msp

Geometry & Topology 20 (2016) 2079–2133

The motive of a classifying space

BURT TOTARO

We give the first examples of finite groups G such that the Chow ring of the classifying space BG depends on the base field, even for fields containing the algebraic closure of $\mathbb{Q}$. As a tool, we give several characterizations of the varieties that satisfy Künneth properties for Chow groups or motivic homology.

We define the (compactly supported) motive of a quotient stack in Voevodsky's derived category of motives. This makes it possible to ask when the motive of BG is mixed Tate, which is equivalent to the motivic Künneth property. We prove that BG is mixed Tate for various "well-behaved" finite groups G, such as the finite general linear groups in cross-characteristic and the symmetric groups.

14C15; 14F42, 14M20, 14A20

The Chow group of algebraic cycles generally does not satisfy the Künneth formula. Nonetheless, there are some schemes X over a field k that satisfy the *Chow Künneth property* that the product $\mathrm{CH}_* X \otimes_{\mathbb{Z}} \mathrm{CH}_* Y \to \mathrm{CH}_*(X \times_k Y)$ is an isomorphism for all separated schemes Y of finite type over k. The Chow Künneth property implies the *weak Chow Künneth property* that $\mathrm{CH}_* X \to \mathrm{CH}_*(X_F)$ is surjective for every finitely generated field F over k (or, equivalently, for every extension field F of k). We characterize several properties of this type. (We also prove versions of all our results with coefficients in a given commutative ring.)

Our characterizations of Künneth properties are as follows. First, a smooth proper scheme X over k satisfies the weak Chow Künneth property if and only if the Chow motive of X is a summand of a finite direct sum of *Tate motives* (Theorem 4.1). (This is related to known results by Bloch, Srinivas, Jannsen, Kimura and others.) A more novel result is about an arbitrary separated scheme X of finite type over k. We say that X satisfies the *motivic Künneth property* if the Künneth spectral sequence converges to the motivic homology groups of $X \times_k Y$ for all Y. (Motivic homology groups are also called *higher Chow groups*; they include the usual Chow groups as a special case.) We show that a k–scheme X satisfies the motivic Künneth property if and only if the motive of X in Voevodsky's derived category of motives is a *mixed Tate motive* (Theorem 7.2). (An example of a scheme with these properties is any *linear scheme*, as discussed in Section 5.) Finally, if a smooth but not necessarily proper k–variety X satisfies the weak Chow Künneth property, then the *birational motive* of X in the

Published: 15 September 2016 DOI: 10.2140/gt.2016.20.2079

sense of Rost and of Kahn and Sujatha is isomorphic to the birational motive of a point (Corollary 2.2).

The last result cannot be strengthened to say that the motive of X is mixed Tate; one has to consider motivic homology groups to get that conclusion. For example, the complement X of a curve of genus 1 in the affine plane has the Chow Künneth property, since $\mathrm{CH}_{i+2}(X \times_k Y) \cong \mathrm{CH}_i Y$ for all separated k–schemes Y of finite type and all i; but the motive of X is not mixed Tate.

As an application of these general results, we disprove the weak Chow Künneth property for some classifying spaces BG. For an affine group scheme G of finite type over a field k, Morel and Voevodsky [40, Section 4.2] and I [55; 57] constructed BG as a direct limit of smooth k–varieties, quotients by G of open subsets of representations of G over k. As a result, the Chow ring of BG makes sense. The Chow ring of BG tensored with the rationals is easy to compute; for example, if G is finite, then $\mathrm{CH}^i(BG) \otimes \mathbb{Q} = 0$ for $i > 0$. The challenge is to understand the integral or mod p Chow ring of BG.

For many finite groups G and fields k, the classifying space BG over k satisfies the Chow Künneth property that $\mathrm{CH}^* BG \otimes_{\mathbb{Z}} \mathrm{CH}_* Y \cong \mathrm{CH}_*(BG \times_k Y)$ for all separated k–schemes Y of finite type. For example, an abelian p–group G of exponent e has the Chow Künneth property when k is a field of characteristic not p that contains the e^{th} roots of unity. The Chow Künneth property also holds for many other groups, such as wreath products of abelian groups [57, Lemma 2.12]. As a result, [57, Chapter 17] asked whether every finite group G has the Chow Künneth property over a field k that contains enough roots of unity. This would imply the weak Chow Künneth property that $\mathrm{CH}^* BG_k \to \mathrm{CH}^* BG_F$ is surjective for every extension field F of k.

In this paper, we give the first examples of finite groups for which the Chow Künneth property fails. For any finite group G such that BG has nontrivial unramified cohomology, there is a finitely generated field F over $\overline{\mathbb{Q}}$ such that $\mathrm{CH}^* BG_{\overline{\mathbb{Q}}} \to \mathrm{CH}^* BG_F$ is not surjective (Corollary 3.1). We also find a field E containing $\overline{\mathbb{Q}}$ such that $\mathrm{CH}^i(BG_E)/p$ is infinite for some i and some prime number p (Corollary 3.2); this answers another question in [57, Chapter 18]. In particular, the ring $\mathrm{CH}^*(BG_E)/p$ is not noetherian in such an example.

As recalled in Section 2, there are groups of order p^5 for any odd prime p, and groups of order 2^6, that have nontrivial unramified cohomology. This is surprisingly sharp. In fact, the Chow ring $\mathrm{CH}^* BG_k$ of a p–group G is independent of the field k containing $\overline{\mathbb{Q}}$, and consists of transferred Euler classes of representations, when G is a p–group of order at most p^4 [57, Theorem 11.1, Theorem 17.4]. Moreover, the weak Chow Künneth property holds for all groups of order 2^5 (Theorem 10.1).

Finally, Section 8 defines the *compactly supported motive*, in Voevodsky's derived category of motives, for a quotient stack over a field. In particular, we get a notion of the compactly supported motive $M^c(BG)$ for an affine group scheme G. Once we have this definition, we can ask when $M^c(BG)$ is a mixed Tate motive. This property is equivalent to the motivic Künneth property for BG, and so it implies the Chow Künneth property for BG. In particular, BG is not mixed Tate for the groups of order p^5 discussed above. On the other hand, we show that many familiar finite groups, such as the finite general linear groups in cross-characteristic and the symmetric groups, are mixed Tate (Theorems 9.11 and 9.12).

The introduction to Section 9 discusses six properties of finite groups. It would be interesting to find out whether all six properties are equivalent, as the known examples suggest. The properties are: stable rationality of BG (say, over the complex numbers), meaning stable rationality of quotient varieties V/G; triviality for the birational motive of BG (or equivalently, of quotient varieties V/G); Ekedahl's class of BG in the Grothendieck ring of varieties being equal to 1 [21]; the weak Chow Künneth property of BG; the Chow Künneth property of BG; and the mixed Tate property of BG.

Acknowledgements I thank Christian Haesemeyer, Tudor Pădurariu, Roberto Pirisi, Yehonatan Sella and a referee for their suggestions. This work was supported by NSF grant DMS-1303105.

1 Notation

A *variety* over a field k means an integral separated scheme of finite type over k. A variety X over k is *geometrically integral* if $X_{\overline{k}} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is an integral scheme (where $\overline{k}$ is an algebraic closure of k), or equivalently if X_E is integral for every extension field E of k [25, Definition IV.4.6.2].

Let X be a scheme of finite type over a field k. The *Chow group* $\mathrm{CH}_i\, X$ is the group of i–dimensional algebraic cycles on X modulo rational equivalence. A good reference is Fulton [23, Chapter 1]. We write $\mathrm{CH}_i(X; R) = \mathrm{CH}_i(X) \otimes_{\mathbb{Z}} R$ for a commutative ring R.

For a smooth scheme X over k, understood to be of finite type over k, we write $\mathrm{CH}^i\, X$ for the Chow group of codimension–i cycles on X. For X smooth over k, the groups $\mathrm{CH}^*\, X$ have a ring structure given by intersecting cycles [23, Chapter 6].

For a field k, let k_s be a separable closure of k, and let M be a torsion $\mathrm{Gal}(k_s/k)$–module with torsion whose order is invertible in k. For a smooth variety X over k, we define the i^{th} *unramified cohomology* group of X with coefficients in M to be

$H^i_{\mathrm{nr}}(k(X)/k, M)$, the subgroup of elements of $H^i(k(X), M)$ that are unramified with respect to all divisorial valuations of $k(X)$ over k. This group is a birational invariant of X. It agrees with the group $H^0(X, H^i_M)$ (sections of the sheafification of the Zariski presheaf $U \mapsto H^i_{\mathrm{et}}(U, M)$) for X proper over k, but not in general [17, Theorem 4.1.1; 47, Corollary 12.10].

For example, if G is an affine group scheme over k, then $H^0(BG, H^i)$ is the group of cohomological invariants of G considered by Serre [24, Part 1], whereas the unramified cohomology of BG in the sense of this paper, $H^i_{\mathrm{nr}}(k(BG), M)$, is a subgroup of that. Both groups can be defined using the variety U/G as a model of BG, where U is any open subset of a k–representation V of G such that G acts freely on U and $V-U$ has codimension at least 2 [24, Part 1, Appendix C].

2 Birational motives

In this section, we give several equivalent characterizations of those smooth proper varieties X whose birational motive in the sense of Rost [33, Appendix RC] and Kahn and Sujatha [32, Equation (2.5)] is isomorphic to the birational motive of a point. The statement includes Merkurjev's theorem that the Chow group of zero-cycles on X is unchanged under field extensions if and only if the unramified cohomology of X in the most general sense is trivial [39]. It seems to be new that these properties are also equivalent to all the Chow groups of X being supported on a divisor. Note that these properties are not equivalent to CH_0 being supported on a divisor; for example, the product of $\mathbb{P}^1$ with a curve of genus at least 1 has CH_0 supported on a divisor, while CH_1 is not supported on a divisor. Also, unlike many earlier results in this area, we work with an arbitrary coefficient ring, not just the rational numbers.

We will use the equivalences of Theorem 2.1 to give the first counterexamples to the Chow Künneth property for the classifying space of a finite group over an algebraically closed field (Section 3).

Theorem 2.1 *Let X be a smooth proper variety over a field k, and let R be a nonzero commutative ring. The following are equivalent:*

(1) *For every finitely generated field F/k, the map $\mathrm{CH}_0(X;R) \to \mathrm{CH}_0(X_F;R)$ is surjective.*

(2) *For every field F/k, the degree homomorphism $\mathrm{CH}_0(X_F;R) \to R$ is an isomorphism.*

(3) *The birational motive of X (in the sense of Kahn and Sujatha) with R coefficients is isomorphic to the birational motive of a point.*

(4) *For every* R*–linear cycle module* M *over* k *(in the sense of Rost [47]), the homomorphism* $M(k) \to M(k(X))_{\mathrm{nr}}$ *is an isomorphism. (That is,* X *has trivial unramified cohomology in the most general sense.)*

(5) *There is a closed subset* $S \subsetneq X$ *such that*

$$\mathrm{CH}_i(X;R)/\mathrm{CH}_i(S;R) \to \mathrm{CH}_i(X_F;R)/\mathrm{CH}_i(S_F;R)$$

is surjective for all finitely generated fields F/k *and all integers* i*. (That is, all the Chow groups of* X *are constant outside a divisor.)*

(6) *The variety* X *is geometrically integral, and there is a closed subset* $S \subsetneq X$ *such that* $\mathrm{CH}_i(X_F;R)/\mathrm{CH}_i(S_F;R) = 0$ *for all fields* F/k *and all* $i < \dim X$*.*

For the coefficient ring $R = \mathbb{Q}$ which has been considered most often, there are other equivalent statements: instead of considering all finitely generated extension fields of k in (1) or (5), one could consider a single algebraically closed field of infinite transcendence degree over k. This gives equivalent conditions, because $\mathrm{CH}_*(X_F;\mathbb{Q}) \to \mathrm{CH}_*(X_E;\mathbb{Q})$ is injective for every scheme X over F and every inclusion of fields $F \hookrightarrow E$. On the other hand, for the coefficient ring $R = \mathbb{F}_p$ which is of most interest for the classifying space of a finite group, it would not be enough to consider algebraically closed extension fields in Theorem 2.1. This follows from Suslin's rigidity theorem: for every extension of algebraically closed fields $F \hookrightarrow E$, every k–scheme X over F, and every prime number p invertible in F, $\mathrm{CH}_*(X_F;\mathbb{F}_p)$ maps isomorphically to $\mathrm{CH}_*(X_E;\mathbb{F}_p)$ [51, Corollary 2.3.3].

Corollary 2.2 *Let* k *be a perfect field which admits resolution of singularities (for example, any field of characteristic zero). Let* U *be a smooth variety over* k*, not necessarily proper. Let* R *be a commutative ring. If* $\mathrm{CH}^*(U;R) \to \mathrm{CH}^*(U_F;R)$ *is surjective for every finitely generated field* F *over* k*, then the birational motive of* U *with coefficients in* R *is isomorphic to the birational motive of a point.*

Proof By resolution of singularities, there is a regular compactification X of U over k, with $U = X - S$ for some closed subset S. Since k is perfect, the regular scheme X is smooth over k. Let us index Chow groups by dimension. We use the localization sequence for Chow groups:

Lemma 2.3 [23, Proposition 1.8] *Let* X *be a scheme of finite type over a field* k*. Let* Z *be a closed subscheme. Then the proper pushforward and flat pullback maps fit into an exact sequence*

$$\mathrm{CH}_i(Z) \to \mathrm{CH}_i(X) \to \mathrm{CH}_i(X-Z) \to 0.$$

In the case at hand, it follows that $\mathrm{CH}_*(U;R)$ is isomorphic to $\mathrm{CH}_*(X;R)/\mathrm{CH}_*(S;R)$. So the assumption on U implies condition (5) in Theorem 2.1. The birational motive of U is (by definition) the same as that of X. By Theorem 2.1, the birational motive of X with coefficients in R is isomorphic to the birational motive of a point. $\square$

Proof of Theorem 2.1 Assume condition (1), ie that $\mathrm{CH}_0(X;R) \to \mathrm{CH}_0(X_F;R)$ is surjective for every finitely generated field F/k. Let n be the dimension of X. The generic fiber of the diagonal Δ in $\mathrm{CH}_n(X \times_k X)$ via projection to the first copy of X is a zero-cycle in $\mathrm{CH}_0\, X_{k(X)}$. By our assumption, the class $[\Delta]$ in $\mathrm{CH}_0(X_{k(X)};R)$ is the image of some zero-cycle $\alpha \in \mathrm{CH}_0(X;R)$. For a variety Y over k, the Chow groups of $X_{k(Y)}$ are the direct limit of the Chow groups of $X \times_k U$, where U runs over all nonempty open subsets of Y. (Note that an i–dimensional cycle on $X \times_k U$ gives a cycle of dimension $i - \dim Y$ on the generic fiber $X_{k(Y)}$.) Therefore, we can write

$$\Delta = X \times \alpha + B$$

in $\mathrm{CH}_n(X \times_k X;R)$, where B is a cycle supported on $S \times X$ for some closed subset $S \subsetneq X$. Here we are using the localization sequence for Chow groups (Lemma 2.3).

As a correspondence, the diagonal Δ induces the identity map from $\mathrm{CH}_i(X;R)$ to $\mathrm{CH}_i(X;R)$ for any i. For this purpose, think of Δ as a correspondence from the first copy of X to the second. It follows that for any extension field F of k and any zero-cycle β in $\mathrm{CH}_0(X_F;R)$, we have

$$\beta = \Delta_*(\beta) = (X \times \alpha)_*(\beta) = \deg(\beta)\alpha.$$

Thus the R–module $\mathrm{CH}_0(X_F;R)$ is generated by α for every field F/k. Moreover, α has degree 1, and so the degree map $\deg\colon \mathrm{CH}_0(X_F;R) \to R$ is an isomorphism. We have proved condition (2).

Condition (3) is immediate from (2). Namely, for any smooth proper varieties X and Y over k, the set of morphisms from the birational motive of X (with R coefficients) to the birational motive of Y is defined to be $\mathrm{CH}_0(Y_{k(X)};R)$ [32, Equation (2.5)]. So, for a point $p = \operatorname{Spec} k$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{bir}}(p,p) &= R,\\
\mathrm{Hom}_{\mathrm{bir}}(X,p) &= \mathrm{CH}_0(\operatorname{Spec} k(X);R) = R,\\
\mathrm{Hom}_{\mathrm{bir}}(p,X) &= \mathrm{CH}_0(X;R),\\
\mathrm{Hom}_{\mathrm{bir}}(X,X) &= \mathrm{CH}_0(X_{k(X)};R).
\end{aligned}$$

By (2), we know that $\mathrm{CH}_0(X;R)$ and $\mathrm{CH}_0(X_{k(X)};R)$ both map isomorphically to R by the degree map; so X has the birational motive of a point. It is now clear that (1), (2) and (3) are equivalent.

When $R = \mathbb{Z}$, Merkurjev proved that (4) is equivalent to (1) and (2) [39, Theorem 2.11]. The proof works with any coefficient ring R. For example, to see that (3) implies (4), it suffices to check that an element of $\mathrm{CH}_0(Y_{k(X)}; R)$ determines a pullback map from unramified cohomology of Y (with coefficients in any R–linear cycle module over k) to unramified cohomology of X.

Now we show that (1) (or, equivalently, (2), (3) or (4)) implies (5) and (6). Given (1), we have a decomposition of the diagonal as above,

$$\Delta = X \times \alpha + B$$

in $\mathrm{CH}_n(X \times_k X; R)$, where B is a cycle supported on $S \times X$ for some closed subset $S \subsetneq X$. Now use the correspondence Δ to pull cycles back from the second copy of X to the first; again, it induces the identity on Chow groups. It follows that for any extension field F of k and any cycle β in $\mathrm{CH}_i(X_F; R)$ with $i < n$, we have $\beta = \Delta^*(\beta) = B^*(\beta)$, which is a cycle supported in S. Thus $\mathrm{CH}_i(S_F; R) \to \mathrm{CH}_i(X_F; R)$ is surjective for all $i < n$.

To prove (6), we also have to show that X is geometrically integral. Since X is smooth and proper over k, $\mathrm{CH}_n(X_{\bar{k}}; R)$ is the free R–module on the set of irreducible components of $X_{\bar{k}}$, and the cycle $[X]$ is the element $(1, \dots, 1)$ in this module. But for any irreducible component Y of $X_{\bar{k}}$, with class $(1, 0, \dots, 0)$ in $\mathrm{CH}_n(X_{\bar{k}}; R)$, we have

$$[Y] = \Delta^*[Y] = (X \times \alpha)^*[Y] \in R \cdot [X] = R \cdot (1, \dots, 1)$$

in $\mathrm{CH}_n(X_{\bar{k}}; R)$. Since the ring R is not zero, it follows that $X_{\bar{k}}$ is irreducible. This proves (6) and hence the weaker statement (5).

Finally, we prove that (5) implies (1), which will complete the proof. This part of the argument seems to be new. We are assuming that there is a closed subset $S \subsetneq X$ such that $\mathrm{CH}_i(X; R)/\mathrm{CH}_i(S; R) \to \mathrm{CH}_i(X_F; R)/\mathrm{CH}_i(S_F; R)$ is surjective for all finitely generated fields F/k and all integers i. Taking $i = n$, it follows that X is geometrically integral (using that R is not zero). As above, let $[\Delta]$ denote the generic fiber in $\mathrm{CH}_0(X_{k(X)}; R)$ of the diagonal Δ in $\mathrm{CH}_n(X \times_k X; R)$. We will show by descending induction on j that, for each $0 \leq j \leq n$, there is a closed subset T_j of X of dimension at most j such that $[\Delta]$ is the image of a zero-cycle α_j on $(T_j)_{k(X)}$. This is clear for $j = n$, by taking $T_n = X$.

Suppose we have a closed subset T_j and a zero-cycle α_j as above, for an integer $1 \leq j \leq n$. Then α_j is the generic fiber (with respect to the first projection) of some n–dimensional cycle A_j on $X \times_k T_j$. Let $T_{j1}, \dots, T_{jm}$ be the irreducible components of dimension j in T_j, and Z the union of any irreducible components of dimension less than j in T_j. We can write A_j in $\mathrm{CH}_n(X \times T_j; R)$ as a sum of cycles A_{jr}

supported on $X \times T_{jr}$, for $r = 1, \dots, m$, and a cycle B_j supported on $X \times Z$. The generic fiber of A_{jr} by the second projection is an $(n-j)$–cycle on $X_{k(T_{jr})}$. By our assumption (5), this cycle is rationally equivalent to the sum of a cycle on $S_{k(T_{jr})}$ and a cycle coming from an $(n-j)$–cycle on X. Therefore, A_{jr} is equivalent to a sum of cycles supported on $X \times Y$ for subvarieties Y of dimension at most $j-1$ and cycles supported on $W \times X$ for closed subsets $W \subsetneq X$ (using that $j > 0$). This proves the inductive step: $[\Delta]$ in $\mathrm{CH}_0(X_{k(X)}; R)$ is the image of a zero-cycle on $(T_{j-1})_{k(X)}$ for some closed subset T_{j-1} of dimension at most $j-1$ in X.

At the end of the induction, we have a zero-dimensional closed subset T_0 of X such that the class of Δ in $\mathrm{CH}_0(X_{k(X)}; R)$ is the image of a zero-cycle α_0 on $(T_0)_{k(X)}$. Here T_0 is a finite union of closed points, which are isomorphic to Spec E for finite extension fields E of k. Because X is geometrically integral, $(\operatorname{Spec} E)_{k(X)} = \operatorname{Spec}(E \otimes_k k(X))$ is the spectrum of a field. So $\mathrm{CH}_0(T_0; R) \to \mathrm{CH}_0((T_0)_{k(X)}; R)$ is an isomorphism. We conclude that the class of Δ in $\mathrm{CH}_0(X_{k(X)}; R)$ is in the image of $\mathrm{CH}_0(X; R)$. This gives a decomposition of the diagonal

$$\Delta = X \times \alpha + B$$

in $\mathrm{CH}_n(X \times_k X; R)$, where α is a zero-cycle on X and B is a cycle supported on $W \times X$ for some closed subset $W \subsetneq X$. This implies statement (2), by the same argument used to show that (1) implies (2). Thus all the conditions are equivalent. □

We now strengthen Theorem 2.1 in a certain direction: if a variety over an algebraically closed field k has nontrivial unramified cohomology, then its Chow groups over extension fields of k have arbitrarily large cardinality (Lemma 2.5). Our proof uses the language of birational motives. One could also give a more bare-hands argument.

Lemma 2.4 *Let k be a field and R a commutative ring. Let W_1 be a variety, W_2 a smooth proper variety, and X a separated scheme of finite type over k. For any integer r, there is a natural pairing*

$$\mathrm{CH}_0((W_2)_{k(W_1)}; R) \otimes_R \mathrm{CH}_r(X_{k(W_2)}; R) \to \mathrm{CH}_r(X_{k(W_1)}; R)$$

which agrees with the obvious pullback when the zero-cycle on $(W_2)_{k(W_1)}$ is the one associated to a dominant rational map $W_1 \dashrightarrow W_2$. As a result, the assignment $W \mapsto \mathrm{CH}_r(X_{k(W)}; R)$ for smooth proper varieties W over k extends to a contravariant functor on the category of birational motives over k with R coefficients.

Proof Let M be an R–linear cycle module over k. The unramified cohomology group $A^0(W; M)$ is defined in Rost [47, Section 5] for k–varieties W. For W smooth proper over k, the group $A^0(W; M)$ is a birational invariant of W, which coincides with

$M(k(W))_{\mathrm{nr}}$ [47, Section 12]. Rost observed that unramified cohomology $A^0(W;M)$ for smooth proper varieties W over k is a contravariant functor on the category of birational motives over k [33, Theorem RC.9]. More precisely, there is a pairing

$$\mathrm{CH}_0((W_2)_{k(W_1)};R)\otimes_R A^0(W_2;M)\to A^0(W_1;M)$$

for any variety W_1 and any smooth proper variety W_2 over k.

It remains to observe that for a separated scheme X of finite type over k and an integer r, there is a cycle module M over k with $A^0(W;M,-r)\cong \mathrm{CH}_r(X_{k(W)};R)$ for all k–varieties W. (The index $-r$ refers to the grading of a cycle module, as in [47, Section 5].) Namely, let $M(F)$ be the R–linear cycle module $A_r(X_F;K_*)$, in the notation of [47, Section 7]. Here F runs over fields F/k, and K_* denotes Milnor K–theory tensored with R. Define the grading of $M(F)$ by saying that elements of $M(F,j)$ are represented by elements of Milnor K_{r+j} of function fields of r–dimensional subvarieties of X_F. Then, by definition,

$$A^0(X;M)=\ker\Bigl(A_r(Y_{k(X)},K_*)\to \bigoplus_{x\in X^{(1)}} A_r(Y_{k(x)};K_*)\Bigr).$$

The group $A_r(Y_{k(X)};K_*,R)$ is the Chow group $\mathrm{CH}_r(Y_{k(X)};R)$. The boundary map takes this graded piece of $A_r(Y_{k(X)};K_*)$ to a zero group (involving K_{-1} of function fields of r–dimensional subvarieties of $Y_{k(x)}$ for codimension-1 points x in X). So $A^0(X;M,-r)\cong \mathrm{CH}_r(Y_{k(X)};R)$, as we want. □

Lemma 2.5 *Let X be a separated scheme of finite type over an algebraically closed field k of characteristic zero, R a commutative ring, and r an integer. Suppose that there is a field E/k such that $\mathrm{CH}_r(X;R)\to \mathrm{CH}_r(X_E;R)$ is not surjective. Then $\mathrm{CH}_r(X_F;R)$ can have arbitrarily large cardinality for fields F/k. In particular, there is a field F/k with $\mathrm{CH}_r(X_F;R)$ not finitely generated as an R–module.*

Proof We can assume that the field E is finitely generated over k. Then E is the function field of some variety W over k. Since k has characteristic zero, we can assume that W is smooth and projective over k. Since k is algebraically closed, W is geometrically integral, and so all powers W^n are varieties over k. Also, since k is algebraically closed, W has a zero-cycle of degree 1, which we can use to give a splitting $M_{\mathrm{bir}}(W)\cong M_{\mathrm{bir}}(k)\oplus T$ for some birational motive T. So, for any natural number n, we have $M_{\mathrm{bir}}(W^n)\cong (M_{\mathrm{bir}}(k)\oplus T)^{\otimes n}$, which contains $M_{\mathrm{bir}}(k)\oplus T^{\oplus n}$ as a summand. By Lemma 2.4 it follows that, for any separated k–scheme X of finite type, we have a canonical splitting

$$\mathrm{CH}_r(X_{k(W^n)};R)\cong \mathrm{CH}_r(X;R)\oplus (\mathrm{CH}_r(X_{k(W)};R)/\mathrm{CH}_r(X;R))^{\oplus n}\oplus(\text{something}).$$

For any set S, let F be the direct limit of the function fields of the varieties W^T over all finite subsets T of S. Then $\mathrm{CH}_r(X_F; R)$ is the direct limit of the Chow groups of the varieties $X_{k(W^T)}$. By the previous paragraph, $\mathrm{CH}_r(X_F; R)$ contains a direct sum of copies of $\mathrm{CH}_r(X_{k(W)}; R)/\mathrm{CH}_r(X; R)$ indexed by the elements of S. Since we assumed that $\mathrm{CH}_r(X_{k(W)}; R)/\mathrm{CH}_r(X; R)$ is not zero, $\mathrm{CH}_r(X_F; R)$ can have arbitrarily large cardinality for fields F/k. □

3 Failure of the weak Chow Künneth property for finite groups

We apply Theorem 2.1 to give the first counterexamples to the Chow Künneth property for the classifying space of a finite group G over an algebraically closed field k, answering a question from [55, Section 6] and [57, Chapter 17]. Namely, if the unramified cohomology of BG in the sense of Section 1 is nontrivial, then the weak Chow Künneth property fails, meaning that there is a finitely generated field F over k with $\mathrm{CH}^* BG_k \to \mathrm{CH}^* BG_F$ not surjective. Examples where BG has nontrivial unramified H^2 were constructed by Saltman and Bogomolov [6]. Correcting Bogomolov's earlier statements, Hoshi, Kang and Kunyavskiĭ gave examples of groups of order p^5 for every odd prime p, and groups of order 2^6, with nontrivial unramified H^2 [26, Theorem 1.13]. These results are sharp for all prime numbers p. Indeed, p–groups of order at most p^4 satisfy the weak Chow Künneth property [57, Theorem 11.1, Theorem 17.4], as do all groups of order 32 (Theorem 10.1).

Chu and Kang showed that for any p–group G of order at most p^4 and exponent e, if k is a field of characteristic not p which contains the e^{th} roots of unity, then BG is stably rational over k [13]. (Concretely, this means that the variety V/G is stably rational over k for every faithful representation V of G over k. The stable birational equivalence class of V/G for a faithful representation V of a finite group G is independent of the representation V, by Bogomolov and Katsylo [8].) For 2–groups of order at most 2^5, BG is again stably rational, by Chu, Hu, Kang and Prokhorov [12]. It is striking that BG has the weak Chow Künneth property for p–groups of order at most p^4, and for groups of order 32, although there is no obvious implication between stable rationality of BG and the weak Chow Künneth property for BG. (If BG can be approximated by quotients $(V-S)/G$ which are linear schemes in the sense of Section 5, then both properties hold; and both properties imply the triviality of unramified cohomology.)

We show in Corollary 3.2 that, for every finite group G such that BG_k has nontrivial unramified cohomology with $\mathbb{F}_p$ coefficients, there is an extension field F of k such that $\mathrm{CH}^i(BG_F)/p$ is infinite for some i. This answers another question from [57, Chapter 18]. In particular, the ring $\mathrm{CH}^*(BG_F)/p$ is not noetherian, and does not consist of transferred Euler classes of representations.

We can still ask whether the abelian group $\mathrm{CH}^i\, BG_F$ is finitely generated for every finite group G and every integer i when F is an algebraically closed field. The question of finiteness is also interesting for other classes of fields, such as finitely generated fields over $\mathbb{Q}$ or $\mathbb{F}_p$. The "motivic Bass conjecture" [31, Conjecture 37] would imply that the Chow groups of every variety over a finitely generated field are finitely generated; that would imply that each group $\mathrm{CH}^i\, BG_F$ is finitely generated for every affine group scheme G over a finitely generated field F.

Finding that the Chow Künneth property fails should be just the beginning. Let G be a group of order p^5 such that BG has nontrivial unramified cohomology. What is the Chow ring of BG over an arbitrary field (say, containing $\overline{\mathbb{Q}}$)? We know that it will depend on the field.

Corollary 3.1 *Let G be an affine group scheme of finite type over a field k. Suppose that k is perfect and k admits resolution of singularities (for example, k could be any field of characteristic zero). Let p be a prime number which is invertible in k. Suppose that the homomorphism $H^i(k,M) \to H^i_{\mathrm{nr}}(k(V/G),M)$ of unramified cohomology is not an isomorphism, for some finite $\mathrm{Gal}(k_s/k)$–module M over $\mathbb{F}_p$, some generically free representation V of G over k, and some integer i. (The stable birational equivalence class of V/G for V generically free is independent of the representation V, and so this hypothesis does not depend on the choice of V.) Then the weak Chow Künneth property with $\mathbb{F}_p$ coefficients fails for BG over k, meaning that $\mathrm{CH}^*(BG)/p \to \mathrm{CH}^*(BG_F)/p$ is not surjective for some finitely generated field F over k.*

To relate the p–groups mentioned earlier to this statement, note that those groups G (of order p^5 for p odd or order 2^6 for $p=2$) are shown to have nontrivial unramified Brauer group $H^2_{\mathrm{nr}}(k(V/G),G_m)$, where k is an algebraically closed field in which p is invertible. This group is p–power torsion, by a transfer argument. By results of Grothendieck, $H^2_{\mathrm{nr}}(k(V/G),\mu_p)$ is isomorphic to the p–torsion subgroup of $H^2_{\mathrm{nr}}(k(V/G),G_m)$ [17, Proposition 4.2.3]. Therefore $H^2_{\mathrm{nr}}(k(V/G),\mu_p)$ is also nonzero, and so Corollary 3.1 applies to these groups G.

Explicitly, for any prime number $p \geq 5$, here is an example of a group G of order p^5 with unramified H^2 (over $\mathbb{C}$) not zero [26, proof of Theorem 2.3]:

$$G = \big\langle f_1, f_2, f_3, f_4, f_5 \mid f_i^p = 1 \text{ for all } i,\ f_5 \text{ central},\ [f_2,f_1]=f_3,\\ [f_3,f_1]=f_4,\ [f_4,f_1]=[f_3,f_2]=f_5,\ [f_4,f_2]=[f_4,f_3]=1\big\rangle.$$

In this presentation, we use the notation $[g,h]=g^{-1}h^{-1}gh$.

Proof of Corollary 3.1 By definition, $\mathrm{CH}^i\,BG$ is isomorphic to $\mathrm{CH}^i(V-S)/G$ for any representation V of G over k and any G–invariant (Zariski) closed subset S such that G acts freely on $V-S$ with quotient a scheme and S has codimension greater than i in V [55, Theorem 1.1; 57, Theorem 2.5]. By the localization sequence for equivariant Chow groups, the homomorphism

$$\mathrm{CH}^*\,BG=\mathrm{CH}^*_G\,V\to\mathrm{CH}^*_G(V-S)=\mathrm{CH}^*(V-S)/G$$

is surjective [20, Proposition 5; 57, Lemma 2.9].

Suppose that $\mathrm{CH}^*(BG)/p\to\mathrm{CH}^*(BG_F)/p$ is surjective for every finitely generated field F over k. Let V be a representation of G with a closed subset $S\subsetneq V$ such that G acts freely on $V-S$ with quotient a separated scheme $U=(V-S)/G$. By the previous paragraph, applied to G and G_F, it follows that $\mathrm{CH}^*(U)/p\to\mathrm{CH}^*(U_F)/p$ is surjective for every finitely generated field F over k. By Corollary 2.2, U has the birational motive of a point, with $\mathbb{F}_p$ coefficients. It follows that the field $k(U)$ over k has trivial unramified cohomology with coefficients in any $\mathbb{F}_p$–linear cycle module over k. Galois cohomology (with p invertible in k, as we assume) is an example of a cycle module. Explicitly, for any finite $\mathrm{Gal}(k_s/k)$–module M killed by p, the assignment $F\mapsto\bigoplus_i H^i(F,M\otimes\mu_p^{\otimes i})$ for finitely generated fields F over k is a cycle module over k [47, Remark 2.5]. That completes the proof. $\square$

The following corollary strengthens Corollary 3.1. We give the first examples of finite groups G and prime numbers p such that the Chow group $\mathrm{CH}^i(BG_F)/p$ is infinite, for some i and some field F. Namely, we can take a group of order p^5 for p odd, or of order 2^6, with nontrivial unramified cohomology.

Corollary 3.2 *Let G be a finite group, and let p be a prime number. Suppose that the unramified cohomology $H^i_{\mathrm{nr}}(\overline{\mathbb{Q}}(V/G),\mathbb{F}_p)$ is not zero, for some generically free representation V of G over $\overline{\mathbb{Q}}$ and some $i>0$. Then there is a field F containing $\overline{\mathbb{Q}}$ and a positive integer r such that $\mathrm{CH}^r(BG_F)/p$ is infinite. It follows that the ring $\mathrm{CH}^*(BG_F)/p$ is not noetherian.*

Proof Corollary 3.1 gives an extension field E of $\overline{\mathbb{Q}}$ such that $\mathrm{CH}^r(BG)/p\to\mathrm{CH}^r(BG_E)/p$ is not surjective for some r. So, for a finite-dimensional approximation $U=(V-T)/G$ to BG with T of codimension greater than r, the map $\mathrm{CH}^r(U)/p\to\mathrm{CH}^r(U_E)/p$ is not surjective. By Lemma 2.5, there is a field $F/\overline{\mathbb{Q}}$ with $\mathrm{CH}^r(U_F)/p$ infinite. Equivalently, $\mathrm{CH}^r(BG_F)/p$ is infinite. Since $\mathrm{CH}^*(BG_F)/p$ is a graded $\mathbb{F}_p$–algebra, it follows that the ring $\mathrm{CH}^*(BG_F)/p$ is not noetherian. $\square$

4 The weak Chow Künneth property for smooth proper k–schemes

In this section, we characterize the smooth proper k–schemes whose Chow groups remain unchanged under arbitrary field extensions: they are the schemes whose Chow motive is a Tate motive. This type of result for smooth proper k–schemes has a long history, including results by Bloch [36, Proposition 3.12; 4, Appendix to Lecture 1], Bloch and Srinivas [5], Jannsen [29, Theorem 3.5] and Kimura [35]. Shinder gave a convenient version of Bloch's argument [49]. One difference from most earlier results is that we consider Chow groups with coefficients in any commutative ring, not just the rational numbers.

In the rest of the paper, Theorem 4.1 is used only to prove Corollary 7.3. Nonetheless, the proof, using the diagonal cycle, helped to suggest the proof of Theorem 7.2 about arbitrary schemes. Theorem 2.1 is a "birational analog" of Theorem 4.1; in particular, the equivalent properties in Theorem 4.1 are not birationally invariant.

Theorem 4.1 *Let M be a Chow motive over a field k with coefficients in a commutative ring R. (For example, M could be the motive $M(X)$ for a smooth proper k–scheme X.) Suppose that M has the weak Chow Künneth property, meaning that the morphism $\mathrm{CH}_*(M) \to \mathrm{CH}_*(M_F)$ is a surjection of R–modules for every finitely generated field F/k. Then M is a summand of a finite direct sum of Tate motives $R(j)[2j]$ for integers j.*

Conversely, suppose that a Chow motive M is a summand of a finite direct sum of Tate motives. Then $\mathrm{CH}_*(M) \to \mathrm{CH}_*(M_F)$ is an isomorphism for every field F/k, and M has the Chow Künneth property that $\mathrm{CH}_*(M) \otimes_R \mathrm{CH}_*(Y; R) \to \mathrm{CH}_*(M \otimes M^c(Y))$ is an isomorphism of R–modules for every separated k–scheme Y of finite type. Also, $\mathrm{CH}_*(M)$ is a finitely generated projective R–module, and $\mathrm{CH}_*(M) \cong H_*(M_{\mathbb{C}}, R)$ if there is an embedding $k \hookrightarrow \mathbb{C}$. Finally, M has the Künneth property for motivic homology in the sense that

$$\mathrm{CH}_*(M) \otimes_R H_*^M(Y, R(*)) \cong H_*^M(M \otimes M^c(Y), R(*))$$

for every separated k–scheme Y of finite type.

The notation $M^c(Y)$ is suggested by Voevodsky's triangulated category of motives (discussed in Section 5), but below we say explicitly what this means.

If R is a PID, then the conditions in the theorem are also equivalent to M being a finite direct sum of Tate motives (without having to take a direct summand). For an arbitrary commutative ring R, it is essential to allow direct summands.

The conclusion cannot be strengthened to say that X is a linear scheme or a rational variety. There are Barlow surfaces over $\mathbb{C}$ whose Chow motive with $\mathbb{Z}$ coefficients is a direct sum of Tate motives, for example by Theorem 4.1 and [2, Proposition 1.9]. It follows that these smooth projective surfaces have the Chow Künneth property, although they are of general type and hence not rational.

Before proving Theorem 4.1, let us define the category of Chow motives over k with coefficients in R. To agree with the conventions in Voevodsky's triangulated category of motives $\mathrm{DM}(k;R)$ (Section 5), we think of the basic functor $X \mapsto M(X)$ from smooth proper k–schemes to Chow motives as being covariant, and we write the motive of $\mathbb{P}^1_k$ as $R \oplus R(1)[2]$. Covariance is only a minor difference from the conventions in Scholl's paper [48], because the category of Chow motives is self-dual. (The "shift" [2] is written in order to agree with the notation in $\mathrm{DM}(k;R)$; it has no meaning by itself in the category of Chow motives.) We will only consider $\mathrm{DM}(k;R)$ when the exponential characteristic of k is invertible in R; in that case, the category of Chow motives is equivalent to a full subcategory of $\mathrm{DM}(k;R)$.

For smooth proper varieties X and Y over k, define the R–module of correspondences of degree r from X to Y as

$$\mathrm{Corr}_r(X,Y) = \mathrm{CH}_{\dim X + r}(X \times_k Y; R).$$

We extend this definition to all smooth proper k–schemes by taking direct sums. For smooth proper k–schemes X, Y, Z, there is a composition of correspondences

$$\mathrm{Corr}_r(X,Y) \otimes_R \mathrm{Corr}_s(Y,Z) \to \mathrm{Corr}_{r+s}(X,Z),$$

written as $f \otimes g \mapsto gf$, given by pulling back the two cycles from $X \times Y$ and $Y \times Z$ to $X \times Y \times Z$, multiplying, and pushing forward to $X \times Z$.

A *Chow motive* over k with coefficients in R, written $(M(X)(a)[2a], p)$, consists of a smooth proper k–scheme X, an integer a, and an idempotent $p = p^2$ in $\mathrm{Corr}_0(X,X)$. The morphisms of Chow motives are given by

$$\mathrm{Hom}\big((M(X)(a)[2a], p), (M(Y)(b)[2b], q)\big) = q\,\mathrm{Corr}_{a-b}(X,Y)\,p \subset \mathrm{Corr}_{a-b}(X,Y).$$

Composition of correspondences makes the Chow motives over k into a category. We write $M(X)$ for the motive $(M(X)(0)[0], \Delta)$, where Δ is the diagonal in $X \times_k X$. Thus $X \mapsto M(X)$ is a covariant functor from smooth proper k–schemes to Chow motives. The *Tate motive* $R(a)[2a]$ is $M(\operatorname{Spec} k)(a)[2a]$. Define the *Chow groups* of a motive M by $\mathrm{CH}_a(M) = \mathrm{Hom}(R(a)[2a], M)$; then the group $\mathrm{CH}_a(M(X))$ is isomorphic to the usual Chow group $\mathrm{CH}_a(X;R)$ of a smooth proper k–scheme X.

The category of Chow motives is symmetric monoidal, with tensor product $\otimes$ such that $M(X)\otimes M(Y)\cong M(X\times_k Y)$ for smooth proper k–schemes X and Y. There is an involution $M\mapsto M^*$ on Chow motives, defined on objects by

$$(M(X)(a)[2a],p)^*=(M(X)(-n-a)[-2n-2a],p^t)$$

for X of pure dimension n. It is immediate that the natural morphism $M\to M^{**}$ is an isomorphism, and that

$$\mathrm{Hom}(M\otimes N,P)\cong\mathrm{Hom}(M,N^*\otimes P)$$

for all Chow motives M,N,P [48, Section 1.1.5]. That is, the category of Chow motives is a rigid additive tensor category, with internal Hom given by $\underline{\mathrm{Hom}}(M,N)=M^*\otimes N$. For a field extension F/k, there is an obvious functor from Chow motives over k to Chow motives over F, taking $M(X)$ to $M(X_F)$ for smooth proper k–schemes X.

Extending the previous notation, for any Chow motive $M=(M(X)(a)[2a],p)$ over k and any k–scheme Y of finite type, we define the Chow groups $\mathrm{CH}_*(M\otimes M^c(Y))$ as the summand of the Chow groups $\mathrm{CH}_*(X\times_k Y;R)$ given by p. (At this point, $M^c(Y)$ has no meaning by itself. In Section 5, $M^c(Y)$ will be used to denote the compactly supported motive of Y in the triangulated category of motives $\mathrm{DM}(k;R)$.)

Proof of Theorem 4.1 Let M be a Chow motive which has the weak Chow Künneth property, meaning that $\mathrm{CH}_*\,M\to\mathrm{CH}_*(M_F)$ is surjective for all finitely generated fields F over k. Then the R–linear map $\mathrm{CH}_*\,M\otimes_R\mathrm{CH}_*\,Y\to\mathrm{CH}_*(M\otimes M^c(Y))$ is surjective for every k–scheme Y of finite type. (In this proof, we write $\mathrm{CH}_*(Y)$ to mean $\mathrm{CH}_*(Y;R)=\mathrm{CH}_*(Y)\otimes_{\mathbb{Z}}R$.) To prove this, do induction on the dimension of Y, using the commutative diagram of exact sequences for any closed subscheme S of Y:

$$\begin{tikzcd}
\mathrm{CH}_*\,M\otimes_R\mathrm{CH}_*\,S \arrow[r]\arrow[d] & \mathrm{CH}_*\,M\otimes_R\mathrm{CH}_*\,Y \arrow[r]\arrow[d] & \mathrm{CH}_*\,M\otimes_R\mathrm{CH}_*(Y-S) \arrow[r]\arrow[d] & 0\\
\mathrm{CH}_*(M\otimes M^c(S)) \arrow[r] & \mathrm{CH}_*(M\otimes M^c(Y)) \arrow[r] & \mathrm{CH}_*(M\otimes M^c(Y-S)) \arrow[r] & 0
\end{tikzcd}$$

Here we use that, for a k–variety Y, $\mathrm{CH}_*(M_{k(Y)})=\varinjlim\mathrm{CH}_*(M\otimes M^c(Y-S))$, where the direct limit runs over all closed subsets $S\subsetneq Y$. It follows that

$$\mathrm{CH}_*\,M\otimes_R\mathrm{CH}_*\,N\to\mathrm{CH}_*(M\otimes N)$$

is surjective for all Chow motives N.

For any Chow motives N and P, we have

$$\mathrm{Hom}(N,P)=\mathrm{Hom}(R\otimes N,P)=\mathrm{Hom}(R,\underline{\mathrm{Hom}}(N,P)).$$

By Lemma 5.5, the identity map on the Chow motive M corresponds to an element $1_M \in \mathrm{Hom}(R, M^* \otimes M) = \mathrm{CH}_0(M^* \otimes M)$. (When M is the motive of a smooth proper variety X, 1_M is the class of the diagonal on $X \times X$.)

For the given motive M, we showed that $\mathrm{CH}_* M \otimes_R \mathrm{CH}_* N \to \mathrm{CH}_*(M \otimes N)$ is surjective for all Chow motives N, and we apply this to $N = M^*$. So we can write $1_M = \sum_{i=1}^r \alpha_i \otimes \beta_i$ in $\mathrm{CH}_0(M^* \otimes M)$ for some $\alpha_1, \dots, \alpha_r \in \mathrm{CH}_*(M^*)$ and $\beta_1, \dots, \beta_r \in \mathrm{CH}_* M$. Here α_i is in $\mathrm{CH}_{-b_i}(M^*)$ and β_i is in $\mathrm{CH}_{b_i} M$ for some integers $b_1, \dots, b_r$. Let $N = \bigoplus_{i=1}^r R(b_i)[2b_i]$. Then $(\beta_1, \dots, \beta_r)$ can be viewed as a morphism $\beta\colon N \to M$, and $(\alpha_1, \dots, \alpha_r)$ can be viewed as a morphism $N^* \to M^*$, or equivalently $\alpha\colon M \to N$. The equation $1_M = \sum \alpha_i \otimes \beta_i$ in $\mathrm{CH}_0(M^* \otimes M)$ means that the composition $M \to N \to M$ is the identity. Since idempotents split in the category of Chow motives, it follows that M is a direct summand of N, which is a finite direct sum of Tate motives. One direction of the theorem is proved.

The converse statements in the theorem are clear for a finite direct sum of Tate motives. That implies the converse statements for any summand of a finite direct sum of Tate motives. □

5 The triangulated category of motives

This section summarizes the properties of Voevodsky's triangulated category of motives $\mathrm{DM}(k; R)$ over a field k. Every separated scheme of finite type over k (not necessarily smooth and proper) determines an object in this category, and Chow groups are given by morphisms from a fixed object (a Tate motive) in this category. So $\mathrm{DM}(k; R)$ is a natural setting for studying Chow groups of k–schemes that need not be smooth and proper.

We use the triangulated category of motives for at least two purposes in this paper. First, we need it even to state the characterization of those schemes of finite type which satisfy the Künneth property for motivic homology groups (Theorem 7.2). The corresponding characterization for smooth proper schemes (Theorem 4.1) used only the more elementary category of Chow motives. Second, we need the triangulated category of motives in order to define the motive $M^c(BG)$ of a classifying space and to study when that motive is mixed Tate (Sections 8 and 9).

Let k be a field. Thanks to recent developments in the theory of motives, k need not be assumed to be perfect or to admit resolution of singularities. We put one restriction on the coefficient ring R, as follows. The *exponential characteristic* of k means 1 if k has characteristic zero, or p if k has characteristic $p > 0$. For the rest of this section, we assume that the exponential characteristic of k is invertible in R. This assumption is used to prove the basic properties of the compactly supported motive of a scheme X over k, $M^c(X)$, such as the localization triangle. (This assumption can be avoided when we have resolution of singularities over k.) This assumption on R should be understood throughout the paper when we discuss motives $M^c(X)$.

A readable introduction to Voevodsky's triangulated categories of motives over k is [59]. Let R be a commutative ring. We primarily use the "big" triangulated category $\mathrm{DM}(k;R)$ of motives with coefficients in R, which contains the direct sum of an arbitrary set of objects. Also, the motive $R(1)$ is invertible in $\mathrm{DM}(k;R)$, as discussed below. (Voevodsky originally considered the subcategory $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ of "bounded-above effective motives", which does not have arbitrary direct sums.) Following Cisinski and Déglise, $\mathrm{DM}(k;R)$ is defined to be the homotopy category of G_m^{tr}–spectra of (unbounded) chain complexes of Nisnevich sheaves with transfers which are A^1–local [46, Section 2.3; 14, Example 6.25]. For k perfect, Röndigs and Østvær showed that the category $\mathrm{DM}(k;\mathbb{Z})$ is equivalent to the homotopy category of modules over the motivic Eilenberg–MacLane spectrum $H\mathbb{Z}$ in Morel and Voevodsky's stable homotopy category $\mathrm{SH}(k)$ [46, Theorem 1]. This is an analog of the equivalence between the derived category $D(\mathbb{Z})$ of abelian groups and the homotopy category of modules over the Eilenberg–MacLane spectrum $H\mathbb{Z}$ in the category of spectra in topology [22, Theorem 8.9].

Let k^{perf} denote the perfect closure of k. That is, k^{perf} is equal to k if k has characteristic zero, and k^{perf} consists of all $(p^r)^{\mathrm{th}}$ roots of elements of k for all $r \geq 0$ if k has characteristic $p>0$. Under our assumption that p is invertible in R, Cisinski and Déglise proved the following convenient result, following a suggestion by Suslin [16, Proposition 8.1].

Theorem 5.1 *The pullback functor* $\mathrm{DM}(k;R) \to \mathrm{DM}(k^{\mathrm{perf}};R)$ *is an equivalence of categories.*

By Theorem 5.1, most results on motives which previously assumed that k is perfect immediately generalize to an arbitrary field k, given our assumption that the exponential characteristic of k is invertible in R. We will mention some examples in what follows.

By definition of a triangulated category (such as $\mathrm{DM}(k;R)$), every morphism $X \to Y$ fits into an exact triangle $X \to Y \to Z \to X[1]$. Here Z is called a *cone* of the morphism $X \to Y$. It is unique up to isomorphism, but not (in general) up to unique isomorphism.

There are two natural functors from schemes to motives, which we write as $M(X)$ and $M^c(X)$. These were defined by Voevodsky when k is a perfect field which admits resolution of singularities (as we know for k of characteristic zero) [59, Section 2.2]. Kelly [34, Lemmas 5.5.2 and 5.5.6] extended these constructions to any perfect field k, under our assumption that the exponential characteristic of k is invertible in R, using Gabber's work on alterations. Finally, these constructions now apply to any field k: given a separated scheme X of finite type over k, we have objects $M(X_{k^{\mathrm{perf}}})$ and $M^c(X_{k^{\mathrm{perf}}})$ of $\mathrm{DM}(k^{\mathrm{perf}}; R)$ by Kelly, hence objects $M(X)$ and $M^c(X)$ of $\mathrm{DM}(k; R)$ by Theorem 5.1.

In more detail, there is a covariant functor $X \mapsto M(X)$ from the category of separated schemes of finite type over k to $\mathrm{DM}(k; R)$. Also, there is a covariant functor $X \mapsto M^c(X)$ (the motive of X "with compact support") from the category of separated schemes of finite type and proper morphisms to $\mathrm{DM}(k; R)$. A flat morphism $X \to Y$ determines a pullback map $M^c(Y) \to M^c(X)$. The motives $M(X)$ and $M^c(X)$ are isomorphic for X proper over k.

The category $\mathrm{DM}(k; R)$ has objects called $R(j)$ for all integers j. The motives $R(j)[2j]$ are called the *Tate motives*. One interpretation of Tate motives is that $M^c(A^j_k) = R(j)[2j]$ for $j \geq 0$. More generally, for an affine bundle $Y \to X$ (a morphism that is locally on X isomorphic to a product with affine space A^r), we have the *homotopy invariance* statements that $M(Y) \cong M(X)$, whereas $M^c(Y) \cong M^c(X)(r)[2r]$.

The category $\mathrm{DM}(k; R)$ is a tensor triangulated category, with a symmetric monoidal product $\otimes$ [14, Example 6.25]. We have $M(X) \otimes M(Y) = M(X \times_k Y)$ and $M^c(X) \otimes M^c(Y) = M^c(X \times_k Y)$ for k–schemes X and Y [59, Proposition 4.1.7; 34, Proposition 5.5.8]. The motive $R = R(0)$ of a point is the identity object for the tensor product. The motive $R(1)$ is invertible in the sense that $R(a) \otimes R(b) \cong R(a+b)$ for all integers a and b.

The category $\mathrm{DM}(k; R)$ has internal Hom objects, with natural isomorphisms

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C))$$

for all motives A, B, C. Moreover, the internal Hom preserves exact triangles in each variable, up to a sign change in the boundary map [27, Definition 6.6.1, Theorem 7.1.11]. (All this is part of Cisinski and Déglise's result that $S \mapsto \mathrm{DM}(S; R)$ is a "premotivic category" for finite-dimensional noetherian schemes S [15, Section 11.1.2].) It follows that, for any motive B in $\mathrm{DM}(k; R)$, the functor $\cdot \otimes B$ is a left adjoint, and therefore preserves arbitrary direct sums.

To understand the two functors, note that the Chow groups $\mathrm{CH}_i X$ are determined by $M^c(X)$, whereas Chow cohomology groups $\mathrm{CH}^i X$ for X smooth over k are determined by $M(X)$. Namely,

$$\mathrm{CH}_i(X) \otimes_{\mathbb{Z}} R = \mathrm{Hom}(R(i)[2i], M^c(X))$$

for any separated scheme X of finite type over k, while

$$\mathrm{CH}^i(X) \otimes_{\mathbb{Z}} R = \mathrm{Hom}(M(X), R(i)[2i])$$

for X also smooth over k [59, Section 2.2]. Voevodsky defined *motivic cohomology* and (Borel–Moore) *motivic homology* for any separated scheme X of finite type over k by

$$H^j_M(X, R(i)) = \mathrm{Hom}(M(X), R(i)[j])$$

and

$$H_j^M(X, R(i)) = \mathrm{Hom}(R(i)[j], M^c(X)).$$

For a separated scheme X of finite type over k and a closed subscheme Z of X, there is an exact triangle in $\mathrm{DM}(k; R)$, the *localization triangle*:

$$M^c(Z) \to M^c(X) \to M^c(X - Z) \to M^c(Z)[1].$$

(This was proved by Voevodsky when k is perfect and admits resolution of singularities [59, Section 2.2], by Kelly for any perfect field k with our assumption on R [34, Proposition 5.5.5], and by Theorem 5.1 for an arbitrary field k.) This triangle induces a long exact sequence of motivic homology groups, called the localization sequence.

Bloch defined higher Chow groups as the homology of an explicit complex of algebraic cycles. Higher Chow groups are essentially the same as motivic homology, but (by tradition) they are numbered by codimension. Namely, for an equidimensional separated scheme X of dimension n over k,

$$\mathrm{CH}^{n-j}(X, i-2j; R) \cong H_i^M(X, R(j)).$$

(For k admitting resolution of singularities and X quasiprojective over k, this is [59, Proposition 4.2.9]. Kelly modified the argument to replace the assumption on resolution of singularities with our assumption on R [34, Theorem 5.6.4]. Finally, the assumption of quasiprojectivity was needed for Bloch's proof of the localization sequence for higher Chow groups [3], but Levine has now proved the localization sequence for the higher Chow groups of all schemes of finite type over a field [38, Theorem 0.7].)

Some higher Chow groups are zero by the definition, because they consist of cycles of negative dimension or negative codimension. It follows that the motivic homology $H_i^M(X, R(j))$ of a separated k–scheme X is zero unless $i \geq 2j$ and $i \geq j$ and $j \leq \dim X$.

For any motive A in $\mathrm{DM}(k; R)$, we define the *motivic homology* groups of A to mean the groups $H_j^M(A, R(i)) = \mathrm{Hom}(R(i)[j], A)$. Note that what we call the motivic homology groups of a separated k–scheme X of finite type are the motivic homology groups of the motive $M^c(X)$, not those of $M(X)$ (although the two motives are isomorphic for X proper over k).

Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums. A *localizing subcategory* of $\mathcal{T}$ means a strictly full triangulated subcategory which is closed under arbitrary direct sums. Following Röndigs and Østvær, the triangulated category $\mathrm{DMT}(k;R)$ of *mixed Tate motives* with coefficients in R is the smallest localizing subcategory of $\mathrm{DM}(k;R)$ that contains $R(j)$ for all integers j [46]. Because the tensor product $\otimes$ on $\mathrm{DM}(k;R)$ is compatible with exact triangles and with arbitrary direct sums, the tensor product of two mixed Tate motives is a mixed Tate motive.

The category of mixed Tate motives is analogous to the category of *cellular* spectra in the stable homotopy category $\mathrm{SH}(k)$ studied by Voevodsky [60] and Dugger and Isaksen [19]. (Actually, Voevodsky says "T–cellular" and Dugger and Isaksen say "stably cellular".) Namely, let T be the suspension spectrum of the pointed k–space $(\mathbb{P}^1_k, \text{point})$; the triangulated category of cellular spectra is defined as the smallest localizing subcategory of $\mathrm{SH}(k)$ that contains T^j for all integers j.

As with motives, there are two natural functors from separated k–schemes X of finite type to $\mathrm{SH}(k)$: the usual functor (which we write as $X \mapsto S(X)$ or $X \mapsto \Sigma^\infty_T X^+$) and a compactly supported version, $X \mapsto S^c(X)$. Explicitly, for any compactification $\bar{X}$ of a k–scheme X, $S^c(X)$ is the spectrum associated to the pointed k–space $\bar{X}/(\bar{X}-X)$. There is a functor from $\mathrm{SH}(k)$ to $\mathrm{DM}(k;R)$, which one can view as smashing with the Eilenberg–MacLane spectrum HR, and this takes $S(X)$ to $M(X)$ and $S^c(X)$ to $M^c(X)$. In particular, the spectrum T goes to the motive $R(1)[2]$.

In a triangulated category with arbitrary direct sums, every idempotent splits [9, Proposition 3.2]. Applying this to the category of mixed Tate motives, it follows that every summand of a mixed Tate motive in $\mathrm{DM}(k;R)$ is a mixed Tate motive.

Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums. An object X of $\mathcal{T}$ is called *compact* if $\mathrm{Hom}(X,\cdot)$ commutes with arbitrary direct sums. The objects $M(X)(a)[b]$ and $M^c(X)(a)[b]$ are compact in $\mathrm{DM}(k;R)$ for every separated k–scheme X of finite type [34, Lemmas 5.5.2 and 5.5.6]. A set $\mathcal{P}$ of objects *generates* $\mathcal{T}$ if every object Y of $\mathcal{T}$ such that $\mathrm{Hom}(P[a],Y)=0$ for all objects P in $\mathcal{P}$ and all integers a is zero. A triangulated category $\mathcal{T}$ is *compactly generated* if it has arbitrary direct sums and it is generated by a set of compact objects.

The following result by Neeman helps to understand the notion of generators for a triangulated category [43, Theorem 2.1].

Lemma 5.2 *Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums, and let $\mathcal{P}$ be a set of compact objects. The following are equivalent:*

(1) *The smallest localizing subcategory of $\mathcal{T}$ that contains $\mathcal{P}$ is equal to $\mathcal{T}$.*

(2) *The set $\mathcal{P}$ generates $\mathcal{T}$. That is, any object X in $\mathcal{T}$ with $\mathrm{Hom}(P[a], X) = 0$ for all P in $\mathcal{P}$ and $a \in \mathbb{Z}$ must be zero.*

Corollary 5.3 *A mixed Tate motive with zero motivic homology must be zero.*

Proof By Lemma 5.2, the objects $R(a)$ for $a \in \mathbb{Z}$ generate the category $\mathrm{DMT}(k; R)$. Since $H_j^M(A, R(i)) = \mathrm{Hom}(R(i)[j], A)$ for a motive A, the corollary is proved. □

Lemma 5.4 *Let k be a field. Then the category $\mathrm{DM}(k; R)$ is compactly generated, with a set of generators given by the compact objects $M(X)(a)$ for X smooth projective over k and a an integer.*

Proof This was proved by Voevodsky when k is perfect and admits resolution of singularities [59, Corollary 3.5.5]. Given our assumption that the exponential characteristic of k is invertible in R, Kelly generalized this result to any perfect field k [34, Proposition 5.5.3]. The generalization to an arbitrary field k follows from Theorem 5.1. □

A reassuring fact is that if the motive $M^c(X)$ in $\mathrm{DM}(k; R)$ of a separated k–scheme X of finite type is mixed Tate, then it is a summand of an object of the smallest strictly full triangulated subcategory of $\mathrm{DM}(k; R)$ that contains $R(j)$ for all integers j. In other words, $M^c(X)$ can be described by a *finite* diagram of objects $R(j)$. This follows from a general result about triangulated categories. Define a *thick* subcategory of a triangulated category to be a strictly full triangulated subcategory that is closed under direct summands. Let $\mathcal{T}$ be a compactly generated triangulated category, and let $\mathcal{P}$ be a set of compact generators. (We have in mind the category of mixed Tate motives, generated by the objects $R(j)$ for integers j.) Then Neeman showed that any compact object in $\mathcal{T}$ belongs to the smallest thick subcategory of $\mathcal{T}$ that contains $\mathcal{P}$ [43, Theorem 2.1].

The category $\mathrm{DM}_{\mathrm{gm}}(k; R)$ of geometric motives is defined as the smallest thick subcategory of $\mathrm{DM}(k; R)$ that contains $M(X)(a)$ for all smooth separated schemes X of finite type over k and all integers a. In fact, it suffices to use $M(X)(a)$ for smooth projective varieties X over k and all integers a, by Lemma 5.4. Another application of Neeman's theorem gives that $\mathrm{DM}_{\mathrm{gm}}(k; R)$ is the subcategory of all compact objects in $\mathrm{DM}(k; R)$.

A *linear scheme* over a field k is defined inductively: affine space A_k^n is a linear scheme for any $n \geq 0$; for any scheme X of finite type over k with a closed subscheme Z, if Z and $X-Z$ are linear schemes, then X is a linear scheme; and if X and Z are linear schemes, then $X-Z$ is a linear scheme. (A slightly narrower class of linear schemes was studied in [56].) Some examples of linear schemes are all toric varieties (not necessarily smooth or compact), the discriminant hypersurface and its complement, and many quotients of affine space by finite group actions. Linear schemes can have torsion in their Chow groups and homology groups, and they can have nonzero rational homology in odd degrees. (To talk about rational homology, assume that the base field is the complex numbers.)

From the localization triangle, a straightforward induction shows that for any linear scheme X over k, the compactly supported motive $M^c(X)$ with any coefficient ring R is a mixed Tate motive. Likewise, for any linear scheme X, the spectrum $S^c(X)$ is cellular in $\mathrm{SH}(k)$. (Dugger and Isaksen asked whether the spectrum $S(X)$ is cellular for linear schemes X, and proved this in some examples [19, Section 1.1]. Arguably, the more natural spectrum associated to a linear scheme X is $S^c(X)$, which is clearly cellular. For X proper over k, $S(X)$ and $S^c(X)$ are isomorphic.)

Let X and Y be smooth proper varieties over k. Then the set of morphisms from $M(X)$ to $M(Y)$ in $\mathrm{DM}(k;R)$ is the Chow group $\mathrm{CH}_{\dim X}(X\times_k Y;R)$ [59, Section 2.2]. Composition of morphisms $M(X)\to M(Y)\to M(Z)$ is given by the composition of correspondences. As a result, the smallest strictly full subcategory of $\mathrm{DM}(k;R)$ that is closed under direct summands and contains $M(X)(a)[2a]$ for all smooth proper varieties X over k and all integers a is equivalent to the category of Chow motives over k with coefficients in R, as defined in Section 4.

We define $N^*=\underline{\mathrm{Hom}}(N,R)$. A version of Poincaré duality says that $M^c(X)\cong M(X)^*(n)[2n]$ for X smooth of pure dimension n over k [34, Theorem 5.5.14]. The internal Hom of motives has a simple description for compact objects, as follows.

Lemma 5.5 *Let M be an object of $\mathrm{DM}_{\mathrm{gm}}(k;R)$, for example the motive $M^c(X)(a)[b]$ for a scheme X of finite type over k and $a,b\in\mathbb{Z}$. Let N be any object of $\mathrm{DM}(k;R)$. Then the morphism $M^*\otimes N\to\underline{\mathrm{Hom}}(M,N)$ is an isomorphism.*

Also, for M in $\mathrm{DM}_{\mathrm{gm}}(k;R)$, the natural map $M\to M^{**}$ is an isomorphism.

Proof At first, let M^* denote the object $\underline{\mathrm{Hom}}_{\mathrm{gm}}(M,R)$ in the subcategory $\mathrm{DM}_{\mathrm{gm}}(k;R)$ of compact objects. Then Voevodsky and Kelly prove that $M\to M^{**}$ is an isomorphism for M compact, and also that $M^*\otimes N\to\underline{\mathrm{Hom}}_{\mathrm{gm}}(M,N)$ is an isomorphism for M and N compact [59, Theorem 4.3.7; 34, Theorem 5.5.14]. That is, the map

$$\mathrm{Hom}(A,B^*\otimes C)\to\mathrm{Hom}(A\otimes B,C)$$

associated to $B^*\otimes B\to R$ is a bijection for all compact objects A,B,C.

For A and B compact, the map of Hom sets above turns arbitrary direct sums of motives C into direct sums, and fits into long exact sequences for any exact triangle of objects C. By Lemmas 5.2 and 5.4, it follows that the map is an isomorphism for A and B compact and C arbitrary. For B compact and C arbitrary, both Hom sets turn arbitrary direct sums of motives A into products, and they fit into long exact sequences for any exact triangle of objects A. Therefore the map is an isomorphism for B compact and A and C any motives. That is, the internal Hom in $\mathrm{DM}(k;R)$ has $\underline{\mathrm{Hom}}(B,C) \cong B^* \otimes C$ for B compact and C arbitrary. In particular, taking $C = R$, we see that the object B^* (which we defined as $\underline{\mathrm{Hom}}_{\mathrm{gm}}(B,R)$ in $\mathrm{DM}_{\mathrm{gm}}(k;R)$) is isomorphic to $\underline{\mathrm{Hom}}(B,R)$ in $\mathrm{DM}(k;R)$. □

6 A Künneth spectral sequence for motivic homology

Dugger and Isaksen proved the following Künneth spectral sequence, which describes the motivic homology of the tensor product of a mixed Tate motive with any motive [19, Proposition 7.10]. Their result applies to modules over any ring spectrum in the stable homotopy category over a field k; the case of the Eilenberg–MacLane spectrum HR in $\mathrm{SH}(k)$ gives the result here, by the identification between the homotopy category of HR–module spectra and $\mathrm{DM}(k;R)$ [46, Theorem 1]. (It is also straightforward to translate Dugger and Isaksen's proof to work directly in $\mathrm{DM}(k;R)$.) In the case of the product of a linear scheme with any scheme over a field, this spectral sequence was constructed by Joshua [30].

Theorem 6.1 *Let k be a field. Let R be a commutative ring. Let X be a mixed Tate motive in $\mathrm{DM}(k;R)$ and Y any motive in $\mathrm{DM}(k;R)$. For each integer j, there is a convergent spectral sequence*

$$E_2^{pq} = \mathrm{Tor}_{-p,-q,j}^{H_*(k,R(*))}(H_*(X,R(*)), H_*(Y,R(*))) \implies H_{-p-q}(X \otimes Y, R(j)).$$

This spectral sequence is concentrated in the left half-plane (columns ≤ 0).

By the discussion after Theorem 7.2, one can define a spectral sequence with the E_2 term above for any motives X and Y in $\mathrm{DM}(k;R)$. It does not always converge to the motivic homology of $X \otimes Y$.

We use cohomological numbering, which means that the differential d_r has bidegree $(r, 1-r)$ for all r.

For bigraded modules M and N over a bigraded ring S, we denote by $\mathrm{Tor}_{a,i,j}^S(M,N)$ the $(i,j)^{\mathrm{th}}$ bigraded piece of $\mathrm{Tor}_a^S(M,N)$. For this purpose, the bigrading of the group $H_i^M(X,R(j))$ is (i,j).

Here $H_i(k, R(j)) \cong H^{-i}(k, R(-j))$, and so the ring $H_*(k, R(*))$ is better known as the motivic cohomology ring of k with coefficients in R. For example, $H_{-1}(k, \mathbb{Z}(-1))$ is isomorphic to k^*. More generally, $\bigoplus_{j \geq 0} H_{-j}(k, \mathbb{Z}(-j))$ is the Milnor K–theory ring, that is, the quotient of the tensor algebra generated by the abelian group k^* by the relation $\{a, 1-a\} = 0$ for each $a \in k - \{0, 1\}$ [45; 54].

If X and Y are k–schemes, viewed as the motives $M^c(X)$ and $M^c(Y)$, then the spectral sequence with $R(j)$ coefficients is concentrated in columns ≤ 0 and rows $\leq -2j$. If we write $H_*(X)$ for the bigraded group $H_*(X, R(*))$, the E_2 term looks like this:

$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
{[\mathrm{Tor}_2^{H_*k}(H_*X, H_*Y)]_{2j,j}} \arrow[drr] & {[\mathrm{Tor}_1^{H_*k}(H_*X, H_*Y)]_{2j,j}} & {[H_*X \otimes_{H_*k} H_*Y]_{2j,j}} & 0 \\
{[\mathrm{Tor}_2^{H_*k}(H_*X, H_*Y)]_{2j+1,j}} & {[\mathrm{Tor}_1^{H_*k}(H_*X, H_*Y)]_{2j+1,j}} & {[H_*X \otimes_{H_*k} H_*Y]_{2j+1,j}} & 0
\end{tikzcd}$$

(Indeed, for a k–scheme X, the group $H_a(X, R(b))$ is zero unless $a \geq 2b$, as mentioned in Section 5. Since this applies to X, Y, and $\operatorname{Spec} k$, the E_2 term for the spectral sequence with $R(j)$ coefficients is concentrated in rows $\leq -2j$.) So there are no differentials into or out of the upper right group, $E_2^{0,-2j}$. We deduce that

$$\mathrm{CH}_*(X \times_k Y; R) \cong \mathrm{CH}_*(X; R) \otimes_R \mathrm{CH}_*(Y; R)$$

if X is a k–scheme with $M^c(X)$ a mixed Tate motive in $\mathrm{DM}(k; R)$ and Y is any separated k–scheme of finite type. I proved this in the special case where X is a linear scheme over k [56], which helped to inspire Joshua's result.

7 The motivic Künneth property

In this section, we prove that a separated scheme X of finite type over a field k satisfies the motivic Künneth property if and only if the motive $M^c(X)$ is a mixed Tate motive. Given the machinery we have developed, the proof is short.

The motivic Künneth property means that the spectral sequence described in Theorem 6.1 converges to the motivic homology of $X \times_k Y$ for every separated k–scheme Y of finite type. (We recall that motivic homology groups are also called higher Chow groups.) There is a neater formulation of the Künneth property in the language of Bousfield localization, to be explained now.

The inclusion of mixed Tate motives $\mathrm{DMT}(k; R)$ into the category $\mathrm{DM}(k; R)$ of all motives has a right adjoint $\mathrm{DM}(k; R) \to \mathrm{DMT}(k; R)$, which we write as $X \mapsto C(X)$.

It associates to any motive a mixed Tate motive with the same motivic homology groups. For X a compact object (a geometric motive), $C(X)$ need not be a compact object. So this construction shows the convenience of "big" categories of motives. The construction is a general application of Bousfield localization, as developed by Neeman for triangulated categories.

Namely, let $\mathcal{T}$ be a triangulated category with arbitrary direct sums. Let $\mathcal{P}$ be a set of compact objects in $\mathcal{T}$. Recall from Section 5 that a localizing subcategory of $\mathcal{T}$ means a full triangulated subcategory which is closed under arbitrary direct sums. Let $\mathcal{S}$ be the smallest localizing category that contains $\mathcal{P}$. Then the inclusion $\mathcal{S} \to \mathcal{T}$ has a right adjoint $C\colon \mathcal{T} \to \mathcal{S}$ known as *colocalization* with respect to $\mathcal{P}$ [43, Theorem 4.1]. By adjointness, there is a canonical morphism $C(X) \to X$, and this morphism induces a bijection $\mathrm{Hom}(P[j], C(X)) \to \mathrm{Hom}(P[j], X)$ for all objects P in $\mathcal{P}$ and all integers j. (The *localization* of an object X with respect to $\mathcal{P}$ means a cone $X/C(X)$, which in this case is defined up to a unique isomorphism.)

The functor $\mathrm{DM}(k; R) \to \mathrm{DMT}(k; R)$ given by $X \mapsto C(X)$, mentioned above, is the colocalization with respect to the compact objects $R(j)$ for $j \in \mathbb{Z}$. The construction implies that $C(X)$ is a mixed Tate motive with a morphism $C(X) \to X$ that induces isomorphisms on motivic homology groups. (That is, $\mathrm{Hom}(R(a)[b], C(X)) \to \mathrm{Hom}(R(a)[b], X)$ is an isomorphism for all integers a and b.) Moreover, $C(X)$ is determined up to a unique isomorphism by this property.

As in any triangulated category with arbitrary direct sums, the *homotopy colimit* $X_\infty = \mathrm{hocolim}(X_0 \to X_1 \to \cdots)$ is defined as a cone of the morphism

$$1 - s\colon \bigoplus_{i \geq 0} X_i \to \bigoplus_{i \geq 0} X_i,$$

where s is the given map from each X_i to X_{i+1} [9].

Here is an explicit construction of the colocalization $C(X)$, modeled on Dugger and Isaksen's analogous construction in the stable homotopy category over k [19, Proposition 7.3]. (They were imitating the usual construction of a *cellular approximation* to any topological space.) Choose a set of generators for all the motivic homology groups $H_b(X, R(a))$ with $a, b \in \mathbb{Z}$. Let C_0 be a direct sum of one motive $R(a)[b]$ for each generator; so we have a morphism $C_0 \to X$ that induces a surjection on motivic homology groups. Next, choose a set of generators for the kernel of $H_*(C_0, R(*)) \to H_*(X, R(*))$, let S_1 be the corresponding direct sum of motives $R(a)[b]$, and let C_1 be a cone of the morphism $S_1 \to C_0$. Then we have a morphism $C_0 \to C_1$, and we can choose an extension of the morphism $C_0 \to X$ to $C_1 \to X$.

Repeating the process, we get a sequence of mixed Tate motives

$$C_0 \to C_1 \to \cdots$$

with a compatible sequence of morphisms $C_i \to X$. These extend to a morphism from the homotopy colimit, $\mathrm{hocolim}_j\ C_j \to X$. This homotopy colimit is a mixed Tate motive, and the morphism induces an isomorphism on motivic homology groups. So the colocalization $C(X)$ is isomorphic to $\mathrm{hocolim}_j\ C_j$.

By Corollary 5.3, any mixed Tate motive with zero motivic homology groups is zero. This is not true for motives in general. In fact, for any motive X, the cone of $C(X) \to X$ has motivic homology groups equal to zero, and it is zero if and only if X is a mixed Tate motive.

Lemma 7.1 *The colocalization functor $X \mapsto C(X)$ from $\mathrm{DM}(k;R)$ to $\mathrm{DMT}(k;R)$ preserves arbitrary direct sums and arbitrary products.*

Proof Because the category $\mathrm{DMT}(k;R)$ is compactly generated, it has arbitrary products [44, Proposition 8.4.6]. (Beware that the inclusion $\mathrm{DMT}(k;R) \to \mathrm{DM}(k;R)$ preserves arbitrary direct sums, but need not preserve arbitrary products [58, Corollary 4.2].) Because the functor $X \mapsto C(X)$ from $\mathrm{DM}(k;R)$ to $\mathrm{DMT}(k;R)$ is a right adjoint, it preserves arbitrary products. Because the functor $X \mapsto C(X)$ is colocalization with respect to a set of compact objects in $\mathrm{DM}(k;R)$ (namely $R(j)$ for integers j), it also preserves arbitrary direct sums [43, Theorem 5.1]. □

For any motives X and Y in $\mathrm{DM}(k;R)$, there is a canonical morphism

$$C(X) \otimes C(Y) \to C(X \otimes Y),$$

which is generally not an isomorphism. Indeed, tensoring the morphisms $C(X) \to X$ and $C(Y) \to Y$ gives a morphism $C(X) \otimes C(Y) \to X \otimes Y$. Since $C(X) \otimes C(Y)$ is a mixed Tate motive, this morphism factors uniquely through $C(X \otimes Y)$, as we want.

Theorem 7.2 *Let k be a field. Let R be a commutative ring. Let X be an object of the category $\mathrm{DM}(k;R)$ of motives (for example, X could be the motive $M^c(W)$ for a separated k–scheme W of finite type, if the exponential characteristic of k is invertible in R). The following are equivalent:*

(1) *X is a mixed Tate motive.*

(2) *X satisfies the motivic Künneth property, meaning that the morphism*

$$C(X) \otimes C(M(Y)) \to C(X \otimes M(Y))$$

of mixed Tate motives is an isomorphism for every smooth projective variety Y over k.

(3) *X satisfies the apparently stronger property that*

$$C(X) \otimes C(Y) \to C(X \otimes Y)$$

is an isomorphism for every motive Y in $\mathrm{DM}(k; R)$*.*

If X belongs to the subcategory $\mathrm{DM}_{\mathrm{gm}}(k; R)$ of geometric motives, for example if $X = M^c(B)$ for some separated k–scheme B of finite type, then (1)–(3) are also equivalent to:

(4) *X is a "small" mixed Tate motive, meaning that X belongs to the smallest thick subcategory of $\mathrm{DM}(k; R)$ that contains $R(j)$ for all integers j.*

Let us explain why properties (2) and (3) deserve to be called Künneth properties of X. Since $C(X) \otimes C(Y)$ and $C(X \otimes Y)$ are both mixed Tate motives, the morphism $C(X) \otimes C(Y) \to C(X \otimes Y)$ is an isomorphism if and only if it induces an isomorphism on motivic homology groups, by Corollary 5.3. The motivic homology groups of $C(X \otimes Y)$ are simply the motivic homology groups of $X \otimes Y$. The motivic homology groups of $C(X) \otimes C(Y)$ are the "output" of the spectral sequence of Theorem 6.1, with E_2 term

$$\begin{aligned}\mathrm{Tor}_*^{H_*(k,*)}&\big(H_*(C(X), R(*)), H_*(C(Y), R(*))\big)\\ &= \mathrm{Tor}_*^{H_*(k,*)}\big(H_*(X, R(*)), H_*(Y, R(*))\big).\end{aligned}$$

So property (3) is saying that this Künneth spectral sequence converges to the motivic homology of $X \otimes Y$.

Proof The Künneth property (2) is preserved under arbitrary direct sums of motives X, since the tensor product $\otimes$ and the functor $X \mapsto C(X)$ (by Lemma 7.1) preserve arbitrary direct sums. Also, if it holds for two of the three motives in an exact triangle, then it holds for the third. Finally, the motives $R(j)$ have the Künneth property. It follows that every mixed Tate motive in $\mathrm{DM}(k; R)$ has the Künneth property. That is, (1) implies (2).

Next, let X be a motive in $\mathrm{DM}(k; R)$ with the Künneth property (2) with respect to smooth projective varieties over k. The statement that the morphism

$$C(X) \otimes C(Y) \to C(X \otimes Y)$$

is an isomorphism is preserved under arbitrary direct sums of motives Y. Also, if it holds for two motives Y in an exact triangle, then it holds for the third. By Lemma 5.4, X satisfies the Künneth property (3) with respect to all motives Y.

We now show that (3) implies (1). As above, the "cellular approximation" $C(X)$ is the unique mixed Tate motive with a morphism $C(X) \to X$ that induces an isomorphism on motivic homology groups. Since $C(X)$ is a mixed Tate motive, it has the Künneth property. Let X_2 be a cone of the morphism $C(X) \to X$. It suffices to show that $X_2 = 0$.

The motivic homology groups of X_2 are equal to zero. Also, X_2 satisfies the Künneth property. So the motivic homology of $X_2 \otimes Y$ is zero for every motive Y in $\mathrm{DM}(k; R)$. In particular, for all smooth projective varieties Y over k and all integers a and b, the motivic homology group $\mathrm{Hom}(R, X_2 \otimes (M(Y)(a)[b])^*)$ is zero. By Lemma 5.5, it follows that $\mathrm{Hom}(M(Y)(a)[b], X_2) = 0$ for all smooth projective varieties Y over k and all integers a and b. By Lemma 5.4, it follows that $X_2 = 0$. We have shown that (3) implies (1).

Finally, if X belongs to the subcategory $\mathrm{DM}_{\mathrm{gm}}(k; R)$ of geometric motives, then we showed after Lemma 5.4 that (1) and (4) are equivalent. □

The following consequence is not surprising, but it seems worth mentioning. Dugger and Isaksen mentioned that it is not immediately clear how to show that a given object in the stable homotopy category $\mathrm{SH}(k)$, for example an elliptic curve over k, is not cellular [19, Section 1.2]. The functor $\mathrm{SH}(k) \to \mathrm{DM}(k; R)$ takes cellular objects to mixed Tate motives. The following result describes which smooth projective varieties have motives that are mixed Tate. As a very special case, we see that elliptic curves are not mixed Tate motives (for any nonzero coefficient ring), and so elliptic curves are not cellular in $\mathrm{SH}(k)$.

Corollary 7.3 *Let X be a smooth proper scheme over a field k. Let R be a commutative ring such that the exponential characteristic of k is invertible in R. If the motive $M(X)$ in $\mathrm{DM}(k; R)$ is a mixed Tate motive, then the Chow motive of X with coefficients in R is a summand of a finite direct sum of Tate motives $R(a)[2a]$. So, for example, $\mathrm{CH}_*(X) \otimes_{\mathbb{Z}} R \to H_*(X_{\mathbb{C}}, R)$ is an isomorphism if there is an embedding $k \hookrightarrow \mathbb{C}$. In particular, $H_*(X_{\mathbb{C}}, R)$ is concentrated in even degrees.*

Proof By Theorem 7.2, X satisfies the Künneth property for motivic homology groups with coefficients in R. By the discussion of the Künneth spectral sequence after Theorem 6.1, it follows that X has the Chow Künneth property: the homomorphism

$$\mathrm{CH}_*(X; R) \otimes_R \mathrm{CH}_*(Y; R) \to \mathrm{CH}_*(X \times_k Y; R)$$

is an isomorphism for every separated k–scheme Y of finite type. By Theorem 4.1, the Chow motive of X with coefficients in R is a summand of a finite direct sum of Tate motives. The theorem includes several consequences of that property, for example that $\mathrm{CH}_*(X; R) \to H_*(X_{\mathbb{C}}, R)$ is an isomorphism if there is an embedding $k \hookrightarrow \mathbb{C}$. □

8 The motive of a quotient stack

Edidin and Graham defined the motivic homology of a quotient stack [20, Sections 2.7 and 5.3]. In this section, we define the compactly supported motive of a quotient stack, in such a way that we recover the same motivic homology groups. One benefit of defining the motive of a quotient stack is that it makes sense to ask whether a given stack, such as BG for an affine group scheme G, is mixed Tate, meaning that the motive $M^c(BG)$ is mixed Tate.

The motive $M(BG)$ (not compactly supported) in $\mathrm{DM}(k;R)$ was already defined, in effect, by Morel and Voevodsky [40, Section 4.2]. Its motivic cohomology is the motivic cohomology of BG. We need to define $M^c(BG)$ because that is the motive relevant to the motivic homology of $BG\times X$ for separated schemes X of finite type over k. To see the difference between the two motives, write G_m for the multiplicative group over k. Then $M(BG_m)$ is the homotopy colimit of the motives $M(\mathbb{P}^j)$, and so $M(BG_m)$ is isomorphic to $\bigoplus_{j\geq 0}\mathbb{Z}(j)[2j]$ in $\mathrm{DM}(k;\mathbb{Z})$. By contrast, $M^c(BG_m)$ is the homotopy limit of the motives $M(\mathbb{P}^{j-1})(-j)[-2j]$ by the definition below, and so $M^c(BG_m)$ is isomorphic to $\prod_{j\leq -1}\mathbb{Z}(j)[2j]$ in $\mathrm{DM}(k;\mathbb{Z})$. This product is isomorphic to the direct sum $\bigoplus_{j\leq -1}\mathbb{Z}(j)[2j]$ by Lemma 8.5, from which we see that $M^c(BG_m)$ is mixed Tate.

Another possible name for the mixed Tate property of BG would be the *motivic Künneth property*. Indeed, by Theorem 7.2, BG is mixed Tate if and only if BG has the motivic Künneth property in the sense that the Künneth spectral sequence

$$E_2^{pq}=\mathrm{Tor}^{H_*(k,*)}_{-p,-q,j}\big(H_*(BG,R(*)),H_*(Y,R(*))\big)\implies H_{-p-q}(BG\times_k Y,R(j))$$

converges to the groups on the right for every separated k–scheme Y of finite type.

Before defining the compactly supported motive of a quotient stack, we recall the definition of homotopy limits. Let

$$\cdots\to X_2\to X_1$$

be a sequence of morphisms in the category $\mathrm{DM}(k;R)$ of motives. Since $\mathrm{DM}(k;R)$ is compactly generated, arbitrary products exist in $\mathrm{DM}(k;R)$ [44, Proposition 8.4.6]. Dualizing Bökstedt and Neeman's definition of homotopy colimits, the *homotopy limit* $\mathrm{holim}_j X_j$ in $\mathrm{DM}(k;R)$ is defined as the fiber of the morphism $f\colon \prod X_j\to\prod X_j$ given by the identity minus the shift map σ [9]. (In other words, the homotopy limit is $\mathrm{cone}(f)[-1]$; so it is well-defined up to isomorphism, but not necessarily up to a unique isomorphism.)

Define a *quotient stack* over a field k to be an algebraic stack over k which is the quotient stack of some quasiprojective scheme Y over k by an action of an affine group scheme G of finite type over k such that there is a G–equivariant ample line bundle on Y. (A short introduction to quotient stacks is [53, Tag 04UV]. It would be more natural to allow quotients of algebraic spaces by affine group schemes, but this definition of quotient stacks is sufficient for our applications.) For Y quasiprojective over k, the assumption that there is a G–equivariant ample line bundle is automatic when G is finite, or when G is smooth over k and Y is normal, by Sumihiro's equivariant completion theorem [50] and [41, Corollary 1.6]. For example, the stack BG means the quotient stack $\operatorname{Spec} k/G$.

The *dimension* of a locally Noetherian stack is defined in such a way that a quotient stack $X = A/G$ has dimension $\dim A - \dim G$ [53, Tag 0AFL]. For example, BG is a smooth stack of dimension $-\dim G$ over k.

We can now define the compactly supported motive of a quotient stack, with coefficients in a given commutative ring R. Let k be a field. Let R be a commutative ring in which the exponential characteristic of k is invertible. Let X be a quotient stack over k. Let $\cdots \twoheadrightarrow V_2 \twoheadrightarrow V_1$ be a sequence of surjections of vector bundles over X. Write n_i for the rank of the bundle V_i. Think of the total space of V_i as a stack over k. For each i, let S_i be a closed substack of V_i such that $V_i - S_i$ is a separated scheme and S_{i+1} is contained in the inverse image of S_i under the morphism $f_i\colon V_{i+1} \twoheadrightarrow V_i$ for all i. Assume that the codimension of S_i in V_i goes to infinity with i. (Such vector bundles V_i and closed subsets S_i exist because X is a quotient stack. In more detail, if we write X as a quotient stack Y/G, then we can use bundles V_i which are given by suitable representations V of G. Take $V_i - S_i$ to be of the form $(Y \times (V-S))/G$ with $(V-S)/G$ a quasiprojective scheme [55, Remark 1.4]. Then $(Y \times (V-S))/G$ is also a quasiprojective scheme by [41, Proposition 7.1], using that Y has a G–equivariant ample line bundle.)

Define the motive $M^c(X)$ in $\mathrm{DM}(k; R)$ to be the homotopy limit of the sequence

$$\cdots \to M^c(V_2 - S_2)(-n_2)[-2n_2] \to M^c(V_1 - S_1)(-n_1)[-2n_1].$$

The morphisms here are the composition

$$\begin{aligned} M^c(V_{i+1} - S_{i+1})(-n_{i+1})[-2n_{i+1}] &\to M^c(V_{i+1} - f_i^{-1}(S_i))(-n_{i+1})[-2n_{i+1}] \\ &\cong M^c(V_i - S_i)(-n_i)[-2n_i], \end{aligned}$$

where the first morphism is the flat pullback associated to an open inclusion, and the isomorphism follows from homotopy invariance for affine bundles. We will show that this motive is independent of the choice of vector bundles V_i and closed substacks S_i.

(Because we are relying on Bökstedt and Neeman's definition of homotopy limits, we have only defined the compactly supported motive of a quotient stack up to isomorphism. With more care, it should be possible to define this motive up to unique isomorphism in $\mathrm{DM}(k;R)$.)

Once we check that this motive is well defined up to isomorphism in Theorem 8.4, it will be immediate that the motivic homology of a quotient stack $X=Y/G$ given by the motive $M^c(X)$ agrees with the motivic homology of X as defined by Edidin and Graham [20, Sections 2.7 and 5.3]. Namely, any given motivic homology group $H_a(\cdot, R(b))$ of the sequence above is eventually constant. In our notation, Edidin and Graham defined $H_a(X,R(b))$ to be equal to

$$H_a\big(((Y\times(V_j-S_j))/G)(-n_j)[-2n_j], R(b)\big)\cong H_{a+2n_j}\big((Y\times(V_j-S_j))/G, R(b+n_j)\big)$$

for any j sufficiently large.

Tudor Păduraru observed that for a smooth quotient stack X of pure dimension n over k, the motive $M(X)$ determines $M^c(X)$ in a simple way: namely, $M^c(X)\cong M(X)^*(n)[2n]$. In particular, $M^c(BG)\cong M(BG)^*(-\dim G)[-2\dim G]$, since BG is a smooth stack of dimension $-\dim G$ over k. For example, it follows that

$$\mathrm{CH}_i\, BG\cong \mathrm{CH}^{-\dim G-i}\, BG.$$

Păduraru's argument uses that the dual of a direct sum in $\mathrm{DM}(k;R)$ is a product, and so the dual of a homotopy colimit is a homotopy limit. By contrast, the dual of a product does not have a simple description in general, and so it is not clear whether $M^c(X)^*$ is isomorphic to $M(X)(-n)[-2n]$ for a smooth quotient stack X of pure dimension n over k.

The following filtration of the category $\mathrm{DM}(k;R)$ is convenient for our arguments. For an integer j, let $D_j(k;R)$ be the smallest localizing subcategory of $\mathrm{DM}(k;R)$ that contains $M^c(X)(a)$ for all separated schemes X of finite type over k and all integers a such that $\dim X+a\le j$. (Another possible notation would be $d_{\le j}\,\mathrm{DM}(k;R)$, by analogy with a notation used for effective motives [28, proof of Corollary 1.9].) Thus we have a sequence of triangulated subcategories

$$\cdots\subset D_{-1}\subset D_0\subset D_1\subset\cdots$$

of $\mathrm{DM}(k;R)$.

For an integer j, let E_j be the smallest localizing subcategory of $\mathrm{DM}(k;R)$ that contains $M(Y)(a)$ for all smooth projective varieties Y over k and all integers $a>j$. This is related to the *slice filtration* of motives; in that setting, E_j would be called $\mathrm{DM}^{\mathrm{eff}}(k;R)(j+1)$ [28, Section 1]. For a triangulated subcategory E of a triangulated category T, the *right orthogonal* to E is the full subcategory $E^{\perp}$ of all objects M such that $\mathrm{Hom}(N, M) = 0$ for every N in E. The right orthogonal $E^{\perp}$ is always a *colocalizing* subcategory of T, meaning a triangulated subcategory that is closed under arbitrary products in T. In the notation of the slice filtration, $E_j^{\perp}$ might be called $\nu_{\leq j}\,\mathrm{DM}(k; R)$ [28, Definition 1.3].

Lemma 8.1 *The subcategory D_j of $\mathrm{DM}(k;R)$ is contained in the right orthogonal $E_j^{\perp}$.*

Proof As mentioned in Section 5, for any separated scheme Z of finite type over k, we have $H_j(Z, R(a)) = 0$ for all integers a and j with $a > \dim Z$. Let Y be a smooth projective variety over k, and let $n = \dim Y$. Then we have $H_j(Y \times Z, R(a)) = 0$ for all integers a and j with $a > n + \dim Z$. Equivalently,

$$\mathrm{Hom}_{\mathrm{DM}(k;R)}(R(a)[b], M(Y) \otimes M^c(Z)) = 0$$

for all integers a and b with $a > n + \dim Z$.

As mentioned in Section 5, we have

$$M(Y)^* \cong M(Y)(-n)[-2n].$$

So $\mathrm{Hom}_{\mathrm{DM}(k;R)}(R(a)[b], M(Y)^* \otimes M^c(Z)) = 0$ for all integers a and b with $a > \dim Z$. By Voevodsky and Kelly's results (see the proof of Lemma 5.5), it follows that $\mathrm{Hom}_{\mathrm{DM}(k;R)}(M(Y)(a)[b], M^c(Z)) = 0$ for all integers a and b with $a > \dim Z$. Since the object $M(Y)(a)$ is compact in $\mathrm{DM}(k; R)$, it follows that $\mathrm{Hom}_{\mathrm{DM}(k;R)}(M(Y)(a)[b], N) = 0$ for all motives N in the subcategory D_j and all integers a and b such that $a > j$. Consequently, D_j is contained in the right orthogonal $E_j^{\perp}$. □

Here is a convenient formal property of the subcategories $E_j^{\perp}$.

Lemma 8.2 *For any integer j, the subcategory $E_j^{\perp}$ of $\mathrm{DM}(k; R)$ is both localizing and colocalizing. That is, it is a triangulated subcategory of $\mathrm{DM}(k; R)$ which is closed under arbitrary direct sums and arbitrary products in $\mathrm{DM}(k; R)$.*

Proof Since E_j is a triangulated subcategory of $\mathrm{DM}(k; R)$, $E_j^{\perp}$ is a triangulated subcategory of $\mathrm{DM}(k; R)$. As is any right orthogonal, $E_j^{\perp}$ is closed under arbitrary products in $\mathrm{DM}(k; R)$. Because E_j is generated by a set of compact objects in $\mathrm{DM}(k; R)$, $E_j^{\perp}$ is also closed under arbitrary direct sums in $\mathrm{DM}(k; R)$ [43, Theorem 5.1]. □

Lemma 8.3 *The intersection of the subcategories $E_j^{\perp}$ of $\mathrm{DM}(k; R)$ for all integers j is zero.*

Proof If a motive N belongs to $E_j^{\perp}$ for all integers j, then

$$\mathrm{Hom}_{\mathrm{DM}(k;R)}(M(Y)(a)[b], N) = 0$$

for all smooth projective varieties Y over k and all integers a and b. Since the triangulated category $\mathrm{DM}(k; R)$ is generated by the objects $M^c(Y)(a)$ for smooth projective varieties W over k and integers a (Section 5), it follows that $N = 0$. Thus $\bigcap_j E_j^{\perp} = 0$. □

Theorem 8.4 *The compactly supported motive of a quotient stack over a field k is well-defined up to isomorphism in* $\mathrm{DM}(k; R)$.

Proof Let X be a quotient stack over k. Let $\cdots \twoheadrightarrow V_2 \twoheadrightarrow V_1$ and $\cdots \twoheadrightarrow W_2 \twoheadrightarrow W_1$ be two sequences of vector bundles over X, viewed as stacks over k, with closed substacks $S_j \subset V_j$ and $T_j \subset W_j$ such that $V_j - S_j$ and $W_j - T_j$ are schemes, S_{j+1} is contained in the inverse image of S_j under $V_{j+1} \twoheadrightarrow V_j$ and likewise for T_{j+1}, and the codimensions of $S_j \subset V_j$ and $T_j \subset W_j$ go to infinity. Let m_j be the rank of the bundle V_j over X and n_j the rank of W_j. We want to define an isomorphism from the motive

$$X_V := \mathrm{holim}_j\, M^c(V_j - S_j)(-m_j)[-2m_j]$$

to

$$X_W := \mathrm{holim}_j\, M^c(W_j - T_j)(-n_j)[-2n_j].$$

Consider the sequence of vector bundles $V_j \oplus W_j$ over X, viewed as stacks over k. (These stacks are the fiber products $V_j \times_X W_j$.) Let Z_j be the union of $S_j \times_X W_j$ and $V_j \times_X T_j$ inside $V_j \oplus W_j$. Then we have flat morphisms of schemes from $(V_j \oplus W_j) - Z_j$ to $V_j - S_j$ and to $W_j - T_j$, for all j. So we can choose morphisms from X_V and from X_W to the homotopy limit

$$X_{VW} := \mathrm{holim}_j\, M^c((V_j \oplus W_j) - Z_j)(-m_j - n_j)[-2m_j - 2n_j],$$

as homotopy limits of flat pullback maps of compactly supported motives. A priori, these morphisms may not be unique, by the nonfunctoriality of fibers in a triangulated category. If we write $X_V = \mathrm{holim}_{j \ge 1} A_j$ and $X_{VW} = \mathrm{holim}_{j \ge 1} B_j$, we ask only that the following diagram commutes:

$$\begin{tikzcd}
\prod_{j\ge 1} A_j[-1] \arrow[r] \arrow[d] & X_V \arrow[r] \arrow[d] & \prod_{j\ge 1} A_j \arrow[r, "1-\sigma"] \arrow[d] & \prod_{j\ge 1} A_j \arrow[d] \\
\prod_{j\ge 1} B_j[-1] \arrow[r] & X_{VW} \arrow[r] & \prod_{j\ge 1} B_j \arrow[r, "1-\sigma"] & \prod_{j\ge 1} B_j
\end{tikzcd}$$

One can choose a map of fibers that satisfies further good properties (such as extending to a 3×3 square of exact triangles), although still without characterizing the map uniquely [42, Example 2.6]. It suffices to show that the chosen morphisms $X_V \to X_{VW}$ and $X_W \to X_{VW}$ are isomorphisms in $\mathrm{DM}(k;R)$.

We will show that $X_V \to X_{VW}$ is an isomorphism; the argument would be the same for X_W. The point is that the morphism $(V_j \oplus W_j) - Z_j \to V_j - S_j$ is the complement of the closed subset $V_j \times_X T_j$ in a vector bundle (with fiber W_j) over the scheme $V_j - S_j$. The vector bundle (of rank n_j) gives an isomorphism

$$M^c((V_j \oplus W_j) - (S_j \times_X W_j))(-m_j - n_j)[-2m_j - 2n_j] \cong M^c(V_j - S_j)(-m_j)[-2m_j].$$

Removing T_j changes this motive by an object in the subcategory $D_{-\operatorname{codim}(T_j \subset W_j)}$, by the localization triangle for compactly supported motives (Section 5). Thus, in the notation above, for each integer m there is an r such that the fiber of $A_j \to B_j$ is in D_m (hence in $E_m^{\perp}$, by Lemma 8.1) for all $j \geq r$. I claim that the fiber F of the morphism $X_V \to X_{VW}$ is in $E_m^{\perp}$. Once we know this claim for all m, we will know that F is zero by Lemma 8.3, and hence that $X_V \to X_{VW}$ is an isomorphism.

To prove that claim, let Z be any motive in the subcategory E_m, and consider the map of long exact sequences of abelian groups (writing $[Z, X]$ for the group of morphisms from Z to X in $\mathrm{DM}(k;R)$):

$$\begin{array}{ccccccc}
\prod_{j\geq 1}[Z, A_j[-1]] & \longrightarrow & [Z, X_V] & \longrightarrow & \prod_{j\geq 1}[Z, A_j] & \xrightarrow{1-\sigma} & \prod_{j\geq 1}[Z, A_j] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\prod_{j\geq 1}[Z, B_j[-1]] & \longrightarrow & [Z, X_{VW}] & \longrightarrow & \prod_{j\geq 1}[Z, B_j] & \xrightarrow{1-\sigma} & \prod_{j\geq 1}[Z, B_j]
\end{array}$$

We know that $[Z, A_j[s]] \to [Z, B_j[s]]$ is an isomorphism for all $j \geq r$ and all integers s, because Z is in E_m and the fiber of $A_j \to B_j$ is in the triangulated subcategory $E_m^{\perp}$ for $j \geq r$. Using that, it is straightforward to check that the homomorphism from the complex of abelian groups $0 \to \prod_{j\geq 1}[Z, A_j[s]] \to \prod_{j \geq 1}[Z, A_j[s]] \to 0$ to the corresponding complex with B_j in place of A_j is a quasi-isomorphism for each s. (That is, the inverse limit and the $\lim^1$ group of a sequence of abelian groups do not change when finitely many groups in the sequence are changed.) By the map of exact sequences above, it follows that $[Z, X_V[s]] \to [Z, X_{VW}[s]]$ is an isomorphism for all integers s. Since Z is any object in E_m, this proves our claim that the fiber F of $X_V \to X_{VW}$ is in $E_m^{\perp}$. $\square$

Our definition of the compactly supported motive of a quotient stack agrees with the standard definition in the special case of a quasiprojective scheme. It would be desirable to make the compactly supported motive a functor from quotient stacks to $\mathrm{DM}(k;R)$, and to prove a localization triangle: for a closed substack Y of a quotient stack X over a field k, there should be an exact triangle $M^c(Y) \to M^c(X) \to M^c(X - Y)$.

Yehonatan Sella pointed out (correcting an error in an earlier version) that this is not clear from the current definition, because it is not clear whether the fiber of a (Bökstedt–Neeman) homotopy limit of morphisms in a triangulated category is the homotopy limit of the fibers. For now, the proof of Lemma 9.1 shows that there is a localization triangle for quotient stacks modulo an "error term" which can be made arbitrarily small, and that suffices for some applications.

We now describe a basic example of the motive of a quotient stack, $M^c(BG_m)$.

Lemma 8.5 *The compactly supported motive of BG_m in $\mathrm{DM}(k;R)$ is isomorphic to $\prod_{j\leq -1} R(j)[2j]$. This is isomorphic to the direct sum $\bigoplus_{j\leq -1} R(j)[2j]$.*

Proof By definition, using the representation of G_m by scalars on an n–dimensional vector space for any given n, $M^c(BG_m)$ is the homotopy limit of the motives

$$M^c(\mathbb{P}^{n-1})(-n)[-2n] \cong \prod_{j=-n}^{-1} R(j)[2j],$$

and so $M^c(BG_m)$ is isomorphic to the product $\prod_{j\leq -1} R(j)[2j]$.

To show that the morphism from the direct sum $\bigoplus_{j\leq -1} R(j)[2j]$ to the product is an isomorphism, it suffices to show that the cone N of this morphism is zero. For any integer $a<0$, N is isomorphic to the cone of the morphism $\bigoplus_{j\leq a} R(j)[2j] \to \prod_{j\leq a} R(j)[2j]$, because finite direct sums are the same as finite products. Because the subcategory $E_a^{\perp}$ is both localizing and colocalizing in $\mathrm{DM}(k;R)$ (Lemma 8.2), both $\bigoplus_{j\leq a} R(j)[2j]$ and $\prod_{j\leq -1} R(j)[2j]$ are in $E_a^{\perp}$. So N is in $E_a^{\perp}$. Since this holds for all negative integers a, N is zero by Lemma 8.3, as we want. □

Lemma 8.6 *Let X be a quotient stack over a field k. Then the motive $M^c(X)$ is in the subcategory $E_{\dim X}^{\perp}$.*

For a quotient stack X, one might ask whether $M^c(X)$ is always in the subcategory $D_{\dim X}$. For example, that is true for $M^c(BG_m) = \prod_{j\leq -1} \mathbb{Z}(j)[2j]$, because that is isomorphic to $\bigoplus_{j\leq -1} \mathbb{Z}(j)[2j]$ by Lemma 8.5, and that direct sum is in D_{-1}. It would be clear that the compactly supported motive of a quotient stack X was in $D_{\dim X}$ if the categories D_m were closed under arbitrary products in $\mathrm{DM}(k;R)$, but in general they are not [58, Theorem 7.1].

Proof of Lemma 8.6 For any separated scheme Z of finite type over k, the compactly supported motive of Z is in the subcategory $D_{\dim Z}$ of $\mathrm{DM}(k;R)$. This is clear if k

has characteristic zero, by resolution of singularities; in general, it follows from Kelly's work [34, proof of Proposition 5.5.3].

It follows that, for a quotient stack X over k, $M^c(X)$ is the homotopy limit of a sequence of motives in $D_{\dim X}$. Since $D_{\dim X}$ is contained in $E^{\perp}_{\dim X}$ (Lemma 8.1) and $E^{\perp}_{\dim X}$ is colocalizing in $\mathrm{DM}(k;R)$ (Lemma 8.2), we conclude that $M^c(X)$ is in $E^{\perp}_{\dim X}$. □

Lemma 8.7 *Let X be a motive in the subcategory $E^{\perp}_m$ of $\mathrm{DM}(k;R)$ for an integer m. Then the colocalization $C(X)$ with respect to Tate motives is in the subcategory D_m, and hence in $E^{\perp}_m$.*

Proof We use the construction of $C(X)$ from Section 7 as a homotopy colimit $\mathrm{hocolim}_j\, C_j$. Since X is in $E^{\perp}_m$, we have $H_b(X, R(a)) = 0$ for all integers a and b with $a > m$. So we can take the motive C_0 in the construction of $C(X)$ to be a direct sum of motives $R(a)[b]$ with $a \leq m$. Then C_0 is in D_m. So $H_b(C_0, R(a)) = 0$ for all integers a and b with $a > m$. By induction, we can choose C_j for all natural numbers j to be in D_m. So $C(X) = \mathrm{hocolim}_j\, C_j$ is in D_m. By Lemma 8.1, $C(X)$ is also in $E^{\perp}_m$. □

Define a motive A in $\mathrm{DM}(k;R)$ to be *mixed Tate modulo dimension m* if the cone of the morphism $C(A) \to A$ is in $E^{\perp}_m$. Also, define a quotient stack X to be *mixed Tate modulo codimension r* if $M^c(X)$ is mixed Tate modulo dimension $\dim X - r$.

Lemma 8.8 *All mixed Tate motives and all motives in $E^{\perp}_m$ are mixed Tate modulo dimension m. Also, the motives that are mixed Tate modulo dimension m form a triangulated subcategory of $\mathrm{DM}(k;R)$.*

Proof It is clear that a mixed Tate motive is mixed Tate modulo dimension m. Also, a motive in $E^{\perp}_m$ is mixed Tate modulo dimension m, by Lemma 8.7. It remains to show that for an exact triangle $X \to Y \to Z$ in $\mathrm{DM}(k;R)$ with X and Y mixed Tate modulo dimension m, Z is also mixed Tate modulo dimension m. We have a morphism of exact triangles:

$$\begin{array}{ccccc} C(X) & \longrightarrow & C(Y) & \longrightarrow & C(Z) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

The 3×3 lemma for triangulated categories gives a map $f\colon C(Z) \to Z$ such that the diagram above extends to a 3×3 square of exact triangles [42, Theorems 1.8 and 2.3]. The map $f\colon C(Z) \to Z$ is not a priori the obvious map, but it does induce an isomorphism on motivic homology, by the diagram. Since $C(Z)$ is the unique mixed Tate motive with a map to Z that induces an isomorphism on motivic homology, f is at least isomorphic to the obvious map.

Thus, the cone of $C(Z) \to Z$ is the cone of a morphism $\mathrm{cone}(C(X) \to X) \to \mathrm{cone}(C(Y) \to Y)$. The latter two cones are in $E_m^{\perp}$, and so the cone of $C(Z) \to Z$ is also in $E_m^{\perp}$. That is, Z is mixed Tate modulo dimension m. □

Corollary 8.9 *Let X be a motive in* $\mathrm{DM}(k;R)$ *which can be approximated by mixed Tate motives in the sense that X is mixed Tate modulo dimension j for every integer j. Then X is a mixed Tate motive.*

Proof The cone of $C(X) \to X$ is in $E_j^{\perp}$ for every integer j, and hence is zero by Lemma 8.3. □

Given more geometric information on a motive N, the following results give better criteria for when N is mixed Tate.

Lemma 8.10 *Let X be a separated scheme of finite type over k. If X is mixed Tate modulo dimension -1, then X is mixed Tate.*

Proof Under our assumption on R [34, Proposition 5.5.5], the motive $M^c(X)$ is in the subcategory E_{-1} of effective motives in $\mathrm{DM}(k;R)$. Let W be the cone of the morphism $C(M^c(X)) \to M^c(X)$. Our assumption that X is mixed Tate modulo dimension -1 means that W is in $E_{-1}^{\perp}$. So the morphism $M^c(X) \to W$ is zero. It follows that $M^c(X)$ is a summand of the mixed Tate motive $C(M^c(X))$. So $M^c(X)$ is a mixed Tate motive. □

There is a "finite-dimensional" criterion for when a quotient stack is mixed Tate, Corollary 8.13. Namely, a quotient stack $X = Y/G$ over k is mixed Tate (meaning that $M^c(X)$ is mixed Tate in $\mathrm{DM}(k;R)$) if and only if the scheme $(Y \times \mathrm{GL}(n))/G$ is mixed Tate, for one or any faithful representation $G \hookrightarrow \mathrm{GL}(n)$ over k.

Here is the main step in proving that.

Lemma 8.11 *Let X be a quotient stack over a field k. Let E be a principal* $\mathrm{GL}(n)$*–bundle over X for some n, viewed as a stack over k. Let r be an integer. Then X is mixed Tate modulo codimension r (in* $\mathrm{DM}(k;R)$*) if and only if E is mixed Tate modulo codimension r.*

Proof By construction of the compactly supported motive of a quotient stack as a homotopy limit (Theorem 8.4), for each integer r there is a morphism

$$f\colon M^c(X) \to M^c(X')(-a)[-2a]$$

with X' a scheme over k such that $\dim X' - a = \dim X$ and the fiber of f is in the subcategory $E^{\perp}_{\dim X-r}$ of $\mathrm{DM}(k;R)$. Moreover, there is a morphism

$$M^c(E) \to M^c(E')(-a)[-2a]$$

with fiber in $E^{\perp}_{\dim E-r}$ such that E' is a principal $\mathrm{GL}(n)$–bundle over the scheme X'. As a result, it suffices to prove the lemma when X is a scheme over k.

First consider the case $n=1$, so that E is a principal G_m–bundle over X. Think of E as the complement of the zero section in a line bundle over X. Since X is a scheme, we have the localization triangle

$$M^c(X) \to M^c(X)(1)[2] \to M^c(E)$$

in $\mathrm{DM}(k;R)$. Consider the morphism of exact triangles:

$$\begin{array}{ccccc} C(M^c(X)) & \longrightarrow & C(M^c(X))(1)[2] & \longrightarrow & C(M^c(E)) \\ \downarrow & & \downarrow & & \downarrow \\ M^c(X) & \longrightarrow & M^c(X)(1)[2] & \longrightarrow & M^c(E) \end{array}$$

The 3×3 lemma for triangulated categories gives a map $f\colon C(M^c(E)) \to M^c(E)$ such that the diagram above extends to a 3×3 square of exact triangles [42, Theorems 1.8 and 2.3]. By the same proof as for Lemma 8.8, f is at least isomorphic to the obvious map $C(M^c(E)) \to M^c(E)$. Let W be the cone of $C(M^c(X)) \to M^c(X)$, and let N be the cone of f. Then we have an exact triangle $W \to W(1)[2] \to N$.

If X is mixed Tate modulo codimension r, then W is in $E^{\perp}_{\dim X-r}$. So $W(1)[2]$ is in $E^{\perp}_{\dim X+1-r}$, and hence N is in $E^{\perp}_{\dim X+1-r} = E^{\perp}_{\dim E-r}$. That is, the scheme E is mixed Tate modulo codimension r, as we want.

Conversely, suppose that E is mixed Tate modulo codimension r. That is, N is in $E^{\perp}_{\dim E-r} = E^{\perp}_{\dim X+1-r}$. By Lemma 8.6, X is in $E^{\perp}_{\dim X}$. By Lemma 8.7, $C(X)$ is also in $E^{\perp}_{\dim X}$, and hence W is in $E^{\perp}_{\dim X}$. We want to show that X is mixed Tate modulo codimension r, meaning that W is in $E^{\perp}_{\dim X-r}$. If not, then there is a smallest integer j such that W is in $E^{\perp}_{j}$; we have $j > \dim X - r$ by assumption. Then $W(-1)$ is in $E^{\perp}_{j-1}$. The exact triangle

$$W(-1)[-2] \to W \to N(-1)[-2]$$

gives that W is in $E_{j-1}^{\perp}$, a contradiction. Thus X is mixed Tate modulo codimension r if and only if the principal G_m–bundle E over X is mixed Tate modulo codimension r.

Now let E be a principal $\mathrm{GL}(n)$–bundle over a scheme X over k, with n arbitrary. Let B be the subgroup of upper-triangular matrices in $\mathrm{GL}(n)$ over k. Then E/B is an iterated projective bundle over X, and so

$$M^c(E/B) \cong \bigoplus_j M^c(X)(a_j)[2a_j],$$

where $a_1, \dots, a_{n!}$ are the dimensions of the Bruhat cells of the flag manifold $\mathrm{GL}(n)/B$. Assume that X is mixed Tate modulo codimension r, that is, modulo dimension $\dim X - r$. Then $M^c(X)(a)[2a]$ is mixed Tate modulo dimension $\dim X - r + a$, for any integer a. It follows that E/B is mixed Tate modulo dimension

$$\dim X - r + \dim G/B = \dim E/B - r.$$

That is, E/B is mixed Tate modulo codimension r. Conversely, if E/B is mixed Tate modulo codimension r, then the summand $M^c(X)(\dim G/B)[2\dim G/B]$ of $M^c(E/B)$ is mixed Tate modulo dimension

$$\dim E/B - r = \dim X + \dim G/B - r,$$

and so $M^c(X)$ is mixed Tate modulo dimension $\dim X - r$, thus modulo codimension r.

Next, let U be the subgroup of strictly upper-triangular matrices in $\mathrm{GL}(n)$ over k. Since $B/U \cong (G_m)^n$, the stack E/U is a principal $(G_m)^n$–bundle over E/B. Applying our result on principal G_m–bundles n times, we deduce that E/U is mixed Tate modulo codimension r if and only if E/B is mixed Tate modulo codimension r, hence if and only if X is mixed Tate modulo codimension r. Finally, U is an extension of copies of the additive group, and so homotopy invariance gives that

$$M^c(E) \cong M^c(E/U)(\dim U)[2\dim U].$$

It follows that E is mixed Tate modulo codimension r if and only if X is mixed Tate modulo codimension r. □

Corollary 8.12 *Let X be a quotient stack over a field k. Let E be a principal $\mathrm{GL}(n)$–bundle over X for some n, viewed as a stack over k. Then E is mixed Tate (in $\mathrm{DM}(k;R)$) if and only if X is mixed Tate.*

Proof This follows from Lemma 8.11, since a motive is mixed Tate if and only if it is mixed Tate modulo dimension r for all integers r (Corollary 8.9). □

Corollary 8.13 *Let Y be a quasiprojective scheme over a field k and G an affine group scheme of finite type over k that acts on Y such that there is a G–equivariant ample line bundle on Y. Let $G \hookrightarrow \mathrm{GL}(n)$ be a faithful representation of G over k. Then (the compactly supported motive of) the stack Y/G over k is mixed Tate if and only if the scheme $(Y \times \mathrm{GL}(n))/G$ over k is mixed Tate.*

Proof The scheme $(Y \times \mathrm{GL}(n))/G$ is a principal $\mathrm{GL}(n)$–bundle over the stack Y/G. So this follows from Corollary 8.12. □

For example, BG is mixed Tate if and only if the scheme $\mathrm{GL}(n)/G$ is mixed Tate, for one or any faithful representation $G \hookrightarrow \mathrm{GL}(n)$ over k.

As a result, we now show that the structure of a classifying space BG is determined in some ways by its properties in low codimension, namely codimension n^2 (roughly), where n is the dimension of a faithful representation of G. Theorem 9.6 reduces the question of whether BG is mixed Tate even further, to properties in codimension n (roughly) together with properties of subgroups of G.

Theorem 8.14 *Let G be an affine group scheme over a field k. Suppose that G has a faithful representation of dimension n over k. If BG is mixed Tate in $\mathrm{DM}(k;R)$ modulo codimension $n^2 - \dim G + 1$, then BG is mixed Tate in $\mathrm{DM}(k;R)$.*

Proof We have a principal $\mathrm{GL}(n)$–bundle $\mathrm{GL}(n)/G \to BG$ of stacks over k. By Lemma 8.11, if BG is mixed Tate modulo codimension $n^2 - \dim G + 1$, then the variety $\mathrm{GL}(n)/G$ is also mixed Tate modulo codimension $n^2 - \dim G + 1$. Since $\mathrm{GL}(n)/G$ has dimension $n^2 - \dim G$, Lemma 8.10 gives that $\mathrm{GL}(n)/G$ is mixed Tate. By Corollary 8.13, BG is mixed Tate. □

9 The mixed Tate property for classifying spaces

The work of Bogomolov and Saltman defines a dichotomy among all finite groups G: is $BG_{\mathbb{C}}$ stably rational? (This means that the variety V/G is stably rational for one, or any, faithful representation V of G over $\mathbb{C}$.) This paper has considered several other dichotomies among finite groups G. Is the birational motive of $BG_{\mathbb{C}}$ trivial? Does $BG_{\mathbb{C}}$ have the weak or strong Chow Künneth property? It would be interesting to know whether these conditions are all equivalent.

Ekedahl defined another property with the same flavor, for a finite group scheme G over a field k. Namely, when does the stack BG have the class of a point in the ring $A = K_0(\mathrm{Var}_k)[L^{-1}, (L^n - 1)^{-1} : n \geq 1]$? Here $K_0(\mathrm{Var}_k)$ denotes the Grothendieck ring of k–varieties and L is the class of A^1. Ekedahl showed that this property is equivalent to the statement (not mentioning stacks) that for one or any faithful representation $G \hookrightarrow \mathrm{GL}(n)$, the variety $\mathrm{GL}(n)/G$ is equal to $\mathrm{GL}(n)$ in the ring A [21, Proposition 3.1]. I do not know any implications between Ekedahl's property and the other properties we have mentioned, but it may be that all these properties are equivalent when the base field k is algebraically closed. In particular, Ekedahl's property fails if G has nontrivial unramified H^2 [21, Theorem 5.1]; for such groups, all the properties we have mentioned fail.

In this section, we consider another dichotomy among finite groups, or more generally among affine group schemes G: is BG mixed Tate, meaning that the motive $M^c(BG)$ is mixed Tate? This property is equivalent to the motivic Künneth property formulated in the introduction to Section 8. It implies the Chow Künneth property, since it gives information about all of motivic homology, not just Chow groups. The mixed Tate property may be equivalent to all the other properties mentioned above, when the base field k is algebraically closed.

We have examples of finite groups which are not mixed Tate (say over $\mathbb{C}$), because they do not even have the weak Chow Künneth property (Corollary 3.1). To justify the concept, we will also give examples of finite groups which are mixed Tate: the symmetric groups (Theorem 9.11), the finite general linear groups in cross-characteristic (Theorem 9.12), and all finite subgroups of GL(2) (Corollary 9.7). It is conceivable that all "naturally occurring" finite groups are mixed Tate over $\mathbb{C}$. For example, Bogomolov conjectured that for every finite simple group G, quotient varieties V/G are stably rational [7]. In that direction, Kunyavskiĭ showed that every finite simple group has unramified H^2 equal to zero [37, Corollary 1.2]. Likewise, I conjecture that all finite simple groups are mixed Tate. By contrast, Kunyavskiĭ showed that there are finite quasisimple groups (central extensions of $\mathrm{PSL}(3,4)$ by $\mathbb{Z}/4$ or $\mathbb{Z}/12$) with unramified H^2 not zero [37, proof of Theorem 1.1].

In order to give examples of finite groups which are mixed Tate, we start by proving some formal properties of mixed Tate stacks. By Corollary 8.13, BG is mixed Tate if and only if the variety $\mathrm{GL}(n)/G$ is mixed Tate for a faithful representation V of G with $\dim V = n$. But $\mathrm{GL}(n)/G$ may be hard to analyze because it has high dimension, namely n^2. Theorem 9.6 gives a sufficient condition for BG to be mixed Tate in terms of the variety $(V-S)/G$, which has dimension only n, together with information on subgroups of G.

Throughout this section, we work in the category $\mathrm{DM}(k;R)$ for a field k and a commutative ring R in which the exponential characteristic of k is invertible.

Lemma 9.1 *Let X be a quotient stack over a field k and Y a closed substack. If two of X, Y, $X-Y$ are mixed Tate, then so is the third. Also, for an integer m, if two of X, Y, $X-Y$ are mixed Tate modulo dimension m, then so is the third.*

Proof By the construction of the compactly supported motive of a quotient stack as a homotopy limit (Theorem 8.4), for each integer m there is an exact triangle $A_m \to B_m \to C_m$ (of shifted compactly supported motives of schemes over k) together with morphisms $M^c(Y) \to A_m$, $M^c(X) \to B_m$ and $M^c(X-Y) \to C_m$ whose fibers are in the subcategory $E_m^{\perp}$ of $\mathrm{DM}(k;R)$. (This works even though we have not shown that there is an exact triangle $M^c(Y) \to M^c(X) \to M^c(X-Y)$.)

As a result, if two of $M^c(X)$, $M^c(Y)$ and $M^c(X-Y)$ are mixed Tate modulo dimension m, then two of A_m, B_m, C_m have that property, by Lemma 8.8. Therefore, the third object of A_m, B_m, C_m is also mixed Tate modulo dimension m, and hence the third object of $M^c(X)$, $M^c(Y)$, $M^c(X-Y)$ has that property. Thus we have shown that if two of $M^c(X)$, $M^c(Y)$, $M^c(X-Y)$ are mixed Tate modulo dimension m, then so is the third. Finally, if two of $M^c(X)$, $M^c(Y)$, $M^c(X-Y)$ are mixed Tate, then the third object is mixed Tate modulo dimension m for every integer m, and so it is mixed Tate by Corollary 8.9. □

Lemma 9.2 *Let k be a field, and let e be the exponential characteristic of k. A quotient stack X over a field k is mixed Tate with $\mathbb{Z}[1/e]$ coefficients (that is, in $\mathrm{DM}(k;\mathbb{Z}[1/e])$) if and only if it is mixed Tate with $\mathbb{Z}_{(p)}$ coefficients for all prime numbers p that are invertible in k.*

Proof Write X as the quotient stack A/G for some affine group scheme G of finite type over k and some quasiprojective scheme A over k with a G–equivariant ample line bundle. Let $G \hookrightarrow \mathrm{GL}(n)$ be a faithful representation over k. Then $E = (A \times \mathrm{GL}(n))/G$ is a quasiprojective scheme over k, and $\mathrm{GL}(n)$ acts on E with quotient stack $E/\mathrm{GL}(n) \cong X$. By Corollary 8.12, $M^c(E)$ is mixed Tate (with any coefficients) if and only if $M^c(X)$ is mixed Tate. So it suffices to show that $M^c(E)$ is mixed Tate in $\mathrm{DM}(k;\mathbb{Z}[1/e])$ if and only if it is mixed Tate in $\mathrm{DM}(k;\mathbb{Z}_{(p)})$ for all prime numbers p that are invertible in k.

For a commutative ring R, E is R–mixed Tate if and only if it has the Künneth property for the R–motivic homology of $E \times Y$ for all separated k–schemes Y of finite type (Theorem 7.2). The motivic homology with R coefficients of a k–scheme is related to motivic homology with $\mathbb{Z}$ coefficients by the universal coefficient theorem. Let p be a prime number that is invertible in k. Since $\mathbb{Z}_{(p)}$ and $\mathbb{Z}[1/e]$ are flat over $\mathbb{Z}$, the Künneth spectral sequence for $E \times Y$ with $\mathbb{Z}_{(p)}$ coefficients is just the localization at p of the spectral sequence with $\mathbb{Z}[1/e]$ coefficients. A homomorphism of $\mathbb{Z}[1/e]$–modules is an isomorphism if and only if it is an isomorphism p–locally for all prime numbers p that are invertible in k. Therefore, X is $\mathbb{Z}[1/e]$–mixed Tate if and only if it is $\mathbb{Z}_{(p)}$–mixed Tate for all prime numbers p that are invertible in k. □

Lemma 9.3 *Let G be a finite group, p a prime number, and H a Sylow p–subgroup of G. Fix a base field k in which p is invertible. Let R be the ring $\mathbb{Z}/p$ or $\mathbb{Z}_{(p)}$. If BH is R–mixed Tate, then BG is R–mixed Tate.*

Proof Use that BG is R–mixed Tate if and only if it has the Künneth property for $BG \times Y$ for all k–schemes Y of finite type. Let R be $\mathbb{Z}/p$ or $\mathbb{Z}_{(p)}$. Using the transfer, the Künneth spectral sequence for $BG \times Y$ is a summand with R coefficients of the spectral sequence for $BH \times Y$. Therefore, if BH satisfies the motivic Künneth property with R coefficients, then so does BG. □

For a representation V of a finite group G and K a subgroup of G, V^K means the linear subspace fixed by K. Following Ekedahl [21], let V_K be the open subset of V^K of points with stabilizer in G equal to K, meaning that $V_K = V^K - \bigcup_{K \subsetneq L} V^L$.

Lemma 9.4 *Let s be a natural number. Let V be a faithful representation of a finite group G over a field k. For each subgroup K of G that occurs as the stabilizer of a point in V, assume that the stack $V_K/N_G(K)$ is mixed Tate in $\mathrm{DM}(k;R)$ modulo codimension $s - \mathrm{codim}(V^K \subset V)$. Then BG is mixed Tate modulo codimension s.*

Proof The stack V/G is a vector bundle over BG. So if we can show that the stack V/G is mixed Tate modulo codimension s, then BG is mixed Tate modulo codimension s, as we want.

The stack V/G is the disjoint union of the locally closed substacks $V_K/N_G(K)$ for all conjugacy classes of stabilizer subgroups K of G. By assumption, each substack $V_K/N_G(K)$ is mixed Tate modulo codimension $s - \mathrm{codim}(V^K \subset V)$, that is, modulo dimension $\dim V - s$. By Lemma 9.1, the stack V/G is mixed Tate modulo dimension $\dim V - s$, that is, modulo codimension s. □

A next step is to express the assumptions on smaller groups in terms of classifying spaces, as follows. This step may not be needed in some examples, but it leads to a neat statement, Theorem 9.6. (We will apply Lemma 9.5 to the subgroups $H = N_G(K)$ acting on V^K in Lemma 9.4, typically not faithfully.)

Lemma 9.5 *Let s be a natural number. Let V be a representation of a finite group H over a field k, not necessarily faithful. Let $K_1 = \ker(H \to \mathrm{GL}(V))$. Consider all chains $K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_r \subset H$, $r \geq 1$, such that if we define $N_i = \bigcap_{j \leq i} N_H(K_j) \subset H$, then K_{i+1} is the stabilizer of a point for N_i acting on V^{K_i}. For every such chain, assume that BN_r is mixed Tate in $\mathrm{DM}(k;R)$ modulo codimension s. (In particular, for $r = 1$, we are assuming that BH is mixed Tate modulo codimension s.) Then the stack V_{K_1}/H is mixed Tate modulo codimension s.*

Proof By our assumption (with $r = 1$), the stack BH is mixed Tate modulo codimension s. So the stack V/H (a vector bundle over BH) is mixed Tate modulo codimension s. The difference $V/H - V_{K_1}/H$ is the disjoint union of the locally closed substacks $(\coprod_{g\in H/N_H(K_2)} V_{gK_2g^{-1}})/H$ for conjugacy classes of stabilizer subgroups K_2 for H acting on V with $K_1 \subsetneqq K_2$. That quotient is isomorphic to the stack $V_{K_2}/N_H(K_2)$. By our assumption (with $r = 2$), $BN_H(K_2) = BN_2$ is mixed Tate modulo codimension s, and so the stack $V^{K_2}/N_H(K_2)$ (a vector bundle over BN_2) is also mixed Tate modulo codimension s. The stack we want is the open substack $V_{K_2}/N_H(K_2)$ of $V^{K_2}/N_H(K_2)$. The complement is the disjoint union of the locally closed substacks

$$\Big(\coprod_{g\in N_2/N_3} V_{gK_3g^{-1}}\Big)/N_2 \cong V_{K_3}/N_3,$$

where K_3 runs over all stabilizer subgroups for N_2 acting on V^{K_2} with $K_1 \subsetneqq K_2 \subsetneqq K_3$, and $N_{N_2}(K_3) = \bigcap_{j\leq 3} N_H(K_j) = N_3$. Since H is finite, the process stops after finitely many steps and gives the statement of the lemma, via Lemma 9.1. □

Combining the previous two lemmas gives the following result. Theorem 9.6 shows that BG is mixed Tate if the variety V_1/G is mixed Tate and BH is mixed Tate for certain proper subgroups H of G. (As in the notation above, V_1 denotes the open subset of V where G acts freely.) Theorem 9.6 was suggested by a similar statement by Ekedahl about his invariant of BG in the Grothendieck ring of varieties [21, Theorem 3.4], but I do not see a direct implication between the two results.

Theorem 9.6 *Let V be a faithful representation of a finite group G over a field k. Consider all chains $1 = K_0 \subsetneqq K_1 \subsetneqq \cdots \subsetneqq K_r \subset G$, $r \geq 1$, such that if we define $N_i = \bigcap_{j\leq i} N_G(K_j) \subset G$, then K_{i+1} is a stabilizer subgroup for N_i acting on V^{K_i}. Suppose that the variety V_1/G is mixed Tate in* $\mathrm{DM}(k;R)$ *and that the stack BN_r is mixed Tate for all such chains with $N_r \neq G$. Then BG is mixed Tate.*

Proof We show by induction on s that BG is mixed Tate modulo codimension s for every natural number s. That will imply that BG is mixed Tate by Corollary 8.9 (or by the stronger Theorem 8.14). Clearly BG is mixed Tate modulo codimension 0. Suppose that BG is mixed Tate modulo codimension s. To show that BG is mixed Tate modulo codimension $s+1$, we use Lemma 9.4. So it suffices to show that for each stabilizer subgroup K_1 of G acting on V, the stack $V_{K_1}/N_G(K_1)$ is mixed Tate modulo codimension $s+1-\mathrm{codim}(V^{K_1} \subset V)$. For $K_1 = 1$, this is true, because we assume that the variety V_1/G is mixed Tate. It remains to consider a stabilizer subgroup $K_1 \neq 1$. We apply Lemma 9.5 to the vector space V^{K_1} with its action of $N_G(K_1)$. If

$N_G(K_1) \neq G$, then Lemma 9.5 and our assumptions imply that the stack $V_{K_1}/N_G(K_1)$ is mixed Tate. Finally, if $K_1 \neq 1$ and $N_G(K_1) = G$, then Lemma 9.5, our assumptions, and the inductive hypothesis that BG is mixed Tate modulo codimension s imply that the stack $V_{K_1}/N_G(K_1)$ is mixed Tate modulo codimension s. This implies that $V_{K_1}/N_G(K_1)$ is mixed Tate modulo codimension $s+1-\operatorname{codim}(V^{K_1} \subset V)$ (as we want), because $\operatorname{codim}(V^{K_1} \subset V) > 0$, since $K_1 \neq 1$ and G acts faithfully on V. The induction is complete. So BG is mixed Tate. □

We now use Theorem 9.6 to give examples of finite groups which are mixed Tate. (The assumption on the field k in Corollary 9.7 could be weakened.) For example, Corollary 9.7 gives that the dihedral groups, generalized quaternion groups, modular 2–groups, and semidihedral groups [1, Section 23.4] are mixed Tate.

Corollary 9.7 *Let k be a field that contains $\overline{\mathbb{Q}}$. Let G be a finite subgroup of* GL(2) *over k. Then BG is mixed Tate in* $\mathrm{DM}(k;\mathbb{Z})$.

Proof Use induction on the order of G. Let V be the given 2–dimensional faithful representation of G. Since BH is mixed Tate for all proper subgroups H of G, Theorem 9.6 shows that BG is mixed Tate if the variety V_1/G is mixed Tate.

The group G acts on the projective space $\mathbb{P}^1$ of lines in V_1. The coarse quotient $\mathbb{P}^1/G$ is a normal projective curve over k, and so it is smooth over k. It is unirational over k, and hence isomorphic to $\mathbb{P}^1$ over k.

It is convenient to observe that the representation V of G can be defined over $\overline{\mathbb{Q}}$. Let S be the closed subset of $\mathbb{P}^1$ where G does not act freely; then $(\mathbb{P}^1 - S)/G$ is isomorphic to $\mathbb{P}^1 - T$ for some closed subset T. Since S and T are defined over $\overline{\mathbb{Q}}$, T is a finite union of copies of $\operatorname{Spec} k$. So $\mathbb{P}^1 - T$ is a linear scheme over k (as defined in Section 5). An open subset of V_1/G is a principal G_m–bundle over $\mathbb{P}^1 - T$, and hence is a linear scheme over k. The complement of this open subset is the union of finitely many curves of the form G_m/H where H is a finite subgroup of G_m; these are isomorphic to G_m and hence are linear schemes over k. So V_1/G is a linear scheme over k. Thus V_1/G is mixed Tate, and so BG is mixed Tate. □

We now show that many wreath product groups are mixed Tate. It will follow that the finite general linear groups in cross-characteristic and the symmetric groups are mixed Tate (Theorems 9.11 and 9.12), since their Sylow p–subgroups are products of iterated wreath products of cyclic groups. This is related to Voevodsky's construction of Steenrod operations on motivic cohomology, which can be viewed as computing the motivic cohomology of the symmetric groups over any field [61, Section 6; 62].

Lemma 9.8 *Let k be a field of characteristic not p that contains the p^{th} roots of unity. Let X be a quasiprojective linear scheme over k (as defined in Section 5). Then the cyclic product $Z^pX = X^p/(\mathbb{Z}/p)$ is a quasiprojective linear scheme over k.*

We assume that X is quasiprojective in order to ensure that the cyclic product Z^pX is a scheme. If we worked with algebraic spaces throughout, then the assumption of quasiprojectivity would be unnecessary.

Proof We start by showing that for any representation V of $\mathbb{Z}/p$ over k, the quotient variety $V/(\mathbb{Z}/p)$ is a linear scheme, following [55, proof of Lemma 8.1]. We use induction on the dimension of V. We can assume that $\mathbb{Z}/p$ acts nontrivially on V. Then we can write $V = W \oplus L$, where L is a nontrivial 1–dimensional representation of $\mathbb{Z}/p$. The quotient variety $V/(\mathbb{Z}/p)$ has a closed subvariety $W/(\mathbb{Z}/p)$, which is a linear scheme by induction. The open complement is a vector bundle (with fiber W) over $(L-0)/(\mathbb{Z}/p) \cong A^1 - 0$. A direct calculation shows that this vector bundle is trivial. So the open complement is isomorphic to $W \times (A^1 - 0)$, which is a linear scheme. Thus $V/(\mathbb{Z}/p)$ is a linear scheme over k, completing the induction.

Next, let Y be a closed subscheme of a scheme X over k, and let $U = X - Y$. Then the cyclic product scheme Z^pX is the disjoint union (as a set) of Z^pY, Z^pU and various products $Y^a \times U^{p-a}$ for $0 \le a \le p$. Suppose that X, Y and U are linear schemes over k. Then all products $Y^a \times U^{p-a}$ are linear schemes. As a result, if any two of Z^pX, Z^pY and Z^pU are linear schemes, then so is the third. By the inductive definition of linear schemes, it follows that for every linear scheme X over k, Z^pX is a linear scheme over k. □

Let G be an affine group scheme of finite type over a field k. We say that BG can be *approximated by linear schemes* over k if, for every natural number r, there is a representation V of G and a closed G–invariant subset S of codimension at least r in V such that G acts freely on $V - S$ and $(V-S)/G$ is a linear scheme over k. If BG can be approximated by linear schemes, then BG is mixed Tate. Indeed, for each r, V, S as just mentioned, the compactly supported motive of the quotient stack S/G is in the subcategory $(E_{\dim S - \dim G})^{\perp}$, by Lemma 8.6. Write V/G for the quotient stack. Then it follows that the cone of the morphism

$$M^c(BG) \cong M^c(V/G)(-\dim V)[-2\dim V] \to M^c(V-S)/G(-\dim V)[-2\dim V]$$

lies in $(E_{\dim S - \dim V - \dim G})^{\perp}$, hence in $(E_{-r-\dim G})^{\perp}$. Since we assumed that r can be arbitrarily large, Corollary 8.9 gives that $M^c(BG)$ is mixed Tate.

For a group G, the *wreath product* $\mathbb{Z}/p \wr G$ means the semidirect product $\mathbb{Z}/p \ltimes G^p$, with $\mathbb{Z}/p$ cyclically permuting the copies of G.

Lemma 9.9 *Let k be a field of characteristic not p that contains the p^{th} roots of unity. Let G be an affine group scheme over k such that BG can be approximated by linear schemes over k. Then $B(\mathbb{Z}/p \wr G)$ can be approximated by linear schemes over k, and hence is mixed Tate.*

Proof Let V be a representation of G over k. Then $V^{\oplus p}$ can be viewed as a representation of $\mathbb{Z}/p \wr G$, where $\mathbb{Z}/p$ permutes the copies of V. If the quotients make sense, then we have $V^{\oplus p}/(\mathbb{Z}/p \wr G) = Z^p(V/G)$. It follows that if BG can be approximated by linear schemes Y, then $B(\mathbb{Z}/p \wr G)$ is approximated by the schemes $Z^p Y$, which are linear schemes by Lemma 9.8. □

Corollary 9.10 *Let G be a group scheme over a field k that satisfies one of the following assumptions:*

(1) *G is the multiplicative group G_m.*

(2) *G is a finite abelian group of exponent e viewed as an algebraic group over k, e is invertible in k, and k contains the e^{th} roots of unity.*

(3) *G is an iterated wreath product $\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p \wr G_m$ over k, p is invertible in k, and k contains the p^{th} roots of unity.*

(4) *G is an iterated wreath product $\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p \wr A$ for a finite abelian group A of exponent e, viewed as an algebraic group over k. Also, p and e are invertible in k and k contains the p^{th} and e^{th} roots of unity.*

Then BG is mixed Tate in $\mathrm{DM}(k;\mathbb{Z})$.

Proof In all these cases, BG can be approximated by linear schemes over k and hence is mixed Tate. First, BG_m can be approximated by the schemes $(A^n - 0)/G_m = \mathbb{P}^{n-1}$ over k as n increases. These are linear schemes. Next, when A is a finite abelian group of exponent e such that e is invertible in k and k contains the e^{th} roots of unity, then A is isomorphic to a product of the group schemes μ_r over k. The classifying space $B\mu_r$ can be approximated by the schemes $(A^n - 0)/\mu_r$ as n increases, where μ_r acts by scalars. This scheme is the total space of the line bundle $O(r)$ minus the zero section over $\mathbb{P}^{n-1}$, and hence is a linear scheme. So BA can be approximated by linear schemes, under our assumption on k. Finally, the statements on wreath products follow from Lemma 9.9. □

Theorem 9.11 *Let n be a positive integer, and let k be a field of characteristic zero that contains the p^{th} roots of unity for all primes p dividing n. Then the symmetric group S_n is mixed Tate over k (with $\mathbb{Z}$ coefficients).*

Proof Let p be a prime number. A Sylow p–subgroup H of $G = S_n$ is a product of iterated wreath products $\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p$. By Corollary 9.10, BH is mixed Tate in $\mathrm{DM}(k;\mathbb{Z})$, hence in $\mathrm{DM}(k;\mathbb{Z}_{(p)})$ by Lemma 9.2. By Lemma 9.3, BG is mixed Tate in $\mathrm{DM}(k;\mathbb{Z}_{(p)})$. Since this holds for all prime numbers p, BG is mixed Tate in $\mathrm{DM}(k;\mathbb{Z})$ by Lemma 9.2. □

Theorem 9.12 *Let n be a positive integer, q a power of a prime number p, and l a prime number different from p. Let r be the order of q in $(\mathbb{Z}/l)^*$, and let ν be the l–adic order of $q^r - 1$. If $l = 2$, assume that $q \equiv 1 \pmod 4$. Let k be a field of characteristic not l that contains the $(l^\nu)^{th}$ roots of unity. Then the finite group $\mathrm{GL}(n,\mathbb{F}_q)$ is mixed Tate in $\mathrm{DM}(k,\mathbb{Z}_{(l)})$.*

Proof If l is odd, or if $l = 2$ and $q \equiv 1 \pmod 4$, then a Sylow l–subgroup of $\mathrm{GL}(n,\mathbb{F}_q)$ is a product of wreath products $\mathbb{Z}/l \wr \cdots \wr \mathbb{Z}/l \wr \mathbb{Z}/l^\nu$ [10; 63]. The result follows from Corollary 9.10 and Lemma 9.3. □

10 Groups of order 32

Let G be a p–group of order at most p^4, for a prime number p. Let e be the exponent of G. Let k be a field of characteristic not p which contains the e^{th} roots of unity. Then the Chow ring of BG consists of transferred Euler classes of representations [57, Theorem 11.1], and this remains true over every extension field of k. All representations of a subgroup of G over an extension field of k can be defined over k, and so it follows that G has the weak Chow Künneth property: $\mathrm{CH}^* BG \to \mathrm{CH}^* BG_E$ is surjective for every extension field E of k.

In this section, we show that groups of order 32 also satisfy the weak Chow Künneth property. It follows that the results after Corollary 3.1 are optimal: there are groups of order 64, and of order p^5 for any odd prime number p, which do not have the weak Chow Künneth property. It is not known whether groups of order 32 are mixed Tate.

Our proof of the weak Chow Künneth property for groups G of order 32 uses the fact that BG is stably rational for these groups, by Chu, Hu, Kang and Prokhorov [12]. We do not know how to relate these two properties in general; as discussed in Section 9, they may be equivalent.

Theorem 10.1 *Let G be a group of order 32. Let e be the exponent of G. Let k be a field of characteristic not 2 which contains the e^{th} roots of unity. Then BG over k satisfies the weak Chow Künneth property.*

Proof For every proper subgroup H of G, the order of H divides 16, and so BH over k satisfies the weak Chow Künneth property, as mentioned above.

Let V be a faithful representation of G over k. Since k does not have characteristic 2, V is a direct sum of irreducible representations, $V = \bigoplus_{i=1}^{c} V_i$. Write $P(W)$ for the space of hyperplanes in a vector space W, so that $P(W^*)$ is the space of lines in W. Then G acts on the product of projective spaces $Y = P(V_1^*) \times \cdots \times P(V_c^*)$. The kernel of the action of G on Y is the center of G, by Schur's lemma. Let D_Y be the closed subset of Y where $G/Z(G)$ does not act freely. Let D be the union of $\bigcup_{i=1}^{c} \bigoplus_{j \neq i} V_j$ with the inverse image of D_Y in V. Then D is a G–invariant finite union of linear subspaces of V, and $D \neq V$.

Lemma 10.2 *Let G be a p–group. Let V be a faithful representation of G over a field k of characteristic not p. Let Y be the product of projective spaces defined above, and define D_Y and D as above. Suppose that the variety $(V - D)/G$ has the weak Chow Künneth property. Also, suppose that for every subgroup $N \neq G$ that is the stabilizer of some intersection of irreducible components of D (as a set), BN has the weak Chow Künneth property. Then BG has the weak Chow Künneth property.*

Lemma 10.2 is analogous to Theorem 9.6 on the mixed Tate property, but the argument for the weak Chow Künneth property is simpler.

Proof of Lemma 10.2 By the localization sequence for Chow groups of quotient stacks [20, Section 2.7], if a quotient stack X over k has the weak Chow Künneth property, then so does every open substack of X. Also, if a closed substack S of X and $X - S$ both have the weak Chow Künneth property, then so does X. We sometimes write CK for Chow Künneth.

We need some variants of these statements. For an integer a, say that a quotient stack X has the weak CK property *in dimension at least a* if $\mathrm{CH}_i X \to \mathrm{CH}_i X_E$ is surjective for all fields E/k and all $i \geq a$. Also, say that X has the weak CK property *in codimension b* if X has the weak CK property in dimension at least $\dim X - b$. By the localization sequence for Chow groups, if X has the weak CK property in codimension b, then so does any open substack of the same dimension as X. Also, if a closed substack S of X and $X - S$ both have the weak CK property in dimension at least a, then so does X.

To prove the lemma, we show by induction on b that BG has the weak Chow Künneth property in codimension b for all b. This is clear for $b = -1$. Suppose that BG has the weak CK property in codimension b. To show that BG has the weak CK property in codimension $b+1$, it is equivalent to show that the stack V/G (a vector bundle over BG) has the weak CK property in codimension $b+1$. We are assuming that the variety $(V-D)/G$ has the weak CK property. Its complement in the stack V/G is a finite disjoint union of locally closed substacks of the form U/N, where U is an open subset of a linear subspace $W \subsetneq V$ and N is the stabilizer in G of W as a set. If $N \neq G$, then we are assuming that BN has the weak CK property. So the stack W/N (a vector bundle over BN) has the weak CK property, and hence the open substack U/N has the weak CK property. On the other hand, if $N = G$, then BG has the weak CK property in codimension b by the inductive assumption, and so the stack W/G and its open substack U/G have the weak CK property in codimension b. Here W has codimension > 0 in V. We conclude that the stack V/G has the weak CK property in codimension $b+1$, thus completing the induction. So BG has the weak CK property. □

We continue the proof of Theorem 10.1. Let G be a group of order 32. Let e be the exponent of G, and let k be a field of characteristic not 2 that contains the e^{th} roots of unity. If G is not isomorphic to $(\mathbb{Z}/2)^5$, then G has a faithful complex representation V of dimension 4. (This can be checked using the free group-theory program GAP [52], or by the methods of Cernele, Kamgarpour and Reichstein [11, proof of Lemma 13].) The group $(\mathbb{Z}/2)^5$ has the weak CK property as we want, and so we can assume that G has a faithful representation of dimension 4. The representation theory of G is the "same" over k as over $\mathbb{C}$, and so G has a faithful representation V of dimension 4 over k. As above, write $V = \bigoplus_{i=1}^{c} V_i$ and $Y = P(V_1^*) \times \cdots \times P(V_c^*)$.

By Lemma 10.2, BG over k has the weak CK property if the k–variety $(V-D)/G$ of dimension 4 has the weak CK property. The variety $(V-D)/G$ is a principal bundle over $(Y-D_Y)/(G/Z(G))$, with fiber $(G_m)^c/Z(G) \cong (G_m)^c$. (The representation V gives an inclusion of the center $Z(G)$ into $(G_m)^c$, which describes the scalar by which an element of the center acts on each irreducible summand V_i.) So the pullback homomorphism

$$\mathrm{CH}^*(Y-D_Y)/(G/Z(G)) \to \mathrm{CH}^*(V-D)/G$$

is surjective. The variety $(Y-D_Y)/(G/Z(G))$ has dimension $4-c$, which is at most 3. As a result, $\mathrm{CH}^*(V-D)/G$ is concentrated in degrees at most $4-c$.

The group $\mathrm{CH}^i BG$ is always generated by Chern classes of representations for $i \leq 2$ [55, Theorem 3.2]. All representations of G over an extension field of k can be defined over k, and so BG has the weak CK property in codimension 2 (meaning that $\mathrm{CH}^i BG \to \mathrm{CH}^i BG_E$ is surjective for any $i \leq 2$ and any field extension E of k). So the stack V/G and hence the variety $(V-D)/G$ have the weak CK property in codimension 2. If $c \geq 2$, meaning that V is reducible, then $\mathrm{CH}^*(V-D)/G$ is concentrated in degrees at most 2 by the previous paragraph. So $(V-D)/G$ has the weak CK property and we are done.

There remains the case where $c=1$, that is, where G has a faithful irreducible representation V of dimension 4 over k. In this case, $\mathrm{CH}^*(V-D)/G$ is concentrated in degrees at most 3, and this remains true over any extension field of k. We know that $(V-D)/G$ has the weak CK property in codimension 2, and we want to show that it has the weak CK property in codimension 3.

We use the fact that BG is stably rational over k for all groups G of order 32, under our assumption on k, by Chu, Hu, Kang and Prokhorov [12]. This means that the variety $(V-D)/G$ is stably rational over k. Since $(V-D)/G$ is a principal G_m–bundle over the 3–fold $(Y-D_Y)/(G/Z(G))$, that 3–fold is also stably rational over k. It follows that $\mathrm{CH}^3(Y-D_Y)/(G/Z(G))$ is generated by a k–rational point on $(Y-D_Y)/(G/Z(G))$, and this remains true over every extension field of k [18, Lemme 1.5]. So $(Y-D_Y)/(G/Z(G))$ has the weak CK property in codimension 3. By the surjection $\mathrm{CH}^3(Y-D_Y)/(G/Z(G)) \twoheadrightarrow \mathrm{CH}^3(V-D)/G$, which remains true over every extension field of k, $(V-D)/G$ has the weak CK property in codimension 3. By what we have said, this completes the proof that BG has the weak CK property. □

References

[1] **M Aschbacher**, *Finite group theory*, 2nd edition, Cambridge Studies in Advanced Mathematics 10, Cambridge Univ. Press (2000) MR1777008

[2] **A Auel**, **J-L Colliot-Thélène**, **R Parimala**, *Universal unramified cohomology of cubic fourfolds containing a plane*, preprint (2013) `arXiv:1310.6705` To appear in "Brauer groups and obstruction problems" (2013 conference proceedings)

[3] **S Bloch**, *The moving lemma for higher Chow groups*, J. Algebraic Geom. 3 (1994) 537–568 MR1269719

[4] **S Bloch**, *Lectures on algebraic cycles*, 2nd edition, New Mathematical Monographs 16, Cambridge Univ. Press (2010) MR2723320

[5] **S Bloch**, **V Srinivas**, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. 105 (1983) 1235–1253 MR714776

[6] **F A Bogomolov**, *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987) 485–516, 688 MR903621 In Russian; translated in Math. USSR Izv. 30 (1988) 455–485

[7] **F A Bogomolov**, *Stable cohomology of groups and algebraic varieties*, Mat. Sb. 183 (1992) 3–28 MR1184307 In Russian; translated in Russian Acad. Sci. Sb. Math. 76 (1993) 1–21

[8] **F A Bogomolov**, **P I Katsylo**, *Rationality of some quotient varieties*, Mat. Sb. 126(168) (1985) 584–589 MR788089 In Russian; translated in Math. USSR Sb. 54 (1986) 571–576

[9] **M Bökstedt**, **A Neeman**, *Homotopy limits in triangulated categories*, Compositio Math. 86 (1993) 209–234 MR1214458

[10] **R Carter**, **P Fong**, *The Sylow 2–subgroups of the finite classical groups*, J. Algebra 1 (1964) 139–151 MR0166271

[11] **S Cernele**, **M Kamgarpour**, **Z Reichstein**, *Maximal representation dimension of finite p–groups*, J. Group Theory 14 (2011) 637–647 MR2818953

[12] **H Chu**, **S-J Hu**, **M-c Kang**, **Y G Prokhorov**, *Noether's problem for groups of order 32*, J. Algebra 320 (2008) 3022–3035 MR2442008

[13] **H Chu**, **M-c Kang**, *Rationality of p–group actions*, J. Algebra 237 (2001) 673–690 MR1816710

[14] **D-C Cisinski**, **F Déglise**, *Local and stable homological algebra in Grothendieck abelian categories*, Homology, Homotopy Appl. 11 (2009) 219–260 MR2529161

[15] **D-C Cisinski**, **F Déglise**, *Triangulated categories of mixed motives*, preprint (2009) arXiv:0912.2110

[16] **D-C Cisinski**, **F Déglise**, *Integral mixed motives in equal characteristic*, Doc. Math. Special volume: Alexander S. Merkurjev's sixtieth birthday (2015) 145–194 MR3404379

[17] **J-L Colliot-Thélène**, *Birational invariants, purity and the Gersten conjecture*, from: "K–theory and algebraic geometry: connections with quadratic forms and division algebras", (B Jacob, A Rosenberg, editors), Proc. Sympos. Pure Math. 58, Amer. Math. Soc., Providence, RI (1995) 1–64 MR1327280

[18] **J-L Colliot-Thélène**, **A Pirutka**, *Hypersurfaces quartiques de dimension 3: non rationalité stable*, Ann. Sci. École Norm. Sup. 49 (2016) 371–397

[19] **D Dugger**, **D C Isaksen**, *Motivic cell structures*, Algebr. Geom. Topol. 5 (2005) 615–652 MR2153114

[20] **D Edidin**, **W Graham**, *Equivariant intersection theory*, Invent. Math. 131 (1998) 595–634 MR1614555

[21] **T Ekedahl**, *A geometric invariant of a finite group*, preprint (2009) arXiv:0903.3148v1

[22] **A D Elmendorf**, **I Kříž**, **M A Mandell**, **J P May**, *Modern foundations for stable homotopy theory*, from: "Handbook of algebraic topology", (I M James, editor), North-Holland, Amsterdam (1995) 213–253 MR1361891

[23] **W Fulton**, *Intersection theory*, 2nd edition, Ergeb. Math. Grenzgeb. 2, Springer, Berlin (1998) MR1644323

[24] **S Garibaldi**, **A Merkurjev**, **J-P Serre**, *Cohomological invariants in Galois cohomology*, University Lecture Series 28, Amer. Math. Soc., Providence, RI (2003) MR1999383

[25] **A Grothendieck**, *Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, II*, Inst. Hautes Études Sci. Publ. Math. 24 (1965) 5–231

[26] **A Hoshi**, **M-C Kang**, **B E Kunyavskii**, *Noether's problem and unramified Brauer groups*, Asian J. Math. 17 (2013) 689–713 MR3152260

[27] **M Hovey**, *Model categories*, Mathematical Surveys and Monographs 63, Amer. Math. Soc., Providence, RI (1999) MR1650134

[28] **A Huber**, **B Kahn**, *The slice filtration and mixed Tate motives*, Compos. Math. 142 (2006) 907–936 MR2249535

[29] **U Jannsen**, *Motivic sheaves and filtrations on Chow groups*, from: "Motives", (U Jannsen, S Kleiman, J-P Serre, editors), Proc. Sympos. Pure Math. 55, Amer. Math. Soc., Providence, RI (1994) 245–302 MR1265533

[30] **R Joshua**, *Algebraic K–theory and higher Chow groups of linear varieties*, Math. Proc. Cambridge Philos. Soc. 130 (2001) 37–60 MR1797730

[31] **B Kahn**, *Algebraic K–theory, algebraic cycles and arithmetic geometry*, from: "Handbook of K–theory, Volume 1", (E M Friedlander, D R Grayson, editors), Springer, Berlin (2005) 351–428 MR2181827

[32] **B Kahn**, **R Sujatha**, *Birational motives, I: Pure birational motives*, preprint (2015) arXiv:0902.4902v5 To appear in Ann. K–Theory

[33] **N A Karpenko**, **A S Merkurjev**, *On standard norm varieties*, Ann. Sci. Éc. Norm. Supér. 46 (2013) 175–214 MR3087392

[34] **S Kelly**, *Triangulated categories of motives in positive characteristic*, preprint (2013) arXiv:1305.5349

[35] **S-i Kimura**, *Surjectivity of the cycle map for Chow motives*, from: "Motives and algebraic cycles", (R de Jeu, J D Lewis, editors), Fields Inst. Commun. 56, Amer. Math. Soc., Providence, RI (2009) 157–165 MR2562457

[36] **S L Kleiman**, *Motives*, from: "Algebraic geometry, Oslo 1970", (F Oort, editor), Wolters-Noordhoff, Groningen (1972) 53–82 MR0382267

[37] **B Kunyavskiĭ**, *The Bogomolov multiplier of finite simple groups*, from: "Cohomological and geometric approaches to rationality problems", (F Bogomolov, Y Tschinkel, editors), Progr. Math. 282, Birkhäuser, Boston (2010) 209–217 MR2605170

[38] **M Levine**, *Techniques of localization in the theory of algebraic cycles*, J. Algebraic Geom. 10 (2001) 299–363 MR1811558

[39] **A Merkurjev**, *Unramified elements in cycle modules*, J. Lond. Math. Soc. 78 (2008) 51–64 MR2427051

[40] **F Morel**, **V Voevodsky**, *$\mathbf{A}^1$–homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. (1999) 45–143 MR1813224

[41] **D Mumford**, **J Fogarty**, **F Kirwan**, *Geometric invariant theory*, 3rd edition, Ergeb. Math. Grenzgeb. 34, Springer, Berlin (1994) MR1304906

[42] **A Neeman**, *Some new axioms for triangulated categories*, J. Algebra 139 (1991) 221–255 MR1106349

[43] **A Neeman**, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. 9 (1996) 205–236 MR1308405

[44] **A Neeman**, *Triangulated categories*, Annals of Mathematics Studies 148, Princeton Univ. Press (2001) MR1812507

[45] **Y P Nesterenko**, **A A Suslin**, *Homology of the general linear group over a local ring, and Milnor's K–theory*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989) 121–146 MR992981 In Russian; translated in Math. USSR Izv. 34 (1990) 121–145

[46] **O Röndigs**, **P A Østvær**, *Modules over motivic cohomology*, Adv. Math. 219 (2008) 689–727 MR2435654

[47] **M Rost**, *Chow groups with coefficients*, Doc. Math. 1 (1996) 319–393 MR1418952

[48] **A J Scholl**, *Classical motives*, from: "Motives", (U Jannsen, S Kleiman, J-P Serre, editors), Proc. Sympos. Pure Math. 55, Amer. Math. Soc., Providence, RI (1994) 163–187 MR1265529

[49] **E Shinder**, *Künneth formula for motivic cohomology* Available at `http://mathoverflow.net/a/12948/703`

[50] **H Sumihiro**, *Equivariant completion, II*, J. Math. Kyoto Univ. 15 (1975) 573–605 MR0387294

[51] **A A Suslin**, *Algebraic K–theory of fields*, from: "Proceedings of the International Congress of Mathematicians, Volume 1", (A M Gleason, editor), Amer. Math. Soc., Providence, RI (1987) 222–244 MR934225

[52] **The GAP group**, *GAP: Groups, algorithms and programming* (2013) Version 4.6.4 Available at `http://www.gap-system.org/`

[53] **The Stacks Project Authors**, *Stacks project* (2014) Available at `http://stacks.math.columbia.edu`

[54] **B Totaro**, *Milnor K–theory is the simplest part of algebraic K–theory*, K–Theory 6 (1992) 177–189 MR1187705

[55] **B Totaro**, *The Chow ring of a classifying space*, from: "Algebraic K–theory", (W Raskind, C Weibel, editors), Proc. Sympos. Pure Math. 67, Amer. Math. Soc., Providence, RI (1999) 249–281 MR1743244

[56] **B Totaro**, *Chow groups, Chow cohomology, and linear varieties*, Forum Math. Sigma 2 (2014) MR3264256

[57] **B Totaro**, *Group cohomology and algebraic cycles*, Cambridge Tracts in Mathematics 204, Cambridge Univ. Press (2014) MR3185743

[58] **B Totaro**, *Adjoint functors on the derived category of motives*, preprint (2015) arXiv:1502.05079

[59] **V Voevodsky**, *Triangulated categories of motives over a field*, from: "Cycles, transfers, and motivic homology theories", (Princeton, editor), Ann. of Math. Stud. 143, Princeton Univ. Press (2000) 188–238 MR1764202

[60] **V Voevodsky**, *Open problems in the motivic stable homotopy theory, I*, from: "Motives, polylogarithms and Hodge theory, Part I", (F Bogomolov, L Katzarkov, editors), Int. Press Lect. Ser. 3, Int. Press, Somerville, MA (2002) 3–34 MR1977582

[61] **V Voevodsky**, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. 98 (2003) 1–57 MR2031198

[62] **V Voevodsky**, *Motivic Eilenberg–Maclane spaces*, Publ. Math. Inst. Hautes Études Sci. 112 (2010) 1–99 MR2737977

[63] **A J Weir**, *Sylow p–subgroups of the classical groups over finite fields with characteristic prime to p*, Proc. Amer. Math. Soc. 6 (1955) 529–533 MR0072143

Department of Mathematics, University of California, Los Angeles
Box 951555, Los Angeles, CA 90095-1555, United States

totaro@math.ucla.edu

http://www.math.ucla.edu/~totaro/

Proposed: Lothar Göttsche | Received: 27 November 2014
Seconded: Jim Bryan, Frances Kirwan | Accepted: 12 September 2015