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Surgery obstructions and Heegaard Floer homology

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Using Taubes' periodic ends theorem, Auckly gave examples of toroidal and hyperbolic irreducible integer homology spheres which are not surgery on a knot in the three-sphere. We use Heegaard Floer homology to give an obstruction to a homology sphere being surgery on a knot, and then use this obstruction to construct infinitely many small Seifert fibered examples.

57M27, 57R58, 57R65

1 Introduction

Background

A classical theorem due to Lickorish [12] and Wallace [26] states that every closed oriented three-manifold can be expressed as surgery on a link in the three-sphere. Therefore, a natural question is which three-manifolds have the simplest surgery presentations. More specifically, we ask which three-manifolds can be represented by Dehn surgery on a knot.

There are a number of obstructions that we can apply (with a range of effectiveness) to attempt to answer this question. Since $H_1(S^3_{p/q}(K);\mathbb{Z}) \cong \mathbb{Z}/p$, we find immediate homological obstructions to a manifold being obtained by surgery on a knot (eg $\mathbb{T}^3$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$ are not surgery on a knot). A more delicate obstruction is the weight of the fundamental group; a three-manifold obtained by surgery on a knot in S^3 has weight-one fundamental group (it is normally generated by a single element). Observe that weight-one groups necessarily have cyclic abelianization. Hence, this obstruction extends the aforementioned homological obstruction.

A more topological obstruction can be found in the prime decomposition of the three-manifold. A theorem of Gordon and Luecke [7] shows that if surgery on a nontrivial knot in S^3 yields a reducible manifold, one of the summands is necessarily a nontrivial lens space. In particular, a reducible integer homology sphere can never be surgery on a knot.

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Boyer and Lines [2] give an infinite family of prime Seifert fibered manifolds with weight-one fundamental group which are not surgery on a knot. Their proof requires two obstructions: the first comes from an extension of the Casson invariant to homology lens spaces and the second comes from the linking form. In particular, having nontrivial first homology is necessary in their proof.

Using the periodic ends theorem of Taubes [25], Auckly constructs examples of irreducible integer homology spheres which are not surgery on a knot in S^3, answering Problem 3.6(C) in Kirby [11]. The first example is toroidal and homology cobordant to $\Sigma(2,3,5)\#-\Sigma(2,3,5)$ (or, equivalently, S^3). In [1], Auckly extends this construction to give a hyperbolic example. However, as far as the authors know, it is unknown whether Auckly's examples have weight-one fundamental group. A negative answer would provide a counterexample to the question of Wiegold (see, for instance, Mazurov and Khukhro [14, Problem 5.52]): does every finitely presented perfect group have weight one?

In [23], Saveliev asks if there are Seifert fibered homology spheres which are not surgery on a knot. Note that every Seifert homology sphere is irreducible, so none of these are ruled out by the Gordon–Luecke criterion. We answer this question affirmatively.

Main results

Theorem 1.1 *For p an even integer at least 8, let Y_p denote the Seifert fibered integer homology sphere $\Sigma(p, 2p-1, 2p+1)$. The manifolds $\{Y_p\}$ satisfy:*

(i) *Y_p is not surgery on a knot in S^3.*

(ii) *Y_p is surgery on a two-component link in S^3.*

(iii) *$\pi_1(Y_p)$ is a weight-one group.*

(iv) *Y_p is not smoothly rationally homology cobordant to any of Auckly's examples nor to any other Y_p (regardless of orientation).*

Theorem 1.1 is proved in two steps. The first step consists of finding an obstruction from Heegaard Floer homology to a homology sphere being surgery on a knot. The second step consists of an analysis (but not complete computation) of the Heegaard Floer homology of the manifolds $\{Y_p\}$.

Before stating these results, we recall from Ozsváth and Szabó [20; 18] that $\mathrm{HF}^+(Y)$, the Heegaard Floer homology of a homology sphere Y, is a $\mathbb{Z}$–graded $\mathbb{F}[U]$–module, where $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$ and U lowers degree by 2. Further, $\mathrm{HF}^+(Y)$ admits a noncanonical decomposition

$$\mathrm{HF}^+(Y) = \mathcal{T}^+_{d(Y)} \oplus \mathrm{HF}^{\mathrm{red}}(Y),$$

in which $\mathcal{T}^+_{d(Y)}$ is the module $\mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$, graded such that $\deg(1) = d(Y)$ and $\mathrm{HF}^{\mathrm{red}}(Y)$ is a finite sum of cyclic modules. The (even) integer $d(Y)$, called the *d–invariant* or *correction term*, is in fact an invariant of smooth rational homology cobordism. The following theorem presents our obstruction for a homology sphere being surgery on a knot.

Theorem 1.2 *Let Y be an oriented integer homology sphere such that $Y = S^3_{1/n}(K)$, for some integer n and some knot $K \subset S^3$. If $d(Y) \leq -8$, then $U \cdot \mathrm{HF}^{\mathrm{red}}_0(Y) \neq 0$.*

Many others have previously used correction terms to obstruct manifolds from being surgery on a knot (see, for instance, Doig [4, Corollary 5], Hoffman and Walsh [8, Theorem 4.4], Ozsváth and Szabó [18, Corollary 9.13, Section 10.2]).

It is known that $d(S^3) = 0$. Since Auckly's surgery obstruction requires the manifold to be homology cobordant to S^3, any manifold one could obstruct from being surgery by Theorem 1.2 could not be obstructed by Auckly's argument, and vice versa.

It is straightforward to generalize Theorem 1.2 to obtain further restrictions of this form on the Heegaard Floer homology of manifolds with highly negative correction terms obtained by surgery on a knot in S^3. Using such a variant, we can also show that the toroidal Seifert fibered homology sphere $\Sigma(2, 5, 19, 21)$ is not obtained by surgery on a knot.

In light of Theorem 1.2, we are interested in analyzing both the d–invariants of Y_p and the U–action on $\mathrm{HF}^{\mathrm{red}}(Y_p)$.

Theorem 1.3 *For p a positive, even integer, let Y_p denote the Seifert fibered homology sphere $\Sigma(p, 2p-1, 2p+1)$, oriented as the boundary of a positive-definite plumbing. Then*

(i) $d(Y_p) = -p$,

(ii) $U \cdot \mathrm{HF}^{\mathrm{red}}_0(Y_p) = 0$.

With this result, we may prove Theorem 1.1.

Proof of Theorem 1.1 (i) Notice that the property of a manifold being surgery on a knot in S^3 is independent of orientation. Therefore, we work with Y_p oriented as in Theorem 1.3. It is clear that for $p \geq 8$, Theorems 1.2 and 1.3 now show that Y_p is not surgery on a knot in S^3.

(ii) Every Seifert fibered space over S^2 with 3 singular fibers can be constructed by surgery on a link with two components. This follows from Eisenbud and Neumann

[5, Proposition 7.3] together with the fact that any Seifert fibered space over S^2 with 2 singular fibers is a lens space. Alternatively, one can directly verify that Y_p is in fact obtained by surgery on the $(2,2p)$ torus link with surgery coefficients $-(p+1)$ and $-(p-1)$ (see Figure 1).

(iii) This part of the proof was shown to us by Cameron Gordon. We will show more generally that the Brieskorn sphere $\Sigma(p,q,r)$ has weight-one fundamental group. Suppose that $Z = \Sigma(p,q,r)$ has normalized Seifert invariants $(e_0; (p,p'), (q,q'), (r,r'))$ (see, for instance, Saveliev [23, Section 1.1.4] or Seifert and Threlfall [24, Theorem 5]). Then we have

$$\pi_1(Z) = \langle x, y, z, h \mid h \text{ is central}, \ x^p = h^{-p'}, \ y^q = h^{-q'}, \ z^r = h^{-r'}, \ xyzh^{e_0} = 1 \rangle.$$

We claim that $\pi_1(Z)$ is normally generated by $h^{e_0}xy$. Let $\langle\langle h^{e_0}xy \rangle\rangle$ denote the normal subgroup of $\pi_1(Z)$ generated by $h^{e_0}xy$. We will show $\pi_1(Z)/\langle\langle h^{e_0}xy \rangle\rangle$ is trivial. In this quotient, $z = 1$, so we have

$$\pi_1(Z)/\langle\langle h^{e_0}xy \rangle\rangle \cong \langle x, y, h \mid h \text{ is central}, \ x^p = h^{-p'}, \ y^q = h^{-q'}, \ h^{-r'} = 1, \ h^{e_0}xy = 1 \rangle.$$

Therefore, we can rewrite this as

$$\pi_1(Z)/\langle\langle h^{e_0}xy \rangle\rangle \cong \langle x, h \mid h \text{ is central}, \ x^p = h^{-p'}, \ (x^{-1}h^{-e_0})^q = h^{-q'}, \ h^{-r'} = 1 \rangle.$$

In particular, $\pi_1(Z)/\langle\langle h^{e_0}xy \rangle\rangle$ is abelian. However, since $\Sigma(p,q,r)$ is an integer homology sphere, $\pi_1(Z)$ is a perfect group, and thus so is $\pi_1(Z)/\langle\langle h^{e_0}xy \rangle\rangle$. Therefore, $\pi_1(Z)/\langle\langle h^{e_0}xy \rangle\rangle$ is a perfect abelian group, and thus trivial. This completes the proof.

(iv) The result will follow quickly from the following two facts about the rational homology cobordism invariant d:

$$d(-Y) = -d(Y) \quad \text{and} \quad d(S^3) = 0.$$

First, recall that Auckly's examples are homology cobordant to $\Sigma(2,3,5) \# -\Sigma(2,3,5)$ and thus to S^3. Now, apply Theorem 1.3(i). □

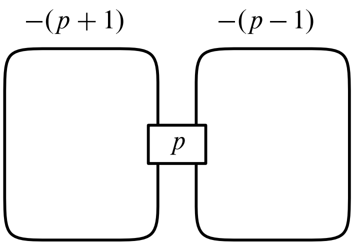

Figure 1: The manifold Y_p presented as surgery on a two-component torus link. The box indicates p positive full twists.

In Theorem 1.4 below, we are able to say something for arbitrary homology spheres. Recall that any reducible homology sphere is not surgery on a knot in S^3. The argument of Gordon and Luecke which is used to prove this result uses that the ambient manifold is S^3. For any homology sphere Y, we are able to construct infinitely many reducible manifolds which cannot be surgery on a knot in Y.

Theorem 1.4 *Let Y be an integer homology sphere and let $\#_k \Sigma(2,3,5)$ denote the connected sum of k Poincaré homology spheres with the same orientation. For $k \gg 0$, the manifold $\#_k \Sigma(2,3,5)$ is not surgery on a knot in Y, regardless of the orientation on Y.*

Remark 1.5 The reducibility of $\#_k \Sigma(2,3,5)$ is not important for Theorem 1.4. What is necessary is a family of integer homology spheres with unbounded d–invariants which are L-spaces (ie $\mathrm{HF}^{\mathrm{red}} = 0$). The only known irreducible homology sphere L-spaces are S^3 and the Poincaré homology sphere.

Organization Theorem 1.2 is proved in Section 2 by using the mapping cone formula for rational surgeries given in Ozsváth and Szabó [21]. In Section 3, we study the plumbing diagrams of the manifolds $\{Y_p\}$ and prove Theorem 1.3(i) using the algorithm of Ozsváth and Szabó [19]. In Section 4, we review the algorithm given in Eisenbud and Neumann [15] and Can and Karakurt [3] to compute the Heegaard Floer homology of Seifert homology spheres. In Section 5, we analyze $\mathrm{HF}^+(Y_p)$ and prove Theorem 1.3(ii). Finally, in Section 6, we prove Theorem 1.4.

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2 Mapping cones

The goal of this section is to prove Theorem 1.2. Let Y be an integer homology sphere with $d(Y) \leq -8$. Recall that we would like to see that if $Y = S^3_{1/n}(K)$, then $U \cdot \mathrm{HF}^{\mathrm{red}}_0(Y) \neq 0$. We first restrict the possible values of n.

Lemma 2.1 *Let Y be an integer homology sphere with $Y = S^3_{1/n}(K)$. If $d(Y) < 0$, then $n > 0$.*

Proof Suppose that $Y = S^3_{1/n}(K)$, where $n < 0$. Then it follows from [1, Figure 8] that Y is the boundary of a negative-definite four-manifold. Since Y is a homology sphere, [18, Corollary 9.8] implies $d(Y) \geq 0$. □

For the rest of this section, we only consider the case of $1/n$–surgery on a knot K for $n > 0$.

The main tool is the rational surgery formula of Ozsváth and Szabó [21]. We refer the reader to [17] for a concise summary. We very briefly recall the main ingredients for notation without much explanation.

As usual, let $\mathcal{T}^+$ denote $\mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$. For each $s \in \mathbb{Z}$, Ozsváth and Szabó associate to K a relatively graded $\mathbb{F}[U]$–module A_s, which is isomorphic to the Heegaard Floer homology of a large positive surgery on K in a certain Spin^c structure. Further, associated to each s, there are two graded, module maps $v_s, h_s \colon A_s \to \mathcal{T}^+$ which represent maps coming from certain Spin^c cobordisms. Each A_s admits a splitting $A_s \cong \mathcal{T}^+ \oplus A_s^{\mathrm{red}}$ where $\mathcal{T}^+$ is the image of U^N for $N \gg 0$ and A_s^{red} is isomorphic to $\bigoplus_{i=1}^m \mathbb{F}[U]/U^{k_i}$. When it will not cause confusion, for $n \geq 0$, we may write U^{-n} to mean the corresponding element of $\mathcal{T}^+ \subset A_s$. Although A_s is not a module over $\mathbb{F}[U, U^{-1}]$, we will further abuse notation and for an element $a \in \mathcal{T}^+ \subset A_s$ write $U^{-k}a$ to mean the unique element in $\mathcal{T}^+ \subset A_s$ such that $U^k \cdot U^{-k} a = a$.

For each s, we have

$$v_s|_{\mathcal{T}^+}(x) = U^{V_s} x$$

for some nonnegative integer V_s. Similarly,

$$h_s|_{\mathcal{T}^+}(x) = U^{H_s} x$$

for some nonnegative integer H_s. Note that each of these maps is surjective. We will need the following important properties of these integers (see [22, Section 7], [9, Lemma 2.5] and [17, Proposition 1.6]):

(2-1) $$H_s = V_{-s},$$

(2-2) $$V_s - 1 \leq V_{s+1} \leq V_s,$$

(2-3) $$H_s = V_s + s,$$

(2-4) $$d(S^3_{1/n}(K)) = -2V_0 = -2H_0.$$

From this information, we can compute the Heegaard Floer homology of $S^3_{p/q}(K)$ for any $p/q \in \mathbb{Q}$. We will restrict our attention to the case of $S^3_{1/n}(K)$, for $n > 0$. For each s, consider n copies of A_s, denoted $A_{s,1}, \dots, A_{s,n}$. Further, for each $s \in \mathbb{Z}$ and $1 \leq i \leq n$, define $B_{s,i} = \mathcal{T}^+$. For an element x in $A_{s,i}$ or $B_{s,i}$, we may write this element as (x,s,i) to keep better track of the indexing. We will also write $i \pmod n$ to refer to the specific representative i between 1 and n. Define the $\mathbb{F}[U]$–module map

$$\Phi_{1/n}\colon \bigoplus_{s\in\mathbb{Z},1\le i\le n} A_{s,i} \to \bigoplus_{s\in\mathbb{Z},1\le i\le n} B_{s,i}$$

by

$$\Phi_{1/n}(x,s,i) = (v_s(x),s,i) + (h_s(x), s+\lfloor i/n\rfloor, i+1 \pmod n).$$

We define an absolute grading on the mapping cone of $\Phi_{1/n}$ (where the $A_{s,i}$ and $B_{s,i}$ are given trivial differential) by requiring that the element $1\in B_{0,1}$ has grading -1 and that $\Phi_{1/n}$ lowers grading by 1. We remark that the indexing we are using is expressed differently than in [21].

Theorem 2.2 (Ozsváth and Szabó, [21, Theorem 1 and Section 7.2]) *The homology of the mapping cone of $\Phi_{1/n}$ is isomorphic to $\mathrm{HF}^+(S^3_{1/n}(K))$. This isomorphism respects the absolute gradings and the $\mathbb{F}[U]$–module structure.*

Note that Theorem 2.2 is not quite stated as in [21]. Ozsváth and Szabó's theorem instead establishes an isomorphism between Heegaard Floer homology and the cone of a chain map whose induced map on homology is $\Phi_{1/n}$. In general, for a nullhomologous knot in an arbitrary three-manifold, one cannot compute Heegaard Floer homology of surgeries by looking at the cone of the induced map on homology. However, for knots in S^3 (or any L-space), one may compute the homology of the cone of $\Phi_{1/n}$ to obtain the desired result (see [6, Section 2]).

Proposition 2.3 *Let $K\subset S^3$ and let $Y = S^3_{1/n}(K)$ for some positive integer n. If $d(Y)\le -8$, then there exist cycles x and y in the cone of $\Phi_{1/n}$ such that*

(i) *x and y are nonzero in homology,*

(ii) $y = Ux$,

(iii) $\mathrm{gr}(x) = 0$,

(iv) *for $N\gg 0$, the element y is not homologous to $U^N w$ for any cycle w.*

Proof For notation, we let X denote the mapping cone of $\Phi_{1/n}$ and denote by $\mathcal{B}$ the submodule $\bigoplus_{s,1\le i\le n} B_{s,i}$. Also, let $d = d(Y)$. Thus $V_0 = H_0 = -d/2$ by (2-4). Since $d\le -8$, it follows that $V_0\ge 4$. By (2-2), we have $V_1\ge 3$ and $V_2\ge 2$.

We first consider the case when $n=1$. In this case, we remove the index i used in the $A_{s,i}$ and $B_{s,i}$. The relevant portion of the mapping cone is shown in the top of Figure 2. Let $x = U^{1-V_2}$ in A_2 and let $y = Ux$. Note that x and y are both nonzero in A_2 since $V_2 \geq 2$. We have $v_2(x) = 0$, since v_2 restricted to $\mathcal{T}^+ \subset A_2$ is multiplication by U^{V_2}. We also have $h_2(x) = 0$, since h_2 restricted to $\mathcal{T}^+ \subset A_2$ is multiplication by U^{H_2} and $H_2 = V_2 + 2$ by (2-3). Hence, x is a cycle in X. Since $\Phi_{1/n}$ is an $\mathbb{F}[U]$–module map, $y = Ux$ must be a cycle in X as well.

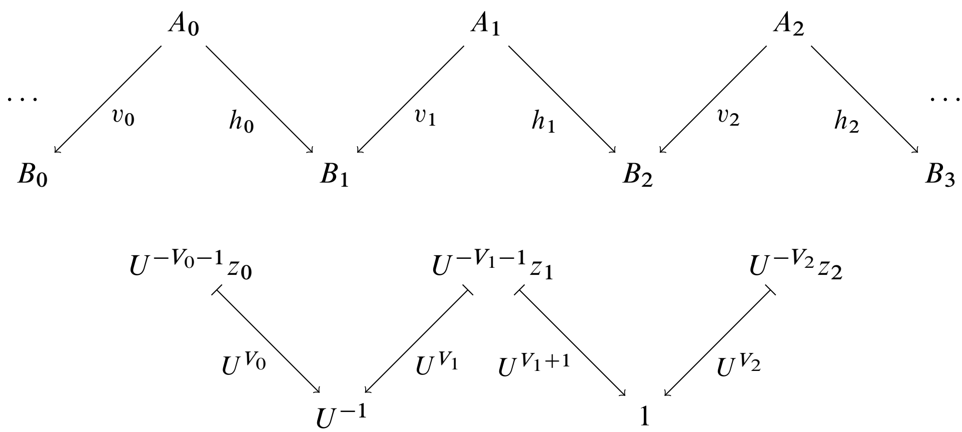

Figure 2: The relevant portion of the mapping cone when $n = 1$ (top), and the images of certain elements in the mapping cone for the case of $n = 1$, from which we can deduce that $\mathrm{gr}(z_2) = 2 - 2V_2$ (bottom).

We now show that x and y satisfy the conditions of the proposition.

(i) Since the image of the differential on X is contained in $\mathcal{B}$ and x is a nontrivial element in A_2, the cycle x is nonzero in the homology of X. Similarly, y is nonzero in the homology of X.

(ii) By the definition of y, we have $y = Ux$.

(iii) Let z_s denote the lowest grading nonzero element of $\mathcal{T}^+ \subset A_s$. Note that $v_s(U^{-V_s} z_s) = h_{s-1}(U^{-H_{s-1}} z_{s-1})$ and this image is the lowest grading nonzero element in B_s. We claim that $\mathrm{gr}(z_0) = -2V_0 = d$. This follows since $v_0(U^{-V_0} z_0)$ is the lowest grading nonzero element in B_0, the map v_0 lowers grading by one, and the grading of the lowest grading nonzero element in B_0 is -1.

We have

$$\mathrm{gr}(z_s) = \mathrm{gr}(z_{s-1}) + 2(H_{s-1} - V_s),$$

since $v_s(U^{-V_s} z_s) = h_{s-1}(U^{-H_{s-1}} z_{s-1})$, and since v_s and h_{s-1} both lower grading by one.

Then

$$\begin{aligned}\mathrm{gr}(z_2) &= \mathrm{gr}(z_1) + 2(H_1 - V_2)\\ &= \mathrm{gr}(z_0) + 2(H_0 - V_1) + 2(H_1 - V_2)\\ &= \mathrm{gr}(z_0) + 2H_0 + 2(H_1 - V_1) - 2V_2\\ &= d - d + 2 - 2V_2\\ &= 2 - 2V_2,\end{aligned}$$

where the penultimate equality follows from (2-3) and (2-4). See the bottom of Figure 2. Since $z_2 = U^{V_2-1}x$ and U lowers grading by two, it follows that $\mathrm{gr}(x) = 0$, as desired.

We would like to show that for large N, the cycle y is not homologous to $U^N w$ for any cycle w in X. Choose N at least

$$\min\{n \mid U^n \cdot A_1^{\mathrm{red}} = U^n \cdot A_2^{\mathrm{red}} = 0\}.$$

Observe that such an N exists since A_s^{red} is finite-dimensional as an $\mathbb{F}$–vector space. Consider $U^{-N}y = U^{2-V_2-N}z_2 \in \mathcal{T}^+ \subset A_2$. If $N \geq 2$, then $U^{-N}y$ is not in the kernel of the differential on X, since $v_2(U^{-N}y) = U^{2-N} \in B_2 \cong \mathcal{T}^+$. We claim that if a cycle contains $U^{-N}y$, then its projection onto A_1 must be $U^{2-H_1-N}z_1$. Recall that $h_1(U^{2-H_1-N}z_1) = U^{2-N} \in B_2$. By our choice of N, the only other element in A_1 or A_2 with the same grading as $U^{-N}y$ is $U^{2-H_1-N}z_1 \in A_1$. Furthermore, for an element not contained in A_1 or A_2, its boundary cannot be contained in B_2. The claim follows.

Now, suppose that w is a cycle in X such that $U^N w$ is homologous to y. Then w has projection onto A_2 given by $U^{-N}y$. Thus the projection of w onto A_1 must be $U^{2-H_1-N}z_1$. Observe that $2 - H_1 < 0$, since $H_1 = V_1 + 1$ and $V_1 \geq 3$ by (2-2), (2-3) and (2-4). Thus we have $U^N \cdot U^{2-H_1-N}z_1 \neq 0$. This implies that $U^N w$ has nontrivial projection to A_1. Since the image of the differential on X is contained in $\mathcal{B}$ and $y \in A_2$, the cycle y cannot be homologous to an element with nontrivial projection to A_1. Hence, y is not homologous to $U^N w$. This completes the proof of the proposition when $n = 1$.

The proof when $n > 1$ is similar. The relevant portion of the mapping cone is shown in Figure 3. Let $x = U^{1-V_1} \in \mathcal{T}^+ \subset A_{1,2}$ and let $y = Ux \in A_{1,2}$. As above, it is straightforward to show that x and y are both nonzero in the homology of X. Thus (i) and (ii) hold. We proceed to show that x and y satisfy (iii) and (iv); the arguments are similar to the $n = 1$ case above.

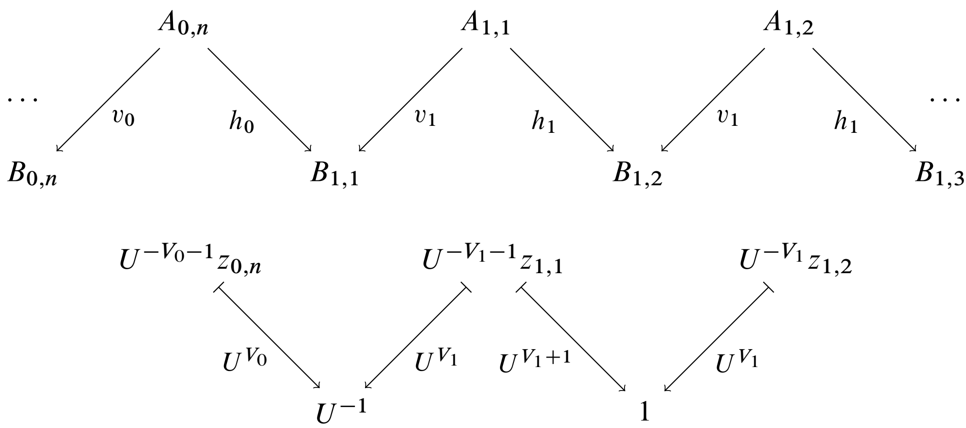

Figure 3: The relevant portion of the mapping cone when $n > 1$ (top), and the images of certain elements in the mapping cone for the case of $n > 1$, from which we can deduce that $\mathrm{gr}(z_{1,2}) = -2V_1 + 2$ (bottom).

(iii) Let $z_{s,i}$ denote the lowest grading element of $\mathcal{T}^+ \subset A_{s,i}$. Note that $\mathrm{gr}(z_{0,1}) = d$. For $1 \le i \le n$,

$$\mathrm{gr}(z_{s,i}) = \mathrm{gr}(z_{s,1}) + 2s(i-1), \tag{2-5}$$

since

- $h_s(U^{-H_s} z_{s,i}) = v_s(U^{-V_s} z_{s,i+1})$ for $1 \le i \le n-1$,
- v_s and h_s both lower grading by one,
- $H_s - V_s = s$.

We also observe

$$\mathrm{gr}(z_{s,1}) = \mathrm{gr}(z_{s-1,n}) + 2(H_{s-1} - V_s), \tag{2-6}$$

since $v_s(U^{-V_s} z_{s,1}) = h_{s-1}(U^{-H_{s-1}} z_{s-1,n})$. Then, by (2-5) and (2-6),

$$\begin{aligned} \mathrm{gr}(z_{1,2}) &= \mathrm{gr}(z_{1,1}) + 2 \\ &= \mathrm{gr}(z_{0,n}) + 2(H_0 - V_1) + 2 \\ &= \mathrm{gr}(z_{0,1}) + 2(H_0 - V_1) + 2 \\ &= d - d - 2V_1 + 2 \\ &= -2V_1 + 2. \end{aligned}$$

Since $z_{1,2} = U^{V_1 - 1} x$, it follows that $\mathrm{gr}(x) = 0$, as desired.

(iv) The proof is the same as the proof for $n=1$, replacing A_2 with $A_{1,2}$ and then showing that if y is homologous to $U^N w$, then w must have nontrivial projection to $A_{1,1}$.

This completes the proof of the proposition. □

With this, we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2 First, we make an observation about $\mathrm{HF}^{\mathrm{red}}$. Recall that $\mathrm{HF}^{\mathrm{red}}(Y)$ is defined to be $\mathrm{HF}^+(Y)/\operatorname{Im}(U^N)$ for $N \gg 0$. Note that if $a \in \mathrm{HF}^+_{k-2}(Y)$ is of the form Ub for some $b \in \mathrm{HF}^+_k(Y)$ and a is not in $\operatorname{Im}(U^N)$ for $N \gg 0$, then $U \cdot \mathrm{HF}^{\mathrm{red}}_k(Y) \neq 0$. Theorem 1.2 now follows by combining this observation with Theorem 2.2 and Proposition 2.3. □

3 Plumbings

Recall that $Y_p = \Sigma(p, 2p-1, 2p+1)$, where we have oriented Y_p such that it bounds a positive-definite plumbing. In this section we determine explicitly the negative-definite plumbing whose boundary is $-Y_p$. We will use this plumbing to compute the correction term of Y_p and hence prove Theorem 1.3(ii).

Proposition 3.1 *For every $p \geq 2$, the manifold $-Y_p$ bounds the four-manifold X_p which is the plumbing of disk bundles over spheres intersecting according to the graph in Figure 4.*

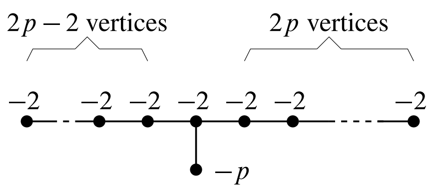

Figure 4: The plumbing graph for $-Y_p$

Proof We follow the recipe given in [16, Sections 4 and 5]. We consider the negative-definite plumbing bounded by $-Y_p$. Since Y_p has three singular fibers, this plumbing graph will have three arms. If a Seifert fibered space with base orbifold S^2 bounding a negative-definite plumbing has normalized Seifert invariants $(e_0; (r_1, r_1'), \dots, (r_k, r_k'))$, the order of H_1 is given by $-r_1 \cdots r_k(e_0 + \sum(r_i'/r_i))$. The number e_0 is the weight of the central vertex in the plumbing. For $-Y_p = M(e_0; (p, p'), (2p-1, q'), (2p+1, r'))$, the unique solution satisfying $0 < p' < p$, $0 < q' < 2p-1$, and $0 < r' < 2p+1$ is $e_0 = -2$, $p' = 1$, $q' = 2p-2$ and $r' = 2p$.

Given integers $m > n > 1$, m/n has a unique continued fraction expansion

$$\frac{m}{n} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_k}}}$$

with $a_i > 1$ for all $i = 1, \dots, k$, denoted $[a_1 : a_2 : \cdots : a_k]$. We now look at the continued fraction expansions of $p/1$, $(2p-1)/(2p-2)$ and $(2p+1)/(2p)$, which determine the negative weights of the vertices on each branch. We have

$$\frac{p}{1} = [p], \quad \frac{2p-1}{2p-2} = [\underbrace{2:2:\cdots:2}_{2p-2}], \quad \frac{2p+1}{2p} = [\underbrace{2:2:\cdots:2}_{2p}]. \qquad \square$$

With this, we are ready to compute the correction term of Y_p.

Theorem 1.3(i) *For p even, we have $d(Y_p) = -p$.*

Proof Let X_p denote the four-manifold given in Figure 4. By Proposition 3.1 we know $\partial X_p = -Y_p$. A result of Ozsváth and Szabó [19, Corollary 1.5] says that the correction term of $-Y_p$ can be computed using the intersection form on $H^2(X_p, \mathbb{Z})$ as follows. Let $\mathrm{Char}(X_p)$ denote the set of all characteristic cohomology classes. Recall that $K \in H^2(X_p, \mathbb{Z})$ is *characteristic* if $K \cdot [v] + [v]^2 \equiv 0 \pmod 2$ for every vertex v of the plumbing graph. Next, we note that the number of vertices in the plumbing graph is $4p$. The correction term of $-Y_p$ at its unique Spin^c structure is given by

$$d(-Y_p) = \max_{K \in \mathrm{Char}(X_p)} \tfrac{1}{4}(K^2 + 4p). \tag{3-1}$$

When p is even, X_p has even intersection form and thus $K = 0$ is a characteristic cohomology class. Note that $K = 0$ maximizes the above expression since the intersection form is negative-definite. Hence, $d(-Y_p) = p$ in this case. Since $d(-Y_p) = -d(Y_p)$ by [18, Proposition 4.2], we have obtained the desired result. □

Remark 3.2 Though we do not need this for our main argument, we would like to point out that $d(Y_p) = -p + 1$ for odd p.

4 Graded roots

The purpose of the present section and the next one is to prove the following result, which finishes the proof of Theorem 1.3.

Theorem 1.3(ii) *For every even integer p, we have $U \cdot \mathrm{HF}_0^{\mathrm{red}}(Y_p) = 0$.*

The proof uses the techniques of graded roots which were introduced by Némethi [15] and extensively studied in [3; 10]. In this section we motivate and explain our strategy to prove Theorem 1.3(ii) and give the necessary background. The proof will be given in the next section.

Background

Definition 4.1 (Némethi [15, Section 3.2]) A *graded root* is a pair (Γ, χ), where Γ is an infinite tree, and χ is an integer-valued function defined on the vertex set of Γ satisfying the following properties:

(i) $\chi(u) - \chi(v) = \pm 1$ if there is an edge connecting u and v.

(ii) $\chi(u) > \min\{\chi(v), \chi(w)\}$ if there are edges connecting u to v and u to w.

(iii) χ is bounded below.

(iv) $\chi^{-1}(k)$ is a finite set for every k.

(v) $\#\chi^{-1}(k) = 1$ for k large enough.

See Figure 5 for an example of a graded root. Up to an overall degree shift, every graded root can be described by a finite sequence as follows. Let $\Delta\colon \{0, \dots, N\} \to \mathbb{Z}$ be a given finite sequence of integers. Let $\tau_\Delta\colon \{0, \dots, N+1\} \to \mathbb{Z}$ be the unique solution of

$$\tau_\Delta(n+1) - \tau_\Delta(n) = \Delta(n) \quad \text{with } \tau_\Delta(0) = 0.$$

For each $n \in \{0, \dots, N+1\}$, let R_n be the infinite graph with vertex set $\mathbb{Z} \cap [\tau_\Delta(n), \infty)$ and edge set $\{[k, k+1] \mid k \in \mathbb{Z} \cap [\tau_\Delta(n), \infty)\}$. We identify, for each $n \in \{0, \dots, N+1\}$, all common vertices and edges in R_n and R_{n+1} to get an infinite tree Γ_Δ. To each vertex v of Γ_Δ, we can assign a grading $\chi_\Delta(v)$ which is the unique integer corresponding to v in any R_n to which v belongs. Note that many different sequences can give the same graded root. For example, the elements $n \in \{0, \dots, N\}$ where $\Delta(n) = 0$ do not affect the resulting graded root.

Associated to a graded root (Γ, χ) is its *homology*, which is a graded $\mathbb{F}[U]$–module denoted by $\mathbb{H}(\Gamma)$; we omit the grading function from the notation. As an $\mathbb{F}$–vector space, $\mathbb{H}(\Gamma)$ is generated by the vertices of Γ. Further, the grading of a vertex v is given by $2\chi(v)$. Finally, $U \cdot v$ is defined to be the sum of vertices w which are connected to v by an edge and satisfy $\chi(w) = \chi(v) - 1$.

A motivating example

To a large family of plumbed manifolds, Némethi associates a graded root whose corresponding module is isomorphic to Heegaard Floer homology up to a grading shift [15]. In [3], Némethi's method is simplified for Seifert homology spheres. Before describing this method in Section 4, we begin with an example to illustrate the process. This will also enable us to explain the strategy for the proof of Theorem 1.3(ii).

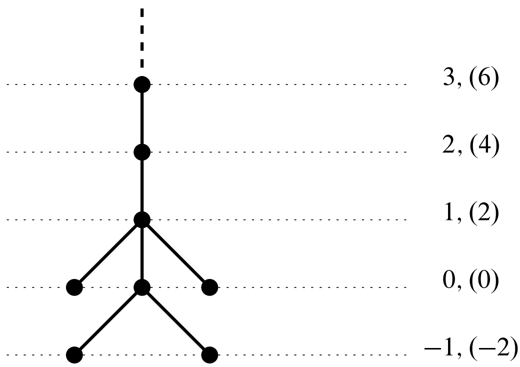

Figure 5: The graded root associated to $Y_3 = \Sigma(3,5,7)$. The grading of each vertex v is written in the form $\chi(v)$, (absolute grading).

For simplicity, we will construct the graded root for $Y_3 = \Sigma(3,5,7)$ and subsequently compute its Heegaard Floer homology. While Y_3 does not have p even, this computation will still lend insight into the family of computations we are interested in. We consider the number $N_{Y_3} = (3\times 5\times 7) - (3\times 5) - (3\times 7) - (5\times 7) = 34$. We look at the elements of the semigroup generated by (3×5), (3×7) and (5×7) that lie in the interval $[0, N_{Y_3}]$. The relevant semigroup elements are $S_{Y_3} = \{0, 15, 21, 30\}$. Let $Q_{Y_3} = \{N_{Y_3} - x \mid x \in S_{Y_3}\} = \{4, 13, 19, 34\}$ and $X_{Y_3} = S_{Y_3} \cup Q_{Y_3}$. We rewrite the ordered set X_{Y_3}, indicating the elements of S_{Y_3} in boldface:

$$\{\mathbf{0}, 4, 13, \mathbf{15}, 19, \mathbf{21}, \mathbf{30}, 34\}.$$

Define the function $\Delta_{Y_3}\colon X_{Y_3} \to \{-1, 1\}$ to have value $+1$ on S_{Y_3} and -1 on Q_{Y_3}. We write Δ_{Y_3} as an ordered set:

$$\Delta_{Y_3} = \langle +\mathbf{1}, -1, -1, +\mathbf{1}, -1, +\mathbf{1}, +\mathbf{1}, -1\rangle.$$

This sequence produces a graded root. To simplify it, we then combine the consecutive positive values and the consecutive negative values to write a new sequence which produces the same graded root:

$$\widetilde{\Delta}_{Y_3} = \langle +\mathbf{1}, -2, +\mathbf{1}, -1, +\mathbf{2}, -1\rangle.$$

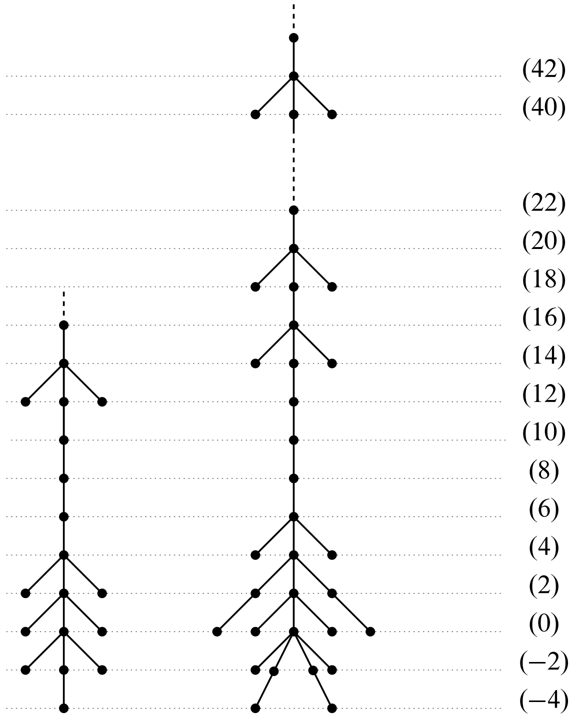

Figure 6: Graded roots associated to Y_4 and Y_5. Gradings shown correspond to the absolute grading on Heegaard Floer homology.

We indicate the graded root Γ_{Y_3} in Figure 5. We can read off the Heegaard Floer homology of Y_3 up to a degree shift from its graded root. As relatively graded modules, we have $\mathrm{HF}^+(Y_3)$ is isomorphic to

$$\mathbb{H}(\Gamma_{Y_3}) = \mathcal{T}^+_{(-2)} \oplus \mathbb{F}_{(-2)} \oplus \mathbb{F}_{(0)} \oplus \mathbb{F}_{(0)}.$$

Since $d(Y_3) = -2$, we do not shift degrees. Hence, we have

$$\mathrm{HF}^+(Y_3) = \mathcal{T}^+_{(-2)} \oplus \mathbb{F}_{(-2)} \oplus \mathbb{F}_{(0)} \oplus \mathbb{F}_{(0)}.$$

We repeat the same process for $p = 4$ and $p = 5$. The resulting delta sequences are

$$\widetilde{\Delta}_{Y_4} = \langle 1, -6, 1, -2, 1, -2, 1, -2, 2, -1, 2, -1, 2, -1, 6, -1 \rangle,$$
$$\widetilde{\Delta}_{Y_5} = \langle 1, -12, 1, -3, 1, -6, 1, -3, 2, -2, 1, -2, 1, -2, 2, -2, 2, -1, 2, -1, 2, -2, 3, -1, 6, -1, 3, -1, 12, -1 \rangle.$$

See Figure 6 for the corresponding graded roots. Since $d(Y_4) = -4$ and $d(Y_5) = -4$, we shift degrees to convert $\mathbb{H}(\Gamma_{Y_p})$ to $\mathrm{HF}^+(Y_p)$.

Let us observe why $U \cdot \mathrm{HF}_0^{\mathrm{red}}(Y_p) = 0$ when $p = 3, 4, 5$ using these graded roots. From the description of the U–action on the homology of a graded root Γ, we see that the dimension of $\ker(U)_n$ is the number of branches ending at degree n, whereas $\dim \mathbb{H}_n(\Gamma)$ is the number of vertices in degree n. From the pictures of the graded roots of Γ_{Y_p} we see that the degree 0 piece of the image of U^N for $N \gg 0$ is 1–dimensional. This is given by the sum of the vertices in degree 0. Each degree 0 vertex other than the central vertex, v, is at the end of a branch, so it must be in the kernel of U. Since $\mathrm{HF}^{\mathrm{red}}(Y_p)$ is the cokernel of U^N for $N \gg 0$, we see that the set of vertices other than v descend to a basis in the quotient $\mathrm{HF}_0^{\mathrm{red}}(Y_p)$. Thus we have $U \cdot \mathrm{HF}_0^{\mathrm{red}}(Y_p) = 0$ for $p = 3, 4, 5$.

In order to prove Theorem 1.3(ii) in general, we need to see a pattern in the graded roots of Y_p. Repeating the graded root computation for a few more values reveals that the bottom of the graded root of Y_p shows one of the patterns indicated in Figure 7, depending on the parity of p. We call these "graded subroots" *creatures* and denote them by Γ_{C_p}. Theorem 1.3(ii) reduces to showing that the bottom of each graded root is the creature Γ_{C_p}. In order to formalize and prove this pattern, we are going to need the abstract delta sequences which were introduced in [10].

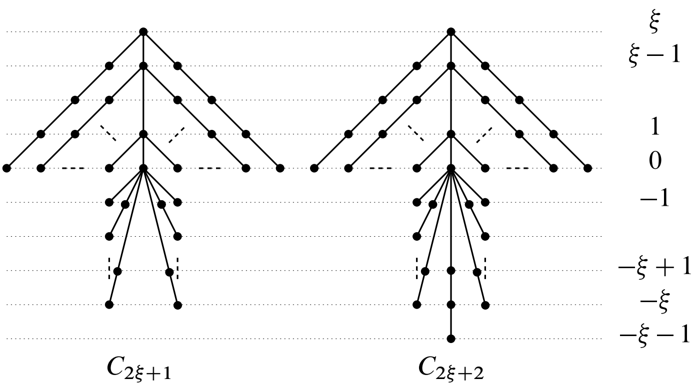

Figure 7: Creatures Γ_{C_p} as graded subroots of the graded roots associated to Y_p. Gradings are χ values. See Definition 5.1.

Abstract delta sequences

We recall the definition of an abstract delta sequence from [10], which codifies graded roots via the method described in Section 4.

Definition 4.2 An *abstract delta sequence* is a pair (X, Δ), where X is a well-ordered finite set, and $\Delta\colon X \to \mathbb{Z} - \{0\}$ is positive at the minimal element of X.

We review the description of the abstract delta sequence (X_Y, Δ_Y) which is associated to an arbitrary Brieskorn sphere $Y = \Sigma(p,q,r)$. Let $N_Y = pqr - pq - pr - qr$, and let S_Y denote the intersection of the interval $[0, N_Y]$ with the semigroup generated by pq, pr and qr. Define the set $Q_Y := \{N_Y - s \mid s \in S_Y\}$, and let $X_Y = S_Y \cup Q_Y$. It turns out that S_Y and Q_Y are disjoint. Define $\Delta_Y\colon X_Y \to \{-1, 1\}$ which takes the value $+1$ on S_Y, and -1 on Q_Y. Thus we have $x \in S_Y$ if and only if $N_Y - x \in Q_Y$ and $\Delta_Y(x) = -\Delta_Y(N_Y - x)$ for $x \in X_Y$. The significance of this abstract delta sequence is the following, which follows from [3, Theorem 1.3].

Theorem 4.3 *Let $Y = \Sigma(p,q,r)$, oriented as the boundary of a positive-definite plumbing. Let Γ_Y be the graded root associated to the abstract delta sequence (X_Y, Δ_Y) defined above. Then, as relatively graded $\mathbb{F}[U]$–modules, $\mathbb{H}(\Gamma_Y) \cong \mathrm{HF}^+(Y)$.*

The rest of the current section is devoted to studying abstract delta sequences in preparation for the proof of Theorem 1.3(ii) in Section 5.

We see an obvious symmetry in Figures 5 and 6. In fact, this symmetry more generally holds for the graded roots of Seifert homology spheres. The purpose of the next definition is to characterize those delta sequences whose graded roots show this symmetry. Write $X = \{x_1, \dots, x_n\}$, where $x_1 < \cdots < x_n$. We shall use the notation $f = \langle k_1, k_2, \dots, k_n\rangle$ to denote the function $f\colon X \to \mathbb{Z}$ satisfying $f(x_i) = k_i$ for each i.

Definition 4.4 Given abstract delta sequences $\Delta = \langle k_1, \dots, k_n\rangle$ and $\Delta' = \langle \ell_1, \dots, \ell_m\rangle$, we define

(i) the *negation* by $-\Delta = \langle -k_1, \dots, -k_n\rangle$,

(ii) the *reverse* by $\overline{\Delta} = \langle k_n, \dots, k_1\rangle$,

(iii) the *join* by $\Delta * \Delta' = \langle k_1, \dots, k_n, \ell_1, \dots, \ell_m\rangle$,

(iv) the *symmetrization* by $\Delta^{\mathrm{Sym}} = \Delta * -\overline{\Delta} = \langle k_1, \dots, k_n, -k_n, \dots, -k_1\rangle$.

Note that neither the negation nor the reverse of an abstract delta sequence need be an abstract delta sequence.

One can define operations on abstract delta sequences which do not change the corresponding graded root. Two such operations, refinement and merging, formalize the transition from Δ_{Y_3} to $\widetilde{\Delta}_{Y_3}$ given in Section 4. Let (X, Δ) be a given abstract delta sequence. Suppose there exists a positive integer $t \geq 2$ and an element z of X such that $|\Delta(z)| \geq t$. Pick integers $n_1, \dots, n_t$, all of which have the same sign as $\Delta(z)$, and satisfy $n_1 + \cdots + n_t = \Delta(z)$. From this we construct a new delta sequence (X', Δ') by replacing $z \in X$ by consecutive elements $z_1, \dots, z_t \in X'$ and setting $\Delta'(x) = \Delta(x)$ for all $x \in X \setminus \{z\}$ and $\Delta'(z_i) = n_i$ for all $i = 1, \dots, t$.

Definition 4.5 We say that the delta sequence (X', Δ') constructed above is a *refinement* of (X, Δ) at z, and conversely (X, Δ) is the *merge* of (X', Δ') at $z_1, \dots, z_t$.

Definition 4.6 An abstract delta sequence is said to be *reduced* if it does not admit any merging (hence there are no consecutive positive or negative values of Δ). An abstract delta sequence is called *expanded* if it does not admit any refinement (hence every value of Δ is ± 1).

Every delta sequence admits unique reduced and expanded forms. Note that the abstract delta sequences of Brieskorn spheres are in expanded form.

Given an element of X, it will be important for us to be able to refer to certain nearby elements in an effective manner. We introduce the following notation.

Definition 4.7 Let (X, Δ) be an abstract delta sequence. Denote the set of all elements in X where Δ is positive (respectively, negative) by S (respectively, Q). For $x \in S$, consider the maximal sequence of adjacent elements in X containing x which sits in S. The biggest (respectively, smallest) element of this sequence will be denoted $\pi_+(x)$ (respectively, $\pi_-(x)$). Similarly, for $y \in Q$, we have the analogous elements $\eta_\pm(y)$. For any element $z \in X$, the *positive successor* $\mathrm{suc}_+(z)$ (respectively, *negative successor* $\mathrm{suc}_-(z)$) is the first element from S (respectively, Q) which comes after z. Should $\mathrm{suc}_\pm(z)$ not exist, we treat it as an auxiliary element which is larger than any element in X. We also have the analogous notions, $\mathrm{pre}_\pm(z)$, which are the *predecessors*.

We may also write $\pi_\pm(x)$ in terms of successors and predecessors by

$$\pi_+(x) = \max\{z \in S \mid z < \mathrm{suc}_-(x)\} \quad \text{and} \quad \pi_-(x) = \min\{z \in S \mid z > \mathrm{pre}_-(x)\}.$$

Similarly, for $y \in Q$,

$$\eta_+(y) = \max\{z \in Q \mid z < \mathrm{suc}_+(y)\} \quad \text{and} \quad \eta_-(y) = \min\{z \in Q \mid z > \mathrm{pre}_+(y)\}.$$

Note (X, Δ) is in reduced form if and only if $x < \mathrm{suc}_-(x) \le \mathrm{suc}_+(x)$ for all $x \in S$ and $x < \mathrm{suc}_+(x) \le \mathrm{suc}_-(x)$ for all $x \in Q$.

We describe an explicit model for the reduced form of (X, Δ), denoted $(\widetilde{X}, \widetilde{\Delta})$, such that $\widetilde{X} \subset X$. This is done as follows. Define $\widetilde{S} = \{\pi_+(x) \mid x \in S\}$ (ie the largest endpoints of each maximal interval of elements with positive values) and define $\widetilde{Q} = \{\eta_-(y) \mid y \in S\}$. We then merge each $x \in [\pi_-(x), \pi_+(x)]$ with $\pi_+(x)$ and each $y \in [\eta_-(y), \eta_+(y)]$ with $\eta_-(y)$. Consequently, if Δ is expanded, we then have $\widetilde{\Delta}(\pi_+(x)) = \#[\pi_-(x), \pi_+(x)]$ and $\widetilde{\Delta}(\eta_-(y)) = -\#[\eta_-(y), \eta_+(y)]$.

When discussing the reduced form of (X, Δ), we will always assume we are working with this explicit model. We will sometimes not distinguish elements of X from their image in $\widetilde{X}$ under the map $x \mapsto \pi_+(x)$ for $x \in S$ and $y \mapsto \eta_-(y)$ for $y \in Q$. The reason for this is that if $x < y$ or $y < x$ in X, for $x \in S$ and $y \in Q$, then the same inequality holds for their images in $\widetilde{X}$.

For a Brieskorn sphere Y, we would like to study the reduced form $(\widetilde{X}_Y, \widetilde{\Delta}_Y)$ of the delta sequence (X_Y, Δ_Y). We have the following:

Proposition 4.8 *Let $Y = \Sigma(p,q,r)$ be a Brieskorn sphere. Then*

(i) $x \in \widetilde{X}_Y$ *if and only if* $N_Y - x \in \widetilde{X}_Y$,

(ii) $\widetilde{\Delta}_Y(N_Y - x) = -\widetilde{\Delta}_Y(x)$ *for* $x \in \widetilde{X}_Y$,

(iii) 0 *and* N_Y *are contained in* $\widetilde{X}_Y \subset X_Y$, *but* $N_Y/2 \notin X_Y$,

(iv) $\widetilde{\Delta}_Y = (\widetilde{\Delta}_Y|_{[0,N_Y/2]})^{\mathrm{Sym}}$.

Proof Item (i) follows from the definition of $\widetilde{X}_Y$. Since $\Delta_Y(N_y - x) = -\Delta_Y(x)$ for $x \in X_Y$, item (ii) holds as well. For (iii), let x_0 denote the minimum of $\{pq, pr, qr\}$. Note that $\mathrm{suc}_+(0) = x_0$. Let y denote the maximal element of $S(pq, pr, qr)$ less than N_Y. Then $\mathrm{suc}_-(0) = N_Y - y$. Since $N_Y \notin S(pq, pr, qr)$, we have $N_Y - y < x_0$ by definition of y. Hence, we have $0 < \mathrm{suc}_-(0) < \mathrm{suc}_+(0)$ which implies $\pi_+(0) = 0$. Therefore, $0 \in \widetilde{X}_Y$. Consequently, $N_Y - 0 = N_Y \in \widetilde{X}_Y$. Since no element of X_Y can be in both S_Y and Q_Y, we see $N_Y/2 \notin X_Y$. Using Definition 4.4, we deduce (iv) directly from the first three items. □

Definition 4.9 Given an abstract delta sequence (X, Δ), define the well-ordered set $X^+ := X \cup \{z^+\}$ where $z^+ > z$ for all $z \in X$, and a function $\tau_\Delta\colon X^+ \to \mathbb{Z}$, as in Section 4, with the following formula:

$$\tau_\Delta(z) = \sum_{\substack{w \in X \\ w < z}} \Delta(w) \quad \text{for all } z \in X^+.$$

We call τ_Δ the *tau function* of the delta sequence (X, Δ).

An important part of the study of abstract delta sequences is to detect where their tau functions attain their absolute minimum. Below we define a class of delta sequences whose tau functions have easily detectable minima.

Definition 4.10 Let (X, Δ) be an abstract delta sequence and let $(\widetilde{X}, \widetilde{\Delta})$ be its reduced form. We say that (X, Δ) is *sinking* if

(i) the maximal element $z_{\max}$ of $\widetilde{X}$ belongs to $\widetilde{Q}$,

(ii) for every element $x \in \widetilde{S}$, we have $\widetilde{\Delta}(x) \leq |\widetilde{\Delta}(\mathrm{suc}_-(x))|$,

(iii) if $\mathrm{suc}_-(x) = z_{\max}$, then the inequality in (ii) is strict.

It follows from the definition that the tau function of a sinking delta sequence attains its absolute minimum at the last element and nowhere else.

We will also need certain dimensional formulas for $\mathbb{H}(\Gamma_\Delta)$, which are independent of whether (X, Δ) is sinking. We find it convenient to work in the reduced form. Let (X, Δ) be a given abstract delta sequence and let $(\widetilde{X}, \widetilde{\Delta})$ be its reduced form. Let $\tilde{\tau}\colon \widetilde{X}^+ \to \mathbb{Z}$ be the tau function of the reduced sequence. For any $z \in \widetilde{X}^+$ other than the minimal element, let $\mathrm{pre}(z)$ denote the immediate predecessor of z in $\widetilde{X}^+$ (ie $\mathrm{pre}_-(z)$ if $z \in \widetilde{S}$ and $\mathrm{pre}_+(z)$ if $z \in \widetilde{Q}$). Denote by $z_{\min}$ the minimal element of $\widetilde{X}^+$.

5 Semigroups and creatures

Having given the necessary background, we are now ready to prove Theorem 1.3(ii). First, we will formally define the creatures given in Figure 7 by their delta sequences. Then we will observe that Theorem 1.3(ii) holds for the creature graded roots, which will be denoted Γ_{C_p}; that is, we will show $U \cdot \mathbb{H}_0^{\mathrm{red}}(\Gamma_{C_p}) = 0$. Finally, we shall prove a technical decomposition lemma which will reduce the proof of Theorem 1.3(ii) for Y_p to checking that it holds for the creatures. It is well known that $\mathrm{HF}^{\mathrm{red}}(Y_2) = 0$; further, in Section 4 we established Theorem 1.3(ii) for the case of $p = 4$. Therefore, throughout this section, we will restrict to the case that p is an even integer with $p \geq 6$. We will often write $p = 2\xi + 2$.

Definition 5.1 For every $p = 2\xi + 2$ with $\xi \geq 2$, the *creature* Γ_{C_p} is the graded root defined by the symmetrization of the abstract delta sequence

$$\Delta_{C_p} = \big\langle \xi, -\xi, (\xi-1), -(\xi-1), \dots, 2, -2, \\ 1, -2, 1, -2, 2, \dots, -(\xi-1), (\xi-1), -\xi, \xi, -(\xi+1)\big\rangle.$$

Let p be given, and consider the creature graded root Γ_{C_p} and its homology $\mathbb{H}(\Gamma_{C_p})$, which is an $\mathbb{F}[U]$–module supported in even degrees.

Proposition 5.2 *For $p = 2\xi + 2$ with $\xi \geq 2$, we have*

$$\dim(\ker U)_0 + 1 = \dim \mathbb{H}_0(\Gamma_{C_p}).$$

Proof The result follows directly from Figure 7. □

Let $Y_p = \Sigma(p, 2p-1, 2p+1)$. Let Δ_{Y_p} denote the corresponding abstract delta sequence as described in Section 4, and let $\widetilde{\Delta}_{Y_p}$ denote its reduced form. The proof of Theorem 1.3(ii) will be a consequence of Proposition 5.2 and the following technical statement about $\widetilde{\Delta}_{Y_p}$.

Lemma 5.3 *For every even integer $p \geq 6$, we have the following decomposition*

$$\widetilde{\Delta}_{Y_p} = (\Delta_{Z_p} * \Delta_{C_p})^{\mathrm{Sym}},$$

where Δ_{C_p} is the creature sequence given in Definition 5.1, and where Δ_{Z_p} is a sinking delta sequence.

Let us first see why the above lemma implies the remaining half of Theorem 1.3.

Proof of Theorem 1.3(ii) Consider the graded root Γ_{Y_p} whose grading is shifted so that it agrees with the absolute grading of $\mathrm{HF}^+(Y_p)$ (see Theorem 4.3). The decomposition in Lemma 5.3 implies that the creature graded root Γ_{C_p} embeds into Γ_{Y_p} as a subgraph. Moreover, Theorem 1.3(i) implies that this embedding is in fact degree preserving. Since Δ_{Z_p} is sinking, the minimum value of $\tau_{\Delta_{Z_p}}$ is uniquely attained at the maximal element of Δ_{Z_p}. As the initial value of $\tau_{\Delta_{C_p}}$ is 0, we conclude $\mathbb{H}_{\leq 0}(\Gamma_{Y_p}) = \mathbb{H}_{\leq 0}(\Gamma_{C_p})$ as graded $\mathbb{F}[U]$–modules. By Theorem 4.3 and Proposition 5.2, we see that $\dim \mathrm{HF}^+_0(Y_p)$ is equal to $\dim(\ker U)_0 + 1$. This implies the desired result. □

The proof of Lemma 5.3

As shown above, Theorem 1.3(ii) will follow from Lemma 5.3. Assuming a few technical results whose details we currently postpone, we provide a proof of this key lemma. For notation, let $r_\pm = p(2p \pm 1)$ and $w = (2p-1)(2p+1)$.

We point out two inequalities which will be used throughout:

$$(p-1)r_- + (p-3)r_+ < N_{Y_p}, \tag{5-1}$$

$$(p-2)r_- + (p-2)r_+ > N_{Y_p}. \tag{5-2}$$

The validity of these two inequalities can be checked directly from the definitions $r_\pm = p(2p \pm 1)$ and $N_{Y_p} = 4p^3 - 8p^2 - p + 1$.

Proof of Lemma 5.3 Recall we want to find a decomposition $\widetilde{\Delta}_{Y_p} = (\Delta_{Z_p} * \Delta_{C_p})^{\mathrm{Sym}}$, such that Δ_{Z_p} is sinking and Δ_{C_p} is the creature sequence from Definition 5.1. Define

$$K = (\xi - 1)r_- + \xi r_+. \tag{5-3}$$

Then

$$K < (p-3)r_+ < N_{Y_p}/2, \tag{5-4}$$

where the first inequality follows from (5-2) and the second from (5-1).

We define

$$\Delta_{Z_p} = \widetilde{\Delta}_{Y_p}|_{\widetilde{X}_{Y_p}\cap[0,K)}, \tag{5-5}$$

$$\Delta_{W_p} = \widetilde{\Delta}_{Y_p}|_{\widetilde{X}_{Y_p}\cap[K,N_{Y_p}/2]}. \tag{5-6}$$

By Proposition 4.8, we have

$$\widetilde{\Delta}_{Y_p} = (\Delta_{Z_p} * \Delta_{W_p})^{\mathrm{Sym}}. \tag{5-7}$$

Notice that K is an element of the semigroup $S(r_-, r_+)$ generated by r_- and r_+. Since $K < (p-3)r_+$, we conclude $K \in \widetilde{S}_{Y_p}$ by Proposition 5.7. Therefore, Δ_{W_p} is an abstract delta sequence. It is clear Δ_{Z_p} is an abstract delta sequence, since $\widetilde{\Delta}_{Y_p}$ is.

The result now follows from (5-7) and Lemma 5.8 (that Δ_{Z_p} is sinking) and Lemma 5.9 (that Δ_{W_p} agrees with Δ_{C_p}) below. □

For the rest of this section we will complete the details of the ingredients used in the proof of Lemma 5.3 above: Lemmas 5.8 and 5.9. In order to study the abstract delta sequence for $\Sigma(p, 2p-1, 2p+1)$, we must work with the semigroup $S(r_-, r_+, w)$ generated by r_-, r_+ and w. Ideally we would like to describe explicitly the elements of $S(r_-, r_+, w)\cap[0, N_{Y_p}]$. This set seems to be too complicated at the moment, but we will only need an explicit description of $S(r_-, r_+, w)\cap[0, (p-1)r_+]$; note $(p-1)r_+ < N_{Y_p}$. We begin with an important subset of $S(r_-, r_+, w) \cap [0, (p-1)r_+]$.

Lemma 5.4 *The intersection $S(r_-, r_+) \cap [0, (p-1)r_+]$ is given by the ordered set*

$$\begin{aligned}
&\{0,\\
&\ r_-, r_+,\\
&\ 2r_-, r_- + r_+, 2r_+,\\
&\ 3r_-, 2r_- + r_+, r_- + 2r_+, 3r_+,\\
&\qquad\qquad\qquad \ddots\\
&\ (p-1)r_-, (p-2)r_- + r_+, \dots, (p-1)r_+\}.
\end{aligned}$$

Here we deliberately break the lines so the pattern of the elements is visible.

Proof First, it is clear that $(p-1)r_+ \in S(r_-, r_+)$, and therefore is the maximal element. Next, note that if $a+b=k$ and $1 \leq a \leq k$ and $1 \leq b \leq k$, then since $r_- < r_+$,

$$ar_- + (b-1)r_+ < (a-1)r_- + br_+.$$

Therefore, to establish the order as given in the statement of the lemma, we just need to show that as long as $k \leq (p-1)$, we have $kr_+ < (k+1)r_-$. This inequality can be checked using the definition $r_\pm = p(2p \pm 1)$. □

Fact 5.5 *Let a and b be positive integers. Observe that since $w = r_- + r_+ - 1$, the sequence of elements of $S(r_-, r_+, w)$*

$$\begin{aligned}
&(a-\min\{a,b\})r_- + (b-\min\{a,b\})r_+ + \min\{a,b\}w,\\
&(a-\min\{a,b\}+1)r_- + (b-\min\{a,b\}+1)r_+ + (\min\{a,b\}-1)w,\\
&\ \vdots\\
&(a-1)r_- + (b-1)r_+ + w,\\
&ar_- + br_+
\end{aligned}$$

is consecutive in $\mathbb{N}$. This will be used frequently throughout the proof.

Before proceeding, we point out that if $x \in S_{Y_p}$ can be written as $x = ar_- + br_+ + cw$ for some nonnegative integers a, b and c, then this decomposition is unique by the Chinese remainder theorem. Suppose now that a is a nonnegative integer and b is a positive integer such that $a + b \leq p - 1$. Fact 5.5, combined with this observation about the unique representability of elements in S_{Y_p}, implies

$$\begin{aligned}
(a+1)r_- + (b-1)r_+ &< (a-\min\{a,b\})r_- + (b-\min\{a,b\})r_+ + \min\{a,b\}w\\
&\ \vdots\\
&< (a-1)r_- + (b-1)r_+ + w\\
&< ar_- + br_+.
\end{aligned} \tag{5-8}$$

Note the top right-hand term of (5-8) is equal to $ar_- + br_+ - \min\{a,b\}$. Combining Lemma 5.4 with (5-8) gives a complete description of $S(r_-, r_+, w) \cap [0, (p-1)r_+]$. That is, for each element $x = ar_- + br_+$ of the pyramid given in Lemma 5.4, we have $\min\{a,b\}$ more consecutive elements preceding x. In particular, there are no elements of $S(r_-, r_+, w)$ between $(k-1)r_+$ and kr_- for $k \leq p-1$.

In order to study Δ_{Y_p}, we will find it more convenient to work with the reduced form given in Section 4. Recall that, in order to determine the reduced form, we must compute $\pi_\pm(x)$ for $x \in S_{Y_p}$. Further, since Δ_{Y_p} is expanded, to compute the values of $\widetilde{\Delta}_{Y_p}$ it suffices to count the number of elements in $S_{Y_p} \cap [\pi_-(x), \pi_+(x)]$. The following lemma therefore allows us to compute a large portion of the reduced form of Δ_{Y_p}.

Lemma 5.6 *Suppose that* $x \in S_{Y_p}$ *satisfies* $x = ar_- + br_+$, *where* $a, b \geq 0$, *and that* $x \leq 2r_- + (p-3)r_+$. *Then*

(i) $x < N_{Y_p} - (p-a-1)r_- - (p-b-3)r_+ < \mathrm{suc}_+(x)$,

(ii) $S_{Y_p} \cap [\pi_-(x), \pi_+(x)] = \begin{cases} \{x - \min\{a,b\}, \dots, x\} & \text{if } x \neq (p-2)r_+, (p-1)r_-, \\ \{(p-2)r_+, (p-1)r_-\} & \text{otherwise.} \end{cases}$

The first part of the above lemma implies that under certain conditions, if $x = ar_- + br_+$, then $\pi_+(x) = x$. Thus, when working with the reduced form, we will be able to restrict attention to elements in $S(r_-, r_+)$ instead of $S(r_-, r_+, w)$. The second part of the lemma determines the explicit values of $\widetilde{\Delta}_{Y_p}$. These statements will be made more precise in Proposition 5.7 below.

Proof (i) First, let

$$x' = \begin{cases} (a-1)r_- + (b+1)r_+ & \text{if } a \geq 1, \\ (b+1)r_- & \text{if } a = 0. \end{cases}$$

One can check that $x' \leq (p-1)r_+$. Hence, we have by Lemma 5.4 and Fact 5.5 that $x < x'$ and that the elements of S_{Y_p} strictly between x and x' are of the form $x' - i$, for $1 \leq i \leq \min\{a-1, b+1\}$, and they are consecutive in X_{Y_p}. Thus

$$\mathrm{suc}_+(x) = \begin{cases} x' - \min\{a-1, b+1\} & \text{if } a \geq 1, \\ x' & \text{if } a = 0. \end{cases}$$

Since the elements between $\mathrm{suc}_+(x)$ and x' are consecutive in X_{Y_p} (and in S_{Y_p}), if $y \in Q_{Y_p}$ satisfies $y < x'$, it must satisfy $y < \mathrm{suc}_+(x)$. However, it is straightforward to verify that (5-1) and (5-2) imply

$$x < N_{Y_p} - (p-a-1)r_- - (p-b-3)r_+ < x', \tag{5-9}$$

where for the second inequality, we also use the observation

$$-r_- + (b+1)r_+ < -r_- + (b+2)r_- = (b+1)r_-.$$

(ii) In fact, when $b < (p-2)$, then $p-a-1 \geq 0$ and $p-b-3 \geq 0$, so we have found an element of Q_{Y_p} between x and $\mathrm{suc}_+(x)$, and thus $\pi_+(x) = x$. When $b \geq (p-2)$, since $x \leq 2r_- + (p-3)r_+$, we have $x = (p-2)r_+$ by Lemma 5.4. By (5-9), we do see $(p-2)r_+ < N_{Y_p} - (p-1)r_- + r_+ < x' = (p-1)r_-$, but it turns out that $N_{Y_p} - (p-1)r_- + r_+$ is not of the form $N_{Y_p} - z$, for any $z \in S_{Y_p}$. We will deal with this exceptional case shortly.

First, we would like to determine the value of $\pi_-(x)$ in the generic case. By Fact 5.5, $\{x-\min\{a,b\},\dots,x\}$ is a consecutive subset of X_{Y_p} which is contained in S_{Y_p}. Let

$$x''=\begin{cases}(a+1)r_-+(b-1)r_+ & \text{if } b\geq 1,\\ (a-1)r_+ & \text{if } b=0.\end{cases}$$

By (5-8) and Lemma 5.4, we have $\mathrm{pre}_+(x-\min\{a,b\})=x''$. Again, Lemma 5.4, (5-1) and (5-2) imply

$$x''<N_{Y_p}-(p-a-2)r_--(p-b-2)r_+<x.$$

Similar to above, when $a<p-1$, we have $N_{Y_p}-(p-a-2)r_--(p-b-2)r_+\in Q_{Y_p}$, since $p-a-2\geq 0$ and $p-b-2\geq 0$. In this case, $\pi_-(x)=x-\min\{a,b\}$. Thus, if x is neither $(p-2)r_+$ nor $(p-1)r_-$, the second claim follows, since $\pi_+(x)=x$.

In order to deal with the exceptional cases, we will prove that there is no element of Q_{Y_p} between $(p-2)r_+$ and $(p-1)r_-$. Since the above arguments show that there exists an element of Q_{Y_p} between $\mathrm{pre}_+((p-2)r_+)$ and $(p-2)r_+$, and an element of Q_{Y_p} between $(p-1)r_-$ and $\mathrm{suc}_+((p-1)r_-)$, this will establish that $S_{Y_p}\cap[\pi_-(x),\pi_+(x)]=\{(p-2)r_+,(p-1)r_-\}$ for $x=(p-2)r_+$ or $(p-1)r_-$. Here, we are using our description of S_{Y_p} to deduce that there are no elements of S_{Y_p} between $(p-2)r_+$ and $(p-1)r_-$.

Suppose $y\in Q_{Y_p}$ satisfies $(p-2)r_+<y<(p-1)r_-$. Then write $y=N_{Y_p}-z$, where $z\in S_{Y_p}$. We therefore have $(p-2)r_++z<N_{Y_p}<(p-1)r_-+z$. By (5-1) and (5-2), we have $(p-3)r_+<z<(p-2)r_-$. However, there is no element of S_{Y_p} between $(p-3)r_+$ and $(p-2)r_-$. This is a contradiction. Thus there are no elements in Q_{Y_p} between $(p-2)r_+$ and $(p-1)r_-$, which is what we needed to show. □

We remark that, more generally, if $x=ar_-+br_+$ and $x\geq(p-1)r_+$, we are still able to deduce $\{x-\min\{a,b\},\dots,x\}\subset S_{Y_p}\cap[\pi_-(x),\pi_+(x)]$. Finally, recall that, for $x\in X_{Y_p}$, we may also write x for the induced element in $\widetilde{X}_{Y_p}$. As promised above, we now use Lemma 5.6 to explicitly compute a great deal of the reduced form $\widetilde{\Delta}_{Y_p}$ of Δ_{Y_p}.

Proposition 5.7 *The reduced form $\widetilde{\Delta}_{Y_p}$ of Δ_{Y_p} has the following properties:*

(i) *As ordered subsets of $\mathbb{N}$,*

$$\widetilde{S}_{Y_p}\cap[0,2r_-+(p-3)r_+]=S(r_-,r_+)\cap[0,2r_-+(p-3)r_+]\setminus\{(p-2)r_+\}.$$

(ii) *Let $x\in S(r_-,r_+)\cap[0,2r_-+(p-3)r_+]\setminus\{(p-2)r_+,(p-1)r_-\}$ be expressed as $x=ar_-+br_+$. Then $\widetilde{\Delta}_{Y_p}(x)=\min\{a,b\}+1$ and $\widetilde{\Delta}_{Y_p}((p-1)r_-)=2$.*

(iii) *Let $x\in\widetilde{S}_{Y_p}$ and suppose that $x<N_{Y_p}-cr_--dr_+<\mathrm{suc}_+(x)$, where $c,d\geq 0$. Then $\widetilde{\Delta}_{Y_p}(\mathrm{suc}_-(x))\leq-\min\{c,d\}-1$.*

Proof Recall from the construction of $\widetilde{X}_{Y_p}$ given in Section 4 that $\widetilde{S}_{Y_p}$ consists of the elements of the form $\pi_+(x)$ for $x \in S_{Y_p}$. For this proof, all predecessors and successors will be taken with respect to X_{Y_p} and never $\widetilde{X}_{Y_p}$. Pick an element $x \in S_{Y_p} \cap [0, 2r_- + (p-3)r_+] \setminus \{(p-2)r_+, (p-1)r_-\}$ and suppose that x is of the form $x = ar_- + br_+$. Lemma 5.6(ii) gives that if $x \neq (p-2)r_+, (p-1)r_-$, then $S_{Y_p} \cap [\pi_-(x), \pi_+(x)] = \{x - \min\{a,b\}, \dots, x\}$ and thus $\pi_+(x) = x$. It is also shown in Lemma 5.6(ii) that $\pi_+((p-2)r_+) = \pi_+((p-1)r_-) = (p-1)r_-$, so $(p-2)r_+$ must be excluded from $\widetilde{S}_{Y_p}$. This finishes the proof of the first claim.

Since Δ_{Y_p} is expanded,

$$\widetilde{\Delta}_{Y_p}(x) = |\{x - \min\{a,b\}, \dots, x\}| = \min\{a,b\} + 1,$$

for every $x \in \widetilde{S}_{Y_p} \cap [0, r_- + (p-2)r_+] \setminus \{(p-1)r_-\}$. In the exceptional case where $x = (p-1)r_-$, we have $\pi_-(x) = (p-2)r_+$ and $\pi_+(x) = (p-1)r_-$, hence $\widetilde{\Delta}_{Y_p}(x) = |\{(p-2)r_+, (p-1)r_-\}| = 2$. This proves the second claim.

It remains to establish the final claim in the proposition. Let $y = N_{Y_p} - cr_- - dr_+$ and suppose that $x < y < \mathrm{suc}_+(x)$. Then we must have $\mathrm{suc}_-(x) = \eta_-(y)$. Since Δ_{Y_p} is expanded, we have $\widetilde{\Delta}_{Y_p}(\mathrm{suc}_-(x)) = -\# X_{Y_p} \cap [\eta_-(y), \eta_+(y)]$. As discussed above, for any $z = sr_- + tr_+ \in S_{Y_p}$, we have $\{z - \min\{s,t\}, \dots, z\} \subset [\pi_-(z), \pi_+(z)]$, regardless of whether $s + t \leq p-1$. From this, we can deduce that

$$\{N_{Y_p} - cr_- - dr_+, \dots, N_{Y_p} - cr_- - dr_+ + \min\{c,d\}\} \subset [\eta_-(y), \eta_+(y)].$$

Therefore, we must have $\widetilde{\Delta}_{Y_p}(\mathrm{suc}_-(x)) \leq -\min\{c,d\} - 1$. □

Lemma 5.8 *The abstract delta sequence Δ_{Z_p} is sinking.*

Proof We must verify all three properties in Definition 4.10 for Δ_{Z_p}. Recall that $\widetilde{\Delta}_{Y_p}$ is in reduced form, and consequently so is the restriction Δ_{Z_p}. Therefore, by the definition of Δ_{Z_p}, the last element of the delta sequence Δ_{Z_p} must have a negative value or else $\widetilde{\Delta}_{Y_p}$ would contain two positive values in a row (the last element of $\widetilde{X}_p \cap [0, K)$ and K), contradicting $\widetilde{\Delta}_{Y_p}$ being in reduced form. This establishes Definition 4.10(i), for Δ_{Z_p}.

Before proceeding further, we set up notation. Throughout this proof, we will denote predecessors and successors taken with respect to $\widetilde{X}_{Y_p}$ by a tilde decoration, and those taken with respect to X_{Y_p} will not receive a tilde decoration. Note that, by the discussion in Section 4, we have

$$\mathrm{suc}_+(x) \leq \widetilde{\mathrm{suc}}_+(x) \quad \text{for every } x \in \widetilde{X}_{Y_p}. \tag{5-10}$$

Next, we show

$$\widetilde{\Delta}_{Y_p}(x) \le -\widetilde{\Delta}_{Y_p}(\widetilde{\mathrm{suc}}_-(x)) \quad \text{for all } x \in \widetilde{S}_{Y_p} \cap [0, K), \tag{5-11}$$

which will prove that the monotonicity condition in Definition 4.10(ii) holds for Δ_{Z_p}. Let $x \in \widetilde{S}_{Y_p} \cap [0, K)$. Then $x \in S(r_-, r_+) \cap [0, (p-3)r_+]$ by (5-4) and Proposition 5.7(i). Writing $x = ar_- + br_+$, we see that $\widetilde{\Delta}_{Y_p}(x) = \min\{a, b\} + 1$ by Proposition 5.7(ii). Let $y = (p-a-1)r_- + (p-b-3)r_+$. By Lemma 5.6(i) and (5-10), we have

$$x < N_{Y_p} - y < \mathrm{suc}_+(x) \le \widetilde{\mathrm{suc}}_+(x).$$

Note that, since $x \in S(r_-, r_+) \cap [0, (p-3)r_+]$, we have $a + b \le p - 3$. Therefore, $(p-a-1) \ge 0$ and $(p-b-3) \ge 0$, and thus $N_{Y_p} - y \in Q_{Y_p}$. By Proposition 5.7(iii),

$$\widetilde{\Delta}_{Y_p}(\widetilde{\mathrm{suc}}_-(x)) \le -\min\{p-a-1, p-b-3\} - 1.$$

Therefore, to prove (5-11), it suffices to show that

$$\min\{a, b\} \le \min\{p-a-1, p-b-3\}. \tag{5-12}$$

However, we have seen that $a + b \le p - 3$. Hence, $a \le p - b - 3$ and $b \le p - a - 3$, proving (5-12).

It remains to show that Δ_{Z_p} satisfies Definition 4.10(iii). That is, we will establish a strict inequality in (5-11). We observe that the last positive value of Δ_{Z_p} occurs at $\widetilde{\mathrm{pre}}_+(K) = \xi r_- + (\xi - 1)r_+$ by Lemma 5.4 and Proposition 5.7(i). Therefore, $\widetilde{\mathrm{suc}}_-(\xi r_- + (\xi - 1)r_+)$ is the maximal element of Z_p. Thus we must prove

$$\widetilde{\Delta}_{Y_p}(\xi r_- + (\xi - 1)r_+) < -\widetilde{\Delta}_{Y_p}(\widetilde{\mathrm{suc}}_-(\xi r_- + (\xi - 1)r_+)). \tag{5-13}$$

We have $a = \xi$ and $b = \xi - 1$, so (5-12) is a strict inequality since $p = 2\xi + 2$. This implies (5-13). □

Lemma 5.9 *As abstract delta sequences, $\Delta_{W_p} \cong \Delta_{C_p}$, where Δ_{C_p} is the abstract delta sequence from Definition 5.1.*

Proof We would like to see that Δ_{W_p} agrees with Δ_{C_p} as abstract delta sequences. To do this, we explicitly compute Δ_{W_p}.

We first list all the elements of $\widetilde{S}_{Y_p} \cap [K, N_{Y_p}/2]$. By (5-4) and (5-2), it follows that

$$K < N_{Y_p}/2 < (p-2)r_+.$$

Now, by Proposition 5.7(i) we have $\widetilde{S}_{Y_p} \cap [K, N_{Y_p}/2] = S(r_-, r_+) \cap [K, N_{Y_p}/2]$. Then, using Lemma 5.4, we see that

$$\begin{aligned}(5\text{-}14)\quad \widetilde{S}_{Y_p} \cap [K, N_{Y_p}/2] = \{&(\xi-1)r_- + \xi r_+, (\xi-2)r_- + (\xi+1)r_+, \dots, \\ &r_- + (2\xi-2)r_+, (2\xi-1)r_+, 2\xi r_-, (2\xi-1)r_- + r_+, \dots, \\ &(\xi+2)r_- + (\xi-2)r_+, (\xi+1)r_- + (\xi-1)r_+\}.\end{aligned}$$

In order to verify that the last element in the above sequence is as indicated, we must show

$$(5\text{-}15)\qquad (\xi+1)r_- + (\xi-1)r_+ < N_{Y_p}/2 < \xi r_- + \xi r_+.$$

These two inequalities follow from (5-1) and (5-2) respectively, since $2\xi + 2 = p$. For the first inequality, we use that $pr_- + (p-4)r_+ < (p-1)r_- + (p-3)r_+$. Hence, (5-14) holds.

Similarly, we determine the elements of $\widetilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K]$ so we may determine the elements of $\widetilde{Q}_{Y_p} \cap [K, N_{Y_p}/2]$. Note that by Lemma 5.4 and (5-2), we have $N_{Y_p}/2 < N_{Y_p} - K < 2r_- + (p-3)r_+$ for $p \geq 4$. By Proposition 5.7(i), we have $\widetilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K] = S(r_-, r_+) \cap [N_{Y_p}/2, N_{Y_p} - K] \setminus \{2\xi r_+\}$. Using Lemma 5.4 and (5-15), we see that

$$\begin{aligned}(5\text{-}16)\quad \widetilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K] = \{&\xi r_- + \xi r_+, (\xi-1)r_- + (\xi+1)r_+, \dots, \\ &r_- + (2\xi-1)r_+, (2\xi+1)r_-, 2\xi r_- + r_+, \dots, \\ &(\xi+3)r_- + (\xi-2)r_+, (\xi+2)r_- + (\xi-1)r_+\}.\end{aligned}$$

Observe that we have purposely omitted $2\xi r_+$. To see that the last element in the above sequence is as written, we must show

$$(5\text{-}17)\qquad (\xi+2)r_- + (\xi-1)r_+ < N_{Y_p} - K < (\xi+1)r_- + \xi r_+.$$

Again these inequalities follow from (5-1) and (5-2). Hence, (5-16) holds.

By the model for the reduced form of delta sequences described in Section 4, the definition of Δ_{Y_p}, and Proposition 5.7, we have

$$\begin{aligned}(5\text{-}18)\quad \widetilde{Q}_{Y_p} &\cap [K, N_{Y_p}/2] \\ &= \{N_{Y_p} - x \mid x \in \widetilde{S}_{Y_p} \cap [N_{Y_p}/2, N_{Y_p} - K]\} \\ &= \{N_{Y_p} - x \mid x \in S(r_-, r_+) \cap [N_{Y_p}/2, N_{Y_p} - K], x \neq 2\xi r_+\}.\end{aligned}$$

In order to determine the sequence $\widetilde{X}_{Y_p} \cap [K, N_{Y_p}/2]$, we need to detect the positions of the elements of $\widetilde{Q}_{Y_p} \cap [K, N_{Y_p}/2]$ relative to the elements of $\widetilde{S}_{Y_p} \cap [K, N_{Y_p}/2]$. Since $\widetilde{X}_{Y_p} \cap [K, N_{Y_p}/2]$ is in reduced form, the elements of $\widetilde{Q}_{Y_p}$ and $\widetilde{S}_{Y_p}$ alternate.

Therefore, for an element of $\widetilde{Q}_{Y_p}$, we would like to determine which elements of $\widetilde{S}_{Y_p}$ it is adjacent to and vice versa. We shall employ the following inequalities:

$$(\xi-1-j)r_- + (\xi+j)r_+ < N_{Y_p} - (\xi+2+j)r_- - (\xi-1-j)r_+ \quad \text{for } j=0,\dots,\xi-1, \tag{5-19}$$

$$N_{Y_p} - (\xi+2+j)r_- - (\xi-1-j)r_+ < (\xi-2-j)r_- + (\xi+1+j)r_+ \quad \text{for } j=0,\dots,\xi-2, \tag{5-20}$$

$$N_{Y_p} - jr_- - (2\xi-j)r_+ < (2\xi-j)r_- + jr_+ \quad \text{for } j=0,\dots,\xi-1, \tag{5-21}$$

$$(2\xi-j)r_- + jr_+ < N_{Y_p} - (j+1)r_- - (2\xi-j-1)r_+ \quad \text{for } j=0,\dots,\xi-1. \tag{5-22}$$

These inequalities again follow from (5-1) and (5-2). Finally, by Lemma 5.4, (5-1) and (5-2), we point out that

$$(2\xi-1)r_+ < N_{Y_p} - (2\xi+1)r_- < N_{Y_p} - 2\xi r_+ < 2\xi r_-. \tag{5-23}$$

It now follows from (5-14), (5-16), (5-18), (5-19), (5-20), (5-21), (5-22) and (5-23) that the sequence $\widetilde{X}_{Y_p} \cap [K, N_{Y_p}/2]$ is given by

$$\begin{aligned}\widetilde{X}_{Y_p} \cap [K, N_{Y_p}/2] = \big\{ &(\xi-1)r_- + \xi r_+, N_{Y_p} - (\xi+2)r_- - (\xi-1)r_+, \\ &(\xi-2)r_- + (\xi+1)r_+, N_{Y_p} - (\xi+3)r_- - (\xi-2)r_+, \dots, \\ &r_- + (2\xi-2)r_+, N_{Y_p} - 2\xi r_- - r_+, (2\xi-1)r_+, \\ &N_{Y_p} - (2\xi+1)r_-, 2\xi r_-, N_{Y_p} - r_- - (2\xi-1)r_+, (2\xi-1)r_- + r_+, \dots, \\ &N_{Y_p} - (\xi-1)r_- - (\xi+1)r_+, (\xi+1)r_- + (\xi-1)r_+, N_{Y_p} - \xi r_- - \xi r_+ \big\}.\end{aligned} \tag{5-24}$$

Again, note that $N_{Y_p} - 2\xi r_+$ is deliberately excluded from the above list, as it is not an element of $\widetilde{X}_{Y_p}$.

It remains to see that the values of $\widetilde{\Delta}_{Y_p}$ on the above sequence are the same as the values of Δ_{C_p}. By Proposition 5.7(ii), since $N_{Y_p}/2 < (p-2)r_+$, we have

$$\widetilde{\Delta}_{Y_p}(cr_- + dr_+) = \min\{c,d\} + 1 \quad \text{for } cr_- + dr_+ \in \widetilde{S}_{Y_p} \cap [K, N_{Y_p}/2]. \tag{5-25}$$

Moreover for every $N_{Y_p} - cr_- - dr_+ \in \widetilde{Q}_{Y_p} \cap [K, N_{Y_p}/2]$ such that $cr_- + dr_+$ is not equal to $(2\xi+1)r_-$, we have

$$\widetilde{\Delta}_{Y_p}(N_{Y_p} - cr_- - dr_+) = -\widetilde{\Delta}_{Y_p}(cr_- + dr_+) = -\min\{c,d\} - 1, \tag{5-26}$$

by Proposition 5.7, since as observed earlier, $cr_- + dr_+ < N_{Y_p} - K < 2r_- + (p-3)r_+$.

Also, by Proposition 5.7, we have $N_{Y_p} - (2\xi + 1)r_- \in \widetilde{Q}_{Y_p}$, and further,

$$\widetilde{\Delta}_{Y_p}(N_{Y_p} - (2\xi + 1)r_-) = -\widetilde{\Delta}_{Y_p}((2\xi + 1)r_-) = -2. \tag{5-27}$$

Computing the value of $\widetilde{\Delta}_{Y_p}$ at each element of the sequence (5-24) using (5-25), (5-26) and (5-27), and comparing the result with Definition 5.1, we see that Δ_{W_p} agrees with Δ_{C_p}. This is what we wanted to show. □

Remark 5.10 Experimental evidence also suggests that for odd p, the structure of HF^+ seems similar to the case of p even (see Figure 7 for the expected form). We expect similar arguments to apply to show that $U \cdot \mathrm{HF}_0^{\mathrm{red}}(Y_p) = 0$. Due to Remark 3.2, such manifolds would also not be surgery on a knot in S^3 for $p \geq 9$.

More generally, it is natural to ask if the obstruction in Theorem 1.2 can be used to construct hyperbolic (or other interesting irreducible) homology spheres that are not surgery on a knot, as Auckly did. Most techniques for computing HF^+ (as opposed to $\widehat{\mathrm{HF}}$) apply only to plumbed manifolds and surgeries on knots in S^3, making this problem difficult. It would be interesting to obtain such examples by applying the mapping cone formula to surgeries on knots in other three-manifolds or by presenting the homology sphere by surgery on a link and applying the link surgery formula of Manolescu and Ozsváth [13].

Finally, we ask if it is possible to refine Theorem 1.2 to obstruct homology spheres from being homology cobordant to surgery on a knot.

6 Knot surgeries in other manifolds

In the proof of Theorem 1.2, the only thing special about S^3 is that it is an integer homology sphere L-space (ie $\mathrm{HF}^{\mathrm{red}} = 0$) and that $d(S^3) = 0$. The following theorem is a slight generalization.

Theorem 6.1 *Let Y and Y' be oriented integer homology spheres. Suppose that $\mathrm{HF}^{\mathrm{red}}(Y) = 0$ and $d(Y') \leq d(Y) - 8$. Then if Y' is obtained by surgery on a knot in Y, then there is a nontrivial element of $\mathrm{HF}^{\mathrm{red}}(Y')$ in degree $d(Y)$ which is not in the kernel of U.*

Proof The proof is the same as Theorem 1.2, where the only difference is that we have to incorporate the d–invariant of Y into some of the statements. The main observation is that for $n > 0$, we have the more general formula $d(Y_{1/n}(K)) = d(Y) - 2V_0$, where V_0 is defined analogously for Y and K as for a knot in S^3. This follows by repeating the arguments in [17, Proposition 1.6] for a knot in an integer homology sphere L-space. □

Proof of Theorem 1.4 Orient $\Sigma(2,3,5)$ such that it is the boundary of a negative-definite plumbing. In this case, $d(\Sigma(2,3,5)) = 2$. Let Y be an integer homology sphere. We will show that for $k \gg 0$, the manifold $\#_k \Sigma(2,3,5)$ is not surgery on a knot in Y, regardless of orientation of Y.

Fix an orientation on Y. Recall that $\mathrm{HF}^{\mathrm{red}}(Y)$ is finite-dimensional over $\mathbb{F}$. Therefore, we may define an integer n_Y by

$$n_Y = |\max\{s \mid \mathrm{HF}^{\mathrm{red}}_s(Y) \neq 0\}|.$$

Choose $k > 0$ such that

$$2k \geq \max\{d(+Y), d(-Y)\} + \max\{n_{+Y}, n_{-Y}\} + 8.$$

Due to the additivity of d under connected-sums [18, Theorem 4.3], we have

$$d(\#_k \Sigma(2,3,5)) = 2k \geq d(\pm Y) + n_{\pm Y} + 8.$$

Therefore, $d(\pm Y) \leq d(\#_k \Sigma(2,3,5)) - 8$. Furthermore, by construction there is no element of $\mathrm{HF}^{\mathrm{red}}(\pm Y)$ in degree $d(\#_k \Sigma(2,3,5))$. Therefore, by Theorem 6.1, neither Y nor $-Y$ can be expressed as surgery on a knot in $\#_k \Sigma(2,3,5)$. Consequently, $\#_k \Sigma(2,3,5)$ cannot be surgery on a knot in Y, regardless of orientation. □

We conclude by pointing out that Theorem 6.1 can also be extended to statements about p/q–surgery where $|p| \geq 2$. (The key observation is that in an integer homology sphere L-space, we can control the integers V_i and H_i from the d–invariants of p/q–surgeries for any $p/q > 0$ by [17, Proposition 1.6].) We can then apply the same arguments as in Theorems 1.2 and 1.4 to show that if Y has cyclic first homology, $\#_k \Sigma(2,3,5)$ is not surgery on a knot in Y for k large. Finally, the analogous statement when Y has noncyclic homology is trivial. Thus, in conclusion, for any three-manifold Y, there exist infinitely many integer homology spheres which are not surgery on a knot in Y.

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