–local ring spectrum with multiplication μ, $A \in \mathrm{Pic}_n$, and $f: A \to X$ an element of $\pi_A(X)$, one can localise X away from f in the same fashion as is usually done for homotopy elements.

Definition 2.4 Define $X[f^{-1}]$ to be the $K(n)$–localisation of the telescope (homotopy colimit) of the diagram

$$X \xrightarrow{m_f} A^{-1} \otimes X \xrightarrow{1 \otimes m_f} A^{-1} \otimes (A^{-1} \otimes X) \xrightarrow{1 \otimes 1 \otimes m_f} \cdots .$$

Here m_f is the composite

$$X = A^{-1} \otimes A \otimes X \xrightarrow{1 \otimes f \otimes 1} A^{-1} \otimes X \otimes X \xrightarrow{1 \otimes \mu} A^{-1} \otimes X,$$

where μ is the multiplication in X.

The $K(n)$–localisation in the definition is necessary because it is not clear that the telescope of m_f, being a homotopy colimit of $K(n)$–local spectra, is itself local.

Proposition 2.5 If X is an E_∞–ring spectrum, then so too is $X[f^{-1}]$.

Proof This is based on [14, Section VIII], which shows that $KU = ku[\beta^{-1}]$ is an E_∞–ring spectrum; we thank the referee for directing us to it. All references in this proof are to [14]; this argument differs from theirs only in the $K(n)$–local aspect. The telescope of m_f is a (right) X–module; Proposition V.2.3 gives it also the weak structure of

X–ring spectrum in the classical homotopy theoretic sense. These descend through the $K(n)$–localisation to give structures on $X[f^{-1}]$ using, for example, Theorem VIII.1.6 for the module structure and the fact that Bousfield localisation is monoidal up to homotopy for the ring structure.

This allows us, for any $K(n)$–local X–module M, to consider the map

$$\lambda\colon M \cong X \otimes_X M \to X[f^{-1}] \otimes_X M$$

given by the unit of $X[f^{-1}]$. This has the property that $\lambda \otimes_X X[f^{-1}]$ is an equivalence, since the multiplication $X[f^{-1}] \otimes_X X[f^{-1}] \to X[f^{-1}]$ is an equivalence (this is the $K(n)$–localisation of Lemma V.1.15). Thus λ is a $X[f^{-1}]$–equivalence, and so $X[f^{-1}] \otimes_X M$ is a $X[f^{-1}]$–Bousfield localisation of M. Applying this to $M = X$, which is an X–algebra, and invoking Theorem VIII.2.2, we see that $X[f^{-1}]$ is an E_∞ X–algebra. $\qquad\square$

Standard properties of homotopy colimits then give:

Proposition 2.6 *The natural map $X \to X[f^{-1}]$ induces an isomorphism*

$$(K_\star X)[f^{-1}] \to K_\star(X[f^{-1}]),$$

where f is regarded as an element of $K_A(X) = [A, X \otimes K]$ by smashing with the unit of K. This induces an isomorphism

$$(K_* X)[f_*^{-1}] \to K_*(X[f^{-1}]),$$

where $f_* \in K_{\dim(A)}(X)$ is the image of a generator of $K_*(A)$ under f. The same holds with K replaced by E_n.

Write η for the unit in X. By construction, there is an equivalence

$$\mu \circ (f \otimes 1)\colon A \otimes X[f^{-1}] \to X[f^{-1}].$$

Let g be the inverse equivalence. Define $f^{-1}\colon A^{-1} \to X[f^{-1}]$ as

$$A^{-1} = A^{-1} \otimes S \xrightarrow{1 \otimes \eta} A^{-1} \otimes X[f^{-1}] \xrightarrow{1 \otimes g} A^{-1} \otimes A \otimes X[f^{-1}] = X[f^{-1}].$$

By construction $\eta = f \cdot f^{-1}\colon S = A \otimes A^{-1} \to X[f^{-1}]$; thus f is indeed invertible in $\pi_\star(X[f^{-1}])$.

2.6 Automorphism groups

For a spectrum A, we will write $\mathrm{end}(A) := F(A, A)$ for the function spectrum of maps from A to itself. This is an associative ring spectrum under composition of functions, and so $\pi_0 \mathrm{end}(A, A) = [A, A]$ forms a ring.

Definition 2.7 Write $\mathrm{Aut}(A) \subseteq \mathrm{End}(A) := \Omega^\infty \mathrm{end}(A)$ for the union of components whose image in the ring $\pi_0 \mathrm{end}(A, A)$ is invertible.

Note that the multiplication in $\mathrm{end}(A)$ equips $\mathrm{Aut}(A)$ with the structure of a grouplike A_∞ monoid. We also note that if $A = S$, the adjoint of the identity on A defines an equivalence of ring spectra $\mathrm{end}(S) \simeq S$, and so $\mathrm{Aut}(S)$ is identified as $\mathrm{GL}_1(S)$, the A_∞ monoid of units in the ring spectrum S:

$$\mathrm{Aut}(S) = \mathrm{GL}_1(S) \subseteq \Omega^\infty S.$$

The following is immediate:

Proposition 2.8 *If A is invertible and B any spectrum, tensoring with the identity of A gives an equivalence*

$$id_A \otimes -: \mathrm{end}(B) \to \mathrm{end}(A \otimes B).$$

Passing to infinite loop spaces yields a natural equivalence of A_∞ monoids $\mathrm{Aut}(B) \to \mathrm{Aut}(A \otimes B)$.

In particular, taking $B = S$, we conclude:

Corollary 2.9 *If A is invertible, smashing with the identity of A defines an equivalence of A_∞ monoids*

$$\mathrm{GL}_1(S) \to \mathrm{Aut}(A).$$

Loosely speaking, A becomes a $\mathrm{GL}_1(S)$–spectrum by its action on the left tensor factor of $S \otimes A = A$.

2.7 Thom spectra

Let X be a topological space, and $\zeta \colon X \to B\,\mathrm{GL}_1(S)$ a continuous map. Ando et al [3] define the Thom spectrum X^ζ as

$$X^\zeta := \Sigma^\infty P_+ \wedge_{\Sigma^\infty \mathrm{GL}_1(S)_+} S,$$

where P is the principal $\mathrm{GL}_1(S)$–bundle over X defined by ζ. Using Corollary 2.9, we may modify and extend this definition in the $K(n)$–local category as follows.

Definition 2.10 For an invertible spectrum A, define the *Thom spectrum* $X^{A\zeta}$ as

$$X^{A\zeta} := \Sigma^\infty P_+ \otimes_{\Sigma^\infty \mathrm{GL}_1(S)_+} A = L_{K(n)}(\Sigma^\infty P_+ \wedge_{\Sigma^\infty \mathrm{GL}_1(S)_+} A).$$

A remark on notation: we have chosen to present this multiplicatively ($X^{A\zeta}$ instead of $X^{A+\zeta}$) to be consistent with the multiplicative nature of the (smash) product in the Picard group. This of course conflicts with the common additive notation $X^{n+\zeta}$ when $A = S^n$; we hope no confusion will result.

Note that even when $A = S$, this differs slightly from the definition in [3] in that we have $K(n)$–localised the Thom spectrum. This extension of the definition of Thom spectra to have any invertible spectrum as "fibre" is convenient, but not very substantial; noticing that the action of $\mathrm{GL}_1(S)$ is on the S factor in $A = S \otimes A$, one can easily show:

Proposition 2.11 $$X^{A\zeta} = X^\zeta \otimes A.$$

The composite

$$X \times X \xrightarrow{\mathrm{pr}_1} X \xrightarrow{\zeta} B\,\mathrm{GL}_1(S)$$

of ζ with the projection onto the first factor defines a Thom spectrum over $X \times X$, which may be identified with $X^{A\zeta} \otimes X_+$. Furthermore, the diagonal map $\Delta\colon X \to X \times X$ is covered by a map of Thom spectra

$$D\colon X^{A\zeta} \to X^{A\zeta} \otimes X_+,$$

since $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_X$. The map D is the *Thom diagonal*, and for any $K(n)$–local ring cohomology theory E, makes $E^\star(X^{A\zeta})$ into a $E^\star(X)$–module.

Similarly, for each point $x \in X$, the inclusion $\{x\} \subseteq X$ is covered by a "fibre inclusion"

$$i_x\colon A = X^{A\zeta}|_{\{x\}} \to X^{A\zeta}.$$

Definition 2.12 A class $u \in E^A(X^{A\zeta})$ is a *Thom class* if, for every $x \in X$, its restriction along $i_x\colon A \to X^{A\zeta}$, $i_x^*(u) \in E^A(A) \cong E_0$, is a unit.

Proposition 2.13 If $X^{A\zeta}$ admits a Thom class u for E, then $E^\star(X^{A\zeta})$ is a free $E^\star(X)$–module of rank one, generated by u.

When $A = S$, this is classical, and is realised in homology by the map

$$E \otimes X^{A\zeta} \xrightarrow{1 \otimes D} E \otimes X^{A\zeta} \otimes X_+ \xrightarrow{1 \otimes u \otimes 1} E \otimes E \otimes A \otimes X \xrightarrow{\mu \otimes 1} E \otimes A \otimes X.$$

See eg [29] or [3, Proposition 5.43]. For general A, this is obtained from that fact and the Pic_n–graded isomorphism

$$E^{\star}(X^{A\xi}) = E^{\star}(X^{\xi} \otimes A) \cong E^{\star}(X^{\xi}) \otimes_{E_{\star}} E_{\star}A.$$

3 Eilenberg–MacLane spaces

3.1 Recollections from Ravenel–Wilson

Working stably, after p–completion, we note the equivalences[5]

$$(3.1.2) \qquad \Sigma^{\infty} K(\mu_{p^{\infty}}, n)_{+} = \varinjlim \Sigma^{\infty} K(\mathbb{Z}/p^{j}, n)_{+} \simeq \Sigma^{\infty} K(\mathbb{Z}_{p}, n+1)_{+}$$
$$\simeq \Sigma^{\infty} K(\mathbb{Z}, n+1)_{+}.$$

(The first equivalence uses the Bockstein.) This will be our main object of study:

Definition 3.1 Let $X = X_n$ denote the $K(n)$–localisation of the unreduced suspension spectrum of $K(\mathbb{Z}_p, n+1)$,

$$X_n = L_{K(n)} \Sigma^{\infty} K(\mathbb{Z}_p, n+1)_{+}.$$

We recall from Ravenel and Wilson [40] the Morava K–theory of this spectrum:

$$K(n)^{*} K(\mathbb{Z}_p, n+1) = K(n)_{*}[\![x]\!] \quad \text{and} \quad K(n)_{*} K(\mathbb{Z}_p, n+1) = \bigotimes_{k \geq 0} R(b_k).$$

Here $|x| = 2g(n)$, where we define $g(n) := (p^n - 1)/(p - 1)$. In the notation of [40, Section 12], x corresponds to the class x_S for $S = (1, 2, \ldots, n-1)$. Also, for each integer $k \geq 0$, $R(b_k)$ is the ring

$$R(b_k) := K(n)_{*}[b_k]/(b_k^p - (-1)^{n-1} v_n^{p^k} b_k) = \mathbb{F}_p[x, v_n^{\pm}]/(b_k^p - (-1)^{n-1} v_n^{p^k} b_k),$$

where the class b_k has dimension $2p^k g(n)$ and is dual to $(-1)^{k(n-1)} x^{p^k}$. The notation b_k is our abbreviation for Ravenel and Wilson's b_J, with $J = (nk, 1, 2, \ldots, n-1)$.

There are similar results for $K(\mathbb{Z}/p^j, n)$:

$$K(n)^{*} K(\mathbb{Z}/p^j, n) = K(n)_{*}[x]/x^{p^j} \quad \text{and} \quad K(n)_{*} K(\mathbb{Z}/p^j, n) = \bigotimes_{k=0}^{j-1} R(b_k).$$

We have normalised these classes to be consistent across the limit in (3.1.2). This is not quite consistent with the notation of [40, Section 11]; there $K(n)_{*} K(\mathbb{Z}/p^j, n)$ is

---

presented as being generated by classes a_I, with

$$I = (nk, n(j-1)+1, n(j-1)+2, \ldots, n(j-1)+n-1)$$

with $0 \leq k < j$. This a_I differs from b_k by a power of v_n.

In the extension

$$K^* X_n = \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} K(n)^*(X_n)[u]/(u^{p^n-1} - v_n),$$

use the $(2p-2)^{\text{nd}}$ root of unity, ξ, to define a new (degree 0) coordinate $y := \xi x u^{g(n)}$. Then

$$K^*(X_n) = K_*[\![y]\!].$$

We may similarly normalise the K–homology; setting

$$c_k = (-1)^{k(n-1)} \xi^{-p^k} b_k u^{-p^k g(n)} = \xi^{-1} b_k u^{-p^k g(n)}$$

we have

$$K_*(X_n) = K_*[c_0, c_1, \ldots]/(c_k^p - c_k)$$

and $\langle c_k, y^{p^j} \rangle = \delta_k^j$.

Proposition 3.2 *The multiplication on X_n equips $K^* X_n$ with the structure of a formal group over K_* which is isomorphic to the formal multiplicative group, $\mathbb{G}_m$.*

Proof The computations described above equip $K^* X_n$ with a coordinate y, and the associative and unital multiplication on X_n coming from the H–space structure on $K(\mathbb{Z}_p, n+1)$ defines a formal group law F on $K^* X_n = K_*[\![y]\!]$.

We may compute the p–series of this formal group law using [40]. The relevant fact is that the Verschiebung, well-defined[6] up to powers of v_n, satisfies $V(x) = (-1)^{n-1} x$. Thus,

$$[p](x) = FV(x) = (-1)^{n-1} v_n^{-1} x^p.$$

Consequently $[p](y) = y^p$, and so F has height 1. Thus there is an isomorphism $f \colon F \cong \mathbb{G}_m$ over $\overline{\mathbb{F}}_p$. In fact f is defined over $\mathbb{F}_p$, for the equation

$$f(y^p) = f([p]_F(y)) = [p]_{\mathbb{G}_m}(f(y)) = f(y)^p$$

implies that if $f(y) = \sum a_i y^i$, then the coefficients a_i satisfy $a_i^p = a_i$, so must lie in $\mathbb{F}_p$. Note, however, that the formal group on $K(n)^* X_n$ is not isomorphic (over $\mathbb{F}_p$) to $\mathbb{G}_m$ when n is even, since the root of unity ξ used in defining y is not present in $\mathbb{F}_p$ in that case. □

---

[6] Because $\mathbb{F}_p = K(n)_*/(v_n - 1)$ is a perfect field, while $K(n)_*$ is not, V is not naturally defined on $K(n)^*(X)$, but on its cyclically graded analogue $\overline{K(n)}^*(X) = K(n)^*(X)/(v_n - 1)$.

We note that this computation allows one to formally define the a–series $[a](y) \in K_*[\![y]\!]$ for any element $a \in \mathbb{Z}_p$.

Definition 3.3 Write a in its p–adic expansion as $a = a_0 + a_1 p + a_2 p^2 + \cdots$, where $0 \le a_i < p$. Then

$$[a](y) := [a_0](y) +_F [a_1]([p](y)) +_F [a_2]([p^2](y)) +_F \cdots .$$

Here, $+_F$ is addition according to the formal group law on $K_*[\![y]\!]$.

The formula gives a well-defined series, since $\deg([p^n](y)) = p^n$ (or equivalently, in this case, the order of vanishing at 0) grows with n.

3.2 Group actions

The p–adic integers $\mathbb{Z}_p$ are a topologically cyclic group with generator $1 \in \mathbb{Z}_p$; that is, the subgroup generated by 1 (ie $\mathbb{Z}$) is dense in $\mathbb{Z}_p$. We will have occasion to write elements of $\mathbb{Z}_p$ in multiplicative notation; then we will write h for the generator 1.

Furthermore, for $p > 2$,

$$\mathbb{Z}_p^\times \cong \mu_{p-1} \oplus (1 + p\mathbb{Z}_p)^\times,$$

and the latter factor is isomorphic to $\mathbb{Z}_p$. Let $\zeta = \zeta_{p-1} \in \mu_{p-1}$ be a primitive $(p-1)^{\text{st}}$ root of unity, ie a generator of μ_{p-1}. A generator for $(1 + p\mathbb{Z}_p)^\times$ is $1 + p$. Then $\mathbb{Z}_p^\times$ is also topologically cyclic, with generator $g := (\zeta, 1 + p)$. We note that $g \bmod p = \zeta \in \mathbb{F}_p^\times$.

The group $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p$, and hence on $X_n = L_{K(n)} \Sigma^\infty K(\mathbb{Z}_p, n + 1)_+$. For an element $a \in \mathbb{Z}_p^\times$, we will denote by ψ^a the map $\psi^a \colon X_n \to X_n$ given by the action of a.

Proposition 3.4 *The action of $\mathbb{Z}_p^\times$ on $K^*(X_n)$ is via ring homomorphisms, and is determined by $\psi^a(y) = [a](y)$.*

Proof This formula in fact holds for every $a \in \mathbb{Z}_p$. Note that this homotopy commutes, for any $m \in \mathbb{N}$:

$$
\begin{array}{ccc}
K(\mathbb{Z}_p, n + 1) & \xrightarrow{\;\Delta\;} & K(\mathbb{Z}_p, n + 1)^{\times m} \\
& \searrow{\scriptstyle \psi^m} & \downarrow{\scriptstyle \text{mult}} \\
& & K(\mathbb{Z}_p, n + 1)
\end{array}
$$

where Δ is the m–fold diagonal, and mult is m–fold multiplication. The path along the upper right carries y to $[m](y)$.

Being defined by space-level maps, the action of $\mathbb{Z}_p$ on $K^*(X_n)$ may be regarded as a *continuous* homomorphism $\mathbb{Z}_p \to \operatorname{End}(K_*\llbracket y \rrbracket)$. Here, the topology on $\operatorname{End}(K^*(X))$ is compact-open with respect to the natural topology on $K_*(Y)$ defined in [27, Section 11]. In that topology, the map $\mathbb{Z}_p \subseteq [X, X] \to \operatorname{End}(K^*(X))$ is continuous. The argument above indicates that it agrees with the action $a \cdot y = [a](y)$ when $a \in \mathbb{N}$. Since the latter is also continuous, and $\mathbb{N}$ is dense in $\mathbb{Z}_p$, these two actions must agree. $\qquad \square$

The group $\mu_{p-1} \cong \mathbb{F}_p^\times$ acts via multiplication on $\mathbb{Z}/p = \mathbb{F}_p$, and therefore on $K(\mathbb{Z}/p, n)$.

Proposition 3.5 *The action of μ_{p-1} on $K^* K(\mathbb{Z}/p, n) \cong K_*[y]/y^p$ is given by $\psi^\zeta(y^m) = \zeta^m y^m$.*

Proof Since the action is space-level, it suffices to show that $\psi^\zeta(y) = \zeta y$. To see this, we note that the following diagram commutes:

$$
\begin{array}{ccc}
K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, 1)^{\times n-1} & \xrightarrow{\ \circ\ } & K(\mathbb{Z}/p, n) \\
{\scriptstyle \psi^\zeta \times 1}\big\downarrow & & \big\downarrow{\scriptstyle \psi^\zeta} \\
K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, 1)^{\times n-1} & \xrightarrow[\ \circ\ ]{} & K(\mathbb{Z}/p, n)
\end{array}
$$

since both paths around the diagram represent ζ times the fundamental class in $H^n(K(\mathbb{Z}/p, 1)^{\times n}; \mathbb{F}_p)$. Thus in $K(n)_*$,

$$
\begin{aligned}
\psi_*^\zeta(b_0) &= \psi_*^\zeta(a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)}) \\
&= \psi_*^\zeta(a_{(0)}) \circ a_{(1)} \circ \cdots \circ a_{(n-1)} \\
&= \zeta a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)} \\
&= \zeta b_0.
\end{aligned}
$$

The third equality uses the claim that $\psi_*^\zeta(a_0) = \zeta a_{(0)} \in K(n)_2 K(\mathbb{Z}/p, 1)$. This follows from the fact that both classes are carried injectively by the Bockstein to the unique class in $K(n)_2 K(\mathbb{Z}_p, 2)$ which is ζ times the Hurewicz image of the fundamental class of $S^2 \to K(\mathbb{Z}_p, 2)$. The claimed result then follows by duality. $\qquad \square$

The commutativity of the diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow{\ p\ } & \mathbb{Z}_p & \xrightarrow{\ \mathrm{mod}\ p\ } & \mathbb{Z}/p & \longrightarrow & 0 \\
& & {\scriptstyle g}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} & & \\
0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow[\ p\ ]{} & \mathbb{Z}_p & \xrightarrow{\ \mathrm{mod}\ p\ } & \mathbb{Z}/p & \longrightarrow & 0
\end{array}
$$

yields a commutative diagram of fibrations:

(3.2.3)
$$
\begin{array}{ccc}
K(\mathbb{Z}/p,n)_{-\psi^\zeta} \xrightarrow{\ \beta\ } K(\mathbb{Z}_p,n+1) \xrightarrow{\ \psi^p\ } K(\mathbb{Z}_p,n+1) \\
\Big\downarrow \qquad\qquad \Big\downarrow {\psi^g} \qquad\qquad \Big\downarrow {\psi^g} \\
K(\mathbb{Z}/p,n) \xrightarrow{\ \beta\ } K(\mathbb{Z}_p,n+1) \xrightarrow{\ \psi^p\ } K(\mathbb{Z}_p,n+1)
\end{array}
$$

Thus, the Bockstein

$$
\beta \colon L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p,n)_+ \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p,n+1)_+ = X_n
$$

is $\mathbb{Z}_p^\times$–equivariant where the action of $\mathbb{Z}_p^\times$ on $L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p,n)_+$ is via the reduction $\mathbb{Z}_p^\times \to \mu_{p-1}$. In particular, we conclude that

$$
\psi^g(y) \bmod y^p = \zeta y.
$$

3.3 Splitting $K(\mathbb{Z}/p,n)$

Now we construct an invertible spectrum Z and decompose $L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p,n)_+$ into powers of Z. It will turn out that Z has order $p-1$ in Pic_n, and is represented in the algebraic Picard group by the character χ given by the composite

$$
\mathbb{G}_n \xrightarrow{\ \det_\pm\ } \mathbb{Z}_p^\times \xrightarrow{\ \mathrm{mod}\ p\ } \mathbb{F}_p^\times \xrightarrow{\ T\ } \mathbb{Z}_p^\times
$$

(T is the Teichmüller character); that is, $E_*^\vee(Z) \cong E_*[\chi]$.

For any p–complete spectrum Y, there is an action (distinct from the action described in the previous section) of $\mathbb{Z}_p^\times$ on Y where group elements act in homotopy by multiplication; we will write the action of ζ simply as ζ.

Definition 3.6 Define an endomorphism π of $L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p,n)_+$ by

$$
\pi = \frac{1}{p-1}\sum_{k=0}^{p-2}\zeta^{-k}(\psi^\zeta)^k = \frac{1}{p-1}\sum_{k=0}^{p-2}\zeta^{-k}\psi^{\zeta^k}.
$$

We thank Tyler Lawson for this construction and the proof of the following proposition, which replaces an incorrect argument in a previous version of this paper.

Proposition 3.7 *The map π is a homotopy idempotent. It thus yields a splitting*

$$
L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p,n)_+ \simeq Z \vee Z^\perp,
$$

where the image of Z in $K_* K(\mathbb{Z}/p,n)$ is the rank one K_*–subspace generated by b_0.

Proof The idempotence follows from the fact that ψ^ζ commutes with ζ and a brief computation. The splitting comes in the standard way, defining Z as the spectrum representing the image of π inside the functor $[-, L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+]$. Consequently, the image of $K_*(Z)$ is the image of π_*. However,

$$\pi_*(b_0^\ell) = \frac{1}{p-1}\sum_{k=0}^{p-2}\zeta^{-k}\psi^{\zeta^k}(b_0^\ell) = \frac{1}{p-1}\sum_{k=0}^{p-2}\zeta^{-k}\zeta^{k\ell}b_0^\ell = \frac{1}{p-1}\sum_{k=0}^{p-2}\zeta^{k(\ell-1)}b_0^\ell,$$

which is precisely b_0 when $\ell = 1$, and 0 otherwise. $\square$

Lemma 3.8 Z *is an element of* Pic_n. *Furthermore,*

$$L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+ \simeq \bigvee_{k=0}^{p-1} Z^{\otimes k}.$$

Lastly, $Z^{\otimes p-1} \simeq S$.

Proof The first claim follows immediately from the previous proposition. Then the map $Z^{\otimes k} \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+$ induced by k–ary multiplication in $K(\mathbb{Z}/p, n)$ is an isomorphism in K_* onto its image, which is the subspace generated by b_0^k. Thus $L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+$ decomposes as indicated into a wedge of tensor powers of Z. The same argument, applied to $k = p$ (along with the fact that $b_0^p = (-1)^{n-1}v_n b_0$) yields $Z^{\otimes p} \simeq Z$. The last result follows by cancelling a factor of Z. $\square$

Lastly, we note that $\pi \circ \psi^\zeta = \zeta \cdot \pi$; thus the action of ψ^ζ on $L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p, n)_+$, when restricted to Z, is simply multiplication by ζ.

3.4 A description of $K_* K(\mathbb{Z}_p, n+1)$ in terms of functions on $\mathbb{Z}_p$

Let G be a profinite group and R a topological ring. Following [26], we define $R[\![G]\!]$, the *completed group ring on* G, as the inverse limit of finite group rings

$$R[\![G]\!] := \varprojlim_U R[G/U],$$

where U ranges over open subgroups of G. We recall that $R[\![G]\!]$ admits the structure of an R–Hopf algebra where each $g \in G$ is grouplike: $\Delta(g) = g \otimes g$.

One may similarly define the R–Hopf algebra $C(G, R)$ to be the ring of continuous R–valued functions on G. The coproduct is the dual of multiplication in G. Theorem 5.4 of [26] and the discussion that follows it give a proof that these two Hopf algebras are dual to each other over R,

$$R[\![G]\!] \cong \mathrm{Hom}_R(C(G, R), R).$$

Remark 3.9 The careful reader will note that in [26], Hovey studies the *twisted* completed group ring $R[\![G]\!]$ in the setting where G acts continuously on R; the multiplication in $R[\![G]\!]$ is deformed by this action. One similarly alters the coproduct (and right unit) in $C(G, R)$; the result is a Hopf *algebroid*. While of course essential to his goal of identifying $E_n^* E_n$ and its dual, for our purposes the untwisted analogues will suffice. We note in particular that the algebra structure on $C(G, R)$ (and coalgebra structure on $R[\![G]\!]$) is the same in the twisted and untwisted setting.

When $G = \mathbb{Z}_p$ and $R = \mathbb{F}_p$, a natural family of functions is given as follows: if $m = m_0 + m_1 p + m_2 p^2 + \cdots \in \mathbb{Z}_p$ with m_i taken to be either 0 or a $p - 1^{\text{st}}$ root of unity in $\mathbb{Z}_p$, define $f_k(m) = m_k \bmod p$. Clearly, $f_k^p = f_k$. In fact, the f_k form a set of generators for $C(\mathbb{Z}_p, \mathbb{F}_p)$, and so

$$C(\mathbb{Z}_p, \mathbb{F}_p) = \mathbb{F}_p[f_0, f_1, f_2, \dots]/(f_k^p - f_k).$$

See eg [39, Section 2] or [26, Section 3.3]. Lastly, if k is any finite extension of $\mathbb{F}_p$, then the natural map $C(\mathbb{Z}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k \to C(\mathbb{Z}_p, k)$ is an isomorphism (this fact is perhaps more evident in the dual Hopf algebra).

Note that $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p$, and hence on $R[\![\mathbb{Z}_p]\!]$.

Proposition 3.10 *There exists a $\mathbb{Z}_p^\times$–equivariant isomorphism of Hopf algebras*

$$\phi \colon K_*[\![\mathbb{Z}_p]\!] \to K^* K(\mathbb{Z}_p, n + 1) \cong K_*[\![y]\!],$$

which carries a topological generator h of $\mathbb{Z}_p$ to $1 + y$.

Proof The ring isomorphism ϕ is a basic fact about the Iwasawa algebra $\mathbb{Z}_p[\![\mathbb{Z}_p]\!]$ (or, in this case, its reduction modulo p, and extension over $\mathbb{F}_{p^n}$). The dual map ϕ^* satisfies

$$\phi^*(c_k)(h^m) = \langle c_k, (1 + y)^m \rangle = \binom{m}{p^k} \langle c_k, y^{p^k} \rangle = \binom{m}{p^k}.$$

If we write m in its p–adic expansion as $m = m_0 + m_1 p + m_2 p^2 + \cdots$, then by Lucas' theorem, $\binom{m}{p^k} \bmod p = m_k$. Therefore the map ϕ^* carries c_k to $f_k \in C(\mathbb{Z}_p, K_*)$. It is evidently a ring isomorphism, and so ϕ is an isomorphism of Hopf algebras.

To see that the action of $\mathbb{Z}_p^\times$ is as claimed, we first note that $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p$ through group homomorphisms; hence it acts on $K_*[\![\mathbb{Z}_p]\!]$ through ring homomorphisms. Thus it suffices to show that for $\gamma \in \mathbb{Z}_p^\times$, $\phi(\gamma \cdot h) = \gamma \cdot \phi(h)$. The left-hand side is $\phi(h^\gamma) = \phi(h)^\gamma = (1 + y)^\gamma$, whereas the right is $\gamma \cdot (1 + y) = 1 + [\gamma](y)$. That these two are equal follows from the fact that y is a coordinate on the formal multiplicative group. $\square$

We note that a more invariant description of this Hopf algebra may be given as follows. Cartier duality gives an isomorphism $\mathrm{Spec}(K_0 K(\mathbb{Z}_p, n+1)) \cong \mathrm{Hom}(F, \mathbb{G}_m)$. Further, $\mathrm{Aut}(\mathbb{G}_m) \cong \mathbb{Z}_p^{\times}$ acts by post-composition.

Proposition 3.2 shows that the formal group F associated to $K^0 K(\mathbb{Z}_p, n+1)$ is isomorphic to the multiplicative formal group $\mathbb{G}_m$. If we *choose* an isomorphism of F with $\mathbb{G}_m$, then $\mathrm{Spec}(K_0 K(\mathbb{Z}_p, n+1))$ is identified with $\mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}_p$ as a $\mathbb{Z}_p^{\times}$ space, which yields the results above. The choice of such an isomorphism corresponds to the choice of coordinate y above. Both points of view on this computation will be useful as we go forward.

3.5 Inverting roots of unity; the spectrum R_n

Define $\alpha = \beta \circ i \colon Z \to X$ as the composite of the Bockstein with the natural map $i \colon Z \to \Sigma^{\infty} K(\mathbb{Z}/p, n)_+$. Consider the localisation of X at this element in the sense of Section 2.5.

Definition 3.11 Write R_n for the E_{∞} ring spectrum $R_n := X[\alpha^{-1}]$.

Corollary 3.12 *The dual isomorphism of Proposition 3.10 localises to a ring isomorphism*
$$\phi^* \colon K_*(R_n) \to C(\mathbb{Z}_p^{\times}, K_*).$$

Proof Proposition 2.6 implies that
$$K_*(X[\alpha^{-1}]) = K_*(X)[\alpha_*^{-1}] = C(\mathbb{Z}_p, K_*)[f_0^{-1}],$$
since the image of a generator of $K_*(Z)$ under α is $b_0 = \xi f_0 u^{g(n)}$. Note that $m \in \mathbb{Z}_p$ lies in $\mathbb{Z}_p^{\times}$ if and only if $f_0(m) = m \bmod p$ is invertible. Therefore $C(\mathbb{Z}_p, K_*)[f_0^{-1}]$ may be identified as $C(\mathbb{Z}_p^{\times}, K_*)$. $\qquad\square$

We would like a $\mathbb{Z}_p^{\times}$–equivariant version of this result. Recall from Proposition 2.5 that $X[\alpha^{-1}]$ may be presented as the Bousfield localisation $X[\alpha^{-1}] \simeq L_{X[\alpha^{-1}]} X$. For each $k \in \mathbb{Z}_p$, the Bousfield localisation of $\psi^k \colon X \to X$ yields a map
$$\psi^k \colon X[\alpha^{-1}] \to X[\alpha^{-1}].$$

Lemma 3.13 *For each $k \in \mathbb{Z}_p^{\times}$, the map $\psi^k \colon X[\alpha^{-1}] \to X[\alpha^{-1}]$ is an E_{∞} map. Its induced map in $K_* X[\alpha^{-1}] = C(\mathbb{Z}_p^{\times}, K_*)$ can be identified with translation of functions by k. Finally, $\psi^k \circ \psi^{\ell} \simeq \psi^{k\ell}$.*

Proof The last claim is simply that Bousfield localisation is functorial up to homotopy. The first claim follows immediately from the fact that the localisation of ψ^k is the

lifted arrow

$$
\begin{array}{ccc}
L_{X[\alpha^{-1}]}X & \xdashrightarrow{\ \psi^k\ } & L_{X[\alpha^{-1}]}X \\
\uparrow & & \uparrow \\
X & \xrightarrow{\ \psi^k\ } & X
\end{array}
$$

provided by [14, Theorem VIII.2.2]. The second claim follows from Proposition 3.10, since that is how ψ^k acts on $K_*(X) = C(\mathbb{Z}_p, K_*)$. $\square$

A caveat is in order: the maps ψ^k defined on $X[\alpha^{-1}]$ in the above fashion make sense for every $k \in \mathbb{Z}_p$, not just $\mathbb{Z}_p^\times$. However, for any $k \in p\mathbb{Z}_p$, the second statement of the above proposition cannot possibly hold, since translation by k carries $\mathbb{Z}_p^\times$ off itself. In fact, the only possible localisation of ψ^k making the above diagram commute is zero in K_*. However, ψ^p is essential on X, as we will see in the next result. Define $j\colon X \to R_n$ to be the natural map from a ring spectrum to its localisation.

Proposition 3.14 *The map $j\colon X \to R_n$ splits; in fact, $j \vee \psi^p\colon X \to R_n \vee X$ is an equivalence.*

Proof Recall that ψ^k acts on $K_*X = C(\mathbb{Z}_p, K_*)$, via multiplication by k on $\mathbb{Z}_p$. Consequently,

$$
(j \vee \psi^p)_*\colon K_*X = C(\mathbb{Z}_p, K_*) \to C(\mathbb{Z}_p^\times, K_*) \oplus C(\mathbb{Z}_p, K_*) = K_*(R_n \vee X)
$$

is induced by the continuous bijection $\mathbb{Z}_p^\times \sqcup \mathbb{Z}_p \to \mathbb{Z}_p$, which is the natural inclusion on $\mathbb{Z}_p^\times$ and multiplication by p on $\mathbb{Z}_p$ (ie $\mathbb{Z}_p = \mathbb{Z}_p^\times \sqcup p\mathbb{Z}_p$). Since both spaces are compact Hausdorff, this is a homeomorphism, so it induces an isomorphism on $C(-, K_*)$. Hence $j \vee \psi^p$ is a $K(n)$–local equivalence. $\square$

In light of Snaith's theorem, for $n = 1$ this says that $KU_p^\wedge$ is a split summand of $L_{K(1)}\Sigma^\infty \mathbb{CP}_+^\infty$. This can alternatively be proven as follows. In [42], Segal proves that BU is a summand of $Q\mathbb{CP}^\infty$ (although not as infinite loop spaces). Apply the Bousfield–Kuhn Φ_1–functor to the splitting map $BU \to Q\mathbb{CP}^\infty$ to get a one-sided inverse $L_{K(1)}KU \to L_{K(1)}\Sigma^\infty \mathbb{CP}_+^\infty$ to j.

3.6 Invertible spectra as homotopy fibres

Proposition 3.15 *Let $\gamma \in \pi_0(R_n)^\times$. The homotopy fibre F_γ of $(\psi^g - \gamma)\colon R_n \to R_n$ is an invertible spectrum. When $\gamma = 1$, F_1 is equivalent to S.*

Proof It is not hard to see that $\psi^g - \gamma$ is surjective in K_*. First note that we may regard the K_*–Hurewicz image of γ as a continuous function $\gamma_* \in K_0(R_n) = C(\mathbb{Z}_p^\times, \mathbb{F}_p)$.

If $\phi\colon \mathbb{Z}_p^\times \to \mathbb{F}_p$ is any continuous map, define $f(g^m) \in \mathbb{F}_p$ for $m \in \mathbb{N}$ inductively by the formulae $f(1) = 1$ and $f(g^{m+1}) = \gamma_*(g^m) f(g^m) + \phi(g^m)$. The function f extends uniquely from the dense subspace $g^{\mathbb{N}}$ to $\mathbb{Z}_p^\times$, since the function f so defined is (uniformly) continuous on $g^{\mathbb{N}} \le \mathbb{Z}_p^\times$. Then, since

$$[\psi^g - \gamma](f)(g^m) = f(g^{m+1}) - \gamma_*(g^m) f(g^m) = \phi(g^m),$$

the equality $(\psi^g - \gamma)_*(f) = \phi$ holds on a dense subset; since both are continuous, they must be equal.

Further, the kernel of $(\psi^g - \gamma)_*$ consists of those functions $f\colon \mathbb{Z}_p^\times \to K_*$ which satisfy $f(gx) = \gamma f(x)$. As such a function is determined by its value on 1, this has rank one over K_*. Thus $K_*(F_\gamma) = \ker((\psi^g - \gamma)_*)$ is rank one over K_*, and hence invertible.

When $\gamma = 1$, the kernel consists of those functions on $\mathbb{Z}_p^\times$ which are invariant under translation by g, namely the constant functions. As in [24, Lemma 2.5], the unit of the ring spectrum R_n — induced by the basepoint inclusion in $K(\mathbb{Z}_p, n+1)$ — lifts to F_1, and carries $K_*(S)$ onto these functions, yielding the desired isomorphism in K_*. $\square$

One should think of the second statement in this proposition as saying that S is the homotopy fixed point spectrum for the action of $\mathbb{Z}_p^\times$ on R_n given in Lemma 3.13.

Definition 3.16 Denote by G the invertible spectrum F_g associated to $g \in \mathbb{Z}_p^\times \le \pi_0(R_n)^\times$.

We will show below that $G = F_g$ is in fact the spectrum $S\langle\det\rangle$ of the introduction. With Theorem 1.2 in hand, this is largely an exercise in notation; until that result is proven, we must apologise for the multiplicity of different symbols for the same object.

Proposition 3.17 *The assignment $\gamma \mapsto F_\gamma$ defines a homomorphism*

$$\pi_0(R_n)^\times \to \mathrm{Pic}_n.$$

Proof We must show that $F_{\gamma\sigma} \simeq F_\gamma \otimes F_\sigma$. Now ψ^g is a map of ring spectra, so this diagram homotopy commutes:

Since $\mu \circ (\gamma \otimes \sigma) \circ i = \gamma\sigma \circ i$, we see that $\mu \circ i$ lifts to $F_{\gamma\sigma}$; this map is easily seen to be an isomorphism in K_*.   □

Corollary 3.18 *The canonical map* $\delta\colon G \to X[\alpha^{-1}]$ *from the homotopy fibre* $G = F_g$ *lifts to a map* $\rho\colon G \to X$:

$$
\begin{array}{ccc}
& X \xrightarrow{\psi^g - g} X \\
{\scriptstyle\rho}\nearrow & \downarrow{\scriptstyle j} & \downarrow{\scriptstyle j} \\
G \xrightarrow{\;\delta\;} X[\alpha^{-1}] \xrightarrow{\psi^g - g} X[\alpha^{-1}]
\end{array}
$$

Furthermore, there are equivalences of ring spectra

$$
X[\rho^{-1}] \xrightarrow{\;\simeq\;} X[\alpha^{-1}][\delta^{-1}] \xleftarrow{\;\simeq\;} X[\alpha^{-1}].
$$

Proof We note that Proposition 3.14 immediately implies the existence of the lift of δ to X.

The right equivalence is the localisation map at δ. By the proof of Proposition 3.15, the image of a generator for G under δ is $f_0(x) = x \bmod p$. As this is clearly invertible in $C(\mathbb{Z}_p^\times, K_*) = K_*(X[\alpha^{-1}])$, m_δ is an equivalence, and thus the directed system defining $X[\alpha^{-1}][\delta^{-1}]$ is constant.

The left equivalence is induced by the map $X \to X[\alpha^{-1}]$ after localisation at ρ (which maps to localisation at δ, since $j\rho = \delta$). By the same argument as above, the image of a generator for G under ρ is a function $f\colon \mathbb{Z}_p \to K_*$ which restricts to f_0 along the inclusion $\mathbb{Z}_p^\times \subseteq \mathbb{Z}_p$. Replacing ρ with $m_\alpha^{p-1} \cdot \rho$ if necessary, we may assume that f vanishes on $\mathbb{Z}_p \backslash \mathbb{Z}_p^\times$, and so $f = f_0$. Thus the localisation map $X[\rho^{-1}] \to X[\alpha^{-1}][\delta^{-1}]$ is a K_*–isomorphism.   □

We note that since $\mathrm{im}(\rho_*)$ is generated by the class[7] f_0, we may conclude that $\dim(G) = \dim(f_0) = 2g(n)$. We additionally record the following for later use.

Proposition 3.19 *The spectrum Z is homotopy equivalent to F_ζ, via a map making the following diagram (whose bottom row is a fibre sequence) commute:*

$$
\begin{array}{ccc}
Z \xrightarrow{\;\alpha\;} X \xrightarrow{\psi^g - \zeta} X \\
\vdots \qquad \downarrow{\scriptstyle j} \qquad \downarrow{\scriptstyle j} \\
F_\zeta \xrightarrow{\quad} X[\alpha^{-1}] \xrightarrow{\psi^g - \zeta} X[\alpha^{-1}].
\end{array}
$$

---

[7]This class is detected by the primitive element $y \in K^*X$. We thank Mike Hopkins, Jacob Lurie, and Eric Peterson for pointing out that this observation may be promoted to the claim that G can be constructed as $\Sigma \operatorname{Cotor}_X(S, S)$; here X is a coalgebra spectrum, and $\operatorname{Cotor}_X(S, S)$ is the associated (reduced) cobar construction. See [37] for details on this point of view and an algebro-geometric interpretation.

Proof Consider the composite

$$L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p,n)_+ \xrightarrow{\pi} L_{K(n)}\Sigma^\infty K(\mathbb{Z}/p,n)_+ \xrightarrow{\beta} X \xrightarrow{\psi^g} X.$$

Using the fact that $\psi^g \circ \beta = \beta \circ \psi^\zeta$, we have

$$\psi^g \circ \beta \circ \pi = \beta \circ \psi^\zeta \circ \left(\frac{1}{p-1} \sum_{k=0}^{p-1} \zeta^{-k} (\psi^\zeta)^k \right) = \beta \circ \zeta \cdot \pi = \zeta \cdot \beta \circ \pi.$$

Thus $(\psi^g - \zeta) \circ j \circ \alpha = 0$, giving rise to the dashed arrow. The fact that it is a K_*–isomorphism (and hence equivalence) follows from inspection of the images of fundamental classes in $K_*(X[\alpha^{-1}])$; they are both generated by f_0. $\qquad\square$

3.7 Splitting R_n

There is an analogue for R_n of the splitting of p–adic K–theory into a wedge of $p-1$ Adams summands.

Proposition 3.20 *There is an equivalence*

$$\bigvee_{k=0}^{p-2} G^{\otimes k} \otimes R_n^{h\mu_{p-1}} \to R_n.$$

Proof There is a natural forgetful map $R_n^{h\mu_{p-1}} \to R_n$. Additionally, one may produce a map $\bigvee_{k=0}^{p-2} G^{\otimes k} \to R_n$ by wedging together powers of δ; the product of these maps gives the desired equivalence.

To see that this map is an isomorphism in K_* (and hence an equivalence), we note that since

$$K_*(R_n) = C(\mathbb{Z}_p^\times, \mathbb{F}_{p^n}) = \mathbb{F}_{p^n}[f_0, f_1, f_2, \dots]/(f_0^{p-1} - 1, f_k^p - f_k),$$

then $K_* R_n^{h\mu_{p-1}}$ is the subspace generated by monomials in the f_i whose total degree is a multiple of $p-1$, since $\psi^\zeta(f_k) = \zeta f_k$. The whole space is a free module over this subalgebra, generated by the classes $\{1, f_0, f_0^2, \dots, f_0^{p-2}\}$, which are the images of $G^{\otimes k}$ under δ^k. $\qquad\square$

3.8 The Morava module of R_n

Proposition 3.21 *There are ring isomorphisms*

$$C(\mathbb{Z}_p, E_{n*}) \to E_n^{\vee}{}_*(X) \quad \text{and} \quad C(\mathbb{Z}_p^\times, E_{n*}) \to E_n^{\vee}{}_*(R_n).$$

In this description, the action of $\mathbb{G}_n$ on both of these Morava modules is induced by $\det_\pm \colon \mathbb{G}_n \to \mathbb{Z}_p^\times$ and the natural action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$.

Proof The proof of the first fact is based on the related computation of $E_n^\vee{}_* E_n$ given in [26, Section 2]. We recall that $K_n^0(X)$ supports a formal group F isomorphic to $\mathbb{G}_m$ (Proposition 3.2), and note that $E_n^0(X)$ supports a deformation $\bar{F}$ of it to E_n^0. The universal deformation of $\mathbb{G}_m$ over $K_n^0 = \mathbb{F}_{p^n}$ is the multiplicative group $\mathbb{G}_m$ over $\mathbb{W}(\mathbb{F}_{p^n})$, and so there is a ring map $\phi\colon \mathbb{W}(\mathbb{F}_{p^n}) \to E_n^0$ and a $*$–isomorphism $f\colon \bar{F} \to \phi^*(\mathbb{G}_m)$. Since the formula for multiplication in $\mathbb{G}_m$ involves only the ring elements 0 and 1, we see that $\phi^*(\mathbb{G}_m) = \mathbb{G}_m$.

Thus $\bar{F}$ is isomorphic to the multiplicative group. Consequently, there is a coordinate $\bar{y} \in E_n^*(X)$ which reduces modulo $\mathfrak{m}$ to $y \in K^*(X)$, and presents $\bar{F}$ as the multiplicative group. Therefore, the action of the Adams operations ψ^k on X gives a continuous, exponential map

$$a\colon \mathbb{Z}_p \to E_n^*(X) \quad \text{by} \quad k \mapsto \psi^k(1+\bar{y}) = 1 + [k](\bar{y}) = (1+\bar{y})^k.$$

This extends linearly over E_{n*} to give a continuous ring homomorphism

$$a\colon E_{n*}[\mathbb{Z}_p] \to E_n^*(X).$$

Writing $\mu\colon X \otimes X \to X$ for the multiplication in X, we have that μ^* is the coproduct on $E_n^*(X)$. Then since $\psi^k\colon X \to X$ is a map of ring spectra and $\bar{y}$ is a coordinate on the multiplicative group,

$$\mu^*(\psi^k(1+\bar{y})) = (\psi^k \otimes \psi^k)(\mu^*(1+\bar{y}))$$
$$= (\psi^k \otimes \psi^k)(1 \otimes 1 + \bar{y} \otimes 1 + 1 \otimes \bar{y} + \bar{y} \otimes \bar{y})$$
$$= \psi^k(1+\bar{y}) \otimes \psi^k(1+\bar{y}).$$

Therefore a is a map of Hopf algebras. Applying the functor $\mathrm{Hom}_{E_{n*}}^c(-, E_{n*})$ of continuous E_{n*}–module homomorphisms into E_{n*} yields a Hopf algebra map

$$a^*\colon \mathrm{Hom}_{E_{n*}}^c(E_n^*(X), E_{n*}) \to \mathrm{Hom}_{E_{n*}}^c(E_{n*}[\mathbb{Z}_p], E_{n*}) \cong C(\mathbb{Z}_p, E_{n*}).$$

The identification of the codomain uses the fact that $E_{n*}[\mathbb{Z}_p]$ is free. Similarly, the domain is

$$\mathrm{Hom}_{E_{n*}}^c(E_n^*(X), \varprojlim_I (E_n/I)_*) = \varprojlim_I \mathrm{Hom}((E_n/I)^*(X), (E_n/I)_*)$$
$$= \varprojlim_I (E_n/I)_*(X)$$
$$= E_n^\vee{}_*(X),$$

where the limit is taken over the ideals $I = (p^{i_0}, u_1^{i_1}, \ldots, u_{n-1}^{i_{n-1}})$. We know that $E_n^\vee{}_*(X)$ is pro-free, concentrated in even dimensions, with reduction modulo $\mathfrak{m}$

isomorphic to $K_*(X) = C(\mathbb{Z}_p, \mathbb{F}_{p^n}[u^{\pm 1}])$. The ring $C(\mathbb{Z}_p, E_{n*})$ is also pro-free, and $a^*\colon E_n^\vee(X) \to C(\mathbb{Z}_p, E_{n*})$ reduces modulo $\mathfrak{m}$ to the isomorphism

$$\phi^*\colon (E_n/\mathfrak{m})_*(X) \to C(\mathbb{Z}_p, \mathbb{F}_{p^n}[u^{\pm 1}])$$

of [Proposition 3.10](#); it is then an isomorphism itself.

The same localisation technique as in the proof of [Corollary 3.12](#) yields the corresponding result for R_n. Specifically, the image of the fundamental class of Z under α is a function $B_0\colon \mathbb{Z}_p \to E_{n*}$ whose reduction modulo $\mathfrak{m}$ is $b_0 = \xi f_0 u^{g(n)}$. As in the residue field, an element $m \in \mathbb{Z}_p$ is invertible if and only if $B_0(m)$ is.

To see that the $\mathbb{G}_n$ action is as claimed, we employ Peterson's adaptation [36] of Ravenel and Wilson's results [40] to identify the formal spectrum $\mathrm{Spf}\, E_n^* K(\mathbb{Z}_p, n+1)$ in terms of the nth exterior power of the p–divisible group associated to E_n. Concretely, we recall that the action of $\mathbb{S}_n$ on $E_{n\,*}^\vee(K(\mathbb{Z}_p, 2)) \cong E_{n\,*}^\vee(K(\mu_{p^\infty}, 1))$ is via the defining action of $\mathcal{O}_n$ on Γ_n. Furthermore the Hopf ring circle product satisfies $a \circ b = -b \circ a$. Therefore, the action of $\mathbb{S}_n$ on classes in $E_{n\,*}^\vee(K(\mathbb{Z}_p, n+1)) \cong E_{n\,*}^\vee(K(\mu_{p^\infty}, n))$ lying in the image of the iterated circle product

$$E_{n\,*}^\vee(K(\mu_{p^\infty}, 1)^{\times n}) \to E_{n\,*}^\vee(K(\mu_{p^\infty}, n))$$

is via the nth exterior power of the defining action. All classes in $K_*(K(\mu_{p^\infty}, n)))$ may be obtained this way, and the above analysis lifts this statement to $E_{n\,*}^\vee$. Therefore the action of $\mathbb{S}_n$ must be via det. Examining the action of elements of $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, we note that if σ is the Frobenius homomorphism, then

$$\sigma \cdot y = \sigma \cdot (\xi x u^{g(n)}) = \xi^p x u^{g(n)} = \xi^{p-1} y = (-1)^{n-1} y.$$

Here, $x u^{g(n)}$ is fixed by the action of Frobenius, since it descends to a class in $K(n)^*(X)$. This yields the result. $\qquad\square$

Theorem 3.22 *The space of E_∞ maps $R_n \to E_n$ is an infinite loop space with contractible components, the set of which is isomorphic to $\mathbb{Z}_p^\times$. Furthermore, the action of $\mathbb{G}_n \simeq \mathrm{Aut}_{E_\infty}(E_n)$ on this set is via the homomorphism $\det_\pm\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$.*

Proof The proof is essentially the same as the main technical result of [41], so we will be brief. The Goerss–Hopkins–Miller obstruction machinery shows that the higher homotopy groups of $\mathrm{Map}_{E_\infty}(R_n, E_n)$ vanish if the cotangent complex for the map $\mathbb{F}_{p^n} \to (E_n^\vee)_0(R_n)/\mathfrak{m}$ is contractible. But the latter is

$$(E_n^\vee)_0(R_n)/\mathfrak{m} \cong C(\mathbb{Z}_p^\times, \mathbb{F}_{p^n}) \cong \mathbb{F}_{p^n}[f_0, f_1, f_2, \dots]/(f_0^{p-1} - 1, f_k^p - f_k).$$

The cotangent complex is contractible because the Frobenius on this ring is evidently an isomorphism.

The set of components of $\mathrm{Map}_{E_\infty}(R_n, E_n)$ is then the set of continuous E_{n*}–algebra homomorphisms $\mathrm{Hom}^c_{E_{n*}-\mathrm{alg}}(E_{n\;*}^\vee R_n, E_{n*})$, which may be identified with $\mathbb{Z}_p^\times = \mathbb{G}_n/S\mathbb{G}_n^\pm$ by the previous result. $\qquad\square$

Phrased in more invariant terms: $E_{n\;*}^\vee R_n$ may be identified as $C(\mathrm{Isom}(F, \mathbb{G}_m), E_{n*})$, in a variant on the comments after the proof of Proposition 3.10. Then the set of components of the E_∞ mapping space is $\mathrm{Isom}(F, \mathbb{G}_m)$. This is evidently a torsor for the group $\mathbb{Z}_p^\times = \mathrm{Aut}(\mathbb{G}_m)$.

Reduction of a chosen generator of the group of such maps $R_n \to E_n$ modulo $\mathfrak{m}$ gives a natural transformation from R_n to K which will be useful in the following sections.

Proposition 3.23 *The cohomology class $1 + y = \varphi \mod \mathfrak{m} \in K^*(X) = [X, K]$ extends over R_n to give a map of A_∞–ring spectra $t\colon R_n \to K$.*

Proof It was shown in [41] that the homotopy class $1 + y$ contains an A_∞ representative using Hochschild cohomology methods. Thus it suffices to show that $(1 + y) \circ \alpha\colon Z \to K$ is invertible. This follows, since the image of the fundamental class of Z under α is b_0, and $\langle 1 + y, b_0 \rangle = \xi$. $\qquad\square$

3.9 R_n as a homotopy fixed point spectrum

We now lift the results of Theorem 3.22 to an equivalence between R_n and $E_n^{hS\mathbb{G}_n^\pm}$. Some setup is required to study such homotopy fixed point spectra. Let H be a closed subgroup of $\mathbb{G}_n$, and let A be a $K(n)$–local E_∞ ring spectrum, with the property that its Morava module is isomorphic to the ring of continuous functions on $\mathbb{G}_n/H$,

$$E_{n\;*}^\vee A \cong C(\mathbb{G}_n/H, E_{n*}).$$

Let $K \leq \mathbb{G}_n$ be another closed subgroup. A consequence of [13] is that the homotopy fixed point spectrum E_n^{hK} admits a model which is an E_∞–ring spectrum. When $K = U$ is open, Devinatz and Hopkins construct a fibrant cosimplicial E_∞–algebra whose totalisation is the spectrum E_n^{hU}. This cosimplicial spectrum is a rectification of an h_∞ cosimplicial E_n–Adams resolution of E_n^{hU}, were it to exist (see also [7]). When K is closed, the homotopy fixed point spectrum is constructed as a colimit over open subgroups U of $\mathbb{G}_n$ containing K: that is, $E_n^{hK} = L_{K(n)}(\mathrm{colim}_U E_n^{hU})$.

Following their lead, but using their result that for any closed K, E_n^{hK} *does* exist, we will consider an actual cosimplicial $K(n)$–local E_n–based Adams resolution of E_n^{hK}. Define

$$B_K^s = E_n^{\otimes s+1} \otimes E_n^{hK}, \quad s \geq -1.$$

Taking $s \geq 0$, one may equip $B_K^\bullet$ with the structure of a cosimplicial spectrum in a familiar fashion, inserting units for coface maps, and applying multiplication for codegeneracies. We may freely replace $B_K^\bullet$ with a fibrant cosimplicial spectrum without changing the homotopy type of the terms B_K^s. The natural coaugmentation $E_n^{hK} = B_K^{-1} \to \mathrm{Tot}(B_K^\bullet)$ is a weak equivalence, as can be seen from the associated Adams spectral sequence, and [13, Theorem 2.(ii)]. Lastly $\mathrm{Tot}(B_K^\bullet)$, being the totalisation of a cosimplicial E_∞ algebra, is itself E_∞; the coaugmentation is an equivalence of E_∞ algebras.

Theorem 3.24 *Let K be a closed subgroup of $\mathbb{G}_n$. The space of maps of E_∞ ring spectra from A to E_n^{hK} has contractible components. Furthermore, there is a bijection from $\pi_0(\mathrm{Map}_{E_\infty}(A, E_n^{hK}))$ to the set*

$$\{xH \in \mathbb{G}_n/H \text{ which conjugate } K \text{ into } H\} \subseteq \mathbb{G}_n/H.$$

Lastly, if K is conjugate to H, each of these maps is an equivalence.

Proof The Morava module of B_K^s may be identified as the ring

$$C(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), E_{n*}).$$

By the same sort of argument as in the previous section, we see that the cotangent complex for the reduction of the degree 0 part of this algebra by $\mathfrak{m}$ (that is, the complex $C(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), \mathbb{F}_{p^n})$) is contractible. Thus the space $\mathrm{Map}_{E_\infty}(A, B_K^s)$ of maps of E_∞ ring spectra from A to B_K^s has contractible components which are in bijection with the set $\mathrm{Map}(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n}$ of continuous maps, equivariant for the action of $\mathbb{G}_n$ (diagonal on the domain):

$$\pi_0 \mathrm{Map}_{E_\infty}(A, B_K^s) \cong \mathrm{Hom}_{E_{n*}-\mathrm{alg}/E_n^\vee {}_* E_n}^c (E_n^\vee {}_*(A), E_n^\vee {}_*(B_K^s))$$
$$= \mathrm{Map}^\delta(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n}.$$

Here, we are equipping this function space with the discrete topology; this is the meaning of the superscript δ. The discrete topology on $\mathrm{Map}(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n}$ is *not* homeomorphic to the natural (compactly generated compact-open) one, which we will denote with a superscript co. However, since $\mathbb{G}_n/H$ is totally disconnected, we note that for all s, the evaluation at any point in Δ^s gives a bijection

$$(*) \qquad \mathrm{Map}(\Delta^s, \mathrm{Map}^{\mathrm{co}}(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n})$$
$$= \mathrm{Map}^{\mathrm{co}}(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n}.$$

Additionally, since Map^δ is equipped with the discrete topology, the same evaluation yields a bijection

$$(**) \quad \mathrm{Map}(\Delta^s, \mathrm{Map}^\delta(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n})$$
$$= \mathrm{Map}^\delta(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n}.$$

We note that $\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K)$ is the sth term of a simplicial $\mathbb{G}_n$–space, the product $E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K = \mathbb{G}_n^{\times \bullet+1} \times \mathbb{G}_n/K$. Therefore $\mathrm{Map}^{\mathrm{co}}(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n}$ is the sth term of the cosimplicial space of $\mathbb{G}_n$–equivariant, continuous functions

$$\mathrm{Map}^{\mathrm{co}}(E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}.$$

Being a function object from a simplicial space, this is a fibrant cosimplicial space.

In contrast, $\mathrm{Map}^\delta(\mathbb{G}_n^{\times s+1} \times (\mathbb{G}_n/K), (\mathbb{G}_n/H))^{\mathbb{G}_n}$ is the sth term of the discretisation of this cosimplicial space, $\mathrm{Map}^\delta(E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}$. It is also fibrant, being a cosimplicial *set*. The identity is a continuous map

$$\mathrm{Tot}(\mathrm{Map}^\delta(E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}) \to \mathrm{Tot}(\mathrm{Map}^{\mathrm{co}}(E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}).$$

We have no expectation that this map is a homeomorphism. However, equations $(*)$ and $(**)$ imply that it *is* a bijection. In fact, this totalisation can be computed as

$$\mathrm{Tot}(\mathrm{Map}^{\mathrm{co}}(E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}) \cong \mathrm{Map}^{\mathrm{co}}(E\mathbb{G}_n \times (\mathbb{G}_n/K), \mathbb{G}_n/H)^{\mathbb{G}_n}$$
$$\cong \mathrm{Map}^{\mathrm{co}}(\mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}.$$

Here $\cong$ indicates homeomorphism. The last homeomorphism is induced by restriction to the basepoint in $E\mathbb{G}_n$; it is a homeomorphism since $E\mathbb{G}_n$ is connected and $\mathbb{G}_n/H$ is totally disconnected. Note that an element $f \in \mathrm{Map}^{\mathrm{co}}(\mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}$ is specified by the element $xH = f(K) \in \mathbb{G}_n/H$. It must be the case that for each $k \in K$,

$$xH = f(K) = f(kK) = kf(K) = kxH,$$

and so $K \subseteq xHx^{-1}$. Thus this set is empty if K is not subconjugate to H, and in fact the set of all such maps f, being determined by x, is given precisely by these cosets:

$$\mathrm{Tot}(\mathrm{Map}^\delta(E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n}) = \{xH \in \mathbb{G}_n/H \text{ which conjugate } K \text{ into } H\}.$$

Now, since $E_n^{hK} \simeq \mathrm{Tot}(B_K^\bullet)$ is (equivalent to) the totalisation of a cosimplicial E_∞–algebra, we have

$$\mathrm{Map}_{E_\infty}(A, E_n^{hK}) \simeq \mathrm{Tot}(\mathrm{Map}_{E_\infty}(A, B_K^\bullet)).$$

Taking $B_K^\bullet$ to be a fibrant cosimplicial E_∞–algebra, $\mathrm{Map}_{E_\infty}(A, B_K^\bullet)$ is a fibrant cosimplicial space. Now, the projection to the set of components,

$$\mathrm{Map}_{E_\infty}(A, B_K^\bullet) \to \pi_0 \mathrm{Map}_{E_\infty}(A, B_K^\bullet) = \mathrm{Map}^\delta(E\mathbb{G}_{n\bullet} \times \mathbb{G}_n/K, \mathbb{G}_n/H)^{\mathbb{G}_n},$$

is a levelwise equivalence of fibrant cosimplicial spaces; the fibrancy of domain and codomain yields an equivalence of totalisations. Therefore $\mathrm{Map}_{E_\infty}(A, E_n^{hK})$ is empty if K is not subconjugate H and has components in bijection with the set of $xH \in \mathbb{G}_n/H$ which conjugate K into H.

Finally, if $K = xHx^{-1}$, the Morava modules of both A and E_n^{hK} are both isomorphic to $C(\mathbb{G}_n/K, E_{n*})$, so the element of the E_∞ mapping space corresponding to multiplication by x implements an isomorphism of Morava modules, and hence an equivalence. $\qquad\square$

Corollary 3.25 *The space of weak equivalence of E_∞ ring spectra $R_n \to E_n^{hS\mathbb{G}_n^\pm}$ has contractible components, the set of which is a torsor for $\mathbb{Z}_p^\times$, where $k \in \mathbb{Z}_p^\times$ acts by post-composition with the Adams operation ψ^k.*

Recall that $\mathbb{Z}_p^\times$ acts up to homotopy on R_n by way of a strict action on

$$L_{K(n)} \Sigma^\infty K(\mathbb{Z}_p, n+1)_+.$$

Any equivalence e constructed in the corollary is equivariant (up to homotopy) for the $\mathbb{Z}_p^\times$–action. This can be seen as follows. For any $k \in \mathbb{Z}_p^\times$, the action of ψ^k on R_n is an E_∞–map, via Lemma 3.13. Comparing the results of Theorem 3.24 (with $H = S\mathbb{G}_n^\pm$) for $K = \{1\}$ and $S\mathbb{G}_n^\pm$, we see that the forgetful map

$$\mathrm{Map}_{E_\infty}(R_n, E_n^{hS\mathbb{G}_n^\pm}) \to \mathrm{Map}_{E_\infty}(R_n, E_n)$$

is a homotopy equivalence (and both spaces have contractible components in bijection with $\mathbb{Z}_p^\times$). So to see that $\psi^k \circ e \simeq e \circ \psi^k$, it suffices to show that their composite with the forgetful map into E_n are homotopic. But this is follows immediately from Proposition 3.21.

3.10 A variant on R_n when n is even

Consider the homomorphism det: $\mathbb{G}_n \to \mathbb{Z}_p^\times$ which is given by the determinant on $\mathbb{S}_n$, and is trivial on $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Let $S\mathbb{G}_n = \ker(\det)$. This group differs slightly from $S\mathbb{G}_n^\pm$ when n is even; we will assume this throughout this section. Let $H = S\mathbb{G}_n \cap S\mathbb{G}_n^\pm$; this is index 2 in both of these groups and normal in $\mathbb{G}_n$. Further, $S\mathbb{G}_n/H$ is generated by $F \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, subject to the relation $F^2 = 1$.

Proposition 3.26 *The spectrum E_n^{hH} splits as the wedge $E_n^{hS\mathbb{G}_n} \vee E_n^{hS\mathbb{G}_n^\pm}$.*

Proof Since p is odd, we define an idempotent $q\colon E_n^{hH} \to E_n^{hH}$ as in Proposition 3.7 by $q = \frac{1}{2}(1 - F)$, and define a spectrum Q which represents the image of this functor inside of E_n^{hH}; then E_n^{hH} splits as a wedge $Q \vee Q^{\perp}$. Note that $Q^{\perp}$ represents the image of the complementary idempotent $1 - q = \frac{1}{2}(1 + F)$. Further, one can see that the composite of the forgetful map and projection

$$E_n^{hS\mathbb{G}_n} \to E_n^{hH} \to Q^{\perp}$$

is an equivalence by examining the induced maps in K_*: $K_*(E_n^{hH}) \cong C(\mathbb{G}_n/H, K_*)$, and the (injective) image of $K_*(E_n^{hS\mathbb{G}_n})$ consists of those functions which factor through $\mathbb{G}_n/S\mathbb{G}_n$. This is precisely

$$K_*(Q^{\perp}) = \ker(q_*) = \{f\colon \mathbb{G}_n/H \to K_* \mid f(Fg) = f(g)\}.$$

A similar argument shows that $Q \simeq E_n^{hS\mathbb{G}_n^{\pm}}$. $\qquad\qquad\square$

Recall the primitive $(2p-2)^{\text{nd}}$ root of unity ξ, which we will regard as an element of $\pi_0 E_n$. Note that it is invariant under $H \leq \mathbb{G}_n$, the relevant fact being that $F(\xi) = -\xi$, so ξ is fixed by

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cap H = \langle F^2 | F^n \rangle.$$

Thus, multiplication by ξ defines an automorphism $\xi\colon E_n^{hH} \to E_n^{hH}$.

Note that both $E_n^{hS\mathbb{G}_n}$ and $E_n^{hS\mathbb{G}_n^{\pm}}$ admit residual actions of $\mathbb{Z}_p^{\times}$ as quotients of $\mathbb{G}_n$.

Proposition 3.27 *The map* $\xi\colon E_n^{hH} \to E_n^{hH}$ *restricts to equivalences*

$$\xi\colon E_n^{hS\mathbb{G}_n} \to E_n^{hS\mathbb{G}_n^{\pm}} \quad \text{and} \quad \xi\colon E_n^{hS\mathbb{G}_n^{\pm}} \to E_n^{hS\mathbb{G}_n}.$$

These are not maps of ring spectra, although they are $\mathbb{Z}_p^{\times}$–equivariant, up to homotopy.

Proof We note that $q \circ \xi = \frac{1}{2}(\xi + \xi F) = \xi \circ (1 - q)$. That is, the automorphism ξ interchanges the two complementary idempotents, and hence the wedge factors Q and $Q^{\perp}$.

There are two natural homomorphisms $\mathbb{G}_n/H \to \mathbb{Z}_p^{\times}$ given by $\det$ and $\det_{\pm}$, respectively. Indeed, $\det$ and the projection onto the Galois factor give an isomorphism

$$\mathbb{G}_n/H \xrightarrow{\det \times \mathrm{proj}} \mathbb{Z}_p^{\times} \times \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)/(H \cap \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)) = \mathbb{Z}_p^{\times} \times \langle F | F^2 \rangle.$$

With respect to this splitting, $\det$ is the projection onto the first factor, and $\det_{\pm}$ is that projection, convolved with the sign character of the second factor.

Let $g \in \mathbb{Z}_p^\times$ be a topological generator and let $h \in \mathbb{G}_n/H$ be the element with $\det(h) = g$ and $\operatorname{proj}(h) = 1$; then also $\det_\pm(h) = g$. Denote by χ^h the action of h on E_n^{hH}, and denote by φ^g the action of g on $E_n^{hS\mathbb{G}_n}$. Then this diagram commutes up to homotopy:

$$
\begin{array}{ccccc}
E_n^{hS\mathbb{G}_n} & \longrightarrow & E_n^{hH} & \longleftarrow & E_n^{hS\mathbb{G}_n^\pm} \\
\varphi^g \downarrow & & \chi^h \downarrow & & \downarrow \psi^g \\
E_n^{hS\mathbb{G}_n} & \longrightarrow & E_n^{hH} & \longleftarrow & E_n^{hS\mathbb{G}_n^\pm}
\end{array}
$$

Since $\operatorname{proj}(h) = 1$, ξ commutes with χ^h. Since ξ permutes the two wedge factors Q and $Q^\perp$, it therefore throws ψ^g onto φ^g, and vice versa. $\qquad\square$

3.11 Gross–Hopkins duality

The previous sections imply that there is a homotopy commutative diagram

$$
\begin{array}{ccccc}
R_n & \xrightarrow{\simeq} & E_n^{hS\mathbb{G}_n^\pm} & \xrightarrow[\xi]{\simeq} & E_n^{hS\mathbb{G}_n} \\
\psi^g - g \downarrow & & \psi^g - g \downarrow & & \downarrow \varphi^g - g \\
R_n & \xrightarrow{\simeq} & E_n^{hS\mathbb{G}_n^\pm} & \xrightarrow[\xi]{\simeq} & E_n^{hS\mathbb{G}_n}
\end{array}
$$

where the action of $\psi^g \in \mathbb{Z}_p^\times$ on $E_n^{hS\mathbb{G}_n^\pm}$ is from the residual $\mathbb{G}_n/S\mathbb{G}_n^\pm$ Morava stabiliser action (and the second square is only required when n is even). Consequently the homotopy fibres of each column are equivalent. In [19], the homotopy fibre of the rightmost arrow was named $S\langle\det\rangle$, and its Morava module was shown to be $E_{n*}^\vee(S\langle\det\rangle) = E_{n*}\langle\det\rangle$.

In [23; 45] it was shown that this is the same Morava module as for $\Sigma^{n-n^2} I_n$, where I_n is the Brown–Comenetz dual of $M_n S^0$, the nth monochromatic layer of the sphere spectrum. It was also shown that there and in [24] that for $2p - 2 \geq \max\{n^2, 2n+2\}$, an invertible spectrum is determined by its Morava module. We conclude:

Corollary 3.28 *There is an equivalence $G \simeq S\langle\det\rangle$. When $2p-2 \geq \max\{n^2, 2n+2\}$, these may also be identified with $\Sigma^{n-n^2} I_n$.*

4 Thom spectra and characteristic classes

4.1 Orientations and cannibalistic classes for R_n

We briefly review the theory of [30; 3] for orientations of Thom spectra, especially from the point of view of [31]. Recall that for an E_∞ ring spectrum R, $\mathrm{gl}_1 R$ is the

spectrum whose infinite loop space is $\mathrm{GL}_1 R$, the space of units in $\Omega^\infty R$. Further, gl_1 is functorial for maps of E_∞–ring spectra.

Let $\eta\colon S \to R_n$ be the unit of R_n, and define $b(S, R_n)$ to be the cofibre of

$$\mathrm{gl}_1 \,\eta\colon \mathrm{gl}_1 S \to \mathrm{gl}_1 R_n.$$

Write $B(S, R_n) = \Omega^\infty b(S, R_n)$; then there is an equivalence

$$B(S, R_n) \simeq B(\ast, \mathrm{GL}_1 S, \mathrm{GL}_1 R_n)$$

between $B(S, R_n)$ and the bar construction for the action of $\mathrm{GL}_1 S$ on $\mathrm{GL}_1 R_n$ via $\mathrm{GL}_1(\eta)$. The fibre sequence of infinite loop spaces associated to this construction is of the form

$$\mathrm{GL}_1 S \xrightarrow{\;\eta\;} \mathrm{GL}_1 R_n \longrightarrow B(S, R_n) \xrightarrow{\;\gamma\;} B\,\mathrm{GL}_1 S.$$

We recall that (homotopy classes of) maps of spaces $\zeta\colon Y \to B\,\mathrm{GL}_1 S$ define Thom spectra with "fibre" S; for such a map ζ, the Thom spectrum Y^ζ is defined as

$$Y^\zeta = \Sigma^\infty P_+ \otimes_{\Sigma^\infty \mathrm{GL}_1 S_+} S,$$

where P is the homotopy fibre of ζ. Furthermore (see [31, Section 3]), the set of lifts of ζ over γ to elements of $[Y, B(S, R_n)]$ are in bijection with R_n–orientations of ζ, that is, Thom classes $u\colon Y^\zeta \to R$.

Consider a space Y and a based map $f\colon Y \to B(S, R_n)$; write $\zeta = \gamma \circ f$. By the above, f consists of the data of an orientation $u\colon Y^\zeta \to R_n$. Let $A \in \mathrm{Pic}_n$, and note that a Thom class for $A\zeta$ is given by

$$1 \otimes u\colon A \otimes Y^\zeta = Y^{A\zeta} \to A \otimes R_n.$$

If we insist that A is a power of G, we may employ the periodicity of R_n to make the following definition.

Definition 4.1 The *normalised* Thom class $\bar{u} = \bar{u}(G^m \zeta) := \delta^m u \in R^0(Y)$ is defined as the composite

$$Y^{G^m \zeta} = G^m \otimes Y^\zeta \xrightarrow{1 \otimes u} G^m \otimes R_n \xrightarrow{\delta^m \otimes 1} R_n \otimes R_n \xrightarrow{\;\mu\;} R_n.$$

Then for any $k \in \mathbb{Z}_p^\times$, define $\theta_k(G^m \zeta) \in R_n^0(Y)$ by the equation

$$\psi^k(\bar{u}) = \theta_k(G^m \zeta) \cdot \bar{u}.$$

Following Adams, we call $\theta_k(G^m \zeta)$ the (Bott) *cannibalistic class* of $G^m \zeta$.

Proposition 4.2 Let ζ and ξ be Thom spectra associated to classes $f, h\colon Y \to B(S, R_n)$. Then:

(1) $\theta_k(0) = 1$

(2) $\theta_k(\zeta + \xi) = \theta_k(\zeta) \cdot \theta_k(\xi)$

(3) $\theta_k(G^m\zeta) = k^m\theta_k(\zeta)$

(4) $\theta_{kl}(\zeta) = \psi^k(\theta_l(\zeta)) \cdot \theta_k(\zeta)$

Proof The proof that θ_k is exponential depends upon the fact that a product of Thom classes for ζ and ξ is a Thom class for $\zeta + \xi$. Note that this implies that $\theta_k(\zeta)^{-1}$ exists, and is equal to $\theta_k(-\zeta)$. The third property uses the fact that $\psi^k(\delta^m \otimes u) = k^m\delta^m \otimes \psi^k(u)$. The last is an application of the equation $\psi^{kl} = \psi^k\psi^l$. □

We note that this proposition implies that $\theta_k(\zeta)$ is invertible; $\theta_k(\zeta) \in R_n^0(Y)^\times$. Further, the last part of this proposition allows us to regard the collection $\{\theta_k(\zeta) : k \in \mathbb{Z}_p^\times\}$ of all of the cannibalistic classes of ζ as a 1–cocycle $\theta(\zeta)\colon \mathbb{Z}_p^\times \to R_n^0(Y)^\times$, where the action of $\mathbb{Z}_p^\times$ on $R_n^0(Y)^\times$ is through Adams operations.

Now for all k, $\psi^k\colon R_n \to R_n$ is a ring map, so there is a well-defined map

$$\mathrm{gl}_1\, \psi^k\colon \mathrm{gl}_1\, R_n \to \mathrm{gl}_1\, R_n.$$

Write $\psi^k/1\colon \mathrm{gl}_1\, R_n \to \mathrm{gl}_1\, R_n$ for the difference (using the *multiplicative* infinite loop space structure on $\mathrm{GL}_1\, R_n$) between this map and the identity. Since $\psi^k \circ \eta = \eta$, we see that $(\psi^k/1) \circ \mathrm{gl}_1\, \eta$ is nullhomotopic. So there is a natural map $c(\psi^k)\colon b(S, R_n) \to \mathrm{gl}_1\, R_n$ making this diagram commute:

$$
\begin{array}{ccccc}
\mathrm{gl}_1\, S & \xrightarrow{\;\mathrm{gl}_1\, \eta\;} & \mathrm{gl}_1\, R_n \longrightarrow b(S, R_n) & \xrightarrow{\;\gamma\;} & \Sigma\, \mathrm{gl}_1\, S \\
& & \searrow \raisebox{0.5em}{$\scriptstyle \psi^k/1$} \qquad \Big\downarrow {\scriptstyle c(\psi^k)} & & \\
& & \mathrm{gl}_1\, R_n & &
\end{array}
$$

Then the cannibalistic classes may be computed via the operation $c(\psi^k)$:

$$\theta_k(\zeta) = c(\psi^k) \circ f$$

(see eg [31, Proposition 3.5]).

4.2 A model for $B(S, R_n)$

Let r denote a topological generator for the p–adic units $\mathbb{Z}_p^\times$. At times it will be useful to take r to be $g = \zeta(1 + p)$, where ζ is a primitive $(p-1)^{\text{st}}$ root of unity. At

others it will be beneficial to choose $r \in \mathbb{N}$; eg any positive integer whose reduction modulo p^2 generates $(\mathbb{Z}/p^2)^\times$. We recall from Proposition 3.15 that for $r = g$ there is a cofibre sequence

$$S \xrightarrow{\eta} R_n \xrightarrow{\psi^r - 1} R_n.$$

This may be shown to be a fibre sequence for other generators r using the fact that $\mathbb{Z}_p^\times$ is topologically cyclic. Alternatively, this follows from the $\mathbb{Z}_p^\times$–equivariant identification of Corollary 3.25,

$$R_n \simeq E_n^{hS\mathbb{G}_n^\pm}.$$

Proposition 4.3 *There is an equivalence between*

$$\Omega^\infty S = \Omega^\infty \operatorname{hofib}(\psi^r - 1 \colon R_n \to R_n)$$

and the homotopy equaliser of the maps ψ^r and $1 \colon \Omega^\infty R_n \to \Omega^\infty R_n$,

$$\operatorname{hoeq}((\psi^r, 1) \colon \Omega^\infty R_n \to \Omega^\infty R_n).$$

Similarly, there is an equivalence between $\operatorname{hofib}(\psi^r/1 \colon \mathrm{GL}_1 R_n \to \mathrm{GL}_1 R_n)$ *and the homotopy equaliser of the maps ψ^r and $1 \colon \mathrm{GL}_1 R_n \to \mathrm{GL}_1 R_n$,*

$$\operatorname{hoeq}((\psi^r, 1) \colon \mathrm{GL}_1 R_n \to \mathrm{GL}_1 R_n).$$

Proof In the first case, an element of the homotopy fibre consists of a pair (x, f), where $x \in \Omega^\infty R_n$, and f is a path in $\Omega^\infty R_n$, starting at the basepoint (the additive unit 0), and ending at $\psi^r(x) - x$. An element of the homotopy equaliser is a path h in $\Omega^\infty R_n$, starting at a point x, and ending at $\psi^r(x)$. An equivalence between these spaces is gotten by sending (x, f) to the path h gotten from f by pointwise addition of x.

The proof of the second claim is the same as for the first, replacing addition with multiplication and 0 with 1. $\qquad\square$

Proposition 4.4 $\mathrm{GL}_1 S$ *is the union of the components of* $\Omega^\infty S$ *lying over* $\mathrm{GL}_1 R_n$.

Proof Equivalently, we must show that for every $x \in \pi_0(S)$, if its image $\eta_*(x) \in \pi_0(R_n)$ is invertible, then so too is x. Let $y = \eta_*(x)^{-1}$; since ψ^r is a ring homomorphism,

$$1 = \psi^r(1) = \psi^r(\eta_*(x))\psi^r(y) = \eta_*(x)\psi^r(y)$$

and so $\psi^r(y) = y$. Thus y lifts to a class $y' \in \pi_0(S)$ with $\eta_*(y') = y$, and so $xy' = 1 + w$, where $w \in \ker(\eta_*) = \operatorname{im}[\partial_* \colon \pi_1 R_n \to \pi_0(S)]$. We note that $w^2 = 0$: w is represented by an element in $E_2^{1,1}$ of the homotopy fixed point spectral sequence

$H^*(\mathbb{Z}_p^\times, \pi_*(R_n)) \implies \pi_*(S)$, and so its square is represented by a class in $E_2^{2,2}$ (modulo terms of *higher* filtration). This is 0, since the homological dimension of $\mathbb{Z}_p^\times$ is 1. Therefore $y'(1-w)$ is a multiplicative inverse for x. $\qquad\square$

Corollary 4.5 *For any topological generator r, the infinite loop map*

$$c(\psi^r)\colon B(S, R_n) \to \mathrm{GL}_1\, R_n$$

is an equivalence on connected components.

Proof The previous two propositions allow us to conclude that there is an equivalence $\mathrm{GL}_1\, S \simeq \mathrm{hofib}(\psi^r/1\colon \mathrm{GL}_1\, R_n \to \mathrm{GL}_1\, R_n)$. This allows us to identify the lower fibre sequence in the following commuting diagram of fibrations:

$$
\begin{array}{ccccccc}
\Omega B(S, R_n) & \xrightarrow{\;\gamma\;} & \mathrm{GL}_1\, S & \xrightarrow{\;\eta\;} & \mathrm{GL}_1\, R_n & \longrightarrow & B(S, R_n) \\
\Big\downarrow{\scriptstyle c(\psi^r)} & & \Big\| & & \Big\| & & \Big\downarrow{\scriptstyle c(\psi^r)} \\
\Omega\, \mathrm{GL}_1\, R_n & \xrightarrow[\;\gamma\;]{} & \mathrm{GL}_1\, S & \xrightarrow[\;\eta\;]{} & \mathrm{GL}_1\, R_n & \xrightarrow[\psi^r/1]{} & \mathrm{GL}_1\, R_n
\end{array}
$$

The leftmost column is an equivalence, since both spaces are homotopy fibres of the same map; this gives the result. $\qquad\square$

Perhaps surprisingly, this allows us to conclude that an R_n–orientation of a Thom spectrum Y^ζ over a connected space Y is uniquely determined by its rth cannibalistic class $\theta_r(\zeta)$ (or equivalently, the cocycle $\theta(\zeta)$). More concretely, examining the exact sequences gotten from mapping a connected Y into to the fibrations above, we have:

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & [Y, \mathrm{GL}_1(R_n)] & \longrightarrow & [Y, B(S, R_n)] & \longrightarrow & [Y, B\,\mathrm{GL}_1(S)] & \longrightarrow \cdots \\
& \Big\| & & {\scriptstyle\theta_r}\Big\downarrow{\scriptstyle\cong} & & \Big\| & \\
\cdots \longrightarrow & R_n^0(Y)^\times & \xrightarrow{\;\psi^r/1\;} & R_n^0(Y)^\times & \longrightarrow & [Y, B\,\mathrm{GL}_1(S)] & \longrightarrow \cdots
\end{array}
$$

Corollary 4.6 *An R_n–oriented Thom spectrum Y^ζ over connected Y is trivial (ie equivalent to $L_{K(n)}\Sigma^\infty Y_+$) if and only if $\theta_r(\zeta) = \psi^r(s)/s$ for some unit $s \in R_n^0(Y)^\times$.*

For $n = 1$, this appears to be a strong form of a $K(1)$–local version of [1, Corollary 5.8]. Further, it says that Y^ζ is trivial if and only if the cocycle $\theta(\zeta)$ is a coboundary, so the map

$$[Y, B(S, R_n)] \to H^1(\mathbb{Z}_p^\times, R_n^0(Y)^\times) \quad \text{given by} \quad \zeta \mapsto \theta(\zeta)$$

is injective. This is perhaps a form of descent for Picard groups in the $\mathbb{Z}_p^\times$–Galois extension $S \to R_n$.

4.3 A Thom spectrum over $K(\mathbb{Z}_p, n+1)$

The following proposition, which is immediate from the connectivity of Y, is an analogue of the fact that a formal power series is invertible if and only if its constant term is.

Proposition 4.7 Let R be an A_∞ ring spectrum. If Y is connected and $\alpha \in R^0(Y) = [Y, \Omega^\infty R]$, then α lies in $R^0(Y)^\times = [Y, \mathrm{GL}_1 R]$ if and only if the restriction of α to a point in Y is a unit, $\alpha|_{\mathrm{pt}} \in R_0^\times$.

We will denote by j the natural localisation map $j \colon X \to R_n = X[\rho^{-1}]$. We will often confuse j with the corresponding element $j \in R_n^0(K(\mathbb{Z}_p, n+1)) \cong R_n^0(X)$. Since $j|_{\mathrm{pt}} = 1$ is the unit of R_n, the previous criterion shows that $j \in R_n^0(X)^\times$, while $j - 1$ is not a unit. Consequently, the following result is not automatic.

Proposition 4.8 Let $r \in \mathbb{N}$ be a topological generator of $\mathbb{Z}_p^\times$. The element $\psi^r(j-1) \in R_n^0(X)$ is divisible by $j-1$, and the quotient $\psi^r(j-1)/(j-1)$ is a unit.

Proof It follows from the proof of Proposition 3.4 that $\psi^r(j)$ is simply j composed with the self map of $K(\mathbb{Z}_p, n+1)$ which is multiplication by r in π_{n+1}. Furthermore, this is implemented by the r–fold diagonal, followed by r–ary multiplication in $K(\mathbb{Z}_p, n+1)$. Since j is multiplicative, we conclude that $\psi^r(j) = j^r$. Thus $\psi^r(j-1) = j^r - 1$ is evidently divisible by $j-1$, with quotient

$$\frac{\psi^r(j-1)}{j-1} = 1 + j + \cdots + j^{r-1}.$$

Restricting j along the unit yields 1; thus the restriction of $\psi^r(j-1)/(j-1)$ along the unit gives $r \in R_0^\times$, so this class is invertible. $\qquad\square$

Definition 4.9 Let $e \colon K(\mathbb{Z}_p, n+1) \to BX_n = B(S, R_n)_{>0}$ be the unique map satisfying

$$c(\psi^r) \circ e = r^{-1} \frac{\psi^r(j-1)}{j-1} \in R_n^0(X)^\times.$$

Here we write $Y_{>m}$ for the m–connected cover of Y. Pulling γ back over e allows us to define a Thom spectrum $K(\mathbb{Z}_p, n+1)^{\gamma \circ e}$ which, for brevity, we will write as X^γ. Then e is defined in such a way that the rth cannibalistic class of γ is $\theta_r(\gamma) = r^{-1} \psi^r(j-1)/(j-1)$.

In fact, this definition of e is independent of our choice of generator r. An induction using part (4) of Proposition 4.2 along with the continuity of the action of the Adams operations gives:

Proposition 4.10 For every $k \in \mathbb{Z}_p^{\times}$,

$$\theta_k(\gamma) = k^{-1} \frac{\psi^k(j-1)}{j-1}.$$

Above, the Thom spectrum X^{γ} has been normalised to have Thom class in dimension 0; classically, this corresponds to $(\mathbb{C}P^{\infty})^{L-1}$, where L is the tautological line bundle. We will devote a lot of attention to the Thom spectrum $X^{G\gamma} = G \otimes X^{\gamma}$, the analogue in our setting of $(\mathbb{CP}^{\infty})^{L} = S^2 \wedge (\mathbb{CP}^{\infty})^{L-1}$. We note then that the cannibalistic class $\theta_g(G\gamma)$ satisfies

$$(j-1) \cdot \theta_g(G\gamma) = \psi^g(j-1).$$

4.4 A monochromatic *J*–homomorphism and an analogue of *MU*

We would like to extend the map $K(\mathbb{Z}_p, n+1) \to B(S, R_n)$ over the inclusion $K(\mathbb{Z}_p, n+1) \to \Omega^{\infty} R_n$. This is possible on the level of spaces; however, the result is not obviously an infinite loop map. If it were, the composite of the associated spectrum maps

$$(R_n)_{>0} \to b(S, R_n) \to \Sigma \operatorname{gl}_1 S$$

could potentially be the connective cover of a map $(R_n)_{\geq 0} \to \operatorname{pic}(S)$ which provides a higher-chromatic analogue of the *J*–homomorphism. Classically (when $n=1$), this is the map which assigns to a vector space V the invertible spectrum $S^V = \Sigma^{\infty}(V \cup \{\infty\})$. The possibility that such a map exists is hinted at by Proposition 3.17 above, which in some sense constructs it in π_0.

In this section, we construct such an infinite loop map; however, we do so at the price of working monochromatically at level n. Let L_n denote localisation with respect to $E(n)$; we recall that the nth monochromatic component of a spectrum Y is fibre $M_n Y$ of the localisation map $L_n Y \to L_{n-1} Y$. Furthermore, $L_n L_{K(n)} Y = L_{K(n)} Y$. From this fact and the chromatic fracture square for Y, it is easy to see that $M_n Y \simeq M_n L_{K(n)} Y$. In particular, $M_n \Sigma^{\infty} K(\mathbb{Z}_p, n+1)_+ \simeq M_n X$.

Let A be a $K(n)$–local E_{∞} ring spectrum. We recall from [4, Theorem 4.11] that the homotopy groups of the homotopy fibre F of the localisation map $\operatorname{gl}_1 A \to L_n \operatorname{gl}_1 A$ (the "discrepancy spectrum") are torsion and concentrated in dimensions $q \leq n$. The p–completion of this fibre sequence is then of the form

$$F_p^{\wedge} \to (\operatorname{gl}_1 A)_p^{\wedge} \to (L_n \operatorname{gl}_1 A)_p^{\wedge} = L_{K(1) \vee \cdots \vee K(n)} \operatorname{gl}_1 A$$

and the homotopy of $F_p^\wedge$ is now concentrated in dimensions $q \le n+1$. Furthermore, the map $\mathrm{gl}_1 A \to (\mathrm{gl}_1 A)_p^\wedge$ is an equivalence on connected covers, since the positive homotopy of $\mathrm{gl}_1 A$ is isomorphic to that of A (which is p–complete).

Take $A = R_n$, and consider the following diagram:

$$
\begin{array}{ccc}
\Sigma^\infty K(\mathbb{Z}_p, n+1)_p^\wedge & \xrightarrow{\;k_p^\wedge\;} & (\mathrm{gl}_1 R_n)_p^\wedge
\end{array}
$$
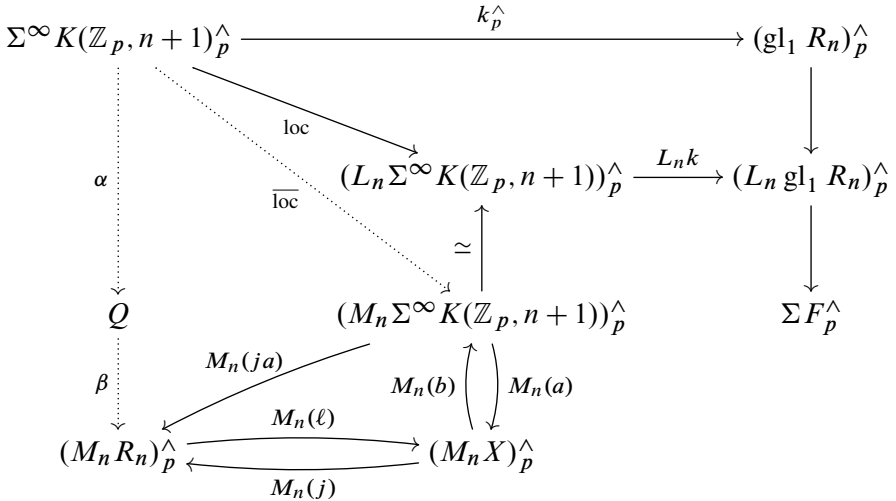
Here $k\colon \Sigma^\infty K(\mathbb{Z}_p, n+1) \to \mathrm{gl}_1 R_n$ is adjoint to the map

$$
r^{-1}\frac{\psi^r(j-1)}{j-1}\colon K(\mathbb{Z}_p, n+1) \to \mathrm{GL}_1 R_n
$$

of the previous section. It follows from [40] that

$$
L_{n-1}\Sigma^\infty K(\mathbb{Z}_p, n+1) = L_{\mathbb{Q}}\Sigma^\infty K(\mathbb{Z}_p, n+1),
$$

so their common p–completion is null. Therefore, the p–complete localisation map

$$
\mathrm{loc}\colon \Sigma^\infty K(\mathbb{Z}_p, n+1)_p^\wedge \to (L_n\Sigma^\infty K(\mathbb{Z}_p, n+1))_p^\wedge
$$

lifts to $(M_n\Sigma^\infty K(\mathbb{Z}_p, n+1))_p^\wedge$. The map $a\colon \Sigma^\infty K(\mathbb{Z}_p, n+1) \to \Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ is the inclusion of the wedge summand, and b is its one-sided inverse. Similarly, the one-sided inverse $\ell\colon R_n \to X$ to j induces a splitting of $M_n X$.

Definition 4.11 Let Q be the $(n+2)^{\mathrm{nd}}$ Moore–Postnikov factorisation of the map $M_n(ja)\circ\overline{\mathrm{loc}}$.

Then Q is n–connected, $\alpha_*\colon \mathbb{Z}_p = \pi_{n+1}\Sigma^\infty K(\mathbb{Z}_p, n+1)_p^\wedge \to \pi_{n+1}Q$ is an isomorphism, and α and β induce an isomorphism

$$
\pi_{n+2}Q \cong \mathrm{im}\,[\pi_{n+2}\Sigma^\infty K(\mathbb{Z}_p, n+1)_p^\wedge \to \pi_{n+2}(M_n R_n)_p^\wedge].
$$

Further, $\beta\colon Q \to M_n R_n$ is an isomorphism in homotopy in degrees greater than $n+2$. As such, we regard Q as a stand-in for $M_n R_n$ (and hence R_n) which interpolates between it and the spectrum $\Sigma^\infty K(\mathbb{Z}_p, n+1)^\wedge_p$. In particular, since $K(n)$–localisation turns co-connected maps into equivalences, we have that

$$\beta\colon L_{K(n)} Q \to L_{K(n)} M_n R_n \simeq R_n$$

is an equivalence.

Write k' for the map $k' = L_n k \circ M_n(b\ell) \circ \beta$. Then we have the commuting diagram of cofibre sequences:

$$
\begin{array}{ccc}
\Sigma^\infty K(\mathbb{Z}_p, n+1)^\wedge_p & \xrightarrow{\;k^\wedge_p\;} & (\mathrm{gl}_1\, R_n)^\wedge_p \\
\alpha \downarrow \quad \; {\scriptstyle K} \quad \nearrow & & \downarrow \\
Q & \xrightarrow[\;k'\;]{} & (L_n\, \mathrm{gl}_1\, R_n)^\wedge_p \\
\downarrow & & \downarrow \\
\mathrm{cofib}(\alpha) & \xrightarrow{\hspace{3cm}} & \Sigma F^\wedge_p
\end{array}
$$

Proposition 4.12 *There exists a map $K\colon Q \to (\mathrm{gl}_1\, R_n)^\wedge_p$ making the above diagram commute. Further, the set of such maps is a torsor for $\pi_{n+1}(F^\wedge_p)$.*

Proof The definition of Q is such that $\mathrm{cofib}(\alpha)$ is $(n+2)$–connected. In contrast, $\Sigma F^\wedge_p$ is $(n+2)$–co-connected, so the bottom map in the diagram is necessarily null, providing the existence of a map K. Further, the set of such lifts is a torsor for the group

$$[Q, F^\wedge_p] = \mathrm{Hom}[\pi_{n+1} Q, \pi_{n+1}(F^\wedge_p)] = \mathrm{Hom}[\mathbb{Z}_p, \pi_{n+1}(F^\wedge_p)] = \pi_{n+1}(F^\wedge_p),$$

where we again employ the co-connectivity of $F^\wedge_p$. □

Note that since Q is connected and $\mathrm{gl}_1\, R_n \to (\mathrm{gl}_1\, R_n)^\wedge_p$ is an equivalence on connected covers, we may regard any choice of K above as actually mapping into $\mathrm{gl}_1\, R_n$.

Definition 4.13 For any choice of K above, we define the nth *monochromatic J–homomorphism*

$$J = J_n\colon Q \to b(S, R_n)$$

to be the unique map with $c(\psi^r) \circ J = K$.

Furthermore, let $BX = BX_n := \Omega^\infty Q$, and define MX_n as the Thom spectrum

$$MX = MX_n := L_{K(n)} BX_n^{\gamma \circ J}$$

associated to the composite $BX_n \xrightarrow{J} B(S, R_n) \xrightarrow{\gamma} B\,\mathrm{GL}_1\,S$.

It is unclear to us the extent to which the choice of the lift K affects these constructions. We will make little use of the spectrum MX_n in this paper; when we do, all our results will be insensitive to the choice of K. We notice that $\gamma \circ J$ is an infinite loop map. Using the methods of [28], we conclude:

Proposition 4.14 *The Thom spectrum MX_n is an E_∞–ring spectrum.*

Note also that by its very nature, $\gamma \circ J$ is R_n–oriented, yielding a Thom isomorphism

$$R_n^\star(MX) \cong R_n^\star BX$$

of $R_n^\star BX$–modules.

Example 4.15 The notation MX_n is meant to evoke thoughts of the complex cobordism Thom spectrum $MU = BU^\gamma$ (where γ is the tautological virtual bundle over BU). Indeed, this is precisely the case when $n = 1$ (at least $K(1)$–locally). There is an equivalence

$$R_1 = X_1[\rho^{-1}] = L_{K(1)}\Sigma^\infty \mathbb{CP}_+^\infty[\beta^{-1}] \simeq L_{K(1)} K \simeq K_p^\wedge,$$

by Snaith's theorem. Then $M_1 R_1$ is the first monochromatic component (or $\mathbb{Q}$–acyclisation) of $K_p^\wedge$; this spectrum has homotopy only in odd degrees, where it is $\mathbb{Q}_p/\mathbb{Z}_p$. The monochromatisation is undone by the p–completion: $(M_1 R_1)_p^\wedge = K_p^\wedge$. Finally, examining the homotopy of Q, we see that the resulting spectrum $Q \simeq (K_p^\wedge)_{>0}$ is equivalent to the p–completion of the connected cover of K. Thus its infinite loop space is $BX_1 = BU_p^\wedge$; and indeed the natural map of $K(1)$–local Thom spectra $L_{K(1)} MU \to MX_1$ is an equivalence by the Thom isomorphism.

In this case, there is a choice of K above so that the monochromatic J–homomorphism $J_1\colon BX_1 \to B(S, R_1)$ is precisely the p–completion of the usual J–homomorphism.

In summary, we have maps

$$K(\mathbb{Z}_p, n+1) \xrightarrow{\alpha} \Omega^\infty Q \xrightarrow{J} B(S, R_n) \xrightarrow{\gamma} B\,\mathrm{GL}_1(S),$$

where all but the first are infinite loop maps. The construction of K makes it evident that for any choice, the resulting map J yields the map e of the previous section upon composite with α. This mirrors the classical situation for $n = 1$:

$$\mathbb{CP}^\infty = BU(1) \times \{1\} \xrightarrow{\subseteq} BU \times \mathbb{Z} \xrightarrow{J} B(S, K) \xrightarrow{\gamma} B\,\mathrm{GL}_1(S^0).$$

The full composite defines the Thom spectrum of the tautological line over $\mathbb{CP}^\infty$; starting at the second term yields MU.

4.5 A zero-section map

Recall that the zero-section of the Thom spectrum $\Sigma^2 MU(1) = \Sigma^2(\mathbb{CP}^\infty)^{L-1}$ is an equivalence. The goal of this section is to give an analogous result in the current setting: a stable "zero-section" $z: X \to X^{G\gamma}$ which is a $K(n)$–local equivalence.

Definition 4.16 For an R_n–oriented Thom spectrum Y^ζ over Y, define $f_g(\zeta)$ as the composite

$$R_n \otimes Y \xrightarrow{1 \otimes \Delta} R_n \otimes Y \otimes Y \xrightarrow{\psi^g \otimes \theta_g(\zeta)^{-1} \otimes 1} R_n \otimes R_n \otimes Y \xrightarrow{\mu \otimes 1} R_n \otimes Y.$$

Proposition 4.17 *Let $Y^{G\zeta}$ be R_n–oriented with Thom class u (and associated Thom isomorphism T_u). Then the following diagram commutes:*

$$
\begin{array}{ccccc}
R_n \otimes Y^{G\zeta} & \xrightarrow{T_u} & G \otimes R_n \otimes Y & \xrightarrow{\delta \otimes 1} & R_n \otimes Y \\
{\scriptstyle \psi^g \otimes 1}\big\downarrow & & {\scriptstyle 1 \otimes f_g(\zeta)}\big\downarrow & & \big\downarrow{\scriptstyle g^{-1} f_g(\zeta)} \\
R_n \otimes Y^{G\zeta} & \xrightarrow{T_u} & G \otimes R_n \otimes Y & \xrightarrow{\delta \otimes 1} & R_n \otimes Y
\end{array}
$$

Proof It follows quickly from this definition and the fact that $\psi^g \circ \delta = g \cdot \delta$ that the right square commutes. Thus the result follows from the commutativity of the left square without the tensor factor of G. Here is an expanded version of that square, which commutes by inspection:
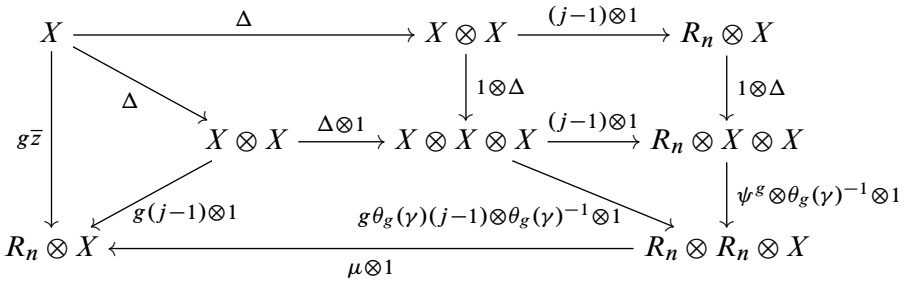
The homotopy fibre of $\psi^g - 1\colon R_n \to R_n$ is S, so we see that $\mathrm{hofib}((\psi^g - 1) \otimes 1) = Y^{G\zeta}$ in the diagram in the statement of Proposition 4.17. Since the horizontal maps are equivalences, we conclude:

Corollary 4.18 $\mathrm{hofib}(g^{-1} f_g(\zeta) - 1) = \mathrm{hofib}(f_g(\zeta) - g) = Y^{G\zeta}$.

We now focus on $X^{G\gamma}$. Define a map $\bar{z}\colon X \to R_n \otimes X$ by $\bar{z} = ((j-1) \otimes 1) \circ \Delta$. Since

$$\psi^g(j-1) = \theta_g(G\gamma) \cdot (j-1) = g\theta_g(\gamma) \cdot (j-1),$$

we see that this commutes:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ \ \Delta\ \ } & X \otimes X & \xrightarrow{(j-1)\otimes 1} & R_n \otimes X \\
& \searrow{\scriptstyle \Delta} & \downarrow{\scriptstyle 1 \otimes \Delta} & & \downarrow{\scriptstyle 1 \otimes \Delta} \\
g\bar{z} \downarrow & X \otimes X \xrightarrow{\Delta \otimes 1} & X \otimes X \otimes X & \xrightarrow{(j-1)\otimes 1} & R_n \otimes X \otimes X \\
& \searrow{\scriptstyle g(j-1)\otimes 1} & {\scriptstyle g\theta_g(\gamma)(j-1)\otimes\theta_g(\gamma)^{-1}\otimes 1} & & \downarrow{\scriptstyle \psi^g \otimes \theta_g(\gamma)^{-1}\otimes 1} \\
R_n \otimes X & \xleftarrow{\ \ \ \ \mu \otimes 1\ \ \ \ } & & & R_n \otimes R_n \otimes X
\end{array}
$$

Passage along the right side of this diagram gives $f_g(\gamma) \circ \bar{z}$, so $f_g(\gamma) \circ \bar{z} = g\bar{z}$. Thus, $\bar{z}$ lifts to a map[8]

$$z\colon X \to X^{G\gamma},$$

which we will regard as the zero-section of the Thom spectrum.

The following result encourages us to regard $j - 1$ as the Euler class of γ over X. Indeed, when $n = 1$, it is precisely the K–theoretic Euler class of the tautological line bundle over $\mathbb{CP}^\infty$. We continue to write $e^*(u) \in R_n^G(X^{G\gamma})$ for the Thom class of $X^{G\gamma}$, and $e^*(\bar{u}) \in R_n^0(X^{G\gamma})$ for its normalisation.

Proposition 4.19 $z^*(e^*(\bar{u})) = j - 1$.

Proof   We first note that one may derive $e^*(\bar{u})$ from the fibre sequence for $f_g(\gamma) - g$; that is, the following diagram commutes:

---

[8]The map z is, of course, not uniquely specified by this computation. However, any lift of $\bar{z}$ will serve for our purposes, as will be evident from the proof of Proposition 4.19.

$$
\begin{array}{c}
X^{G\gamma} \\
\end{array}
$$

Here, $p\colon X \to S$ is induced by the projection of $K(\mathbb{Z}_p, n+1)$ to a point. The commutativity of the upper right triangle follows from the diagram of fibre inclusions

since passage along the lower left defines $\delta e^*(u) = e^*(\overline{u})$. Then

$$z^*(e^*(\overline{u})) = (1 \otimes p) \circ \overline{z} = (1 \otimes p) \circ ((j-1) \otimes 1) \circ \Delta = j - 1. \qquad \square$$

Recall that $K^*(X) = K_*[\![y]\!]$, and let $t\colon R_n \to K$ be the natural transformation of Proposition 3.23. We note that $t_*(e^*(u))$ is the K–Thom class of $X^{G\gamma}$. It satisfies $z^*(t_*(e^*(u))) = y$, for

$$z^*(t_*(e^*(u))) = t \circ (z^*(e^*(u))) = t \circ (j-1) = (1+y) - 1 = y.$$

We conclude:

Proposition 4.20 *The composite of the Thom isomorphism for $X^{G\gamma}$ and z^*, as a map from $K^*(X) \cong K^{*+2g(n)}(X^{G\gamma})$ to $K^{*+2g(n)}X$ is multiplication by y.*

Note that the image of z^* may be identified as the (split) subspace $yK_*[\![y]\!] = \widetilde{K}^*(K(\mathbb{Z}_p, n+1))$. Since z^* is evidently injective, we obtain the main result of this section:

Corollary 4.21 *The zero-section restricts to a $K(n)$–local equivalence*

$$z\colon L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1) \to X^{G\gamma}.$$

5 Higher orientations for chromatic homotopy theory

5.1 *n*–orientations and formal group laws

The map $\rho\colon G \to X$ allows us to extend the notion of a complex orientation of a cohomology theory to the $K(n)$–local category.

Definition 5.1 An *n–orientation* of a $K(n)$–local ring spectrum E is a class

$$x \in E^G(X) = [X, G \otimes E]$$

with the property that $\rho^*(x) \in E^G(G) = \pi_0 E = E_0$ is a unit.

We note that since $\rho_1\colon L_{K(1)}S^2 \to X_1$ is the $K(1)$–localisation of the inclusion $\mathbb{CP}^1 \subseteq \mathbb{CP}^\infty$, a 1–orientation is precisely a $K(1)$–local complex orientation.

Examples 5.2 The following are *n*–oriented spectra:

(1) The spectrum $R_n = X[\rho^{-1}]$ is naturally *n*–oriented. Define $x \in R_n^G(X)$ by

$$x = \delta^{-1} \otimes (j-1)\colon X \to G \otimes X[\rho^{-1}].$$

Then $\rho^*(x) = \delta^{-1} \cdot \rho^*(j-1) = \delta^{-1} \cdot \delta = 1$.

(2) K is *n*–oriented, via the conveniently named class $x \in K^{2g(n)}(X) = K^G(X)$. The image of ρ in $K_*(X_n)$ is b_0, and so $\rho^*(x) = x(b_0) = 1$. More generally, any power series $f(x) \in K_*[\![x]\!]$ which begins with a unit multiple of x gives an *n*–orientation of K.

(3) E_n is *n*–oriented by a lift of the previous orientation. More carefully,

$$E_n^G(X) \cong E_n^{2g(n)}(X) = u^{g(n)} \cdot E_n^0(X) = u^{g(n)} \cdot \mathbb{W}(\mathbb{F}_{p^n})[\![u_1, \dots, u_{n-1}]\!][\![\mathbb{Z}_p]\!].$$

Then an orientation is given by the class of $u^{g(n)} \cdot g$, since the image of a fundamental class under ρ in $E_{n*}(X) \cong C(\mathbb{Z}_p, E_{n*})$ is a function which carries g to a unit.

(4) MX_n is *n*–oriented via the map $x := (\alpha^\gamma \otimes 1) \circ z$, as can be seen from the diagram

$$
\begin{array}{ccccccc}
X_n & \xrightarrow{\;z\;} & X^{G\gamma} & \xrightarrow{\;=\;} & X^\gamma \otimes G & \xrightarrow{\;\alpha^\gamma \otimes 1\;} & MX_n \otimes G \\
\big\uparrow{\scriptstyle\rho} & & & & \big\uparrow{\scriptstyle\eta \otimes 1} & & \\
G & \xleftarrow{\quad=\quad} & & & S \otimes G & &
\end{array}
$$

which commutes by the proof of Proposition 4.19. Thus $\rho^*(x) = \alpha^\gamma \circ \eta$ is the unit of the ring spectrum MX_n.

Theorem 5.3 *An n–orientation x of E gives an isomorphism $E^{\star}(X) \cong E_{\star}[\![x]\!]$, and the multiplication in X equips this ring with a formal group law $F(x, y) \in E_{\star}[\![x, y]\!]$.*

Proof We note that an n–orientation of E, $x \in E^G(X)$ defines, via the equivalence

$$z \colon L_{K(n)}\Sigma^{\infty} K(\mathbb{Z}_p, n+1) \to X^{G\gamma},$$

a Thom class $u = (z^*)^{-1}(x) \in E^G(X^{G\gamma})$, since restricting u to each fibre gives the same class as restricting x to G, namely, a unit. Furthermore, extending z to all of $X = L_{K(n)}\Sigma^{\infty} K(\mathbb{Z}_p, n+1) \vee S$, we see that the augmentation ideal of $E^{\star}X$ is $\widetilde{E}^{\star} K(\mathbb{Z}_p, n+1) = z^* E^{\star} X^{G\gamma}$. Moreover, by the Thom isomorphism, the latter is the cyclic ideal generated by $z^*u = x$.

We conclude two things: that the augmentation ideal of $E^{\star}X$ is generated by x, and that it is itself isomorphic to $E^{\star}X^{G\gamma}$, and hence $E^{\star}X$ via the Thom isomorphism. We thereby inductively observe that the quotients of the filtration by powers of the augmentation ideal are free $E^{\star}$–modules of rank 1 generated by the powers of x, and so $E^{\star}X \cong E_{\star}[\![x]\!]$.

The Künneth spectral sequence for $E^{\star}(X \otimes X)$ collapses to $E^{\star}(X_n) \otimes_{E_{\star}} E^{\star}(X_n) = E_{\star}[\![x, y]\!]$, since each factor has free $E^{\star}$–cohomology. The properties of the formal group law all derive from the unital, associative multiplication on X. □

5.2 A remark on $(n-1)$–gerbes

It is natural to ask what sort of object an n–orientation is orienting. We recall that a complex orientation of E yields a theory of Chern classes for E in which the orientation is the first Chern class of the tautological bundle. Furthermore, the formal group law of the cohomology theory encodes the Chern class of a tensor product of line bundles for the cohomology theory.

We will consider a p–adic $(n-1)$–gerbe V over a space Y to be a map $f_V \colon Y \to K(\mathbb{Z}_p, n+1)$. Here we are purposefully confusing a gerbe with its Dixmier–Douady type of characteristic class. For an n–oriented cohomology theory E, the orientation class x defines a first Chern class for V by the formula

$$c_1(V) := f_V^*(x) \in E^{S\langle \det \rangle}(Y).$$

The formal group law on $E^{\star}$ then allows one to compute

$$c_1(V \otimes W) = F(c_1(V), c_1(W)).$$

It is not apparent to the author how to extend this notion to higher rank (ie nonabelian) $(n-1)$–gerbes or higher Chern classes.

5.3 Multiplicative n–oriented spectra

Definition 5.4 An n–oriented spectrum R is *multiplicative* if there is a unit $t \in R_\star$ such that the formal group law that R supports on $R^\star(X_n)$ is given by the formula

$$F(x, y) = x + y + txy.$$

Theorem 5.5 *The spectrum R_n with its natural n–orientation is the universal (ie initial) multiplicative n–oriented spectrum.*

Proof This is argument closely follows that of Spitzweck and Østvær [44] for the motivic analogue of Snaith's theorem.

First, $X_n[\rho^{-1}]$ is multiplicative. We note that j is a map of ring spectra, as it is induced by the identity on X_n. Thus if m denotes multiplication in X_n,

$$m^*(j) = j \otimes j \in R_n^\star(X_n \otimes X_n) = R_{n\star}[\![x, y]\!],$$

since the tensor product is multiplication in the cohomology of the smash product. Definitionally, $j \otimes 1 = 1 + \delta x$, and $1 \otimes j = 1 + \delta y$, so $j \otimes j = 1 + \delta(x + y + \delta xy)$. Thus,

$$F(x, y) = m^*(x) = m^*(\delta^{-1}(j - 1)) = \delta^{-1}(j \otimes j - 1) = x + y + \delta xy.$$

Loosely, $R_n = X_n[\rho^{-1}]$ is universal because j is initial amongst maps from X_n to spectra in which ρ is invertible. More carefully, let E be an n–oriented, multiplicative spectrum, with orientation v, and whose formal group satisfies $F(v, w) = v + w + tvw$ for some $t \in E_G$. Then there is a map of ring spectra[9] $\phi\colon X_n \to E$ defined by $\phi = 1 + tv$. To check that ϕ is multiplicative, we need to see that $m^*(\phi) = \phi \otimes \phi \in E^\star(X_n \otimes X_n) = E_\star[\![x, y]\!]$. But

$$m^*(\phi) = 1 + tm^*(v) = 1 + tF(v, w) = 1 + tv + tw + t^2 vw = (1 + tv)(1 + tw) = \phi \otimes \phi.$$

Similarly, ϕ is unital, since t restricts to 0 over S^0.

We note that

$$\phi_*(\rho) = (1 + tv) \circ \rho = tv \circ \rho = t \cdot (\rho^* v)$$

is a product of units, so ϕ extends over j to a map of ring spectra

$$\Phi\colon R_n = X_n[\rho^{-1}] \to E.$$

---

[9]We are not claiming that this map is highly structured, only that it preserves multiplication and units up to homotopy.

Since E is n–oriented, the map $\delta\colon G \to X_n$ defines a function

$$\delta^*\colon E_\star[\![v]\!]_G = E^G(X_n) \to E^G(G) = E_0$$

which carries a power series in v to the coefficient of v. Therefore $\phi \circ \delta = \delta^*(\phi) = t$. We see, then, that Φ is orientation preserving (ie $\Phi_*(x) = v$), since

$$v = t^{-1}(1 + tv - 1) = \Phi_*(\delta^{-1})(\phi - 1) = \Phi_*(\delta^{-1}(j - 1)) = \Phi_*(x).$$

Let $\Psi\colon X_n[\rho^{-1}] \to E$ be any other orientation-preserving map. Since it is orientation preserving, it must preserve the formal group law, so $\Psi_*(\delta) = t$. Consider the composite map $\psi := \Psi \circ j\colon X_n \to E$. Then

$$v = \Psi_*(x) = \Psi_*(\delta^{-1}(j - 1)) = t^{-1}(\psi - 1),$$

giving $\psi = 1 + tv = \phi$, and so $\Psi = \Phi$.

Therefore, for any multiplicative, n–oriented spectrum E, there exists an orientation-preserving map $\Psi\colon X_n[\rho^{-1}] \to E$, unique up to homotopy. $\qquad\square$

5.4 Identifying the coefficients of R_n

There is an "integral lift" $\mathbb{W}(K)$ of the cohomology theory K with homotopy groups

$$\mathbb{W}(K)_* = \mathbb{W}(\mathbb{F}_{p^n})[u^{\pm 1}].$$

One may define $\mathbb{W}(K)$ as the E_n–algebra

$$\mathbb{W}(K) := E_n/(u_1, \dots, u_{n-1}).$$

Note that reduction modulo p gives a natural transformation $\mathbb{W}(K) \to K$. Furthermore, the reduction map $E_n \to \mathbb{W}(K)$, being a ring homomorphism, carries an n–orientation of E_n to an n–orientation for $\mathbb{W}(K)$.

Proposition 5.6 *The formal group law on $\mathbb{W}(K)^{G^*}(X)$ is the universal multiplicative formal group law in the category of $\mathbb{W}(\mathbb{F}_{p^n})$–algebras.*

Proof K supports the multiplicative formal group over $\mathbb{F}_{p^n}$, via Proposition 3.2; $\mathbb{W}(K)$ is its universal deformation. But the universal deformation of the multiplicative group is the multiplicative group over $\mathbb{W}(\mathbb{F}_{p^n})$, which is clearly initial in the indicated category. $\qquad\square$

Recall that the Picard-graded homotopy of E_n associated to powers of G are

$$(E_n)_{G^*} = \mathbb{W}(\mathbb{F}_{p^n})[\![u_1, \dots, u_{n-1}]\!][s^{\pm 1}],$$

where $|s| = \dim(G) = 2g(n)$, following the discussion in Example 2.2. Thus

$$\mathbb{W}(K)_{G*} = \mathbb{W}(\mathbb{F}_{p^n})[s^{\pm 1}].$$

We note that $\mathbb{Z}_p^\times$ acts on the multiplicative group $\mathbb{G}_m$, and therefore on $\mathbb{W}(K)_{G*}$, since the previous proposition implies that $\mathbb{W}(K)_{G*}$ co-represents $\mathbb{G}_m$ in the category of $\mathbb{W}(\mathbb{F}_{p^n})$–algebras. The action (through $\mathbb{W}(\mathbb{F}_{p^n})$–algebra homomorphisms) is easily seen to be determined by the formula $\gamma \cdot s = \gamma s$.

For the next result, we recall that R_n is also equipped with an action of $\mathbb{Z}_p^\times = \mathbb{G}_n / S\mathbb{G}_n^\pm$.

Corollary 5.7 *As a $\mathbb{Z}_p^\times$–representation, $\pi_{G*} R_n$ contains $\mathbb{Z}_p[\rho^{\pm 1}]$ as a split summand.*

Proof We will write σ for a lift of a primitive $(p^n - 1)^{\text{st}}$ root of unity in $\mathbb{F}_{p^n}$ to $\mathbb{W}(\mathbb{F}_{p^n})$, so that $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\sigma]$.

Since R_n is the universal multiplicative n–oriented spectrum, the multiplicativity of the orientation of $\mathbb{W}(K)$ gives us a unique oriented map $\Phi\colon R_n \to \mathbb{W}(K)$, which induces $\Phi_{G*}\colon \pi_{G*}(R_n) \to \pi_{G*}(\mathbb{W}(K))$. The last proposition gives us a unique oriented map $\Xi\colon \pi_{G*}\mathbb{W}(K) \to \pi_{G*} R_n[\sigma]$. It must then be the case that $\Phi_{G*} \circ \Xi = \mathrm{id}$ once we extend Φ_{G*} over $\mathbb{W}(\mathbb{F}_{p^n}) = \mathbb{Z}_p[\sigma]$. Thus $\pi_{G*}(R_n)[\sigma]$ contains $\mathbb{W}(\mathbb{F}_{p^n})[\Xi(s)^{\pm 1}]$ as a split summand. As in the proof of Theorem 5.5, we must have $\Xi(s) = \rho$. As both maps were equivariant by universality properties, this splitting is equivariant.

We note that $\pi_0(S)$ does not contain any of the roots of unity in $\mu_{p^n - 1}$ not lying in μ_{p-1}. If it contained one such τ, then for any $K(n)$–local ring spectrum Y, $\pi_0(Y)$ would be a $\mathbb{Z}_p[\tau]$–algebra. In particular, $\pi_0(K(n)) = \mathbb{F}_p$ would be an $\mathbb{F}_p[\tau] = \mathbb{Z}_p[\tau]/p$–vector space, which is false if $\tau \notin \mu_{p-1} = \mathbb{F}_p^\times$.

Suppose now that $\tau \in \pi_0(R_n)$. Then the action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p[\tau] \subseteq \pi_0(R_n)$ is trivial since it is trivial upon extending further to $\mathbb{Z}_p[\sigma] = \mathbb{W}(K)_0$. Examine the long exact sequence

$$\cdots \longrightarrow \pi_0(S) \xrightarrow{\eta} \pi_0(R_n) \xrightarrow{\psi^g - 1} \pi_0(R_n) \longrightarrow \cdots .$$

Then $\mathbb{Z}_p[\tau] \subseteq \ker(\psi^g - 1) \subseteq \pi_0(S)$, a contradiction.

Knowing that $\pi_{G*} R_n[\sigma]$ contains $\mathbb{W}(\mathbb{F}_{p^n})[\rho^{\pm 1}]$ as a split summand, and that $\pi_0 R_n$ does not contain σ, we see that $\pi_{G*} R_n$ contains $\mathbb{Z}_p[\rho^{\pm 1}]$ as a split summand. $\square$

5.5 R_n for large primes

Let

$$q := g^{p-1} = \zeta^{p-1}(1+p)^{p-1} = (1+p)^{p-1};$$

q is a topological generator of $(1 + p\mathbb{Z}_p)^\times$. Consequently, the equivalence $S = R_n^{h\mathbb{Z}_p^\times}$

may be factorised (following [12]) as

$$S = (R_n^{h\mu_{p-1}})^{h(1+p\mathbb{Z}_p)^\times} = \mathrm{hofib}(\psi^q - 1\colon R_n^{h\mu_{p-1}} \to R_n^{h\mu_{p-1}}).$$

A variation on a standard sparseness result for the Adams–Novikov spectral sequence (see eg [17]) yields the following.

Proposition 5.8 *If $n^2 < 2p - 3$, the long exact sequence in homotopy associated to*

$$S \xrightarrow{\ \eta\ } R_n^{h\mu_{p-1}} \xrightarrow{\ \psi^q - 1\ } R_n^{h\mu_{p-1}}$$

splits in a range of degrees, giving

$$\mathbb{Z}_p = \pi_0(S) \cong \pi_0(R_n^{h\mu_{p-1}}) \quad \text{and} \quad \pi_*(S) \cong \pi_*(R_n^{h\mu_{p-1}}) \oplus \pi_{*+1}(R_n^{h\mu_{p-1}})$$

when $-2p + 1 \leq * < 0$.

Proof Equivalently, we may show these facts for $E_n^{h\mathbb{G}_n^1} \simeq R_n^{h\mu_{p-1}}$. The homotopy of this spectrum is computed via the spectral sequence

$$H^s(\mathbb{G}_n^1, (E_n)_t) \Longrightarrow \pi_{t-s}(E_n^{h\mathbb{G}_n^1}).$$

Since p is odd, our assumption implies that $(p-1)$ and p do not divide n, and thus $\mathbb{G}_n$ is a p–adic analytic Lie group of dimension n^2 with no p–torsion; thus, its cohomological dimension is $n^2 + 1$ (see [34]). Similarly, $\mathbb{G}_n^1$ has cohomological dimension n^2. So the only contribution to π_* comes from $H^s(\mathbb{G}_n^1, (E_n)_{s+*})$, where $0 \leq s \leq n^2$.

However, one may compute this group cohomology as

$$H^s(\mathbb{G}_n^1/\mu_{p-1}, (E_n)_{s+*}^{\mu_{p-1}}) = H^s(S\mathbb{G}_n^\pm, (E_n)_{s+*}^{\mu_{p-1}}),$$

which vanishes when $s + *$ is not a multiple of $2(p-1)$. Assuming that $n^2 < 2(p-1)$ and that $-2(p-1) < * \leq 0$ ensures that $-2(p-1) < * \leq s + * \leq s \leq n^2 < 2(p-1)$, so the only possible contribution is when $s = -*$ (so $s + * = 0$). So for $* \leq 0$,

$$\pi_*(R_n^{h\mu_{p-1}}) = H^{-*}(\mathbb{G}_n^1, (E_n)_0).$$

The same analysis holds (for $n^2 + 1 < 2(p-1)$) to show that when $-2(p-1) < * \leq 0$, $\pi_*(S) = H^{-*}(\mathbb{G}_n, (E_n)_0)$.

Now, since $p \nmid n$, the reduced determinant is split, giving an isomorphism $\mathbb{G}_n \cong \mathbb{G}_n^1 \times \mathbb{Z}_p$, and the $\mathbb{Z}_p$ factor acts trivially on $(E_n)_0$ (and hence $H^*(\mathbb{G}_n^1, (E_n)_0)$). Thus

the (collapsing) Lyndon–Hochschild–Serre spectral sequence gives $H^0(\mathbb{G}_n, (E_n)_0) \cong H^0(\mathbb{G}_n^1, (E_n)_0)$, and if $* < 0$,

$$H^{-*}(\mathbb{G}_n, (E_n)_0) \cong H^1(\mathbb{Z}_p, H^{-1-*}(\mathbb{G}_n^1, (E_n)_0)) \oplus H^0(\mathbb{Z}_p, H^{-*}(\mathbb{G}_n^1, (E_n)_0))$$

$$\cong H^{-1-*}(\mathbb{G}_n^1, (E_n)_0) \oplus H^{-*}(\mathbb{G}_n^1, (E_n)_0). \qquad \square$$

This implies that for $0 \le m \le 2p - 1$,

$$1 = \psi^q \colon \pi_{-m} R_n^{h\mu_{p-1}} \to \pi_{-m} R_n^{h\mu_{p-1}},$$

since the unit map $S \to R_n^{h\mu_{p-1}}$ is surjective in π_{-m}, and fixed by ψ^q. Now, the G–periodicity of $\pi_\star(R_n)$ yields a $G^{\otimes p-1}$–periodicity of $\pi_\star(R_n^{h\mu_{p-1}})$, so

$$[G^{\otimes k(p-1)}, \Sigma^m R_n^{h\mu_{p-1}}] \cong \pi_{-m}(R_n^{h\mu_{p-1}}).$$

Corollary 5.9 *If $n^2 < 2p - 3$, and $0 \le m \le 2p - 1$, the endomorphism ψ^q of $[G^{\otimes k(p-1)}, \Sigma^m R_n^{h\mu_{p-1}}]$ is multiplication by q^k.*

Proof It suffices to observe that for $f \in \pi_{-m}(R_n^{h\mu_{p-1}})$ this commutes:

$$\Sigma^{-m} G^{\otimes k(p-1)} = G^{\otimes k(p-1)} \otimes S^{-m} \xrightarrow{1 \otimes f} G^{\otimes k(p-1)} \otimes R_n^{h\mu_{p-1}} \xrightarrow{\delta^{k(p-1)}} R_n^{h\mu_{p-1}}$$

with downward maps $g^{k(p-1)} \otimes \psi^q$ and ψ^q, and diagonal $q^k \otimes f$:

$$G^{\otimes k(p-1)} \otimes R_n^{h\mu_{p-1}} \xrightarrow{\delta^{k(p-1)}} R_n^{h\mu_{p-1}} \qquad \square$$

5.6 Analogues of the image of J

Corollary 5.10 *Let $k \in \mathbb{Z}$, and write $k = p^s m$, where m is coprime to p. Then $[G^{\otimes k(p-1)}, S^1]$ contains a subgroup isomorphic to $\mathbb{Z}/p^{s+1}$. Furthermore, if $n^2 < 2p - 3$, there is an exact sequence*

$$0 \to \mathbb{Z}/p^{s+1} \to [G^{\otimes k(p-1)}, S^1] \to N_{s+1} \to 0,$$

where $N_{s+1} \le \pi_{-1}(S)$ is the subgroup of p^{s+1}–torsion elements.

Proof Without assumptions on n, Corollary 5.7 implies that $[G^{\otimes i}, R_n]$ has a split summand $\mathbb{Z}_p$ upon which the action of ψ^g is through multiplication by g^i. Taking $i = k(p-1)$, we conclude that the action of $\psi^q = \psi^{g^{p-1}}$ on the corresponding summand $\mathbb{Z}_p \le [G^{\otimes k(p-1)}, R_n^{h\mu_{p-1}}] \cong \pi_0(R_n)$ is by multiplication by q^k.

Consider the long exact sequence obtained by applying $[G^{\otimes k(p-1)}, -]$ to the fibre sequence

$$\cdots \longrightarrow R_n^{h\mu_{p-1}} \xrightarrow{\psi^q - 1} R_n^{h\mu_{p-1}} \longrightarrow S^1 \longrightarrow \Sigma R_n^{h\mu_{p-1}} \xrightarrow{\psi^q - 1} \Sigma R_n^{h\mu_{p-1}} \longrightarrow \cdots.$$

The first map contains a factor which is given by multiplication by $q^k - 1$ on $\mathbb{Z}_p$. Since $q^k - 1$ generates the procyclic subgroup $p^{s+1}\mathbb{Z}_p \subseteq \mathbb{Z}_p$, we see that $[G^{\otimes k(p-1)}, S^1]$ contains the indicated cokernel as a subgroup.

Now, if $n^2 < 2p - 3$, the results of the previous section give us a very precise computation of this sequence:

$$\mathbb{Z}_p \xrightarrow{\psi^q - 1} \mathbb{Z}_p \longrightarrow [G^{\otimes k(p-1)}, S^1] \longrightarrow [G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}]$$
$$\xrightarrow{\psi^q - 1} [G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}].$$

Corollary 5.9 implies that $\psi^q - 1$ is again multiplication by a unit multiple of p^{s+1}. Lastly, the identification $[G^{\otimes k(p-1)}, \Sigma R_n^{h\mu_{p-1}}] \cong \pi_{-1}(R_n^{h\mu_{p-1}})$ as a summand of $\pi_{-1}(S)$ (with complement $\mathbb{Z}_p$) yields the desired short exact sequence. $\qquad\square$

6 Redshift

We will write $M(p)$ for the $K(n)$–local mod p Moore spectrum; ie the cofibre of the map $p \colon S \to S$. When $p > 2$, $M(p)$ is a ring spectrum, homotopy associative if $p > 3$; we will assume the latter throughout this section.

Proposition 6.1 *There is a map* $v \colon S\langle \det \rangle \to Z \otimes M(p)$ *which induces an isomorphism in* E_n *after multiplying by the identity on* $M(p)$:

$$v_* \colon E_{n*}(S\langle \det \rangle \otimes M(p)) \xrightarrow{\;\cong\;} E_{n*}(Z \otimes M(p)).$$

Thus v *induces an equivalence* $S\langle \det \rangle \otimes M(p) \simeq Z \otimes M(p)$.

Proof Notice that after tensoring with $M(p)$, the element $g \in \mathbb{Z}_p \subseteq [X_n, X_n]$ becomes homotopic to $\zeta \in \mathbb{F}_p \subseteq [X_n \otimes M(p), X_n \otimes M(p)]$. Thus the indicated lift in the following diagram exists:

$$
\begin{array}{ccc}
S\langle \det \rangle & \xdashrightarrow{\;v\;} & Z \otimes M(p) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \alpha \otimes 1} \\
R_n & \xrightarrow{\;1 \otimes \eta\;} & R_n \otimes M(p) \\
\downarrow{\scriptstyle \psi^g - g} & & \downarrow{\scriptstyle (\psi^g - \zeta) \otimes 1} \\
R_n & \xrightarrow{\;1 \otimes \eta\;} & R_n \otimes M(p)
\end{array}
$$

To see that v_* is an isomorphism, we note that the image in

$$E_{n*}(R_n \otimes M(p)) = C(\mathbb{Z}_p, E_{n*}/p) = C(\mathbb{Z}_p, \mathbb{F}_{p^n}[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}])$$

of $\alpha \otimes 1$ and $(1 \otimes \eta) \circ \delta$ are both generated by the function

$$f_0 \colon \mathbb{Z}_p \to \mathbb{F}_{p^n}[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}])$$

given by $f_0(x) = x \bmod p$. □

This fact allows us to multiply any element of the Picard-graded homotopy of $Y \otimes M(p)$ with v, for any spectrum Y.

Proposition 6.2 *The map $\beta_n^{p-1} \colon S\langle\det\rangle^{\otimes p-1} \to L_{K(n)}K(E_{n-1})$ is, after tensoring with $M(p)$, homotopic to multiplication of the unit $\eta \colon S \to L_{K(n)}K(E_{n-1})$ by v^{p-1}.*

Proof The statement of the proposition amounts to the claim that the following diagram commutes:

$$
\begin{array}{ccccc}
S\langle\det\rangle^{\otimes p-1} & \xrightarrow{\ \rho_n^{p-1}\ } & X_n & \xrightarrow{\ i \circ B\varphi_{n-1}\ } & L_{K(n)}K(E_{n-1}) \\
\Big\downarrow{\scriptstyle v^{\otimes p-1}} & & \Big\downarrow{\scriptstyle 1 \otimes \eta} & & \Big\downarrow{\scriptstyle 1 \otimes \eta} \\
S \otimes M(p) & \xrightarrow{\ \eta \otimes 1\ } & X_n \otimes M(p) & \xrightarrow{\ i \circ B\varphi_{n-1} \otimes 1\ } & L_{K(n)}K(E_{n-1}) \otimes M(p)
\end{array}
$$

since the lower composite is the unit of $L_{K(n)}K(E_{n-1})$ tensored with the identity on $M(p)$. The first square commutes by construction, and the second is evident. □

Corollary 6.3 *Multiplication by $\beta_n \in \pi_{S\langle\det\rangle}L_{K(n)}K(E_{n-1})$ is an equivalence.*

Proof Since $v^{p-1} \colon S\langle\det\rangle^{\otimes p-1} \otimes M(p) \to Z^{\otimes p-1} \otimes M(p) = M(p)$ is an equivalence by Proposition 6.1, smashing it with the identity of $L_{K(n)}K(E_{n-1})$ is also an equivalence. By the previous proposition, then multiplication by β_n^{p-1},

$$[\beta_n^{p-1}] = [v^{p-1}] \colon (E_n)_\star(K(E_{n-1}) \otimes M(p)) \to (E_n)_{\star + S\langle\det\rangle^{\otimes p-1}}(K(E_{n-1}) \otimes M(p))$$

is an isomorphism. However, this implies that β_n^{p-1} is an isomorphism in K_* and hence an equivalence, since K_n is a module spectrum for $E_n \otimes M(p)$, and hence in the same Bousfield class. □

One may go slightly further. One of the chromatic redshift conjectures of [6] (specifically Conjecture 4.4) proposes an equivalence

$$L_{K(n)}K(\Omega_{n-1}) \simeq E_n,$$

where Ω_{n-1} is a suitable interpretation of the algebraic closure of the fraction field of E_{n-1}. In particular, this should restrict to a unital E_∞ map

$$r_n: L_{K(n)}K(E_{n-1}) \to E_n.$$

When $n = 1$, this is a map $L_{K(1)}K(\mathbb{Q}_p) \to KU_p^\wedge$ corresponding to a choice of embedding $\mathbb{Q}_p \to \mathbb{C}$. Let us presume only the existence of such a map r_n. The previous results indicate the commutativity of the solid part of the following diagram:

$$
\begin{array}{ccccc}
L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+ & \xrightarrow{B\varphi_{n-1}} & L_{K(n)}\Sigma^\infty B\,\mathrm{GL}_1(E_{n-1})_+ & \xrightarrow{i} & L_{K(n)}K(E_{n-1}) \\
\downarrow & & & & \vdots\ r_n \\
R_n & \xrightarrow{\simeq} & E_n^{hS\mathbb{G}_n^\pm} & \longrightarrow & E_n
\end{array}
$$

Now, the path along the lower left is the map φ_n. The path along the upper right (including the conjectural redshift map r_n) is an E_∞ map, and thus by [Theorem 3.22](), differs from φ_n by at worst an Adams operation. So up to a suitable re-embedding of $E_n^{hS\mathbb{G}_n^\pm}$ into E_n, the diagram must commute if r_n is to exist. This provides a falsifiable condition upon any candidate redshift map r_n: it must give a factorisation of the homotopy fixed point inclusion $E_n^{hS\mathbb{G}_n^\pm} \to E_n$.

7 Questions

This section is purely speculative, and is intended as a collection of vaguely posed questions which the reader is invited to either clarify or (more likely) disprove.

Perhaps the most evident thing lacking in this paper is a geometric definition of the cohomology theory R_n. Despite all of the analogies with K–theory given above, we do not have a description of $R_n^*(X)$ akin to the Grothendieck group of (some generalisation of) vector bundles over X. The discussion of orientations on p–adic $(n-1)$–gerbes indicates a possible direction with this question, but it is not apparent to the author what the corresponding analogue of higher rank vector bundles should be (although the work of Michael Murray and others on bundle gerbe modules [35; 10] may point a way forward).

A related deficiency is the fact that we have described a $K(n)$–local analogue of the *image* of the J–homomorphism, but not precisely the homomorphism itself. We do have a monochromatic formulation of the homomorphism, but it would be ideal to extend this to a geometric construction defined on R_n, not the Moore–Postnikov factorisation Q. Classically, one can think of the J–homomorphism as built from

either the functions $O(m) \to \Omega^m S^m$, or (in families, after delooping) the function that assigns to a vector bundle the associated sphere bundle. Neither of these has an obvious analogue in our construction. Such a description would be enlightening. In connection with the first question this raises the obvious question: does an n–bundle gerbe have a geometrically defined "sphere" bundle, where $S\langle \det \rangle$ is our replacement for the spherical fibre?

This, in turn, raises the question of whether there is a good unstable description of the map $\rho_n \colon S\langle \det \rangle \to L_{K(n)} \Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ analogous to the inclusion $\mathbb{CP}^1 \to \mathbb{CP}^\infty$. It follows from [9] that $S\langle \det \rangle$ is the $K(n)$–localisation of a suspension spectrum, but a more geometric description of this spectrum and map is desirable.

Further, the definition of an n–oriented spectrum raises the question of whether it is possible to repeat the whole program of analysing complex-oriented spectra by their formal group laws, but in the $K(n)$–local category, with $K(\mathbb{Z}_p, n+1)$ replacing $\mathbb{CP}^\infty$. All of our examples, however, have resulted in essentially multiplicative formal groups. One standout is the spectrum MX_n for which we haven't even the beginnings of a computation of the associated formal group when $n > 1$. We would like to believe that it plays a universal role analogous to MU in the complex-oriented case, but do not have any evidence to back this up.

Lastly, we have no examples in hand of n–oriented spectra whose associated formal group law is *additive* (ie the analogue of singular homology). Does such a theory exist? If so, and if it could be made part of an Atiyah–Hirzebruch spectral sequence for $K(n)$–local theories, the rather convoluted construction of the formal group law in Theorem 5.3 (requiring the entirety of section 4) could be replaced with a direct analogue of the argument familiar to most homotopy theorists for the proof that $h^*(\mathbb{CP}^\infty) \cong h_*[\![x]\!]$ for complex-oriented theories h. However, when $n = 1$, this argument requires singular cohomology, a non-$K(1)$–local theory. So to mimic this argument, our definition of n–oriented spectrum would need to be extended beyond the $K(n)$–local setting to a larger class of spectra.

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