Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds

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For metric measure spaces satisfying the reduced curvature–dimension condition $CD^*(K, N)$ we prove a series of sharp functional inequalities under the additional "essentially nonbranching" assumption. Examples of spaces entering this framework are (weighted) Riemannian manifolds satisfying lower Ricci curvature bounds and their measured Gromov Hausdorff limits, Alexandrov spaces satisfying lower curvature bounds and, more generally, $RCD^*(K, N)$ spaces, Finsler manifolds endowed with a strongly convex norm and satisfying lower Ricci curvature bounds.

In particular we prove the Brunn–Minkowski inequality, the p–spectral gap (or equivalently the p–Poincaré inequality) for any $p \in [1, \infty)$, the log-Sobolev inequality, the Talagrand inequality and finally the Sobolev inequality.

All the results are proved in a sharp form involving an upper bound on the diameter of the space; all our inequalities for essentially nonbranching $CD^*(K, N)$ spaces take the same form as the corresponding sharp ones known for a weighted Riemannian manifold satisfying the curvature–dimension condition $CD(K, N)$ in the sense of Bakry and Émery. In this sense our inequalities are sharp. We also discuss the rigidity and almost rigidity statements associated to the p–spectral gap.

In particular, we have also shown that the sharp Brunn–Minkowski inequality in the *global* form can be deduced from the *local* curvature–dimension condition, providing a step towards (the long-standing problem of) globalization for the curvature–dimension condition $CD(K, N)$.

To conclude, some of the results can be seen as answers to open problems proposed in Villani's book *Optimal transport*.

1 Introduction

The theory of metric measure spaces satisfying a synthetic version of lower curvature and upper dimension bounds is nowadays a rich and well-established theory; nevertheless some important functional and geometric inequalities are in some cases still not proven and in others not proven in a sharp form. The scope of this note is to generalize several functional inequalities known for Riemannian manifolds satisfying a lower

bound on the Ricci curvature to the more general case of metric measure spaces satisfying the so-called curvature–dimension condition $CD(K, N)$, as defined by Lott and Villani [43] and Sturm [59; 60]. More precisely, our results will hold under the *reduced* curvature dimension condition $CD^*(K, N)$ introduced by Bacher and Sturm [7] (which is, a priori, a weaker assumption than the classic $CD(K, N)$) coupled with an essentially nonbranching assumption on geodesics. We refer to Section 2.1 for the precise definitions; here let us recall that remarkable examples of essentially nonbranching $CD^*(K, N)$ spaces are (weighted) Riemannian manifolds satisfying lower Ricci curvature bounds and their measured Gromov Hausdorff limits (for the theory of Ricci limit spaces, see Cheeger and Colding [20; 21; 22; 23] and Colding and Naber [24]), Alexandrov spaces satisfying lower curvature bounds and, more generally, $RCD^*(K, N)$ spaces, Finsler manifolds endowed with a strongly convex norm and satisfying lower Ricci curvature bounds; see Ohta [52].

Remark 1.1 To avoid technicalities in the introduction, all the results will be stated for $N > 1$; nevertheless everything holds (and will be proved in the paper) also for $N = 1$, but in this case $CD^*(K, N)$ has to be replaced by $CD_{loc}(K, N)$. The two conditions are equivalent for $N > 1$ and for $N = 1$, $K \geq 0$, but when $N = 1$ and $K < 0$ the $CD_{loc}(K, N)$ condition is strictly stronger (see Section 2.1 for more details).

Before committing a paragraph to each of the functional inequalities we will consider in this note, we underline that most of the proofs contained in this note are based on L^1 optimal transportation theory and in particular on one-dimensional localization. This technique, having its roots in work of Payne and Weinberger [54] and developed by Gromov and Milman [30], Lovász and Simonovits [44] and Kannan, Lovász and Simonovits [34], consists in reducing an n–dimensional problem to a one-dimensional one via tools of convex geometry. Recently Klartag [37] found an L^1–optimal transportation approach leading to a generalization of these ideas to Riemannian manifolds; the authors [14], via a careful analysis avoiding any smoothness assumption, generalized this approach to metric measure spaces.

It is also convenient to introduce here the family of one-dimensional measures that will be used several times for comparison:

$$\mathcal{F}_{K,N,D} := \big\{ \mu \in \mathcal{P}(\mathbb{R}) \mid \mathrm{supp}(\mu) \subset [0, D],\ \mu = h_\mu \cdot \mathcal{L}^1,\ h_\mu \in C^2((0, D)),$$
$$(\mathbb{R}, |\cdot|, \mu) \in CD(K, N) \big\},$$

where $(\mathbb{R}, |\cdot|, \mu) \in CD(K, N)$ means the metric measure space $(\mathbb{R}, |\cdot|, \mu)$ verifies $CD(K, N)$ or, equivalently,

$$(h_\mu^{1/(N-1)})'' + \frac{K}{N-1} h_\mu^{1/(N-1)} \leq 0.$$

1.1 Brunn–Minkowski inequality

The celebrated Brunn–Minkowski inequality estimates from below the measure of the t–intermediate points between two given subsets A_0 and A_1 of X for $t \in [0, 1]$. For metric measure spaces satisfying the reduced curvature–dimension condition $\mathrm{CD}^*(K, N)$ (see Section 2.1 for a brief account of different versions of the curvature–dimension condition), almost by definition, for any A_0, $A_1 \subset X$,

$$(1\text{-}1) \qquad \mathfrak{m}(A_t)^{1/N} \geq \sigma_{K,N}^{(1-t)}(\theta)\, \mathfrak{m}(A_0)^{1/N} + \sigma_{K,N}^{(t)}(\theta)\, \mathfrak{m}(A_1)^{1/N},$$

where A_t is the set of t–intermediate points between A_0 and A_1, that is,

$$A_t = \mathrm{e}_t\big(\{\gamma \in \mathrm{Geo}(X) \mid \gamma_0 \in A_0,\ \gamma_1 \in A_1\}\big),$$

(see Section 2 for the definition of e) θ is the minimal/maximal length of geodesics from A_0 to A_1,

$$\theta := \begin{cases} \inf_{(x_0, x_1) \in A_0 \times A_1} \mathrm{d}(x_0, x_1) & \text{if } K \geq 0, \\ \sup_{(x_0, x_1) \in A_0 \times A_1} \mathrm{d}(x_0, x_1) & \text{if } K < 0, \end{cases}$$

and $\sigma_{K,N}^{(t)}(\theta)$ is as defined in (2-3). Nevertheless, (1-1) is not sharp. Indeed, if $(X, \mathrm{d}, \mathfrak{m})$ is a weighted Riemannian manifold satisfying $\mathrm{CD}^*(K, N)$, then (1-1) holds but with better interpolation coefficients, that is, with $\tau_{K,N}^{(t)}(\theta)$ and $\tau_{K,N}^{(1-t)}(\theta)$ replacing $\sigma_{K,N}^{(t)}(\theta)$ and $\sigma_{K,N}^{(1-t)}(\theta)$, respectively. Indeed for a weighted Riemannian manifold the two (a priori) different definitions of $\mathrm{CD}^*(K, N)$ and $\mathrm{CD}(K, N)$ coincide and then again, almost by definition — see Sturm [60] — one can obtain the improved (and sharp) Brunn–Minkowski inequality (let us mention that a direct proof of the Brunn–Minkowski inequality in the smooth setting was done earlier by Cordero-Erausquin, McCann and Schmuckenschläger [25]; see also Milman and Rotem [49]).

A first main result of this paper is to establish the sharp inequality for essentially nonbranching $\mathrm{CD}^*(K, N)$ metric measure spaces.

Theorem 3.1 *Let $(X, \mathrm{d}, \mathfrak{m})$ with $\mathfrak{m}(X) < \infty$ verify $\mathrm{CD}_{\mathrm{loc}}(K, N)$ for some $K, N \in \mathbb{R}$ and $N \in (1, \infty)$. Assume moreover that $(X, \mathrm{d}, \mathfrak{m})$ is essentially nonbranching. Then it satisfies the following sharp Brunn–Minkowski inequality: for any A_0, $A_1 \subset X$,*

$$\mathfrak{m}(A_t)^{1/N} \geq \tau_{K,N}^{(1-t)}(\theta)\, \mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\, \mathfrak{m}(A_1)^{1/N},$$

where A_t is the set of t–intermediate points between A_0 and A_1 and θ is the minimal/maximal length of geodesics from A_0 to A_1.

Remark 1.2 The surprising feature of Theorem 3.1 is that the sharp Brunn–Minkowski inequality in the *global* form can be deduced from the *local* curvature–dimension

condition, providing a step towards (the long-standing problem of) globalization for the curvature–dimension condition $CD(K, N)$. For an account and for partial results about this problem we refer to Ambrosio, Mondino and Savaré [6], Bacher and Sturm [7], Cavalletti [13], Cavalletti and Sturm [16], Rajala [56] and Villani [62].

1.2 p–spectral gap

In the smooth setting, a spectral gap inequality establishes a bound from below on the first eigenvalue of the Laplacian. More generally, for any $p \in (1, \infty)$ one can define the positive real number $\lambda^{1,p}_{(X,d,m)}$ as follows:

$$\lambda^{1,p}_{(X,d,m)} := \inf \left\{ \frac{\int_X |\nabla f|^p \, \mathfrak{m}}{\int_X |f|^p \, \mathfrak{m}} \ \middle| \ f \in \text{Lip}(X) \cap L^p(X, \mathfrak{m}), \ f \neq 0, \ \int_X f |f|^{p-2} \, \mathfrak{m} = 0 \right\},$$

where $|\nabla f|$ is the slope (also called the local Lipschitz constant) of the Lipschitz function f. The name is motivated by the fact that if $(X, d, \mathfrak{m})$ is the m.m.s. corresponding to a smooth compact Riemannian manifold then $\lambda^{1,p}_{(X,d,m)}$ coincides with the first positive eigenvalue of the problem

$$\Delta_p f = \lambda |f|^{p-2} f,$$

on $(X, d, \mathfrak{m})$, where $\Delta_p f := -\text{div}(|\nabla f|^{p-2} \nabla f)$ is the so-called p–Laplacian.

We now state the main theorem of this paper on p–spectral gap inequality:

Theorem 4.4 *Let $(X, d, \mathfrak{m})$ be a metric measure space satisfying $CD^*(K, N)$ for some K, $N \in \mathbb{R}$ with $N \in (1, \infty)$ and assume moreover it is essentially nonbranching. Let $D \in (0, \infty)$ be the diameter of X.*

Then, for any $p \in (1, \infty)$,

$$\lambda^{1,p}_{(X,d,m)} \geq \lambda^{1,p}_{K,N,D},$$

where $\lambda^{1,p}_{K,N,D}$ is defined by

$$\lambda^{1,p}_{K,N,D} := \inf_{\mu \in \mathcal{F}_{K,N,D}} \inf \left\{ \frac{\int_{\mathbb{R}} |u'|^p \, \mu}{\int_{\mathbb{R}} |u|^p \, \mu} \ \middle| \ u \in \text{Lip}(\mathbb{R}) \cap L^p(\mu), \ \int_{\mathbb{R}} u|u|^{p-2} \mu = 0, \ u \neq 0 \right\}.$$

In other terms, for any Lipschitz function $f \in L^p(X, \mathfrak{m})$ with $\int_X f |f|^{p-2} \, \mathfrak{m}(dx) = 0$,

$$\lambda^{1,p}_{K,N,D} \int_X |f(x)|^p \, \mathfrak{m}(dx) \leq \int_X |\nabla f|^p(x) \, \mathfrak{m}(dx).$$

For more about the quantity $\lambda^{1,p}_{K,N,D}$ the reader is referred to Section 4.1, where the model spaces are discussed in detail. From the last formulation of the statement, it is clear that the sharp p–spectral gap above is equivalent to a sharp p–Poincaré inequality.

Let us now give a brief (and incomplete) account on the huge literature about the spectral gap.

When the ambient metric measure space is a smooth Riemannian manifold equipped with the volume measure, the study of the first eigenvalue of the Laplace–Beltrami operator has a long history, going back to Lichnerowicz [41], Cheeger [19], Li and Yau [40], for example. For an overview the reader can consult for instance the book by Chavel [17], the survey by Ledoux [39] or Schoen and Yau [58, Chapter 3] and references therein.

We mention that the estimate of Theorem 4.4 in the case $p = 2$ started with Payne and Weinberger [54] for convex domains in $\mathbb{R}^n$, where a diameter-improved spectral gap inequality for the Laplace operator was originally proved. Later this was generalized to Riemannian manifolds with nonnegative Ricci curvature by Zhong and Yang [65], and by Bakry and Qian [9] for manifolds with densities. The generalization to arbitrary $p \in (1, \infty)$ has been proved by Valtorta [61] for $K = 0$ and Naber and Valtorta [51] for any $K \in \mathbb{R}$. All of these results hold for Riemannian manifolds.

Regarding metric measure spaces, the sharp Lichnerowicz spectral gap for $p = 2$ was proved by Lott and Villani [42] under the $CD(K, N)$ condition. Jiang and Zhang [33] recently showed, still for $p = 2$, that the improved version under an upper diameter bound holds for $RCD^*(K, N)$ metric measure spaces. For Ricci limit spaces, in the case $K > 0$ and $D = \pi \sqrt{(N-1)/K}$, the p–spectral gap above has recently been obtained by Honda [32] via proving the stability of $\lambda^{1,p}$ under measured Gromov–Hausdorff (mGH) convergence of compact Riemannian manifolds; this approach was inspired by the celebrated work of Cheeger and Colding [23], where, in particular, it was shown the stability of $\lambda^{1,2}$ under mGH convergence. We also obtain the *almost rigidity* for the p–spectral gap: if an almost equality in the p–spectral gap holds, then the space must have almost maximal diameter.

Theorem 4.5 Let $N > 1$ and $p \in (1, \infty)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N, p)$ such that the following holds:

Let (X, d, m) be an essentially nonbranching metric measure space that satisfies $CD^*(N - 1 - \delta, N + \delta)$. If $\lambda^{1,p}_{(X,d,m)} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta$, then $\operatorname{diam}(X) \geq \pi - \varepsilon$.

As a consequence, by a compactness argument and using the maximal diameter theorem proved recently for $RCD^*(K, N)$ by Ketterer [35], we have the following p–Obata and almost p–Obata theorems:

Corollary 1.3 (p–Obata theorem) Let (X, d, m) be an $RCD^*(N - 1, N)$ space for some $N \geq 2$, and let $1 < p < \infty$. If

$$\lambda^{1,p}_{(X,d,m)} = \lambda^{1,p}_{N-1,N,\pi} (= \lambda^{1,p}(S^N) \text{ for integer } N),$$

then $(X, \mathrm{d}, \mathfrak{m})$ *is a spherical suspension, ie there exists an* $\mathrm{RCD}^*(N-2, N-1)$ *space* $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ *such that* $(X, \mathrm{d}, \mathfrak{m})$ *is isomorphic to* $[0, \pi] \times_{\sin}^{N-1} Y$.

Corollary 1.4 (almost p–Obata theorem) *Let* $N \geq 2$ *and* $p \in (1, \infty)$ *be fixed. Then for every* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, N, p) > 0$ *such that the following holds:*

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}^*(N-1-\delta, N+\delta)$ *space. If*

$$\lambda_{(X, \mathrm{d}, \mathrm{m})}^{1, p} \leq \lambda_{N-1, N, \pi}^{1, p} + \delta,$$

then there exists an $\mathrm{RCD}^*(N-2, N-1)$ *space* $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ *such that*

$$\mathrm{d}_{\mathrm{mGH}}\big((X, \mathrm{d}, \mathfrak{m}), [0, \pi] \times_{\sin}^{N-1} Y\big) \leq \varepsilon.$$

Let us mention that the classical Obata's theorem for $\mathrm{RCD}^*(K, N)$ spaces, ie the version of Corollary 1.3 for $p = 2$, was recently obtained by Ketterer [36] (see also Jiang and Zhang [33]) with different methods.

Finally we recall that the case $p = 1$ can be attacked using the identity $h_{(X, \mathrm{d}, \mathrm{m})} = \lambda_{(X, \mathrm{d}, \mathrm{m})}^{1, 1}$, where $h_{(X, \mathrm{d}, \mathrm{m})}$ is the so-called Cheeger isoperimetric constant; see Section 5.1. Therefore Theorems 4.4 and 4.5 and Corollaries 1.3 and 1.4 for the case $p = 1$ follow from the analogous results proved for the isoperimetric profile in Cavalletti and Mondino [14]. Nevertheless, for the reader's convenience, the case $p = 1$ will be discussed in detail in Section 5.

1.3 Log-Sobolev and Talagrand inequalities

Given a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$, we say that it supports the log-Sobolev inequality with constant $\alpha > 0$ if, for any Lipschitz function $f \colon X \to [0, \infty)$ with $\int_X f(x) \, \mathfrak{m}(dx) = 1$,

$$(1\text{-}2) \qquad\qquad 2\alpha \int_X f \log f \, \mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \mathfrak{m}.$$

The largest constant α such that (1-2) holds for any Lipschitz function $f \colon X \to [0, \infty)$ with $\int_X f(x) \, \mathfrak{m}(dx) = 1$ will be called the log-Sobolev constant of $(X, \mathrm{d}, \mathfrak{m})$ and denoted by $\alpha_{(X, \mathrm{d}, \mathrm{m})}^{LS}$. The log-Sobolev inequality is already known — see Villani [62, Theorem 30.22] — for essentially nonbranching metric measure spaces satisfying $\mathrm{CD}(K, \infty)$ for $K > 0$ with sharp constant $\alpha = K$, but it is an open problem (see for instance Villani [62, Open Problem 21.6]) to get the sharp dimensional constant $\alpha_{K,N} = KN/(N-1)$ for metric measure spaces with N–Ricci curvature bounded below by K. This is the goal of the next result.

As already done above, let us introduce the model constant for the one-dimensional case. Given $K \in \mathbb{R}$, $N \geq 1$ and $D \in (0, +\infty)$ we denote by $\alpha_{K,N,D}^{LS} > 0$ the maximal constant α such that

$$(1\text{-}3) \qquad 2\alpha \int_{\mathbb{R}} f \log f \, \mu \leq \int_{\{f>0\}} \frac{|f'|^2}{f} \mu \quad \text{for all } \mu \in \mathcal{F}_{K,N,D}$$

for every Lipschitz $f \colon \mathbb{R} \to [0, \infty)$ with $\int f \, \mu = 1$.

Remark 1.5 If $K > 0$ and $D = \pi \sqrt{(N-1)/K}$, it is known that the corresponding optimal log-Sobolev constant is $KN/(N-1)$ (for more details see the discussion in Section 6.1). It is an interesting open problem, which we don't address here, to give an explicit expression of the quantity $\alpha_{K,N,D}^{LS}$ for general $K \in \mathbb{R}$, $N \geq 1$ and $D \in (0, \infty)$.

Theorem (sharp log-Sobolev inequality; see Theorem 6.2 and Corollary 6.3) *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space with diameter $D \in (0, \infty)$ satisfying $\mathsf{CD}^*(K, N)$ for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Assume moreover it is essentially nonbranching.*

Then, for any Lipschitz function $f \colon X \to [0, \infty)$ with $\int_X f \, \mathfrak{m} = 1$,

$$2\alpha_{K,N,D}^{LS} \int_X f \log f \, \mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \mathfrak{m}.$$

In other terms, $\alpha_{(X,\mathsf{d},\mathfrak{m})}^{LS} \geq \alpha_{K,N,D}^{LS}$.

As a consequence, if $K > 0$ and no diameter upper bound is assumed or $D = \pi \sqrt{(N-1)/K}$, then $\alpha_{K,N}^{LS} = KN/(N-1)$, that is, for any Lipschitz function $f \colon X \to [0, \infty)$ with $\int_X f \, \mathfrak{m} = 1$,

$$\frac{2KN}{N-1} \int_X f \log f \, \mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \mathfrak{m}.$$

In order to state the Talagrand inequality let us recall that the relative entropy functional $\mathrm{Ent}_{\mathfrak{m}} \colon \mathcal{P}(X) \to [0, +\infty]$ with respect to a given $\mathfrak{m} \in \mathcal{P}(X)$ is defined to be

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \int_X \varrho \log \varrho \, \mathfrak{m} \quad \text{if } \mu = \varrho \mathfrak{m}$$

and $+\infty$ otherwise. Otto and Villani [53] proved that for smooth Riemannian manifolds the log-Sobolev inequality with constant $\alpha > 0$ implies the Talagrand inequality with constant $2/\alpha$ preserving sharpness. The result was then generalized to arbitrary metric measure spaces by Gigli and Ledoux [28].

Combining this result with Theorem 6.2 gives the following corollary, which improves the Talagrand constant $2/K$ — which is sharp for $\mathsf{CD}(K, \infty)$ spaces — by a factor

$(N-1)/N$ if the dimension is bounded above by N. This constant is sharp for $CD^*(K,N)$ (or $CD_{loc}(K,N)$) spaces, indeed it is sharp already in the smooth setting; see Villani [62, Remark 22.43]. Since both our proof of the sharp log-Sobolev inequality and the proof of Theorem 6.4 are essentially optimal transport based, the following can be seen as an answer to Villani [62, Open Problem 22.44].

Theorem 1.6 (sharp Talagrand inequality) *Let $(X, d, \mathfrak{m})$ be a metric measure space with diameter $D \in (0,\infty)$ satisfying $CD^*(K,N)$ for some $K \in \mathbb{R}$ and $N \in (1,\infty)$, and assume moreover it is essentially nonbranching and $\mathfrak{m}(X) = 1$.*

Then it supports the Talagrand inequality with constant $2/\alpha_{K,N,D}^{LS}$, where $\alpha_{K,N,D}^{LS}$ was defined in (1-3), *ie*

$$W_2^2(\mu, \mathfrak{m}) \le \frac{2}{\alpha_{K,N,D}^{LS}} \mathrm{Ent}_{\mathfrak{m}}(\mu) \quad \text{for all } \mu \in \mathcal{P}(X).$$

In particular, if $K > 0$ and no upper bound on the diameter is assumed or $D = \pi\sqrt{(N-1)/K}$, then

$$W_2^2(\mu, \mathfrak{m}) \le \frac{2(N-1)}{KN} \mathrm{Ent}_{\mathfrak{m}}(\mu) \quad \text{for all } \mu \in \mathcal{P}(X),$$

the constant in the last inequality being sharp.

1.4 Sobolev inequality

Sobolev inequalities have been studied in many different contexts and many papers and books are devoted to this family of inequalities. Here we only mention two references mainly dealing with them in the Riemannian manifold case and the smooth CD condition case, respectively; see Hebey [31] and Ledoux [38].

We say that $(X, d, \mathfrak{m})$ supports a (p,q)–Sobolev inequality with constant $\alpha^{p,q}$ if, for any $f: X \to \mathbb{R}$ Lipschitz function,

$$(1\text{-}4) \qquad \frac{\alpha^{p,q}}{p-q} \left\{ \left(\int_X |f|^p \, \mathfrak{m} \right)^{\frac{q}{p}} - \int_X |f|^q \, \mathfrak{m} \right\} \le \int_X |\nabla f|^q \, \mathfrak{m},$$

and the largest constant $\alpha^{p,q}$ such that (1-4) holds for any Lipschitz function f will be called the (p,q)–Sobolev constant of $(X, d, \mathfrak{m})$ and will be denoted by $\alpha_{(X,d,\mathfrak{m})}^{p,q}$.

A Sobolev inequality is known to hold for essentially nonbranching m.m.s. satisfying $CD(K,N)$, provided $K < 0$ — see Villani [62, Theorem 30.23] — and other Sobolev-type inequalities have been obtained in Lott and Villani [42] for $CD(K,N)$ spaces. Let us also mention Profeta [55], where the sharp $(2^*, 2)$–Sobolev inequality has been

established for $\mathsf{RCD}^*(K, N)$ spaces with $K > 0$ and $N \in (2, \infty)$. The goal here is to give a Sobolev inequality with sharp constant for essentially nonbranching $\mathsf{CD}^*(K, N)$ spaces for $K \in \mathbb{R}$ and $N > 1$, also taking into account an upper diameter bound.

Theorem 7.1 (sharp Sobolev inequality) *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space with diameter $D \in (0, \infty)$ and satisfying $\mathsf{CD}^*(K, N)$ for some $K \in \mathbb{R}, N \in (1, \infty)$. Assume moreover it is essentially nonbranching.*

Then, for any Lipschitz function,

$$\frac{\alpha_{K,N,D}^{p,q}}{p-q}\left\{\left(\int_X |f(x)|^p \, \mathfrak{m}(dx)\right)^{\frac{q}{p}} - \int_X |f(x)|^q \, \mathfrak{m}(dx)\right\} \le \int_X |\nabla f(x)|^q \, \mathfrak{m}(dx),$$

where $\alpha_{K,N,D}^{p,q}$ is defined as the supremum among $\alpha > 0$ such that

$$\frac{\alpha}{p-q}\left\{\left(\int_X |f|^p \, \mu\right)^{\frac{q}{p}} - \int_X |f|^q \, \mu\right\} \le \int_X |\nabla f|^q \, \mu \quad \text{for all } f \in \text{Lip}, \mu \in \mathcal{F}_{K,N,D}.$$

In particular, if $K > 0$, $N > 2$ and no upper bound on the diameter is assumed or $D = \pi\sqrt{(N-1)/K}$, then, for any Lipschitz function f,

$$\frac{KN}{(p-2)(N-1)}\left\{\left(\int_X |f|^p \, \mathfrak{m}\right)^{\frac{2}{p}} - \int_X |f|^2 \, \mathfrak{m}\right\} \le \int_X |\nabla f|^2 \, \mathfrak{m}$$

for any $2 < p \le 2N/(N-2)$; in other terms, $\alpha_{(X,\mathsf{d},\mathfrak{m})}^{p,2} \ge KN/(N-1)$.

This last result can be seen as a solution to Villani [62, Open Problem 21.11].

2 Prerequisites

In what follows we say that a triple $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space (m.m.s. for short) if (X, d) is a complete and separable metric space and $\mathfrak{m}$ has positive Radon measure over X. For this note we will only be concerned with m.m.s. with $\mathfrak{m}$ a probability measure, that is, $\mathfrak{m}(X) = 1$, or at most with $\mathfrak{m}(X) < \infty$, which will be reduced to the probability case by a constant rescaling. The space of all Borel probability measures over X will be denoted by $\mathcal{P}(X)$.

A metric space is a geodesic space if and only if, for each x, $y \in X$, there exists $\gamma \in \mathrm{Geo}(X)$ such that $\gamma_0 = x$ and $\gamma_1 = y$, with

$$\mathrm{Geo}(X) := \{\gamma \in C([0, 1], X) \mid \mathsf{d}(\gamma_s, \gamma_t) = (s - t)\mathsf{d}(\gamma_0, \gamma_1), \, s, t \in [0, 1]\}.$$

Recall that, for complete geodesic spaces, local compactness is equivalent to properness (a metric space is proper if every closed ball is compact). We directly assume the ambient space (X, d) to be proper. Hence, from now on we assume the following: the ambient metric space (X, d) is geodesic, complete, separable and proper and $\mathfrak{m}(X) = 1$.

We denote by $\mathcal{P}_2(X)$ the space of probability measures with finite second moment endowed with the L^2–Wasserstein distance W_2 defined as follows: for μ_0, $\mu_1 \in \mathcal{P}_2(X)$ we set

$$(2\text{-}1) \qquad\qquad W_2^2(\mu_0, \mu_1) = \inf_{\pi} \int_X \mathsf{d}^2(x, y) \, \pi(dx \, dy),$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal. Assuming the space (X, d) to be geodesic, the space $(\mathcal{P}_2(X), W_2)$ is also geodesic.

Any geodesic $(\mu_t)_{t \in [0,1]}$ in $(\mathcal{P}_2(X), W_2)$ can be lifted to a measure $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ such that $(\mathrm{e}_t) \sharp \nu = \mu_t$ for all $t \in [0, 1]$. Here e_t, for any $t \in [0, 1]$, denotes the evaluation map

$$\mathrm{e}_t \colon \mathrm{Geo}(X) \to X, \quad \mathrm{e}_t(\gamma) := \gamma_t.$$

Given μ_0, $\mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ for which $(\mathrm{e}_0, \mathrm{e}_1) \sharp \nu$ realizes the minimum in (2-1). If (X, d) is geodesic, then the set $\mathrm{OptGeo}(\mu_0, \mu_1)$ is nonempty for any μ_0, $\mu_1 \in \mathcal{P}_2(X)$. It is worth also introducing the subspace of $\mathcal{P}_2(X)$ formed by all those measures absolutely continuous with respect to $\mathfrak{m}$; it is denoted by $\mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$.

2.1 Geometry of metric measure spaces

Here we briefly recall the synthetic notions of lower Ricci curvature bounds; for more detail we refer to [7; 43; 59; 60; 62].

In order to formulate curvature properties for $(X, \mathsf{d}, \mathfrak{m})$ we introduce the following distortion coefficients: given two numbers K, $N \in \mathbb{R}$ with $N \geq 1$, we set, for $(t, \theta) \in [0, 1] \times \mathbb{R}_+$,

$$(2\text{-}2) \quad \sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2, \\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ and } N = 0, \text{ or if } K\theta^2 = 0, \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 \leq 0 \text{ and } N > 0. \end{cases}$$

We also set, for $N \geq 1$, $K \in \mathbb{R}$ and $(t, \theta) \in [0, 1] \times \mathbb{R}_+$,

$$(2\text{-}3) \qquad \tau_{K,N}^{(t)}(\theta) := t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}.$$

As we will consider only the case of essentially nonbranching spaces, we recall:

Definition 2.1 A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is *essentially nonbranching* if and only if, for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ which are absolutely continuous with respect to $\mathfrak{m}$, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of nonbranching geodesics.

A set $F \subset \mathrm{Geo}(X)$ is a set of nonbranching geodesics if and only if, for any $\gamma^1, \gamma^2 \in F$,

$$\exists \bar{t} \in (0, 1) \ \forall t \in (0, \bar{t}) \quad \gamma_t^1 = \gamma_t^2 \quad \Longrightarrow \quad \forall s \in [0, 1] \ \gamma_s^1 = \gamma_s^2.$$

After [15] we can give the following equivalent definition:

Definition 2.2 (CD condition) An essentially nonbranching m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathrm{CD}(K, N)$ if and only if, for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that

$$(2\text{-}4) \qquad \varrho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0, \gamma_1))\varrho_0^{-1/N}(\gamma_0) + \tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0, \gamma_1))\varrho_1^{-1/N}(\gamma_1)$$

for ν–ae $\gamma \in \mathrm{Geo}(X)$ for all $t \in [0, 1]$, where $e_t \sharp \nu = \varrho_t \mathfrak{m}$.

For the general definition of $\mathrm{CD}(K, N)$ see [43; 59; 60]. It is worth recalling that, if (M, g) is a Riemannian manifold of dimension n and $h \in C^2(M)$ with $h > 0$, then the m.m.s. $(M, g, h \,\mathrm{vol})$ verifies $\mathrm{CD}(K, N)$ with $N \geq n$ if and only if (see [60, Theorem 1.7])

$$\mathrm{Ric}_{g,h,N} \geq Kg, \quad \mathrm{Ric}_{g,h,N} := \mathrm{Ric}_g - (N-n)\frac{\nabla_g^2 h^{1/(N-n)}}{h^{1/(N-n)}}.$$

In particular, if $N = n$ the generalized Ricci tensor $\mathrm{Ric}_{g,h,N} = \mathrm{Ric}_g$ makes sense only if h is constant. In particular, if $I \subset \mathbb{R}$ is any interval, $h \in C^2(I)$ and $\mathcal{L}^1$ is the

one-dimensional Lebesgue measure, the m.m.s. $(I, |\cdot|, h\mathcal{L}^1)$ verifies $\mathrm{CD}(K, N)$ if and only if

$$(2\text{-}5) \qquad (h^{1/(N-1)})'' + \frac{K}{N-1} h^{1/(N-1)} \leq 0,$$

and verifies $\mathrm{CD}(K, 1)$ if and only if h is constant.

We also mention the more recent Riemannian curvature dimension condition RCD^* introduced in the infinite-dimensional case in [3; 4; 1] and analyzed in the finite-dimensional case in [26; 5]. We refer to these papers and references therein for a general account on the synthetic formulation of Ricci curvature lower bounds for metric measure spaces. Here we only mention that $\mathrm{RCD}^*(K, N)$ condition is an enforcement of the so-called reduced curvature dimension condition, denoted by $\mathrm{CD}^*(K, N)$, that has been introduced in [7]; in particular, the additional condition is that the Sobolev space $W^{1,2}(X, \mathfrak{m})$ is a Hilbert space; see [2; 3].

The reduced $\mathrm{CD}^*(K, N)$ condition asks for the same inequality (2-4) of $\mathrm{CD}(K, N)$ but the coefficients $\tau_{K,N}^{(t)}(\mathrm{d}(\gamma_0, \gamma_1))$ and $\tau_{K,N}^{(1-t)}(\mathrm{d}(\gamma_0, \gamma_1))$ are replaced by $\sigma_{K,N}^{(t)}(\mathrm{d}(\gamma_0, \gamma_1))$ and $\sigma_{K,N}^{(1-t)}(\mathrm{d}(\gamma_0, \gamma_1))$, respectively.

Hence, while the distortion coefficients of the $\mathrm{CD}(K, N)$ condition are formally obtained by imposing one direction with linear distortion and $N - 1$ directions affected by curvature, the $\mathrm{CD}^*(K, N)$ condition imposes the same volume distortion in all N directions.

It was proved in [57] that the $\mathrm{RCD}^*(K, N)$ condition implies the essentially nonbranching property, so this is a fairly natural assumption in the framework of m.m.s. satisfying lower Ricci bounds.

For both the CD and CD^* definitions there is a local version that is of some relevance for our analysis. Here we state only the local formulation $\mathrm{CD}(K, N)$, the one for $\mathrm{CD}^*(K, N)$ being similar.

Definition 2.3 ($\mathrm{CD}_{\mathrm{loc}}$ condition) An essentially nonbranching m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ satisfies $\mathrm{CD}_{\mathrm{loc}}(K, N)$ if, for any point $x \in X$, there exists a neighborhood $X(x)$ of x such that, for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathrm{d}, \mathfrak{m})$ supported in $X(x)$, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that (2-4) holds true for all $t \in [0, 1]$. The support of $\mathrm{e}_t \sharp \nu$ is not necessarily contained in the neighborhood $X(x)$.

One of the main properties of the reduced curvature dimension condition is the globalization one: under the essentially nonbranching property, $\mathrm{CD}_{\mathrm{loc}}^*(K, N)$ and $\mathrm{CD}^*(K, N)$ are equivalent (see [7, Corollary 5.4]). Let us mention that the local-to-global property is satisfied also by the $\mathrm{RCD}^*(K, N)$ condition; see [6].

Remark 2.4 ($CD^*(K, N)$ vs $CD_{loc}(K, N)$) Results of [7] imply the following chain of implications: if (X, d, m) is a proper, essentially nonbranching, metric measure space, then

$$CD_{loc}(K-, N) \iff CD^*_{loc}(K-, N) \iff CD^*(K, N),$$

provided $K, N \in \mathbb{R}$ with $N > 1$ or $N = 1$ and $K \geq 0$. On the other hand, $CD^*(K, 1)$ does not imply $CD_{loc}(K, 1)$ for $K < 0$: indeed it is possible to check that $(X, d, m) = ([0, 1], |\cdot|, c \sinh(\cdot) \mathcal{L}^1)$ satisfies $CD^*(-1, 1)$ but not $CD_{loc}(-1, 1)$, which would require the density to be constant. Hence $CD^*(K, N)$ and $CD_{loc}(K, N)$ are equivalent if $1 < N < \infty$ or $N = 1$ and $K \geq 0$, but for $N = 1$ and $K < 0$ the $CD_{loc}(K, N)$ condition is strictly stronger than $CD^*(K, N)$.

Lastly, for $K > 0$, $CD^*(K, N)$ implies $CD(K^*, N)$, where $K^* = K(N-1)/N$. For a deeper analysis on the interplay between CD^* and CD we refer to [13; 16].

2.2 Measured Gromov–Hausdorff convergence and stability of $RCD^*(K, N)$

Let us first recall the notion of measured Gromov–Hausdorff convergence (mGH for short). Since in this work we will apply it to compact m.m.s. endowed with probability measures having full support, we will restrict to this framework for simplicity (for a more general treatment see for instance [29]).

Definition 2.5 A sequence (X_j, d_j, m_j) of compact m.m.s. with $\operatorname{supp}(m_j) = X_j$ and $m_j(X_j) = 1$ is said to converge in the mGH topology to a compact m.m.s. $(X_\infty, d_\infty, m_\infty)$ with $\operatorname{supp}(m_\infty) = X_\infty$ and $m_\infty(X) = 1$ if and only if there exists a separable metric space (Z, d_Z) and isometric embeddings $\{\iota_j \colon (X, d_j) \to (Z, d_Z)\}_{i \in \overline{\mathbb{N}}}$ such that, for every $\varepsilon > 0$, there exists j_0 such that, for every $j > j_0$,

$$\iota_\infty(X_\infty) \subset B^Z_\varepsilon[\iota_j(X_j)] \quad \text{and} \quad \iota_j(X_j) \subset B^Z_\varepsilon[\iota_\infty(X_\infty)],$$

where $B^Z_\varepsilon[A] := \{z \in Z \mid d_Z(z, A) < \varepsilon\}$ for every subset $A \subset Z$, and

$$\int_Z \varphi((\iota_j)_\sharp(m_j)) \to \int_Z \varphi((\iota_\infty)_\sharp(m_\infty)) \quad \text{for all } \varphi \in C_b(Z),$$

where $C_b(Z)$ denotes the set of real-valued bounded continuous functions in Z.

The following theorem summarizes the compactness/stability properties we will use in the proof of the almost rigidity result (notice these hold more generally for every $K \in \mathbb{R}$ by replacing mGH with *pointed* mGH convergence).

Theorem 2.6 (metrizability and compactness) *Let $K > 0$ and $N > 1$ be fixed. Then the mGH convergence restricted to (isomorphism classes of) $\mathrm{RCD}^*(K, N)$ spaces is metrizable by a distance function $\mathrm{d}_{\mathrm{mGH}}$. Furthermore, every sequence $(X_j, \mathrm{d}_j, \mathfrak{m}_j)$ of $\mathrm{RCD}^*(K, N)$ spaces admits a subsequence which mGH-converges to a limit $\mathrm{RCD}^*(K, N)$ space.*

The compactness follows by the standard argument of Gromov; indeed, for fixed $K > 0$ and $N > 1$, the spaces have uniformly bounded diameter, moreover the measures of $\mathrm{RCD}^*(K, N)$ spaces are uniformly doubling, hence the spaces are uniformly totally bounded and thus compact in the GH topology; the weak compactness of the measures follows using the doubling condition again and the fact that they are normalized. For the stability of the $\mathrm{RCD}^*(K, N)$ condition under mGH convergence see for instance [7; 26; 29]. The metrizability of mGH convergence restricted to a class of uniformly doubling normalized m.m.s. having uniform diameter bounds is also well known; see for instance [29].

2.3 Warped product

Given two geodesic m.m.s. $(B, \mathrm{d}_B, \mathfrak{m}_b)$ and $(F, \mathrm{d}_F, \mathfrak{m}_F)$ and a Lipschitz function $f \colon B \to \mathbb{R}_+$, one can define a length function on the product $B \times F$: for any absolutely continuous curve $\gamma \colon [0, 1] \to B \times F$ with $\gamma = (\alpha, \beta)$, define

$$L(\gamma) := \int_0^1 \left(|\dot{\alpha}|^2(t) + (f \circ \alpha)^2(t) |\dot{\beta}|^2(t) \right)^{1/2} dt$$

and define accordingly the pseudodistance

$$|(p, x), (q, y)| := \inf\{ L(\gamma) \mid \gamma_0 = (p, x), \ \gamma_1 = (q, y) \}.$$

Then the warped product of B with F is defined as

$$B \times_f F := (B \times F / \sim, |\cdot, \cdot|),$$

where $(p, x) \sim (q, y)$ if and only if $|(p, x), (q, y)| = 0$. One can also associate a measure and obtain the object

$$B \times_f^N F := (B \times_f F, \mathfrak{m}_C), \quad \mathfrak{m}_C := f^N \mathfrak{m}_B \otimes \mathfrak{m}_F.$$

Then $B \times_f^N F$ will be a metric measure space called a measured warped product. For a general picture on the curvature properties of warped products, we refer to [35].

2.4 Localization method

The next theorem represents the key technical tool of the present paper. The roots of such a result, known in the literature as a localization technique, can be traced back to work of Payne and Weinberger [54], further developed in the Euclidean space by Gromov and Milman [30], Lovász and Simonovits [44] and Kannan, Lovász and Simonovits [34]. The basic idea consists in reducing an n–dimensional problem to a one-dimensional one via tools of convex geometry. Recently Klartag [37] found an L^1–optimal transportation approach, leading to a generalization of these ideas to Riemannian manifolds; the authors [14], via a careful analysis avoiding any smoothness assumption, generalized this approach to metric measure spaces.

Theorem 2.7 *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially nonbranching metric measure space with $\mathfrak{m}(X) = 1$ satisfying $\mathsf{CD}_{\mathrm{loc}}(K, N)$ for some K, $N \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f \colon X \to \mathbb{R}$ be $\mathfrak{m}$–integrable such that $\int_X f \, \mathfrak{m} = 0$ and assume the existence of $x_0 \in X$ such that $\int_X |f(x)| \, \mathsf{d}(x, x_0) \, \mathfrak{m}(dx) < \infty$.*

Then the space X can be written as the disjoint union of two sets Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q \in Q}$, where each X_q is the image of a geodesic; moreover, there exists a family of probability measures $\{\mathfrak{m}_q\}_{q \in Q} \subset \mathcal{P}(X)$ with the following properties:

- *For any $\mathfrak{m}$–measurable set $B \subset \mathcal{T}$,*

$$\mathfrak{m}(B) = \int_Q \mathfrak{m}_q(B) \, \mathfrak{q}(dq),$$

  where $\mathfrak{q}$ is a probability measure over $Q \subset X$.

- *For $\mathfrak{q}$–almost every $q \in Q$, the set X_q is a geodesic with strictly positive length and $\mathfrak{m}_q$ is supported on it. Moreover $q \mapsto \mathfrak{m}_q$ is a $\mathsf{CD}(K, N)$ disintegration, that is, $\mathfrak{m}_q = g(q, \cdot) \sharp (h_q \cdot \mathcal{L}^1)$, with*

(2-6) $h_q((1-s)t_0 + s t_1)^{1/(N-1)}$

$$\geq \sigma_{K,N-1}^{(1-s)}(t_1 - t_0) h_q(t_0)^{1/(N-1)} + \sigma_{K,N-1}^{(s)}(t_1 - t_0) h_q(t_1)^{1/(N-1)}$$

  for all $s \in [0, 1]$ and for all t_0, $t_1 \in \mathrm{Dom}(g(q, \cdot))$ with $t_0 < t_1$, where $g(q, \cdot)$ is the isometry with range X_q. If $N = 1$, for $\mathfrak{q}$–ae $q \in Q$ the density h_q is constant.

- *For $\mathfrak{q}$–almost every $q \in Q$, we have $\int_{X_q} f \, \mathfrak{m}_q = 0$ and $f = 0$ $\mathfrak{m}$–ae in Z.*

Remark 2.8 Inequality (2-6) is the weak formulation of the following differential inequality on h_{q,t_0,t_1}:

$$(2\text{-}7) \qquad (h_{q,t_0,t_1}^{1/(N-1)})'' + (t_1 - t_0)^2 \frac{K}{N-1} h_{q,t_0,t_1}^{1/(N-1)} \leq 0$$

for all $t_0 < t_1 \in \mathrm{Dom}(g(q,\cdot))$, where $h_{q,t_0,t_1}(s) := h_q((1-s)t_0 + st_1)$. It is easy to observe that the differential inequality (2-7) on h_{q,t_0,t_1} is equivalent to the following differential inequality on h_q:

$$(h_q^{1/(N-1)})'' + \frac{K}{N-1} h_q^{1/(N-1)} \leq 0,$$

which is precisely (2-5). Then Theorem 2.7 can be alternatively stated as follows: If $(X, d, \mathfrak{m})$ is an essentially nonbranching m.m.s. verifying $\mathsf{CD}_{\mathrm{loc}}(K, N)$ and $\varphi \colon X \to \mathbb{R}$ is a 1–Lipschitz function, then the corresponding decomposition of the space in maximal rays $\{X_q\}_{q \in Q}$ produces a disintegration $\{\mathfrak{m}_q\}_{q \in Q}$ of $\mathfrak{m}$ such that, for $\mathfrak{q}$–ae $q \in Q$,

the m.m.s. $(\mathrm{Dom}(g(q,\cdot)), |\cdot|, h_q \mathcal{L}^1)$ verifies $\mathsf{CD}(K, N)$.

Accordingly, from now on we will say that the disintegration $q \mapsto \mathfrak{m}_q$ is a $\mathsf{CD}(K, N)$ disintegration.

A few comments on Theorem 2.7 are in order. From (2-6) it follows that

$$(2\text{-}8) \quad \{t \in \mathrm{Dom}(g(q,\cdot)) \mid h_q(t) > 0\} \text{ is convex}$$

$$\text{and} \quad t \mapsto h_q(t) \text{ is locally Lipschitz continuous.}$$

The measure $\mathfrak{q}$ is the quotient measure associated to the partition $\{X_q\}_{q \in Q}$ of $\mathcal{T}$ and Q is its quotient set; see [14] for details.

3 Sharp Brunn–Minkowski inequality

In this section we prove the sharp Brunn–Minkowski inequality for m.m.s. satisfying $\mathsf{CD}_{\mathrm{loc}}(K, N)$. It follows from Theorem 2.7 that the same result holds under $\mathsf{CD}^*(K, N)$ for any $K, N \in \mathbb{R}$ provided $N \in (1, \infty)$ or $N = 1$ and $K \geq 0$. See also Remark 1.1. The same will hold for all the inequalities proved in the paper.

Theorem 3.1 Let $(X, d, \mathfrak{m})$ with $\mathfrak{m}(X) < \infty$ verify $\mathsf{CD}_{\mathrm{loc}}(K, N)$ for some $N, K \in \mathbb{R}$ and $N \in [1, \infty)$. Assume moreover $(X, d, \mathfrak{m})$ to be essentially nonbranching. Then it satisfies the following sharp Brunn–Minkowski inequality: for any $A_0, A_1 \subset X$,

$$(3\text{-}1) \qquad \mathfrak{m}(A_t)^{1/N} \geq \tau_{K,N}^{(1-t)}(\theta) \mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta) \mathfrak{m}(A_1)^{1/N},$$

where A_t is the set of t–intermediate points between A_0 and A_1, that is,

$$A_t = e_t\big(\{\gamma \in \mathrm{Geo}(X) \mid \gamma_0 \in A_0, \, \gamma_1 \in A_1\}\big)$$

and θ is the minimal/maximal length of geodesics from A_0 to A_1:

$$\theta := \begin{cases} \inf_{(x_0, x_1) \in A_0 \times A_1} \mathsf{d}(x_0, x_1) & \text{if } K \geq 0, \\ \sup_{(x_0, x_1) \in A_0 \times A_1} \mathsf{d}(x_0, x_1) & \text{if } K < 0. \end{cases}$$

Before starting the proof of Theorem 3.1 we recall the classical result of Borell [11] and Brascamp and Lieb [12] characterizing one-dimensional measures satisfying the Brunn–Minkowski inequality.

Lemma 3.2 *Let η be a Borel measure defined on $\mathbb{R}$ admitting the representation $\eta = h \cdot \mathcal{L}^1$. The following are equivalent:*

(i) *The density h is (K, N)–concave on its convex support, that is,*

$$(h^{1/(N-1)})'' + \frac{K}{N-1} h^{1/(N-1)} \leq 0$$

in the weak sense; see (2-6).

(ii) *For any subsets A_0, A_1 of $\mathbb{R}$,*

$$\eta(A_t) \geq \tau_{K,N}^{(1-t)}(\theta) \, \eta(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta) \, \eta(A_1)^{1/N},$$

where $A_t := \{(1-t)x + ty \mid x \in A_0, \, y \in A_1\}$ and θ is the minimal/maximal length of geodesics from A_0 to A_1:

$$\theta := \begin{cases} \mathrm{ess\,inf}_{(x_0, x_1) \in A_0 \times A_1} \mathsf{d}(x_0, x_1) & \text{if } K \geq 0, \\ \mathrm{ess\,sup}_{(x_0, x_1) \in A_0 \times A_1} \mathsf{d}(x_0, x_1) & \text{if } K < 0. \end{cases}$$

For the reader's convenience we include here a proof that (i) implies (ii), which is the implication we will use later.

Proof Consider the N–entropy: for any $\mu = \rho \cdot \eta$

$$\mathcal{S}_N(\mu|\eta) := -\int \rho^{-1/N}(x) \, \mu(dx).$$

Observe that (ii) is implied by displacement convexity of $\mathcal{S}_N$ with respect to the L^2–Wasserstein distance over $(\mathbb{R}, |\cdot|)$. Just consider $\mu_0 := \eta(A_0)^{-1} \eta_{\llcorner A_0}$ and $\mu_1 := \eta(A_1)^{-1} \eta_{\llcorner A_1}$ and use Jensen's inequality. Consider therefore a geodesic curve

$$[0, 1] \ni t \mapsto \rho_t \, \eta \in W_2(\mathbb{R}, |\cdot|), \quad T_t \sharp \rho_0 \eta = \rho_t \eta,$$

where $T_t = \mathrm{id}(1-t) + tT$ and T is the (μ_0–essentially) unique monotone rearrangement such that $T \sharp \mu_0 = \mu_1$. Thanks to the approximate differentiability of T, one can use the change of variable formula

$$\rho_t(T_t(x))h(T_t(x))|(1-t)+tT'|(x) = \rho_0(x)h(x)$$

and obtain the chain of equalities

$$\int_{\mathrm{supp}(\mu_t)} \rho_t(x)^{(N-1)/N} \, \eta(dx)$$

$$= \int_{\mathrm{supp}(\mu_t)} \rho_t(x)^{(N-1)/N} h(x) \, dx$$

$$= \int_{\mathrm{supp}(\mu_0)} \rho_t(T_t(x))^{(N-1)/N} h(T_t(x))|(1-t)+tT'| \, dx$$

$$= \int_{\mathrm{supp}(\mu_0)} \rho_0(x)^{(N-1)/N} \left(\frac{h(T_t(x))}{h(x)} \right)^{\frac{1}{N}} |(1-t)+tT'|(x)^{1/N} \, \eta(dx).$$

Hence the claim has become to prove that $t \mapsto J_t(x)^{1/N}$ is concave, where J_t is the Jacobian of T_t with respect to η and

$$J_t(x) = J_t^G(x) \cdot J_t^W(x), \quad J_t^G(x) = |(1-t)+tT'|(x), \quad J_t^W(x) = \frac{h(T_t(x))}{h(x)},$$

where J^G is the geometric Jacobian and J^W the weighted Jacobian. Since $t \mapsto J_t^G(x)$ is linear, using Hölder's inequality the claim follows straightforwardly from the (K, N)–convexity of h. $\qquad\square$

We can now move to the proof of Theorem 3.1.

Proof of Theorem 3.1 First of all notice that, up to replacing $\mathfrak{m}$ with the normalized measure $(1/\mathfrak{m}(X))\,\mathfrak{m}$, we can assume that $\mathfrak{m}(X) = 1$. Let A_0, $A_1 \subset X$ be two given Borel sets of positive $\mathfrak{m}$–measure.

Step 1 Consider the function $f := \chi_{A_0}/\mathfrak{m}(A_0) - \chi_{A_1}/\mathfrak{m}(A_1)$ and observe that $\int_X f \, \mathfrak{m} = 0$. From Theorem 2.7, the space X can be written as the disjoint union of two sets Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q \in Q}$ and a corresponding disintegration of $\mathfrak{m} \llcorner \mathcal{T}$, namely $\{\mathfrak{m}_q\}_{q \in Q}$, such that

$$\mathfrak{m} \llcorner \mathcal{T} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(dq),$$

where $\mathfrak{q}$ is the quotient measure, for $\mathfrak{q}$–almost every $q \in Q$ the set X_q is a geodesic, $\mathfrak{m}_q$ is supported on it and $q \mapsto \mathfrak{m}_q$ is a $\mathsf{CD}(K, N)$ disintegration. Finally, for $\mathfrak{q}$–almost

every $q \in Q$, we have $\int_{X_q} f\, \mathfrak{m}_q = 0$ and $f = 0$ $\mathfrak{m}$–ae in Z. We can also consider the trivial disintegration of $\mathfrak{m}$ restricted to Z where each equivalence class is a single point:

$$\mathfrak{m}_{\llcorner Z} = \int_Z \delta_z\, \mathfrak{m}(dz),$$

where δ_z stands for the Dirac delta in z. Then define $\hat{\mathfrak{q}} := \mathfrak{q} + \mathfrak{m}_{\llcorner Z}$ and $\hat{\mathfrak{m}}_q = \mathfrak{m}_q$ if $q \in Q$ and $\hat{\mathfrak{m}}_q = \delta_q$ if $q \in Z$. Since $Q \cap Z = \varnothing$, the previous definitions are well posed and we have the decomposition of $\mathfrak{m}$ on the whole space

$$\mathfrak{m} = \int_{Q \cup Z} \hat{\mathfrak{m}}_q\, \hat{\mathfrak{q}}(dq).$$

Step 2 Use the notation $A_{0,q} := A_0 \cap X_q$, $A_{1,q} := A_1 \cap X_q$ and the set of t–intermediate points between $A_{0,q}$ and $A_{1,q}$ in X_q is denoted with $A_{t,q} \subset X_q$. Then, from Lemma 3.2, for $\hat{\mathfrak{q}}$–ae $q \in Q$,

$$\mathfrak{m}_q(A_{t,q}) \geq \big(\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}_q(A_{0,q})^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}_q(A_{1,q})^{1/N}\big)^N.$$

Since $\int f\,\mathfrak{m}_q = 0$ implies $\mathfrak{m}_q(A_{0,q})/\mathfrak{m}(A_0) = \mathfrak{m}_q(A_{1,q})/\mathfrak{m}(A_1)$, it follows that

$$(3\text{-}2) \qquad \mathfrak{m}_q(A_{t,q}) \geq \frac{\mathfrak{m}_q(A_{0,q})}{\mathfrak{m}(A_0)}\big(\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N}\big)^N.$$

We now show that (3-2) holds also for $\hat{\mathfrak{q}}$–ae (or equivalently $\mathfrak{m}$–ae) $q \in Z$. Note that in this case $\mathfrak{m}_q$ has to be replaced by δ_q. Since, by construction, $0 = f = \chi_{A_0}/\mathfrak{m}(A_0) - \chi_{A_1}/\mathfrak{m}(A_1)$ on Z, then necessarily

$$\mathfrak{m}\big(Z \setminus \big((A_0 \cap A_1) \cup X \setminus (A_0 \cup A_1)\big)\big) = 0.$$

It follows that if Z does not have $\mathfrak{m}$–measure zero, we have two possibilities,

$$\mathfrak{m}\big(Z \cap (X \setminus (A_0 \cup A_1))\big) > 0, \quad \text{or} \quad \mathfrak{m}(A_0) = \mathfrak{m}(A_1) \text{ and } \mathfrak{m}(Z \cap (A_0 \cap A_1)) > 0.$$

Therefore, if $\mathfrak{m}(Z) > 0$, for $\hat{\mathfrak{q}}$–ae (or equivalently $\mathfrak{m}$–ae) $q \in Z$ we have two possibilities:

$$q \in X \setminus (A_0 \cup A_1), \quad \text{or} \quad q \in A_0 \cap A_1.$$

Interpreting the intermediate points as the point itself, in the first case, (3-2) (with $\mathfrak{m}_q$ replaced by δ_q) holds trivially (ie we get $0 \geq 0$). In the second case it reduces to show that

$$(\tau_{K,N}^{(1-t)}(\theta) + \tau_{K,N}^{(t)}(\theta))^N \leq 1.$$

For $K \geq 0$, since we are in the case $\mathfrak{m}(A_0 \cap A_1) > 0$, it follows that $\theta = 0$ and therefore $\tau_{K,N}^{(t)}(\theta) = t$, proving the previous inequality. For $K < 0$, recalling that

$K \to \sigma_{K,N}^{(t)}(\theta)$ is nondecreasing (see [7, Remark 2.2]) by Hölder's inequality,

$$(\tau_{K,N}^{(1-t)}(\theta) + \tau_{K,N}^{(t)}(\theta))^N \le (1 - t + t) \cdot (\sigma_{K,N-1}^{(1-t)}(\theta) + \sigma_{K,N-1}^{(t)}(\theta))^{N-1} \le 1,$$

as desired. We have therefore proved that

$$(3\text{-}3) \qquad \hat{\mathfrak{m}}_q(A_{t,q}) \ge \frac{\hat{\mathfrak{m}}_q(A_{0,q})}{\mathfrak{m}(A_0)} (\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N})^N$$

for $\hat{\mathfrak{q}}$–ae $q \in Q \cup Z$. Taking the integral of (3-3) in $q \in Q \cup Z$ one obtains that

$$
\begin{aligned}
\mathfrak{m}(A_t) &= \int_{Q \cup Z} \hat{\mathfrak{m}}_q(A_t \cap X_q) \, \hat{\mathfrak{q}}(dq) \\
&\ge \int_{Q \cup Z} \hat{\mathfrak{m}}_q(A_{t,q}) \, \hat{\mathfrak{q}}(dq) \\
&\ge (\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N})^N \int_{Q \cup Z} \frac{\hat{\mathfrak{m}}_q(A_{0,q})}{\mathfrak{m}(A_0)} \, \hat{\mathfrak{q}}(dq) \\
&= (\tau_{K,N}^{(1-t)}(\theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\theta)\mathfrak{m}(A_1)^{1/N})^N,
\end{aligned}
$$

and the claim follows. $\qquad \square$

4 p–spectral gap

Given a metric space (X, d), we denote by $\mathrm{Lip}(X)$ (resp. $\mathrm{Lip}_c(X)$) the vector space of real-valued Lipschitz functions (resp. with compact support). For a Lipschitz function $f \colon X \to \mathbb{R}$ the local Lipschitz constant $|\nabla f|$ is defined by

$$|\nabla f|(x) = \begin{cases} \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} & \text{if } x \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

For a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, for every $p \in (1, \infty)$ we define *the first eigenvalue* $\lambda_{1,p}(X, \mathsf{d}, \mathfrak{m})$ *of the p–Laplacian* by

$$(4\text{-}1) \quad \lambda_{(X,\mathsf{d},\mathfrak{m})}^{1,p} := \inf \left\{ \frac{\int_X |\nabla f|^p \, \mathfrak{m}}{\int_X |f|^p \, \mathfrak{m}} \; \middle| \; f \in \mathrm{Lip}(X) \cap L^p(X, \mathfrak{m}), \; f \ne 0, \atop \int_X f|f|^{p-2} \, \mathfrak{m} = 0 \right\}.$$

4.1 p–spectral gap for m.m.s. over $(\mathbb{R}, |\cdot|)$: the model spaces

Consider the family of probability measures

$$(4\text{-}2) \quad \mathcal{F}_{K,N,D}^s := \big\{ \mu \in \mathcal{P}(\mathbb{R}) \mid \mathrm{supp}(\mu) \subset [0, D], \; \mu = h_\mu \mathcal{L}^1, \; h_\mu \equiv \text{const if } N = 1$$
$$h_\mu \text{ verifies (2-6) and is continuous if } N \in (1, \infty) \big\},$$

where $D \in (0, \infty)$ and the corresponding *synthetic* first nonnegative eigenvalue of the p–Laplacian is

$$^s\lambda_{K,N,D}^{1,p} := \inf_{\mu \in \mathcal{F}_{K,N,D}^s} \inf\left\{ \frac{\int_{\mathbb{R}} |u'|^p \, \mu}{\int_{\mathbb{R}} |u|^p \, \mu} \,\Big|\, u \in \mathrm{Lip}(\mathbb{R}) \cap L^p(\mu), \int_{\mathbb{R}} u|u|^{p-2}\mu = 0, \, u \neq 0 \right\}.$$

The term synthetic refers to $\mu \in \mathcal{F}_{K,N,D}^s$, meaning that the Ricci curvature bound is satisfied in its synthetic formulation: if $\mu = h \cdot \mathcal{L}^1$, then h verifies (2-6).

The first goal of this section is to prove that $^s\lambda_{K,N,D}^{1,p}$ coincides with its smooth counterpart $\lambda_{K,N,D}^{1,p}$, defined by

(4-3) $\quad \lambda_{K,N,D}^{1,p} := \inf_{\mu \in \mathcal{F}_{K,N,D}} \inf\left\{ \frac{\int_{\mathbb{R}} |u'|^p \, \mu}{\int_{\mathbb{R}} |u|^p \, \mu} \,\Big|\, u \in \mathrm{Lip}(\mathbb{R}) \cap L^p(\mu), \right.$

$$\left. \int_{\mathbb{R}} u|u|^{p-2}\mu = 0, \, u \neq 0 \right\},$$

where now $\mathcal{F}_{K,N,D}$ denotes the set of $\mu \in \mathcal{P}(\mathbb{R})$ such that $\mathrm{supp}(\mu) \subset [0, D]$ and $\mu = h \cdot \mathcal{L}^1$ with $h \in C^2((0, D))$ satisfying

(4-4) $$(h^{1/(N-1)})'' + \frac{K}{N-1} h^{1/(N-1)} \leq 0.$$

It is easily verified that $\mathcal{F}_{K,N,D} \subset \mathcal{F}_{K,N,D}^s$.

In order to prove that $^s\lambda_{K,N,D}^{1,p} = \lambda_{K,N,D}^{1,p}$, the following approximation result, proved in [14, Lemma 6.2], will play a key role. In order to state it let us recall that a standard mollifier in $\mathbb{R}$ is a nonnegative $C^\infty(\mathbb{R})$ function ψ with compact support in $[0, 1]$ such that $\int_{\mathbb{R}} \psi = 1$.

Lemma 4.1 *Let $D \in (0, \infty)$ and let $h: [0, D] \to [0, \infty)$ be a continuous function. Fix $N \in (1, \infty)$ and, for $\varepsilon > 0$, define*

(4-5) $\quad h_\varepsilon(t) := [h^{1/(N-1)} * \psi_\varepsilon(t)]^{N-1} := \left[\int_{\mathbb{R}} h(t-s)^{1/(N-1)} \psi_\varepsilon(s) \, ds \right]^{N-1}$

$$= \left[\int_{\mathbb{R}} h(s)^{1/(N-1)} \psi_\varepsilon(t-s) \, ds \right]^{N-1},$$

where $\psi_\varepsilon(x) = (1/\varepsilon)\psi(x/\varepsilon)$ and ψ is a standard mollifier function. The following properties hold:

(1) *h_ε is a nonnegative C^∞ function with support in $[-\varepsilon, D + \varepsilon]$.*

(2) *$h_\varepsilon \to h$ uniformly as $\varepsilon \downarrow 0$; in particular, $h_\varepsilon \to h$ in L^1.*

(3) *If h satisfies the convexity condition (2-6) corresponding to the above fixed $N > 1$ and some $K \in \mathbb{R}$, then also h_ε does. In particular, h_ε satisfies the differential inequality (4-4).*

Proposition 4.2 *For every $p \in (1, +\infty)$, $N \in [1, \infty)$, $K \in \mathbb{R}$ and $D \in (0, \infty)$, we have ${}^s\lambda^{1,p}_{K,N,D} = \lambda^{1,p}_{K,N,D}$.*

Proof First of all observe that, for $N = 1$, clearly we have $\mathcal{F}_{K,N,D} = \mathcal{F}^s_{K,N,D}$, since the density h_μ has to be constant. We can then assume without loss of generality that $N \in (1, \infty)$.

Since $\mathcal{F}_{K,N,D} \subset \mathcal{F}^s_{K,N,D}$, clearly ${}^s\lambda^{1,p}_{K,N,D} \leq \lambda^{1,p}_{K,N,D}$.

Assume by contradiction the inequality is strict. Then there exists a measure $\mu = h \cdot \mathcal{L}^1 \in \mathcal{F}^s_{K,N,D}$ and $\delta > 0$ such that

$$\lambda^{1,p}_{(\mathbb{R},|\cdot|,\mu)} \leq \lambda^{1,p}_{K,N,D} - 2\delta.$$

Therefore, by the very definition of $\lambda^{1,p}_{(\mathbb{R},|\cdot|,\mu)}$, there exists $u \in \mathrm{Lip}(\mathbb{R})$ such that $u \neq 0$, $\int_\mathbb{R} u|u|^{p-2} h \, ds = 0$ and

$$(4\text{-}6) \qquad \int_\mathbb{R} |u'(s)|^p h(s) \, ds \leq \left(\lambda^{1,p}_{K,N,D} - \tfrac{3}{2}\delta\right) \int_\mathbb{R} |u(s)|^p h(s) \, ds.$$

Now, Lemma 4.1 gives a sequence $h_k \in C^\infty(\mathbb{R})$ such that

$$(4\text{-}7) \qquad \mathrm{supp}(h_k) \subset \left[-\tfrac{1}{k}, D + \tfrac{1}{k}\right], \quad \mu_k := h_k \cdot \mathcal{L}^1 \in \mathcal{F}_{K,N,D+2/k},$$
$$h_k \to h \text{ uniformly on } [0, D].$$

Now if we define $u_k := u - c_k \in \mathrm{Lip}(\mathbb{R}) \cap L^p(\mathbb{R}, h_k\mathcal{L}^1)$ with $c_k \in \mathbb{R}$ such that $\int_\mathbb{R} u_k|u_k|^{p-2} h_k \, ds = 0$, thanks to (4-7) we have $c_k \to 0$ and thus

$$\int_\mathbb{R} |u_k(s)|^p h_k(s) \, ds \to \int_\mathbb{R} |u(s)|^p h(s) \, ds,$$
$$\int_\mathbb{R} |u'_k(s)|^p h_k(s) \, ds \to \int_\mathbb{R} |u'(s)|^p h(s) \, ds.$$

Therefore (4-6), combined with the continuity of $\varepsilon \mapsto \lambda^{1,p}_{K,N,D+\varepsilon}$ (see Theorem 4.3 below), implies that for k large enough one has

$$\int_\mathbb{R} |u'_k(s)|^p h_k(s) \, ds \leq \left(\lambda^{1,p}_{K,N,D} - \delta\right) \int_\mathbb{R} |u_k(s)|^p h_k(s) \, ds$$
$$\leq \left(\lambda^{1,p}_{K,N,D+2/k} - \tfrac{1}{2}\delta\right) \int_\mathbb{R} |u_k(s)|^p h_k(s) \, ds,$$

contradicting the definition of $\lambda^{1,p}_{K,N,D+2/k}$ given in (4-3). $\qquad \square$

The next goal of the section is to understand the quantity $\lambda_{K,N,D}^{1,p}$. Since the density of the reference probability measure is now smooth, we enter into a more classical framework, where a number of people contributed. The sharp p–spectral gap in case $K > 0$ and without upper bounds on the diameter was obtained by Matei [45]. The case $K = 0$ and the diameter is bounded above was obtained in the sharp form by Valtorta [61]. Finally the case $K < 0$ and diameter bounded above was obtained in the sharp form by Naber and Valtorta [51]. Actually, as explained in their paper, the arguments in [51] hold in the general case $K \in \mathbb{R}$ and $N \in [1, \infty)$ provided one identifies the correct model space. As usual, to describe the model space one has to examine separately the cases $K < 0$, $K = 0$ and $K > 0$; in order to unify the presentation let us denote by $\tan_{K,N}(t)$ the function

(4-8) $$\tan_{K,N}(t) := \begin{cases} \sqrt{-K/(N-1)} \tanh(\sqrt{-K/(N-1)}\, t) & \text{if } K < 0, \\ 0 & \text{if } K = 0, \\ \sqrt{K/(N-1)} \tan(\sqrt{K/(N-1)}\, t) & \text{if } K > 0. \end{cases}$$

Now, for each $K \in \mathbb{R}$, $N \in [1, \infty)$ and $D \in (0, \infty)$, let $\hat{\lambda}_{K,N,D}^{1,p}$ denote the first positive eigenvalue on $\left[-\frac{1}{2}D, \frac{1}{2}D\right]$ of the eigenvalue problem

(4-9) $$\frac{d}{dt}(\dot{w}^{(p-1)}) + (N-1)\tan_{K,N}(t)\dot{w}^{(p-1)} + \hat{\lambda}_{K,N,D}^{1,p} w^{(p-1)} = 0.$$

It is possible to show (see [51]) that $\hat{\lambda}_{K,N,D}^{1,p}$ is the unique value of $\hat{\lambda}$ such that the solution of

$$\begin{cases} \dot{\phi} = \left(\dfrac{\hat{\lambda}}{p-1}\right)^{1/p} + \dfrac{N-1}{p-1}\tan_{K,N}(t)\cos_p^{(p-1)}(\phi)\sin_p(\phi), \\ \phi(0) = 0, \end{cases}$$

satisfies $\phi\left(\frac{1}{2}D\right) = \frac{1}{2}\pi_p$, where π_p, $\cos_p$ and $\sin_p$ are defined as follows:

For every $p \in (1, \infty)$ the positive number π_p is defined by

$$\pi_p := \int_{-1}^{1} \frac{ds}{(1-s^p)^{1/p}} = \frac{2\pi}{p\sin(\pi/p)}.$$

The $C^1(\mathbb{R})$ function $\sin_p \colon \mathbb{R} \to [-1, 1]$ is defined implicitly on $\left[-\frac{1}{2}\pi_p, \frac{3}{2}\pi_p\right]$ by

$$\begin{cases} t = \int_0^{\sin_p(t)} ds/(1-s^p)^{1/p} & \text{if } t \in \left[-\frac{1}{2}\pi_p, \frac{1}{2}\pi_p\right], \\ \sin_p(t) = \sin_p(\pi_p - t) & \text{if } t \in \left[\frac{1}{2}\pi_p, \frac{1}{3}3\pi_p\right], \end{cases}$$

and is periodic on $\mathbb{R}$. Set also, by definition, $\cos_p(t) = d\sin_p(t)/dt$. The usual fundamental trigonometric identity can be generalized by $|\sin_p(t)|^p + |\cos_p(t)|^p = 1$, and so it is easily seen that $\cos_p^{(p-1)} \in C^1(\mathbb{R})$. Clearly, if $p = 2$ one finds the usual quantities: $\pi_2 = \pi$, $\sin_2 = \sin$ and $\cos_2 = \cos$.

Theorem 4.3 [45; 61; 51] *Let $K \in \mathbb{R}$, $N \in [1, \infty)$ and $D \in (0, \infty)$. Then:*

(1) $\lambda_{K,N,D}^{1,p} = \hat{\lambda}_{K,N,D}^{1,p}$, *where $\lambda_{K,N,D}^{1,p}$ was defined in* (4-3) *and $\hat{\lambda}_{K,N,D}^{1,p}$ in* (4-9).

(2) *For every fixed $p \in (1, \infty)$, the map $K, N, D \mapsto \lambda_{K,N,D}^{1,p}$ is continuous.*

(3) *If $K > 0$ then, for every $D \in (0, \pi\sqrt{N-1/K}]$,*

$$\lambda_{K,N,D}^{1,p} \geq \lambda_{K,N,\pi\sqrt{N-1/K}}^{1,p}$$

and equality holds if and only if $D = \pi\sqrt{N-1/K}$. If, moreover, $N \in \mathbb{N}$, then

$$\lambda_{K,N,\pi\sqrt{N-1/K}}^{1,p} = \lambda^{1,p}(S^N(\sqrt{N-1/K})),$$

ie $\lambda_{K,N,\pi\sqrt{N-1/K}}^{1,p}$ coincides with the first eigenvalue of the p–Laplacian on the round sphere of radius $\sqrt{N-1/K}$.

(4) *If $K = 0$ then $\lambda_{0,N,D}^{1,p} = (p-1)(\pi_p/D)^p$.*

For $K \neq 0$ and $p \neq 2$, it is not easy to give an explicit expression of the lower bound $\lambda_{K,N,D}^{1,p}$. At least one can give some lower bounds; for instance recently Li and Wang [63] obtained that

$$(4\text{-}10) \qquad \lambda_{K,N,D}^{1,p} \geq \frac{1}{(p-1)^{p-1}}\left(\frac{NK}{N-1}\right)^{p/2} \quad \text{for } K > 0, \ p \geq 2.$$

4.2 p–spectral gap for $\mathrm{CD}_{\mathrm{loc}}(K, N)$ spaces

Theorem 4.4 *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying $\mathrm{CD}_{\mathrm{loc}}(K, N)$ for some $K, N \in \mathbb{R}$ with $N \geq 1$, and assume moreover it is essentially nonbranching. Let $D \in (0, \infty)$ be the diameter of X and fix $p \in (1, \infty)$. Then, for any Lipschitz function $f \in L^p(X, \mathfrak{m})$ with $\int_X f|f|^{p-2}\,\mathfrak{m}(dx) = 0$,*

$$(4\text{-}11) \qquad \lambda_{K,N,D}^{1,p} \int_X |f(x)|^p\,\mathfrak{m}(dx) \leq \int_X |\nabla f|^p(x)\,\mathfrak{m}(dx).$$

In other terms, $\lambda_{(X,\mathrm{d},\mathfrak{m})}^{1,p} \geq \lambda_{K,N,D}^{1,p}$. Notice that, for $D = \pi\sqrt{(N-1)/K}$ and $N \in \mathbb{N}$, it follows that

$$\lambda_{(X,\mathrm{d},\mathfrak{m})}^{1,p} \geq \lambda^{1,p}(S^N((N-1)/K)).$$

Proof Since the space (X, d) is bounded, the $\mathrm{CD}_{\mathrm{loc}}(K, N)$ condition implies that $\mathfrak{m}(X) < \infty$. Noting that the inequality (4-11) is invariant under multiplication of $\mathfrak{m}$ by a positive constant, we can assume without loss of generality that $\mathfrak{m}(X) = 1$. Observing that the function

$$(4\text{-}12) \qquad\qquad \tilde{f} := f|f|^{p-2} \in \mathrm{Lip}(X)$$

verifies the hypothesis of Theorem 2.7, we can write $X = Y \cup \mathcal{T}$ with

$$\widetilde{f}(x) = 0 \quad \text{for } \mathfrak{m}\text{–ae } y \in Y, \qquad \mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(dq),$$

with $\mathfrak{m}_q = g(q, \cdot) \sharp (h_q \cdot \mathcal{L}^1)$, where the density h_q verifies (2-6) for $\mathfrak{q}$–ae $q \in Q$ and

$$0 = \int_X \widetilde{f}(z) \, \mathfrak{m}_q(dz) = \int_{\mathrm{Dom}(g(q,\cdot))} \widetilde{f}(g(q,t)) \cdot h_q(t) \, \mathcal{L}^1(dt)$$

$$= \int_{\mathrm{Dom}(g(q,\cdot))} f(g(q,t)) |f(g(q,t))|^{p-2} \cdot h_q(t) \, \mathcal{L}^1(dt)$$

for $\mathfrak{q}$–ae $q \in Q$. Now consider the map $t \mapsto f_q(t) := f(g(q,t))$ and note that it is Lipschitz. Since $\mathrm{diam}(\mathrm{Dom}(g(q,\cdot))) \leq D$, from the definition of $\mathcal{F}^s_{K,N,D}$ and of $\lambda^{1,p}_{K,N,D}$ we deduce that

$$\lambda^{1,p}_{K,N,D} \int_{\mathbb{R}} |f_q(t)|^p h_q(t) \, \mathcal{L}^1(dt) \leq \int_{\mathbb{R}} |f'_q(t)|^p h_q(t) \, \mathcal{L}^1(dt).$$

Noticing that $|f'_q(t)| \leq |\nabla f|(g(q,t))$ one obtains that

$$\begin{aligned}
\lambda^{1,p}_{K,N,D} \int_X |f(x)|^p \, \mathfrak{m}(dx) &= \lambda^{1,p}_{K,N,D} \int_{\mathcal{T}} |f(x)|^p \, \mathfrak{m}(dx) \\
&= \lambda^{1,p}_{K,N,D} \int_Q \left(\int_X |f(x)|^p \, \mathfrak{m}_q(dx) \right) \mathfrak{q}(dq) \\
&= \lambda^{1,p}_{K,N,D} \int_Q \left(\int_{\mathrm{Dom}(g(q,\cdot))} |f_q(t)|^p h_q(t) \, \mathcal{L}^1(dt) \right) \mathfrak{q}(dq) \\
&\leq \int_Q \left(\int_{\mathrm{Dom}(g(q,\cdot))} |f'_q(t)|^p h_q(t) \, \mathcal{L}^1(dt) \right) \mathfrak{q}(dq) \\
&\leq \int_Q \left(\int_X |\nabla f|^p(x) \, (g(q,\cdot)) \sharp (h_q(t) \mathcal{L}^1)(dx) \right) \mathfrak{q}(dq) \\
&= \int_X |\nabla f|^p(x) \, \mathfrak{m}(dx),
\end{aligned}$$

and the claim follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

4.3 Almost rigidity for the p–spectral gap

Theorem 4.5 (almost equality in the p–spectral gap implies almost maximal diameter) *Let $N > 1$ and $p \in (1, \infty)$ be fixed. Then, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, N, p)$ such that the following holds:*

Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially nonbranching metric measure space that satisfies $\mathrm{CD}^*(N - 1 - \delta, N + \delta)$. If $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta$, then $\mathrm{diam}(X) \geq \pi - \varepsilon$.

Proof As above, without loss of generality we can assume $\mathfrak{m}(X) = 1$. Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ we can find an essentially

nonbranching metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfying $\mathrm{CD}^*(N-1-\delta, N+\delta)$, with $\mathfrak{m}(X) = 1$, such that $\mathrm{diam}(X) \leq \pi - \varepsilon_0$ but $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta$.

The definition of $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})}$ implies that there exists a function $f \in \mathrm{Lip}(X)$, with $\int_X f|f|^{p-2}\mathfrak{m} = 0$ and $\int_X |f|^p \, \mathfrak{m}(dx) = 1$, such that

$$(4\text{-}13) \qquad \int_X |\nabla f|^p(x) \, \mathfrak{m}(dx) \leq \lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} + \delta \leq \lambda^{1,p}_{N-1,N,\pi} + 2\delta.$$

On the other hand, Theorem 4.3 ensures that there exists $\eta > 0$ such that

$$\lambda^{1,p}_{N-1,N,D} \geq \lambda^{1,p}_{N-1,N,\pi} + 2\eta \quad \text{for all } D \in [0, \pi - \varepsilon_0].$$

Moreover, the continuity of $K, N, D \mapsto \lambda^{1,p}_{K,N,D}$ guarantees that for every $D_0 \in (0,1)$ there exists $\delta_0 = \delta_0(N, D_0)$ such that

$$\lambda^{1,p}_{N-1-\delta,N+\delta,D} \geq \lambda^{1,p}_{N-1,N,D} - \eta \quad \text{for all } \delta \in [0, \delta_0], \ D \in [D_0, 2\pi].$$

Since clearly by definition we have that $\lambda^{1,p}_{K,N,D} \geq \lambda^{1,p}_{0,N,D}$ for every $K > 0$, $N \geq 1$ and $p \in (1, \infty)$, Theorem 4.3 gives that

$$\lim_{D \downarrow 0} \lambda^{1,p}_{N-1-\delta,N+\delta,D} \geq \lim_{D \downarrow 0} \lambda^{1,p}_{0,N+\delta,D} = +\infty$$

uniformly for $\delta \in [0, \delta_0(N)]$. The combination of the last two estimates yields

$$(4\text{-}14) \quad \lambda^{1,p}_{N-1-\delta,N+\delta,D} \geq \lambda^{1,p}_{N-1,N,\pi} + \eta \quad \text{for all } D \in [0, \pi - \varepsilon_0], \ \delta \in [0, \delta_0(N)].$$

By repeating the proof of Theorem 4.4, and observing that, by construction, we have $\mathrm{diam}(\mathrm{Dom}(g(q, \cdot)) \leq \pi - \varepsilon_0$, we then obtain

$$
\begin{aligned}
\int_X |\nabla f|^p(x) \, \mathfrak{m}(dx) &= \int_Q \left(\int_X |\nabla f|^p(x) \, (g(q, \cdot)) \sharp (h_q(t) \mathcal{L}^1)(dx) \right) \mathfrak{q}(dq) \\
&\geq \int_Q \left(\int_{\mathrm{Dom}(g(q,\cdot))} |f_q'(t)|^p \, h_q(t) \, \mathcal{L}^1(dt) \right) \mathfrak{q}(dq) \\
&\geq \int_Q \lambda^{1,p}_{N-1-\delta,N+\delta,\mathrm{diam}(\mathrm{Dom}(g(q,\cdot)))} \\
&\qquad\qquad \times \left(\int_{\mathrm{Dom}(g(q,\cdot))} |f_q(t)|^p \, h_q(t) \, \mathcal{L}^1(dt) \right) \mathfrak{q}(dq) \\
&\geq (\lambda^{1,p}_{N-1,N,\pi} + \eta) \int_Q \left(\int_{\mathrm{Dom}(g(q,\cdot))} |f_q(t)|^p \, h_q(t) \, \mathcal{L}^1(dt) \right) \mathfrak{q}(dq) \\
&= \lambda^{1,p}_{N-1,N,\pi} + \eta,
\end{aligned}
$$

contradicting (4-13), once we choose $\delta < \tfrac{1}{2}\eta$. $\qquad\qquad \square$

Corollary 4.6 (almost equality in the p–spectral gap implies mGH-closeness to a spherical suspension) *Let $N \geq 2$ and $p \in (1, \infty)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N, p) > 0$ such that the following holds:*

Let $(X, \mathsf{d}, \mathfrak{m})$ be an RCD$^*(N - 1 - \delta, N + \delta)$ *space. If*

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + \delta,$$

then there exists an RCD$^*(N - 2, N - 1)$ *space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ such that*

$$\mathsf{d}_{\mathrm{mGH}}((X, \mathsf{d}, \mathfrak{m}), [0, \pi] \times^{N-1}_{\sin} Y) \leq \varepsilon.$$

Proof Fix $N \in [2, \infty)$ and $p \in (1, \infty)$ and assume by contradiction there exist $\varepsilon_0 > 0$ and a sequence $(X_j, \mathsf{d}_j, \mathfrak{m}_j)$ of RCD$^*(N - 1 - 1/j, N + 1/j)$ spaces such that $\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} \leq \lambda^{1,p}_{N-1,N,\pi} + 1/j$, but

(4-15) $$\mathsf{d}_{\mathrm{mGH}}(X_j, [0, \pi] \times^{N-1}_{\sin} Y) \geq \varepsilon_0 \quad \text{for every } j \in \mathbb{N}$$

and every RCD$^*(N - 2, N - 1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y) = 1$. Observe that [Theorem 4.5](#) yields

(4-16) $$\mathrm{diam}((X_j, \mathsf{d}_j)) \to \pi.$$

By the compactness/stability property of RCD$^*(K, N)$ spaces recalled in [Theorem 2.6](#) we get that up to subsequences the spaces X_j mGH-converge to a limit RCD$^*(N-1, N)$ space $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$. Since the diameter is continuous under mGH convergence of uniformly bounded spaces, [(4-16)](#) implies that $\mathrm{diam}((X_\infty, \mathsf{d}_\infty)) = \pi$. But then by the maximal diameter theorem [35] we get that $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is isomorphic to a spherical suspension $[0, \pi] \times^{N-1}_{\sin} Y$ for some RCD$^*(N - 2, N - 1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y) = 1$. Clearly this contradicts [(4-15)](#) and the thesis follows. $\square$

Corollary 4.7 (p–Obata theorem) *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$^*(N - 1, N)$ *space for some $N \geq 2$, and let $1 < p < \infty$. If*

$$\lambda^{1,p}_{(X,\mathsf{d},\mathfrak{m})} = \lambda^{1,p}_{N-1,N,\pi},$$

which equals $\lambda^{1,p}(S^N)$ if $N \in \mathbb{N}$, then $(X, \mathsf{d}, \mathfrak{m})$ is a spherical suspension, ie there exists an RCD$^*(N - 2, N - 1)$ *space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ such that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to $[0, \pi] \times^{N-1}_{\sin} Y$.*

Proof [Theorem 4.5](#) implies that $\mathrm{diam}((X, \mathsf{d})) = \pi$ and the thesis then follows by the maximal diameter theorem [35]. $\square$

Remark 4.8 Obata's theorem for $p = 2$ in $\mathrm{RCD}^*(N-1, N)$ spaces has been recently obtained by Ketterer [35] by different methods (see also [33]); the approach proposed here has the double advantage of length and of being valid for every $p \in (1, \infty)$.

5 The case $p = 1$ and the Cheeger constant

It is well known (see for instance [32; 64]) that an alternative way of defining $\lambda^{1,p}_{(X,\mathrm{d},\mathfrak{m})}$ which extends also to $p = 1$ is the following. For every $p \in [1, \infty)$ and every $f \in L^p(X)$ let

$$c_p(f) := \inf_{c \in \mathbb{R}} \left(\int_X |f - c|^p \, \mathfrak{m} \right)^{1/p}.$$

For every $p \in (1, \infty)$ it holds — see [32, Corollary 2.11] — that

$$\lambda^{1,p}_{(X,\mathrm{d},\mathfrak{m})} = \inf\left\{ \int_X |\nabla f|^p \, \mathfrak{m} \ \middle|\ f \in \mathrm{Lip} \cap L^p(X), \ c_p(f) = \|f\|_{L^p} = 1 \right\}.$$

It is then natural to set

$$(5\text{-}1) \qquad \lambda^{1,1}_{(X,\mathrm{d},\mathfrak{m})} = \inf\left\{ \int_X |\nabla f| \, \mathfrak{m} \ \middle|\ f \in \mathrm{Lip} \cap L^1(X), \ c_1(f) = \|f\|_{L^1} = 1 \right\}.$$

Assuming that $\mathfrak{m}(X) = 1$, recall that a number $M_f \in \mathbb{R}$ is a median for f if and only if

$$\mathfrak{m}(\{f \geq M_f\}) \geq \tfrac{1}{2} \quad \text{and} \quad \mathfrak{m}(\{f \leq M_f\}) \geq \tfrac{1}{2}.$$

It is not difficult to check that (see for instance [18, Section VI]) for every $f \in L^1(X)$ there exists a median of f, and moreover

$$\int_X |f - M_f| \, \mathfrak{m} = c_1(f)$$

holds for every median M_f of f. This link between $c_1(f)$ and M_f is useful to prove the equivalence between the Cheeger constant and $\lambda^{1,1}_{(X,\mathrm{d},\mathfrak{m})}$. Recall that the Cheeger constant $h_{(X,\mathrm{d},\mathfrak{m})}$ is defined by

$$h_{(X,\mathrm{d},\mathfrak{m})} := \inf\left\{ \frac{\mathfrak{m}^+(E)}{\mathfrak{m}(E)} \ \middle|\ E \subset X \text{ is Borel and } \mathfrak{m}(E) \in \left(0, \tfrac{1}{2}\right] \right\},$$

where

$$\mathfrak{m}^+(E) := \liminf_{\varepsilon \downarrow 0} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon}$$

is the (outer) Minkowski content. As usual $E^\varepsilon := \{x \in X \mid \mathrm{d}(x, y) < \varepsilon \text{ for some } y \in E\}$ is the ε–neighborhood of E with respect to the metric d. The next result, due to

Maz'ya [46] and Federer and Fleming [27] (see also [10] for a careful derivation, and [47, Lemma 2.2; 32, Proposition 2.13] for the present formulation), rewrites Cheeger's isoperimetric inequality in functional form.

Proposition 5.1 *Assume that $(X, \mathrm{d}, \mathfrak{m})$ is a m.m.s. with $\mathfrak{m}(\{x\}) = 0$ for every $x \in X$, ie $\mathfrak{m}$ is atomless. Then*

$$h_{(X,\mathrm{d},\mathfrak{m})} = \lambda^{1,1}_{(X,\mathrm{d},\mathfrak{m})}.$$

It is then clear that the comparison and almost rigidity theorems for $\lambda^{1,1}$ will be based on the corresponding isoperimetric ones obtained by the authors in [14]. To this aim, in the next subsection we briefly recall the model Cheeger constant for the comparison.

5.1 The model Cheeger constant $h_{K,N,D}$

If $K > 0$ and $N \in \mathbb{N}$, by the Lévy–Gromov isoperimetric inequality we know that, for N–dimensional smooth manifolds having Ricci curvature bounded below by K, the Cheeger constant i is bounded below by the one of the N–dimensional round sphere of the suitable radius. In other words the *model* Cheeger constant is the one of $\mathbb{S}^N$. For $N \geq 1$ and $K \in \mathbb{R}$ arbitrary real numbers the situation is more complicated, and just recently E Milman [48] discovered what the model Cheeger constant is (more precisely he discovered the model isoperimetric profile, which in turn implies the model Cheeger constant). In this short section we recall its definition.

Given $\delta > 0$, set

$$s_\delta(t) := \begin{cases} \sin(\sqrt{\delta}\,t)/\sqrt{\delta} & \text{if } \delta > 0, \\ t & \text{if } \delta = 0, \\ \sinh(\sqrt{-\delta}\,t)/\sqrt{-\delta} & \text{if } \delta < 0. \end{cases} \qquad c_\delta(t) := \begin{cases} \cos(\sqrt{\delta}\,t) & \text{if } \delta > 0, \\ 1 & \text{if } \delta = 0, \\ \cosh(\sqrt{-\delta}\,t) & \text{if } \delta < 0. \end{cases}$$

Given a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ with $f(0) \geq 0$, we denote by $f_+ \colon \mathbb{R} \to \mathbb{R}^+$ the function coinciding with f between its first nonpositive and first positive roots, and vanishing everywhere else, ie $f_+ := f\chi_{[\xi_-,\xi_+]}$ with $\xi_- = \sup\{\xi \leq 0 \mid f(\xi) = 0\}$ and $\xi_+ = \inf\{\xi > 0 \mid f(\xi) = 0\}$.

Given $H, K \in \mathbb{R}$ and $N \in [1, \infty)$, set $\delta := K/(N-1)$ and define the following (Jacobian) function of $t \in \mathbb{R}$:

$$J_{H,K,N}(t) := \begin{cases} \chi_{\{t=0\}} & \text{if } N = 1, \, K > 0, \\ \chi_{\{Ht \geq 0\}} & \text{if } N = 1, \, K \leq 0, \\ \left(c_\delta(t) + \dfrac{H}{N-1} s_\delta(t)\right)^{N-1}_+ & \text{if } N \in (1, \infty). \end{cases}$$

As last piece of notation, given a nonnegative integrable function f on a closed interval $L \subset \mathbb{R}$, we denote by $\mu_{f,L}$ the probability measure supported in L with density (with respect to the Lebesgue measure) proportional to f there. In order to simplify the notation a bit we will write $h_{(L,f)}$ in place of $h_{(L,|\cdot|,\mu_{f,L})}$.

The model Cheeger constant for spaces having Ricci curvature bounded below by $K \in \mathbb{R}$, dimension bounded above by $N \geq 1$ and diameter at most $D \in (0, \infty]$ is then defined by

$$(5\text{-}2) \qquad h_{K,N,D} := \inf_{H \in \mathbb{R}, a \in [0,D]} h_{([-a,D-a], J_{H,K,N})}.$$

The formula above has the advantage of considering all the possible cases in just one equation, but probably it is also instructive to isolate the different cases in a more explicit way. Indeed one can check [48, Section 4] that:

Case 1 If $K > 0$ and $D < \sqrt{(N-1)/K}\,\pi$,

$$h_{K,N,D} = \inf_{\xi \in [0, \sqrt{(N-1)/K}\,\pi - D]} h_{([\xi, \xi+D], \sin(\sqrt{K/(N-1)}\,t)^{N-1})}.$$

Case 2 If $K > 0$ and $D \geq \sqrt{(N-1)/K}\,\pi$,

$$h_{K,N,D} = h_{([0, \sqrt{(N-1)/K}\,\pi], \sin(\sqrt{K/(N-1)}\,t)^{N-1})}.$$

Case 3 If $K = 0$ and $D < \infty$,

$$h_{K,N,D} = \min\left\{ \inf_{\xi \geq 0} h_{([\xi, \xi+D], t^{N-1})},\ h_{([0,D],1)} \right\}$$

$$= \frac{N}{D} \inf_{\xi \geq 0, v \in (0,1/2)} \frac{\left(\min(v, 1-v)(\xi+1)^N + \max(v, 1-v)\xi^N \right)^{(N-1)/N}}{v[(\xi+1)^N - \xi^N]}.$$

Case 4 If $K < 0$ and $D < \infty$,

$$h_{K,N,D} = \min\left\{ \inf_{\xi \geq 0} h_{([\xi, \xi+D], \sinh(\sqrt{-K/(N-1)}\,t)^{N-1})},\ h_{([0,D], \exp(\sqrt{-K(N-1)}\,t))}, \right.$$

$$\left. \inf_{\xi \in \mathbb{R}} h_{([\xi, \xi+D], \cosh(\sqrt{-K/(N-1)}\,t)^{N-1})} \right\}.$$

In all the remaining cases, the model Cheeger constant trivializes: $h_{K,N,D} = 0$.

5.2 Sharp comparison and almost rigidity for $\lambda^{1,1} = h$

Theorem 5.2 *Let $(X, \mathrm{d}, \mathfrak{m})$ be an essentially nonbranching $\mathrm{CD}_{\mathrm{loc}}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in [1, \infty)$, with $\mathfrak{m}(X) = 1$ and having diameter $D \in (0, +\infty]$. Then*

$$(5\text{-}3) \qquad h_{(X, \mathrm{d}, \mathfrak{m})} \geq h_{K,N,D}.$$

Moreover, for $K > 0$ the following holds: for every $N > 1$ and $\varepsilon > 0$ there exists $\bar{\delta} = \bar{\delta}(K, N, \varepsilon)$ such that, for every $\delta \in [0, \bar{\delta}]$, if $(X, d, \mathfrak{m})$ is an essentially nonbranching $CD^*(K - \delta, N + \delta)$ space such that

$$(5\text{-}4) \qquad h_{(X,d,\mathfrak{m})} \le h_{K,N,\pi\sqrt{(N-1)/K}} + \delta,$$

which equals $h(S^N(\sqrt{(N-1)/K})) + \delta$ if $N \in \mathbb{N}$, then $\operatorname{diam}(X) \ge \pi\sqrt{(N-1)/K} - \varepsilon$.

Proof Recall that the isoperimetric profile of $(X, d, \mathfrak{m})$ is the largest function $\mathcal{I}_{(X,d,\mathfrak{m})}$: $[0, 1] \to \mathbb{R}^+$ such that for every Borel subset $E \subset X$ we have $\mathfrak{m}^+(E) \ge \mathcal{I}_{(X,d,\mathfrak{m})}(\mathfrak{m}(E))$. As discovered in [48] (see also [14, Section 2.5] for the present notation), for every $K \in \mathbb{R}$, $N \in [1, \infty)$ and $D \in (0, \infty]$ there exists a model isoperimetric profile $\mathcal{I}_{K,N,D} \colon [0, 1] \to \mathbb{R}^+$; it is straightforward to check that

$$h_{(X,d,\mathfrak{m})} = \inf_{v \in (0,1/2)} \frac{\mathcal{I}_{(X,d,\mathfrak{m})}(v)}{v} \quad \text{and} \quad h_{K,N,D} = \inf_{v \in (0,1/2)} \frac{\mathcal{I}_{K,N,D}(v)}{v}.$$

Since in our previous paper [14, Theorem 1.2] we proved that, for every $v > 0$,

$$(5\text{-}5) \qquad \mathcal{I}_{(X,d,\mathfrak{m})}(v) \ge \mathcal{I}_{K,N,D}(v),$$

the first claim (5-3) follows.

In order to prove the second part of the theorem, note (5-4) implies that there exists $\bar{v} \in \left(0, \frac{1}{2}\right)$ such that

$$\frac{\mathcal{I}_{(X,d,\mathfrak{m})}(\bar{v})}{\bar{v}} \le h_{(X,d,\mathfrak{m})} + \delta \le h_{K,N,\pi\sqrt{(N-1)/K}} + 2\delta \le \frac{\mathcal{I}_{K,N,\pi\sqrt{(N-1)/K}}(\bar{v})}{\bar{v}} + 2\delta.$$

Multiplying by $\bar{v}$, we get

$$\mathcal{I}_{(X,d,\mathfrak{m})}(\bar{v}) \le \mathcal{I}_{K,N,\pi\sqrt{(N-1)/K}}(\bar{v}) + 2\delta\bar{v} \le \mathcal{I}_{K,N,\pi\sqrt{(N-1)/K}}(\bar{v}) + \delta.$$

The thesis then follows by direct application of [14, Theorem 1.5]. $\qquad\square$

Before stating the result let us observe that if $(X, d, \mathfrak{m})$ is an $RCD^*(K, N)$ space for some $K > 0$ then, letting $d' := \sqrt{K/(N-1)}\, d$, we have that $(X, d', \mathfrak{m})$ is $RCD^*(N - 1, N)$; in other words, if the Ricci lower bound is $K > 0$ then up to scaling we can assume it is actually equal to $N - 1$.

Arguing as in the proof of Corollaries 4.6–4.7 we get the following result:

Corollary 5.3 *For every $N \in [2, \infty)$ and $\varepsilon > 0$, there exists $\bar{\delta} = \bar{\delta}(N, \varepsilon) > 0$ such that the following hold: For every $\delta \in [0, \bar{\delta}]$, if $(X, d, \mathfrak{m})$ is an $RCD^*(N - 1 - \delta, N + \delta)$ space with $\mathfrak{m}(X) = 1$ satisfying*

$$h_{(X,d,\mathfrak{m})} \le h_{N-1,N,\pi} + \delta,$$

which equals $h(S^N) + \delta$ if $N \in \mathbb{N}$, then there exists an RCD*$(N-2, N-1)$ space (Y, d_Y, m_Y) with $m_Y(Y) = 1$ such that

$$d_{\mathrm{mGH}}(X, [0, \pi] \times^{N-1}_{\sin} Y) \leq \varepsilon.$$

In particular, if (X, d, m) is an RCD*$(N-1, N)$ space satisfying

$$h_{(X,d,m)} = h_{N-1,N,\pi} = h(S^N),$$

then it is isomorphic to a spherical suspension, ie there exists an RCD*$(N-2, N-1)$ space (Y, d_Y, m_Y) with $m_Y(Y) = 1$ such that (X, d, m) is isomorphic to $[0, \pi] \times^{N-1}_{\sin} Y$.

6 Sharp log-Sobolev and Talagrand inequalities

6.1 Sharp log-Sobolev in diameter–curvature–dimensional form

Recall that a m.m.s. (X, d, m) supports the log-Sobolev inequality with constant $\alpha > 0$ if, for any Lipschitz function $f \colon X \to [0, \infty)$ with $\int_X f \, m = 1$,

$$(6\text{-}1) \qquad\qquad 2\alpha \int_X f \log f \, m \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \, m.$$

The largest constant α such that (6-1) holds for any Lipschitz function $f \colon X \to [0, \infty)$ with $\int_X f \, m = 1$, will be called the log-Sobolev constant of (X, d, m) and denoted by $\alpha^{LS}_{(X,d,m)}$.

As before we will reduce to the one-dimensional case. Given $K \in \mathbb{R}$, $N \geq 1$ and $D \in (0, +\infty]$, we denote by $\alpha^{LS}_{K,N,D} > 0$ the maximal constant α such that

$$(6\text{-}2) \qquad\qquad 2\alpha \int_{\mathbb{R}} f \log f \, \mu \leq \int_{\{f>0\}} \frac{|f'|^2}{f} \, \mu \quad \text{for all } \mu \in \mathcal{F}^s_{K,N,D}$$

for every Lipschitz $f \colon \mathbb{R} \to [0, \infty)$ with $\int f \, \mu = 1$.

Remark 6.1 If $K > 0$ and $D = \pi\sqrt{(N-1)/K}$, it is known that the corresponding optimal log-Sobolev constant is $KN/(N-1)$ (see the discussion below). It is an interesting open problem, which we don't address here, to give an explicit expression of the quantity $\alpha^{LS}_{K,N,D}$ for general $K \in \mathbb{R}$, $N \geq 1$ and $D \in (0, \infty)$.

Theorem 6.2 Let (X, d, m) be a metric measure space with diameter $D \in (0, \infty)$ and satisfying $\mathrm{CD}_{\mathrm{loc}}(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Assume moreover it

is essentially nonbranching. Then, for any Lipschitz function $f : X \to [0, \infty)$ with $\int_X f \, \mathfrak{m} = 1$,

$$2 \alpha^{LS}_{K,N,D} \int_X f \log f \, \mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \, \mathfrak{m}.$$

In other terms, $\alpha^{LS}_{(X,\mathrm{d},\mathfrak{m})} \geq \alpha^{LS}_{K,N,D}$.

Proof Since $\mathrm{CD}_{\mathrm{loc}}(K, N)$ implies that the measure is locally doubling, the finiteness of the diameter implies that $\mathfrak{m}(X) < \infty$. Observing that the log-Sobolev inequality (6-1) is invariant under a multiplication of $\mathfrak{m}$ by a constant, we can then assume without loss of generality that $\mathfrak{m}(X) = 1$. Consider any Lipschitz function with $\int_X f \, \mathfrak{m} = 1$ and apply Theorem 2.7 to $\hat{f} := 1 - f$. Hence we can write $X = Y \cup \mathcal{T}$ with

$$f(y) = 1 \quad \text{for } \mathfrak{m}\text{–ae } y \in Y, \qquad \mathfrak{m} \llcorner \mathcal{T} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(dq),$$

with $\mathfrak{m}_q = g(q, \cdot) \sharp (h_q \cdot \mathcal{L}^1)$, the density h_q verifies (2-6) for $\mathfrak{q}$–ae $q \in Q$ and

$$1 = \int_X f(z) \, \mathfrak{m}_q(dz) = \int_{\mathrm{Dom}(g(q,\cdot))} f(g(q,t)) \cdot h_q(t) \, \mathcal{L}^1(dt)$$

for $\mathfrak{q}$–ae $q \in Q$. Now consider the map $t \mapsto f_q(t) := f(g(q,t))$ and note that it is Lipschitz. Since $\mathrm{diam}(\mathrm{Dom}(g(q, \cdot))) \leq D$, from the definition of $\mathcal{F}^s_{K,N,D}$ and of $\alpha^{LS}_{K,N,D}$ we deduce that

$$2 \alpha^{LS}_{K,N,D} \int_{\mathbb{R}} f_q(t) \log(f_q(t)) \, h_q(t) \, \mathcal{L}^1(dt) \leq \int_{\{f_q(\cdot)>0\}} \frac{|f_q'(t)|^2}{f_q(t)} \, h_q(t) \, \mathcal{L}^1(dt).$$

Noticing that $|f_q'(t)| \leq |\nabla f|(g(q,t))$ and that $f \log f$ vanishes over Y, one obtains that

$$2 \alpha^{LS}_{K,N,D} \int_X f \log f \, \mathfrak{m}(dx)$$

$$= 2 \alpha^{LS}_{K,N,D} \int_{\mathcal{T}} f \log f \, \mathfrak{m}(dx)$$

$$= 2 \alpha^{LS}_{K,N,D} \int_Q \left(\int_X f \log f \, \mathfrak{m}_q(dx) \right) \mathfrak{q}(dq)$$

$$= 2 \alpha^{LS}_{K,N,D} \int_Q \left(\int_{\mathrm{Dom}(g(q,\cdot))} f_q(t) \log(f_q(t)) \, h_q(t) \, \mathcal{L}^1(dt) \right) \mathfrak{q}(dq)$$

$$\leq \int_Q \left(\int_{\mathrm{Dom}(g(q,\cdot)) \cap \{f_q(\cdot)>0\}} \frac{|f_q'(t)|^2}{f_q(t)} \, h_q(t) \, \mathcal{L}^1(dt) \right) \mathfrak{q}(dq)$$

$$\leq \int_Q \left(\int_{\{f>0\}} \frac{|\nabla f|^2}{f} (g(q,\cdot)) \, \sharp \, (h_q(t) \, \mathcal{L}^1)(dx) \right) \mathfrak{q}(dq)$$

$$\leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \, \mathfrak{m}(dx),$$

and the claim follows. □

If $K > 0$, by the Bonnet–Myers diameter bound, we know that if $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}_{\mathrm{loc}}(K, N)$ then $\mathrm{diam}(X) \leq \pi \sqrt{(N-1)/K}$. Recalling definition (6-2), we then set $\alpha_{K,N}^{LS} := \alpha_{K,N,\pi \sqrt{(N-1)/K}}^{LS}$ for the log-Sobolev constant without an upper diameter bound. By applying the regularization of the measures $h \mathcal{L}^1 \in \mathcal{F}_{K,N,\pi \sqrt{(N-1)/K}}^s$ discussed in Lemma 4.1 and arguing analogously to the proof of Proposition 4.2, we get that in the definition of $\alpha_{K,N}^{LS}$ it is equivalent to take the inf among measures in $\mathcal{F}_{K,N,\pi \sqrt{(N-1)/K}}$, defined in (4-4). But now if $\mu \in \mathcal{F}_{K,N,\pi \sqrt{(N-1)/K}}$ is a probability measure on $\mathbb{R}$ with smooth density satisfying the $\mathsf{CD}_{\mathrm{loc}}(K, N)$ condition for $K > 0$, it is known that the sharp log-Sobolev constant is $\alpha_{K,N}^{LS} = KN/(N-1)$ (see for instance [8, Proposition 6.6]). More precisely, as proved by Mueller and Weissler [50], for every $K > 0$ and $N \geq 1$, the sharp constant is attained by the usual model probability measure on the interval $[0, \sqrt{(N-1)/K} \, \pi]$ proportional to $\sin(\sqrt{K/(N-1)} \, t)^{N-1}$; notice that for $N \in \mathbb{N}$ it corresponds to the round sphere of radius $\sqrt{(N-1)/K}$. We then have the following corollary:

Corollary 6.3 (sharp log-Sobolev under $\mathsf{CD}_{\mathrm{loc}}(K, N)$, $K > 0$ and $N > 1$) *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space satisfying $\mathsf{CD}_{\mathrm{loc}}(K, N)$ for some $K > 0$ and $N > 1$, and assume moreover it is essentially nonbranching. Then, for any Lipschitz function $f \colon X \to [0, \infty)$ with $\int_X f \, \mathfrak{m} = 1$,*

$$\frac{2KN}{N-1} \int_X f \log f \, \mathfrak{m} \leq \int_{\{f>0\}} \frac{|\nabla f|^2}{f} \, \mathfrak{m}.$$

In other terms, $\alpha_{(X,\mathsf{d},\mathfrak{m})}^{LS} \geq KN/(N-1)$.

Let us mention that, since the reduction to a 1–dimensional problem is done via an L^1–optimal transportation argument, Corollary 6.3 can be seen as a solution to [62, Open Problem 21.6].

6.2 From sharp log-Sobolev to sharp Talagrand

First of all let us recall that the relative entropy functional $\mathrm{Ent}_{\mathfrak{m}} \colon \mathcal{P}(X) \to [0, +\infty]$ with respect to a given $\mathfrak{m} \in \mathcal{P}(X)$ is defined to be

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \begin{cases} \int_X \varrho \log \varrho \, \mathfrak{m} & \text{if } \mu = \varrho \mathfrak{m}, \\ +\infty & \text{otherwise.} \end{cases}$$

Otto and Villani [53] proved that for smooth Riemannian manifolds the log-Sobolev inequality with constant $\alpha > 0$ implies the Talagrand inequality with constant $2/\alpha$ preserving sharpness. The result was then generalized to arbitrary metric measure spaces by Gigli and Ledoux [28], so that we can state:

Theorem 6.4 (from log-Sobolev to Talagrand [53; 28]) *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space supporting the log-Sobolev inequality with constant $\alpha > 0$. Then it also supports the Talagrand inequality with constant $2/\alpha$, ie*

$$W_2^2(\mu, \mathfrak{m}) \leq \frac{2}{\alpha} \mathrm{Ent}_{\mathfrak{m}}(\mu)$$

for all $\mu \in \mathcal{P}(X)$.

Combining Theorem 6.2 with Theorem 6.4 we get Theorem 1.6, which improves the Talagrand constant $2/K$, which is sharp for $\mathrm{CD}(K, \infty)$ spaces, by a factor $(N-1)/N$ if the dimension is bounded above by N. This constant is sharp for $\mathrm{CD}_{\mathrm{loc}}(K, N)$ spaces, indeed it is sharp already in the smooth setting [62, Remark 22.43]. Since both our proof of the sharp log-Sobolev inequality and the proof of Theorem 6.4 are essentially optimal transport based, this be seen as an answer to [62, Open Problem 22.44].

Remark 6.5 (sharpness and estimates of the best constants) Recall that, for weighted smooth manifolds, the log-Sobolev inequality implies the Talagrand inequality, which in turn implies the Poincaré inequality every step without any loss in the constants [62, Theorem 22.17]. Since, when we compute the comparison log-Sobolev constant $\alpha_{K,N,D}^{LS}$ and the comparison first eigenvalue $\lambda_{K,N,D}^{1,2}$, we work with the smooth measures $\mathcal{F}_{K,N,D}$ on $\mathbb{R}$, we always have the estimate

$$(6\text{-}3) \qquad\qquad \alpha_{K,N,D}^{LS} \leq \lambda_{K,N,D}^{1,2}.$$

Notice that for $K > 0$ and $D = \sqrt{(N-1)/K}\,\pi$ they actually coincide:

$$(6\text{-}4) \qquad \frac{KN}{N-1} = \alpha_{K,N,\sqrt{(N-1)/K}\,\pi}^{LS} = \lambda_{K,N,\sqrt{(N-1)/K}\,\pi}^{1,2}.$$

An interesting question we do not address here is if this is always the case, ie if in (6-3) equality holds for every $K \in \mathbb{R}$, $N \geq 1$ and $D \in (0, \infty)$. Since the value of $\lambda_{K,N,D}^{1,2}$ is known in many cases, it would have as a consequence the determination of the explicit value of the best constant in both the log-Sobolev and the Talagrand inequalities in the curvature–dimension–diameter forms. This would also imply rigidity and almost-rigidity statements attached to the log-Sobolev and Talagrand inequalities, as proven here for the Poincaré inequality. Let us note that for the almost rigidity to hold for both the log-Sobolev and Talagrand inequalities it would be enough to prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha_{K,N,D}^{LS} \geq \alpha_{K,N,\sqrt{(N-1)/K}\,\pi}^{LS} + \delta = KN/(N-1) + \delta$ if $D \in [0, \sqrt{(N-1)/K}\,\varepsilon - \delta]$.

7 Sharp Sobolev inequalities

Recall that $(X, \mathrm{d}, \mathfrak{m})$ supports a (p, q)–Sobolev inequality with constant $\alpha^{p,q}$ if, for any Lipschitz function $f \colon X \to \mathbb{R}$,

$$(7\text{-}1) \qquad \frac{\alpha^{p,q}}{p-q} \left\{ \left(\int_X |f|^p \, \mathfrak{m} \right)^{\frac{q}{p}} - \int_X |f|^q \, \mathfrak{m} \right\} \le \int_X |\nabla f|^q \, \mathfrak{m}.$$

The largest constant $\alpha^{p,q}$ such that (7-1) holds for any Lipschitz function f will be called the (p, q)–Sobolev constant of $(X, \mathrm{d}, \mathfrak{m})$ and will be denoted by $\alpha^{p,q}_{(X,\mathrm{d},\mathfrak{m})}$.

Again we consider the one-dimensional case and given $K \in \mathbb{R}$, $N \ge 1$ and $D \in (0, \infty]$ we define ${}^s\alpha^{p,q}_{K,N,D}$ to be the maximal constant α such that

$$\frac{\alpha}{p-q} \left\{ \left(\int_X |f|^p \, \mu \right)^{\frac{q}{p}} - \int_X |f|^q \, \mu \right\} \le \int_X |\nabla f|^q \, \mu \quad \text{for all } \mu \in \mathcal{F}^s_{K,N,D}$$

for every Lipschitz function $f \colon \mathbb{R} \to \mathbb{R}$. Restricting the maximization to $\mu \in \mathcal{F}_{K,N,D}$, we obtain the constant $\alpha^{p,q}_{K,N,D}$. Using the approximation Lemma 4.1 and reasoning as in Proposition 4.2 one obtains that

$$ {}^s\alpha^{p,q}_{K,N,D} = \alpha^{p,q}_{K,N,D}.$$

Theorem 7.1 *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space with diameter $D \in (0, \infty)$ and satisfying $\mathrm{CD}_{\mathrm{loc}}(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Assume moreover it is essentially nonbranching. Then, for any Lipschitz function,*

$$\frac{\alpha^{p,q}_{K,N,D}}{p-q} \left\{ \left(\int_X |f(x)|^p \, \mathfrak{m}(dx) \right)^{\frac{q}{p}} - \int_X |f(x)|^q \, \mathfrak{m}(dx) \right\} \le \int_X |\nabla f(x)|^q \, \mathfrak{m}(dx),$$

In other terms, it holds $\alpha^{p,q}_{(X,\mathrm{d},\mathfrak{m})} \ge \alpha^{p,q}_{K,N,D}$.

Proof First of all note that $\mathrm{CD}_{\mathrm{loc}}(K, N)$ coupled with the finiteness of the diameter implies $\mathfrak{m}(X) < \infty$.

Step 1 (the case $p > q$) With a slight abuse of notation, q will denote both the exponent in the Sobolev embedding and the index in the disintegration; there should be no confusion between the clearly different roles. Fix any Lipschitz function f and consider the function $\hat{f}(x) := 1 - c|f(x)|^p$, with $c := 1/(\int |f|^p \mathfrak{m})$. Therefore $\int \hat{f} \, \mathfrak{m} = 0$ and we can invoke Theorem 2.7. Hence $X = Y \cup \mathcal{T}$ with

$$\hat{f}(y) = 0 \quad \text{for } \mathfrak{m}\text{–ae } y \in Y, \qquad \mathfrak{m}_{\llcorner \mathcal{T}} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(dq),$$

with $\mathfrak{m}_q = g(q, \cdot) \sharp (h_q \cdot \mathcal{L}^1)$, the density h_q verifies (2-6) for $\mathfrak{q}$–ae $q \in Q$ and

$$0 = \int_X \hat{f}(z) \, \mathfrak{m}_q(dz) = \int_{\mathrm{Dom}(g(q, \cdot))} \hat{f}(g(q, t)) \cdot h_q(t) \, \mathcal{L}^1(dt)$$

for $\mathfrak{q}$–ae $q \in Q$.

Now consider the map $t \mapsto f_q(t) := f(g(q, t))$ and note that it is Lipschitz. Since $\mathrm{diam}(\mathrm{Dom}(g(q, \cdot))) \leq D$, from the definition of $\mathcal{F}^s_{K,N,D}$ and of $\alpha^{p,q}_{K,N,D}$ we deduce that

$$\left(\int_{\mathbb{R}} |f_q(t)|^p h_q(t) \, \mathcal{L}^1(dt) \right)^{\frac{q}{p}}$$
$$\leq \int_{\mathbb{R}} |f_q(t)|^q h_q(t) \, \mathcal{L}^1(dt) + \frac{p-q}{\alpha^{p,q}_{K,N,D}} \int_{\mathbb{R}} |f'(t)|^q h_q(t) \, \mathcal{L}^1(dt).$$

Since for $\mathfrak{q}$–ae $q \in Q$ we have $\int \hat{f} \, \mathfrak{m}_q = 0$, it follows that

$$\int_X |f(x)|^p \, \mathfrak{m}_q(dx) = \frac{1}{c} = \int_X |f(x)|^p \, \mathfrak{m}(dx).$$

Therefore the previous inequality reads as

$$1 \leq \left(\frac{1}{\int |f(x)|^p \, \mathfrak{m}(dx)} \right)^{\frac{q}{p}} \left(\int_X |f_q|^q \, \mathfrak{m}_q + \frac{p-q}{\alpha^{p,q}_{K,N,D}} \int_X |f'|^q \, \mathfrak{m}_q \right).$$

Noticing that $|f'_q(t)| \leq |\nabla f|(g(q, t))$, integrating over Q one obtains that

$$(7\text{-}2) \quad \mathfrak{m}(\mathcal{T}) \leq \left(\frac{1}{\int |f(x)|^p \, \mathfrak{m}(dx)} \right)^{\frac{q}{p}} \int_{\mathcal{T}} |f(x)|^q \, \mathfrak{m}(dx)$$
$$+ \frac{p-q}{\alpha^{p,q}_{K,N,D}} \int_{\mathcal{T}} |\nabla f(x)|^q \, \mathfrak{m}(dx).$$

To complete the argument one should prove that, for each $y \in Y$,

$$1 \leq \left(\frac{1}{\int |f|^p \, \mathfrak{m}} \right)^{\frac{q}{p}} \left(|f(y)|^q + \frac{p-q}{\alpha^{p,q}_{K,N,D}} |\nabla f(y)|^q \right).$$

As for $\mathfrak{m}$–ae $y \in Y$ one has $|f(y)|^p = \int_X |f|^p \, \mathfrak{m}$, this last inequality holds trivially. Integrating this last inequality over Y and adding it to (7-2), we obtain the claim.

Step 2 (the case $p < q$) This follows repeating the previous localization argument and writing the Sobolev inequality in the form

$$\left(\int_X |f(x)|^p \, \mathfrak{m}(dx) \right)^{\frac{q}{p}} \geq \int_X |f(x)|^q \, \mathfrak{m}(dx) - \frac{q-p}{\alpha} \int_X |\nabla f(x)|^q \, \mathfrak{m}(dx). \qquad \square$$

As already observed, if $K > 0$ then $\text{diam}(X) \leq \pi\sqrt{(N-1)/K}$ and therefore one can define

$$\alpha^{p,q}_{K,N} := \alpha^{p,q}_{K,N,\pi\sqrt{(N-1)/K}},$$

the (p,q)–Sobolev inequality with no diameter upper bound. If $\mu \in \mathcal{F}_{K,N,\pi\sqrt{(N-1)/K}}$ with $K > 0$, it is known that the sharp $(p,2)$–Sobolev constant verifies (see for instance [38, Theorem 3.1])

$$\alpha^{p,2}_{K,N} \geq \frac{KN}{N-1} \quad \text{for } 1 \leq p \leq \frac{2N}{N-2}.$$

Moreover, for $N \in \mathbb{N}$ it is attained on the round sphere of radius $\sqrt{(N-1)/K}$. We then have the following corollary:

Corollary 7.2 Let $(X, d, \mathfrak{m})$ be a metric measure space satisfying $\text{CD}^*(K, N)$ for some $K > 0$ and $N \in (2, \infty)$, and assume moreover it is essentially nonbranching. Then, for any Lipschitz function f,

$$\frac{KN}{(p-2)(N-1)}\left\{\left(\int_X |f|^p \, \mathfrak{m}\right)^{\frac{2}{p}} - \int_X |f|^2 \, \mathfrak{m}\right\} \leq \int_X |\nabla f|^2 \, \mathfrak{m}$$

for any $2 < p \leq 2N/(N-2)$. In other terms, $\alpha^{p,2}_{(X,d,\mathfrak{m})} \geq KN/(N-1)$.

Corollary 7.2 can be seen as a solution to [62, Open Problem 21.11].

Appendix

All the inequalities we have presented here rely on the general scheme of applying one-dimensional localization to a big family of inequalities, called 4–functions inequalities (see for instance the work of Kannan, Lovász and Simonovits [34]).

The argument goes as follows. Suppose we are interested in proving that, for integrable functions f_1, f_2, f_3 and f_4 and α, $\beta > 0$,

$$\text{(A-1)} \qquad \left(\int_X f_1 \, \mathfrak{m}\right)^\alpha \left(\int_X f_2 \, \mathfrak{m}\right)^\beta \leq \left(\int_X f_3 \, \mathfrak{m}\right)^\alpha \left(\int_X f_4 \, \mathfrak{m}\right)^\beta.$$

Then consider the one-dimensional localization induced by $g := f_3 - cf_1$, with $c = \left(\int f_3 \, \mathfrak{m}\right)/\left(\int f_1 \, \mathfrak{m}\right)$,

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(dq),$$

where $X = \mathcal{T} \cup Y$ and on Y we have $g(x) = 0$ for $\mathfrak{m}$–ae $x \in Y$. Then it is sufficient to prove that

$$\left(\int_X f_1 \, \mathfrak{m}_q\right)^\alpha \left(\int_X f_2 \, \mathfrak{m}_q\right)^\beta \leq \left(\int_X f_3 \, \mathfrak{m}_q\right)^\alpha \left(\int_X f_4 \, \mathfrak{m}_q\right)^\beta \quad \text{for } \mathfrak{q}\text{–ae } q \in Q,$$
$$f_2(x) \leq c^{\alpha/\beta} f_4(x) \qquad\qquad \text{for } \mathfrak{m}\text{–ae } x \in Y.$$

Indeed from the localization it follows that $\int g \, \mathfrak{m}_q = 0$ for $\mathfrak{q}$–ae $q \in Q$ and therefore

$$\int_X f_2(x) \, \mathfrak{m}_q(dx) \leq c^{\alpha/\beta} \int_X f_4(x) \, \mathfrak{m}_q(dq) \quad \text{for } \mathfrak{q}\text{–ae } q \in Q.$$

Integrating over Q and adding the integral over Y, (A-1) follows.

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