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Slices of hermitian K–theory and Milnor's conjecture on quadratic forms

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We advance the understanding of K–theory of quadratic forms by computing the slices of the motivic spectra representing hermitian K–groups and Witt groups. By an explicit computation of the slice spectral sequence for higher Witt theory, we prove Milnor's conjecture relating Galois cohomology to quadratic forms via the filtration of the Witt ring by its fundamental ideal. In a related computation we express hermitian K–groups in terms of motivic cohomology.

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1 Introduction

Suppose that F is a field of characteristic char $F \neq 2$. Milnor [32] defines the K–theory of F in terms of generators and relations by

$$K^M_*(F) = T^*F^\times/(a \otimes (1-a)), \quad a \neq 0, 1.$$

Here $T^*F^\times$ is the tensor algebra of the multiplicative group of units $F^\times$. In degrees zero, one and two these groups agree with Quillen's K–groups, but for higher degrees they differ in general. Milnor [32] proposed two conjectures relating $k^M_*(F) = K^M_*(F)/2K^M_*(F)$ to the mod-2 Galois cohomology ring $H^*(F;\mathbb{Z}/2)$ and the graded Witt ring $\mathrm{GW}_*(F) = \bigoplus_{q \geq 0} I(F)^q/I(F)^{q+1}$ for the fundamental ideal $I(F)$ of even-dimensional forms, via the two homomorphisms

$$\begin{tikzcd} & k^M_*(F) \arrow[dl, "s^F_*"'] \arrow[dr, "h^F_*"] & \\ \mathrm{GW}_*(F) & & H^*(F;\mathbb{Z}/2). \end{tikzcd}$$

The solutions of the Milnor conjectures on Galois cohomology — h^F_* is an isomorphism (see Voevodsky [61]) — and on quadratic forms — s^F_* is an isomorphism (see Orlov, Vishik and Voevodsky [38]) — are two striking applications of motivic homotopy theory. For background and influence of these conjectures and also for the history of their

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proofs we refer to Bass [4], Friedlander [11], Kahn [21], Levine [26], Milnor [33], Morel [34], Pfister [42] and Suslin [53]. In this paper we give an alternate proof of Milnor's conjecture on quadratic forms by explicitly computing the slice spectral sequence for the higher Witt theory spectrum **KT**. Our method of proof applies also to smooth semilocal rings containing a field of characteristic zero.

Let $X \in \mathrm{Sm}_F$ be a smooth scheme of finite type over a field F. We refer to Grayson [12] for a survey of the known constructions of the first-quadrant convergent spectral sequence

$$\mathbf{MZ}^{\star}(X) \Longrightarrow \mathbf{KGL}_{*}(X) \tag{1}$$

relating motivic cohomology to algebraic K–theory. From the viewpoint of motivic homotopy theory (see Voevodsky [59]), the problem of constructing (1) reduces to identifying the slices $\mathsf{s}_q(\mathbf{KGL})$ of the motivic spectrum **KGL** representing algebraic K–theory. Voevodsky introduced the slice spectral sequence and conjectured that

$$\mathsf{s}_q(\mathbf{KGL}) \cong \Sigma^{2q,q}\mathbf{MZ}. \tag{2}$$

The formula (2) was proven for fields of characteristic zero by Voevodsky [60; 63], and (invoking different methods) for perfect fields by Levine [27]. By base change the same holds for all fields.

When $\operatorname{char} F \neq 2$ we are interested in the analogues of (1) and (2) for the motivic spectra **KQ** and **KT** representing hermitian K–groups and Witt groups on Sm_F, respectively; see Hornbostel [15]. Theorem 4.18 shows the slices of hermitian K–theory are given by infinite wedge product decompositions

$$\mathsf{s}_q(\mathbf{KQ}) \cong \begin{cases} \Sigma^{2q,q}\mathbf{MZ} \vee \bigvee_{i<q/2} \Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 0 \bmod 2, \\ \bigvee_{i<(q+1)/2} \Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 1 \bmod 2. \end{cases} \tag{3}$$

Moreover, in Theorem 4.28 we compute the slices of higher Witt theory, namely

$$\mathsf{s}_q(\mathbf{KT}) \cong \bigvee_{i\in\mathbb{Z}} \Sigma^{2i+q,q}\mathbf{MZ}/2. \tag{4}$$

The summand $\Sigma^{2q,q}\mathbf{MZ}$ in (3) is detected by showing that $\mathsf{s}_q(\mathbf{KGL})$ is a retract of $\mathsf{s}_q(\mathbf{KQ})$ if q is even. We deduce that $\mathsf{s}_q(\mathbf{KQ})$ is a wedge sum of $\Sigma^{2q,q}\mathbf{MZ}$ and some **MZ**–module, ie a motive, which we identify with the infinite wedge summand in (3). Our first results show there is an additional "mysterious summand" $\Sigma^{2q,q}\mathbf{M}\mu$ of $\mathsf{s}_q(\mathbf{KQ})$. We show $\mathbf{M}\mu$ is trivial by using base change and the solution of the homotopy fixed point problem for hermitian K–theory of the prime fields; see Berrick and Karoubi [5] and Friedlander [10], and compare Berrick, Karoubi, Schlichting and

Østvær [6] and Hu, Kriz and Ormsby [19]. As conjectured in Hornbostel's foundational paper [15], Theorem 3.4 shows there is a homotopy cofiber sequence

$$\Sigma^{1,1}\mathbf{KQ} \xrightarrow{\eta} \mathbf{KQ} \longrightarrow \mathbf{KGL} \tag{5}$$

relating the algebraic and hermitian K–theories. The stable Hopf map η is induced by the canonical map $\mathbb{A}^2 \smallsetminus \{0\} \to \mathbb{P}^1$. We show this over any finite-dimensional regular noetherian base scheme S equipped with the trivial involution and with 2 invertible in its ring of regular functions, ie $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. A closely related statement is obtained in Schlichting [48]. The sequence (5) is employed in our computation of the slices of $\mathbf{KQ}$.

The algebraic K–theory spectrum $\mathbf{KGL}$ affords an action by the stable Adams operation Ψ^{-1}. For the associated homotopy orbit spectrum $\mathbf{KGL}_{hC_2}$ there is a homotopy cofiber sequence

$$\mathbf{KGL}_{hC_2} \longrightarrow \mathbf{KQ} \longrightarrow \mathbf{KT}. \tag{6}$$

In (6), $\mathbf{KGL}_{hC_2} \longrightarrow \mathbf{KQ}$ is induced by the hyperbolic map $\mathbf{KGL} \longrightarrow \mathbf{KQ}$, while $\mathbf{KQ} \longrightarrow \mathbf{KT}$ is the natural map from hermitian K–theory into the homotopy colimit of the tower

$$\mathbf{KQ} \xrightarrow{\eta} \Sigma^{-1,-1}\mathbf{KQ} \xrightarrow{\Sigma^{-1,-1}\eta} \Sigma^{-2,-2}\mathbf{KQ} \xrightarrow{\Sigma^{-2,-2}\eta} \cdots. \tag{7}$$

We use the formulas $\mathsf{s}_0(\Psi^{-1}) = \mathrm{id}$ and $\Sigma^{2,1}\Psi^{-1} = -\Psi^{-1}$ to identify the slices

$$\mathsf{s}_q(\mathbf{KGL}_{hC_2}) \cong \begin{cases} \Sigma^{2q,q}\mathbf{MZ} \vee \bigvee_{i\geq 0}\Sigma^{2(i+q)+1,q}\mathbf{MZ}/2, & q \equiv 0 \bmod 2, \\ \bigvee_{i\geq 0}\Sigma^{2(i+q),q}\mathbf{MZ}/2, & q \equiv 1 \bmod 2. \end{cases} \tag{8}$$

By combining the slice computations in (3) and (8) with the homotopy cofiber sequence in (6) we deduce the identification of the slices of the Witt theory spectrum $\mathbf{KT}$ in (4). Alternatively, this follows from (3), (7) and Spitzweck's result that slices commutes with homotopy colimits [49].

Our next goal is to determine the first differentials in the slice spectral sequences as maps of motivic spectra. Because of the special form the slices of $\mathbf{KQ}$ and $\mathbf{KT}$ have, this involves the motivic Steenrod squares Sq^i constructed by Voevodsky [61] and further elaborated on in Hoyois, Kelly and Østvær [18]. According to (4) the differential

$$\mathbf{d}_1^{\mathbf{KT}}(q)\colon \mathsf{s}_q(\mathbf{KT}) \longrightarrow \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KT})$$

is a map of the form

$$\bigvee_{i\in\mathbb{Z}}\Sigma^{2i+q,q}\mathbf{MZ}/2 \longrightarrow \Sigma^{2,1}\bigvee_{j\in\mathbb{Z}}\Sigma^{2j+q,q}\mathbf{MZ}/2.$$

Let $\mathbf{d}_1^{\mathbf{KT}}(q,i)$ denote the restriction of $\mathbf{d}_1^{\mathbf{KT}}(q)$ to the i^{th} summand $\Sigma^{2i+q,q}\mathbf{MZ}/2$ of $\mathsf{s}_q(\mathbf{KT})$. By comparing with motivic cohomology operations of weight one, it suffices to consider

$$\mathbf{d}_1^{\mathbf{KT}}(q,i)\colon \Sigma^{2i+q,q}\mathbf{MZ}/2 \longrightarrow \Sigma^{2i+q+4,q+1}\mathbf{MZ}/2 \vee \Sigma^{2i+q+2,q+1}\mathbf{MZ}/2 \vee \Sigma^{2i+q,q+1}\mathbf{MZ}/2.$$

In Theorem 5.3 we show the closed formula

$$\mathbf{d}_1^{\mathbf{KT}}(q,i) = \begin{cases} (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2, 0), & i-2q \equiv 0 \bmod 4, \\ (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2+\rho\mathsf{Sq}^1, \tau), & i-2q \equiv 2 \bmod 4. \end{cases} \tag{9}$$

The classes $\tau \in h^{0,1}$ and $\rho \in h^{1,1}$ are represented by $-1 \in F$; here $h^{p,q}$ is shorthand for the mod-2 motivic cohomology group in degree p and weight q. We denote integral motivic cohomology groups by $H^{p,q}$. This sets the stage for our proof of Milnor's conjecture on quadratic forms formulated in Milnor [32, Question 4.3]. For fields of characteristic zero this conjecture was shown by Orlov, Vishik and Voevodsky in [38], and by Morel [35; 36] using different approaches.

According to (4) the slice spectral sequence for $\mathbf{KT}$ fills out the upper half-plane. A strenuous computation using (9), Adem relations, and the action of the Steenrod squares on the mod-2 motivic cohomology ring $h^\star$ of F shows that it collapses. We read off the isomorphisms

$$E^2_{p,q}(\mathbf{KT}) = E^\infty_{p,q}(\mathbf{KT}) \cong \begin{cases} h^{q,q}, & p \equiv 0 \bmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

To connect this computation with quadratic form theory, we show the spectral sequence converges to the filtration of the Witt ring $W(F)$ by the powers of the fundamental ideal $I(F)$ of even-dimensional forms. By identifying motivic cohomology with Galois cohomology for fields we arrive at the following result.

Theorem 1.1 *If* char $F \neq 2$ *the slice spectral sequence for* **KT** *converges and furnishes a complete set of invariants*

$$\bar{e}_F^q\colon I(F)^q/I(F)^{q+1} \xrightarrow{\cong} H^q(F;\mathbb{Z}/2)$$

for quadratic forms over F with values in the mod-2 Galois cohomology ring.

If $X \in \mathrm{Sm}_F$ is a semilocal scheme and F a field of characteristic zero, our computations and results extend to the Witt ring $W(X)$ with fundamental ideal $I(X)$ and the mod-2 motivic cohomology of X. Our reliance on the Milnor conjecture for Galois cohomology (see Voevodsky [61]) can be replaced by Levine's generalized Milnor conjecture on étale cohomology of semilocal rings [25], as shown in Hoobler

[14, Section 2.2] and Kerz [23, Theorem 7.8]; compare Lichtenbaum [30]. We refer to the end of Section 6 for further details.

In Section 7 we perform computations in the slice spectral sequence for **KQ** in low degrees. We formulate our computation of the second orthogonal K–group, and refer to the main body of the paper for the more complicated results on other hermitian K–groups.

Theorem 1.2 *If* char $F \neq 2$ *there is a naturally induced isomorphism*

$$\mathrm{KO}_2(F) \xrightarrow{\cong} \ker(\tau \circ \mathrm{pr} + \mathsf{Sq}^2\colon H^{2,2} \oplus h^{0,2} \longrightarrow h^{2,3}).$$

Throughout the paper we employ the following notation:

- S denotes a finite-dimensional regular and separated noetherian base scheme.
- Sm_S is the collection of smooth schemes of finite type over S.
- $S^{m,n}$, $\Omega^{m,n}$ and $\Sigma^{m,n}$ denote the motivic (m,n)–sphere, (m,n)–loop space and (m,n)–suspension, respectively.
- **SH** and $\mathbf{SH}^{\mathrm{eff}}$ denote the motivic and effective motivic stable homotopy categories, respectively.
- E and $\mathbf{1} = S^{0,0}$ denote the generic motivic spectrum and the motivic sphere spectrum, respectively.

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2 Slices and the slice spectral sequence

Let $i_q\colon \Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}} \hookrightarrow \mathbf{SH}$ be the full inclusion of the localizing subcategory generated by $\Sigma^{2q,q}$–suspensions of smooth schemes. We denote by r_q the right adjoint of i_q and set $\mathsf{f}_q = i_q \circ r_q$. The q^{th} slice of E is characterized up to unique isomorphism by the distinguished triangle

$$\mathsf{f}_{q+1}(\mathsf{E}) \longrightarrow \mathsf{f}_q(\mathsf{E}) \longrightarrow \mathsf{s}_q(\mathsf{E}) \longrightarrow \Sigma^{1,0}\mathsf{f}_{q+1}(\mathsf{E}) \tag{10}$$

in **SH** [59]. When it is helpful to emphasize the base scheme S we shall write $\mathsf{f}^S_q(\mathsf{E})$ and $\mathsf{s}^S_q(\mathsf{E})$. Smashing with the motivic sphere $\Sigma^{1,1}$ has the following effect on the slice filtration.

Lemma 2.1 *For all $q \in \mathbb{Z}$ there are natural isomorphisms*

$$\mathsf{f}_q(\Sigma^{1,1}\mathsf{E}) \xrightarrow{\cong} \Sigma^{1,1}\mathsf{f}_{q-1}(\mathsf{E}) \quad \textit{and} \quad \mathsf{s}_q(\Sigma^{1,1}\mathsf{E}) \xrightarrow{\cong} \Sigma^{1,1}\mathsf{s}_{q-1}(\mathsf{E})$$

which are compatible with the natural transformations occurring in (10).

Proof Let m and q be integers. The suspension functor $\Sigma^{m,m}$ restricts to a functor

$$\Sigma^{m,m}_q -\colon \Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}} \longrightarrow \Sigma^{2(q+m),q+m}\mathbf{SH}^{\mathrm{eff}}$$

satisfying $\Sigma^{m,m} i_q(\mathsf{E}) = i_{q+m} \circ (\Sigma^{m,m}_q \mathsf{E})$ for all $\mathsf{E} \in \Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}}$. This equality induces a unique natural isomorphism

$$r_q(\Sigma^{-m,-m}\mathsf{E}) \cong \Sigma^{-m,-m}_{q+m} r_{q+m}(\mathsf{E})$$

on the respective right adjoints. In particular, this results in a natural isomorphism

$$\mathsf{f}_q(\Sigma^{1,1}\mathsf{E}) = i_q r_q(\Sigma^{1,1}\mathsf{E}) \cong i_q \Sigma^{1,1}_{q-1} r_{q-1}(\mathsf{E}) = \Sigma^{1,1} i_{q-1} r_{q-1}(\mathsf{E}) = \Sigma^{1,1}\mathsf{f}_q(\mathsf{E}).$$

In order to conclude the same for s_q, observe that the inclusion i_{q+1} factors as $i_q \circ i^q_{q+1}$, where

$$i^q_{q+1}\colon \Sigma^{2(q+1),q+1}\mathbf{SH}^{\mathrm{eff}} \longrightarrow \Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}}$$

is the natural inclusion. The functor i^q_{q+1} has a right adjoint r^q_{q+1} for the same reason that i_q does, and the natural transformation $\mathsf{f}_{q+1} \longrightarrow \mathsf{f}_q$ is obtained from the counit $i^q_{q+1} \circ r^q_{q+1} \longrightarrow \mathrm{Id}$. As before, the equality

$$\Sigma^{m,m}_q i^q_{q+1}(\mathsf{E}) = i^{q+m}_{q+m+1} \circ (\Sigma^{m,m}_{q+1}\mathsf{E})$$

induces a unique natural isomorphism on right adjoints, which serves to show that the diagram

$$\begin{tikzcd}
\mathsf{f}_{q+1}(\Sigma^{1,1}\mathsf{E}) \arrow[r, "\cong"] \arrow[d] & \Sigma^{1,1}\mathsf{f}_q(\mathsf{E}) \arrow[d] \\
\mathsf{f}_q(\Sigma^{1,1}\mathsf{E}) \arrow[r, "\cong"] & \Sigma^{1,1}\mathsf{f}_{q-1}(\mathsf{E})
\end{tikzcd}$$

commutes. The remaining statements follow. □

Let $\eta\colon S^{1,1} \longrightarrow \mathbf{1}$ denote the Hopf map induced by the canonical map $\mathbb{A}^2 \smallsetminus \{0\} \longrightarrow \mathbb{P}^1$. Since every motivic spectrum E is a module over the motivic sphere spectrum $\mathbf{1}$, multiplication with η defines, by abuse of notation, a map $\eta\colon \Sigma^{1,1}\mathsf{E} \longrightarrow \mathsf{E}$.

Lemma 2.2 *For every* E *and* $q \in \mathbb{Z}$ *there is a naturally induced commutative diagram:*

$$\begin{tikzcd} \mathsf{f}_{q+1}(\Sigma^{1,1}\mathsf{E}) \arrow[r, "\cong"] \arrow[d, "\mathsf{f}_{q+1}(\eta)"'] & \Sigma^{1,1}\mathsf{f}_q(\mathsf{E}) \arrow[d, "\eta"] \\ \mathsf{f}_{q+1}\mathsf{E} \arrow[r] & \mathsf{f}_q\mathsf{E} \end{tikzcd}$$

Proof This follows from Lemma 2.1 by naturality. □

Example 2.3 The Hopf map induces a periodicity isomorphism $\eta\colon \Sigma^{1,1}\mathbf{KT} \xrightarrow{\cong} \mathbf{KT}$; see the definition (18). In particular, the vertical map on the left hand side in the diagram of Lemma 2.2 is an isomorphism. It also implies the isomorphism $\mathsf{s}_q(\mathbf{KT}) \cong \Sigma^{q,q}\mathsf{s}_0(\mathbf{KT})$.

The slices $\mathsf{s}_q(\mathsf{E})$ are modules over the motivic ring spectrum $\mathsf{s}_0(\mathbf{1})$; see [40, Theorem 3.6.22] and [13, Section 6(v)]. If the base scheme S is a perfect field, then $\mathsf{s}_0(\mathbf{1})$ is the Eilenberg–MacLane spectrum $\mathbf{M}\mathbb{Z}$ by the works of Levine [27] and Voevodsky [63]. We set $D^1_{p,q,n} = \pi_{p,n}\mathsf{f}_q(\mathsf{E})$ and $E^1_{p,q,n} = \pi_{p,n}\mathsf{s}_q(\mathsf{E})$. The exact couple

$$\begin{tikzcd} D^1_* \arrow[rr, "{(0,-1,0)}"] & & D^1_* \arrow[dl, "{(0,0,0)}"] \\ & E^1_* \arrow[ul, "{(-1,1,0)}"] & \end{tikzcd}$$

gives rise to the slice spectral sequence

$$E^1_{p,q,n} \Longrightarrow \pi_{p,n}(\mathsf{E}). \tag{11}$$

Our notation does not connote any information on the convergence of (11). The d_1–differential

$$d_1^{\mathsf{E}}(p,q,n)\colon \pi_{p,n}\mathsf{s}_q(\mathsf{E}) \longrightarrow \pi_{p-1,n}\mathsf{s}_{q+1}(\mathsf{E})$$

of (11) is induced on homotopy groups $\pi_{p,n}$ by the composite map

$$\mathbf{d}_1^{\mathsf{E}}(q)\colon \mathsf{s}_q(\mathsf{E}) \longrightarrow \Sigma^{1,0}\mathsf{f}_{q+1}(\mathsf{E}) \longrightarrow \Sigma^{1,0}\mathsf{s}_{q+1}(\mathsf{E})$$

of motivic spectra. The r^{th} differential has tri-degree $(-1, r, 0)$. By construction of the q^{th} slice, if $n > q$ the maps in

$$\cdots \longrightarrow \pi_{p,n}\mathsf{f}_{q+1}(\mathsf{E}) \longrightarrow \pi_{p,n}\mathsf{f}_q(\mathsf{E}) \longrightarrow \cdots \tag{12}$$

are isomorphisms, and $\pi_{p,n}\mathsf{s}_q(\mathsf{E})$ is trivial. Thus only finitely many nontrivial differentials enter each tri-degree, so that (11) is a half-plane spectral sequence with entering differentials. Let $\mathsf{f}_q\pi_{p,n}(\mathsf{E})$ denote the image of $\pi_{p,n}\mathsf{f}_q(\mathsf{E})$ in $\pi_{p,n}(\mathsf{E})$. The terms

$\mathsf{f}_q\pi_{p,n}(\mathsf{E})$ form an exhaustive filtration of $\pi_{p,n}(\mathsf{E})$. Moreover, E is called convergent with respect to the slice filtration if

$$\bigcap_{i\geq 0}\mathsf{f}_{q+i}\pi_{p,n}\mathsf{f}_q(\mathsf{E})=0$$

for all $p,q,n\in\mathbb{Z}$ [59, Definition 7.1]. That is, the filtration $\{\mathsf{f}_q\pi_{p,n}(\mathsf{E})\}$ of $\pi_{p,n}(\mathsf{E})$ is Hausdorff. When E is convergent, the spectral sequence (11) converges [59, Lemma 7.2]. Precise convergence properties of the slice spectral sequence are unclear in general [28].

When comparing slices along field extensions we shall appeal to the following base change property of the slice filtration.

Lemma 2.4 *Let $\alpha\colon X\longrightarrow Y$ be a smooth map. For $q\in\mathbb{Z}$ there are natural isomorphisms*

$$\mathsf{f}^X_q\alpha^*\xrightarrow{\cong}\alpha^*\mathsf{f}^Y_q\quad\text{and}\quad\mathsf{s}^X_q\alpha^*\xrightarrow{\cong}\alpha^*\mathsf{s}^Y_q.$$

Proof Any map $\alpha\colon X\longrightarrow Y$ between base schemes yields a commutative diagram

$$\begin{tikzcd}
\Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}}(Y) \arrow[r, "\alpha^*"] \arrow[d, "i^Y_q"'] & \Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}}(X) \arrow[d, "i^X_q"] \\
\mathbf{SH}(Y) \arrow[r, "\alpha^*"] & \mathbf{SH}(X).
\end{tikzcd}$$

Let α_* be the right adjoint of α^*. By uniqueness of adjoints, up to unique isomorphism, there exists a natural isomorphism of triangulated functors

$$r^Y_q\alpha_*\xrightarrow{\cong}\alpha_*r^X_q.$$

Since α is smooth, the functor α^* has a left adjoint $\alpha_\sharp$ and there is a commutative diagram

$$\begin{tikzcd}
\Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}}(X) \arrow[r, "\alpha^\sharp"] \arrow[d, "i^X_q"'] & \Sigma^{2q,q}\mathbf{SH}^{\mathrm{eff}}(Y) \arrow[d, "i^Y_q"] \\
\mathbf{SH}(X) \arrow[r, "\alpha^\sharp"] & \mathbf{SH}(Y).
\end{tikzcd}$$

By uniqueness of adjoints, there is an isomorphism

$$r^X_q\alpha^*\xrightarrow{\cong}\alpha^*r^Y_q,$$

and hence

$$\mathsf{f}^X_q\alpha^*=r^X_q i^X_q\alpha^*=r^X_q\alpha^*i^Y_q\cong\alpha^*r^Y_q i^Y_q=\alpha^*\mathsf{f}^Y_q. \tag{13}$$

The desired isomorphism for slices follows since, by uniqueness of adjoints, (13) is compatible with the natural transformation $\mathsf{f}_{q+1} \longrightarrow \mathsf{f}_q$. □

Theorem 2.5 *Suppose that $\mathcal{I}$ is a filtered partially ordered set and $D\colon \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{Sm}_Y$, $i \longmapsto X_i$ is a diagram of Y–schemes with affine bonding maps. Let*

$$\alpha\colon X \equiv \lim_{i\in\mathcal{I}} X_i \longrightarrow Y$$

be the naturally induced morphism. For every $q \in \mathbb{Z}$ there are natural isomorphisms

$$\mathsf{s}_q^X \alpha^* \xrightarrow{\cong} \alpha^* \mathsf{s}_q^Y \quad \text{and} \quad \mathsf{f}_q^X \alpha^* \xrightarrow{\cong} \alpha^* \mathsf{f}_q^Y.$$

Proof Let $\mathcal{G}$ be a compact generator of the triangulated category $\Sigma^{2q+2,q+1}\mathbf{SH}^{\mathrm{eff}}(X)$ and $\mathsf{E} \in \mathbf{SH}(Y)$. According to [41, Theorem 2.12] it suffices to show $[\mathcal{G}, \alpha^*\mathsf{s}_q^Y(\mathsf{E})]$ is trivial. Note that $\mathcal{G}$ is of the form $\alpha(j)^*\mathcal{G}_j$ for $\mathcal{G}_j$ a compact generator of the triangulated category $\Sigma^{2q+2,q+1}\mathbf{SH}^{\mathrm{eff}}(X_j)$. We consider the overcategory of an arbitrary element $j \in \mathcal{I}$ and the composite diagram

$$D \downarrow j\colon \mathcal{I}^{\mathrm{op}} \downarrow j \longrightarrow \mathcal{I}^{\mathrm{op}} \xrightarrow{D} \mathrm{Sm}_Y, \quad (j \to i) \mapsto X_i.$$

We claim the functor $\Phi\colon \mathcal{I}^{\mathrm{op}} \downarrow j \longrightarrow \mathcal{I}^{\mathrm{op}}$ yields an isomorphism $\operatorname{colim} D \downarrow j \longrightarrow \operatorname{colim} D$. For $i \in \mathcal{I}$ there exists an object i' and maps $i \to i' \leftarrow j$, since $\mathcal{I}$ is filtered. Thus $\Phi \downarrow i$ is nonempty. For zig-zags $i \to i' \leftarrow j$ and $i \to i'' \leftarrow j$ there exist maps $i' \to \ell \leftarrow i''$ such that the induced maps from i to ℓ coincide, and similarly for j ($\mathcal{I}$ is a partially ordered set). Hence $\Phi \downarrow i$ is connected. For $j \to i$ in $\mathcal{I}^{\mathrm{op}} \downarrow j$, let $e(i)\colon X_i \longrightarrow X_j$ be the structure map of the diagram. Thus the map

$$\operatorname*{hocolim}_{j\downarrow\mathcal{I}} e(i)_* e(i)^* \mathsf{F} \longrightarrow \alpha(j)_*\alpha(j)^*\mathsf{F}$$

is an isomorphism in $\mathbf{SH}(X_j)$ for all $\mathsf{F} \in \mathbf{SH}(X_j)$. Since $\mathcal{G}_j$ is compact and slices commute with base change along smooth maps by Lemma 2.4, the group

$$\begin{aligned}
[\mathcal{G}, \alpha^*\mathsf{s}_q^Y(\mathsf{E})] &= [\alpha(j)^*\mathcal{G}_j, \alpha^*\mathsf{s}_q^Y(\mathsf{E})] \\
&\cong [\mathcal{G}_j, \alpha(j)_*\alpha(j)^*\beta(j)^*\mathsf{s}_q^Y(\mathsf{E})] \\
&\cong \Big[\mathcal{G}_j, \operatorname*{hocolim}_{j\downarrow\mathcal{I}} e(i)_*e(i)^*\beta(j)^*\mathsf{s}_q^Y(\mathsf{E})\Big] \\
&\cong \operatorname*{colim}_{j\downarrow\mathcal{I}} [e(i)^*\mathcal{G}_j, e(i)^*\beta(j)^*\mathsf{s}_q^Y(\mathsf{E})]
\end{aligned}$$

is trivial. With reference to [41, Remark 2.13] the second isomorphism follows in the same way. □

Lemma 2.6 *If $\alpha\colon A \longrightarrow B$ is a regular ring map then $\mathsf{s}_q^B\alpha^* \cong \alpha^*\mathsf{s}_q^A$.*

Proof By Popescu's general Néron desingularization theorem [43] the regularity assumption on α is equivalent to B being the colimit of a filtered diagram of smooth A–algebras of finite type. The result follows now from Theorem 2.5. □

Corollary 2.7 *If $\alpha\colon F \longrightarrow E$ is a separable field extension then $\mathsf{s}_q^E\alpha^* \cong \alpha^*\mathsf{s}_q^F$.*

3 Algebraic and hermitian K–theory

Let $\mathbf{KGL} = (K, \dots)$ denote the motivic spectrum representing algebraic K–theory [56, Section 6.2] over the base scheme S. Here $K\colon \mathrm{Sm}_S \longrightarrow \mathbf{Spt}$ sends a smooth S–scheme to its algebraic K–theory spectrum. The structure maps of $\mathbf{KGL}$ are given by the Bott periodicity operator

$$\beta\colon K \xrightarrow{\simeq} \Omega^{2,1}K.$$

Suppose 2 is invertible in the ring of regular functions on S. Let $\mathrm{KO}\colon \mathrm{Sm}_S \longrightarrow \mathbf{Spt}$ denote the functor sending a smooth S–scheme to the (non-connective) spectrum representing the hermitian K–groups for the trivial involution on S and sign of symmetry $\varepsilon = +1$. Similarly, when $\varepsilon = -1$, we use the notation $\mathrm{KSp}\colon \mathrm{Sm}_S \longrightarrow \mathbf{Spt}$.

There are natural maps $f_0\colon \mathrm{KO} \longrightarrow K$ and $f_2\colon \mathrm{KSp} \longrightarrow K$ induced by the forgetful functors. Taking the homotopy fibers of these forgetful maps yields the homotopy fiber sequences

$$\begin{gathered}\Omega^{1,0}K \xrightarrow{h_3'} \mathrm{VQ} \xrightarrow{\mathrm{can}} \mathrm{KO} \xrightarrow{f_0} K,\\ \Omega^{1,0}K \xrightarrow{h_1'} \mathrm{VSp} \xrightarrow{\mathrm{can}} \mathrm{KSp} \xrightarrow{f_2} K.\end{gathered} \tag{14}$$

Moreover, there are natural maps

$$h_0\colon K \longrightarrow \mathrm{KO} \quad \text{and} \quad h_2\colon K \longrightarrow \mathrm{KSp}$$

induced by the hyperbolic functors. Taking the homotopy fibers of these hyperbolic maps yields the homotopy fiber sequences

$$\begin{gathered}\Omega^{1,0}\mathrm{KO} \xrightarrow{\mathrm{can}} \mathrm{UQ} \xrightarrow{f_3} K \xrightarrow{h_0} \mathrm{KO},\\ \Omega^{1,0}\mathrm{KSp} \xrightarrow{\mathrm{can}} \mathrm{USp} \xrightarrow{f_1} K \xrightarrow{h_2} \mathrm{KSp}.\end{gathered} \tag{15}$$

Karoubi's fundamental theorem in hermitian K–theory [22] can be formulated as follows.

Theorem 3.1 (Karoubi) *There are natural weak equivalences*

$$\phi\colon \Omega^{1,0}\mathrm{USp} \xrightarrow{\simeq} \mathrm{VQ} \quad \text{and} \quad \psi\colon \Omega^{1,0}\mathrm{UQ} \xrightarrow{\simeq} \mathrm{VSp}.$$

In their foundational paper on localization in hermitian K–theory of rings [17, Section 1.8], Hornbostel and Schlichting show the following result.

Theorem 3.2 (Hornbostel and Schlichting) *The homotopy cofiber of the maps*

$$\mathrm{KO} \longrightarrow \mathrm{KO}(\mathbb{A}^1 \smallsetminus \{0\} \times_S -) \quad \textit{and} \quad \mathrm{KSp} \longrightarrow \mathrm{KSp}(\mathbb{A}^1 \smallsetminus \{0\} \times_S -)$$

induced by the map $\mathbb{A}^1 \smallsetminus \{0\} \longrightarrow S$ *are naturally weakly equivalent to* $\Sigma^{1,0}\,\mathrm{UQ}$ *and* $\Sigma^{1,0}\,\mathrm{USp}$*, respectively.*

The homotopy cofiber sequences in Theorem 3.2 split by the unit section $1 \in \mathbb{A}^1 \smallsetminus \{0\}(S)$. Since $\Omega^{1,1}$ is the homotopy fiber with respect to the unit section, Theorem 3.2 implies the natural weak equivalences

$$\Omega^{2,1}\,\mathrm{KO} \sim \Omega^{1,0}\Sigma^{1,0}\,\mathrm{UQ} \xleftarrow{\simeq} \mathrm{UQ} \quad \text{and} \quad \Omega^{2,1}\,\mathrm{KSp} \sim \Omega^{1,0}\Sigma^{1,0}\,\mathrm{USp} \xleftarrow{\simeq} \mathrm{USp}.$$

In [15] Hornbostel shows that hermitian K–theory is represented by the motivic spectrum

$$\mathbf{KQ} = (\mathrm{KO}, \mathrm{USp}, \mathrm{KSp}, \mathrm{UQ}, \mathrm{KO}, \mathrm{USp}, \dots).$$

Our notation emphasizes the connection between hermitian K–theory and quadratic forms. The structure maps of $\mathbf{KQ}$ are the adjoints of the weak equivalences

$$\begin{aligned} &\mathrm{KO} \xrightarrow{\simeq} \Omega^{2,1}\,\mathrm{USp}, \qquad && \mathrm{USp} \xrightarrow{\simeq} \Omega^{2,1}\,\mathrm{KSp}, \\ &\mathrm{KSp} \xrightarrow{\simeq} \Omega^{2,1}\,\mathrm{UQ}, \qquad && \mathrm{UQ} \xrightarrow{\simeq} \Omega^{2,1}\,\mathrm{KO}. \end{aligned} \tag{16}$$

Proposition 3.3 *There are commutative diagrams:*

$$\begin{tikzcd} \mathrm{KO} \arrow[r, "f_0"] \arrow[d, "\sim"] & K \arrow[d, "\beta", "\sim"'] \\ \Omega^{2,1}\,\mathrm{USp} \arrow[r, "\Omega^{2,1} f_1"] & \Omega^{2,1} K \end{tikzcd} \qquad \begin{tikzcd} \mathrm{USp} \arrow[r, "f_1"] \arrow[d, "\sim"] & K \arrow[d, "\beta", "\sim"'] \\ \Omega^{2,1}\,\mathrm{KSp} \arrow[r, "\Omega^{2,1} f_2"] & \Omega^{2,1} K \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{KSp} \arrow[r, "f_2"] \arrow[d, "\sim"] & K \arrow[d, "\beta", "\sim"'] \\ \Omega^{2,1}\,\mathrm{UQ} \arrow[r, "\Omega^{2,1} f_3"] & \Omega^{2,1} K \end{tikzcd} \qquad \begin{tikzcd} \mathrm{UQ} \arrow[r, "f_3"] \arrow[d, "\sim"] & K \arrow[d, "\beta", "\sim"'] \\ \Omega^{2,1}\,\mathrm{KO} \arrow[r, "\Omega^{2,1} f_0"] & \Omega^{2,1} K \end{tikzcd}$$

The forgetful map $f\colon \mathbf{KQ} \longrightarrow \mathbf{KGL}$ is given by the sequence $(f_0, f_1, f_2, f_3, f_0, \dots)$ of maps of pointed motivic spaces displayed in diagrams (14) and (15). Similarly, the hyperbolic map $h\colon \mathbf{KGL} \longrightarrow \mathbf{KQ}$ is given by the sequence $(h_0, h_1, h_2, h_3, h_0, \dots)$. Here h_0 and h_2 have been introduced before, and h_1 and h_3 are defined by the weak equivalences from Theorem 3.1 and the canonical maps h'_1 and h'_3 introduced with the construction of VQ and VSp in diagram (14). By inspection of the structure maps of $\mathbf{KQ}$ determined by (16), devising a map $\mathbf{KQ} \longrightarrow \Omega^{1,1}\mathbf{KQ}$ of motivic spectra is tantamount to giving a compatible sequence of maps between motivic spaces

$$\mathrm{KO} \longrightarrow \Sigma^{1,0}\,\mathrm{UQ}, \quad \mathrm{USp} \longrightarrow \Sigma^{1,0}\,\mathrm{KO}, \quad \mathrm{KSp} \longrightarrow \Sigma^{1,0}\,\mathrm{USp}, \quad \mathrm{UQ} \longrightarrow \Sigma^{1,0}\,\mathrm{KSp}, \ldots.$$

Next we show the homotopy cofiber sequence (5) relating algebraic and hermitian K–theory via the stable Hopf map, as reviewed in the introduction. A closely related statement is obtained in [48, Theorem 6.1].

Theorem 3.4 *The stable Hopf map and the forgetful map yield a homotopy cofiber sequence*

$$\Sigma^{1,1}\mathbf{KQ} \xrightarrow{\eta} \mathbf{KQ} \xrightarrow{f} \mathbf{KGL} \longrightarrow \Sigma^{2,1}\mathbf{KQ}.$$

The connecting map factors as $\mathbf{KGL} \xrightarrow{\cong} \Sigma^{2,1}\mathbf{KGL} \xrightarrow{\Sigma^{2,1}h} \Sigma^{2,1}\mathbf{KQ}$.

Proof We show the map $\mathbf{KQ} \longrightarrow \Omega^{1,1}\mathbf{KQ}$ induced by η is determined by the canonical maps

$$\begin{aligned} &\mathrm{KO} \longrightarrow \Sigma^{1,0}\,\mathrm{UQ}, && \mathrm{USp} \sim \Sigma^{1,0}\,\mathrm{VQ} \longrightarrow \Sigma^{1,0}\,\mathrm{KO}, \\ &\mathrm{KSp} \longrightarrow \Sigma^{1,0}\,\mathrm{USp}, && \mathrm{UQ} \sim \Sigma^{1,0}\,\mathrm{VSp} \longrightarrow \Sigma^{1,0}\,\mathrm{KSp}\,. \end{aligned}$$

To that end, it suffices to describe the maps

$$\begin{aligned} &\mathrm{KO}(\mathbb{P}^1) \longrightarrow \mathrm{KO}(\mathbb{A}^2 \smallsetminus \{0\}), && \mathrm{USp}(\mathbb{P}^1) \longrightarrow \mathrm{USp}(\mathbb{A}^2 \smallsetminus \{0\}), \\ &\mathrm{KSp}(\mathbb{P}^1) \longrightarrow \mathrm{KSp}(\mathbb{A}^2 \smallsetminus \{0\}), && \mathrm{UQ}(\mathbb{P}^1) \longrightarrow \mathrm{UQ}(\mathbb{A}^2 \smallsetminus \{0\}) \end{aligned}$$

induced by the unstable Hopf map $\mathbb{A}^2 \smallsetminus \{0\} \longrightarrow \mathbb{P}^1$, or equivalently (see [37, page 73 in Section 3.3 and Example 7.26]) by the Hopf construction applied to the map

$$\Upsilon\colon (\mathbb{A}^1 \smallsetminus \{0\}) \times_S (\mathbb{A}^1 \smallsetminus \{0\}) \longrightarrow \mathbb{A}^1 \smallsetminus \{0\}, \quad (x,y) \longmapsto xy^{-1}.$$

We note there is an isomorphism of schemes

$$\theta\colon (\mathbb{A}^1 \smallsetminus \{0\}) \times_S (\mathbb{A}^1 \smallsetminus \{0\}) \longrightarrow (\mathbb{A}^1 \smallsetminus \{0\}) \times_S (\mathbb{A}^1 \smallsetminus \{0\}), \quad (x,y) \longmapsto (xy,y).$$

Now the composite $\Upsilon\theta$ is the projection map on the first factor. Thus for KO, USp, KSp and UQ, the homotopy cofibers of the maps induced by Υ and the projection map coincide up to weak equivalence. The latter homotopy cofibers are given in Theorem 3.2. □

Lemma 3.5 *The unit map* $\mathbf{1} \longrightarrow \mathbf{KGL}$ *factors as* $\mathbf{1} \longrightarrow \mathbf{KQ} \xrightarrow{f} \mathbf{KGL}$.

Proof The unit map $\mathbf{1} \longrightarrow \mathbf{KGL}$ is given by the trivial line bundle over the base scheme S. The latter is obtained by forgetting the standard nondegenerate quadratic form on it, which provides the factorization. □

Let $\epsilon\colon \mathbf{1} \longrightarrow \mathbf{1}$ be the endomorphism of the sphere spectrum induced by the commutativity isomorphism on the smash product $S^{1,1} \wedge S^{1,1}$.

Lemma 3.6 *The composition* $\mathbf{KQ} \xrightarrow{f} \mathbf{KGL} \xrightarrow{h} \mathbf{KQ}$ *coincides with multiplication by* $1-\epsilon$.

Proof Since $1-\epsilon = 1 + \langle -1\rangle$ is the hyperbolic plane [37, page 53] the unit map for $\mathbf{KQ}$ induces a commutative diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, "1-\epsilon"'] & \mathbf{KQ} \arrow[d, "h\circ f"] \\ \mathbf{1} \arrow[r] & \mathbf{KQ}. \end{tikzcd}$$

By smashing with $\mathbf{KQ}$ and employing its multiplicative structure we obtain the diagram

$$\begin{tikzcd} \mathbf{KQ} \arrow[r] \arrow[d, "\mathbf{KQ}\wedge(1-\epsilon)"'] & \mathbf{KQ}\wedge\mathbf{KQ} \arrow[r] \arrow[d] & \mathbf{KQ} \arrow[d, "h\circ f"] \\ \mathbf{KQ} \arrow[r] & \mathbf{KQ}\wedge\mathbf{KQ} \arrow[r] & \mathbf{KQ}. \end{tikzcd}$$

The middle vertical map is $\mathbf{KQ}\wedge(h\circ f)$, while the two composite horizontal maps coincide with the identity on $\mathbf{KQ}$. The right-hand square commutes because $h\circ f$ is a map of $\mathbf{KQ}$–modules. □

4 Slices

This section contains a determination of the slices of hermitian K–theory and Witt theory over any field of characteristic not two. These are the first examples of non-orientable motivic spectra for which all slices are explicitly known. Our starting point is the computation of the slices of algebraic K–theory.

4.1 Algebraic K–theory

We recall and augment previous work on the slices of algebraic K–theory.

Theorem 4.1 (Levine, Voevodsky) *The unit map* $\mathbf{1} \longrightarrow \mathbf{KGL}$ *induces an isomorphism of zero slices*

$$\mathsf{s}_0(\mathbf{1}) \longrightarrow \mathsf{s}_0(\mathbf{KGL}).$$

Hence there is an isomorphism $\mathsf{s}_q(\mathbf{KGL}) \cong \Sigma^{2q,q}\mathbf{MZ}$ *for all* $q \in \mathbb{Z}$.

Proof By [27], [60], and [63], and base change to any field as in [18], the unit $\mathbf{1} \longrightarrow \mathbf{KGL}$ induces a map

$$\mathbf{MZ} \cong \mathsf{s}_0(\mathbf{1}) \longrightarrow \mathsf{s}_0(\mathbf{KGL}) \cong \mathbf{MZ}$$

corresponding to multiplication by an integer $i \in \mathbf{MZ}_{0,0} \cong \mathbb{Z}$. Now $\mathsf{s}_0(\mathbf{1}) \longrightarrow \mathsf{s}_0(\mathbf{KGL})$ is a map of ring spectra by multiplicativity of the slice filtration [40; 13, Theorem 5.19]. It follows that $i = 1$. Moreover, the q^{th} power $\beta^q \colon S^{2q,q} \longrightarrow \mathbf{KGL}$ of the Bott map induces an isomorphism

$$\Sigma^{2q,q}\mathbf{MZ} \cong \mathsf{s}_q(S^{2q,q}) \longrightarrow \mathsf{s}_q(\mathbf{KGL}). \qquad \square$$

4.2 Homotopy orbit K–theory

Let $\mathbb{P}^1$ be pointed at ∞. The general linear group scheme $\mathbb{GL}(2n)$ acquires an involution given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix}^{-1}.$$

Geometrically, this corresponds to a strictification of the pseudo-involution obtained by sending a vector bundle of rank n to its dual. These involutions induce the inverse-transpose involution on the infinite general linear group scheme $\mathbb{GL}$ and its classifying space $\mathrm{B}\mathbb{GL}$ [55]. Letting C_2 operate trivially on the first factor in $\mathbb{Z} \times \mathrm{B}\mathbb{GL}$, the involution coincides sectionwise with the unstable Adams operation Ψ^{-1} on the motivic space representing algebraic K–theory. The stable Adams operation

$$\Psi^{-1}_{\mathrm{st}} \colon \mathbf{KGL} \longrightarrow \mathbf{KGL}$$

is determined by the structure map

$$K \longrightarrow \Omega^{2,1}K = \mathrm{hofib}(K(- \times \mathbb{P}^1) \xrightarrow{\infty^*} K(-))$$

obtained from multiplication by the class $1 - [\mathcal{O}(-1)] \in K_0(\mathbb{P}^1)$ of the hyperplane section. Note that

$$\Psi^{-1}(1-[\mathcal{O}(-1)]) = 1-[\mathcal{O}(1)] = 1-(1+(1-[\mathcal{O}(-1)])) = [\mathcal{O}(-1)]-1 = -(1-[\mathcal{O}(-1)]).$$

Thus, up to homotopy, the stable operation $\Psi^{-1}_{\mathrm{st}}\colon \mathbf{KGL} \longrightarrow \mathbf{KGL}$ is given levelwise on motivic spaces by the formula

$$\Psi^{-1}_{\mathrm{st},n} = \begin{cases} \Psi^{-1}, & n \equiv 0 \bmod 2, \\ -\Psi^{-1}, & n \equiv 1 \bmod 2. \end{cases}$$

Since smashing with $S^{2,1}$ shifts motivic spectra by one index we get

$$\Sigma^{2,1}\Psi^{-1}_{\mathrm{st}} = -\Psi^{-1}_{\mathrm{st}}. \tag{17}$$

When forming homotopy fixed points and homotopy orbits of $\mathbf{KGL}$ we implicitly make use of a naive C_2–equivariant motivic spectrum, ie a non-equivariant motivic spectrum with a C_2–action (given by Ψ^{-1}_{st}) which maps by a levelwise weak equivalence to $\mathbf{KGL}$. Recall the Witt theory spectrum $\mathbf{KT}$ is the homotopy colimit of the sequential diagram

$$\mathbf{KQ} \xrightarrow{\eta} \Sigma^{-1,-1}\mathbf{KQ} \xrightarrow{\Sigma^{-1,-1}\eta} \Sigma^{-2,-2}\mathbf{KQ} \xrightarrow{\Sigma^{-2,-2}\eta} \cdots . \tag{18}$$

Note that $\mathbf{KT}$ is a motivic ring spectrum equipped with an evident $\mathbf{KQ}$–algebra structure. Our notation follows [15] and reminds us of the fact that $\mathbf{KT}$ is an example of a "Tate spectrum" or more precisely "geometric fixed point spectrum" via the homotopy cofiber sequence relating it to $\mathbf{KQ}$ and homotopy orbit algebraic K–theory [24].

Theorem 4.2 (Kobal) *There is a homotopy cofiber sequence*

$$\mathbf{KGL}_{hC_2} \longrightarrow \mathbf{KQ} \longrightarrow \mathbf{KT}.$$

The involution on $\mathbf{KGL}$ induces an $\mathbf{MZ}$–linear involution on the slices $\mathsf{s}_q(\mathbf{KGL}) \cong \Sigma^{2q,q}\mathbf{MZ}$. Suspensions of $\mathbf{MZ}$ allow only two possible involutions, namely the identity (trivial involution) and the multiplication by -1 map (nontrivial involution). This follows because $\mathbf{MZ}^{\times}_{0,0} \cong \{\pm 1\}$.

Proposition 4.3 *Let* $(\mathbf{KGL}, \Psi^{-1}_{\mathrm{st}})$ *be the motivic K–theory spectrum with its Adams involution. The induced involution on* $\mathsf{s}_q(\mathbf{KGL}) \cong \Sigma^{2q,q}\mathbf{MZ}$ *is nontrivial if q is odd and trivial if q is even.*

Proof This follows from (17) and the equality $\mathsf{s}_0(\Psi^{-1}) = \mathrm{id}_{\mathbf{MZ}}$. For the latter, note that the Adams involution and the unit map of $\mathbf{KGL}$ yield commutative diagrams:

$$\begin{tikzcd} & \mathbf{KGL} \arrow[d, "\Psi^{-1}_{\mathrm{st}}"] \\ \mathbf{1} \arrow[ur, "\iota"] \arrow[dr, "\iota"'] & \\ & \mathbf{KGL} \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{s}_0(\mathbf{KGL}) \arrow[d, "\mathsf{s}_0(\Psi^{-1}_{\mathrm{st}})"] \\ \mathsf{s}_0(\mathbf{1}) \arrow[ur, "\mathsf{s}_0(\iota)"] \arrow[dr, "\mathsf{s}_0(\iota)"'] & \\ & \mathsf{s}_0(\mathbf{KGL}) \end{tikzcd}$$

By reference to Theorem 4.1 it follows that $\mathsf{s}_0(\Psi^{-1}_{\mathrm{st}}) = \mathrm{id}_{\mathbf{MZ}}$. □

Next we consider the composite of the hyperbolic and forgetful maps

$$f \circ h = \Psi^1_{\mathrm{st}} + \Psi^{-1}_{\mathrm{st}} \colon \mathbf{KGL} \longrightarrow \mathbf{KGL}. \tag{19}$$

Proposition 4.4 *The endomorphism $\mathsf{s}_q(fh)$ of $\mathsf{s}_q(\mathbf{KGL}) \cong \Sigma^{2q,q}\mathbf{MZ}$ is multiplication by 2 if q is even and the trivial map if q is odd.*

Proof This follows from Proposition 4.3, the additivity of s_q and the equality $\mathsf{s}_q(\Psi^1_{\mathrm{st}}) = \mathsf{s}_q(\mathrm{id}_{\mathbf{KGL}}) = \mathrm{id}_{\mathsf{s}_q(\mathbf{KGL})}$. □

Lemma 4.5 *The slice functor s_q commutes with homotopy colimits for all $q \in \mathbb{Z}$.*

Proof By a general result for Quillen adjunctions between stable pointed model categories shown by Spitzweck [49, Lemma 4.4], the conclusion follows because s_q commutes with sums. □

The idea is now to combine Theorem 4.1, Proposition 4.3 and Lemma 4.5 to identify the slices of homotopy orbit K–theory. To start with, we compute the homotopy orbit spectra of $\mathbf{MZ}$ for the trivial involution and the unique nontrivial involution over a base scheme essentially smooth over a field. These computations are parallel to the corresponding computations for the topological integral Eilenberg–MacLane spectrum $H\mathbb{Z}$, as the proofs suggest. In the case where F is a subfield of the complex numbers, complex realization — which is compatible with homotopy colimits, and sends $\mathbf{MZ}$ to $H\mathbb{Z}$ by [29, Lemma 5.6] — maps the motivic computation to the topological one.

Lemma 4.6 *With the trivial involution on motivic cohomology there is an isomorphism*

$$(\mathbf{MZ}, \mathrm{id})_{hC_2} \cong \mathbf{MZ} \vee \textstyle\bigvee_{i=0}^{\infty} \Sigma^{2i+1,0}\mathbf{MZ}/2.$$

Proof If E is equipped with an involution then $\mathsf{E}_{hC_2} \cong (\mathsf{E} \wedge (EC_2)_+)_{C_2}$, where EC_2 is a contractible simplicial set with a free C_2–action considered as a constant motivic space. In the case of $(\mathbf{MZ}, \mathrm{id})$,

$$(\mathbf{MZ}, \mathrm{id})_{hC_2} \cong (\mathbf{MZ} \wedge (EC_2)_+)_{C_2} \cong \mathbf{MZ} \wedge (EC_2)_+/C_2 \cong \mathbf{MZ} \wedge \mathbb{RP}^\infty_+.$$

Since $S^{n,0} \longrightarrow \mathbb{RP}^n \longrightarrow \mathbb{RP}^{n+1}$ is a homotopy cofiber sequence, so is

$$\Sigma^{n,0}\mathbf{MZ} \longrightarrow \mathbf{MZ} \wedge \mathbb{RP}^n \longrightarrow \mathbf{MZ} \wedge \mathbb{RP}^{n+1}. \tag{20}$$

Clearly $\mathbf{MZ} \wedge \mathbb{RP}^2 \cong \Sigma^{1,0}\mathbf{MZ}/2$. Proceeding by induction on (20) using Lemma A.3, we obtain

$$\mathbf{MZ} \wedge \mathbb{RP}^n \cong \begin{cases} \bigvee_{i=0}^{n-2} \Sigma^{2i+1,0}\mathbf{MZ}/2, & n \equiv 0 \bmod 2, \\ \Sigma^{n,0}\mathbf{MZ} \vee \bigvee_{i=0}^{n-3} \Sigma^{2i+1,0}\mathbf{MZ}/2, & n \equiv 1 \bmod 2. \end{cases}$$
□

Lemma 4.7 *With the nontrivial involution σ on motivic cohomology there is an isomorphism*

$$(\mathbf{MZ},\sigma)_{hC_2} \cong \bigvee_{i=0}^{\infty} \Sigma^{2i,0}\mathbf{MZ}/2.$$

Proof The nontrivial involution σ on $\mathbf{MZ}$ is determined levelwise by the nontrivial involution on

$$\mathbf{MZ}_n = K(\mathbb{Z},2n,n) = \mathbb{Z}^{\mathrm{tr}}(S^{2n,n}).$$

Here, $\mathbb{Z}^{\mathrm{tr}}$ sends a motivic space $\mathcal{X}$ to the motivic space with transfers freely generated by $\mathcal{X}$; see [9, Example 3.4]. On the level of motivic spaces the nontrivial involution is induced by a degree -1 pointed map of the simplicial circle $S^{1,0}$. Since $\mathbb{Z}^{\mathrm{tr}}$ commutes with homotopy colimits [46, Section 2] we are reduced to identifying

$$\operatorname*{hocolim}_{C_2} S^{2n,n}.$$

When $n=0$ the homotopy colimit is contractible. When $n>0$ there are isomorphisms

$$\operatorname*{hocolim}_{C_2} S^{2n,n} \cong \operatorname*{hocolim}_{C_2} \Sigma^{2n-1,n} S^{1,0} \cong \Sigma^{2n-1,n}(\operatorname*{hocolim}_{C_2} S^{1,0}) \cong \Sigma^{2n-1,n}\mathbb{RP}^{\infty}.$$

By passing to the sphere spectrum the above yields an isomorphism

$$\operatorname*{hocolim}_{C_2} \mathbf{1} \cong \Sigma^{-1,0}\mathbb{RP}^{\infty}.$$

Finally, applying transfers and arguing as in the proof of Lemma 4.6, we deduce the isomorphism

$$\operatorname*{hocolim}_{hC_2}(\mathbf{MZ},\sigma) \cong \bigvee_{i=0}^{\infty} \Sigma^{2i,0}\mathbf{MZ}/2. \qquad \square$$

Theorem 4.8 *Suppose F is a field equipped with the trivial involution and* char $F \neq 2$. *The slices of homotopy orbit K–theory are given by*

$$\mathsf{s}_q(\mathbf{KGL}_{hC_2}) \cong \begin{cases} \Sigma^{2q,q}\mathbf{MZ} \vee \bigvee_{i=\frac{q}{2}}^{\infty} \Sigma^{q+2i+1,q}\mathbf{MZ}/2, & q \equiv 0 \bmod 2, \\ \bigvee_{i=(q-1)/2}^{\infty} \Sigma^{q+2i+1,q}\mathbf{MZ}/2, & q \equiv 1 \bmod 2. \end{cases}$$

Proof This follows from Theorem 4.1, Proposition 4.3 and Lemmas 4.5–4.7. $\square$

4.3 Hermitian K–theory

Throughout this section F is a field of char $F \neq 2$.

Corollary 4.9 *There is a splitting of* $\mathbf{MZ}$*–modules*

$$\mathsf{s}_0(\mathbf{KQ}) \cong \mathsf{s}_0(\mathbf{KGL}) \vee \mu$$

which identifies $\mathsf{s}_0(f)$ with the projection map onto $\mathsf{s}_0(\mathbf{KGL})$.

Proof Lemma 3.5 shows the unit of $\mathbf{KQ}$ and the forgetful map to $\mathbf{KGL}$ furnish a factorization

$$\mathbf{1} \xrightarrow{\iota} \mathbf{KQ} \xrightarrow{f} \mathbf{KGL}$$

of the unit of $\mathbf{KGL}$. By Theorem 4.1 the composite

$$\mathsf{s}_0(\mathbf{1}) \xrightarrow{\mathsf{s}_0(\iota)} \mathsf{s}_0(\mathbf{KQ}) \xrightarrow{\mathsf{s}_0(f)} \mathsf{s}_0(\mathbf{KGL})$$

is an isomorphism. The desired splitting follows since retracts are direct summands in $\mathbf{SH}$. □

Proposition 4.10 *The composite* $\mathbf{KQ} \xrightarrow{f} \mathbf{KGL} \xrightarrow{h} \mathbf{KQ}$ *induces the multiplication by* 2 *map on* $\mathsf{s}_q(\mathbf{KQ})$ *for all* $q \in \mathbb{Z}$.

Proof The unit of $\mathbf{MZ}$ induces an isomorphism on zero slices by Theorem 4.1. Since $\mathbf{MZ} \in \mathbf{SH}^{\mathrm{eff}}$ is an effective motivic spectrum, the counit $\mathsf{f}_0(\mathbf{MZ}) \longrightarrow \mathbf{MZ}$ is an isomorphism. Thus there is a canonical isomorphism of ring spectra

$$\mathbf{MZ} \cong \mathsf{f}_0(\mathbf{MZ}) \longrightarrow \mathsf{s}_0(\mathbf{MZ}) \cong \mathbf{MZ}.$$

It follows that smashing with $\mathbf{MZ}$, ie passing to motives [45; 46], induces a canonical ring map

$$[\mathbf{1},\mathbf{1}] \longrightarrow [\mathsf{s}_0(\mathbf{1}),\mathsf{s}_0(\mathbf{1})] \cong [\mathbf{MZ},\mathbf{MZ}].$$

Lemma 3.6 shows the composite hf coincides with $1-\epsilon$. Its image in motives is multiplication by 2 because the twist isomorphism of $\mathbb{Z}(1)\otimes\mathbb{Z}(1)$ is the identity map [58, Corollary 2.1.5]. □

Corollary 4.11 *In the splitting* $\mathsf{s}_0(\mathbf{KQ}) \cong \mathsf{s}_0(\mathbf{KGL}) \vee \mu$ *the homotopy groups* $\pi_{s,t}\mu$ *are modules over* $\pi_{0,0}\mathbf{MZ}/2 \cong \mathbb{F}_2$.

Proof This follows from Corollary 4.9 and Proposition 4.10. □

The slice functors are triangulated. Thus the homotopy cofiber sequence in Theorem 3.4 induces homotopy cofiber sequences of slices.

Lemma 4.12 *There is a distinguished triangle of* $\mathbf{MZ}$*–modules*

$$\Sigma^{1,1}\mathbf{MZ} \xrightarrow{\Sigma^{1,1}(2,0)} \Sigma^{1,1}(\mathbf{MZ}\vee\mu) \longrightarrow \mathsf{s}_1(\mathbf{KQ}) \longrightarrow \Sigma^{2,1}\mathbf{MZ},$$

where $2 \in \mathbb{Z} \cong \mathbf{MZ}_{0,0}$ *and* $0 \in \pi_{0,0}\mu$.

Proof The distinguished triangle follows from Theorems 3.4 and 4.1, Lemma 2.1 and Corollary 4.9. Since induced maps between slices are module maps over $\mathsf{s}_0(\mathbf{1}) \cong \mathbf{MZ}$, we have

$$\mathrm{Hom}_{\mathbf{MZ}}(\Sigma^{1,1}\mathbf{MZ}, \Sigma^{1,1}(\mathbf{MZ} \vee \mu)) \cong \mathrm{Hom}_{\mathbf{SH}}(\mathbf{1}, \mathbf{MZ} \vee \mu) \cong \mathbf{MZ}_{0,0} \oplus \pi_{0,0}\mu.$$

This shows $\mathbf{MZ} \longrightarrow \mathbf{MZ} \vee \mu$ is of the form $(a, \alpha) \in \mathbb{Z} \oplus \pi_{0,0}\mu$. By the proof of Corollary 4.9, $\mathsf{s}_0(f)$ is the projection map onto the direct summand $\mathsf{s}_0(\mathbf{KGL})$. The connecting map in the distinguished triangle identifies with the $(1,1)$–shift of the hyperbolic map h. It follows that $\mathsf{s}_{-1}(h) = 0$. By Proposition 4.10, $\mathsf{s}_0(hf)$ is multiplication by 2. Thus $a = 2$ and $\pi_{0,0}\mathsf{s}_0(\mathbf{KGL}) \longrightarrow \pi_{0,0}\mathsf{s}_0(\mathbf{KQ})$ is injective. Consider the short exact sequence

$$0 \longrightarrow \pi_{0,0}\mathsf{s}_0(\mathbf{KGL}) \longrightarrow \pi_{0,0}\mathsf{s}_0(\mathbf{KQ}) \longrightarrow \pi_{0,0}\operatorname{cone}(2,\alpha) \longrightarrow \pi_{-1,0}\mathsf{s}_0\mathbf{KGL} = 0.$$

Proposition 4.10 shows that $\mathsf{s}_1(\mathbf{KQ}) \longrightarrow \mathsf{s}_1(\mathbf{KGL}) \longrightarrow \mathsf{s}_1(\mathbf{KQ})$ is multiplication by 2. Hence the same holds for the induced map

$$\pi_{0,0}\operatorname{cone}(2,\alpha) \longrightarrow \pi_{0,0}\Sigma^{1,0}\mathbf{MZ} = 0 \longrightarrow \pi_{0,0}\operatorname{cone}(2,\alpha).$$

It follows that $\pi_{0,0}\operatorname{cone}(2,\alpha)$ is an $\mathbb{F}_2$–module. We note that the image of $(1,\alpha)$ has order 4 unless $\alpha = 0$: Write $\pi_{0,0}\operatorname{cone}(2,\alpha)$ as the cokernel of $(2,\alpha)\colon \mathbb{Z} \longrightarrow \mathbb{Z} \oplus A$, A an $\mathbb{F}_2$–module. Its subgroup generated by $\overline{(1,\alpha)}$ is

$$\{\overline{(1,\alpha)}, \overline{(2,0)}, \overline{(3,\alpha)}, \overline{(4,0)} = 2\overline{(2,\alpha)} = 0\},$$

and if $\alpha \neq 0$, then $\overline{(2,0)} \neq 0$. □

Corollary 4.13 *There is an isomorphism* $\mathsf{s}_1(\mathbf{KQ}) \cong \Sigma^{1,1}(\mathbf{MZ}/2 \vee \mu)$, *and* $\mathsf{s}_1(f)$ *coincides with the composite map*

$$\mathsf{s}_1(\mathbf{KQ}) \cong \Sigma^{1,1}(\mathbf{MZ}/2 \vee \mu) \xrightarrow{\mathrm{pr}} \Sigma^{1,1}\mathbf{MZ}/2 \xrightarrow{\delta} \Sigma^{2,1}\mathbf{MZ} \cong \mathsf{s}_1(\mathbf{KGL}).$$

Proof This follows from Lemma 4.12. □

Lemma 4.14 *The map* $\mathsf{s}_1(h)\colon \mathsf{s}_1(\mathbf{KGL}) \longrightarrow \mathsf{s}_1\mathbf{KQ}$ *is trivial.*

Proof The first component of

$$\mathsf{s}_1(h)\colon \mathsf{s}_1(\mathbf{KGL}) \longrightarrow \mathsf{s}_1(\mathbf{KQ}) \cong \Sigma^{1,1}(\mathbf{MZ}/2 \vee \mu)$$

is trivial for degree reasons by Lemma A.3. Thus $\mathsf{s}_1(h)$ corresponds to the transpose of the matrix $(0\ \beta)$. By Corollary 4.13, $\mathsf{s}_1(f)$ corresponds to the matrix $(\delta\ 0)$.

Proposition 4.10 implies that

$$\mathsf{s}_1(hf) = \begin{pmatrix} 0 & 0 \\ 0 & \beta\delta \end{pmatrix}$$

is the multiplication by 2 map, hence trivial on $\Sigma^{1,1}(\mathbf{MZ}/2 \vee \mu)$. We conclude that $\beta\delta = 0$. Thus β factors as

$$\Sigma^{2,1}\mathbf{MZ} \xrightarrow{2} \Sigma^{2,1}\mathbf{MZ} \xrightarrow{\beta'} \Sigma^{1,1}\mu,$$

which coincides with

$$\Sigma^{2,1}\mathbf{MZ} \xrightarrow{\beta'} \Sigma^{1,1}\mu \xrightarrow{2} \Sigma^{1,1}\mu.$$

Since the multiplication by 2 map on μ is trivial, the result follows. □

Corollary 4.15 *There is an isomorphism*

$$\mathsf{s}_2(\mathbf{KQ}) \xrightarrow{\cong} \Sigma^{4,2}\mathbf{MZ} \vee \Sigma^{2,2}(\mathbf{MZ}/2 \vee \mu)$$

which identifies $\mathsf{s}_2(f)$ with the projection map onto $\Sigma^{4,2}\mathbf{MZ}$.

Proof Theorem 4.1 and Corollary 4.13 show that, up to isomorphism, Theorem 3.4 gives rise to the distinguished triangle

$$\Sigma^{3,2}\mathbf{MZ} \xrightarrow{\Sigma^{1,1}\mathsf{s}_1(h)} \Sigma^{2,2}(\mathbf{MZ}/2 \vee \mu) \longrightarrow \mathsf{s}_2(\mathbf{KQ}) \xrightarrow{\mathsf{s}_2(f)} \Sigma^{4,2}\mathbf{MZ}.$$

Lemma 4.14 implies that $\mathsf{s}_1(h)$ is the trivial map. □

Lemma 4.16 *The map $\mathsf{s}_2(h)\colon \mathsf{s}_2(\mathbf{KGL}) \longrightarrow \mathsf{s}_2(\mathbf{KQ})$ coincides with the composite*

$$\Sigma^{4,2}\mathbf{MZ} \xrightarrow{(2,0)} \Sigma^{4,2}\mathbf{MZ} \vee \Sigma^{2,2}(\mathbf{MZ}/2 \vee \mu).$$

Proof Corollary 4.15 identifies $\mathsf{s}_2(f)$ with the projection map onto $\Sigma^{4,2}\mathbf{MZ}$. Since $\mathsf{s}_2(fh) = 2$ by Proposition 4.4, $\mathsf{s}_2(h) = (2, \gamma_1, \gamma_2)$, where

$$(\gamma_1, \gamma_2)\colon \Sigma^{4,2}\mathbf{MZ} \longrightarrow \Sigma^{2,2}(\mathbf{MZ}/2 \vee \mu).$$

Note that $\gamma_1 = 0$ by Lemma A.3. Similarly $\mathsf{s}_2(hf) = 2$ by Proposition 4.10. Using the matrix

$$\mathsf{s}_2(hf) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_2 & 0 & 0 \end{pmatrix}$$

we conclude that $\gamma_2 = 0$. □

Corollary 4.17 *There is an isomorphism*

$$\mathsf{s}_3(\mathbf{KQ}) \cong \Sigma^{5,3}\mathbf{MZ}/2 \vee \Sigma^{3,3}(\mathbf{MZ}/2 \vee \mu).$$

Moreover, $\mathsf{s}_3(f)$ coincides with the composite map

$$\mathsf{s}_3(\mathbf{KQ}) \cong \Sigma^{5,3}\mathbf{MZ}/2 \vee \Sigma^{3,3}(\mathbf{MZ}/2 \vee \mu) \xrightarrow{\mathrm{pr}} \Sigma^{5,3}\mathbf{MZ}/2 \xrightarrow{\delta} \Sigma^{6,3}\mathbf{MZ} \cong \mathsf{s}_3(\mathbf{KGL}).$$

Proof Apply s_3 to the Hopf cofiber sequence in Theorem 3.4 and use Lemma 4.16. □

Theorem 4.18 *The slices of the hermitian K–theory spectrum* $\mathbf{KQ}$ *are given by*

$$\mathsf{s}_q(\mathbf{KQ}) = \begin{cases} \Sigma^{2q,q}\mathbf{MZ} \vee \Sigma^{q,q}\mathbf{M}\mu \vee \bigvee_{i<q/2}\Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 0 \bmod 2, \\ \Sigma^{q,q}\mathbf{M}\mu \vee \bigvee_{i<(q+1)/2}\Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 1 \bmod 2. \end{cases}$$

Here $\mathbf{M}\mu \cong \Sigma^{4,0}\mathbf{M}\mu$ and $\mathbf{M}\mu_{s,t}$ is an $\mathbb{F}_2$–module for all integers s and t.

Proof Corollary 4.17 identifies the third slice

$$\mathsf{s}_3(\mathbf{KQ}) \cong \Sigma^{5,3}\mathbf{MZ}/2 \vee \Sigma^{3,3}(\mathbf{MZ}/2 \vee \mu).$$

On the other hand, Karoubi periodicity and Corollary 4.9 imply the isomorphism

$$\mathsf{s}_3(\mathbf{KQ}) \cong \mathsf{s}_3(\Sigma^{8,4}\mathbf{KQ}) \cong \Sigma^{8,4}s_{-1}(\mathbf{KQ}) \cong \Sigma^{7,3}\mu.$$

It follows that $\Sigma^{-2,0}\mathbf{MZ} \vee \Sigma^{-4,0}\mathbf{MZ}$ is a direct summand of μ. Iterating the procedure furnishes an isomorphism

$$\mu \cong \mathbf{M}\mu \vee \bigvee_{i<0}\Sigma^{-2i,0}\mathbf{MZ}/2,$$

where $\mathbf{M}\mu$ is simply the complementary summand. The result follows. □

In the next section we show the mysterious summand $\mathbf{M}\mu$ of $\mathsf{s}_0(\mathbf{KQ})$ is trivial. The following summarizes some of the main observations in this section.

Proposition 4.19 *The map*

$$\mathsf{s}_{2q}(f)\colon \mathsf{s}_{2q}(\mathbf{KQ}) \longrightarrow \mathsf{s}_{2q}(\mathbf{KGL})$$

is the projection onto $\Sigma^{4q,2q}\mathbf{MZ}$, the map

$$\mathsf{s}_{2q+1}(f)\colon \mathsf{s}_{2q+1}(\mathbf{KQ}) \longrightarrow \mathsf{s}_{2q+1}(\mathbf{KGL})$$

is the projection map onto $\Sigma^{4q+1,2q+1}\mathbf{MZ}/2$ composed with

$$\delta\colon \Sigma^{4q+1,2q+1}\mathbf{MZ}/2 \longrightarrow \Sigma^{4q+2,2q+1}\mathbf{MZ}.$$

The map

$$\mathsf{s}_i(h)\colon \mathsf{s}_i(\mathbf{KGL}) \longrightarrow \mathsf{s}_i(\mathbf{KQ})$$

is multiplication by 2 *composed with the inclusion of* $\Sigma^{4q,2q}\mathbf{MZ}$ *if* $i = 2q$, *and trivial if* i *is odd.*

4.4 The mysterious summand is trivial

To prove $\mathbf{M}\mu$ is trivial we use the solution of the homotopy limit problem for hermitian K–theory of prime fields due to Berrick and Karoubi [5] and Friedlander [10]. Their results have been generalized by Hu, Kriz and Ormsby [19] and Berrick, Karoubi, Schlichting and Østvær [6], where the following is shown. Here $\mathsf{vcd}_2(F) = \mathsf{cd}_2(F(\sqrt{-1}))$ denotes the virtual mod-2 cohomological dimension of F.

Theorem 4.20 *If* $\mathsf{vcd}_2(F) < \infty$ *there is a canonical weak equivalence* $\mathbf{KQ}/2 \longrightarrow \mathbf{KGL}^{hC_2}/2$.

To make use of Theorem 4.20 we need to understand how slices compare with mod-2 reductions of motivic spectra and formation of homotopy fixed points. The former is straightforward because the slice functors are triangulated.

Lemma 4.21 *Let* E *be a motivic spectrum. There is a canonical isomorphism* $\mathsf{s}_q(\mathsf{E})/2 \cong \mathsf{s}_q(\mathsf{E}/2)$.

Lemma 4.22 *There is a naturally induced commutative diagram*

$$\begin{tikzcd}
(\mathbf{MZ},\mathrm{id})^{hC_2} \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{MZ} \vee \bigvee_{i<0}\Sigma^{2i,0}\mathbf{MZ}/2 \arrow[dl, "\mathrm{pr}"] \\
& \mathbf{MZ} &
\end{tikzcd}$$

Proof With the trivial C_2–action on $\mathbf{MZ}$ the homotopy fixed points spectrum is given by

$$(\mathbf{MZ},\mathrm{id})^{hC_2} = \mathbf{sSet}_*(\mathbb{RP}^\infty_+, \mathbf{MZ}) \cong \mathbf{MZ} \vee \mathbf{sSet}_*(\mathbb{RP}^\infty, \mathbf{MZ}).$$

The canonical map $(\mathbf{MZ},\mathrm{id})^{hC_2} \longrightarrow \mathbf{MZ}$ corresponds to the map induced by

$$\mathrm{Spec}(F)_+ \hookrightarrow \mathbb{RP}^\infty_+.$$

Induction on n shows there are isomorphisms

$$\mathbf{sSet}_*(\mathbb{RP}^n, \mathbf{MZ}) \cong \begin{cases} \Sigma^{-n,0}\mathbf{MZ} \times \prod_{i=1}^{(n-1)/2} \Sigma^{-2i,0}\mathbf{MZ}/2 & n \equiv 1 \bmod 2, \\ \prod_{i=1}^{n/2} \Sigma^{-2i,0}\mathbf{MZ}/2 & n \equiv 0 \bmod 2. \end{cases}$$

Writing $\mathbf{sSet}_*(\mathbb{RP}^\infty, \mathbf{MZ}) = \operatorname{colim}_n \mathbf{sSet}_*(\mathbb{RP}^n, \mathbf{MZ})$, we deduce

$$\mathbf{sSet}_*(\mathbb{RP}^\infty, \mathbf{MZ}) \cong \prod_{i<0} \Sigma^{2i,0}\mathbf{MZ}/2.$$

Proposition A.5 identifies the latter with $\bigvee_{i<0}\Sigma^{2i,0}\mathbf{MZ}/2$. □

Lemma 4.23 *The nontrivial involution on* $\mathbf{MZ}$ *and the connecting map*

$$\delta\colon \Sigma^{-1,0}\mathbf{MZ}/2 \longrightarrow \mathbf{MZ}$$

(on the 0^{th} *summand below) give rise to the commutative diagram:*

$$\begin{tikzcd}
(\mathbf{MZ},\sigma)^{hC_2} \arrow[rr, "\cong"] \arrow[dr] & & \bigvee_{i\leq 0}\Sigma^{2i-1,0}\mathbf{MZ}/2 \arrow[dl, "\delta"] \\
& \mathbf{MZ} &
\end{tikzcd}$$

Proof Let (G,σ) be an involutive simplicial abelian group with $\sigma(g) = -g$ and EC_2 a contractible simplicial set with a free C_2–action. Then C_2 acts on the space of fixed points $\mathbf{sSet}(EC_2, G)^{C_2}$ by conjugation, ie on the simplicial set of maps from EC_2 to (G,σ). Now choose EC_2 such that its skeletal filtration

$$(S^0, g) \subseteq (S^1, g) \subseteq \cdots \subseteq (S^n, g) \subseteq \cdots$$

comprises spheres equipped with the antipodal C_2–action. Here (S^{n+1}, g) arises from (S^n, g) by attaching a free C_2–cell of dimension $n+1$ along the C_2–map

$$C_2 \times S^n \longrightarrow (S^n, g)$$

adjoint to the identity on S^n. Thus the inclusion $(S^n, g) \subseteq (S^{n+1}, g)$ yields a pullback square

$$\begin{tikzcd}
\mathbf{sSet}\big((S^{n+1},g),(G,\sigma)\big)^{C_2} \arrow[r] \arrow[d] & \mathbf{sSet}\big((S^{n},g),(G,\sigma)\big)^{C_2} \arrow[d] \\
\mathbf{sSet}(D^{n+1}, G) \arrow[r] & \mathbf{sSet}(S^n, G)
\end{tikzcd} \tag{21}$$

of simplicial sets. By specializing to $n = 0$, (21) extends to the commutative diagram

$$\begin{tikzcd}
\mathbf{sSet}\big((S^{1},g),(G,\sigma)\big)^{C_2} \arrow[r] \arrow[d] & \mathbf{sSet}\big((S^{0},g),(G,\sigma)\big)^{C_2} \cong G \arrow[r] \arrow[d, "x\mapsto (x,-x)"'] & 0 \arrow[d] \\
\mathbf{sSet}(D^1, G) \arrow[r, "\phi"] & \mathbf{sSet}(S^0, G) \cong G \times G \arrow[r, "+"] & G.
\end{tikzcd}$$

Here ϕ is a fibrant replacement of the diagonal $G \longrightarrow G \times G$. The right-hand square is a pullback, and addition is a Kan fibration. Hence $\mathbf{sSet}\big((S^1, g), (G, \sigma)\big)^{C_2}$ is the homotopy fiber of multiplication by 2 on G. On the other hand, there is a homotopy cofiber sequence

$$S^1 \xrightarrow{2} S^1 \cong \mathbb{RP}^1 \longrightarrow \mathbb{RP}^2$$

and $\mathbf{sSet}\big((S^1, g), (G, \sigma)\big)^{C_2}$ has a canonical basepoint. Thus there is a homotopy equivalence

$$\Omega\mathbf{sSet}\big((S^1, g), (G, \sigma)\big)^{C_2} \simeq \mathbf{sSet}_*(\mathbb{RP}^2, G).$$

By induction on (21), we find for every $n \geq 0$ a homotopy equivalence

$$\Omega\mathbf{sSet}\big((S^n, g), (G, \sigma)\big)^{C_2} \simeq \mathbf{sSet}_*(\mathbb{RP}^{n+1}, G),$$

which implies

$$\Omega(G, \sigma)^{hC_2} \simeq \mathbf{sSet}_*(\mathbb{RP}^\infty, G). \tag{22}$$

A levelwise and sectionwise application of (22) yields the weak equivalences

$$\Omega^{1,0}(\mathbf{MZ}, \sigma)^{hC_2} \cong \mathbf{sSet}_*(\mathbb{RP}^\infty, \mathbf{MZ}) \cong \prod_{i<0} S^{2i,0}\mathbf{MZ}/2.$$

(For the second weak equivalence see the proof of Lemma 4.6.) The canonical map from $(\mathbf{MZ}, \sigma)^{hC_2}$ to $\mathbf{MZ}$ corresponds to the map induced by $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^\infty$. Proposition A.5 concludes the proof. $\square$

Proposition 4.24 *Suppose that* $\mathsf{vcd}_2(F) < \infty$. *The canonical map*

$$\mathsf{s}_q(\mathbf{KGL}^{hC_2}) \longrightarrow \mathsf{s}_q(\mathbf{KGL})^{hC_2}$$

is an isomorphism for every integer q.

Proof Let F be an arbitrary field of characteristic not 2. Lemma 4.26 shows that $\mathbf{KGL}^{hC_2}$ satisfies the properties which were used to determine the slices of $\mathbf{KQ}$. As in the proof of Theorem 4.18, this results in a splitting

$$\mathsf{s}_q(\mathbf{KGL}^{hC_2}) = \begin{cases} (\Sigma^{2q,q}\mathbf{MZ}) \vee (\Sigma^{q,q}\mathbf{M}\nu) \vee \bigvee_{i<q/2} \Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 0 \bmod 2, \\ (\Sigma^{q,q}\mathbf{M}\nu) \vee \bigvee_{i<(q+1)/2} \Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 1 \bmod 2. \end{cases}$$

Here $\mathbf{M}\nu \cong \Sigma^{4,0}\mathbf{M}\nu$ and $\mathbf{M}\nu_{s,t}$ is an $\mathbb{F}_2$-module for all integers s and t. Hence the vanishing of $\mathbf{M}\nu$ can be deduced from a computation of the zero slice of $\mathbf{KGL}^{hC_2}/2 = (\mathbf{KGL}/2)^{hC_2}$. Moreover, the homotopy norm cofiber sequence

$$\mathbf{KGL}_{hC_2} \longrightarrow \mathbf{KGL}^{hC_2} \longrightarrow \widehat{\mathbf{KGL}}^{C_2} \longrightarrow \Sigma^{1,0}\mathbf{KGL}_{hC_2}$$

from [19, Diagram (20)] and Theorem 4.8 imply that $\mathbf{M}\nu$ is also a direct summand of $\mathsf{s}_0(\widehat{\mathbf{KGL}}^{C_2})$.

Suppose that $\mathsf{vcd}_2(F) < \infty$. Theorem 4.20 shows that the canonical maps $\mathbf{KQ}/2 \longrightarrow \mathbf{KGL}^{hC_2}/2$ and $\mathbf{KT}/2 \longrightarrow \widehat{\mathbf{KGL}/2}^{C_2}$ are equivalences. Thus the Tate motivic spectrum $\widehat{\mathbf{KGL}/2}^{C_2}$ is cellular in the sense of [8], since $\mathbf{KQ}$ is cellular [47]. (The latter is deduced using the model for $\mathbf{KQ}$ from [39].) Hence the zero slice of $\widehat{\mathbf{KGL}/2}^{C_2}$ is the motivic Eilenberg–MacLane spectrum associated to a chain complex of abelian groups. This chain complex does not depend on the base field. Thus it suffices to compute the zero slice of $\widehat{\mathbf{KGL}/2}^{C_2}$ (or, equivalently, of $\mathbf{KT}/2$) over an algebraically closed field of characteristic not two. For this one can employ the specific cell presentation given in [16, Theorem 3.2] (see also [44, Section 4]), which extends to any algebraically closed field of characteristic not two via the base change techniques used in the proof of Lemma 5.1. It shows that $\mathbf{KT}$ may be obtained in two steps from the η–inverted sphere $\mathbf{1}[1/\eta]$. The first step produces a canonical map $\mathsf{D}_\infty \longrightarrow \mathbf{KT}$ from the colimit D_∞ of the sequential diagram

$$\mathbf{1}[1/\eta] = \mathsf{D}_1 \longrightarrow \mathsf{D}_2 \longrightarrow \cdots \longrightarrow \mathsf{D}_n \longrightarrow \cdots,$$

where $\mathsf{D}_n \longrightarrow \mathsf{D}_{n+1}$ is the canonical map to the homotopy cofiber of the unique nontrivial map $\Sigma^{4n-1,0}\mathbf{1}[1/\eta] \longrightarrow \mathsf{D}_n$. The second step shows that $\mathsf{D}_\infty \longrightarrow \mathbf{KT}$ is obtained by inverting the unique nontrivial element $\kappa \in \pi_{4,0}\mathsf{D}_\infty$. The computation of the slices of the sphere spectrum [29, Section 8] leads to the computation

$$\mathsf{s}_0(\mathbf{1}[1/\eta]) \cong \mathbf{MZ}/2 \vee \bigvee_{n\geq 2} \Sigma^{n,0}\mathbf{MZ}/2,$$

stated as [44, Theorem 4.12]. The sequential diagram above implies that

$$\mathsf{s}_0(\mathsf{D}_\infty) \cong \bigvee_{n\geq 0} \Sigma^{2n,0}\mathbf{MZ}/2,$$

whence, using $\mathbf{KT} \cong \mathsf{D}_\infty[1/\kappa]$, the zero slice of $\mathbf{KT}$ is

$$\mathsf{s}_0(\mathbf{KT}) \cong \mathsf{s}_0(\mathsf{D}_\infty[1/\kappa]) \cong \mathsf{s}_0(\mathsf{D}_\infty)[1/\mathsf{s}_0\kappa] \cong \bigvee_{n\in\mathbb{Z}} \Sigma^{2n,0}\mathbf{MZ}/2.$$

It follows that $\mathbf{M}\nu$ is contractible, which proves the statement. □

Theorem 4.25 *Suppose that $\mathsf{vcd}_2(F) < \infty$. The slices of homotopy fixed point algebraic K–theory $\mathbf{KGL}^{hC_2}$ are given by*

$$\mathsf{s}_q(\mathbf{KGL}^{hC_2}) = \begin{cases} \Sigma^{q,q}\big(\Sigma^{q,0}\mathbf{MZ} \vee \bigvee_{i<q/2}\Sigma^{2i,0}\mathbf{MZ}/2\big), & q \equiv 0 \bmod 2, \\ \Sigma^{q,q}\bigvee_{i<(q+1)/2}\Sigma^{2i,0}\mathbf{MZ}/2, & q \equiv 1 \bmod 2. \end{cases}$$

Proof This follows from Theorem 4.1, Lemmas 4.22–4.23 and Proposition 4.24. □

Lemma 4.26 *The map* $\mathbf{KQ} \longrightarrow \mathbf{KGL}^{hC_2}$ *induces the projection map on all slices.*

Proof This is clear from the following claims.

(1) The map $\theta\colon \mathbf{KQ} \longrightarrow \mathbf{KGL}^{hC_2}$ fits into a commutative diagram

$$\begin{tikzcd}
\Sigma^{1,1}\mathbf{KQ} \arrow[r, "\eta"] \arrow[d, "\Sigma^{2,1}\theta"'] & \mathbf{KQ} \arrow[r, "f"] \arrow[d, "\theta"'] & \mathbf{KGL} \arrow[r, "\Sigma^{2,1}h\circ\beta"] \arrow[d, "\mathrm{id}"] & \Sigma^{2,1}\mathbf{KQ} \arrow[d, "\Sigma^{2,1}\theta"] \\
\Sigma^{1,1}\mathbf{KGL}^{hC_2} \arrow[r, "\eta"] & \mathbf{KGL}^{hC_2} \arrow[r, "f'"] & \mathbf{KGL} \arrow[r, "\Sigma^{2,1}h'\circ\beta"] & \Sigma^{2,1}\mathbf{KGL}^{hC_2}
\end{tikzcd}$$

of homotopy cofiber sequences.

(2) The map $s_0\theta\colon s_0\mathbf{KQ} \longrightarrow s_0\mathbf{KGL}^{hC_2}$ is the identity on the summand $\mathbf{MZ}$.

(3) The map $h'\circ f'\colon \mathbf{KGL}^{hC_2} \longrightarrow \mathbf{KGL}^{hC_2}$ is multiplication with $1-\epsilon$.

(4) The map $f'\circ h'\colon \mathbf{KGL} \longrightarrow \mathbf{KGL}$ coincides with $1+\Psi^{-1}_{\mathrm{st}}$.

To prove the first claim, consider the homotopy fiber of the canonical map $\theta\colon \mathbf{KQ} \longrightarrow \mathbf{KGL}^{hC_2}$. The Tate diagram [19, Diagram (20)] shows that it coincides with the homotopy fiber of the canonical map $\mathbf{KT} \longrightarrow \widehat{\mathbf{KGL}}^{C_2}$. Since η acts invertibly on these two motivic spectra, it acts invertibly on the homotopy fiber of θ. This implies that the diagram

$$\begin{tikzcd}
\Sigma^{1,1}\mathbf{KQ} \arrow[r, "\eta"] \arrow[d, "\Sigma^{2,1}\theta"'] & \mathbf{KQ} \arrow[d, "\theta"'] \\
\Sigma^{1,1}\mathbf{KGL}^{hC_2} \arrow[r, "\eta"] & \mathbf{KGL}^{hC_2}
\end{tikzcd}$$

is a homotopy pullback square, and hence the first claim. See also [20, Remark 5.9].

The second claim follows from the factorization of the unit map for algebraic K–theory

$$\mathbf{1} \xrightarrow{\text{unit}} \mathbf{KQ} \xrightarrow{\theta} \mathbf{KGL}^{hC_2} \xrightarrow{f'} \mathbf{KGL}.$$

By the proof of Lemma 3.6 the third claim follows from the commutative diagram

$$\begin{tikzcd}
\mathbf{KQ} \arrow[r, "f"] \arrow[d, "\theta"'] & \mathbf{KGL} \arrow[r, "h"] \arrow[d, "\mathrm{id}"'] & \mathbf{KQ} \arrow[d, "\theta"] \\
\mathbf{KGL}^{hC_2} \arrow[r, "f'"] & \mathbf{KGL} \arrow[r, "h'"] & \mathbf{KGL}^{hC_2}.
\end{tikzcd}$$

The previous commutative diagram and (19) imply the fourth and final claim.

Using the above one may now compute the slices of $\mathbf{KGL}^{hC_2}$ in the same way as for $\mathbf{KQ}$; see Section 4.3. That is, one obtains an identification

$$\mathsf{s}_q(\mathbf{KGL}^{hC_2}) = \begin{cases} \Sigma^{2q,q}\mathbf{MZ} \vee \Sigma^{q,q}\mathbf{M}\nu \vee \bigvee_{i<q/2}\Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 0 \bmod 2, \\ \Sigma^{q,q}\mathbf{M}\nu \vee \bigvee_{i<(q+1)/2}\Sigma^{2i+q,q}\mathbf{MZ}/2, & q \equiv 1 \bmod 2. \end{cases}$$

Here $\mathbf{M}\nu \cong \Sigma^{4,0}\mathbf{M}\nu$ and $\mathbf{M}\nu_{s,t}$ is an $\mathbb{F}_2$–module for all integers s and t. Moreover, it follows from our first claim above that $\theta\colon \mathbf{KQ} \longrightarrow \mathbf{KGL}^{hC_2}$ splits as (id, ζ), where id is the identity on the non-mysterious summands of the respective slices, and $\zeta\colon \mathbf{M}\mu \longrightarrow \mathbf{M}\nu$ up to suspension with $S^{1,1}$. By comparison with the identification of the slices of $\mathbf{KGL}^{hC_2}$ in Theorem 4.25, the summand $\mathbf{M}\nu$ is contractible, which completes the proof. □

Theorem 4.27 *The mysterious summand $\mathbf{M}\mu$ is trivial.*

Proof By Corollary 2.7 it suffices to consider prime fields. Theorems 4.20 and 4.25 and Lemma 4.26 show that the mysterious summand is the homotopy fiber of a weak equivalence. □

4.5 Higher Witt theory

By combining Theorems 4.2, 4.8, 4.18 and 4.27 we have enough information to identify the slices of Witt theory. An alternate proof with no mention of homotopy orbit K–theory follows by Lemma 4.5 and the identification of $\mathsf{s}_q(\eta)\colon \mathsf{s}_q(\Sigma^{1,1}\mathbf{KQ}) \longrightarrow \mathsf{s}_q(\mathbf{KQ})$ worked out in Section 4.3.

Theorem 4.28 *The slices of Witt theory are given by*

$$\mathsf{s}_q(\mathbf{KT}) \cong \Sigma^{q,q}\bigvee_{i\in\mathbb{Z}}\Sigma^{2i,0}\mathbf{MZ}/2.$$

Let $u\colon \mathbf{KQ} \longrightarrow \mathbf{KT}$ be the canonical map. The next result follows now by an easy inspection.

Proposition 4.29 *Restricting the map $\mathsf{s}_{2q}(u)\colon \mathsf{s}_{2q}(\mathbf{KQ}) \longrightarrow \mathsf{s}_{2q}(\mathbf{KT})$ to the summand $\Sigma^{4q,2q}\mathbf{MZ}$ yields the projection $\Sigma^{4q,2q}\mathbf{MZ} \longrightarrow \Sigma^{4q,2q}\mathbf{MZ}/2$ composed with the inclusion into $\mathsf{s}_{2q}(\mathbf{KT})$. Restricting the same map to a suspension of $\mathbf{MZ}/2$ yields the inclusion into $\mathsf{s}_{2q}(\mathbf{KT})$. On odd slices, $\mathsf{s}_{2q+1}(u)\colon \mathsf{s}_{2q+1}(\mathbf{KQ}) \longrightarrow \mathsf{s}_{2q+1}(\mathbf{KT})$ is the inclusion.*

5 Differentials

Having determined the slices we now turn to the problem of computing the first differentials in the slice spectral sequences for $\mathbf{KGL}_{hC_2}$, $\mathbf{KQ}$ and $\mathbf{KT}$.

5.1 Mod-2 algebraic K–theory

Theorem 4.1 and Lemma 4.21 show there is an isomorphism

$$\mathsf{s}_q(\mathbf{KGL}/2) \cong \Sigma^{2q,q}\mathbf{MZ}/2.$$

Hence the differential

$$d_1^{\mathbf{KGL}/2}\colon \mathsf{s}_q(\mathbf{KGL}/2) \longrightarrow \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KGL}/2)$$

corresponds to a bidegree $(3,1)$ element in the mod-2 motivic Steenrod algebra. By Bott periodicity it is independent of q. Next we resolve the mod-2 version of a question stated in [57, Remark 3.12].

Lemma 5.1 *If* char $F \neq 2$ *the differential*

$$d_1^{\mathbf{KGL}/2}\colon \mathbf{MZ}/2 \longrightarrow \Sigma^{3,1}\mathbf{MZ}/2$$

equals the Milnor operation $\mathsf{Q}_1 = \mathsf{Sq}^3 + \mathsf{Sq}^2\mathsf{Sq}^1$.

Proof By Lemma A.2 and the Adem relation $\mathsf{Sq}^3 = \mathsf{Sq}^1\mathsf{Sq}^2$ there exist $a, b \in \mathbb{Z}/2$ such that

$$d_1^{\mathbf{KGL}/2} = a\mathsf{Sq}^3 + b\mathsf{Sq}^2\mathsf{Sq}^1. \tag{23}$$

Using the Adem relations

$$\mathsf{Sq}^3\mathsf{Sq}^3 = \mathsf{Sq}^5\mathsf{Sq}^1, \quad \mathsf{Sq}^2\mathsf{Sq}^3 = \mathsf{Sq}^5 + \mathsf{Sq}^4\mathsf{Sq}^1 \quad \text{and} \quad \mathsf{Sq}^1\mathsf{Sq}^1 = 0$$

we find

$$\begin{aligned}(a\mathsf{Sq}^3 + b\mathsf{Sq}^2\mathsf{Sq}^1)^2 &= a^2\mathsf{Sq}^3\mathsf{Sq}^3 + ab\mathsf{Sq}^3\mathsf{Sq}^2\mathsf{Sq}^1 + ab\mathsf{Sq}^2\mathsf{Sq}^1\mathsf{Sq}^3 + b^2\mathsf{Sq}^2\mathsf{Sq}^1\mathsf{Sq}^2\mathsf{Sq}^1\\ &= (a^2+b^2)\mathsf{Sq}^5\mathsf{Sq}^1.\end{aligned}$$

Since $d_1^{\mathbf{KGL}/2}$ squares to zero this implies $a = b$. Recall the classes $0 \neq \tau \in h^{0,1}$ and $\rho \in h^{1,1}$ represented by $-1 \in F$. Over $\mathbb{R}$, and hence $\mathbb{Q}$, $d_1^{\mathbf{KGL}/2}(\tau^2) = \rho^3$ by Suslin's computation of the algebraic K–theory of the real numbers [51], while

$$d_1^{\mathbf{KGL}/2}(\tau^2) = a\mathsf{Sq}^3(\tau^2) + b\mathsf{Sq}^2\mathsf{Sq}^1(\tau^2) = a\rho^3$$

by (23) and Corollary 6.2. These computations imply that $a=b=1$ by base change for all fields of characteristic zero. To extend this result to fields of odd characteristic, we consider the following diagram of motivic spectra over $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$:

$$\mathbf{KGL}/2 \longleftarrow \mathbf{MGL}/(2,x_2,x_3,\dots)\mathbf{MGL} \longrightarrow \mathbf{MGL}/(2,x_1,x_2,\dots)\mathbf{MGL} \longrightarrow \mathbf{MZ}/2. \tag{24}$$

Here **MGL** denotes Voevodsky's algebraic cobordism spectrum, x_n denotes the canonical image of Lazard's generator in $\mathbf{MGL}_{*,*}$, $\mathbf{MZ}/2$ denotes Spitzweck's motivic cohomology spectrum with $\mathbb{F}_2$–coefficients [50], and the maps are induced by the respective canonical orientations. All maps in diagram (24) induce equivalences on zero slices. For the map pointing to the left, this follows from the description of **KGL** as $\big(\mathbf{MGL}/(x_2,x_3,\dots)\mathbf{MGL}\big)[x_1^{-1}]$ [49, Theorem 5.2]. For the map in the middle, this follows from its construction. By [50, Theorem 11.3], the rightmost map in diagram (24) is an equivalence, and in particular after applying the zero slice functor. This results in a commutative diagram

$$\begin{tikzcd} \mathsf{s}_0\mathbf{KGL}/2 \arrow[r, "\mathbf{d}_1"] \arrow[d, "\cong"'] & \Sigma^{3,1}\mathsf{s}_0\mathbf{KGL}/2 \arrow[d, "\cong"] \\ \mathsf{s}_0\mathbf{MZ}/2 \arrow[r] & \Sigma^{3,1}\mathsf{s}_0\mathbf{MZ}/2, \end{tikzcd} \tag{25}$$

where the map on the top is defined as over a field. By its construction, the motivic spectrum $\mathbf{MZ}/2$ satisfies $\mathsf{f}_1\mathbf{MZ}/2=*$, and the aforementioned [50, Theorem 11.3] implies that $\mathbf{MZ}/2$ is effective over $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$. This results in a canonical isomorphism $\mathsf{s}_0\mathbf{MZ}/2\cong\mathbf{MZ}/2$ which is a map of motivic ring spectra by [13, Theorem 5.19]. Base change via $f\colon \operatorname{Spec}\mathbb{Q}\longrightarrow\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ maps diagram (25) to the following diagram over $\operatorname{Spec}\mathbb{Q}$, by Lemma 2.6, [50, Lemma 7.5] and the previous argument:

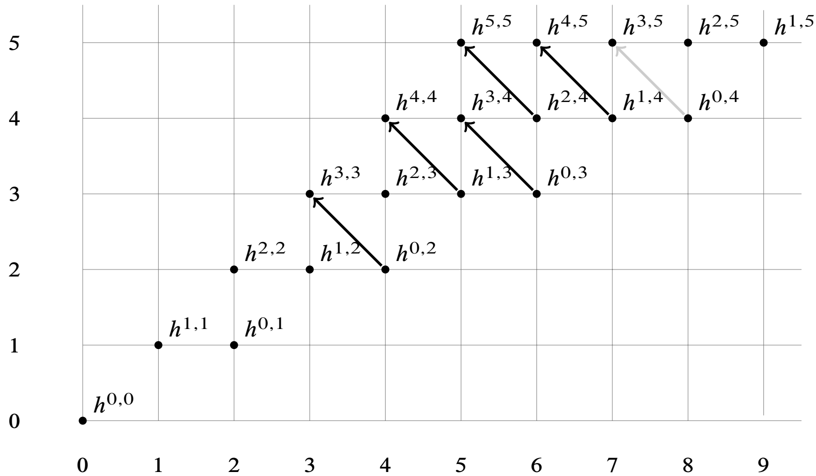
$$\begin{tikzcd} \mathsf{s}_0\mathbf{KGL}/2 \arrow[r, "\mathbf{d}_1"] \arrow[d, "\cong"'] & \Sigma^{3,1}\mathsf{s}_0\mathbf{KGL}/2 \arrow[d, "\cong"] \\ \mathbf{MZ}/2 \arrow[r, "\mathsf{Q}_1"] & \Sigma^{3,1}\mathbf{MZ}/2 \end{tikzcd}$$

If p is an odd prime, base change via $i\colon \operatorname{Spec}\mathbb{F}_p \hookrightarrow \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ maps diagram (25) to the diagram

$$\begin{tikzcd} i^*\mathsf{s}_0\mathbf{KGL}/2 \arrow[r, "i^*\mathbf{d}_1"] \arrow[d, "\cong"'] & \Sigma^{3,1}i^*\mathsf{s}_0\mathbf{KGL}/2 \arrow[d, "\cong"] \\ i^*\mathbf{MZ}/2 \arrow[r] & \Sigma^{3,1}i^*\mathbf{MZ}/2. \end{tikzcd}$$

By [50, Theorem 9.19], $i^*\mathbf{MZ}/2$ is Voevodsky's Eilenberg–MacLane motivic ring spectrum over $\mathbb{F}_p$, and the bottom map in the last diagram is the first Milnor operation by [50, Theorem 11.24]. $\square$

Thus the E^1–page of the 0^{th} slice spectral sequence for $\mathbf{KGL}/2$ takes the form



The differential $h^{0,4} \longrightarrow h^{3,5}$ has a nontrivial group as source and a potentially nontrivial group as target, but is always zero by Corollary 6.2.

Corollary 5.2 *If $\sqrt{-1} \in F$ then $d_1^{\mathbf{KGL}/2}\colon \pi_{p,n}\mathsf{s}_q(\mathbf{KGL}/2) \longrightarrow \pi_{p-1,n}\mathsf{s}_{q+1}(\mathbf{KGL}/2)$ is trivial.*

Proof By assumption $\rho = 0$, so the claim follows from Lemma 5.1 and Corollary 6.2. $\square$

5.2 Hermitian K–theory, I

In what follows we make of use the identification of $d_1^{\mathbf{KGL}/2}$ with the first Milnor operation Q_1 to give formulas for the $\mathbf{d}_1$–differential in the slice spectral sequence for $\mathbf{KQ}$. To begin with, consider the commutative diagram

(26)
$$\begin{tikzcd}
\mathsf{s}_q(\mathbf{KQ}) \arrow[r, "d_1^{\mathbf{KQ}}(q)"] \arrow[d] & \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KQ}) \arrow[d] \\
\mathsf{s}_q(\mathbf{KGL}) \arrow[r, "d_1^{\mathbf{KGL}}(q)"] \arrow[d] & \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KGL}) \arrow[d] \\
\mathsf{s}_q(\mathbf{KGL}/2) \arrow[r, "\mathsf{Q}_1"] & \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KGL}/2).
\end{tikzcd}$$

To proceed from here requires two separate arguments depending on whether the $\mathbf{d}_1$–differential exits an even or an odd slice. First we analyze the even slices.

Theorems 4.18 and 4.27 show that $\mathbf{d}_1^{\mathbf{KQ}}(2q)$ is a map from

$$\mathsf{s}_{2q}(\mathbf{KQ}) = \Sigma^{2q,2q}\big(\Sigma^{2q,0}\mathbf{MZ} \vee \bigvee_{i<0}\Sigma^{2q+2i,0}\mathbf{MZ}/2\big)$$

to

$$\Sigma^{1,0}\mathsf{s}_{2q+1}(\mathbf{KQ}) = \Sigma^{2q+1,2q+1}\bigvee_{i\leq 0}\Sigma^{2q+2i+1,0}\mathbf{MZ}/2.$$

Up to suspension with $S^{4q,2q}$, Proposition 4.19 shows

$$\mathsf{s}_{2q}(\mathbf{KQ}) \longrightarrow \mathsf{s}_{2q}(\mathbf{KGL}) \longrightarrow \mathsf{s}_{2q}(\mathbf{KGL}/2)$$

in (26) is the projection map

$$\mathbf{MZ} \vee \bigvee_{i<0}\Sigma^{2i,0}\mathbf{MZ}/2 \longrightarrow \mathbf{MZ} \longrightarrow \mathbf{MZ}/2.$$

Likewise, up to suspension with $S^{4q,2q}$, the composite of the maps on the right-hand side of (26) equals

$$\Sigma^{3,1}\bigvee_{i<0}\Sigma^{2i+1,0}\mathbf{MZ}/2 \xrightarrow{\mathrm{pr}} \Sigma^{2,1}\mathbf{MZ}/2 \xrightarrow{\delta} \Sigma^{3,1}\mathbf{MZ} \xrightarrow{\mathrm{pr}} \Sigma^{3,1}\mathbf{MZ}/2.$$

By Lemma A.4, $\mathsf{Sq}^2 \circ \mathrm{pr}$ is the only element of $[\mathbf{MZ}, \Sigma^{2,1}\mathbf{MZ}/2]$ whose composition with Sq^1 and the projection $\mathbf{MZ} \longrightarrow \mathbf{MZ}/2$ yields $\mathsf{Q}_1 \circ \mathrm{pr} = \mathsf{Sq}^1\mathsf{Sq}^2 \circ \mathrm{pr}$. When restricting to $\mathbf{MZ}$ this shows

$$d_1^{\mathbf{KQ}}(2q) = (0, \mathsf{Sq}^2 \circ \mathrm{pr}, a(2q)\tau \circ \mathrm{pr}); \quad a(2q) \in h^{0,0}.$$

On odd slices the first differential $\mathbf{d}_1^{\mathbf{KQ}}(2q+1)$ is a map from

$$\mathsf{s}_{2q+1}(\mathbf{KQ}) = \Sigma^{2q+1,2q+1}\bigvee_{i\leq 0}\Sigma^{2q+2i,0}\mathbf{MZ}/2$$

to

$$\Sigma^{1,0}\mathsf{s}_{2q+2}(\mathbf{KQ}) = \Sigma^{2q+2,2q+2}\big(\Sigma^{2q+3,0}\mathbf{MZ} \vee \bigvee_{i\leq 0}\Sigma^{2q+2i+1,0}\mathbf{MZ}/2\big).$$

Restricting $\mathbf{d}_1^{\mathbf{KQ}}(2q+1)$ to the top summand $\mathbf{MZ}/2$ corresponding to $i=0$ in the infinite wedge product yields — up to suspension with $S^{4q+1,2q+1}$ — a map

$$\mathbf{d}_1^{\mathbf{KQ}}(2q+1,0)\colon \mathbf{MZ}/2 \longrightarrow \Sigma^{4,1}\mathbf{MZ} \vee \Sigma^{2,1}\mathbf{MZ}/2 \vee \Sigma^{0,1}\mathbf{MZ}/2.$$

Lemma A.4 shows $\delta\mathsf{Sq}^2\mathsf{Sq}^1$ is the only element of $[\mathbf{MZ}/2, \Sigma^{4,1}\mathbf{MZ}]$ whose composition with the projection map pr equals $\mathsf{Q}_1\mathsf{Sq}^1 = \mathsf{Sq}^3\mathsf{Sq}^1$. Thus by commutativity of (26) we obtain

$$\mathbf{d}_1^{\mathbf{KQ}}(2q+1) = \big(\delta\mathsf{Sq}^2\mathsf{Sq}^1, b(2q+1)\mathsf{Sq}^2 + \phi(2q+1)\mathsf{Sq}^1, a(2q+1)\tau\big),$$

where $a(2q+1)$, $b(2q+1) \in h^{0,0}$ and $\phi(2q+1) \in h^{1,1}$.

In what follows we use the computations in this section to explicitly identify $\mathbf{d}_1^{\mathbf{KT}}$ and $\mathbf{d}_1^{\mathbf{KQ}}$ in terms of motivic cohomology classes and Steenrod operations.

5.3 Higher Witt theory

Up to suspension with $S^{q,q}$, Section 4.5 shows that the differential

$$\mathbf{d}_1^{\mathbf{KT}}(q)\colon \mathsf{s}_q(\mathbf{KT}) \longrightarrow \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KT})$$

takes the form

$$\bigvee_{i\in\mathbb{Z}}\Sigma^{2i,0}\mathbf{MZ}/2 \longrightarrow \Sigma^{2,1}\bigvee_{j\in\mathbb{Z}}\Sigma^{2j,0}\mathbf{MZ}/2.$$

Lemma 2.1 and $(1,1)$–periodicity $\Sigma^{1,1}\mathbf{KT}\cong\mathbf{KT}$ for $\mathbf{KT}$ imply $\mathbf{d}_1^{\mathbf{KT}}(q)=\Sigma^{q,q}\mathbf{d}_1^{\mathbf{KT}}(0)$. Let $\mathbf{d}_1^{\mathbf{KT}}(q,2i)$ denote the restriction of $\mathbf{d}_1^{\mathbf{KT}}(q)$ to the summand $\Sigma^{q+2i,q}\mathbf{MZ}/2$ of $\mathsf{s}_q(\mathbf{KT})$. The $(4,0)$–periodicity $\Sigma^{4,0}\mathbf{KT}\cong\mathbf{KT}$ shows $\mathbf{d}_1^{\mathbf{KT}}(0)$ is determined by its values on the summands $\mathbf{MZ}/2$ and $\Sigma^{2,0}\mathbf{MZ}/2$. That is, $\mathbf{d}_1^{\mathbf{KT}}(q,2i)$ is uniquely determined by $\mathbf{d}_1^{\mathbf{KT}}(0,0)$ or $\mathbf{d}_1^{\mathbf{KT}}(0,2)$. By Proposition A.5 the first differential is determined by its value on each summand in its target.

Theorem 5.3 *The $\mathbf{d}_1$–differential in the slice spectral sequence for $\mathbf{KT}$ is given by*

$$\mathbf{d}_1^{\mathbf{KT}}(q,i)=\begin{cases}(\mathsf{Sq}^3\mathsf{Sq}^1,\mathsf{Sq}^2,0), & i-2q\equiv 0 \bmod 4,\\ (\mathsf{Sq}^3\mathsf{Sq}^1,\mathsf{Sq}^2+\rho\mathsf{Sq}^1,\tau), & i-2q\equiv 2 \bmod 4.\end{cases}$$

The motivic cohomology classes $0\neq\tau\in h^{0,1}$ and $\rho=[-1]\in h^{1,1}$ are represented by $-1\in F$.

Proof We have reduced to computing the maps

$$\mathbf{d}_1^{\mathbf{KT}}(0,0)\colon \mathbf{MZ}/2\longrightarrow\Sigma^{4,1}\mathbf{MZ}/2\vee\Sigma^{2,1}\mathbf{MZ}/2\vee\Sigma^{0,1}\mathbf{MZ}/2,$$
$$\mathbf{d}_1^{\mathbf{KT}}(0,2)\colon \Sigma^{2,0}\mathbf{MZ}/2\longrightarrow\Sigma^{6,1}\mathbf{MZ}/2\vee\Sigma^{4,1}\mathbf{MZ}/2\vee\Sigma^{2,1}\mathbf{MZ}/2.$$

To proceed we invoke the commutative diagram

$$\begin{tikzcd}
\mathsf{s}_q(\mathbf{KQ}) \arrow[r] \arrow[d, "\mathbf{d}_1^{\mathbf{KQ}}(q)"'] & \mathsf{s}_q(\mathbf{KT}) \arrow[d, "\mathbf{d}_1^{\mathbf{KT}}(q)"] \\
\Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KQ}) \arrow[r] & \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KT}).
\end{tikzcd}$$

By combining Proposition 4.29 with the computations in Section 5.2 we find

$$\mathbf{d}_1^{\mathbf{KT}}(0,0)=(\mathsf{Sq}^3\mathsf{Sq}^1,\mathsf{Sq}^2+\phi(0)\mathsf{Sq}^1,a(0)\tau),$$
$$\mathbf{d}_1^{\mathbf{KT}}(0,2)=(\mathsf{Sq}^3\mathsf{Sq}^1,\mathsf{Sq}^2+\phi(2)\mathsf{Sq}^1,a(2)\tau).$$

Here $a(0)$, $a(2) \in h^{0,0}$ and $\phi(0)$, $\phi(2) \in h^{1,1}$. Since $\mathbf{d}_1^{\mathbf{KT}}$ squares to zero, we extract the equations

$$a(2)a(0) = a(0)\phi(2) = a(0)(\phi(0)+\phi(2)+\rho) = 0, \quad a(0)+a(2) = 1,$$
$$a(0)\rho+\phi(0) = a(2)\phi(0) = a(2)(\phi(0)+\phi(2)+\rho) = a(2)\rho+\phi(2) = 0.$$

The proof of Proposition 5.4 discusses the two possible values for $a(2)$ in the case of an algebraically closed field. In particular, having $a(2)=0$ would result in the group $\pi_{2,0}\mathbf{KT}$ being nontrivial, which would contradict the vanishing of Balmer's higher Witt groups of fields in degrees not congruent to zero modulo four [3, Theorem 98]. Hence $a(2)=1$ for algebraically closed fields, which extends to all fields by base change to an algebraic closure. This implies $a(0)=0$, $\phi(0)=0$, and $\phi(2)=\rho$. □

Proposition 5.4 *Suppose F has mod-2 cohomological dimension zero. Then the slice filtration for $\mathbf{KT}$ coincides with the fundamental ideal filtration of $W(F)$. Moreover, there are isomorphisms*

$$\pi_{p,0}\mathsf{f}_q(\mathbf{KT}) \cong \begin{cases} h^{0,q}, & p \equiv q \bmod 4,\ q \geq 0, \\ h^{0,0}, & p \equiv 0 \bmod 4,\ q < 0, \\ 0, & \text{otherwise.} \end{cases}$$

The first two isomorphisms are induced by the canonical map $\mathsf{f}_q(\mathbf{KT}) \longrightarrow \mathsf{s}_q(\mathbf{KT})$.

Proof By the $(1,1)$– and $(4,0)$–periodicities of $\mathbf{KT}$ and Theorem 4.28, it suffices to consider the filtration

$$\begin{tikzcd}
\cdots & h^{0,n+1} \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \pi_{p,-n}\mathsf{f}_2(\mathbf{KT}) \arrow[r] \arrow[d] & \pi_{p,-n}\mathsf{f}_1(\mathbf{KT}) \arrow[r] \arrow[d] & \pi_{p,-n}\mathsf{f}_0(\mathbf{KT}) \arrow[d] \\
\cdots & h^{0,n+2} & 0 & h^{0,n}
\end{tikzcd}$$

for p even, and

$$\begin{tikzcd}
\cdots & 0 \arrow[d] & h^{0,n} \arrow[d] & \\
\cdots \arrow[r] & \pi_{p,-n}\mathsf{f}_2(\mathbf{KT}) \arrow[r] \arrow[d] & \pi_{p,-n}\mathsf{f}_1(\mathbf{KT}) \arrow[r] \arrow[d] & \pi_{p,-n}\mathsf{f}_0(\mathbf{KT}) \arrow[d] \\
\cdots & 0 & h^{0,n+1} & 0
\end{tikzcd}$$

for p odd. When $n = p = 0$, $\pi_{0,0}\mathsf{f}_0(\mathbf{KT}) \longrightarrow h^{0,0}$ is a ring map [13; 40], hence an isomorphism. It follows that $\pi_{0,0}\mathsf{f}_1(\mathbf{KT}) = 0$, hence $\mathsf{f}_q\pi_{0,0}(\mathbf{KT}) = 0$ for all $q > 0$. In particular, the slice filtration is Hausdorff, and it coincides with the (trivial) filtration on the Witt ring given by the fundamental ideal. Theorem 5.3 and Lemma 6.1 leave us with two possibilities:

(1) The first differential

$$E^1_{p,q} = h^{0,q} \longrightarrow h^{0,q+1} = E^1_{p-1,q+1}$$

is an isomorphism if $q \equiv p + 2 \bmod 4$, and trivial otherwise.

(2) The first differential

$$E^1_{p,q} = h^{0,q} \longrightarrow h^{0,q+1} = E^1_{p-1,q+1}$$

is an isomorphism if $q \equiv p \bmod 4$, and trivial otherwise.

In both cases we find $E^2 = E^\infty$. Our case distinctions can be recast in the following way.

(1) $E^\infty_{p,q} \cong h^{0,0}$ if $q = 0$ and $p \equiv 0 \bmod 4$, and trivial otherwise.

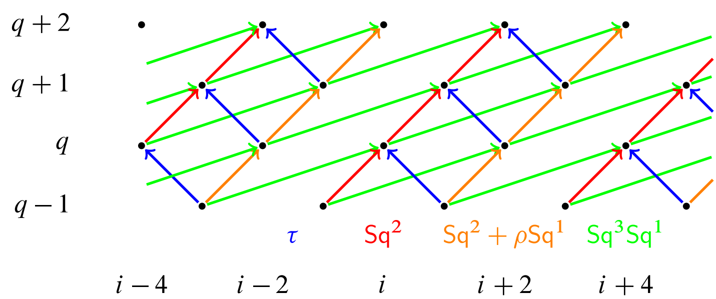
(2) $E^\infty_{p,q} \cong h^{0,0}$ if $q = 0$ and $p \equiv 2 \bmod 4$, and trivial otherwise.

The second case contradicts the vanishing of $\pi_{2,0}(\mathbf{KT})$. Computing with the first condition yields the desired result on the filtration. One extends to arbitrary n by using the commutative diagram

$$\begin{tikzcd}
\pi_{p,n}\mathsf{f}_{q+i}(\mathbf{KT}) \arrow[r] \arrow[d, "\cong"'] & \pi_{p,n}\mathsf{f}_q(\mathbf{KT}) \arrow[d, "\cong"] \\
\pi_{p-n,0}\mathsf{f}_{q-n+i}(\mathbf{KT}) \arrow[r] & \pi_{p-n,0}\mathsf{f}_{q-n}(\mathbf{KT}).
\end{tikzcd}$$

□

The following image depicts the first differential when $2q \equiv i \bmod 4$ with degrees along the horizontal axis and weights along the vertical axis. Each dot is a suspension of $\mathbf{MZ}/2$.



5.4 Hermitian K–theory, II

Next we determine $\mathbf{d}_1^{\mathbf{KQ}}$ by combining Proposition 4.29 with the formula for $\mathbf{d}_1^{\mathbf{KT}}$ in Theorem 5.3. Recall the identifications

$$\mathsf{s}_{2q}(\mathbf{KQ}) \cong \Sigma^{2q,2q}\big(\Sigma^{2q,0}\mathbf{MZ} \vee \textstyle\bigvee_{i<q}\Sigma^{2i,0}\mathbf{MZ}/2\big),$$
$$\mathsf{s}_{2q+1}(\mathbf{KQ}) \cong \Sigma^{2q+1,2q+1}\textstyle\bigvee_{i\leq q}\Sigma^{2i,0}\mathbf{MZ}/2.$$

Let $\mathbf{d}_1^{\mathbf{KQ}}(q,2i)$ denote the restriction of $\mathbf{d}_1^{\mathbf{KQ}}(q)$ to the $2i^{\text{th}}$ summand $\Sigma^{2i,0}\mathbf{MZ}/2$ of $\mathsf{s}_q(\mathbf{KQ})$, where $i \leq \lfloor q/2 \rfloor$. There are canonical maps $\delta\colon \mathbf{MZ}/2 \longrightarrow \Sigma^{1,0}\mathbf{MZ}$ and $\mathrm{pr}\colon \mathbf{MZ} \longrightarrow \mathbf{MZ}/2$; see the appendix. Inspection of the computations in Section 5.2 and Theorem 5.3 yields:

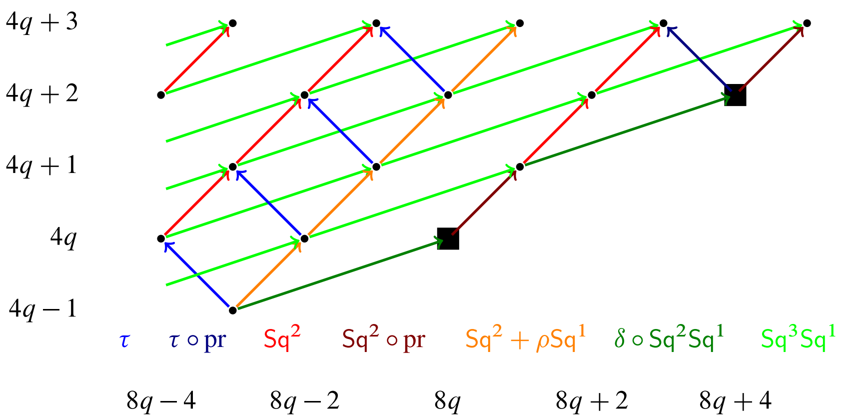
Theorem 5.5 *The $\mathbf{d}_1$–differential in the slice spectral sequence for* $\mathbf{KQ}$ *is given by*

$$\mathbf{d}_1^{\mathbf{KQ}}(q,i) = \begin{cases} (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2, 0), & q-1 > i \equiv 0 \bmod 4, \\ (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \tau), & q-1 > i \equiv 2 \bmod 4, \end{cases}$$

$$d_1^{\mathbf{KQ}}(q,q) = \begin{cases} (0, \mathsf{Sq}^2 \circ \mathrm{pr}, 0), & q \equiv 0 \bmod 4, \\ (0, \mathsf{Sq}^2 \circ \mathrm{pr}, \tau \circ \mathrm{pr}), & q \equiv 2 \bmod 4, \end{cases}$$

$$d_1^{\mathbf{KQ}}(q,q-1) = \begin{cases} (\delta\mathsf{Sq}^2\mathsf{Sq}^1, \mathsf{Sq}^2, 0), & q \equiv 1 \bmod 4, \\ (\delta\mathsf{Sq}^2\mathsf{Sq}^1, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \tau), & q \equiv 3 \bmod 4. \end{cases}$$

The following image depicts the first differential for $\mathbf{KQ}$ with degrees along the horizontal axis and weights along the vertical axis. Each small dot is a suspension of $\mathbf{MZ}/2$ while a large square is a suspension of $\mathbf{MZ}$.

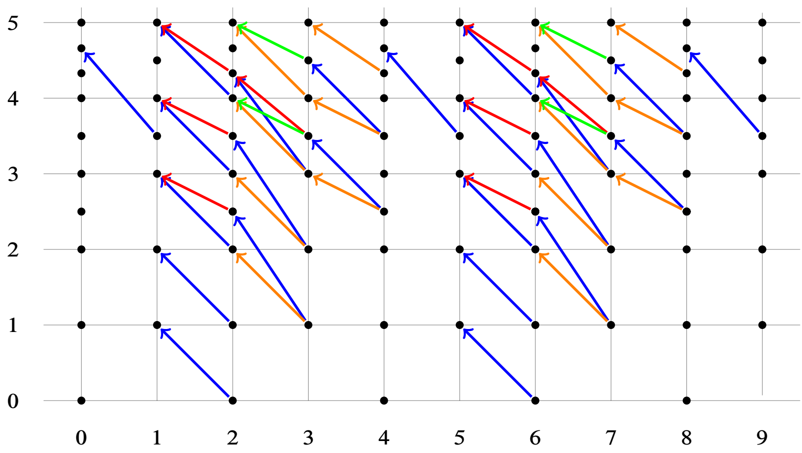


6 Milnor's conjecture on quadratic forms

In this section we compute the slice spectral sequence of **KT** over any field F of char $F \neq 2$. Note that **KT** acquires a ring spectrum structure from hermitian K–theory **KQ** and the tower (7). Recall from Theorem 4.28 (see Example 2.3) the identification

$$\mathsf{s}_0(\mathbf{KT}) \cong \bigvee_{i\in\mathbb{Z}} \Sigma^{2i,0}\mathbf{MZ}/2.$$

By the periodicity $\Sigma^{1,1}\mathbf{KT} \cong \mathbf{KT}$ induced by multiplication with the Hopf map, this determines all slices of **KT**. Recall that the first differential $\mathsf{s}_q(\mathbf{KT}) \longrightarrow \Sigma^{1,0}\mathsf{s}_{q+1}(\mathbf{KT})$ is determined by the motivic Steenrod operations in (9), a formula proven in Theorem 5.3. With the elements $\boldsymbol{\tau}$, $\mathsf{Sq}^{\mathbf{2}}$, $\mathsf{Sq}^{\mathbf{2}} + \boldsymbol{\rho}\mathsf{Sq}^{\mathbf{1}}$, and $\mathsf{Sq}^{\mathbf{3}}\mathsf{Sq}^{\mathbf{1}}$ corresponding to their given colors, the first differentials can be represented as follows. The group in bidegree (p,q) is a direct sum of mod-2 motivic cohomology groups positioned on the vertical line above p and in between the horizontal lines corresponding to weights q and $q+1$. The number of direct summands increases linearly with the weight:



The next statement is an immediate consequence of Voevodsky's proof of Milnor's conjecture on Galois cohomology [61]. Recall the classes $\tau \in h^{0,1}$ and $\rho \in h^{1,1}$ are represented by $-1 \in F$.

Lemma 6.1 *For $0 \leq p \leq q$ cup-product with τ yields an isomorphism $\tau\colon h^{p,q} \xrightarrow{\cong} h^{p,q+1}$.*

Corollary 6.2 *If $a \in h^{p,q}$, where $0 \leq p \leq q$, write $a = \tau^{q-p}c$, where $c \in h^{p,p}$, and let $n = q-p$. The Steenrod squares of weight ≤ 1 act on the mod-2 motivic* *cohomology ring $h^{*,*}$ by*

$$\mathsf{Sq}^1(\tau^n c) = \begin{cases} \rho\tau^{n-1}c, & n \equiv 1 \bmod 2, \\ 0, & n \equiv 0 \bmod 2, \end{cases}$$
$$\mathsf{Sq}^2(\tau^n c) = \begin{cases} \rho^2\tau^{n-1}c, & n \equiv 2, 3 \bmod 4, \\ 0, & n \equiv 0, 1 \bmod 4, \end{cases}$$
$$\mathsf{Sq}^2\mathsf{Sq}^1(\tau^n c) = \begin{cases} \rho^3\tau^{n-2}c, & n \equiv 3 \bmod 4, \\ 0, & n \equiv 0, 1, 2 \bmod 4, \end{cases}$$
$$\mathsf{Sq}^3\mathsf{Sq}^1(\tau^n c) = \begin{cases} \rho^4\tau^{n-3}c, & n \equiv 3 \bmod 4, \\ 0, & n \equiv 0, 1, 2 \bmod 4. \end{cases}$$

Proof This follows from Lemma 6.1, the computation of $\mathsf{Sq}^i(\tau^n)$ for $i \in \{1, 2\}$ and the Cartan formula [62, Proposition 9.6]. □

We consider an element $\phi \in h^{m,n}$ as a stable motivic cohomology operation of bidegree (m, n) with the same name via multiplication with ϕ on the left. Only elements of bidegrees $(0, 1)$, $(1, 1)$, and $(1, 2)$ will be relevant here. The Adem relations in weight ≤ 2 are given by

$$\mathsf{Sq}^1\mathsf{Sq}^1 = 0, \quad \mathsf{Sq}^1\tau = \tau\mathsf{Sq}^1 + \rho, \quad \mathsf{Sq}^1\rho = \rho\mathsf{Sq}^1, \quad \mathsf{Sq}^1\mathsf{Sq}^2 = \mathsf{Sq}^3,$$
$$\mathsf{Sq}^1\mathsf{Sq}^3 = 0, \quad \mathsf{Sq}^2\tau = \tau\mathsf{Sq}^2 + \tau\rho\mathsf{Sq}^1, \quad \mathsf{Sq}^2\rho = \rho\mathsf{Sq}^2, \quad \mathsf{Sq}^2\mathsf{Sq}^2 = \tau\mathsf{Sq}^3\mathsf{Sq}^1,$$
$$\mathsf{Sq}^2\mathsf{Sq}^3 = \mathsf{Sq}^5 + \mathsf{Sq}^4\mathsf{Sq}^1, \quad \mathsf{Sq}^3\mathsf{Sq}^2 = \rho\mathsf{Sq}^3\mathsf{Sq}^1, \quad \mathsf{Sq}^3\mathsf{Sq}^3 = \mathsf{Sq}^5\mathsf{Sq}^1.$$

This concludes the prerequisites for our proof of the following result.

Theorem 6.3 *The 0^{th} slice spectral sequence for* **KT** *collapses at its E^2–page, and*

$$E^\infty_{p,q}(\mathbf{KT}) \cong \begin{cases} h^{q,q}, & p \equiv 0 \bmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Proof The E^1–page takes the form

$$E^1_{p,q} = \pi_{p,0}\mathsf{s}_q(\mathbf{KT}) = \bigoplus_{i\in\mathbb{Z}} h^{2i+(q-p),q} = \begin{cases} \bigoplus_{j=0}^{\lfloor q/2\rfloor} h^{2j,q}, & q \equiv p \bmod 2, \\ \bigoplus_{j=0}^{\lfloor (q-1)/2\rfloor} h^{2j+1,q}, & q \not\equiv p \bmod 2. \end{cases}$$

The group $h^{2i+(q-p),q}$ in the sum $\bigoplus_{i\in\mathbb{Z}} h^{2i+(q-p),q}$ arises from the $2i^{\text{th}}$ summand of $\mathsf{s}_q(\mathbf{KT})$. The vanishing of $h^{p,q}$ for $p < 0$ and $p > q$ shows the sum is finite. Hence for every element $a \in E^1_{p,q}$ there exists a unique collection of elements $\{a_j \in h^{j,q}\}$

such that

$$a = \begin{cases} (a_q, a_{q-2}, \dots, a_0) \in E^1_{p,q}, & q \equiv 0 \equiv p \bmod 2, \\ (a_{q-1}, a_{q-3}, \dots, a_0) \in E^1_{p,q}, & q \equiv 1 \equiv p \bmod 2, \\ (a_q, a_{q-2}, \dots, a_1) \in E^1_{p,q}, & q \equiv 1 \not\equiv p \bmod 2, \\ (a_{q-1}, a_{q-3}, \dots, a_1) \in E^1_{p,q}, & q \equiv 0 \not\equiv p \bmod 2. \end{cases}$$

By inspection of (9) the components of $d_1^{\mathbf{KT}}$ are given by

$$d_1^{\mathbf{KT}}(a)_j = \begin{cases} \mathsf{Sq}^3\mathsf{Sq}^1 a_{j-4} + \mathsf{Sq}^2 a_{j-2} + \rho\mathsf{Sq}^1 a_{j-2}, & j \equiv q-p \bmod 4, \\ \mathsf{Sq}^3\mathsf{Sq}^1 a_{j-4} + \mathsf{Sq}^2 a_{j-2} + \tau a_j, & j \equiv q-p+2 \bmod 4. \end{cases} \tag{27}$$

Here j is the dimension index of the target group. The formula can be read off from the parity of the dimension index of the source group. Note that $\rho\mathsf{Sq}^1$, Sq^2 and τ shift the dimension by 2, 2 and 0, respectively. We compute the E^2–page by repeatedly using the Steenrod square computations and Adem relations given in the beginning of Section 6.

$p \equiv 0$ mod 4 For $a = (a_q, a_{q-2}, \dots)$, Corollary 6.2 implies

$$d_1^{\mathbf{KT}}(a)_j = \begin{cases} \mathsf{Sq}^2 a_{j-2}, & j \equiv q \bmod 4, \\ \tau a_j, & j \equiv q+2 \bmod 4. \end{cases}$$

If $d_1^{\mathbf{KT}}(a) = 0$ the injectivity part of Lemma 6.1 implies $0 = a_{q-2} = a_{q-6} = \cdots$, so that

$$\ker(d_1^{\mathbf{KT}}) = h^{q,q} \oplus h^{q-4,q} \oplus \cdots.$$

We may assume $q > 0$. For $b \in E^1_{p+1,q-1}$ the entering differential is given by

$$d_1^{\mathbf{KT}}(b)_j = \begin{cases} \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \mathsf{Sq}^2 b_{j-2} + \tau b_j, & j \equiv q-p \bmod 4, \\ \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \mathsf{Sq}^2 b_{j-2} + \rho\mathsf{Sq}^1 b_{j-2}, & j \equiv q-p+2 \bmod 4. \end{cases}$$

Corollary 6.2 simplifies this formula to

$$d_1^{\mathbf{KT}}(b)_j = \begin{cases} \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \tau b_j, & j \equiv q-p \bmod 4, \\ 0, & j \equiv q-p+2 \bmod 4. \end{cases}$$

For example, if $p \equiv 0 \bmod 4$, then $j \equiv q-p \bmod 4$ implies $j \equiv q \bmod 4$. Thus for $b_{j-2} \in h^{j-2,q-1}$ one has $q-1-(j-2) \equiv 1 \bmod 4$, whence $\mathsf{Sq}^2 b_{j-2} = 0$. It follows that $E^2_{p,q}$ is the homology of the complex

$$h^{q-4,q-1} \oplus h^{q-8,q-1} \oplus \cdots \xrightarrow{\alpha} h^{q,q} \oplus h^{q-4,q} \oplus \cdots \longrightarrow 0,$$

where $\alpha(b_{q-4}, b_{q-8}, \dots, b_m) = (\mathsf{Sq}^3\mathsf{Sq}^1 b_{q-4}, \mathsf{Sq}^3\mathsf{Sq}^1 b_{q-8} + \tau b_{q-4}, \dots, \tau b_m)$. Here $m \equiv q \bmod 4$ and $0 \le m \le 3$. Lemma 6.1 implies α is split injective by mapping

$(a_q, a_{q-4}, \dots, a_m)$ to

$$\big(\tau^{-1}a_{q-4} + \phi(a_{q-4} + \tau\phi(a_{q-8} + \cdots + \tau\phi(a_m))), \dots, \tau^{-1}a_{m+4} + \phi(a_m), \tau^{-1}a_m\big),$$

where ϕ is the composite map

$$h^{p,q} \xrightarrow{\tau^{-1}} h^{p,q-1} \xrightarrow{\mathsf{Sq}^3\mathsf{Sq}^1} h^{p+4,q} \xrightarrow{\tau^{-1}} h^{p+4,q-1}.$$

It follows that $E^2_{p,q} \cong h^{q,q}$ for all $p \equiv 0 \bmod 4$.

***p* ≡ 1 mod 4** For $a = (a_{q-1}, a_{q-3}, \dots)$, Corollary 6.2 implies

$$d_1^{\mathbf{KT}}(a)_j = \begin{cases} 0, & j \equiv q-1 \bmod 4, \\ \mathsf{Sq}^3\mathsf{Sq}^1 a_{j-4} + \tau a_j, & j \equiv q+1 \bmod 4. \end{cases}$$

If $d_1^{\mathbf{KT}}(a) = 0$ then $0 = a_{q-3} = a_{q-7} = \cdots$ by applying inductively the injectivity statement in Lemma 6.1. Thus we have

$$\ker(d_1^{\mathbf{KT}}) = h^{q-1,q} \oplus h^{q-5,q} \oplus \cdots .$$

For $b \in E^1_{p+1,q-1}$ the entering differential is given by

$$d_1^{\mathbf{KT}}(b)_j = \begin{cases} \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \mathsf{Sq}^2 b_{j-2} + \tau b_j, & j \equiv q-p \bmod 4, \\ \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \mathsf{Sq}^2 b_{j-2} + \rho\mathsf{Sq}^1 b_{j-2}, & j \equiv q-p+2 \bmod 4. \end{cases}$$

Corollary 6.2 simplifies this formula to

$$d_1^{\mathbf{KT}}(b)_j = \begin{cases} \mathsf{Sq}^2 b_{j-2} + \tau b_j, & j \equiv q-p \bmod 4, \\ 0, & j \equiv q-p+2 \bmod 4. \end{cases}$$

Thus $E^2_{p,q}$ is the homology of the complex

$$h^{q-1,q-1} \oplus h^{q-3,q-1} \oplus \cdots \xrightarrow{d_1^{\mathbf{KT}}} h^{q-1,q} \oplus h^{q-5,q} \oplus \cdots \longrightarrow 0.$$

Since the restriction of $d_1^{\mathbf{KT}}$ to $h^{q-1,q-1} \oplus h^{q-5,q-1} \oplus \cdots$ is surjective by Lemma 6.1, $E^2_{p,q} = 0$.

***p* ≡ 2 mod 4** For $a = (a_q, a_{q-2}, \dots)$, Corollary 6.2 implies

$$d_1^{\mathbf{KT}}(a)_j = \begin{cases} 0, & j \equiv q-2 \bmod 4, \\ \mathsf{Sq}^2 a_{j-2} + \tau a_j, & j \equiv q \bmod 4. \end{cases}$$

Hence the subgroup $\ker(d_1^{\mathbf{KT}})$ can be identified with

$$\{(a_q, a_{q-2}, \dots) \in h^{q,q} \oplus h^{q-2,q} \oplus \cdots : \tau a_j = \mathsf{Sq}^2 a_{j-2} \text{ for all } j \equiv q \bmod 4,\ 0 \le j \le q\}.$$

For $b \in E^1_{p+1,q-1}$ the entering differential is given by

$$d_1^{\mathbf{KT}}(b)_j = \begin{cases} \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \mathsf{Sq}^2 b_{j-2} + \rho\mathsf{Sq}^1 b_{j-2}, & j \equiv q \bmod 4, \\ \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \mathsf{Sq}^2 b_{j-2} + \tau b_j, & j \equiv q-2 \bmod 4. \end{cases}$$

Corollary 6.2 simplifies this formula to

$$d_1^{\mathbf{KT}}(b)_j = \begin{cases} \mathsf{Sq}^3\mathsf{Sq}^1 b_{j-4} + \rho\mathsf{Sq}^1 b_{j-2}, & j \equiv q \bmod 4, \\ \mathsf{Sq}^2 b_{j-2} + \tau b_j, & j \equiv q-2 \bmod 4. \end{cases}$$

If $a \in \ker(d_1^{\mathbf{KT}})$, Lemma 6.1 shows there exist elements $b_{j,q-1} \in h^{j,q-1}$ for all $0 \le j < q$ where $j \equiv q-2 \bmod 4$ and $\tau b_{j,q-1} = a_j$. For these indices j, the Adem relation $\mathsf{Sq}^2\tau = \tau\mathsf{Sq}^2 + \tau\rho\mathsf{Sq}^1$ and $d_1^{\mathbf{KT}}(a) = 0$ imply

$$\tau a_{j+2} = \mathsf{Sq}^2 a_j = \mathsf{Sq}^2\tau b_j = \tau\mathsf{Sq}^2 b_j + \tau\rho\mathsf{Sq}^1 b_j = \tau\rho\mathsf{Sq}^1 b_j.$$

Thus $\rho\mathsf{Sq}^1 b_j = a_{j+2}$ by Lemma 6.1. It follows that

$$\begin{aligned} d_1^{\mathbf{KT}}(b_{q-2}, 0, b_{q-6}, \dots) &= (\rho\mathsf{Sq}^1 b_{q-2}, \tau b_{q-2}, \rho\mathsf{Sq}^1 b_{q-6}, \tau b_{q-6}) \\ &= (a_q, a_{q-2}, a_{q-4}, a_{q-6}, \dots). \end{aligned}$$

This shows that $E^2_{p,q}$ is trivial.

***p* ≡ 3 mod 4** For $a = (a_{q-1}, a_{q-3}, \dots)$ the exiting differential simplifies to

$$d_1^{\mathbf{KT}}(a)_j = \begin{cases} \mathsf{Sq}^3\mathsf{Sq}^1 a_{j-4} + \rho\mathsf{Sq}^1 a_{j-2}, & j \equiv q-3 \bmod 4, \\ \mathsf{Sq}^2 a_{j-2} + \tau a_j, & j \equiv q-1 \bmod 4 \end{cases}$$

by Corollary 6.2. Applying Sq^2 to $\mathsf{Sq}^2 a_{j-2} + \tau a_j$ yields

$$\mathsf{Sq}^2(\mathsf{Sq}^2 a_{j-2} + \tau a_j) = \tau\mathsf{Sq}^3\mathsf{Sq}^1 a_{j-2} + \tau\mathsf{Sq}^2 a_j + \tau\rho\mathsf{Sq}^1 a_j.$$

Thus $\ker(d_1^{\mathbf{KT}})$ is comprised of tuples $(a_{q-1}, a_{q-3}, \dots) \in h^{q-1,q} \oplus h^{q-3,q} \oplus \cdots$ for which $\tau a_j = \mathsf{Sq}^2 a_{j-2}$ whenever $j \equiv q-1 \bmod 4$, $0 \le j < q$. For $b \in E^1_{p+1,q-1}$ the entering differential simplifies to

$$d_1^{\mathbf{KT}}(b)_j = \begin{cases} \tau b_j, & j \equiv q-3 \bmod 4, \\ \mathsf{Sq}^2 b_{j-2}, & j \equiv q-1 \bmod 4. \end{cases}$$

If $a \in \ker(d_1^{\mathbf{KT}})$, Lemma 6.1 shows there exist elements $b_{j,q-1} \in h^{j,q-1}$ for all $0 \le j < q$ where $j \equiv q-3 \bmod 4$ and $\tau b_{j,q-1} = a_j$. For these j, $d_1^{\mathbf{KT}}(a) = 0$ implies

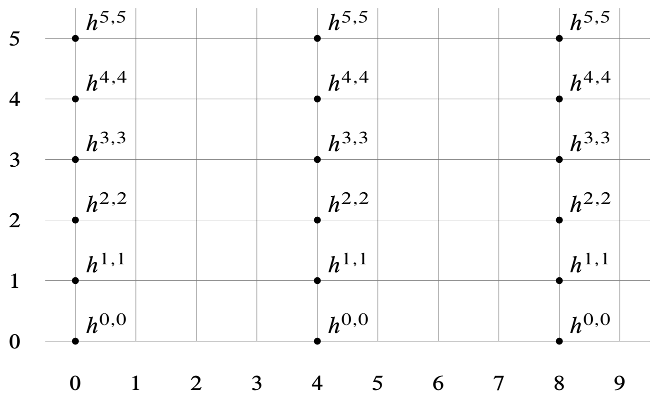
$$\tau a_{j+2} = \mathsf{Sq}^2 a_j = \mathsf{Sq}^2\tau b_j = \tau\mathsf{Sq}^2 b_j + \tau\rho\mathsf{Sq}^1 b_j = \tau\mathsf{Sq}^2 b_j.$$

Hence $\mathsf{Sq}^2 b_j = a_{j+2}$ by Lemma 6.1. It follows that

$$\begin{aligned} d_1^{\mathbf{KT}}(b_{q-3}, 0, b_{q-7}, \dots) &= (\mathsf{Sq}^2 b_{q-3}, \tau b_{q-3}, \mathsf{Sq}^2 b_{q-7}, \tau b_{q-7}) \\ &= (a_{q-1}, a_{q-3}, a_{q-5}, a_{q-7}, \dots). \end{aligned}$$

This shows that $E^2_{p,q}$ is trivial.

An inspection of the E^2–page shows that the 0^{th} slice spectral sequence for $\mathbf{KT}$ collapses with $E^2 = E^\infty$–page



□

Let $I(F)$ denote the fundamental ideal of even-dimensional quadratic forms in the Witt ring $W(F)$. To conclude our proof of Milnor's conjecture on quadratic forms, it remains to identify the slice filtration on the Witt ring with the filtration given by the fundamental ideal.

Lemma 6.4 *The slice filtration of* $\mathbf{KT}$ *induces a commutative diagram*

$$\begin{tikzcd} \pi_{0,0}\mathsf{f}_1(\mathbf{KT}) \arrow[r, hook] \arrow[d, "\cong"'] & \pi_{0,0}\mathsf{f}_0(\mathbf{KT}) \arrow[d, "\cong"] \\ I(F) \arrow[r, hook] & W(F). \end{tikzcd}$$

Proof By [13, Corollary 5.18] the canonical map $\mathsf{f}_0(\mathbf{KT}) \longrightarrow \mathsf{s}_0(\mathbf{KT})$ induces a ring homomorphism

$$W(F) \cong \pi_{0,0}\mathsf{f}_0(\mathbf{KT}) \longrightarrow \pi_{0,0}\mathsf{s}_0(\mathbf{KT}) \cong h^{0,0} \cong \mathbb{F}_2.$$

It is natural with respect to separable field extensions according to Corollary 2.7. Comparing with an algebraic closure of F shows this ring homomorphism is induced by sending a quadratic form to its rank. Since the group $\pi_{1,0}\mathsf{s}_0(\mathbf{KT}) = h^{1,0}$ is trivial, the map

$$\pi_{0,0}\mathsf{f}_1(\mathbf{KT}) \longrightarrow \pi_{0,0}\mathsf{f}_0(\mathbf{KT})$$

is injective, which proves the result. □

Corollary 6.5 *The identification $\pi_{0,0}\mathbf{KT} \cong W(F)$ induces an inclusion $I(F)^q \subseteq \mathsf{f}_q\pi_{0,0}\mathbf{KT}$.*

Proof Since $\mathbf{KT}$ is a ring spectrum the claim follows from the multiplicative structure of the slice filtration [13, Theorem 5.15], [40, Theorem 3.6.9], and Lemma 6.4. □

Any rational point $u \in \mathbb{A}^1 \smallsetminus \{0\}(F)$ defines a map of motivic spectra $[u]\colon \mathbf{1} \longrightarrow S^{1,1}$. We are interested in the effect of the map $[u]$ on motivic cohomology and $\mathbf{KT}$.

Lemma 6.6 *The map*

$$H^{0,0} = [\mathbf{1}, \mathbf{MZ}] = [\mathbf{1}, \mathbf{1} \wedge \mathbf{MZ}] \xrightarrow{([u]\wedge\mathbf{MZ})_*} [\mathbf{1}, \Sigma^{1,1}\mathbf{MZ}] = H^{1,1}$$

sends 1 to $u \in \mathbb{A}^1 \smallsetminus \{0\}(F) = F^\times \cong H^{1,1}$.

Proof Let $\mathbb{Z}(1)$ denote the Tate object in the derived category of motives DM_F over F. The assertion follows from the canonically induced diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Sm}_F}(F, \mathbb{A}^1 \smallsetminus \{0\}) \arrow[r] \arrow[d] & {[\mathbf{1}, \Sigma^{1,1}\mathbf{MZ}]} \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathrm{DM}_F}(\mathbb{Z}, \mathcal{O}^\times) & \mathrm{Hom}_{\mathrm{DM}_F}(\mathbb{Z}, \mathbb{Z}(1)[1]). \arrow[l, "\cong"'] \end{tikzcd}$$

The lower horizontal isomorphism follows from [31, Theorem 4.1]. □

Corollary 6.7 *Suppose $u_1, \dots, u_q \in \mathbb{A}^1 \smallsetminus \{0\}(F)$ are rational points. The map $H^{0,0} \longrightarrow H^{q,q}$ induced by the smash product $[u_1]\wedge\cdots\wedge[u_q]$ sends 1 to $\{u_1, \dots, u_q\} \in K^M_q \cong H^{q,q}$, and likewise for $h^{0,0} \longrightarrow h^{q,q}$.*

Lemma 6.8 *The composition $\mathbf{1} \xrightarrow{[u]} S^{1,1} \xrightarrow{\eta} \mathbf{1}$ induces multiplication by $\langle u\rangle - 1$ on $\pi_{0,0}\mathbf{1}$, where $\langle u\rangle$ is the class of the rank-one quadratic form in the Grothendieck–Witt ring defined by u.*

Proof This follows from [37, Corollary 1.24]. □

The unit map for $\mathbf{KT}$ induces the canonical map from the Grothendieck–Witt ring to the Witt ring $\pi_{0,0}\mathbf{1} \longrightarrow \pi_{0,0}\mathbf{KT}$. By definition, multiplication by η is an isomorphism on $\pi_{*,*}\mathbf{KT}$. Thus the "multiplication by $[u]$" map on $\pi_{0,0}\mathbf{KT}$ is determined by its effect on the Grothendieck–Witt ring $\pi_{0,0}\mathbf{1}$. Recall that $\mathsf{f}_q\pi_{p,n}\mathsf{E}$ denotes the image of the canonical map $\pi_{p,n}\mathsf{f}_q(E) \longrightarrow \pi_{p,n}\mathsf{E}$.

Lemma 6.9 *There is a canonically induced short exact sequence*

$$0 \longrightarrow \mathsf{f}_{q+1}\pi_{0,0}\mathbf{KT} \xrightarrow{j_q} \mathsf{f}_q\pi_{0,0}\mathbf{KT} \longrightarrow h^{q,q} \longrightarrow 0.$$

Proof Theorem 6.3 shows the exact sequence in the proof of [59, Lemma 7.2] takes the form

$$0 \longrightarrow \mathsf{f}_q\pi_{0,0}\mathbf{KT}/\mathsf{f}_{q+1}\pi_{0,0}\mathbf{KT} \xrightarrow{\alpha} h^{q,q} \longrightarrow \bigcap_{i\geq 1} \mathsf{f}_{q+i}\pi_{-1,0}\mathsf{f}_q(\mathbf{KT}) \longrightarrow \bigcap_{i\geq 0} \mathsf{f}_{q+i}\pi_{-1,0}\mathbf{KT} \longrightarrow 0.$$

Moreover, $\pi_{-1,0}\mathbf{KT}$ is the trivial group. Corollary 6.5 furnishes a map

$$I(F)^q/I(F)^{q+1} \xrightarrow{\beta} \mathsf{f}_q\pi_{0,0}\mathbf{KT}/\mathsf{f}_{q+1}\pi_{0,0}\mathbf{KT}.$$

Combined with the canonical surjective map $k_q^M \xrightarrow{\gamma} I(F)^q/I(F)^{q+1}$ from the q^{th} mod-2 Milnor K–theory group defined in [32], we obtain the composite map

$$\alpha\circ\beta\circ\gamma\colon k_q^M \longrightarrow h^{q,q}. \tag{28}$$

Lemmas 6.6 and 6.8 show that (28) coincides with Suslin's isomorphism between Milnor K–theory and the diagonal of motivic cohomology [31, Lecture 5]. In particular, α is surjective. □

The above shows that γ is injective, which gives an alternate proof of the main result in [38].

Theorem 6.10 *The canonical map $k_q^M \longrightarrow I(F)^q/I(F)^{q+1}$ is an isomorphism for $q\geq 0$.*

Corollary 6.11 *The identification $\pi_{0,0}\mathbf{KT}\cong W(F)$ induces an equality $I(F)^q = \mathsf{f}_q\pi_{0,0}\mathbf{KT}$ for $q\geq 0$.*

Proof By the definition of β there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I(F)^{q+1} \arrow[r] \arrow[d] & I(F)^q \arrow[r] \arrow[d] & I(F)^q/I(F)^{q+1} \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathsf{f}_{q+1}\pi_{0,0}\mathbf{KT} \arrow[r, "j_q"] & \mathsf{f}_q\pi_{0,0}\mathbf{KT} \arrow[r] & h^{q,q} \arrow[r] & 0.
\end{tikzcd}$$

Note that β is an isomorphism by Lemma 6.9, Theorem 6.10, and the isomorphism (28). Thus the result follows by induction using the identification

$$I(F) = \mathsf{f}_1\pi_{0,0}\mathbf{KT} = \pi_{0,0}\mathsf{f}_1\mathbf{KT}$$

in Lemma 6.4. □

This finishes our proof of Milnor's conjecture on quadratic forms.

Theorem 6.12 *The image of $\pi_{4p+q,q}\mathsf{f}_n(\mathbf{KT})$ in $\pi_{4p+q,q}(\mathbf{KT}) \cong W(F)$ coincides with $I^{n-q}(F)$, where $I(F) \subseteq W(F)$ is the fundamental ideal. Thus the slice spectral sequence for $\mathbf{KT}$ converges to the filtration of the Witt ring given by the fundamental ideal.*

Proof This follows from Corollary 6.11 and the main result in [1], which shows that the filtration of $W(F)$ by $I(F)$ is Hausdorff. For completeness we analyze the map $\pi_{p,0}\mathsf{f}_{q+1}(\mathbf{KT}) \longrightarrow \pi_{p,0}\mathsf{f}_q(\mathbf{KT})$ for columns $p \equiv 1, 2, 3 \bmod 4$ in Lemma 6.13. □

Lemma 6.13 *Let $q \geq 0$. The canonical map $\mathsf{f}_{q+1}(\mathbf{KT}) \longrightarrow \mathsf{f}_q(\mathbf{KT})$ induces the trivial map*

$$\pi_{p,0}\mathsf{f}_{q+1}(\mathbf{KT}) \longrightarrow \pi_{p,0}\mathsf{f}_q(\mathbf{KT})$$

for $p \equiv 1, 2, 3 \bmod 4$.

Proof The statement is clear for $q = 0$. Suppose that $p \equiv 1, 2 \bmod 4$ and $x \in \pi_{p,0}\mathsf{f}_{q+1}(\mathbf{KT})$. Its image $y \in \pi_{p,0}\mathsf{f}_q(\mathbf{KT})$ lies in the kernel of the map $\pi_{p,0}\mathsf{f}_q(\mathbf{KT}) \longrightarrow \pi_{p,0}\mathsf{s}_q(\mathbf{KT})$. By induction, the map $\pi_{p+1,0}\mathsf{s}_{q-1}(\mathbf{KT}) \longrightarrow \pi_{p,0}\mathsf{f}_q(\mathbf{KT})$ is surjective. Hence there is an element $z \in \pi_{p+1,0}\mathsf{s}_{q-1}(\mathbf{KT})$ mapping to y. Thus z lies in the kernel of $d_1(p+1, q-1)$. Theorem 6.3 shows the kernel of $d_1(p+1, q-1)$ coincides with the image of $d_1(p+2, q)$. In particular, there is an element in $\pi_{p+2,q}\mathsf{s}_q(\mathbf{KT})$ whose image is z, showing that $y = 0$.

Suppose now that $p \equiv 3 \bmod 4$ and $x \in \pi_{p,0}\mathsf{f}_{q+1}(\mathbf{KT})$. Consider its image $y \in \pi_{p,0}\mathsf{s}_{q+1}(\mathbf{KT})$. Since its image under $d_1(p, q+1)$ is trivial, Theorem 6.3 furnishes an element $z \in \pi_{p+1,0}\mathsf{s}_{q+2}(\mathbf{KT})$ whose image under $d_1(p+1, q+2)$ is precisely y. Consider the difference $x - w$, where w is the image of z in $\pi_{p,0}\mathsf{f}_{q+1}(\mathbf{KT})$. The image of $x - w$ in $\pi_{p,0}\mathsf{f}_q(\mathbf{KT})$ then coincides with the image of x. Since the image of $x - w$ in $\pi_{p,0}\mathsf{s}_{q+1}(\mathbf{KT})$ is zero, there is an element $v \in \pi_{p,0}\mathsf{f}_{q+2}(\mathbf{KT})$ mapping to $x - w$. Proceeding inductively yields an element

$$e \in \bigcap_{i \geq 1} \mathsf{f}_{q+i}\pi_{p,0}\mathsf{f}_q(\mathbf{KT}).$$

However, using that $\pi_{p,0}\mathbf{KT}=0$, this group is trivial by the exact sequence

$$0\longrightarrow \mathsf{f}_q\pi_{p+1,0}\mathbf{KT}/\mathsf{f}_{q+1}\pi_{p+1,0}\mathbf{KT}\longrightarrow h^{q,q}\\ \longrightarrow \bigcap_{i\geq 1}\mathsf{f}_{q+i}\pi_{p,0}\mathsf{f}_q(\mathbf{KT})\longrightarrow \bigcap_{i\geq 0}\mathsf{f}_{q+i}\pi_{p,0}\mathbf{KT}\longrightarrow 0$$

from the proof of [59, Lemma 7.2]. Hence $e=0$, and the image of x in $\pi_{p,0}\mathsf{f}_q(\mathbf{KT})$ is zero. □

Corollary 6.14 *Let $q\geq 0$ and $p\equiv 2,3 \bmod 4$. There is a canonically induced split short exact sequence of $\mathbb{F}_2$–modules*

$$0\longrightarrow \pi_{p,0}\mathsf{f}_q(\mathbf{KT})\longrightarrow \pi_{p,0}\mathsf{s}_q(\mathbf{KT})\longrightarrow \pi_{p-1,0}\mathsf{f}_{q+1}(\mathbf{KT})\longrightarrow 0.$$

Corollary 6.15 *For $q\geq 0$ there are canonically induced isomorphisms*

$$\pi_{p,0}\mathsf{f}_q(\mathbf{KT})\cong\begin{cases} h^{q-1,q}\oplus h^{q-5,q}\oplus\cdots, & p\equiv 1 \bmod 4,\\ h^{q-2,q}\oplus h^{q-6,q}\oplus\cdots, & p\equiv 2 \bmod 4,\\ h^{q-3,q}\oplus h^{q-7,q}\oplus\cdots, & p\equiv 3 \bmod 4.\end{cases}$$

Proof Use Theorem 6.3, Lemma 6.13, and Corollary 6.14. □

Corollary 6.16 *For $q\geq 0$ the canonical map $\Sigma^{1,0}\mathsf{s}_q(\mathbf{KT})\longrightarrow \mathsf{f}_{q+1}(\mathbf{KT})$ induces a split short exact sequence*

$$0\longrightarrow h^{q-3,q}\oplus h^{q-7,q}\oplus\cdots\longrightarrow \pi_{0,0}\mathsf{f}_{q+1}(\mathbf{KT})\longrightarrow \mathsf{f}_{q+1}\pi_{0,0}\mathbf{KT}=I(F)^{q+1}\longrightarrow 0.$$

Moreover, the map

$$\pi_{0,0}\mathsf{f}_q(\mathbf{KT})\longrightarrow \pi_{0,0}\mathsf{f}_{q-1}(\mathbf{KT})$$

is injective on the image of $\pi_{0,0}\mathsf{f}_{q+1}(\mathbf{KT})$.

Proof The latter claim follows by a diagram chase and Theorem 6.3, since $E^2_{p,q}=0$ if $p\equiv 1 \bmod 4$. Hence the exact sequence

$$\cdots\xrightarrow{\beta}\pi_{1,0}\mathsf{s}_q(\mathbf{KT})\xrightarrow{\alpha}\pi_{0,0}\mathsf{f}_{q+1}(\mathbf{KT})\longrightarrow \pi_{0,0}\mathsf{f}_q(\mathbf{KT})\longrightarrow\cdots$$

induces the short exact sequence

$$0\longrightarrow \pi_{1,0}\mathsf{s}_q(\mathbf{KT})/\mathrm{Ker}(\alpha)\longrightarrow \pi_{0,0}\mathsf{f}_{q+1}(\mathbf{KT})\longrightarrow \mathsf{f}_{q+1}\pi_{0,0}(\mathbf{KT})\longrightarrow 0.$$

Since $\mathrm{Ker}(\alpha)=\mathrm{Im}(\beta)=\mathrm{Im}(d_{2,q-1})=h^{q-1,q}\oplus h^{q-5,q}\oplus\cdots$ by Theorem 6.3, the sequence is short exact. It splits by Lemma 6.1, since the composition of

$$h^{q-3,q}\oplus h^{q-7,q}\oplus\cdots\longrightarrow \pi_{0,0}\mathsf{f}_{q+1}$$

and

$$\pi_{0,0}\mathsf{f}_{q+1}(\mathbf{KT}) \longrightarrow \pi_{0,0}\mathsf{s}_{q+1}(\mathbf{KT}) \cong h^{q+1,q+1} \oplus h^{q-1,q+1} \oplus \cdots \xrightarrow{\mathrm{pr}} h^{q-3,q+1} \oplus h^{q-7,q+1} \oplus \cdots$$

is given by multiplication with $\tau \in h^{0,1}$. □

Theorems 6.3 and 6.12 imply Theorem 1.1 stated in the introduction. If $X \in \mathrm{Sm}_F$ is a semilocal scheme and F a field of characteristic zero, our computations and results extend to the Witt ring $W(X)$ with fundamental ideal $I(X)$ and the mod-2 motivic cohomology of X. Our reliance on the Milnor conjecture for Galois cohomology [61] can be replaced by [14, Section 2.2] or [23, Theorem 7.8], while the isomorphism (28) holds for X by [23, Theorem 7.6]. The rest of the proof is identical to the one given for fields. Kerz proved a closely related result in [23, Theorem 7.10]. By periodicity of $\mathbf{KT}$ there is an evident variant of Theorem 6.3 for the n^{th} slice spectral sequence of $\mathbf{KT}$ for every $n \in \mathbb{Z}$. We note the following result from [2] is transparent from our computation of the slice spectral sequence for $\mathbf{KT}$.

Corollary 6.17 *If $X \in \mathrm{Sm}_F$ is a semilocal scheme of geometric origin then $W(X)$ contains no elements of odd order. If X is not formally real then $W(X)$ is a 2–primary torsion group.*

7 Hermitian K–groups

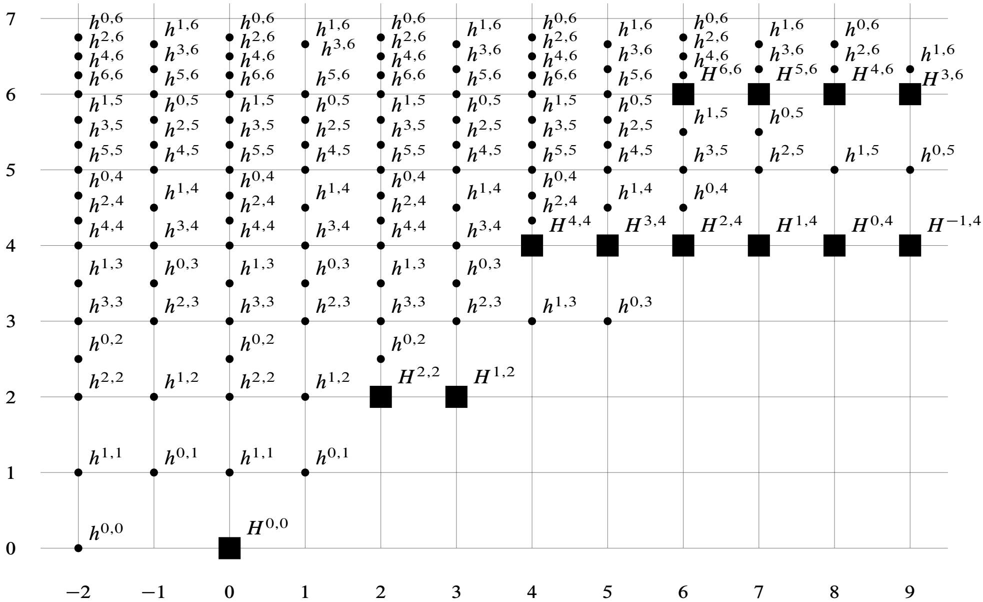
According to Theorem 4.18 the E^1–page of the 0^{th} slice spectral sequence for $\mathbf{KQ}$ takes the form

$$E^1_{p,q} = \pi_{p,0}\mathsf{s}_q(\mathbf{KQ}) = \begin{cases} H^{2q-p,q} \oplus \bigoplus_{i<q/2} h^{2i+(q-p),q}, & q \equiv 0 \bmod 2, \\ \bigoplus_{i \leq (q-1)/2} h^{2i+(q-p),q}, & q \equiv 1 \bmod 2. \end{cases}$$

Using the formula in Theorem 5.5 for the first differentials we are ready to perform low-degree computations in the slice spectral sequence for $\mathbf{KQ}$. We assume throughout that F is a field of char $F \neq 2$.

The Beilinson–Soulé vanishing conjecture predicts the integral motivic cohomology group $H^{2q-p,q}$ is trivial if $p > 2q$, which holds for instance for finite fields and number fields. (The same group is uniquely divisible if $p \geq 2q$, except when $p = q = 0$.) We note that $H^{4-p,2} = 0$ for $p \geq 4$ by comparison with Lichtenbaum's weight-2 motivic complex $\Gamma(2)$ [7, Section 7; 30].

In the following table for the E^1–page of the slice spectral sequence for **KQ**, the group $E^1_{p,q}$ in bidegree (p,q) is a direct sum of motivic cohomology groups positioned on the vertical line above p and in between the horizontal lines corresponding to the weights q and $q+1$.



A comparison with **KT** via Proposition 4.29 implies:

Lemma 7.1 *Suppose that $p \leq 3$. Then the canonical map induces an identification $E^2_{p,q}(\mathbf{KQ}) = E^2_{p,q}(\mathbf{KT})$ for all $q \geq p+1$ if p is even and for all $q \geq p+2$ if p is odd. If F satisfies the Beilinson–Soulé vanishing conjecture, the identification holds also for $p > 3$.*

Lemma 7.1 combined with the classical computation of $\pi_{0,0}\mathbf{KQ}$ as the Grothendieck–Witt group of F basically determines the 0^{th} and 1^{st} columns of the 0^{th} slice spectral sequence of **KQ**. In the 1^{st} column we note that there are no entering or exiting differentials in weight $q \leq 2$. In weight 3 the exiting differential has kernel $h^{2,3}$ by Corollary 6.2. Moreover, the differential

$$E^1_{2,2}(\mathbf{KQ}) = H^{2,2} \oplus h^{0,2} \longrightarrow h^{2,2} \oplus h^{0,2} \xrightarrow{\tau + \mathsf{Sq}^2} h^{2,3}$$

is surjective by Lemma 6.1 and Corollary 6.2. All d_r–differentials exiting $E^r_{p,q}(\mathbf{KQ})$ for $p=0$, $p=1$ and $r \geq 2$ are trivial.

Lemma 7.2 *There are isomorphisms*

$$E^{\infty}_{p,q}(\mathbf{KQ}) \cong \begin{cases} H^{0,0}, & p=q=0, \\ h^{q,q}, & p=0,\ q>0, \\ h^{0,1}, & p=q=1, \\ h^{1,2}, & p=1,\ q=2, \\ 0, & p=1,\ q\neq 1,2. \end{cases}$$

Remark 7.3 There are isomorphisms $H^{0,0} \cong \mathbb{Z}$, $h^{0,1} \cong \mathbb{Z}/2$, and $h^{1,2} \cong F^{\times}/2$.

By computing in the 2[nd] column we obtain the motivic cohomological description of the second orthogonal K–group $\mathrm{KO}_2(F)$ stated in Theorem 1.2.

Lemma 7.4 *There are isomorphisms*

$$E^{\infty}_{2,q}(\mathbf{KQ}) \cong \begin{cases} \ker(\tau\circ \mathrm{pr} + \mathsf{Sq}^2\colon H^{2,2}\oplus h^{0,2} \longrightarrow h^{2,3}), & q=2, \\ 0, & q\neq 2. \end{cases}$$

Proof The claim for $q=2$ follows from Theorem 5.5. We note that the kernel of the surjection $\tau\circ\mathrm{pr}+\mathsf{Sq}^2$ is the preimage of the subgroup $\{0,\rho^2\}\subseteq h^{2,2}$ under $\mathrm{pr}\colon H^{2,2}\longrightarrow h^{2,2}$. If $\rho^2=0$, this group coincides with $2H^{2,2}$. In weight 3, the kernel of

$$\tau+\mathsf{Sq}^2\colon h^{3,3}\oplus h^{1,3}\longrightarrow h^{3,4}$$

is isomorphic to $h^{1,1}$ via $\phi\longmapsto(\rho^2\phi,\tau^2\phi)$. The image of the entering differential corresponds to the image of $H^{1,1}$ in $h^{1,1}$ under this isomorphism, hence it coincides with $h^{1,1}$. The remaining vanishing follows from Lemma 7.1. □

Remark 7.5 There are isomorphisms $H^{2,2}\cong K_2(F)$, $h^{0,2}\cong\mathbb{Z}/2$, and $h^{2,3}\cong {}_2\mathrm{Br}(F)$ (the 2–torsion subgroup of the Brauer group of equivalence classes of central simple F–algebras).

Lemma 7.6 *There is a short exact sequence*

$$0\longrightarrow h^{0,3}\longrightarrow \mathrm{KO}_3(F)\longrightarrow 2H^{1,2}\longrightarrow 0$$

and isomorphisms

$$E^{\infty}_{3,q}(\mathbf{KQ}) \cong \begin{cases} 2H^{1,2}, & q=2, \\ h^{0,3}, & q=3, \\ 0, & q\neq 2,3. \end{cases}$$

Proof Note that Sq^2 acts trivially on $h^{1,2}$ by Corollary 6.2. In weight 2 we look at the kernel of

$$\tau \circ \mathrm{pr}\colon H^{1,2} \longrightarrow h^{1,3}.$$

By Lemma 6.1 this equals the kernel of pr, ie $2H^{1,2} \subseteq H^{1,2}$. In weight 3, the kernel of the exiting differential is isomorphic to $h^{0,0}$ via

$$c \longmapsto (\tau\rho^2 c, \tau^3 c).$$

In weight 4, the kernel of the exiting differential is isomorphic to $h^{1,1}$ via

$$\phi \longmapsto (\tau\rho^2\phi, \tau^3\phi).$$

The entering differential surjects onto the kernel of the latter map. In weight 5, the group coincides with the corresponding group for $\mathbf{KT}$, as the change from $h^{4,4}$ to $H^{4,4}$ does not affect the spectral sequence, due to the triviality of the differential exiting bidegree $(4,4)$. The remaining claims follow from Lemma 7.1. □

Remark 7.7 There are isomorphisms $H^{1,2} \cong K_3^{\mathrm{ind}}(F)$ and $h^{0,3} \cong \mathbb{Z}/2$; here $K_3^{\mathrm{ind}}(F)$ is the cokernel of $K_3^M(F) \longrightarrow K_3(F)$, ie the K_3–group of indecomposable elements. By Lichtenbaum's weight-2 motivic complex $\Gamma(2)$ [30] there is a short exact sequence

$$0 \longrightarrow h^{0,2} \longrightarrow H^{1,2} \longrightarrow 2H^{1,2} \longrightarrow 0.$$

By comparing with the forgetful map $\mathbf{KQ} \to \mathbf{KGL}$ and identifying $h^{0,2}$ with $h^{0,3}$ one concludes that $\mathrm{KO}_3(F)$ is isomorphic to $K_3^{\mathrm{ind}}(F)$.

Lemma 7.8 *There are isomorphisms*

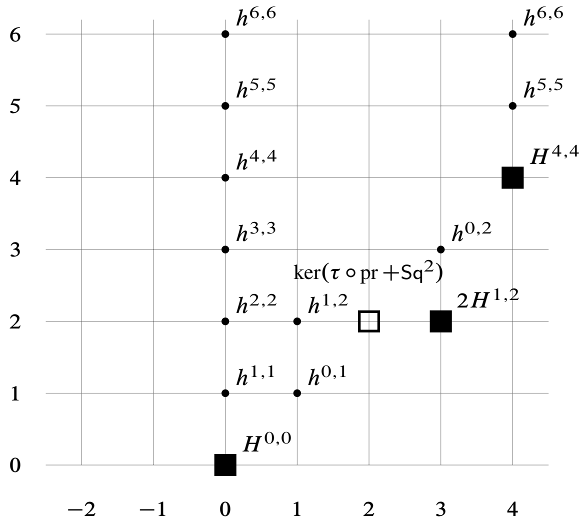
$$E^2_{4,q}(\mathbf{KQ}) \cong \begin{cases} 0, & q = 2,3, \\ H^{4,4}, & q = 4, \\ h^{q,q}, & q \geq 5. \end{cases}$$

Proof As noted above the group $H^{0,2}$ is trivial. Lemma 6.1 shows the exiting differential in weight 3 is injective. In weight 4, the kernel of the exiting differential is $H^{4,4} \oplus h^{0,4}$. The nontrivial element in the image of the entering differential is $(\delta(\rho^3\tau), \tau^4)$. Thus the quotient can be identified with $H^{4,4}$. In weight 5, this follows from the case of $\mathbf{KT}$ because

$$\mathsf{Sq}^2 \circ \mathrm{pr}\colon H^{3,4} \longrightarrow h^{3,4} \longrightarrow h^{5,5}$$

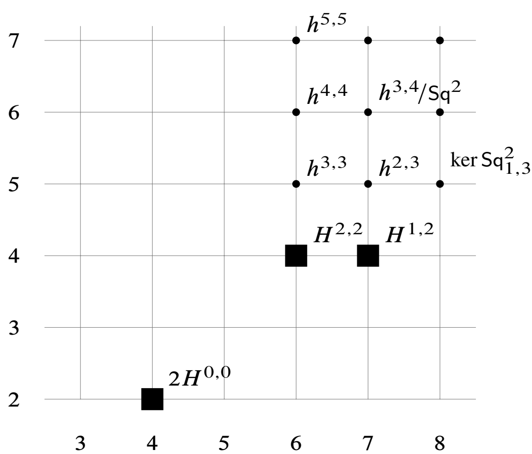
is trivial. Lemma 7.1 finishes the proof. □

In principle one can continue with a similar analysis of the next columns. To summarize the computations above, we note that in low degrees the E^2–page takes the form



Corollary 7.9 *The group* $\mathrm{KO}_4(F)$ *surjects onto* $H^{4,4} \cong K_4^M$*. If* $K_4^M = 0$ *then* $\mathrm{KO}_4(F)$ *is the trivial group.*

The symplectic K–groups $\mathrm{KSp}_*(F)$ of F are the filtered target groups of the second slice spectral sequence for $\mathbf{KQ}$ on account of the isomorphism $\pi_{p,2}\mathbf{KQ} \cong \mathrm{KSp}_{p-4}(F)$. Computations similar to the above yield the E^2–page



We read off that $\mathrm{KSp}_0(F) \cong 2H^{0,0}$ is infinite cyclic and $\mathrm{KSp}_1(F)$ is the trivial group. It follows that all the classes in the sixth column are infinite cycles and $E^2_{p,q} = E^\infty_{p,q}$

when $(p,q)=(6,4),(6,5)$. Hence we obtain a surjection $\mathrm{KSp}_2(F)\longrightarrow H^{2,2}$. Its kernel is isomorphic to $I(F)^3$ as shown by Suslin [52, Section 6]. If $\mathsf{cd}_2(F)\le 2$, so that $h^{p,q}=0$ when $p\ge 3$, the seventh column degenerates to a short exact sequence $0\to h^{2,3}\to \mathrm{KSp}_3(F)\to H^{1,2}\to 0$.

Remark 7.10 The computations in this section hold for smooth semilocal rings containing a field of characteristic zero; see the generalization of Milnor's conjecture on quadratic forms discussed in the introduction.

Appendix: Maps between motivic Eilenberg–MacLane spectra

Throughout this section the base scheme is essentially smooth over a field of characteristic unequal to 2 [18, Definition 2.9]. In the following series of results, we identify weight-0 and weight-1 endomorphisms of motivic Eilenberg–MacLane spectra in **SH** in terms of Steenrod operations Sq^i and the motivic cohomology classes ρ, τ. When the base scheme is a field of characteristic zero, these identifications follow from Voevodsky's work on the motivic Steenrod algebra [62], while the generalization to our set-up relies on [18]. We use square brackets to denote maps in **SH**.

Lemma A.1
$$[\mathbf{MZ}/2,\Sigma^{p,0}\mathbf{MZ}/2]=\begin{cases}\mathbb{F}_2, & p=0,\\ \mathbb{F}_2\{\mathsf{Sq}^1\}, & p=1,\\ 0, & \text{otherwise.}\end{cases}$$

Recall that Sq^1 is the canonical composite map $\mathbf{MZ}/2\xrightarrow{\delta}\Sigma^{1,0}\mathbf{MZ}\xrightarrow{\mathrm{pr}}\Sigma^{1,0}\mathbf{MZ}/2$.

Lemma A.2
$$[\mathbf{MZ}/2,\Sigma^{p,1}\mathbf{MZ}/2]=\begin{cases}\tau h^{0,0}, & p=0,\\ h^{1,1}\oplus h^{0,0}\{\tau\mathsf{Sq}^1\}, & p=1,\\ h^{1,1}\mathsf{Sq}^1\oplus h^{0,0}\{\mathsf{Sq}^2\}, & p=2,\\ h^{0,0}\{\mathsf{Sq}^2\mathsf{Sq}^1\}\oplus h^{0,0}\{\mathsf{Sq}^1\mathsf{Sq}^2\}, & p=3,\\ h^{0,0}\{\mathsf{Sq}^1\mathsf{Sq}^2\mathsf{Sq}^1\}, & p=4,\\ 0, & \text{otherwise.}\end{cases}$$

Lemma A.3 *We have*
$$[\mathbf{MZ},\Sigma^{p,0}\mathbf{MZ}/2]=\begin{cases}\mathbb{F}_2\{\mathrm{pr}\}, & p=0,\\ 0, & p\neq 0,\end{cases}\qquad [\mathbf{MZ}/2,\Sigma^{p,0}\mathbf{MZ}]=\begin{cases}\mathbb{F}_2\{\delta\}, & p=1,\\ 0, & p\neq 1.\end{cases}$$

Lemma A.4 *We have*

$$[\mathbf{MZ},\Sigma^{p,1}\mathbf{MZ}/2]=\begin{cases}\tau h^{0,0}\circ \mathrm{pr}, & p=0,\\ h^{1,1}\circ \mathrm{pr}, & p=1,\\ h^{0,0}\{\mathsf{Sq}^2\circ \mathrm{pr}\}, & p=2,\\ h^{0,0}\{\mathsf{Sq}^1\mathsf{Sq}^2\circ \mathrm{pr}\}, & p=3,\\ 0, & \text{otherwise,}\end{cases}$$

$$[\mathbf{MZ}/2,\Sigma^{p,1}\mathbf{MZ}]=\begin{cases}\delta\circ\tau h^{0,0}, & p=1,\\ \delta\circ h^{1,1}, & p=2,\\ \mathbb{F}_2\{\delta\circ\mathsf{Sq}^2\}, & p=3,\\ \mathbb{F}_2\{\delta\circ\mathsf{Sq}^2\mathsf{Sq}^1\}\quad, & p=4,\\ 0, & \text{otherwise.}\end{cases}$$

Here $\mathsf{Sq}^1(\tau)=\rho$, $\mathsf{Sq}^2(\tau)=0$ and $\mathsf{Sq}^2(\tau^2)=\tau\rho^2$. It follows that

$$\mathsf{Sq}^1(\tau^n)=\begin{cases}\rho\tau^{n-1}, & n\equiv 1 \bmod 2,\\ 0, & n\equiv 0 \bmod 2,\end{cases}\quad\text{and}\quad \mathsf{Sq}^2(\tau^n)=\begin{cases}\rho^2\tau^{n-1}, & n\equiv 2,3 \bmod 4,\\ 0, & n\equiv 0,1 \bmod 4.\end{cases}$$

Proposition A.5 *For every subset $A\subseteq\mathbb{Z}$ there is a canonical weak equivalence*

$$\alpha\colon \textstyle\bigvee_{i\in A}\Sigma^{i,0}\mathbf{MZ}/2\longrightarrow\displaystyle\prod_{i\in A}\Sigma^{i,0}\mathbf{MZ}/2.$$

Proof It suffices to show $[\Sigma^{p,q}X_+,\alpha]$ is an isomorphism for all $p,q\in\mathbb{Z}$ and $X\in\mathrm{Sm}_F$. This is the canonical map

$$\bigoplus_{i\in A}h^{i-p,-q}(X_+)\longrightarrow\prod_{i\in A}h^{i-p,-q}(X_+).$$

Work of Suslin and Voevodsky [54] shows the group $h^{i-p,q}(X_+)$ is nonzero only if $0\le i-p\le 2q$ (see [18, Corollary 2.14]) for base schemes essentially smooth over a field. □

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