Geometry & Topology 19 (2015) 525–724

Bimodules in bordered Heegaard Floer homology

ROBERT LIPSHITZ
PETER S OZSVÁTH
DYLAN P THURSTON

Bordered Heegaard Floer homology is a three-manifold invariant which associates to a surface F an algebra $\mathcal{A}(F)$ and to a three-manifold Y with boundary identified with F a module over $\mathcal{A}(F)$. In this paper, we establish naturality properties of this invariant. Changing the diffeomorphism between F and the boundary of Y tensors the bordered invariant with a suitable bimodule over $\mathcal{A}(F)$. These bimodules give an action of a suitably based mapping class group on the category of modules over $\mathcal{A}(F)$. The Hochschild homology of such a bimodule is identified with the knot Floer homology of the associated open book decomposition. In the course of establishing these results, we also calculate the homology of $\mathcal{A}(F)$. We also prove a duality theorem relating the two versions of the 3–manifold invariant. Finally, in the case of a genus-one surface, we calculate the mapping class group action explicitly. This completes the description of bordered Heegaard Floer homology for knot complements in terms of the knot Floer homology.

57R57; 53D40

1 Introduction

Bordered Heegaard Floer homology is an invariant associated to a three-manifold with boundary (see the authors [21]), depending on some additional data. More specifically, let F be a closed, oriented surface of genus k. A bordered three-manifold with boundary F is a compact, oriented three-manifold Y equipped with an orientation-preserving diffeomorphism $\phi\colon F \to \partial Y$. Bordered Heegaard Floer homology associates to F (and some extra data; see below) a differential graded (*dg*) algebra $\mathcal{A}(F)$. If Y is a bordered three-manifold with boundary F, the theory associates to Y a right $\mathcal{A}_\infty$–module $\widehat{CFA}(Y)$ over $\mathcal{A}(F)$, the *type A module of* Y, whose quasi-isomorphism type depends only on the underlying diffeomorphism type of Y. To a three-manifold with boundary the theory also associates a left $\mathcal{A}_\infty$–module over $\mathcal{A}(-F)$, $\widehat{CFD}(Y)$, the *type D module of* Y.

Published: 10 April 2015 DOI: 10.2140/gt.2015.19.525

Bordered Heegaard Floer homology is related to Heegaard Floer homology $\widehat{HF}(Y)$ via a pairing theorem: if Y is a three-manifold which is divided into Y_1 and Y_2 by a separating surface F, then $\widehat{CF}(Y)$, a chain complex whose homology calculates $\widehat{HF}(Y)$, is obtained as the $\mathcal{A}_\infty$–tensor product of $\widehat{CFA}(Y_1)$ with $\widehat{CFD}(Y_2)$. In other words,

$$\widehat{HF}(Y) \cong \mathrm{Tor}_{\mathcal{A}(F)}(\widehat{CFA}(Y_1), \widehat{CFD}(Y_2)).$$

1.1 Reparametrization and the bordered Floer invariants

A key goal of this paper is to study how the bordered Heegaard Floer invariants change under reparametrization of the boundary. More precisely, we fix a closed surface F and a preferred disk $D \subset F$, together with a point $z \in \partial D$. Consider the space of diffeomorphisms of F which preserve the disk D and the point $z \in \partial D$. This topological group will be called the *strongly based diffeomorphism group of* (F, D, z). Its group of path components is called the *strongly based mapping class group* of F, and two diffeomorphisms in the same path component are called *strongly isotopic*. This agrees with the usual mapping class group of $F \setminus \mathrm{int}(D)$ (fixing the boundary).

Consider next a handle decomposition of F with one zero-handle and where D is the unique two-handle. We will mark in addition a basepoint z on the boundary of $F \setminus D$. This data can be combinatorially encoded in the form of a *pointed matched circle* (see Definition 3.1 below). Bordered Floer homology associates to a pointed matched circle $\mathcal{Z}$ a differential-graded algebra $\mathcal{A}(\mathcal{Z})$. Modules over these algebras are independent of the decomposition $\mathcal{Z}$ in the following sense:

Theorem 1 *If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are two pointed matched circles representing the same underlying surface, then the derived categories of dg $\mathcal{A}(\mathcal{Z}_1)$– and $\mathcal{A}(\mathcal{Z}_2)$–modules are equivalent.*

For the purpose of this introduction, we will typically suppress the pointed matched circle $\mathcal{Z}$ from the notation, referring somewhat imprecisely to $\mathcal{A}(F)$. Theorem 1 provides some justification for this practice.

A *bordered three-manifold* is a quadruple (Y, Δ, z_1, ψ), where Y is an oriented three-manifold with boundary, Δ is a disk in ∂Y, z_1 is a point on $\partial\Delta$ and

$$\psi\colon (F, D, z) \to (\partial Y, \Delta, z)$$

is a diffeomorphism from F to ∂Y sending D to Δ and z to z_1.

The strongly based diffeomorphism group of F acts on the set of bordered three-manifolds by composition:

$$\phi \cdot (Y, \Delta, z_1, \psi) = (Y, \Delta, z_1, \psi \circ \phi^{-1}).$$

There are bimodules which encode this action, using the $\mathcal{A}_\infty$–tensor product. Specifically, let M be a right $\mathcal{A}_\infty$–module over the *dg* algebra $\mathcal{A}$ and N be an $\mathcal{A}_\infty$–bimodule over $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{B}$ is another *dg* algebra. Then we can form the derived (or $\mathcal{A}_\infty$) tensor product $M \mathbin{\widetilde{\otimes}}_{\mathcal{A}} N$, to obtain a right $\mathcal{A}_\infty$–module over $\mathcal{B}$. We have bimodules associated to reparameterizing the boundary, as given in the following:

Theorem 2 *Given a strongly based diffeomorphism $\phi\colon (F_1, D, z) \to (F_2, D, z)$ between surfaces F_1 and F_2 (corresponding to possibly different pointed matched circles), there are associated bimodules:*

$$\widehat{CFAA}(\phi)_{\mathcal{A}(-F_1),\mathcal{A}(F_2)}, \quad {}_{\mathcal{A}(F_1)}\widehat{CFDA}(\phi)_{\mathcal{A}(F_2)}, \quad {}_{\mathcal{A}(F_1),\mathcal{A}(-F_2)}\widehat{CFDD}(\phi).$$[1]

If $(Y_1, \Delta_1, z_1, \psi_1\colon F_1 \to \partial Y_1)$ and $(Y_2, \Delta_2, z_2, (-\psi_2)\colon -F_2 \to \partial Y_2)$ (so $\psi_2\colon F_2 \to -\partial Y_2$) are bordered 3–manifolds then

$$\begin{aligned}
\widehat{CFA}(Y_1, \psi_1) \mathbin{\widetilde{\otimes}}_{\mathcal{A}(F_1)} \widehat{CFDA}(\phi) &\simeq \widehat{CFA}(Y_1, \psi_1 \circ \phi^{-1}),\\
\widehat{CFAA}(\phi) \mathbin{\widetilde{\otimes}}_{\mathcal{A}(F_2)} \widehat{CFD}(Y_2, \psi_2) &\simeq \widehat{CFA}(Y_2, -(\psi_2 \circ \phi)),\\
\widehat{CFA}(Y_1, \psi_1) \mathbin{\widetilde{\otimes}}_{\mathcal{A}(F_1)} \widehat{CFDD}(\phi) &\simeq \widehat{CFD}(Y_1, -(\psi_1 \circ \phi^{-1})),\\
\widehat{CFDA}(\phi) \mathbin{\widetilde{\otimes}}_{\mathcal{A}(F_2)} \widehat{CFD}(Y_2, \psi_2) &\simeq \widehat{CFD}(Y_2, \psi_2 \circ \phi).
\end{aligned}$$

(This is proved in Section 7.1. See particularly Figure 17 for a schematic illustrating why the parametrizations are as given.)

The bimodules satisfy the following invariance property:

Theorem 3 *If ϕ and ϕ' are strongly isotopic diffeomorphisms of F then their associated bimodules are quasi-isomorphic.*

(This is proved in Section 6.4.)

The bimodules also behave functorially under composition, according to the following two results:

Theorem 4 *The type DA bimodule associated to identity map from F to itself, $\widehat{CFDA}(\mathbb{I}_F)$, is quasi-isomorphic to $\mathcal{A}(F)$ as an $(\mathcal{A}(F), \mathcal{A}(F))$–bimodule.*

(This is proved in Section 8.1.) Note that $\mathcal{A}(F)$ is the identity for the tensor product operation.

[1] Given algebras A and B, we write ${}_A M_B$ to denote a module with a left action of A and a right action of B. We write ${}_{A,B}M$ to denote a module with commuting left actions of A and B, and $M_{A,B}$ to denote a module with commuting right actions of A and B.

Theorem 5 *Given two strongly based diffeomorphisms $\phi_1\colon F_1 \to F_2$, $\phi_2\colon F_2 \to F_3$, we have that*

$$\widehat{CFDA}(\phi_1) \widetilde{\otimes}_{\mathcal{A}(F_2)} \widehat{CFDA}(\phi_2) \simeq \widehat{CFDA}(\phi_2 \circ \phi_1).$$

(This is proved in Section 7.1.)

Together, Theorems 3, 4 and 5 can be summarized by saying that the bimodules induce an action of the based mapping class group of a surface F on the module category of $\mathcal{A}(F)$; see Theorem 15 in Section 8 for a more precise statement. Actions of mapping class groups — particularly, braid groups — on categories have arisen in other contexts; see for example Khovanov and Thomas [16] and the references therein.

Theorem 2 is also interesting in the case where ϕ is the identity map. In that case, the theorem allows one to convert a type D module into a type A module, and vice versa. The statement is given in the following corollary, which can be thought of as exhibiting a kind of Koszul duality (see Priddy [32]) between the algebra of a surface and its orientation reverse:

Corollary 1.1 *There are dualizing modules $\widehat{CFAA}(\mathbb{I})$ and $\widehat{CFDD}(\mathbb{I})$ (bimodules associated to the identity map on F), which can be used to convert $\widehat{CFD}$–modules to $\widehat{CFA}$–modules and vice versa, in the sense that*

$$\widehat{CFAA}(\mathbb{I}) \widetilde{\otimes}_{\mathcal{A}(-F)} \widehat{CFD}(Y, -\psi) \simeq \widehat{CFA}(Y, \psi),$$
$$\widehat{CFA}(Y, \psi) \widetilde{\otimes}_{\mathcal{A}(F)} \widehat{CFDD}(\mathbb{I}) \simeq \widehat{CFD}(Y, -\psi).$$

The modules $\widehat{CFAA}(\mathbb{I})$ and $\widehat{CFDD}(\mathbb{I})$ uniquely determine one another, as explained in Section 9. Indeed, this description, along with the above corollary, quickly leads to the following result which allows one to express type A modules entirely in terms of type D modules:

Theorem 6 *Let Y be a bordered three-manifold with boundary F. Then $\widehat{CFA}(Y)$ is quasi-isomorphic, as a right $\mathcal{A}_\infty$–module over $\mathcal{A}(F)$, to the chain complex of maps from $\widehat{CFDD}(\mathbb{I})$ to $\widehat{CFD}(Y)$.*

(This theorem is stated more precisely and proved in Section 9, as Theorem 16.)

In fact, $\widehat{CFDD}(\mathbb{I})$, and hence also $\widehat{CFAA}(\mathbb{I})$, can be calculated explicitly; see the authors [20].

The bimodule ${}_{\mathcal{A}(F)}\widehat{CFDA}(\phi)_{\mathcal{A}(F)}$ of a strongly based diffeomorphism

$$\phi\colon (F, D, z) \to (F, D, z)$$

is a bimodule over a single algebra $\mathcal{A}(F)$ on both sides; hence it is natural to study its Hochschild homology. We give this operation a topological interpretation. To state this interpretation, recall that a strongly based diffeomorphism $\phi\colon (F, D, z) \to (F, D, z)$ naturally gives rise to a three-manifold with an open book decomposition. More precisely, consider the three-manifold with torus boundary, defined as the quotient of $[0, 1] \times (F \setminus D)$ by the equivalence relation $(0, \phi(x)) \sim (1, x)$. This manifold is equipped with an embedded, closed curve on the boundary, $([0, 1] \times \{z\})/(0, z) \sim (1, z)$. By filling along the curve on the boundary, we get a closed three-manifold, equipped with a knot K induced from $\{0\} \times \partial D$. We denote the resulting three-manifold by $Y(\phi)$, and let $K \subset Y(\phi)$ be the canonical knot in it. This presentation of the three-manifold Y underlying $Y(\phi)$ is called an *open book decomposition* of Y, and K is called the *binding*.

Theorem 7 *Let ϕ be a strongly-based diffeomorphism, and consider its associated $\mathcal{A}(F)$–bimodule $\widehat{CFDA}(\phi)$. The Hochschild homology of the bimodule $\widehat{CFDA}(\phi)$ is the knot Floer homology of $Y(\phi)$ with respect to its binding K, $\widehat{HFK}(Y(\phi), K)$.*

(This is proved in Section 7.2.)

Remark 1.2 Let K be a fibered knot in S^3, F a fiber surface for K and $\phi\colon F \to F$ the monodromy. A classical theorem states that the Alexander polynomial of K is the characteristic polynomial of $\phi_*\colon H_1(F) \to H_1(F)$. Theorem 7 categorifies this in the following sense. On the one hand, by the second author and Szabó [28, Equation (1)], the Euler characteristic of $\widehat{HFK}(Y(\phi), K)$ is $\Delta_K(t)$. On the other hand, the (modulo 2) Grothendieck group of the category of $\mathcal{A}(F)$–modules is $\Lambda^* H_1(F; \mathbb{F}_2)$ and the functor $\widehat{CFDA}(\phi) \mathbin{\widetilde{\otimes}} \cdot$ decategorifies to $\phi_*\colon \Lambda^* H_1(F; \mathbb{F}_2) \to \Lambda^* H_1(F; \mathbb{F}_2)$; compare the authors [23]. Hochschild homology decategorifies to the graded trace; and the graded trace of $\phi_*\colon \Lambda^* H_1(F; \mathbb{F}_2) \to \Lambda^* H_1(F; \mathbb{F}_2)$ is (by definition) the characteristic polynomial of $\phi_*\colon H_1(F; \mathbb{F}_2) \to H_1(F; \mathbb{F}_2)$. (To obtain the decategorification with $\mathbb{Z}$–coefficients would require an absolute $\mathbb{Z}/2$–grading on the bimodules, which we do not construct.)

1.2 A more general framework

The bimodules associated to mapping classes come from a more general construction, which gives an invariant for a bordered three-manifold Y_{12} with two boundary components F_1 and F_2, and extra data specified by a *strong framing*, which is a parameterization of the boundary components of Y_{12} and a framed arc connecting the two boundary components of Y_{12}. More precisely, we have the following.

Definition 1.3 Fix two connected surfaces (F_1, D_1, z_1) and (F_2, D_2, z_2) equipped with preferred disks and basepoints on the boundaries of the disks. A *strongly bordered three-manifold with boundary* $F_1 \amalg F_2$ is an oriented three manifold Y_{12} with two boundary components, equipped with

- preferred disks Δ_1 and Δ_2 on its two boundary components,
- basepoints $z_i' \in \partial\Delta_i$,
- diffeomorphisms $\psi\colon (F_1 \amalg F_2, D_1 \amalg D_2, z_1 \amalg z_2) \to (\partial Y_{12}, \Delta_1 \amalg \Delta_2, z_1' \amalg z_2')$,
- an arc γ connecting z_1' to z_2', and
- a framing of γ, pointing into Δ_i at z_i' for $i = 1, 2$.

(See also Definition 5.1.) Note that we can remove a neighborhood of γ from Y_{12} to obtain a three-manifold M with boundary $F_1 \# F_2$. The trivialization of the normal bundle of γ is the additional data needed to construct a parameterization of the boundary of M from the parameterization of ∂Y.

To obtain a bimodule, we must fix further data: we mark each boundary component of Y with an A or a D. This determines whether the corresponding boundary component is treated as a type D module or a type A module (and hence, whether the underlying algebra is associated to the boundary component with its induced orientation, or the opposite of its induced orientation). More explicitly, we have the following:

Theorem 8 *Let Y_{12} be a bordered three-manifold with two boundary components F_1 and F_2 and a strong framing. Then, we can associate the following bimodules to Y_{12}:*

$$\widehat{CFAA}(Y_{12})_{\mathcal{A}(F_1),\mathcal{A}(F_2)}, \quad {}_{\mathcal{A}(-F_1)}\widehat{CFDA}(Y_{12})_{\mathcal{A}(F_2)}, \quad {}_{\mathcal{A}(-F_1),\mathcal{A}(-F_2)}\widehat{CFDD}(Y_{12}).$$

The quasi-isomorphism types of these bimodules are diffeomorphism invariants of the bordered three-manifold Y_{12} with its strong boundary framing.

(A graded refinement of this theorem is proved in Section 6, as Theorem 10.)

The bimodules associated to an automorphism of F are gotten as a special case of the above construction, where $Y_{12} = [0,1] \times F$, with the identity parametrization on one component and ϕ on the other component. In Section 7, we shall prove general versions of the pairing theorem, from which Theorems 2 and 5 follow as corollaries.

There are other versions of the pairing theorem, including a "double-pairing theorem", where we glue two three-manifolds with two boundary components together, as in Theorem 14. Theorem 7 is the specialization of this to the case where we are self-gluing $[0,1] \times F$.

After building the basic background, we turn to the particular case of a surface of genus one. In this case, we calculate the bimodules associated to an arbitrary mapping class, hence giving an explicit description of the dependence of the bordered Heegaard Floer homology of a three-manifold with torus boundary on the parameterization of the boundary. This also completes the description of $\widehat{CFD}$ in terms of knot Floer homology for a knot in S^3. Specifically, we showed in [21, Chapter 11] how to calculate the type D module of a knot complement in terms of the knot Floer homology of the knot when the framing on the knot is sufficiently large. With the help of the torus mapping class group calculations and Theorem 2, we are now able to calculate the type D module for arbitrary framings, as promised in [21].

1.3 Organization

This paper is organized as follows. In Section 2 we recall the algebraic background on $\mathcal{A}_\infty$–algebras and modules used throughout this paper. These include the familiar notions of $\mathcal{A}_\infty$–modules, which we call here *type A structures* to distinguish them from *type D structures*, which are a variant of projective modules (see Corollary 2.3.25; compare also Remarks 2.2.36 and 2.2.37). These are then combined to give various notions of bimodules. We discuss various operations on modules and bimodules — in particular, the tensor product, Hom and Hochschild homology functors — and review some category theory. Section 2 concludes with a discussion of group-valued gradings.

In Section 3 we recall the notion of a *pointed matched circle* $\mathcal{Z}$, which is effectively the handle decomposition of a surface used to define its associated algebra. We then recall the definition of the algebra $\mathcal{A}(\mathcal{Z})$ (introduced in [21]). In Section 4, we turn to the calculation of the homology of the algebra $\mathcal{A}(\mathcal{Z})$. This calculation is used significantly in the subsequent proof of Theorem 4 in Section 8.1. It is interesting to note that, as a consequence of this calculation, we obtain a smaller differential graded algebra $\mathcal{A}'$ (a quotient of $\mathcal{A}$ by a differential ideal), which can be used in place of $\mathcal{A}$ for the purposes of the invariant; see Proposition 4.2. (Although we do not pursue this further in the present work, it does considerably simplify computations in practice.)

In Section 5, we study Heegaard diagrams associated to strongly framed three-manifolds, and recall their existence and uniqueness properties. We also turn our attention to a case of particular importance: strongly framed three-manifolds associated to strongly based surface automorphisms. We show how to construct explicitly the corresponding Heegaard diagrams in this case. In Section 6, we turn to the construction of bordered bimodules for strongly based three-manifolds, and verify their invariance properties, verifying Theorems 8 and 3.

In Section 7, we turn to the various pairing theorems which these bimodules enjoy. Theorems 2, 5 and 7 are deduced from these pairing theorems. In Section 8, we employ the calculations from Section 4, together with a suitable variant of the pairing theorem, to deduce that the type *DA* bimodule associated to the identity map is $\mathcal{A}(F)$; see Theorem 4. Thus armed, we complete the proof that the derived category of modules over the algebra associated to a pointed matched circle depends only on the homeomorphism type of the underlying surface; see Theorem 1. This information also allows us to construct an action of the mapping class group on the derived category of $\mathcal{A}(F)$–modules. Theorem 4 is also the main ingredient we use in Section 9 to prove the duality theorem, Theorem 6.

Finally, in Section 10 we compute the bimodules associated to torus automorphisms.

Acknowledgements We are grateful for the supportive and stimulating mathematical environment provided by the Columbia mathematics department, where most of this work was done. We also wish to thank MSRI for its hospitality during the completion of this project. In addition, the third author thanks the Center for the Topology and Quantization of Moduli Spaces at Aarhus University for its hospitality.

The authors thank M Khovanov and A Lauda for substantial help with the categorical aspects of this paper, particularly with formulating Theorem 15. The authors would also like to thank R Zarev for sharing with us his insights during the course of this work. We thank C Manolescu for a correction to a previous version. Finally, we thank the referee for many helpful comments.

The first author was supported by an NSF Mathematical Sciences Postdoctoral Fellowship, NSF grant number DMS-0905796, and a Sloan Research Fellowship. The second author was supported by NSF grant number DMS-0505811. The third author was supported by NSF grant number DMS-1008049 and a Sloan Research Fellowship.

2 $\mathcal{A}_\infty$–algebras and modules

In this section we recall various notions from the theory of $\mathcal{A}_\infty$–algebras and derived categories. Most of these results are standard (see Keller [13], Bernstein and Lunts [5], Lefèvre-Hasegawa [19] and Seidel [36]), and are collected here for the reader's convenience. Our treatment is slightly nonstandard in that we use extensively a certain algebraic object defined over $\mathcal{A}_\infty$–algebras which we call "type D structures." The reader is encouraged to think of these as projective modules over the algebra. These arise naturally in the context of bordered Floer theory — the bordered invariant $\widehat{CFD}(Y)$ of [21] is a type D structure — and are convenient for various algebraic constructions. (In fact, type D structures have appeared under various guises elsewhere; see Remarks 2.2.36 and 2.2.37.)

Convention 2.0.1 Throughout, ($\mathcal{A}_\infty$–) algebras will be algebras over the a ring $\boldsymbol{k}$, which is either $\mathbb{F}_2$ or, more generally, $\bigoplus_{i=1}^N \mathbb{F}_2$. Unless otherwise stated, tensor products denote tensor products over $\boldsymbol{k}$. (When we need to refer to a second such ground ring we will denote it $\boldsymbol{j}$.)

Note that for most of this paper it is enough to consider *dg* algebras rather than more general $\mathcal{A}_\infty$–algebras, so the reader could skip Section 2.1 if desired. More general $\mathcal{A}_\infty$–algebras do appear in Section 4.2.

In one unusual twist, our $\mathcal{A}_\infty$–algebras and modules are graded by noncommutative groups, as explained in Section 2.5.

2.1 $\mathcal{A}_\infty$–algebras

2.1.1 Definition of $\mathcal{A}_\infty$–algebras

Definition 2.1.1 An $\mathcal{A}_\infty$–algebra $\mathcal{A}$ over $\boldsymbol{k}$ is a $\mathbb{Z}$–graded $\boldsymbol{k}$–bimodule A, equipped with degree-0 $\boldsymbol{k}$–linear multiplication maps

$$\mu_i\colon A^{\otimes i} \to A[2-i] \tag{2.1.2}$$

defined for all $i \geq 1$, satisfying the compatibility conditions that, for each $n \geq 1$ and elements $a_1, \dots, a_n$,

$$\sum_{i+j=n+1} \sum_{\ell=1}^{n-j+1} \mu_i\big(a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes a_{\ell+j} \otimes \cdots \otimes a_n\big) = 0. \tag{2.1.3}$$

Here $A^{\otimes i}$ denotes the $\boldsymbol{k}$–bimodule $\overbrace{A \otimes_{\boldsymbol{k}} \cdots \otimes_{\boldsymbol{k}} A}^{i}$ and $[2-i]$ denotes a degree shift. We use $\mathcal{A}$ for the $\mathcal{A}_\infty$–algebra and A for its underlying $\boldsymbol{k}$–bimodule.

An $\mathcal{A}_\infty$–algebra is *strictly unital* (or just *unital*) if there is an element $1 \in A$ with the property that $\mu_2(a,1) = \mu_2(1,a) = a$ and $\mu_i(a_1, \dots, a_i) = 0$ if $i \neq 2$ and $a_j = 1$ for some j. For a unital $\mathcal{A}_\infty$–algebra, the unit gives a preferred map $\iota\colon \boldsymbol{k} \to \mathcal{A}$.

An *augmentation* of an $\mathcal{A}_\infty$–algebra is a map $\epsilon\colon A \to \boldsymbol{k}$, satisfying the conditions that

$$\begin{aligned} \epsilon(1) &= 1, \\ \epsilon(\mu_2(a_1,a_2)) &= \epsilon(a_1)\epsilon(a_2), \\ \epsilon(\mu_k(a_1,\dots,a_k)) &= 0 \quad \text{for } k \neq 2. \end{aligned} \tag{2.1.4}$$

This gives an *augmentation ideal* $A_+ = \ker \epsilon$.

One could consider a more general notion of augmentation, where ϵ is an $\mathcal{A}_\infty$ homomorphism in the sense of Section 2.1.2. We do not do this here, as we do not need this level of generality for our present purposes, and indeed, it would cause undue complication, especially in Section 2.3.5.

Note that in particular μ_1 gives A the structure of a chain complex. A differential graded algebra over $\boldsymbol{k}$ is an $\mathcal{A}_\infty$–algebra with $\mu_i = 0$ for all $i > 2$.

Convention 2.1.5 Throughout this paper, $\mathcal{A}_\infty$–algebras will be assumed strictly unital and augmented.

We can think about the $\mathcal{A}_\infty$–relation (2.1.3) graphically. First, we associate operations to graphs.

Definition 2.1.6 An $\mathcal{A}_\infty$*–operation tree* Γ is a finite, directed tree embedded in the plane such that each nonleaf vertex of Γ has exactly one outgoing edge.

In every $\mathcal{A}_\infty$–operation tree Γ there is a unique leaf that is a sink, along with n source leaves. Then given an $\mathcal{A}_\infty$–algebra $\mathcal{A}$ we can associate to Γ an operation

$$\mu_\Gamma\colon A^{\otimes n} \to A$$

(with some grading shift) as follows. To compute $\mu_\Gamma(a_1 \otimes \cdots \otimes a_n)$, start with $a_1, \dots, a_n$ at the source leaves (labeled in order, clockwise around the boundary of Γ). Flow these elements along Γ, and when a string of elements enter a vertex of valence $k > 1$, apply μ_k. The element at the sink is the output.

In these terms, the basic $\mathcal{A}_\infty$ relation states that the sum of μ_Γ over $\mathcal{A}_\infty$–operation trees Γ with exactly two nonleaf vertices is zero.

Given an $\mathcal{A}_\infty$–algebra $\mathcal{A} = (A, \{\mu_i\})$ we can form the tensor algebra

$$T^*(A[1]) := \bigoplus_{n=0}^{\infty} A^{\otimes n}[n].$$

Setting $\mu_0 = 0$, we can combine all the μ_i to form a single map

$$\mu\colon T^*(A[1]) \to A[2]. \tag{2.1.7}$$

Defining $\bar{D}^{\mathcal{A}}\colon T^*(A[1]) \to T^*(A[1])$ by

$$\bar{D}^{\mathcal{A}}(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^{n} \sum_{\ell=1}^{n-j+1} a_1 \otimes \cdots \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes \cdots \otimes a_n,$$

the $\mathcal{A}_\infty$ compatibility relations are encoded in the relation $\mu \circ \bar{D}^{\mathcal{A}} = 0$ or, equivalently, $\bar{D}^{\mathcal{A}} \circ \bar{D}^{\mathcal{A}} = 0$.

Our $\mathcal{A}_\infty$–algebras also need to be appropriately bounded. Before giving the general case, we start with the version for *dg* algebras:

Definition 2.1.8 We say that an augmented *dg* algebra $\mathcal{A}$ has *nilpotent augmentation ideal* if there exists an n so that $(A_+)^n = 0$. We will also abuse terminology and say that $\mathcal{A}$ itself is *nilpotent*.

Note that this is stronger than saying that every element of A_+ is nilpotent. Also, a unital algebra can never be nilpotent in the strict sense.

Definition 2.1.9 An augmented $\mathcal{A}_\infty$–algebra $\mathcal{A} = (A, \{\mu_i\})$ is called *nilpotent* if there exists an n so that for any $i > n$, any elements $a_1, \dots, a_i \in A_+$ and any $\mathcal{A}_\infty$–operation tree Γ, $\mu_\Gamma(a_1 \otimes \cdots \otimes a_i) = 0$.

(An equivalent definition of nilpotent would be to require that there are only finitely many Γ for which μ_Γ is not identically zero on inputs from A_+. The definitions of "operationally bounded" for modules and bimodules, below, can be reformulated similarly. Also, the fact that the two definitions of nilpotent agree in the case of *dg* algebras involves using the Leibniz rule to push any instances of μ_1 onto the inputs and then using $(\mu_1)^2 = 0$.)

Remark 2.1.10 In [21] we assumed a weaker condition on our algebra: that $\mu_i = 0$ for i sufficiently large. The stronger condition of Definition 2.1.9 is used to ensure that our smaller model $\boxtimes$ of the $\mathcal{A}_\infty$–tensor product of bimodules (rather than just modules) is well defined in all cases (Proposition 2.3.10) and for some of the categorical aspects of this paper (in particular, Proposition 2.3.18). See also Remark 2.3.12.

Convention 2.1.11 With the exception of Section 2.1.3, all of the $\mathcal{A}_\infty$–algebras that show up in this paper will be assumed (or proved) to be nilpotent.

2.1.2 Definition of $\mathcal{A}_\infty$–algebra maps

Definition 2.1.12 Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{A}_\infty$–algebras. An *$\mathcal{A}_\infty$–homomorphism from $\mathcal{A}$ to $\mathcal{B}$* is a collection of degree-0 maps $\phi = \{\phi_i\colon \mathcal{A}[1]^{\otimes i} \to \mathcal{B}[1]\}$, $i \geq 1$, satisfying a compatibility condition which we state in terms of an auxiliary map $F^\phi = T^*(\mathcal{A}[1]) \to T^*(\mathcal{B}[1])$, defined by

$$F^\phi(a_1 \otimes \cdots \otimes a_n) = \sum_{i_1+\cdots+i_k=n} \phi_{i_1}(a_1 \otimes \cdots \otimes a_{i_1}) \otimes \phi_{i_2}(a_{i_1+1} \otimes \cdots \otimes a_{i_1+i_2}) \\ \otimes \cdots \otimes \phi_{i_k}(a_{n-i_k+1} \otimes \cdots \otimes a_n).$$

The compatibility condition is that

$$\overline{D}^{\mathcal{B}} \circ F^{\phi} = F^{\phi} \circ \overline{D}^{\mathcal{A}}. \tag{2.1.13}$$

Note that if $\phi = \{\phi_i\}$ is an $\mathcal{A}_\infty$–homomorphism then ϕ_1 is a chain map.

Composition of $\mathcal{A}_\infty$–homomorphisms is characterized by the property that $F^{\phi\circ\psi} = F^{\phi} \circ F^{\psi}$; we leave it to the reader to verify that such a composition exists.

It turns out that $\mathcal{A}_\infty$–algebra *isomorphisms* are just homomorphisms with ϕ_1 invertible:

Lemma 2.1.14 *Let $\mathcal{A}$ be an $\mathcal{A}_\infty$–algebra and $\phi\colon \mathcal{A} \to \mathcal{B}$ an $\mathcal{A}_\infty$–algebra homomorphism such that ϕ_1 is an isomorphism. Then ϕ is invertible, ie there is an $\mathcal{A}_\infty$–algebra homomorphism $\psi\colon \mathcal{B} \to \mathcal{A}$ such that $\phi \circ \psi = \mathbb{I}_{\mathcal{B}}$ and $\psi \circ \phi = \mathbb{I}_{\mathcal{A}}$.*

Proof It suffices to show that ϕ has both left and right inverses; we will show that ϕ has a left inverse. Set $\psi_1 = \phi_1^{-1}$. Observe that ψ satisfies the first relation for $\mathcal{A}_\infty$–homomorphisms ($\mu_1^{\mathcal{A}} \circ \psi_1 + \psi_1 \circ \mu_1^{\mathcal{B}} = 0$); moreover, for any way of completing ψ to an $\mathcal{A}_\infty$–homomorphism, $(\psi \circ \phi)_1 = \mathbb{I}_{\mathcal{A}}$.

Now, assume inductively that we have found ψ_i for $i < n$ so that for $1 < i < n$, $(\psi \circ \phi)_i = 0$. Observe that

$$(\psi \circ \phi)_n(a_1 \otimes \cdots \otimes a_n) = \psi_n(\phi_1(a_1) \otimes \cdots \otimes \phi_1(a_n)) + \sum_{i=1}^{n-1} \psi_i(F_i^{\phi}(a_1 \otimes \cdots \otimes a_n)).$$

(Here F_i^{ϕ} is the component of F^{ϕ} that lands in $\mathcal{B}^{\otimes i} \subset T^*\mathcal{B}$.) So, if we set

$$\psi_n(b_1 \otimes \cdots \otimes b_n) = \sum_{i=1}^{n-1} \psi_i(F_i^{\phi}(\phi_1^{-1}(b_1) \otimes \cdots \otimes \phi_1^{-1}(b_n))),$$

we have $(\psi \circ \phi)_n = 0$. Continuing in this way, we construct a map ψ so that $\psi \circ \phi = \mathbb{I}_{\mathcal{A}}$. We can also find a map ψ' so that $\phi \circ \psi' = \mathbb{I}_{\mathcal{B}}$ similarly; it follows, of course, that $\psi = \psi'$.

It remains to check that ψ satisfies the relation for $\mathcal{A}_\infty$–homomorphisms, or equivalently that F^{ψ} is a chain map. But we already know that $F^{\psi} = (F^{\phi})^{-1}$, so the result follows from the fact that the inverse of a chain isomorphism is a chain map. □

Definition 2.1.15 An $\mathcal{A}_\infty$–algebra homomorphism $\phi\colon \mathcal{A} \to \mathcal{B}$ is called a *quasi-isomorphism* if ϕ_1 induces an isomorphism from the homology of A with respect to $\mu_1^{\mathcal{A}}$ to the homology of B with respect to $\mu_1^{\mathcal{B}}$.

2.1.3 Induced $\mathcal{A}_\infty$–algebra structures The following is the main lemma of homological perturbation theory; see [36, Section (1i)], Kontsevich and Soibelman [18, Section 6.4] and Keller [13, Section 3.3; 11, page 4]. For the proof we refer the reader to the references.

Proposition 2.1.16 *Let $\mathcal{A}$ be an $\mathcal{A}_\infty$–algebra and B_* a chain complex over $\boldsymbol{k}$.*

(1) *If $f\colon B_* \to \mathcal{A}$ is a homotopy equivalence of chain complexes then there is an $\mathcal{A}_\infty$–algebra structure $\{\mu_i\}$ on B_* and maps $f_i\colon B^{\otimes i} \to \mathcal{A}[1-i]$ ($i \geq 1$) so that*

- *μ_1 is the differential on B_*,*
- *f_1 is the given chain map f, and*
- *$\{f_i\}\colon (B_*, \{\mu_i\}) \to \mathcal{A}$ is a quasi-isomorphism.*

Moreover, if B_* is a dg algebra and f_1 is an algebra map, we can choose $\{\mu_2\}$ to be the multiplication on B_*.

(2) *If $g\colon \mathcal{A} \to B_*$ is a homotopy equivalence of chain complexes then there is an $\mathcal{A}_\infty$–algebra structure $\{\mu_i\}$ on B_* and maps $g_i\colon \mathcal{A}^{\otimes i} \to B[1-i]$ so that*

- *μ_1 is the differential on B_*,*
- *g_1 is the given chain map g, and*
- *$\{g_i\}\colon \mathcal{A} \to (B_*, \{\mu_i\})$ is a quasi-isomorphism.*

Remark 2.1.17 Note that over $\boldsymbol{k}$ there is no distinction between homotopy equivalences and quasi-isomorphisms of chain complexes.

Corollary 2.1.18 *Let $\mathcal{A}$ be an $\mathcal{A}_\infty$–algebra and $\mathcal{H}$ the homology of $\mathcal{A}$, which inherits an associative algebra structure from $\mathcal{A}$. Let $i\colon \mathcal{H} \to \mathcal{A}$ denote an inclusion choosing a representative for each homology class and $p\colon \mathcal{A} \to \mathcal{H}$ any projection that sends each cycle to its homology class. Then there is an $\mathcal{A}_\infty$–algebra structure $\mathcal{H}$ consisting of maps $\{\mu_i\}$ on $\mathcal{H}$ such that there are $\mathcal{A}_\infty$–quasi-isomorphisms $f\colon \mathcal{H} \to \mathcal{A}$ and $g\colon \mathcal{A} \to \mathcal{H}$ extending i and p respectively.*

Proof The existence of $\{\mu_i\}$ and either f or g is immediate from Proposition 2.1.16; it remains to show that the $\mathcal{A}_\infty$–algebra structures on $\mathcal{H}$ given by the two parts of Proposition 2.1.16 can be chosen to be the same.

Proposition 2.1.16 gives us two $\mathcal{A}_\infty$–algebra structures $\mathcal{H}$ and $\mathcal{H}'$ on H and $\mathcal{A}_\infty$–quasi-isomorphisms $f\colon \mathcal{H} \to \mathcal{A}$, $g\colon \mathcal{A} \to \mathcal{H}'$. Observe that $g_1 \circ f_1$ is the identity map on H. So, by Lemma 2.1.14, $g \circ f\colon \mathcal{H} \to \mathcal{H}'$ is an isomorphism. Then $f\colon \mathcal{H} \to \mathcal{A}$ and $(g \circ f)^{-1} \circ g\colon \mathcal{A} \to \mathcal{H}$ are the desired maps. □

Remark 2.1.19 Even if $\mathcal{A}$ is nilpotent, the induced $\mathcal{A}_\infty$–structure on the homology $\mathcal{H}$ of $\mathcal{A}$ may not be nilpotent (or even satisfy the weaker condition that $\mu_i = 0$ for sufficiently large i). However, for the algebras $\mathcal{A}(\mathcal{Z})$ of interest in this paper, the induced $\mathcal{A}_\infty$–structures on $\mathcal{H}$ will be nilpotent; see also Remark 4.9.

Definition 2.1.20 The quasi-isomorphisms $f\colon \mathcal{H} \to \mathcal{A}$ given by Corollary 2.1.18 will be called *standard quasi-isomorphisms*.

Note that the higher products on $\mathcal{H}$ depended on some choices. In certain situations, however, they are canonically defined; we discuss one instance of this (which will play a role in Section 4). Let $\mathcal{A}$ be an $\mathcal{A}_\infty$–algebra, with products μ_i, and $\mathcal{H}$ be its homology algebra, with products $\overline{\mu}_i$.

Definition 2.1.21 A sequence $\alpha_1, \dots, \alpha_m \in \mathcal{H}$ is said to be *Massey admissible* if for any $1 \le i < j \le m$ with $(i, j) \ne (1, m)$, we have $\overline{\mu}_{j-i+1}(\alpha_i, \alpha_{i+1}, \dots, \alpha_j) = 0$ and $\mathcal{H}_{g(i,j)+1} = 0$, where $g(i, j)$ is the grading of $\overline{\mu}_{j-i+1}(\alpha_i, \dots, \alpha_j)$, ie $g(i, j) = j - i - 1 + \mathrm{gr}(\alpha_i) + \cdots + \mathrm{gr}(\alpha_j)$.

Lemma 2.1.22 *Let $\mathcal{A}$ be a dg algebra, and let $\mathcal{H}$ denote its homology. Suppose $\alpha_1, \dots, \alpha_m \in \mathcal{H}$ is Massey admissible. Then, there are elements $\xi_{i,j} \in \mathcal{A}$ for $0 \le i < j \le m$ and $(i, j) \ne (0, m)$ such that*

$$d\xi_{i,j} = \sum_{i<k<j} \xi_{i,k} \cdot \xi_{k,j},$$

and where, for $i = 1, \dots, m$, $\xi_{i-1,i}$ is a cycle representing the homology class α_i. Moreover, $\overline{\mu}_m(\alpha_1, \dots, \alpha_m)$ is represented by the cycle

$$\sum_{0<k<m} \xi_{0,k} \cdot \xi_{k,m}. \tag{2.1.23}$$

The homology class of this cycle is independent of the choices of the $\xi_{i,j}$.

Proof Fix a standard quasi-isomorphism $f\colon \mathcal{H} \to \mathcal{A}$ and consider the $\mathcal{A}_\infty$ relation for the map f, with inputs $\alpha_1, \dots, \alpha_m$. Since the target is a *dg* algebra, this relation contains no trees with more than two nodes labeled by f. Indeed, there are the following four types of trees:

(1) Trees where there are two nodes labeled by f, whose two outputs get multiplied in $\mathcal{A}$.
(2) Trees with one node which is a multiplication $\overline{\mu}_i$ of some proper (consecutive) subsequence of $\alpha_1, \dots, \alpha_m$, followed by a node labeled f_{m-i+1}.

(3) The tree representing $f_1(\overline{\mu}_m(\alpha_1, \dots, \alpha_m))$.

(4) The tree representing $\mu_1(f_m(\alpha_1, \dots, \alpha_m))$.

Now set $\xi_{i,j} = f_{j-i}(\alpha_{i+1}, \dots, \alpha_j)$. We can interpret the sum of trees of type (1) as the sum appearing in (2.1.23). Massey admissibility guarantees that higher multiplication $\overline{\mu}_i$ on a proper consecutive subsequence vanishes, and hence that terms of type (2) vanish. The term of type (3) gives a cycle representing $\mu_m(\alpha_1, \dots, \alpha_m)$. The term of type (4) evidently gives a coboundary. It follows that for one choice of the $\xi_{i,j}$ the lemma holds.

It remains to show that the homology class is independent of all the choices made, and hence that, for a Massey admissible sequence, $\overline{\mu}_m(\alpha_1, \dots, \alpha_m)$ is independent of the choice of compatible $\mathcal{A}_\infty$–algebra structure on $H_*(\mathcal{A})$. To this end, we show that if for some c, we exchange exactly one of the $\xi_{a,b}$ for $b - a \leq c$ by $\xi'_{a,b}$, then we can complete this to a system of $\xi'_{i,j}$ so that the final result changes by a coboundary. Specifically, note that $\xi_{a,b} - \xi'_{a,b}$ is a cycle, and it is supported in grading $g(a,b)+1$. Hence, by Massey admissibility, in fact $\xi_{a,b} - \xi'_{a,b} = d\eta_{a,b}$ for some choice of $\eta_{a,b}$. Now, define, for all $i \leq a$ and $j \geq b$,

$$\begin{aligned}
\xi'_{a,j} &= \xi_{a,j} + \eta_{a,b} \cdot \xi_{b,j}, \\
\xi'_{i,b} &= \xi_{i,b} + \xi_{i,a} \cdot \eta_{a,b},
\end{aligned}$$

and all other $i < j$, $\xi'_{i,j} = \xi_{i,j}$. It is straightforward to check that the $\xi'_{i,j}$ satisfy the same equations as the original $\xi_{i,j}$, and also that

$$\sum_{0<k<m} \xi'_{i,k} \cdot \xi'_{k,j} - \sum_{0<k<m} \xi_{i,k} \cdot \xi_{k,j} = \begin{cases} d(\eta_{0,m}) & \text{if } 1 = a,\ b = m, \\ d(\eta_{0,b} \cdot \xi_{b,j}) & \text{if } 1 = a,\ b < m, \\ d(\xi_{0,a} \cdot \eta_{a,m}) & \text{if } 1 < a,\ b = m, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that we can go between any two solutions $\{\xi_{i,j}\}$ and $\{\xi'_{i,j}\}$ by a sequence of moves of the above type, and each step leaves the homology class of the expression from (2.1.23) unchanged. □

Remark 2.1.24 Equation (2.1.23) is the traditional definition of the Massey product, which is typically defined only up to some indeterminacy. The Massey admissibility condition guarantees that there is no ambiguity in its definition.

2.2 Modules over $\mathcal{A}_\infty$–algebras

In this section we define various notions of $\mathcal{A}_\infty$–modules and bimodules which are used throughout the paper. As noted earlier, our treatment is nonstandard in that we introduce an object which we call a "type D structure", which the reader can think of as a type of projective module; see Corollary 2.3.25.

Another slightly unusual feature of our treatment is that we will define our categories of modules as *dg* categories. We start by reviewing this notion.

2.2.1 Background on *dg* categories The material in this section is standard, but perhaps unfamiliar to the low-dimensional topology community. Our treatment is drawn from Keller [14], to which we refer the reader for more details and further results.

Definition 2.2.1 A *differential graded category* is a category $\mathscr{C}$ such that the morphism spaces are chain complexes and composition of morphisms is bilinear and commutes with the differential, ie such that composition of functions gives chain maps $\circ\colon \mathrm{Mor}(y,z)\otimes_{\boldsymbol{k}}\mathrm{Mor}(x,y)\to\mathrm{Mor}(x,z)$.

The prototypical example is the category of chain complexes:

Example 2.2.2 The *dg category of chain complexes* has objects chain complexes C_* and morphism spaces

$$\mathrm{Mor}(C_*,D_*)_n=\{f=(f_i\colon C_i\to D_{n+i})_{i=-\infty}^{\infty}\}$$

with differential defined by

$$(\partial f)(x)=\partial_D(f(x))+f(\partial_C x).$$

Note that the 0–cycles in $(\mathrm{Mor}(C_*,D_*),\partial)$ are exactly the degree-0 chain maps between C_* and D_* and the boundaries are the nullhomotopic chain maps. The homology of $(\mathrm{Mor}(C_*,D_*),\partial)$ is the group of chain maps modulo homotopy.

Definition 2.2.3 Given a *dg* category $\mathscr{C}$, let $\mathsf{Z}(\mathscr{C})$ (respectively $\mathsf{Z}_*(\mathscr{C})$) denote the category with the same objects as $\mathscr{C}$ and morphisms $\mathrm{Hom}_{\mathsf{Z}(\mathscr{C})}(x,y)=Z_0(\mathrm{Mor}_{\mathscr{C}}(x,y))$ (respectively $\mathrm{Hom}_{\mathsf{Z}_*(\mathscr{C})}(x,y)=Z_*(\mathrm{Mor}_{\mathscr{C}}(x,y))$) the degree-0 cycles (respectively cycles of any degree) in the morphism space of $\mathscr{C}$. We call the morphisms in $\mathsf{Z}_*(\mathscr{C})$ the *homomorphisms in* $\mathscr{C}$, and sometimes denote the set of homomorphisms simply by Hom (as distinct from the set of all morphisms Mor).

Let $\mathsf{H}(\mathscr{C})$ (respectively $\mathsf{H}_*(\mathscr{C})$) denote the category with the same objects as $\mathscr{C}$ and morphisms $\mathrm{Hom}_{\mathsf{H}(\mathscr{C})}(x,y)=H_0(\mathrm{Mor}_{\mathscr{C}}(x,y))$ (respectively $\mathrm{Hom}_{\mathsf{H}_*(\mathscr{C})}(x,y)=H_*(\mathrm{Mor}_{\mathscr{C}}(x,y))$), the degree-0 homology (respectively total homology) of the morphism space of $\mathscr{C}$.

Example 2.2.4 For $\mathscr{C}$ the *dg* category of chain complexes, $\mathsf{Z}(\mathscr{C})$ is the usual category of chain complexes, in which the morphism spaces are the degree-0 chain maps. The category $\mathsf{H}(\mathscr{C})$ is the homotopy category of chain complexes.

The category $\mathsf{H}(\mathscr{C})$ is naturally a triangulated category; see [14, Section 3.4]. One could go further and invert quasi-isomorphisms, but for our purposes this will not be necessary.

Definition 2.2.5 Let $\mathscr{C}$ be a *dg* category. Morphisms $f, g \in \mathrm{Mor}_{\mathscr{C}}(x, y)$ are called *homotopic* if there is a morphism $h \in \mathrm{Mor}_{\mathscr{C}}(x, y)$ so that $(\partial h) = f - g$; in this case we write $f \sim g$. A cycle $f \in \mathrm{Mor}_{\mathscr{C}}(x, y)$ is a *homotopy equivalence* if there is a cycle $g \in \mathrm{Mor}_{\mathscr{C}}(y, x)$ so that $g \circ f \sim \mathbb{I}_x$ and $f \circ g \sim \mathbb{I}_y$.

Definition 2.2.6 For $f \in \mathrm{Mor}(x, y)$, let

$$
\begin{aligned}
f_{*w} &\colon \mathrm{Mor}(w, x) \to \mathrm{Mor}(w, y), \\
f^{*z} &\colon \mathrm{Mor}(y, z) \to \mathrm{Mor}(x, z),
\end{aligned}
$$

be the maps obtained by pre- and post-composing with f. We will sometimes write f_* or f^* if the spaces are clear from context.

Lemma 2.2.7 *If $f, g \in \mathrm{Mor}_{\mathscr{C}}(x, y)$ are homotopic morphisms then $f_{*w} \sim g_{*w}$ and $f^{*z} \sim g^{*z}$.*

Similarly, if $f \in \mathrm{Mor}_{\mathscr{C}}(x, y)$ is a homotopy equivalence then the maps f_{*w} and f^{*z} are chain homotopy equivalences.

The proof is straightforward.

Definition 2.2.8 Let $\mathscr{C}$ and $\mathscr{D}$ be *dg* categories. A *dg functor* $\mathcal{F}\colon \mathscr{C} \to \mathscr{D}$ is a functor $\mathscr{C} \to \mathscr{D}$ such that for any objects x and y of $\mathscr{C}$, $\mathcal{F}(\mathrm{Mor}(x, y))$ is a degree-0 *dg* module homomorphism.

Lemma 2.2.9 *If $\mathcal{F}$ is a dg functor and $f, g \in \mathrm{Mor}(x, y)$ are homotopic then $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are homotopic. If f is a homotopy equivalence then $\mathcal{F}(f)$ is a homotopy equivalence.*

The proof is immediate from the definitions.

Definition 2.2.10 Let $\mathscr{C}$ and $\mathscr{D}$ be *dg* categories and $\mathcal{F},\mathcal{G}\colon \mathscr{C}\to\mathscr{D}$ *dg* functors. We say $\mathcal{F}$ is *homotopic to* $\mathcal{G}$ if for each $x\in\mathrm{Ob}(\mathscr{C})$ there are homotopy equivalences $\eta_x\colon \mathcal{F}(x)\to\mathcal{G}(x)$ so that for any $x,y\in\mathrm{Ob}(\mathscr{C})$ the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathrm{Mor}_{\mathscr{C}}(x,y) \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{G}"'] & \mathrm{Mor}_{\mathscr{D}}(\mathcal{F}(x),\mathcal{F}(y)) \arrow[d, "(\eta_y)_*"] \\
\mathrm{Mor}_{\mathscr{D}}(\mathcal{G}(x),\mathcal{G}(y)) \arrow[r, "(\eta_x)^*"'] & \mathrm{Mor}_{\mathscr{D}}(\mathcal{F}(x),\mathcal{G}(y))
\end{tikzcd}$$

A *dg* functor $\mathcal{F}\colon \mathscr{C}\to\mathscr{D}$ is a *homotopy equivalence* if there is a functor $\mathcal{G}\colon \mathscr{D}\to\mathscr{C}$ so that $\mathcal{G}\circ\mathcal{F}$ is homotopic to $\mathbb{I}_{\mathscr{C}}$ and $\mathcal{F}\circ\mathcal{G}$ is homotopic to $\mathbb{I}_{\mathscr{D}}$.

(In Definition 2.2.10 we have not required any coherence for the homotopies in the diagram; one could formulate stronger notions with such coherence built in.)

Definition 2.2.11 Let $\mathscr{C}$ and $\mathscr{D}$ be *dg* categories. A functor $\mathcal{F}\colon \mathscr{C}\to\mathscr{D}$ is a *quasiequivalence* if

- for all $x,y\in\mathrm{Ob}(\mathscr{C})$, the map $\mathcal{F}(x,y)\colon \mathrm{Mor}_{\mathscr{C}}(x,y)\to\mathrm{Mor}_{\mathscr{D}}(\mathcal{F}(x),\mathcal{F}(y))$ is a quasi-isomorphism, and
- the induced map $\mathsf{H}(\mathcal{F})\colon \mathsf{H}(\mathscr{C})\to\mathsf{H}(\mathscr{D})$ is an equivalence of categories.

(See [14, Section 2.3] for more details.)

Proposition 2.2.12 *If $\mathcal{F}\colon \mathscr{C}\to\mathscr{D}$ is a homotopy equivalence then $\mathcal{F}$ is a quasiequivalence.*

Again, the proof is straightforward.

Just as one can generalize the notion of *dg* algebras to $\mathcal{A}_\infty$–algebras, one can generalize the notion of *dg* categories and *dg* functors to $\mathcal{A}_\infty$–categories and $\mathcal{A}_\infty$–functors. Some of the categories studied in this paper (in particular, the category of type D modules) are $\mathcal{A}_\infty$–categories, and some of the functors (in particular, $\boxtimes$) are $\mathcal{A}_\infty$–functors. Most of the additional complications are, however, not important for the applications in this paper: when working with *dg* algebras (rather than $\mathcal{A}_\infty$–algebras), the categories we consider are honest *dg* categories (see Remark 2.2.28). So we will not spell out the notions of $\mathcal{A}_\infty$–categories and $\mathcal{A}_\infty$–functors, trusting the reader to provide them if desired, or to consult Seidel [35].

Remark 2.2.13 The reader might find the following analogy helpful for understanding the role of these *dg* categories. It has been understood for some time that when working with complexes, rather than taking homology of complexes it is often better to pass to the derived category, ie to invert morphisms which induce isomorphisms on homology. For example, operations like tensor product and Hom are better behaved with respect to inverting quasi-isomorphisms than with respect to taking homology, giving rise to the derived functors Tor and Ext. The language of $\mathcal{A}_\infty$–algebras allows one to view the derived category of R–modules itself as the homology of a *dg* category. Then, if one is interested in studying categories of modules, it is better to work with *dg* categories and invert quasiequivalences of categories rather than take homology and work with derived categories.

2.2.2 The category of $\mathcal{A}_\infty$–modules

Definition 2.2.14 A *(right)* $\mathcal{A}_\infty$*–module* $M_{\mathcal{A}}$ *over* $\mathcal{A} = (A, \{\mu_i\})$ is a $\mathbb{Z}$–graded $\boldsymbol{k}$–module M together with degree-0 $\boldsymbol{k}$–linear maps $m_{j+1}\colon M \otimes A[1]^{\otimes j} \to M[1]$ ($j = 0, \dots, \infty$) such that for each $i = 0, \dots, \infty$, $x \in M$ and $a_1, \dots, a_i \in A$,

$$0 = \sum_{j=0}^{i-1} m_{i-j}(m_{j+1}(x, a_1, \dots, a_j), a_{j+1}, \dots, a_i) + \sum_{j=1}^{i} \sum_{k=1}^{i-j+1} m_{i-j+1}(x, a_1, \dots, a_{k-1}, \mu_j(a_k, \dots, a_{k+j}), a_{k+j+1}, \dots, a_i). \tag{2.2.15}$$

An $\mathcal{A}_\infty$–module is *strictly unital* if $m_2(x, 1) = x$ and $m_{i+1}(x, a_1, \dots, a_i) = 0$ if $i \neq 1$ and one of the $a_i \in \boldsymbol{k}$.

We will sometimes refer to $\mathcal{A}_\infty$–modules as *type A modules*, to place them on equal footing with the type D modules which will appear later (Definition 2.2.23). The bordered invariant $\widehat{CFA}(Y)$ is an $\mathcal{A}_\infty$–module.

Convention 2.2.16 All $\mathcal{A}_\infty$–modules will be assumed strictly unital.

As with the $\mathcal{A}_\infty$–relation for algebras (2.1.3), (2.2.15) has an interpretation in terms of trees. In this interpretation, there are two types of strands: those corresponding to algebra elements, and distinguished (dotted) ones corresponding to module elements. Precisely:

Definition 2.2.17 A (right) $\mathcal{A}_\infty$*–module operation tree* is a finite directed tree Γ embedded in the plane so that all edges point downwards, with the edges of two types, the algebra edges (labeled by A) or module edges (labeled by M) and nonleaf vertices marked either μ or m, and so that:

- Each nonleaf vertex has exactly one outgoing edge.
- Each μ vertex touches only algebra edges.
- At each m vertex, the leftmost incoming edge and the outgoing edge are module edges, and the other incoming edges are algebra edges.
- There is at least one module edge.

To an $\mathcal{A}_\infty$–module operation tree we can associate an operation $m_\Gamma\colon M \otimes A^{\otimes n} \to M$ by flowing along the edges as before, applying μ_i or m_i depending on the label at the vertex. Then (2.2.15) says that the sum of m_Γ over all $\mathcal{A}_\infty$–module operation trees Γ with two vertices and a fixed number of inputs vanishes.

An $\mathcal{A}_\infty$–module operation tree is called *spinal* if each node is labeled m, ie the line of module edges goes through each node.

Definition 2.2.18 We say that an $\mathcal{A}_\infty$–module $M_{\mathcal{A}}$ is *operationally bounded* (or just *bounded*) if, for all $x \in M$ there exists an n so that for any $i > n$ and any spinal $\mathcal{A}_\infty$–module operation tree Γ with $i+1$ input edges, $m_\Gamma(x \otimes \cdot\,))$ vanishes on $(A_+)^{\otimes i}$.

Graphically, this says that for each x there exists an n so that if $i_1 + \cdots + i_k > n$ then for any $a_{1,1}, \dots, a_{k,i_k} \in A_+$,

$$
\begin{tikzcd}
x \arrow[d, dashed] & a_{1,1} \otimes \cdots \otimes a_{1,i_1} \arrow[dl, Rightarrow] & \cdots & a_{k,1} \otimes \cdots \otimes a_{k,i_k} \arrow[dddlll, Rightarrow] \\
\mu_{i_1+1} \arrow[d, dashed] & & & \\
\vdots \arrow[d, dashed] & & & \\
\mu_{i_k+1} \arrow[d, dashed] & & & \\
0. & & &
\end{tikzcd}
$$

The notion of operationally bounded modules is different from more traditional definitions of boundedness; for instance, it does not imply any bound on degrees which have nonzero homology groups.

Remark 2.2.19 As in Remark 2.1.10, this is a stronger condition than the condition called "operationally bounded" in our previous paper [21], where we just assumed that $m_i = 0$ for i sufficiently large. But note that a *dg* module over a nilpotent *dg* algebra is automatically operationally bounded in the stronger sense above. The reason we want the stronger condition relates to subtleties of the box tensor product of bimodules, not modules.

We can combine the multiplications m_i on an $\mathcal{A}_\infty$–module $M_{\mathcal{A}}$ into a single map $m\colon M \otimes T^*(A[1]) \to M[1]$. Because we assume $M_{\mathcal{A}}$ is strictly unital, all the information is contained in a map $M \otimes T^*(A_+[1]) \to M[1]$, also denoted m. If $M_{\mathcal{A}}$ is operationally bounded then the map m extends to a map from the completed tensor product: $m\colon M \otimes \bar{T}^*(A[1]) \to M[1]$. (Here, $\bar{T}^*(A[1]) = \prod_{i=0}^{\infty} A[1]^{\otimes i}$.)

Definition 2.2.20 The category $\mathsf{Mod}_{\mathcal{A}}$ of (right) $\mathcal{A}_\infty$–modules over $\mathcal{A}$ is the *dg* category whose objects are $\mathcal{A}_\infty$–modules $M_{\mathcal{A}}$ and whose morphism spaces are defined as follows. The $\mathbb{Z}$–graded vector spaces underlying the morphism spaces are the vector spaces of $\boldsymbol{k}$–module homomorphisms

$$\mathrm{Mor}_{\mathcal{A}}(M_{\mathcal{A}}, N_{\mathcal{A}}) := \mathrm{Hom}_{\boldsymbol{k}}(M \otimes T^*(A_+[1]), N),$$

where A_+ is the augmentation ideal of A. Here we think of the homomorphisms between two $\mathbb{Z}$–graded $\boldsymbol{k}$–modules V and W as a graded $\mathbb{F}_2$ vector space, with

$$\mathrm{Hom}_{\boldsymbol{k}}(V, W)_i := \bigoplus_j \mathrm{Hom}_{\boldsymbol{k}}(V_j, W_{i+j}).$$

If $h \in \mathrm{Mor}_{\mathcal{A}}(M_{\mathcal{A}}, N_{\mathcal{A}})$, denote the component parts by $h_i\colon M \otimes (A_+[1])^{\otimes(i-1)} \to N$.

The differential on morphisms is given by

$$\begin{aligned}
(\partial h)(x, a_1, \dots, a_n)&\\
:= \sum_{i+j=n} & h_{i+1}(m_{j+1}(x, a_1, \dots, a_j), a_{j+1}, \dots, a_n)\\
&+ \sum_{i+j=n} m_{i+1}(h_{j+1}(x, a_1, \dots, a_j), a_{j+1}, \dots, a_n)\\
&+ \sum_{i+j=n} \sum_{k=1}^{n-j} h_{i+1}(x, a_1, \dots, a_{k-1}, \mu_{j+1}(a_k, \dots, a_{k+j}), a_{k+j+1}, \dots, a_n).
\end{aligned}$$

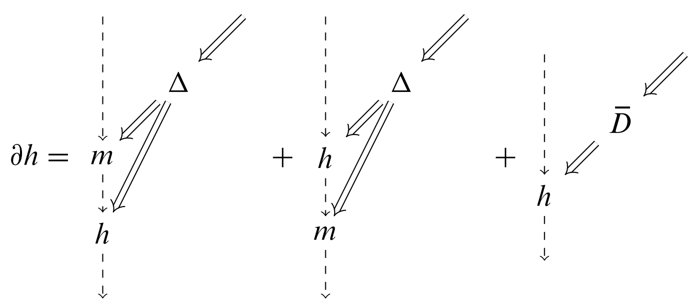
The $\mathcal{A}_\infty$*–homomorphisms* from $M_{\mathcal{A}}$ to $N_{\mathcal{A}}$ in the usual sense (see for example [13]) are the cycles in the morphism complex; cf Definition 2.2.3.

Remark 2.2.21 Strict unitality is built into the definition of $\mathcal{A}_\infty$–morphisms, since we use the augmentation ideal in the definition of the morphism spaces.

We can again give a graphical notation for operations built from morphisms, as follows. Let h be an $\mathcal{A}_\infty$–morphism from $M_\mathcal{A}$ to $N_\mathcal{A}$. An *$\mathcal{A}_\infty$–module morphism tree* Γ is a planar directed tree satisfying all the conditions of an $\mathcal{A}_\infty$–module operation tree, except that the module edges may be marked either M or N, and there is one node labeled by h with an M input and N output which is otherwise like an m node. Again, we can define a map $h_\Gamma\colon M \otimes (A_+[1])^{\otimes n} \to N$ by applying at each vertex μ_i, h_i, or m_i in either M or N, as appropriate.

Definition 2.2.22 An $\mathcal{A}_\infty$–morphism h from $M_\mathcal{A}$ to $N_\mathcal{A}$ is *operationally bounded* if for each $x \in M$ there is an n so that $h_\Gamma(x \otimes \cdot\,))$ vanishes on $(A_+)^{\otimes i}$ for all spinal $\mathcal{A}_\infty$–module morphism trees Γ with $i > n$ inputs.

We can also view a morphism $(h_j)_{j=1}^\infty$ as a single map $h\colon M \otimes T^*(A[1]) \to N$. Writing $\Delta\colon T^*(V) \to T^*(V) \otimes T^*(V)$ for the natural comultiplication for any vector space V, we can also draw the differential of a morphism h as:



We have used dashed lines for module elements, and solid lines for algebra elements, either in $\mathcal{A}$ in $\mathcal{A}_+$. A doubled arrow denotes elements of $T^*(A[1])$ or $T^*(A_+[1])$.

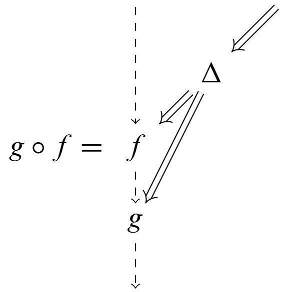
Given $f \in \mathrm{Mor}_\mathcal{A}(M,N)$ and $g \in \mathrm{Mor}_\mathcal{A}(N,L)$, we define the composite morphism $g \circ f \in \mathrm{Mor}_\mathcal{A}(M,L)$ by

$$(g \circ f)(x \otimes \alpha) = g \circ (f \otimes \mathbb{I}_{T^*(\mathcal{A}[1])})(x \otimes \Delta(\alpha)).$$

This induces a chain map

$$\mathrm{Mor}_\mathcal{A}(N,L) \otimes \mathrm{Mor}_\mathcal{A}(M,N) \to \mathrm{Mor}_\mathcal{A}(M,L).$$

We can draw composition as



2.2.3 The category of type *D* structures Throughout this subsubsection, let $\mathcal{A} = (A, \{\mu_i\})$ be an $\mathcal{A}_\infty$–algebra satisfying Convention 2.1.5.

Definition 2.2.23 A (left) *type D structure* over $\mathcal{A}$ is an object ${}^{\mathcal{A}}N$ consisting of a graded $\boldsymbol{k}$–module N equipped with a degree-0 linear map $\delta^1 \colon N \to A[1] \otimes N$, satisfying a compatibility condition which is best described after introducing an auxiliary construction.

Define maps $\delta^i \colon N \to (A[1])^{\otimes i} \otimes N$ $(i \geq 2)$ inductively by

$$\delta^{i+1} := (\mathbb{I}_{\mathcal{A}^{\otimes i}} \otimes \delta^1) \circ \delta^i$$

and define $\delta \colon N \to \overline{T}^*(A[1]) \otimes N$ by

$$\delta(x) := \sum_{i=0}^{\infty} \delta^i. \tag{2.2.24}$$

(Here $\overline{T}^*(A[1]) = \prod_{i=0}^{\infty} \mathcal{A}[1]^{\otimes i}$.) Then the compatibility condition is that for any $x \in N$,

$$(\mu \otimes \mathbb{I}_N) \circ \delta(x) = 0. \tag{2.2.25}$$

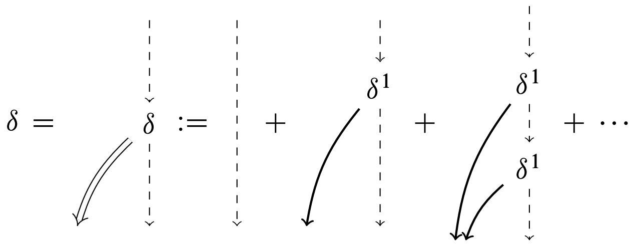
Here μ is the sum of the structure maps of $\mathcal{A}$, as in (2.1.7).

We say that ${}^{\mathcal{A}}N$ is *operationally bounded* if for each $x \in {}^{\mathcal{A}}N$, there is a constant $n = n(x)$ with the property that for all $i > n$, $\delta^i(x) = 0$. This is equivalent to saying that the above δ factors thorough the inclusion of $T^*(A[1])$ in $\overline{T}^*(A[1])$.

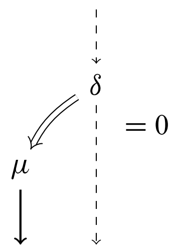
When more than one module is present, we will often write $\delta^{N,i}$ for the operation δ^i on the type D structure N and δ^N to denote the map δ on N. This conflict of notation with δ^i should not cause confusion.

Sometimes we refer to type D structures as *type D modules*.

Note that the defining (2.2.25) makes sense since $\mathcal{A}$ is nilpotent. We can represent Definition 2.2.23 graphically by:



Then, (2.2.25) takes the form:



We can also form type D structures into a *dg* or an $\mathcal{A}_\infty$ category ${}^{\mathcal{A}}_{\mathrm{u}}\mathsf{Mod}$ where $\mathrm{Mor}({}^{\mathcal{A}}M, {}^{\mathcal{A}}N)$ is the chain complex whose underlying space consists of maps

$$h^1\colon M \to A \otimes N.$$

Such a map can be upgraded to a map to the tensor algebra: define $h^i\colon M \to A^{\otimes i} \otimes N$ by

$$h^i := \sum_{j=0}^{i-1} (\mathbb{I}_{A^{\otimes j+1}} \otimes \delta^{N,i-j-1}) \circ (\mathbb{I}_{A^{\otimes j}} \otimes h^1) \circ \delta^{M,j}$$

and $h\colon M \to \overline{T}^* A \otimes N$ by $h := \sum_{i=1}^{\infty} h^i$. The symbol u in the notation ${}^{\mathcal{A}}_{\mathrm{u}}\mathsf{Mod}$ indicates that objects in the category of type D structures are not required to be bounded (ie they are possibly unbounded; see Definition 2.2.29).

The degree of an element of $\mathrm{Mor}({}^{\mathcal{A}}M, {}^{\mathcal{A}}N)$ is the degree of the corresponding map of vector spaces $M \to A \otimes N$.

The boundary operator on morphisms is defined by

$$(\partial h)^1 := (\mu \otimes \mathbb{I}_N) \circ h,$$

as depicted on the left in Figure 1.

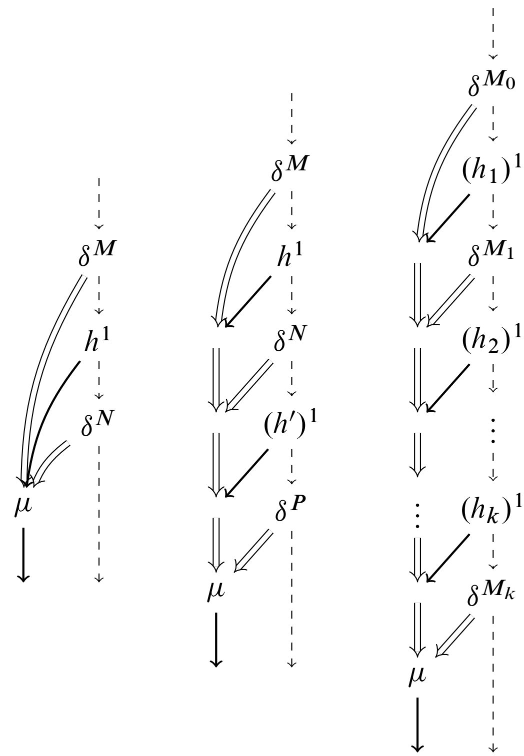


Figure 1: Operations on type D morphisms: from left to right, we have the differential on $\mathrm{Mor}({}^{\mathcal{A}}M, {}^{\mathcal{A}}N)$, the composition of morphisms and the higher ($\mathcal{A}_\infty$) compositions.

The composition of two morphisms $h^1 \in \mathrm{Mor}({}^{\mathcal{A}}M, {}^{\mathcal{A}}N)$ and $(h')^1 \in \mathrm{Mor}({}^{\mathcal{A}}N, {}^{\mathcal{A}}P)$ is defined by

$$(h' \circ h)^1 := (\mu \otimes \mathbb{I}_N) \circ (\mathbb{I}_{T^*} \otimes \delta^P) \circ (\mathbb{I}_{T^*} \otimes (h')^1) \circ (\mathbb{I}_{T^*} \otimes \delta^N) \circ (\mathbb{I}_{T^*} \otimes h^1) \circ \delta^M; \tag{2.2.26}$$

this is illustrated in the middle in Figure 1.

It is straightforward to verify that this is a map of chain complexes, ie $\partial(h \circ h') = \partial h \circ h' + h \circ \partial h'$.

The attentive reader will notice that the composition of morphisms is not associative but associative only up to homotopy. In fact:

Lemma 2.2.27 *There are higher composition maps which make ${}^{\mathcal{A}}_{\mathrm{u}}\mathsf{Mod}$ into an $\mathcal{A}_\infty$–category.*

Proof Let ${}^{\mathcal{A}}M_0,\dots,{}^{\mathcal{A}}M_k$ be type D structures over $\mathcal{A}$. Define a higher composition map

$$\circ_k\colon \mathrm{Mor}({}^{\mathcal{A}}M_0,{}^{\mathcal{A}}M_1)\otimes\cdots\otimes\mathrm{Mor}({}^{\mathcal{A}}M_{k-1},{}^{\mathcal{A}}M_k)\to\mathrm{Mor}({}^{\mathcal{A}}M_0,{}^{\mathcal{A}}M_k)$$

by the diagram on the right in Figure 1.

It is easy to check that the higher composition maps $\circ_k$ satisfy the $\mathcal{A}_\infty$–relation. □

Remark 2.2.28 In the case where $\mathcal{A}$ is a *dg* algebra, the higher composition maps of Lemma 2.2.27 vanish, so ${}^{\mathcal{A}}_{\mathfrak{u}}\mathsf{Mod}$ is an honest *dg* category.

We call the cycles in $\mathrm{Mor}({}^{\mathcal{A}}M,{}^{\mathcal{A}}N)$ *type D module homomorphisms*.

Note that, since composition of morphisms in $\mathrm{Mor}({}^{\mathcal{A}}_{\mathfrak{u}}\mathsf{Mod})$ is associative up to homotopy, the homotopy category $\mathsf{H}({}^{\mathcal{A}}_{\mathfrak{u}}\mathsf{Mod})$ is an honest category.

Finally, we can also form the category of bounded type D structures:

Definition 2.2.29 We call a type D morphism $h\colon {}^{\mathcal{A}}M\to{}^{\mathcal{A}}N$ *bounded* if for any $x\in{}^{\mathcal{A}}M$, there is a constant $n=n(x)$ such that for all $i\geq n$, $(\mathbb{I}_{A^{\otimes i}}\otimes h)\circ\delta^{M,i}(x)=0$ and $(\mathbb{I}_A\otimes\delta^{N,i})\circ h(x)=0$.

Let ${}^{\mathcal{A}}_{\mathfrak{b}}\mathsf{Mod}$ denote the category of bounded type D structures and bounded morphisms.

In fact, it is clear from the definition that:

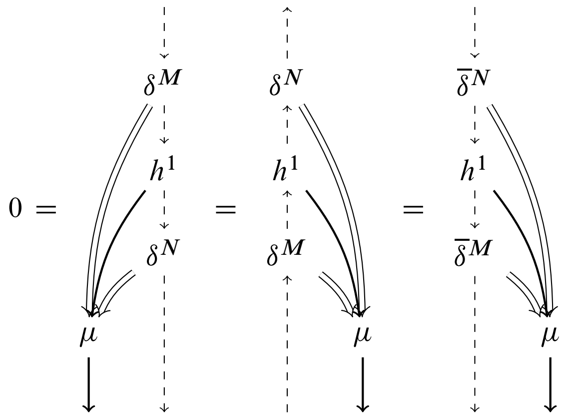
Lemma 2.2.30 *Any morphism between bounded type D structures is bounded.*

One can, of course, define right type D structures similarly; we denote the category of right type D structures over $\mathcal{A}$ by $\mathsf{Mod}^{\mathcal{A}}_{\mathfrak{u}}$.

Definition 2.2.31 Given a type D structure $({}^{\mathcal{A}}M,\delta^1)$, define the *opposite* type D structure $(\overline{M}^{\mathcal{A}},\overline{\delta}^1)$ as follows. First suppose that $\mathcal{A}$ and M are finite-dimensional. As a $\boldsymbol{k}$–module, $\overline{M}$ is just $M^*=\mathrm{Hom}_{\boldsymbol{k}}(M,\boldsymbol{k})$. The map δ^1 on M is an element of $\mathrm{Hom}_{\boldsymbol{k}}(M,A\otimes M)\simeq M^*\otimes A\otimes M\simeq\mathrm{Hom}_{\boldsymbol{k}}(M^*,M^*\otimes A)$, and $\overline{\delta}^1$ is obtained by viewing δ^1 as lying in $\mathrm{Hom}_{\boldsymbol{k}}(M^*,M^*\otimes A)$. If $\mathcal{A}$ or M are not finite-dimensional, $\mathrm{Hom}_{\boldsymbol{k}}(M,A\otimes M)$ is a subspace of $\mathrm{Hom}_{\boldsymbol{k}}(M^*,M^*\otimes A)$, and $\overline{\delta}^1$ is the image of δ^1 under the inclusion.

Lemma 2.2.32 *Given a type D structure (M,δ^1), the opposite structure $((\overline{M})^{\mathcal{A}},\overline{\delta}^1)$ satisfies the type D structure equation (2.2.25).*

Proof The two structure equations are the same, as can be seen most easily graphically in the finite-dimensional case:



In the last diagram, the dotted arrow represents the dual space M^*, for which the maps travel in the opposite direction. $\square$

Lemma 2.2.30 of course says that ${}^{\mathcal{A}}_{\mathfrak{b}}\mathsf{Mod}$ is a full subcategory of ${}^{\mathcal{A}}_{\mathfrak{u}}\mathsf{Mod}$. We will find it most convenient to work in a category which is in between the two:

Definition 2.2.33 Let ${}^{\mathcal{A}}\mathsf{Mod} \subset {}^{\mathcal{A}}_{\mathfrak{u}}\mathsf{Mod}$ denote the full subcategory whose objects are type D structures N which are homotopy equivalent to bounded type D structures.

See also Proposition 2.3.24 for an alternate characterization, in terms of bar resolutions. Our reason for preferring the above category is that it is quasiequivalent to the category of type A modules; see Proposition 2.3.18.

A type D structure ${}^{\mathcal{A}}M$ can be seen as a way to generalize the notion of "projective module" to modules over an $\mathcal{A}_\infty$–algebra. In particular, we will see in Section 2.3.2 how to make $\mathcal{A} \otimes_{\boldsymbol{k}} M$ into an $\mathcal{A}_\infty$–module over $\mathcal{A}$, which, when $\mathcal{A}$ is a differential graded algebra and ${}^{\mathcal{A}}M$ is bounded, is a projective module (Corollary 2.3.25).

Remark 2.2.34 The notion of boundedness is not invariant under isomorphisms of type D structures. For example, consider the algebra $\mathcal{A} = \mathcal{A}(\mathbb{T}, 0)$ discussed in Section 3.3 and the following type D structures over $\mathcal{A}$:

- ${}^{\mathcal{A}}M$ given by $M = \mathbb{F}_2\langle a, b\rangle$ with $\delta^1(a) = \rho_{12} \otimes a + (\rho_1 + \rho_3) \otimes b$ and $\delta^1(b) = \rho_{23} \otimes b$.
- ${}^{\mathcal{A}}N$ given by $M = \mathbb{F}_2\langle c, d\rangle$ with $\delta^1(c) = (\rho_1 + \rho_3) \otimes d$ and $\delta^1(d) = 0$.

(These are both models for $\widehat{CFD}$ of the (-1)–framed solid torus; compare [21, Section 11.2].) Clearly, ${}^{\mathcal{A}}N$ is bounded but ${}^{\mathcal{A}}M$ is not. The map $f\colon {}^{\mathcal{A}}N \to {}^{\mathcal{A}}M$ given by $f^1(c) = a$ and $f^1(d) = b + \rho_2 \otimes a$ is an isomorphism.

Remark 2.2.35 Let $\mathrm{Cob}(\mathcal{A}) = T^*(A_+[1]^*)$ be the tensor algebra on the dual of A_+ (the "cobar resolution" of $\mathcal{A}$) and $\widehat{\mathrm{Cob}}(\mathcal{A})$ its completion with respect to the length filtration. Then the operation $\bar{D}^{\mathcal{A}}$ dualizes to endow $\mathrm{Cob}(\mathcal{A})$ and $\widehat{\mathrm{Cob}}(\mathcal{A})$ with a differential. One can show that an $\mathcal{A}_\infty$–module over $\mathcal{A}$ is exactly a type D structure over $\widehat{\mathrm{Cob}}(\mathcal{A})$.

Remark 2.2.36 In a similar vein to Remark 2.2.35, we have the following reformulation of bounded type D structures in terms of differential comodules. Consider the bar resolution $\mathrm{Bar}(\mathcal{A}) = T^*(\mathcal{A}_+[1])$, endowed with its differential, which sums over all the ways of grouping together k consecutive elements and applying μ_k to them. This can be thought of as an associative coalgebra equipped with the comultiplication Δ, which sums over all the ways of splitting up an element $a_1 \otimes \cdots \otimes a_n$ as a tensor product of two elements of $\mathrm{Bar}(\mathcal{A})$, $(a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$. Bounded type D structures over $\mathcal{A}$ are precisely differential comodules over $\mathrm{Bar}(\mathcal{A})$. Specifically, fix a differential comodule

$$\Delta\colon N \to \mathrm{Bar}(\mathcal{A}) \otimes N$$

equipped with compatible differential $D\colon N \to N$. Letting Π^i denote the natural projection map $\Pi^i\colon T^*(\mathcal{A}_+[1]) \to \mathcal{A}_+[1]^{\otimes i}$, we can define the associated type D structure by $\delta^1 = D + (\Pi^1 \otimes \mathbb{I}_N) \circ \Delta$. Conversely, given a type D structure δ^1, the comodule structure Δ is given by $\Delta = [T^*(\mathbb{I}_A - \epsilon) \otimes \mathbb{I}_M] \circ \delta$, where δ is given in (2.2.24), and the differential is given by $(\epsilon \otimes \mathbb{I}_M) \circ \delta^1$. Indeed, it is interesting to compare $\boxtimes$ with the twisted tensor product of [19]. (See also Keller [12].)

Remark 2.2.37 Given the *dg* algebra (respectively $\mathcal{A}_\infty$ algebra) $\mathcal{A}$, we can form the *dg* category (respectively $\mathcal{A}_\infty$–category) $C_0(\mathcal{A})$ whose objects correspond to elementary idempotents of $\mathcal{A}$. Given idempotents I and J, let $\mathrm{Mor}(I, J)$ be the algebra elements with $J \cdot a \cdot I = a$ (endowed with the natural differential), and let composition correspond to multiplication in the algebra (respectively higher composition correspond to higher multiplications in the algebra). A type D structure over $\mathcal{A}$ can be thought of as a twisted complex over the additive closure of $C_0(\mathcal{A})$. (For the definition of twisted complexes, see Bondal and Kapranov [7] and Kontsevich [17].)

2.2.4 Bimodules of various types

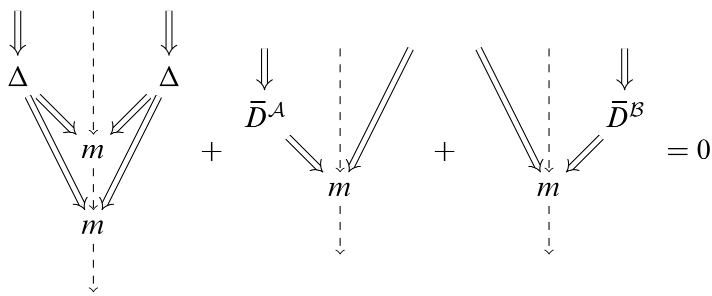
Just as there are two notions of module over an $\mathcal{A}_\infty$–algebra — those of an $\mathcal{A}_\infty$–module (or type A module) and a type D structure — there are four notions of bimodule: type *DD*, *AA*, *DA* and *AD*. Actually, there are subtleties about defining type *DD* modules over general $\mathcal{A}_\infty$–algebras, so for these we restrict to *dg* algebras.

Definition 2.2.38 Let $\mathcal{A}$ and $\mathcal{B}$ be (strictly unital, augmented) $\mathcal{A}_\infty$–algebras over $\boldsymbol{k}$ and $\boldsymbol{j}$ respectively. Then an *$\mathcal{A}_\infty$–bimodule* or *type AA bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ over $\mathcal{A}$ and $\mathcal{B}$* consists of a graded $(\boldsymbol{k},\boldsymbol{j})$–bimodule M and degree-0 maps

$$m_{i,1,j}\colon A[1]^{\otimes i} \otimes M \otimes B[1]^{\otimes j} \to M[1]$$

such that, for $m = \sum_{i,j} m_{i,1,j}$, the following analogue of (2.2.15) is satisfied:

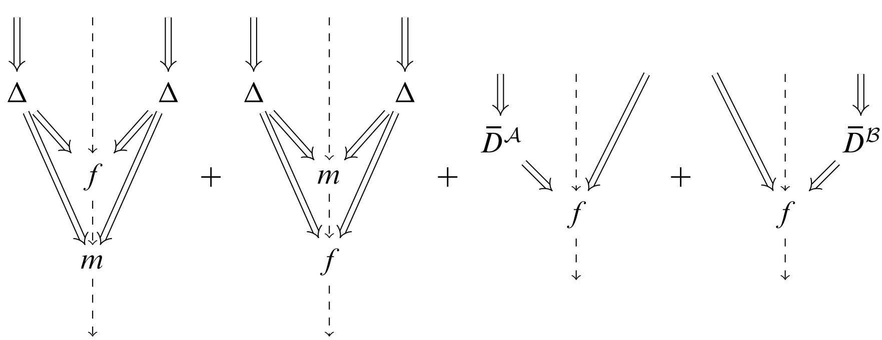
(2.2.39)



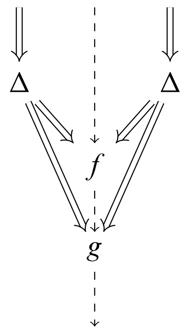
An $\mathcal{A}_\infty$–bimodule is *strictly unital* if $m_{1,1,0}(1,x) = x = m_{0,1,1}(x,1)$ for any x and if $m_{i,1,j}(a_1,\dots,a_i,x,b_1,\dots,b_j)$ vanishes if $i+j>1$ and one of the a_k or b_ℓ lies in $\boldsymbol{k}$ or $\boldsymbol{j}$. We shall always work with strictly unital $\mathcal{A}_\infty$–bimodules.

A morphism ${}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}N_{\mathcal{B}}$ is a collection of maps $f_{i,1,j}\colon A_+[1]^{\otimes i} \otimes M \otimes B_+[1]^{\otimes j} \to N$. The set of morphisms is naturally a graded $\mathbb{F}_2$ vector space; compare Definition 2.2.20. Moreover, the set of morphisms forms a chain complex: writing f to denote the total map $f\colon T^*(A_+[1]) \otimes M \otimes T^*(B_+[1]) \to N$, the differential of such a morphism f is:

$\partial f =$



Given another morphism $g\colon {}_{\mathcal{A}}N_{\mathcal{B}} \to {}_{\mathcal{A}}P_{\mathcal{B}}$ define:

$g \circ f =$ 

We let ${}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ denote the *dg* category of type *AA* modules over $\mathcal{A}$ and $\mathcal{B}$. The cycles in $\mathrm{Mor}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}N_{\mathcal{B}})$ are the *type AA bimodule homomorphisms*.

Example 2.2.40 If $\mathcal{A}$ and $\mathcal{B}$ are *dg* algebras and M is a type *AA* bimodule such that $m_{i,1,j}$ vanishes whenever $i + j > 1$ then M is an ordinary *dg* bimodule.

An *AA module operation tree* is like an $\mathcal{A}_\infty$–module operation tree, except that the algebra edges are now labeled either A or B, and nodes labeled m the incoming module edge need not be the leftmost edge, but edges to the left of the module edge are labeled A and edges to the right are labeled B. Similarly for *AA* morphism trees.

Definition 2.2.41 An $\mathcal{A}_\infty$–bimodule ${}_{\mathcal{A}}M_{\mathcal{B}}$ is *(operationally) bounded* if for each $x \in M$ there is an n so that for $i + j > n$ and any spinal *AA* module operation tree Γ with $i+1+j$ total inputs, i labeled A (to the left of the module edge) and j labeled B (to the right of the module edge), $m_\Gamma(\,\cdot \otimes x \otimes \cdot\,)$ vanishes on $(A_+[1])^{\otimes i} \otimes (B_+[1])^{\otimes j}$. It is *left* (respectively *right*) *(operationally) bounded* if for each $x \in M$ and each i there exists an n so that for all spinal *AA* module operation trees Γ with i right (respectively left) inputs and $j > n$ left (respectively right) inputs, $m_\Gamma(\,\cdot \otimes x \otimes \cdot\,)$ vanishes on $(A_+[1])^{\otimes j} \otimes (B_+[1])^{\otimes i}$ (respectively $(A_+[1])^{\otimes i} \otimes (B_+[1])^{\otimes j}$).

Similarly, a morphism $f\colon {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}N_{\mathcal{B}}$ is called *bounded* if for each $x \in M$ there is an n so that for any spinal *AA* morphism tree Γ with $i + 1 + j > n$ total inputs, $f_\Gamma(\,\cdot \otimes x \otimes \cdot\,)$ vanishes on $(A_+[1])^{\otimes i} \otimes (B_+[1])^{\otimes j}$. It is *left* (respectively *right*) *bounded* if for each $x \in M$ and each i there exists an n so that for any spinal *AA* morphism tree Γ with i right (respectively left) inputs and $j > n$ left (respectively right) inputs, $f_\Gamma(\,\cdot \otimes x \otimes \cdot\,)$ vanishes on $(A_+[1])^{\otimes j} \otimes (B_+[1])^{\otimes i}$ (respectively $(A_+[1])^{\otimes i} \otimes (B_+[1])^{\otimes j}$).

(See also Lemma 2.3.11 for a mild reformulation of these notions and unification with the notion of boundedness for type *DA* and *DD* structures, defined below.)

Example 2.2.42 An $\mathcal{A}_\infty$–algebra $\mathcal{A}$ can be viewed as an $\mathcal{A}_\infty$ bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ over itself, with

$$m_{i,1,j}(a_1,\dots,a_m,x,b_1,\dots,b_n)=\mu_{i+j+1}(a_1,\dots,a_i,x,b_1,\dots,b_j).$$

This bimodule is operationally bounded if $\mathcal{A}$ is nilpotent (as we are assuming throughout), but not usually otherwise.

Definition 2.2.43 Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{A}_\infty$–algebras over $\boldsymbol{k}$ and $\boldsymbol{j}$ respectively. Then a *type DA bimodule ${}^{\mathcal{A}}N_{\mathcal{B}}$ over $\mathcal{A}$ and $\mathcal{B}$* consists of a graded $(\boldsymbol{k},\boldsymbol{j})$–bimodule N and degree-0, $(\boldsymbol{k},\boldsymbol{j})$–linear maps

$$\delta^1_{1+j}\colon N\otimes B[1]^{\otimes j}\to A[1]\otimes N.$$

The compatibility condition is as follows. Let $\delta^1=\sum_j\delta^1_j$. Define maps $\delta^i\colon N\otimes T^*(B[1])\to A[1]^{\otimes i}\otimes N$ inductively by

$$\begin{aligned}\delta^0&=\mathbb{I}_N,\\ \delta^{i+1}&=(\mathbb{I}_{A^{\otimes i}}\otimes\delta^1)\circ(\delta^i\otimes\mathbb{I}_{T^*B})\circ(\mathbb{I}_N\otimes\Delta),\end{aligned}$$

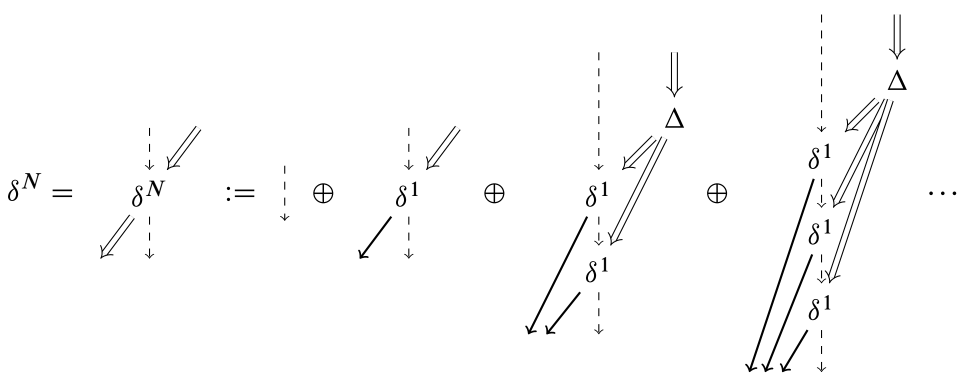
where $\Delta\colon T^*(B)\to T^*(B)\otimes T^*(B)$ is the canonical comultiplication. Let

$$\delta^N\colon N\otimes T^*(B)\to\overline{T}^*(A[1])\otimes N$$

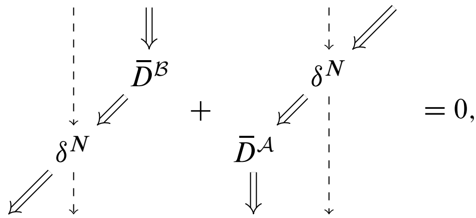
be the map defined by

$$\delta^N=\sum_{i=0}^{\infty}\delta^i.$$

That is, graphically:



Then, the compatibility condition is given graphically by



or symbolically by

$$\delta^N \circ (\mathbb{I}_N \otimes \bar{D}^{\mathcal{B}}) + (\bar{D}^{\mathcal{A}} \otimes \mathbb{I}_N) \circ \delta^N = 0. \tag{2.2.44}$$

(Compare (2.2.25).)

A type *DA* structure ${}^{\mathcal{A}}M_{\mathcal{B}}$ is called *strictly unital* if $\delta_2^1(x, 1) = 1 \otimes x$ for any $x \in M$ and $\delta_{1+i}^1(x, b_1, \dots, b_i) = 0$ if $i > 1$ and some $b_\ell \in \boldsymbol{j}$, so δ_{1+i}^1 is induced by a map from $N \otimes B_+[1]^{\otimes i}$ to $A[1] \otimes N$, which we also denote δ_{1+i}^1. We will assume our type *DA* structures are strictly unital.

A morphism of type *DA* structures $f^1\colon {}^{\mathcal{A}}M_{\mathcal{B}} \to {}^{\mathcal{A}}N_{\mathcal{B}}$ is a collection of maps $f_{1+j}^1\colon M \otimes B_+[1]^{\otimes j} \to A \otimes N$. The set of morphisms is naturally a graded vector space; compare Definition 2.2.20. Moreover, the set of morphisms forms a chain complex: the differential of a morphism f^1 is shown in Figure 2 (left). Define higher composition maps $\circ_k(f_1, \dots, f_k)$ as in Figure 2 (right). These composition maps make the collection of type *DA* structures over $\mathcal{A}$ and $\mathcal{B}$ into an $\mathcal{A}_\infty$ category, which we denote ${}^{\mathcal{A}}_{\mathrm{u}}\mathsf{Mod}_{\mathcal{B}}$. The cycles in $\mathrm{Mor}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{B}})$ are called *type DA structure homomorphisms.*

Let

$$\delta_j^i = \delta^i\,\big|_{N \otimes B_+^{\otimes(j-1)}}, \quad \delta_j^N = \delta^N\,\big|_{N \otimes B_+^{\otimes(j-1)}}$$

be the parts of δ^i and δ^N taking j inputs.

Definition 2.2.45 A *DA module operation graph* Γ consists of:

- A connected, directed graph G.
- An embedding $\iota\colon G \to \bar{\mathbb{D}}^2$ of G in the disk, so that all edges point downwards.
- A labeling of the vertices of G mapped by ι to the interior of the disk (the *interior vertices*) by μ, δ, or ϵ.
- A marking of each edge of G as either a module edge (labeled M) or an algebra edge (labeled A or B).

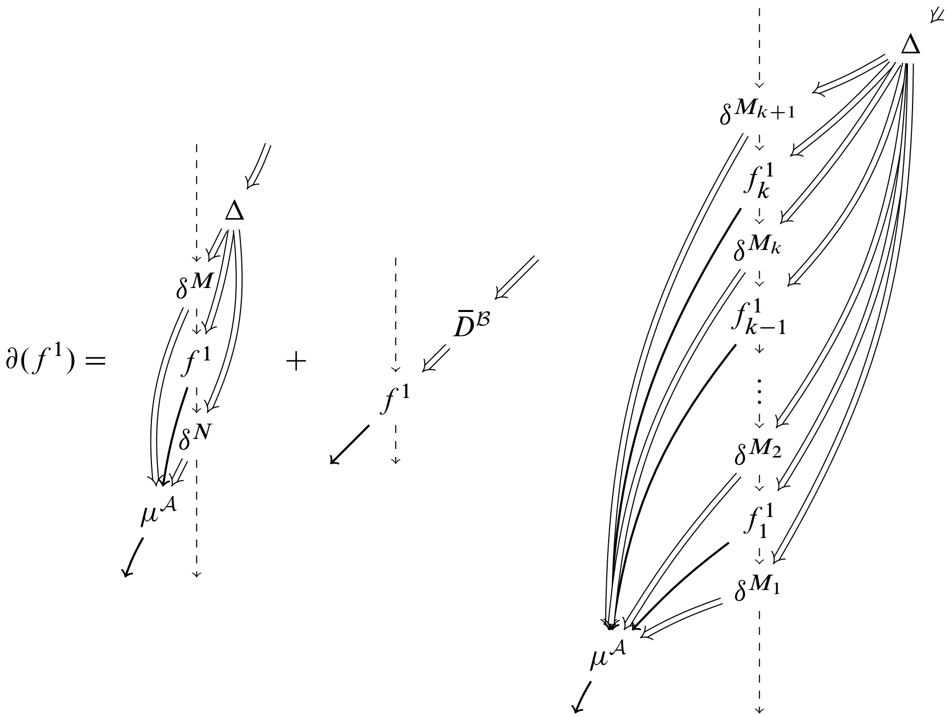


Figure 2: The differential and composition of *DA* morphisms: on the left, we have the differential of a type *DA* morphism, which can also be thought of as the composition map $\circ_1$; on the right, we have the composition $\circ_k$ of morphisms $f_1, \dots, f_k$.

This data is required to satisfy:

- The directed graph G has no oriented cycles.
- The vertices mapped by ι to $\partial\bar{\mathbb{D}}^2$ (the *exterior vertices*) are leaves.
- At each μ vertex, there is at least one incoming and exactly one outgoing edge, all algebra edges with the same label.
- At each ϵ vertex, there is one incoming algebra edge and no outgoing edges.
- At each δ vertex, there is at least one incoming and exactly two outgoing edges, such that the leftmost incoming and right outgoing edge are module edges, and with the other incoming edges marked B and the other outgoing edge marked A.
- There is a module edge.

Call an exterior vertex an *in* (respectively *out*) vertex if it is a source (respectively sink). Note that all in (respectively out) vertices are consecutive with respect to the cyclic order on $\partial\overline{\mathbb{D}}^2$. It also follows from the conditions that the edge from the leftmost in (respectively rightmost out) vertex is a module edge, and the edges from all other in (respectively out) vertices are algebra edges.

Associated to a *DA* module operation graph with $(i+1)$ in and $(j+1)$ out vertices is a map

$$\delta_\Gamma\colon M \otimes B_+^{\otimes i} \to A^{\otimes j} \otimes M,$$

defined in the obvious way.

A *DA* module operation graph is *spinal* if it has no μ nodes.

Definition 2.2.46 A type *DA* structure ${}^{\mathcal{A}}M_{\mathcal{B}}$ is called *(operationally) bounded* if for each $x \in M$ there is an n so that for all $i+j>n$ and spinal *DA* module operation graphs Γ with i (right) algebra inputs and j (left) algebra outputs, $\delta_\Gamma(x \otimes \cdot\,)$ vanishes on $(B_+[1])^{\otimes i}$. It is *right (operationally) bounded* if for each x and each j there is an n so that for spinal *DA* module operation graphs Γ with $i>n$ algebra inputs and j algebra outputs, $\delta_\Gamma(x \otimes \cdot\,))$ vanishes on $(B_+[1])^{\otimes i}$. It is *left (operationally) bounded* if for each $x \in {}^{\mathcal{A}}M_{\mathcal{B}}$ and each i there is an n so that for all *DA* module operation graphs Γ with i algebra inputs and $j>n$ algebra outputs, $\delta_\Gamma(x \otimes \cdot\,)$ vanishes on $(B_+[1])^{\otimes i}$. Boundedness for morphisms is defined similarly.

We denote the category of bounded type *DA* structures and bounded type *DA* morphisms by ${}^{\mathcal{A}}_{\mathfrak{B}}\mathsf{Mod}_{\mathcal{B}}$.

Remark 2.2.47 One reason that ϵ appears in the boundedness conditions for type *DA* structures is that we want the forgetful functor from type *DA* structures to type *A* modules (see Section 2.3.1) to take right bounded structures to bounded modules.

Definition 2.2.48 Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{A}_\infty$–algebras over $\boldsymbol{k}$. Given an $\mathcal{A}_\infty$–morphism $\phi\colon \mathcal{B} \to \mathcal{A}$, defined by maps $\phi_k\colon B^{\otimes k} \to A$, define a bimodule ${}^{\mathcal{A}}[\phi]_{\mathcal{B}}$ with underlying space a free rank-1 module over $\boldsymbol{k}$ and structure maps given by the ϕ_k. That is, let $\iota \in [\phi]$ be the generator and define

$$\delta^1_{1+k}(\iota, b_1, \dots, b_k) = \phi_k(b_1, \dots, b_k) \otimes \iota.$$

As a special case, given an $\mathcal{A}_\infty$–algebra $\mathcal{A}$ we have the module ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$. As a $\boldsymbol{k}$–module, ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ is isomorphic to $\boldsymbol{k}$. For $k \neq 2$, $\delta^1_k = 0$, while

$$\delta^1_2(\iota, a) = a \otimes \iota,$$

where ι is the generator of ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$. We call ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ the *identity bimodule*.

Remark 2.2.49 The identity bimodule ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ is typically not (operationally) bounded: in fact, it is bounded if and only if T^*A_+ is finite-dimensional. The module ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ is always left and right (operationally) bounded, however.

In fact, the modules from Definition 2.2.48 have the following easy characterization:

Lemma 2.2.50 *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{A}_\infty$–algebras over $\boldsymbol{k}$. Let ${}^{\mathcal{A}}M_{\mathcal{B}}$ be a type DA bimodule whose underlying $\boldsymbol{k}$–bimodule is $\boldsymbol{k}$. Assume furthermore that $\delta_1^1 = 0$. Then there is an $\mathcal{A}_\infty$–algebra morphism $\phi\colon \mathcal{B} \to \mathcal{A}$ with the property that ${}^{\mathcal{A}}M_{\mathcal{B}} \cong {}^{\mathcal{A}}[\phi]_{\mathcal{B}}$.*

Proof Given ${}^{\mathcal{A}}M_{\mathcal{B}}$ with generator ι as a $\boldsymbol{k}$–module, the homomorphism ϕ is uniquely characterized by

$$\delta^1_{n+1}(\iota, b_1, \dots, b_n) = \phi_n(b_1, \dots, b_n) \otimes \iota.$$

The hypothesis that $\delta_1^1 = 0$ ensures that $\phi_0 = 0$, so we drop it. We must verify that $\phi = \{\phi_i\}_{i=1}^\infty$ satisfies the $\mathcal{A}_\infty$–relation (2.1.13). Define $\tau\colon \boldsymbol{k} \otimes T^*(\mathcal{B}) \simeq T^*(\mathcal{B}) \otimes \boldsymbol{k}$ to be the canonical identification (as both are isomorphic to $T^*(\mathcal{B})$). Then we have $\delta = (F^\phi \otimes \mathbb{I}_{\boldsymbol{k}}) \circ \tau$. So, by the $\mathcal{A}_\infty$–relation for δ,

$$\begin{aligned} 0 &= \delta \circ (\mathbb{I}_{\boldsymbol{k}} \otimes \bar{D}^{\mathcal{B}}) + (\bar{D}^{\mathcal{A}} \otimes \mathbb{I}_{\boldsymbol{k}}) \circ \delta \\ &= (F^\phi \otimes \mathbb{I}_{\boldsymbol{k}}) \circ \tau \circ (\mathbb{I}_{\boldsymbol{k}} \otimes \bar{D}^{\mathcal{B}}) + (\bar{D}^{\mathcal{A}} \otimes \mathbb{I}_{\boldsymbol{k}}) \circ (F^\phi \otimes \mathbb{I}_{\boldsymbol{k}}) \circ \tau \\ &= ((F^\phi \circ \bar{D}^{\mathcal{B}}) \otimes \mathbb{I}_{\boldsymbol{k}}) \circ \tau + ((\bar{D}^{\mathcal{A}} \circ F^\phi) \otimes \mathbb{I}_{\boldsymbol{k}}) \circ \tau, \end{aligned}$$

hence $F^\phi \circ \bar{D}^{\mathcal{B}} + \bar{D}^{\mathcal{A}} \circ F^\phi = 0$, as desired. □

Remark 2.2.51 The hypothesis that $\delta_1^1 = 0$ can be dropped, if we allow for more general types of $\mathcal{A}_\infty$–algebra morphisms, ie those which contain a term ϕ_0. The hypotheses of Lemma 2.2.50 are satisfied in the case we use it (Theorem 4).

Definition 2.2.52 Let ${}^{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ denote the full subcategory of ${}^{\mathcal{A}}_{\mathrm{u}}\mathsf{Mod}_{\mathcal{B}}$ consisting of type *DA* bimodules which are homotopy equivalent to bounded type *DA* bimodules.

This is equivalent to the category of type *DA* bimodules which are homotopy equivalent to *left* bounded type *DA* bimodules; see Proposition 2.3.24 below.

One can define *type AD modules* similarly, by reflecting all of the pictures. For instance, given $\phi\colon \mathcal{B} \to \mathcal{A}$, one can define a module ${}_{\mathcal{B}}[\phi]^{\mathcal{A}}$ analogous to the one from Definition 2.2.48.

Like for the D structures, type *DA* modules have opposite type *AD* modules. We will explain this operation only under some finiteness assumptions:

Definition 2.2.53 Suppose that $({}^{\mathcal{A}}M_{\mathcal{B}}, \delta^1)$ is a type *DA* structure and $\mathcal{A}$, $\mathcal{B}$ and M are finite-dimensional. Define the *opposite* type *AD* structure $({}_{\mathcal{B}}\overline{M}^{\mathcal{A}}, \overline{\delta}^1)$ as follows. As a $(\boldsymbol{k}, \boldsymbol{j})$–bimodule, $\overline{M}$ is just $M^* = \mathrm{Hom}_{\boldsymbol{k}\otimes\boldsymbol{j}}(M, \boldsymbol{k}\otimes\boldsymbol{j}) \cong \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$. The map δ^1_k on M is an element of $\mathrm{Hom}_{\boldsymbol{k}}(M \otimes B_+^{\otimes k}, A \otimes M) \simeq \mathrm{Hom}_{\boldsymbol{k}}(B_+^{\otimes k} \otimes M^*, M^* \otimes A)$, and $\overline{\delta}^1$ is obtained by viewing δ^1 as lying in the right-hand side.

Lemma 2.2.54 *Given a type DA structure ${}^{\mathcal{A}}M_{\mathcal{B}}$, the opposite type AD structure ${}_{\mathcal{B}}\overline{M}^{\mathcal{A}}$ satisfies the type AD structure equation.*

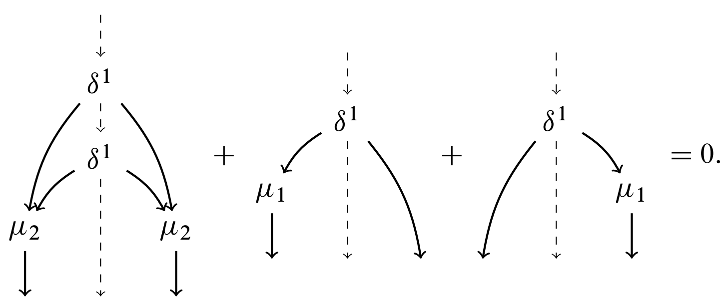
Proof We leave the verification, which is similar to the proof of Lemma 2.2.32, to the reader. □

Next, we turn to our final notion of bimodule, a type *DD* structure. We will only define these when $\mathcal{A}$ and $\mathcal{B}$ are *dg* algebras.

Definition 2.2.55 Let $\mathcal{A}$ and $\mathcal{B}$ be *dg* algebras over $\boldsymbol{k}$ and $\boldsymbol{j}$. We define the *category of type DD structures over $\mathcal{A}$ and $\mathcal{B}$*, ${}^{\mathcal{A}}_{\mathrm{u}}\mathsf{Mod}^{\mathcal{B}}_{\mathrm{u}}$, to be the category of type D structures over $\mathcal{A} \otimes_{\mathbb{F}_2} \mathcal{B}^{\mathrm{op}}$, that is, ${}^{\mathcal{A}\otimes_{\mathbb{F}_2}\mathcal{B}^{\mathrm{op}}}_{\mathrm{u}}\mathsf{Mod}$.

We denote a type *DD* structure M by ${}^{\mathcal{A}}M^{\mathcal{B}}$. The cycles in $\mathrm{Mor}({}^{\mathcal{A}}M^{\mathcal{B}}, {}^{\mathcal{A}}N^{\mathcal{B}})$ are the *type DD structure homomorphisms.*

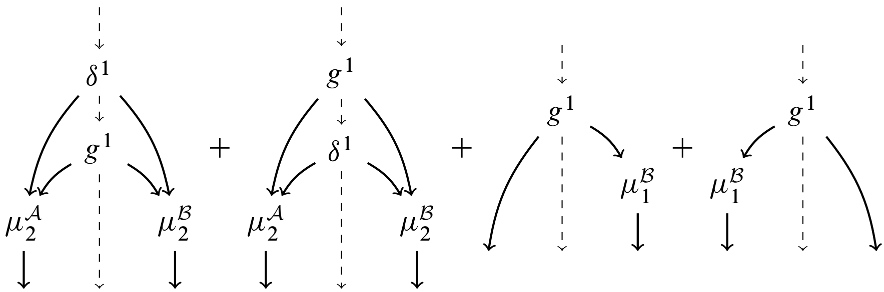
We think of the data of a type *DD* structure as a graded $(\boldsymbol{k}, \boldsymbol{j})$–bimodule M and a degree-0 map $\delta^1\colon M \to A \otimes M \otimes B[1]$, such that the following compatibility condition holds:



A morphism $g^1\colon {}^{\mathcal{A}}M^{\mathcal{B}} \to {}^{\mathcal{A}}N^{\mathcal{B}}$ of type *DD* structures is a map $g^1\colon M \to A \otimes N \otimes B$. The set of morphisms is naturally a graded vector space, as in Definition 2.2.20. Moreover, the set of morphisms forms a chain complex: the differential of a morphism

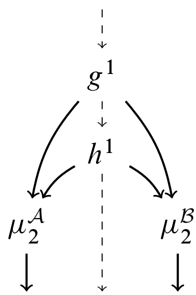
g^1 is:

$\partial(g^1) =$



Composition in ${}_{\mathrm{u}}^{\mathcal{A}}\mathsf{Mod}_{\mathrm{u}}^{\mathcal{B}}$ is given by: for $g^1\colon {}^{\mathcal{A}}M^{\boldsymbol{B}} \to {}^{\mathcal{A}}N^{\boldsymbol{B}}$ and $h^1\colon {}^{\mathcal{A}}N^{\boldsymbol{B}} \to {}^{\mathcal{A}}P^{\boldsymbol{B}}$,

$h^1 \circ g^1 =$



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Given a type *DD* structure (M, δ^1) define maps δ^i inductively by

$$\delta^i = (\mathbb{I}_{\mathcal{A}^{\otimes(i-1)}} \otimes \delta^1 \otimes \mathbb{I}_{\mathcal{B}^{\otimes(i-1)}}) \circ \delta^{i-1}\colon M \to \mathcal{A}^{\otimes i} \otimes M \otimes \mathcal{B}^{\otimes i}$$

and set $\delta = \bigoplus_{i=0}^{\infty} \delta^i$.

Similarly, given a morphism $g^1\colon {}^{\mathcal{A}}M^{\mathcal{B}} \to {}^{\mathcal{A}}N^{\mathcal{B}}$, define $g^i\colon M \to \mathcal{A}^{\otimes i} \otimes N \otimes B^{\otimes i}$ by

$$g^i = \sum_{j=1}^{i} (\mathbb{I}_{\mathcal{A}^{\otimes j}} \otimes \delta^{N,i-j} \otimes \mathbb{I}_{\boldsymbol{B}^{\otimes j}}) \circ (\mathbb{I}_{\mathcal{A}^{\otimes(j-1)}} \otimes g^1 \otimes \mathbb{I}_{\boldsymbol{B}^{\otimes(j-1)}}) \circ \delta^{M,j-1}$$

and let $g = \sum_{i=1}^{\infty} g^i$.

Note that the augmentation $\epsilon\colon A \to \boldsymbol{k}$ of $\mathcal{A}$ extends to a map $\epsilon\colon T^*A \to \boldsymbol{k}$ by $\epsilon(a_1 \otimes \cdots \otimes a_k) = \epsilon(a_1)\cdots\epsilon(a_k)$.

Definition 2.2.56 We call a type *DD* structure ${}^{\mathcal{A}}M^{\mathcal{B}}$ *left* (respectively *right*) *(operationally) bounded* if for each $x \in {}^{\mathcal{A}}M^{\mathcal{B}}$, there is a constant n with the property that for all $i > n$, $(\mathbb{I}_{\boldsymbol{A}^{\otimes i}} \otimes \mathbb{I}_M \otimes \epsilon_{\mathcal{B}}) \circ \delta^i = 0$ (respectively $(\epsilon_{\mathcal{A}} \otimes \mathbb{I}_M \otimes \mathbb{I}_{\boldsymbol{B}^{\otimes i}}) \circ \delta^i = 0$). We call ${}^{\mathcal{A}}M^{\mathcal{B}}$ *(operationally) bounded* if for each x, n can be chosen so that $\delta^i(x) = 0$ for $i > n$. Boundedness for morphisms of type *DD* structures is defined similarly.

As usual, the condition of being operationally bounded is stronger than the condition of being both left and right bounded.

Definition 2.2.57 Call a type *DD* structure *separated* if the map δ^1 can be written as $\delta^{1L} + \delta^{1R}$, where $\delta^{1L}\colon M \to \mathcal{A} \otimes M \otimes \boldsymbol{j}$ and $\delta^{1R}\colon M \to \boldsymbol{k} \otimes M \otimes \mathcal{B}$.

Remark 2.2.58 In general, over $\mathcal{A}_\infty$–algebras, a type *DD* bimodule ${}^{\mathcal{A}}M^{\mathcal{B}}$ should be a type D module over $\mathcal{A} \otimes_{\mathbb{F}_2} \mathcal{B}^{\mathrm{op}}$. The difficulty is in defining the tensor product of $\mathcal{A}_\infty$–algebras. This has been done (see Saneblidze and Umble [33], Markl and Shnider [26] and Loday [24]) but is somewhat complicated and is unnecessary for this paper.

Definition 2.2.59 Given *dg* algebras $\mathcal{A}$ and $\mathcal{B}$, let ${}^{\mathcal{A}}_{\mathfrak{B}}\mathsf{Mod}^{\mathcal{B}}$ denote the category whose objects consist of bounded type *DD* bimodules. Similarly, we define ${}^{\mathcal{A}}\mathsf{Mod}^{\mathcal{B}}$ to be the full subcategory of ${}^{\mathcal{A}}_{\mathfrak{u}}\mathsf{Mod}^{\mathcal{B}}_{\mathfrak{u}}$ consisting of type *DD* bimodules which are homotopy equivalent to bounded ones.

See also Proposition 2.3.24.

So far, we have discussed bimodules with a single left and a single right action. One can also consider bimodules with two left actions or two right actions, and, indeed, it is most natural to define the invariant $\widehat{CFAA}$ (respectively $\widehat{CFDD}$) of Section 6 as a bimodule with two right (respectively left) actions. Obviously, there are no new mathematical difficulties in this theory. Moreover, the notation extends easily; for example, $M_{\mathcal{A},\mathcal{B}}$ denotes a type *AA* bimodule with two right actions and $\mathsf{Mod}_{\mathcal{A},\mathcal{B}}$ denotes the *dg* category of such bimodules.

2.3 Operations on bimodules

2.3.1 Forgetful functors In defining the tensor product and one sided Mor operations, it will be convenient to invoke forgetful functors between certain of our categories. In particular, there are forgetful functors

$$\mathcal{F}\colon {}^{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}} \to {}^{\mathcal{A}}\mathsf{Mod},$$
$$\mathcal{F}\colon {}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}} \to {}_{\mathcal{A}}\mathsf{Mod},$$

gotten by $\mathcal{F}({}^{\mathcal{A}}M_{\mathcal{B}}, \{\delta^1_i\}) = ({}^{\mathcal{A}}M, \overline{\delta}^1)$ where ${}^{\mathcal{A}}M$ is isomorphic to M as a $(\boldsymbol{k}, \boldsymbol{j})$–bimodule and $\overline{\delta}^1 = \delta^1_1$; and similarly $\mathcal{F}({}_{\mathcal{A}}M_{\mathcal{B}}, \{m_{i,1,j}\}) = ({}_{\mathcal{A}}M, \overline{m}_i)$ where $\overline{m}_{i+1} = m_{i,1,0}$. (The forgetful functor is defined similarly on morphisms.)

Similarly, there are forgetful functors

$$\mathcal{F}\colon {}^{\mathcal{A}}\mathsf{Mod}^{\mathcal{B}} \to {}^{\mathcal{A}}\mathsf{Mod},$$
$$\mathcal{F}\colon {}_{\mathcal{A}}\mathsf{Mod}^{\mathcal{B}} \to {}_{\mathcal{A}}\mathsf{Mod},$$

gotten by $\mathcal{F}({}^{\mathcal{A}}M^{\mathcal{B}}, \{\delta^1\}) = ({}^{\mathcal{A}}M, \overline{\delta}^1)$ where ${}^{\mathcal{A}}M$ is isomorphic to M as a $(\boldsymbol{k}, \boldsymbol{j})$–bimodule and $\overline{\delta}^1 = (\mathbb{I}_{\mathcal{A}} \otimes M \otimes \epsilon_{\mathcal{B}}) \circ \delta^1$; and similarly $\mathcal{F}({}_{\mathcal{A}}M^{\mathcal{B}}, \{\delta^1_i\}) = ({}_{\mathcal{A}}M, \overline{m}_i)$ where $\overline{m}_{i+1} = (M \otimes \epsilon_{\mathcal{B}}) \circ \delta^1_i$. As before, $\epsilon_{\mathcal{B}}\colon \mathcal{B} \to \boldsymbol{j}$ denotes the augmentation.

These forgetful functors interact well with our definition of boundedness:

Lemma 2.3.1 *Let M be a bimodule of any type. If M is left bounded then $\mathcal{F}(M)$ is bounded.*

We leave the proof to the reader; it is not hard, but involves several cases. Note that this lemma is actually implicit in writing, say, that $\mathcal{F}\colon {}^{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}} \to {}^{\mathcal{A}}\mathsf{Mod}$, since ${}^{\mathcal{A}}\mathsf{Mod}$ consists of type D structures homotopy equivalent to bounded ones.

There are, of course, also functors which forget the left action; we will denote these by $\mathcal{F}$ as well.

2.3.2 Tensor products In the present section, we define a pairing between type A modules and type D modules (and their generalizations to bimodules), which gives a model for the derived tensor product of $\mathcal{A}_\infty$–modules. This model for the derived product comes up naturally when one studies the gluing problems for pseudoholomorphic curves (see Section 7). Indeed, this model typically has smaller rank than the usual derived tensor product (though its differentials are correspondingly more complicated).

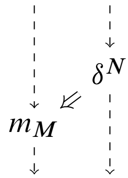
We start by considering modules with a single action, and then proceed to bimodules.

Definition 2.3.2 For $\mathcal{A}$ an $\mathcal{A}_\infty$–algebra, $M_{\mathcal{A}} \in \mathsf{Mod}_{\mathcal{A}}$, and ${}^{\mathcal{A}}N \in {}^{\mathcal{A}}\mathsf{Mod}$, with at least one of $M_{\mathcal{A}}$ or ${}^{\mathcal{A}}N$ bounded, define $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ to be chain complex with underlying space $M \otimes_{\boldsymbol{k}} N$ and boundary operator

$$\partial := (m_M \otimes \mathbb{I}_N) \circ (\mathbb{I}_M \otimes \delta^N).$$

The boundedness hypothesis implies that the tensor product is well-defined as follows. If $M_{\mathcal{A}}$ is bounded then the operations m_i vanish for sufficiently large i, so they sum to give a map $M \otimes \overline{T}^*(A) \to M$. If ${}^{\mathcal{A}}N$ is bounded then the image of δ^N lies in $T^*A \otimes M$, to which we can apply $\sum m_i$.

Graphically, the differential on $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ is given by:



The fact that $\partial^2 = 0$ is verified in [21, Lemma 2.30].

We will see presently that $\boxtimes$ induces a bifunctor on the level of derived categories. Functoriality for $\boxtimes$ on the *dg* level is somewhat subtle, however. Given a $f\colon M_{\mathcal{A}} \to M'_{\mathcal{A}}$ and $g^1\colon {}^{\mathcal{A}}N \to {}^{\mathcal{A}}N'$, there are two natural diagrams that one might use to define $f \boxtimes g^1$, shown in the left of Figure 3.

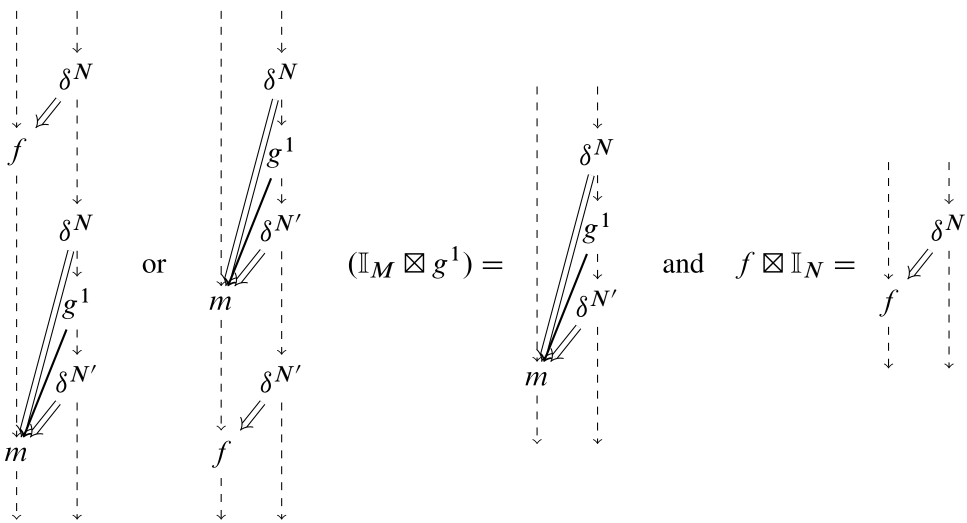


Figure 3: Diagrams for defining the box product of morphisms: on the left, we have two options for defining $f \boxtimes g$; on the right, we have box products with the identity morphisms.

In other words, if we define $(\mathbb{I} \boxtimes g^1)$ and $(f \boxtimes \mathbb{I})$ as in the right of Figure 3 then the two different choices correspond to $(\mathbb{I}_{M'} \boxtimes g^1) \circ (f \boxtimes \mathbb{I}_N)$ and $(f \boxtimes \mathbb{I}_{N'}) \circ (\mathbb{I}_M \boxtimes g^1)$ respectively. (Of course, for the diagram defining $\mathbb{I}_M \boxtimes g^1$ to make sense we either need $M_{\mathcal{A}}$ or both ${}^{\mathcal{A}}N$ and ${}^{\mathcal{A}}N'$ to be operationally bounded, and for the diagram defining $f \boxtimes \mathbb{I}_N$ to be defined we need either the map f or the module ${}^{\mathcal{A}}N$ to be operationally bounded.)

The two choices above are homotopic. Indeed:

Lemma 2.3.3 *Fix $M_{\mathcal{A}}, N_{\mathcal{A}}, L_{\mathcal{A}} \in \mathsf{Mod}_{\mathcal{A}}$ and ${}^{\mathcal{A}}P, {}^{\mathcal{A}}Q, {}^{\mathcal{A}}R \in {}^{\mathcal{A}}\mathsf{Mod}$ with ${}^{\mathcal{A}}P, {}^{\mathcal{A}}Q$ and ${}^{\mathcal{A}}R$ bounded or $M_{\mathcal{A}}, N_{\mathcal{A}}, L_{\mathcal{A}}$ and the morphisms between them bounded. Then the maps*

$$(\,\cdot \boxtimes \mathbb{I}_P)\colon \mathrm{Mor}_{\mathcal{A}}(M_{\mathcal{A}}, N_{\mathcal{A}}) \to \mathrm{Mor}_{\boldsymbol{k}}(M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}P, N_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}P),$$
$$(\mathbb{I}_M \boxtimes \,\cdot\,))\colon \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}P, {}^{\mathcal{A}}Q) \to \mathrm{Mor}_{\boldsymbol{k}}(M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}P, M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}Q),$$

defined above are chain maps. Further:

(1) *The maps $(\mathbb{I}_M \boxtimes \,\cdot\,))$ and $(\,\cdot \boxtimes \mathbb{I}_P)$ are functorial under composition in sense that the square*

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{A}}(M,N) \otimes \mathrm{Mor}_{\mathcal{A}}(N,L) \arrow[r, "f\otimes g\mapsto(f\boxtimes\mathbb{I}_P)\otimes(g\boxtimes\mathbb{I}_P)"] \arrow[d, "(f\otimes g)\mapsto g\circ f"'] & \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes P, N\boxtimes P) \otimes \mathrm{Mor}_{\boldsymbol{k}}(N\boxtimes P, L\boxtimes P) \arrow[d, "(k\otimes l)\mapsto l\circ k"] \\
\mathrm{Mor}_{\mathcal{A}}(M,L) \arrow[r, "h\mapsto(h\boxtimes\mathbb{I}_P)"'] & \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes P, L\boxtimes P)
\end{tikzcd}$$

commutes while the following square commutes up to homotopy:

$$\begin{tikzcd}
\mathrm{Mor}^{\mathcal{A}}(P,Q) \otimes \mathrm{Mor}^{\mathcal{A}}(Q,R) \arrow[r, "f^1\otimes g^1\mapsto(\mathbb{I}_M\boxtimes f^1)\otimes(\mathbb{I}_M\boxtimes g^1)"] \arrow[d, "(f^1\otimes g^1)\mapsto g^1\circ f^1"'] & \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes P, M\boxtimes Q) \otimes \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes Q, M\boxtimes R) \arrow[d, "(k\otimes l)\mapsto l\circ k"] \\
\mathrm{Mor}^{\mathcal{A}}(P,R) \arrow[r, "h^1\mapsto(\mathbb{I}_M\boxtimes h^1)"'] & \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes P, M\boxtimes R)
\end{tikzcd}$$

(2) *The maps $(\cdot \boxtimes \mathbb{I})$ and $(\mathbb{I} \boxtimes \,\cdot\,))$ commute in the sense that the square*

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{A}}(M,N) \otimes \mathrm{Mor}^{\mathcal{A}}(P,Q) \arrow[r, "f\otimes g^1\mapsto(\mathbb{I}_M\boxtimes g^1)\otimes(f\boxtimes\mathbb{I}_Q)"] \arrow[d, "(f\otimes g^1)\mapsto(f\boxtimes\mathbb{I}_P)\otimes(\mathbb{I}_N\boxtimes g^1)"'] & \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes P, M\boxtimes Q) \otimes \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes Q, N\boxtimes Q) \arrow[d, "(k\otimes l)\mapsto l\circ k"] \\
\mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes P, N\boxtimes P) \otimes \mathrm{Mor}_{\boldsymbol{k}}(N\boxtimes P, N\boxtimes Q) \arrow[r, "(k\otimes l)\mapsto l\circ k"'] & \mathrm{Mor}_{\boldsymbol{k}}(M\boxtimes P, N\boxtimes Q)
\end{tikzcd}$$

commutes up to homotopy.

Proof We leave as exercises that the $(\cdot \boxtimes \mathbb{I}_P)$ and $(\mathbb{I}_M \boxtimes \cdot))$ are chain maps. The fact that the first square commutes is straightforward. The homotopy H for the second square

$$H\colon \mathrm{Mor}(P,Q) \otimes \mathrm{Mor}(Q,R) \to \mathrm{Mor}(M \boxtimes P, M \boxtimes R)$$

is defined by

$$H(f^1 \otimes g^1) = (\mu \otimes \mathbb{I}_R) \circ (\mathbb{I}_M \otimes ((\mathbb{I}_{\overline{T}^*(\mathcal{A}[1])} \otimes \delta^R) \circ (\mathbb{I}_{\overline{T}^*(\mathcal{A}[1])} \otimes g^1) \circ (\mathbb{I}_{\overline{T}^*(\mathcal{A}[1])} \otimes \delta^Q) \circ (\mathbb{I}_{\overline{T}^*(\mathcal{A}[1])} \otimes f^1) \circ \delta^P)). \tag{2.3.4}$$

Note the similarity of the right-hand side to the definition of the composition $f^1 \circ g^1$ in (2.2.26).

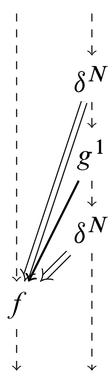
The homotopy K for the third square

$$K\colon \mathrm{Mor}(M,N) \otimes \mathrm{Mor}(P,Q) \to \mathrm{Mor}(M \boxtimes P, N \boxtimes Q)$$

is furnished by

$$K(f, g^1)(x \otimes p) = (f \otimes \mathbb{I}_Q)(x \otimes g(p)),$$

or pictorially:



This completes the proof. □

Let us choose, arbitrarily, to define $f \boxtimes g^1 = (f \boxtimes \mathbb{I}) \circ (\mathbb{I} \boxtimes g^1)$. Then:

Corollary 2.3.5 (1) *The operation $\boxtimes$ induces a chain map*

$$\mathrm{Mor}_{\mathcal{A}}(M,N) \otimes \mathrm{Mor}^{\mathcal{A}}(P,Q) \to \mathrm{Mor}(M \boxtimes P, N \boxtimes Q).$$

(2) *The operation $\boxtimes$ is functorial up to homotopy. That is, $(f \boxtimes g^1) \circ (f' \boxtimes (g')^1)$ is homotopic to $(f \circ f') \boxtimes (g^1 \circ (g')^1)$.*

(3) *If* $f \in \mathrm{Mor}(M_{\mathcal{A}}, N_{\mathcal{A}})$ *and* $g^1 \in \mathrm{Mor}({}^{\mathcal{A}}M, {}^{\mathcal{A}}N)$ *are cycles then* $f \boxtimes g^1$ *is a cycle.*

(4) *If* $f \in \mathrm{Mor}(M_{\mathcal{A}}, N_{\mathcal{A}})$ *and* $g^1 \in \mathrm{Mor}({}^{\mathcal{A}}M, {}^{\mathcal{A}}N)$ *are cycles and either* f *or* g^1 *is nullhomotopic then* $f \boxtimes g$ *is nullhomotopic.*

(5) *The operation* $\boxtimes$ *descends to bifunctors*

$$\boxtimes\colon \mathsf{H}(\mathsf{Mod}^{\mathfrak{b}}_{\mathcal{A}}) \times \mathsf{H}({}^{\mathcal{A}}\mathsf{Mod}) \to \mathsf{H}(\mathsf{Mod}_{\boldsymbol{k}}),$$
$$\boxtimes\colon \mathsf{H}(\mathsf{Mod}_{\mathcal{A}}) \times \mathsf{H}({}^{\mathcal{A}}\mathsf{Mod}^{\mathfrak{b}}) \to \mathsf{H}(\mathsf{Mod}_{\boldsymbol{k}}).$$

Proof The fact that $\boxtimes$ induces a chain map on morphism spaces follows from Lemma 2.3.3: it is defined as a composite of two chain maps. To verify part (2), note that

$$\begin{aligned}(f \boxtimes g^1) \circ (f' \boxtimes (g')^1) &= [(f \boxtimes \mathbb{I}) \circ (\mathbb{I} \boxtimes g^1)] \circ [(f' \boxtimes \mathbb{I}) \circ (\mathbb{I} \boxtimes (g'^1))] \\ &\sim [(f \boxtimes \mathbb{I}) \circ (f' \boxtimes \mathbb{I})] \circ [\mathbb{I} \boxtimes g^1 \circ \mathbb{I} \boxtimes (g')^1] \\ &\sim (f \circ f' \boxtimes \mathbb{I}) \circ (\mathbb{I} \boxtimes g^1 \circ (g')^1) \\ &= (f \circ f') \boxtimes (g^1 \circ (g')^1),\end{aligned}$$

where the first homotopy uses part (2) of Lemma 2.3.3 while the second homotopy uses part (1) of Lemma 2.3.3.

Parts (3) and (4) are easy to verify. Part (5) then follows formally. □

Corollary 2.3.6 *If* $f^1 \in \mathrm{Mor}^{\mathcal{A}}(P, Q)$ *is a chain homotopy equivalence of type* D *structures, then* $\mathbb{I}_M \boxtimes f^1 \in \mathrm{Mor}(M \boxtimes P, M \boxtimes Q)$ *is a chain homotopy equivalence of complexes. Similarly, if* $\phi \in \mathrm{Mor}_{\mathcal{A}}(M, N)$ *is a chain homotopy equivalence of* $\mathcal{A}$*–modules, then* $\phi \boxtimes \mathbb{I}_P\colon \mathrm{Mor}(M \boxtimes P, N \boxtimes P)$ *is a chain homotopy equivalence of complexes.*

While commuting $f \boxtimes \mathbb{I}$ and $\mathbb{I} \boxtimes g$ was somewhat subtle, ($\mathcal{A}_\infty$) functoriality of $\boxtimes$ in each factor is more straightforward:

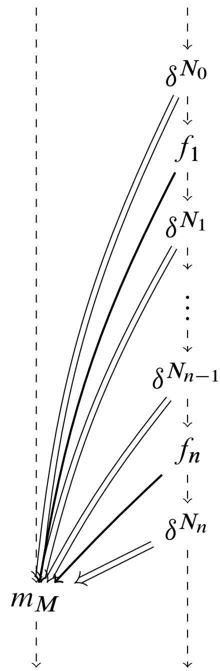
Lemma 2.3.7 *Let* $M_{\mathcal{A}}$ *be an* $\mathcal{A}_\infty$*–module and* ${}^{\mathcal{A}}N$ *a type* D *structure. Then*

(1) *the operation* $\cdot_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ *gives a dg functor* $\mathsf{Mod}_{\mathcal{A}} \to \mathsf{Mod}_{\boldsymbol{k}}$ *and*

(2) *the operation* $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\cdot$ *extends to an* $\mathcal{A}_\infty$*–functor* ${}^{\mathcal{A}}\mathsf{Mod} \to \mathsf{Mod}_{\boldsymbol{k}}$.

Proof The first part is straightforward. For the second, if ${}^{\mathcal{A}}N_i$, $i = 0, \dots, n$ are type D structures and $f_i\colon {}^{\mathcal{A}}N_i \to {}^{\mathcal{A}}N_{i+1}$ are morphisms, define:

$$(M_{\mathcal{A}} \boxtimes \cdot))_{1,n}(\mathbb{I}_M, f_1, \dots, f_n) =$$



It is straightforward to verify that this makes $M_{\mathcal{A}} \boxtimes \cdot$ into an $\mathcal{A}_\infty$–functor. ☐

Remark 2.3.8 Even when $\mathcal{A}$ is a *dg* algebra, so ${}^{\mathcal{A}}\mathsf{Mod}$ is an honest *dg* category, the operation $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\cdot$ is still only an $\mathcal{A}_\infty$–functor.

Next we turn to the behavior of $\boxtimes$ for bimodules. Since we have various kinds of bimodules, there are various cases of the tensor product:

- $DA \boxtimes DD$ is a type DD module.
- $AA \boxtimes DD$ is a type AD module.
- $DA \boxtimes DA$ is a type DA module.
- $AA \boxtimes DA$ is a type AA module.

(In each case, we assume one of the factors in the tensor product is appropriately bounded; see Proposition 2.3.10.)

Definition 2.3.9 Let ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{B}}N_{\mathcal{C}}$ (respectively ${}^{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{B}}N_{\mathcal{C}}$, ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{B}}N^{\mathcal{C}}$, ${}^{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{B}}N^{\mathcal{C}}$) by a type *AA* and *DA* (respectively *DA* and *DA*, *AA* and *DD*, *DA* and *DD*)[2] bimodules with either M right bounded or N left bounded. As a chain complex, define

$$ {}_{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}N_{\mathcal{C}} = \mathcal{F}({}_{\mathcal{A}}M_{\mathcal{B}})_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathcal{F}({}^{\mathcal{B}}N_{\mathcal{C}}) $$

with *AA* (respectively *DA*, *AD*, *DD*) structure map given as in Figure 4 far left (respectively center left, center right, far right).

(In the figure, we use the following notation: given elements $b_1, \dots, b_k$ in a *dg* algebra $\mathcal{B}$, $\Pi(b_1, \dots, b_k) = b_1 \cdots b_k$ denotes their product. Note that *DD* bimodules by assumption involve *dg* algebras.)

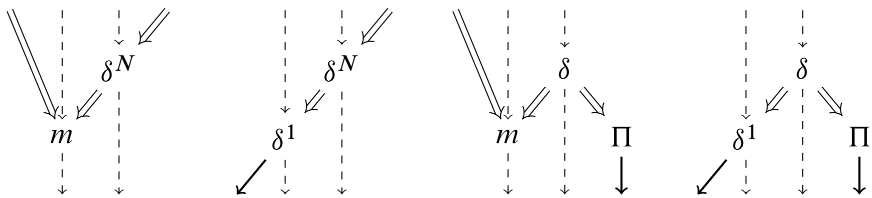


Figure 4: Bimodule structure on $\boxtimes$–tensor products of bimodules: on the far left, we have $AA \boxtimes DA$; center left, $DA \boxtimes DA$; center right, $AA \boxtimes DD$; far right, $DA \boxtimes DD$. Here, for elements $a_1, \dots, a_n \in A$ of a *dg* algebra we let $\Pi(a_1 \otimes \cdots \otimes a_n)$ denote the product of the elements, $a_1 \cdots a_n$.

Since most of the results in all of these cases are quite similar, we will often use the ambiguous notation M or N to refer to any consistent way of placing superscripts and subscripts.

Proposition 2.3.10 *The condition that M be right bounded or N be left bounded guarantees that the sums defining the structure maps for $M \boxtimes N$ in Figure 4 are finite. In this case they satisfy the corresponding structure equations. Moreover:*

(1) *If M and N are both left bounded (respectively right bounded) then $M \boxtimes N$ is left bounded (respectively right bounded).*

(2) *If M (respectively N) is bounded then $M \boxtimes N$ is left bounded (respectively right bounded).*

(3) *If M (respectively N) is bounded and N (respectively M) is right bounded (respectively left bounded) then $M \boxtimes N$ is bounded.*

[2] In the last two cases we assume that $\mathcal{B}$ and $\mathcal{C}$ are *dg* algebras. In general, we implicitly add this hypothesis any time a type *DD* structure is mentioned.

The reader might find it interesting to compare Proposition 2.3.10 with Lemma 5.7.

In order to prove Proposition 2.3.10 without checking all the cases individually, we will reformulate and unify the various definitions of boundedness. Given an operation graph of one of the types considered above (planar, directed graphs with labeled nodes and edges, obeying certain restrictions), we can restrict the inputs and outputs to lie in A_+ and B_+. (For the outputs, this means applying $(1-\epsilon)$ to each output edge.) This gives a map

$$m_\Gamma^+\colon (A_+)^{\otimes k_1}\otimes M\otimes (B_+)^{\otimes k_2}\to (A_+)^{\otimes l_1}\otimes M\otimes (B_+)^{\otimes l_2}$$

for appropriate values of k_1, k_2, l_1 and l_2. (Some of k_1, k_2, l_1 and l_2 will necessarily be 0, depending on the type of bimodule.)

Lemma 2.3.11 *A bimodule M (of any type) is bounded if and only if for each $x\in M$ there is a bound on the number of leaves of bimodule operation trees Γ for which the corresponding operation m_Γ^+ is nonzero when applied to x. It is left bounded if and only if for any x and any bound on the number of right inputs/outputs there is a bound on the number of left inputs/outputs. Similar statements hold for right bounded bimodules and for bimodule morphisms.*

Proof sketch This is very close to the definition of boundedness or left/right boundedness in each case, with the exception of the restriction to spinal graphs (without μ nodes). Let Γ' be the graph obtained from Γ by pushing all ϵ nodes upstream through μ nodes, using the relations in (2.1.4), as far as possible, and then taking the subgraph formed by the m or δ nodes and any adjacent ϵ nodes. Because A_+ and B_+ are nilpotent, there is a bound on the number of inputs to Γ that can contribute to an input to Γ'. Similarly, there is a bound on the number of outputs from Γ' that can contribute to an output of Γ. Thus, bounds on the inputs/outputs of Γ give bounds on the inputs/outputs of Γ', and vice versa. □

Proof sketch of Proposition 2.3.10 All of the bimodule structure operations in Figure 4 can be expanded out so that each algebra edge carries an element of A_+, B_+ or C_+, rather than just A, B or C, simply by taking a sum of terms where we apply ϵ or $(1-\epsilon)$ on each edge. Now suppose N is left bounded and we wish to compute a structure map on $M\boxtimes N$ with some fixed number of outputs/inputs on the right and left. Then there is a bound on the number of B_+ edges leaving the dotted line corresponding to N, which immediately gives a bound on the number of terms contributing to the definition of the structure map.

A similar argument works if M is right bounded.

If M and N are both left bounded and we have a bound on the number of terms on the right of the entire diagram, then left-boundedness of N gives us a bound on the number of algebra edges communicating between the two dotted lines, and left-boundedness of M then gives a bound on the number on the number of algebra edges on the left of the diagram, as desired to show that $M \boxtimes N$ is left bounded.

The other cases are similar. $\square$

Remark 2.3.12 If we drop the assumption that the algebras involved are nilpotent, Lemma 2.3.11 becomes false, but most cases of Proposition 2.3.10 remain true. However, in a tensor product where the right factor is a *DD* module (ie ${}_{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}N^{\mathcal{C}}$ or ${}^{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}N^{\mathcal{C}}$), if we only assume that N is left bounded it does not follow that the sums in Figure 4 are finite. A strengthening of the definition of left/right boundedness for *DD* bimodules fixes this case. However, if $\mathcal{A}$ is not nilpotent, ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is not usually left or right bounded which, for instance, breaks Proposition 2.3.18.

As with tensoring type D and A modules, the tensor product for bimodules is not strictly functorial. Again, we define the box product of two morphisms in terms of the box product of a morphism with the identity morphism. There are now eight cases:

- Given $f_{AA}\colon {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}M'_{\mathcal{B}}$ and ${}^{\mathcal{B}}N_{\mathcal{C}}$ define $f_{AA} \boxtimes \mathbb{I}_N$ as in Figure 5(a).
- Given $f_{DA}\colon {}^{\mathcal{A}}M_{\mathcal{B}} \to {}^{\mathcal{A}}M'_{\mathcal{B}}$ and ${}^{\mathcal{B}}N_{\mathcal{C}}$ define $f_{DA} \boxtimes \mathbb{I}_N$ as in Figure 5(b).
- Given $f_{AA}\colon {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}M'_{\mathcal{B}}$ and ${}^{\mathcal{B}}N^{\mathcal{C}}$ define $f_{AA} \boxtimes \mathbb{I}_N$ as in Figure 5(c).
- Given $f_{DA}\colon {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}M'_{\mathcal{B}}$ and ${}^{\mathcal{B}}N^{\mathcal{C}}$ define $f_{DA} \boxtimes \mathbb{I}_N$ as in Figure 5(d).
- Given ${}_{\mathcal{A}}M_{\mathcal{B}}$ and $g_{DA}\colon {}^{\mathcal{B}}N_{\mathcal{C}} \to {}^{\mathcal{B}}N'_{\mathcal{C}}$ define $\mathbb{I}_M \boxtimes g_{DA}$ as in Figure 5(e).
- Given ${}^{\mathcal{A}}M_{\mathcal{B}}$ and $g_{DA}\colon {}^{\mathcal{B}}N_{\mathcal{C}} \to {}^{\mathcal{B}}N'_{\mathcal{C}}$ define $\mathbb{I}_M \boxtimes g_{DA}$ as in Figure 5(f).
- Given ${}_{\mathcal{A}}M_{\mathcal{B}}$ and $g_{DD}\colon {}^{\mathcal{B}}N^{\mathcal{C}} \to {}^{\mathcal{B}}(N')^{\mathcal{C}}$ define $\mathbb{I}_M \boxtimes g_{DD}$ as in Figure 5(g).
- Given ${}^{\mathcal{A}}M_{\mathcal{B}}$ and $g_{DD}\colon {}^{\mathcal{B}}N^{\mathcal{C}} \to {}^{\mathcal{B}}(N')^{\mathcal{C}}$ define $\mathbb{I}_M \boxtimes g_{DD}$ as in Figure 5(h).

In all cases, define $f \boxtimes g$ to be $(f \boxtimes \mathbb{I}) \circ (\mathbb{I} \boxtimes g^1)$.

With these definitions, the obvious analogue of Lemma 2.3.3 holds. Moreover, similarly to Corollary 2.3.5 and Lemma 2.3.7 we have:

Lemma 2.3.13 (1) $\boxtimes$: $\mathrm{Mor}(M, M') \otimes \mathrm{Mor}(N, N') \to \mathrm{Mor}(M \boxtimes N, M' \boxtimes N')$ *is a chain map.*

(2) *The operation $\boxtimes$ is functorial up to homotopy. That is, $(f \boxtimes g) \circ (f' \boxtimes g')$ is homotopic to $(f \circ f') \boxtimes (g \circ g')$.*

(3) *The operations $M_{\mathcal{A}} \boxtimes \cdot$ and $\cdot \boxtimes {}^{\mathcal{A}}N$ extend to $\mathcal{A}_\infty$–functors, and so*

(4) *The operations $\boxtimes$ descend to bifunctors of homotopy categories.*

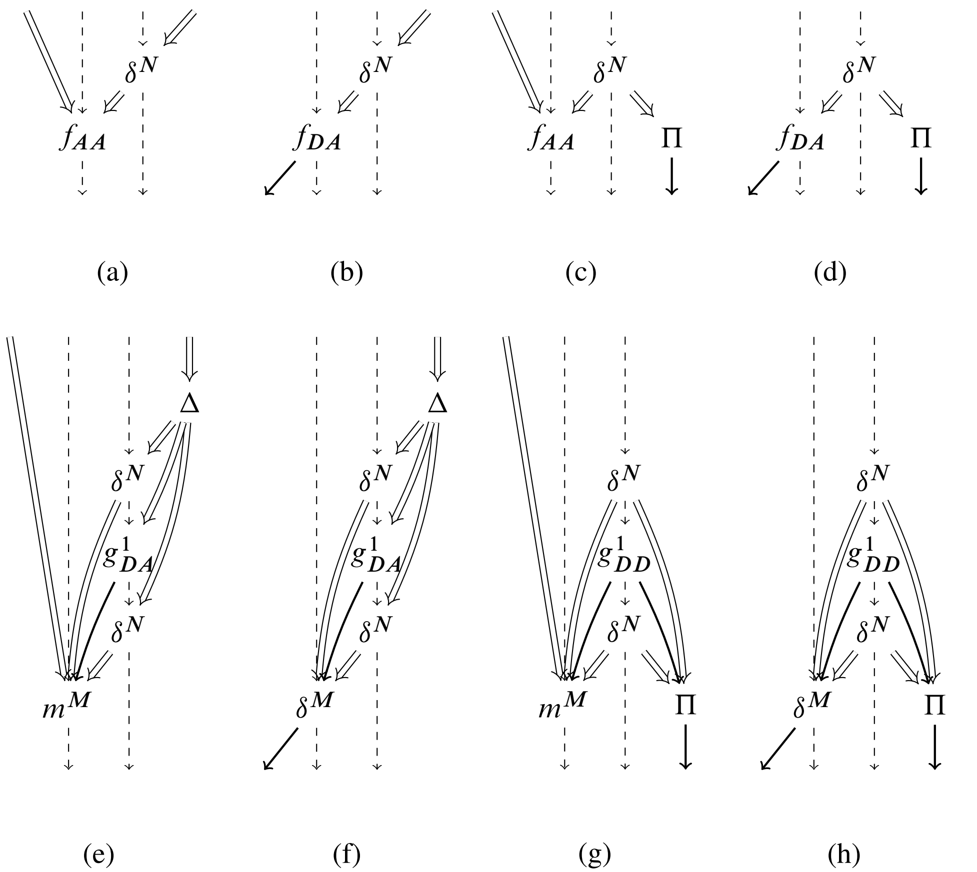


Figure 5: The box product of a bimodule morphism with the identity morphism

Next we turn to the question of associativity of tensor product. Like functoriality on the *dg* level this is somewhat subtle, but it is straightforward in several cases:

Lemma 2.3.14 (1) *Let ${}_{\mathcal{A}}N_{\mathcal{B}}$ be a type AA module and $M^{\mathcal{A}}$ and ${}^{\mathcal{B}}P$ type D structures. Then there is a canonical isomorphism*

$$(M^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}N_{\mathcal{B}}) \boxtimes {}^{\mathcal{B}}P \cong M^{\mathcal{A}} \boxtimes ({}_{\mathcal{A}}N_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P).$$

(2) *Let ${}^{\mathcal{A}}N_{\mathcal{B}}$ be a type DA structure, $M_{\mathcal{A}}$ a type A module and ${}^{\mathcal{B}}P$ a type D structure. Then there is a canonical isomorphism*

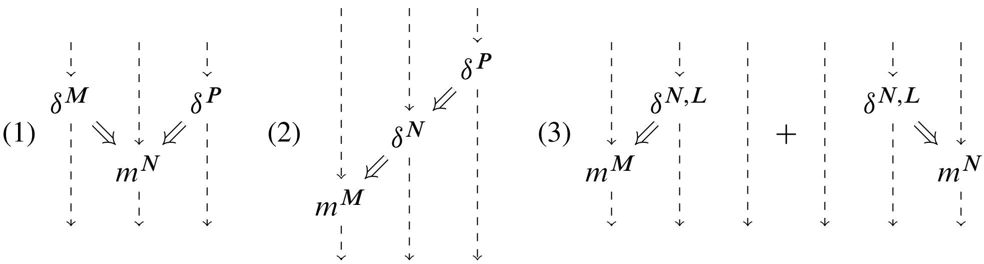
$$(M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{B}}) \boxtimes {}^{\mathcal{B}}P \cong M_{\mathcal{A}} \boxtimes ({}^{\mathcal{A}}N_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P).$$

(3) *Let ${}^{\mathcal{A}}N^{\mathcal{B}}$ be a separated type DD structure as in Definition 2.2.57. Then for any $M_{\mathcal{A}}$ and ${}_{\mathcal{B}}P$ there is a canonical isomorphism*

$$(M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N^{\mathcal{B}})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}P \cong M_{\mathcal{A}} \boxtimes {}^{\mathcal{B}}({}^{\mathcal{A}}N^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}P).$$

Similar statements hold if M and/or P is a bimodule (of any type compatible with the $\boxtimes$ products).

Proof The differential on the triple box product in the three cases is given by the diagrams



respectively, independently of which way one associates. (In the separated type *DD* module, the map δ^L is defined from δ^{1L} in the same way δ is defined from δ^1. The fact that these are the only two terms in the differential follows from strict unitality of the modules.) The same holds for bimodules, with slightly extended diagrams. □

For N a nonseparated type *DD*, the box tensor product is not strictly associative as in Lemma 2.3.14. However, associativity does hold up to homotopy equivalence:

Proposition 2.3.15 *Let $\mathcal{A}$ and $\mathcal{B}$ be dg algebras and $M_{\mathcal{A}}$, ${}^{\mathcal{A}}N^{\mathcal{B}}$ and ${}_{\mathcal{B}}P$ be right type A, type DD, and left type A structures respectively. Suppose moreover that ${}^{\mathcal{A}}N^{\mathcal{B}}$ is homotopy equivalent to a bounded type DD structure. Then $(M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N^{\mathcal{B}}) \boxtimes {}_{\mathcal{B}}P$ is homotopy equivalent to $M_{\mathcal{A}} \boxtimes ({}^{\mathcal{A}}N^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}P)$. The analogous statements hold if M is a type AA or DA module and/or P is a type AA or AD module.*

We will prove this in Section 2.3.3, after introducing the bar resolution.

2.3.3 Bar resolutions of modules

Definition 2.3.16 For $\mathcal{A}$ a *dg* algebra, ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}$ is the type *DD* bimodule with underlying $\boldsymbol{k}$–module $T^*(\mathcal{A}[1])$, with basis written $[a_1|\cdots|a_k]$ for $k \geq 0$, and structure maps

$$\begin{aligned}\delta^1[a_1|\cdots|a_k] := a_1\otimes[a_2|\cdots|a_k]\otimes 1 + 1\otimes[a_1|\cdots|a_{k-1}]\otimes a_k\\ +\sum_{1\le i\le k} 1\otimes[a_1|\cdots|\mu_1(a_i)|\cdots|a_k]\otimes 1\\ +\sum_{1\le i\le k-1} 1\otimes[a_1|\cdots|\mu_2(a_i,a_{i+1})|\cdots|a_k]\otimes 1.\end{aligned}$$

Note that ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}$ is bounded as a type *DD* structure.

We can use the bar resolution to define the tensor product of $\mathcal{A}_\infty$–modules:

Definition 2.3.17 Given $\mathcal{A}_\infty$–modules $M_{\mathcal{A}}$ and ${}_{\mathcal{A}}N$ over a *dg* algebra $\mathcal{A}$, define the *$\mathcal{A}_\infty$–tensor product of M and N* to be

$$M_{\mathcal{A}}\,\widetilde{\otimes}\,{}_{\mathcal{A}}N := M_{\mathcal{A}}\boxtimes{}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}\boxtimes{}_{\mathcal{A}}N.$$

The module ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}$ is separated, so by Lemma 2.3.14 it is okay that we have not parenthesized the triple box product. This definition agrees with the standard definition (eg [13, Section 6.3]).

Proposition 2.3.18 *Let $\mathcal{A}$ be a dg algebra. The map ${}^{\mathcal{A}}N\mapsto{}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}\boxtimes{}^{\mathcal{A}}N$ induces an $\mathcal{A}_\infty$–functor ${}^{\mathcal{A}}\mathsf{Mod}\to{}_{\mathcal{A}}\mathsf{Mod}$. Similarly, the map $M_{\mathcal{A}}\mapsto M_{\mathcal{A}}\boxtimes{}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}$ induces an $\mathcal{A}_\infty$–functor ${}_{\mathcal{A}}\mathsf{Mod}\to{}^{\mathcal{A}}\mathsf{Mod}$. These two functors are homotopy inverses to one another (and the homotopy is canonical). They entwine the tensor products $\boxtimes$ and $\widetilde{\otimes}$ in the sense that there is a canonical homotopy equivalence*

$$M_{\mathcal{A}}\boxtimes{}^{\mathcal{A}}N\simeq(M_{\mathcal{A}})\,\widetilde{\otimes}\,({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}\boxtimes{}^{\mathcal{A}}N).$$

In particular, the categories ${}^{\mathcal{A}}\mathsf{Mod}$ and ${}_{\mathcal{A}}\mathsf{Mod}$ are quasiequivalent, and hence their derived categories are equivalent.

Corresponding statements hold for the categories of type DD, DA, and AA modules.

We will prove Proposition 2.3.18 presently. The proposition justifies the following abuse of notation: given a type D structure ${}^{\mathcal{A}}M$, let ${}_{\mathcal{A}}M={}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}\boxtimes{}^{\mathcal{A}}M$. Similarly, given a type A module ${}_{\mathcal{A}}N$, let ${}^{\mathcal{A}}N={}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}\boxtimes{}_{\mathcal{A}}N$. This notation extends in an obvious way to bimodules. The statement in Proposition 2.3.18 about tensor products becomes

$$M_{\mathcal{A}}\boxtimes{}^{\mathcal{A}}N\simeq M_{\mathcal{A}}\,\widetilde{\otimes}\,{}_{\mathcal{A}}N.$$

The proof of Proposition 2.3.18 is based on the following key lemma:

Lemma 2.3.19 *For any dg algebra $\mathcal{A}$, the type DA module ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is homotopy equivalent to ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$. Moreover, the homotopy equivalence $\kappa\colon {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ is bounded.*

(See Example 2.2.42 and Definition 2.2.48 for definitions of ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ and ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$. Also, the homotopy inverse to κ is not necessarily bounded.)

Proof This lemma is a version of the standard fact that the bar resolution is a resolution. (See [21, Proposition 2.16] for a version that is not far from the one we give below.) We translate the proof into our language. For convenience, write ${}^{\mathcal{A}}M_{\mathcal{A}}$ or just M for ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$. Now define $\kappa\colon M \to \mathbb{I}$ by

$$\kappa^1_{1+l}([a_1|\cdots|a_k]b, c_1, \dots, c_l) := \begin{cases} 0 & k > 0 \text{ or } l > 0, \\ b \otimes 1 & k = l = 0. \end{cases}$$

Define $\psi\colon \mathbb{I} \to M$ by

$$\psi^1_{1+l}(1, a_1, \dots, a_l) := 1 \otimes [a_1|\cdots|a_l]\,1.$$

It is elementary to check that κ and ψ are homomorphisms of type *DA* structures (ie cycles in their respective morphism spaces) and that $\kappa \circ \psi$ is the identity. The other composition $\psi \circ \kappa$ is not the identity, but it is homotopic to the identity by the map $h\colon M \to M$ defined by

$$h^1_{1+l}([a_1|\cdots|a_k]b, c_1, \dots, c_l) := 1 \otimes [a_1|\cdots|a_k|b|c_1|\cdots|c_l]\,1. \qquad \square$$

Lemma 2.3.20 *Let $\mathcal{A}$ be a dg algebra and $M_{\mathcal{A}}$ an $\mathcal{A}_\infty$–module over $\mathcal{A}$. Then $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ is canonically isomorphic to $M_{\mathcal{A}}$. Similarly, if ${}^{\mathcal{A}}N$ is a type D structure then ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ is canonically isomorphic to ${}^{\mathcal{A}}N$. Similar statements hold when M is a type AA or DA module and when N is a type DA or DD module.*

Proof This is immediate from the definitions. $\square$

Proof of Proposition 2.3.18 The fact that ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes \cdot$ and ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes \cdot$ are functorial is part of Lemma 2.3.13. To see that these two functors are homotopy inverses, note that

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes ({}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes \cdot) \cong ({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}) \boxtimes \cdot \simeq {}_{\mathcal{A}}[\mathbb{I}]^{\mathcal{A}} \boxtimes \cdot,$$

where the isomorphism uses Lemma 2.3.14 and the homotopy equivalence uses Lemmas 2.3.19 and 2.3.13. But ${}_{\mathcal{A}}[\mathbb{I}]^{\mathcal{A}} \boxtimes \cdot$ is an equivalence of categories. Similar reasoning applies to the composition ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes ({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes \cdot)$, proving the result.

The corresponding statements about bimodules follow similarly. The fact that the functors intertwine $\boxtimes$ and $\widetilde{\otimes}$ is obvious from the definitions. $\square$

Definition 2.3.21 Given an $\mathcal{A}_\infty$–module $M_{\mathcal{A}}$ over a *dg* algebra define its *bar resolution* to be

$$\mathrm{Bar}(M_{\mathcal{A}}) = M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}.$$

Similarly, if ${}_{\mathcal{A}}M_{\mathcal{B}}$ is a type *AA* structure then define its *bar resolution* to be

$$\mathrm{Bar}({}_{\mathcal{A}}M_{\mathcal{B}}) = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}.$$

By taking bar resolutions, over a *dg* algebra, every $\mathcal{A}_\infty$–module is $\mathcal{A}_\infty$–homotopy equivalent to an honest *dg* module:

Proposition 2.3.22 *Let $\mathcal{A}$ be a dg algebra and $M_{\mathcal{A}}$ an $\mathcal{A}_\infty$–module over $\mathcal{A}$. Then:*

(1) $\mathrm{Bar}(M_{\mathcal{A}})$ *is $\mathcal{A}_\infty$–homotopy equivalent to $M_{\mathcal{A}}$.*

(2) $\mathrm{Bar}(M_{\mathcal{A}})$ *is an honest dg module.*

Similarly, if $\mathcal{B}$ is another dg algebra and ${}_{\mathcal{A}}M_{\mathcal{B}}$ is a type AA structure then:

(1) $\mathrm{Bar}({}_{\mathcal{A}}M_{\mathcal{B}})$ *is $\mathcal{A}_\infty$–homotopy equivalent to ${}_{\mathcal{A}}M_{\mathcal{B}}$.*

(2) $\mathrm{Bar}({}_{\mathcal{A}}M_{\mathcal{B}})$ *is an honest dg module.*

Proof This is a combination of Lemmas 2.3.19 and 2.3.20. Lemma 2.3.19 furnishes a homotopy equivalence between left-bounded type *DA* modules $\kappa\colon {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$. This induces a homotopy equivalence

$$\mathbb{I}_M \boxtimes \kappa\colon M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}},$$

while Lemma 2.3.20 furnishes the isomorphism $M_{\mathcal{A}} \cong M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$. □

It is immediate from Lemma 2.3.19 that:

Corollary 2.3.23 *If $\mathcal{A}$ is a dg algebra, then:*

- $M_{\mathcal{A}} \mathbin{\widetilde{\otimes}} {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ *is chain-homotopy equivalent to $M_{\mathcal{A}}$.*
- $(M_{\mathcal{A}} \mathbin{\widetilde{\otimes}} {}_{\mathcal{A}}N_{\mathcal{B}}) \mathbin{\widetilde{\otimes}} {}_{\mathcal{B}}P = M_{\mathcal{A}} \mathbin{\widetilde{\otimes}} ({}_{\mathcal{A}}N_{\mathcal{B}} \mathbin{\widetilde{\otimes}} {}_{\mathcal{B}}P)$.
- $M_{\mathcal{A}} \mathbin{\widetilde{\otimes}} ({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N) \simeq M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$.

Thus, for instance, $\widetilde{\otimes}$ turns $\mathsf{H}({}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{A}})$ into a monoidal category.

The bar resolution can be used to give an alternate characterization of the subcategory ${}^{\mathcal{A}}\mathsf{Mod} \subset {}^{\mathcal{A}}_{\mathrm{u}}\mathsf{Mod}$:

Proposition 2.3.24 *The following conditions on a type D structure are equivalent:*

(D-1) *${}^{\mathcal{A}}N$ is homotopy equivalent to a bounded type D structure.*

(D-2) *The canonical map ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N \to {}^{\mathcal{A}}N$ (induced by combining maps from Lemmas 2.3.19 and 2.3.20) is a homotopy equivalence.*

Analogously, the following are equivalent for a type DA structure:

(DA-1) *${}^{\mathcal{A}}N_{\mathcal{B}}$ is homotopy equivalent to a left bounded type DA structure.*

(DA-2) *${}^{\mathcal{A}}N_{\mathcal{B}}$ is homotopy equivalent to a bounded type DA structure.*

(DA-3) *The canonical map ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{B}} \to {}^{\mathcal{A}}N_{\mathcal{B}}$ is a homotopy equivalence.*

(DA-4) *The canonical map ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \to {}^{\mathcal{A}}N_{\mathcal{B}}$ is a homotopy equivalence.*

The following conditions on a type DD bimodule are equivalent:

(DD-1) *${}^{\mathcal{A}}N^{\mathcal{B}}$ is homotopy equivalent to a bounded type DD structure.*

(DD-2) *The canonical map*

$$ {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}} \to {}^{\mathcal{A}}N^{\mathcal{B}} $$

is a homotopy equivalence.

Proof We start with the case of a type D module ${}^{\mathcal{A}}N$.

(D-1)$\Rightarrow$(D-2) Let ${}^{\mathcal{A}}N'$ be a bounded type D structure, and $\phi\colon N \to N'$ be a homotopy equivalence. According to Lemma 2.3.13, we have a homotopy commutative diagram:

$$\begin{tikzcd}
{}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N \arrow[r, "\kappa \boxtimes \mathbb{I}_N"] \arrow[d, "\mathbb{I}_{\mathrm{Bar}(\mathcal{A})\boxtimes\mathcal{A}} \boxtimes \phi"'] & {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N \arrow[r, "\cong"] & {}^{\mathcal{A}}N \arrow[d, "\phi"] \\
{}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N' \arrow[r, "\kappa \boxtimes \mathbb{I}_{N'}"'] & {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N' \arrow[r, "\cong"'] & {}^{\mathcal{A}}N'
\end{tikzcd}$$

All arrows with the possible exception of $\kappa \boxtimes \mathbb{I}_N$ are homotopy equivalences. It follows that $\kappa \boxtimes \mathbb{I}_N$ is a homotopy equivalence as well.

(D-2)$\Rightarrow$(D-1) Since ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is a bounded type *AA* structure, we can form ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$. Moreover, since ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}}$ is a bounded type *DD* structure, ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ is a bounded type D structure.

The type *DA* case is similar. The type *DD* case follows from the type D case. $\square$

Corollary 2.3.25 *Let $\mathcal{A}$ be a dg algebra. Suppose ${}^{\mathcal{A}}M$ is homotopy equivalent to a bounded type D structure (ie ${}^{\mathcal{A}}M$ is an object in ${}^{\mathcal{A}}\mathsf{Mod}$). Then ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M$ is a projective $\mathcal{A}$–module.*

Proof By Lemma 2.3.13 and Proposition 2.3.24, ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M$ is homotopy equivalent to its bar resolution. But the bar resolution of any module is projective [5, Proposition 10.12.2.6], and projectivity is preserved by homotopy equivalences. $\square$

Remark 2.3.26 The condition that ${}^{\mathcal{A}}M$ is homotopy equivalent to a bounded type D structure is essential. For instance, let $\mathcal{A} = \mathbb{F}_2[t]/t^2$, and ${}^{\mathcal{A}}M$ have one generator x with $\delta^1 x = t \otimes x$, then ${}_{\mathcal{A}}M = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M$ is acyclic but, since $(\mathcal{A}/t) \otimes {}_{\mathcal{A}}M$ has homology, we can conclude that ${}_{\mathcal{A}}M$ is not homotopy equivalent to the trivial module. Thus ${}_{\mathcal{A}}M$ is not projective.

Note in particular that this implies that not every type D structure is homotopy equivalent to a bounded type D structure.

Finally, we turn to the proof of Proposition 2.3.15, which is based on the following observation:

Lemma 2.3.27 *If ${}^{\mathcal{A}}N^{\mathcal{B}}$ is homotopy equivalent to a bounded type DD structure, then it is homotopy equivalent to a separated one.*

Proof Let ${}^{\mathcal{A}}M^{\mathcal{B}}$ be a type *DD* module. Then ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}}$ is a separated type *DD* module, and is homotopy equivalent to ${}^{\mathcal{A}}M^{\mathcal{B}}$ by Proposition 2.3.24. $\square$

Proof of Proposition 2.3.15 Let ${}^{\mathcal{A}}\widetilde{N}^{\mathcal{B}}$ be a separated type *DD* module homotopy equivalent to ${}^{\mathcal{A}}N^{\mathcal{B}}$. Then

$$(M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N^{\mathcal{B}}) \boxtimes_{\mathcal{B}} P \simeq (M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\widetilde{N}^{\mathcal{B}}) \boxtimes_{\mathcal{B}} P \simeq M_{\mathcal{A}} \boxtimes ({}^{\mathcal{A}}\widetilde{N}^{\mathcal{B}} \boxtimes_{\mathcal{B}} P) \simeq M_{\mathcal{A}} \boxtimes ({}^{\mathcal{A}}N^{\mathcal{B}} \boxtimes_{\mathcal{B}} P),$$

where the outside equivalences use Lemma 2.3.13 and the middle equivalence uses Lemma 2.3.14. $\square$

Remark 2.3.28 The results of this section generalize in a transparent way to the case of $\mathcal{A}_\infty$–algebras; the only obstruction is that we have not defined the bar resolution except over *dg* algebras, an obstruction which is more terminological than mathematical. The key observation is that the difficulties mentioned in Remark 2.2.58 do not arise for separated *DD* modules like ${}^{\mathcal{A}}M^{\mathcal{A}}$.

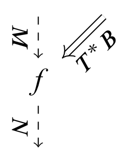
2.3.4 More Mor

Recall from Section 2.3.1 that there are functors $\mathcal{F}$ which forget one of the actions on a bimodule.

Definition 2.3.29 Define $\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}})$ to be the complex $\mathrm{Mor}(\mathcal{F}(M)_{\mathcal{B}}, \mathcal{F}(N)_{\mathcal{B}})$. We define a $(\mathcal{C},\mathcal{A})$–bimodule module structure on $\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}})$ as follows. The operation $m_{i,1,j} = 0$ unless either $i = 0$ or $j = 0$. The operation $m_{0,1,0}$ is, of course, the differential on the Mor complex. Finally, for exactly one of i and j nonzero, define

$$\begin{aligned}
&m_{i,1,0}(c_1,\dots,c_i,f)_{k+1}(x,b_1,\dots,b_k)\\
&\qquad\qquad = \sum_{p+q=k} m^N_{i,1,q}(c_1,\dots,c_i,f_{p+1}(x,b_1,\dots,b_p),b_{p+1},\dots,b_k),\\
&m_{0,1,j}(f,a_c,\dots,a_j)_{k+1}(x,b_1,\dots,b_k)\\
&\qquad\qquad = \sum_{p+q=k} f_{q+1}(m^M_{j,1,p}(a_1,\dots,a_j,x,b_1,\dots,b_p),b_{p+1},\dots,b_k).
\end{aligned}$$

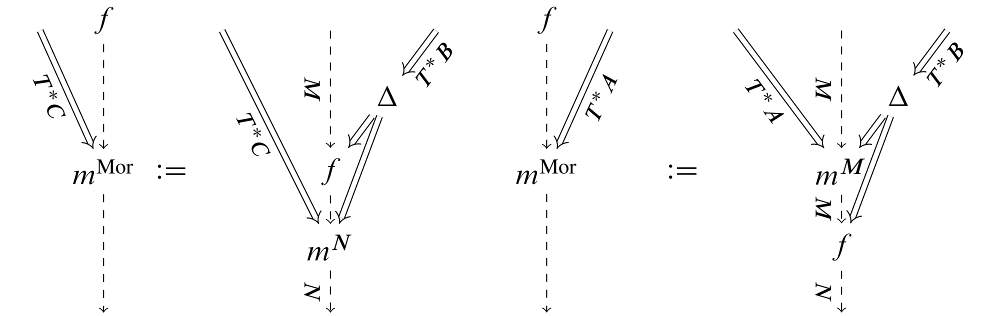
Graphically, if we draw f as



then the module structure on $\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}})$ is given as in Figure 6.

Lemma 2.3.30 *The structure defined in Definition 2.3.29 satisfies the $\mathcal{A}_\infty$–bimodule relation (2.2.39).*

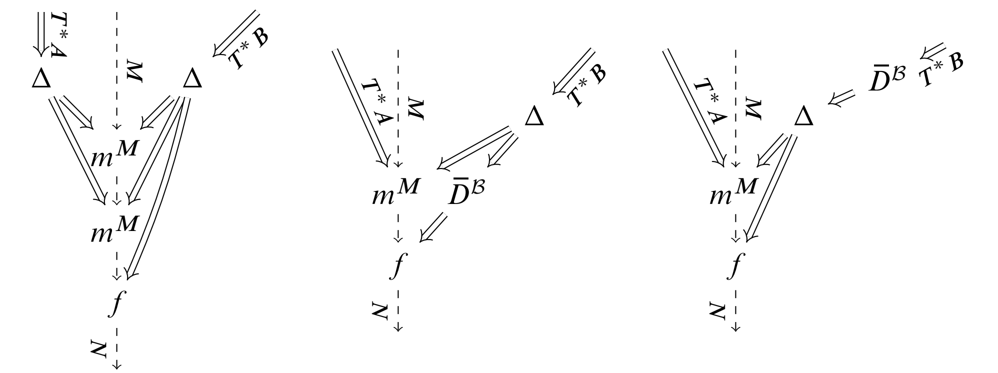
Proof This is a straightforward, if tedious, verification. It is clear from the diagrams that the left and right actions commute. The $\mathcal{A}_\infty$–relation for the right action involves

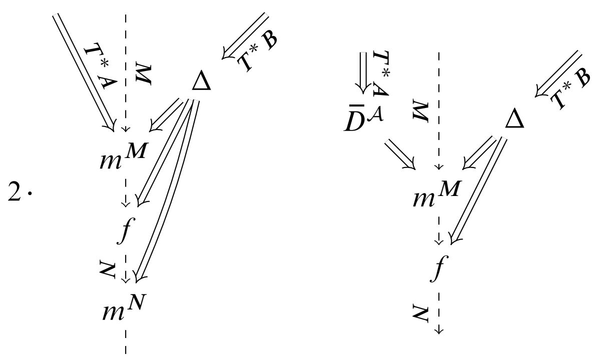


$m_{i,1,0}(c_1, \dots, c_i, f)$ $\qquad$ $m_{0,1,j}(f, a_1, \dots, a_j)$

Figure 6: Module structure on the Mor–complex

the following terms:





Here, the first four terms come from $m(m(f,a_1,\dots,a_j),a_{j+1},\dots,a_k)$; the first one is the generic case, the second is part of the $j=0$ case, the third is part of the $j=k$ case, and the fourth occurs in both the $j=0$ and $j=k$ cases. The fifth term comes from $m(f,\bar{D}^{\mathcal{A}}(a_1,\dots,a_k))$. Applying the $\mathcal{A}_\infty$–relation for M, these terms cancel.

A similar argument applies to the left action. $\square$

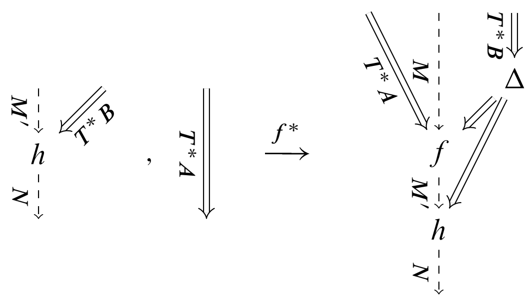
Bimodule morphisms $f\colon {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}M'_{\mathcal{B}}$ and $g\colon {}_{\mathcal{C}}N_{\mathcal{B}} \to {}_{\mathcal{C}}N'_{\mathcal{B}}$ induce $\mathcal{A}_\infty$–maps

$$f^*\colon \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M'_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}) \to \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}),$$
$$g_*\colon \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}) \to \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N'_{\mathcal{B}}),$$

as follows. Define $f^*_{i,1,j}$ to be zero if $i>0$ and define

$$f^*\colon \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M'_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}) \otimes T^*(A) \to \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}})$$

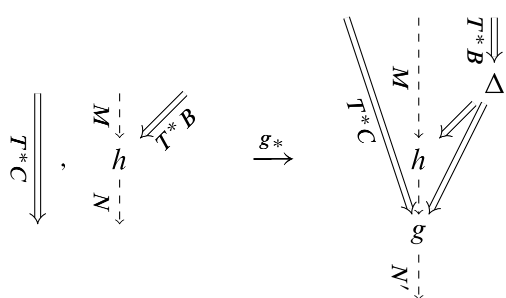
by:



Similarly, define $g_{*,i,1,j}$ to be zero if $j>0$ and

$$g_*\colon T^*(C) \otimes \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}) \to \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N'_{\mathcal{B}})$$

by:



Proposition 2.3.31 *The assignments $f \mapsto f^*$ and $g \mapsto g_*$ define chain maps*

$$\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}M'_{\mathcal{B}}) \to {}_{\mathcal{C}}\mathrm{Mor}_{\mathcal{A}}(\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M'_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}), \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}})),$$
$$\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{C}}N_{\mathcal{B}}, {}_{\mathcal{C}}N'_{\mathcal{B}}) \to {}_{\mathcal{C}}\mathrm{Mor}_{\mathcal{A}}(\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}), \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N'_{\mathcal{B}})).$$

In particular:

(1) *If f and g are $\mathcal{A}_\infty$ bimodule homomorphisms then f^* and g_* respect the bimodule structures on $\mathrm{Mor}_{\mathcal{B}}$.*

(2) *If $f'\colon {}_{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}M'_{\mathcal{B}}$ is homotopic to f then $(f')^*$ is homotopic to f^*. If $g'\colon {}_{\mathcal{C}}N_{\mathcal{B}} \to {}_{\mathcal{C}}N'_{\mathcal{B}}$ is homotopic to g then g'_* is homotopic to g_*.*

(3) *If f is a homotopy equivalence then f^* is a homotopy equivalence. If g is a homotopy equivalence then g_* is a homotopy equivalence.*

Moreover, these maps are functorial: given $f'\colon {}_{\mathcal{A}}M'_{\mathcal{B}} \to {}_{\mathcal{A}}M''_{\mathcal{B}}$ and $g'\colon {}_{\mathcal{C}}N'_{\mathcal{B}} \to {}_{\mathcal{C}}N''_{\mathcal{B}}$, $(f' \circ f)^* = f^* \circ (f')^*$ and $(g' \circ g)_* = g'_* \circ g_*$.

Proof The statement that the assignments $f \mapsto f^*$ and $g \mapsto g_*$ are chain maps follows from a straightforward computation once you sort through the definitions. The functoriality statement is also verified directly; this verification is less painful. □

Recall that the operation $\boxtimes$ is functorial (up to homotopy). In particular, given a morphism $f \in \mathrm{Mor}(M_{\mathcal{B}}, N_{\mathcal{B}})$ and a type D structure ${}^{\mathcal{B}}P$ there is an associated morphism $(f \boxtimes \mathbb{I}_{P}) \in \mathrm{Mor}_{\boldsymbol{k}}(M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P, N_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P)$, defined in Figure 3. If M, N and P are bimodules ${}_{\mathcal{A}}M_{\mathcal{B}}$, ${}_{\mathcal{C}}N_{\mathcal{B}}$ and ${}^{\mathcal{B}}P_{\mathcal{E}}$ then we may view $f \boxtimes \mathbb{I}$ as an element of $\mathrm{Mor}_{\mathcal{E}}({}_{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P_{\mathcal{E}}, {}_{\mathcal{C}}N_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P_{\mathcal{E}})$.

Proposition 2.3.32 *Let ${}_{\mathcal{A}}M_{\mathcal{B}}$, ${}_{\mathcal{C}}N_{\mathcal{B}}$ and ${}^{\mathcal{B}}P_{\mathcal{E}}$ be bimodules. Then "tensoring with the identity map"*

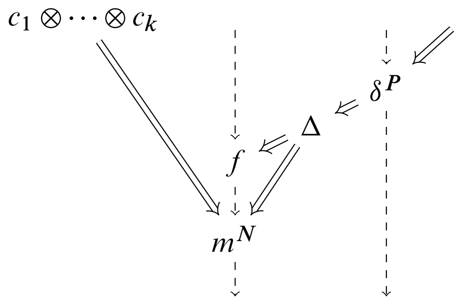
$$\cdot \boxtimes \mathbb{I}_{P}\colon \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}}) \to \mathrm{Mor}_{\mathcal{E}}({}_{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P_{\mathcal{E}}, {}_{\mathcal{C}}N_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}P_{\mathcal{E}})$$

is a map of $(\mathcal{C}, \mathcal{A})$–bimodules.

An analogous result holds if P is a type AA module, with $\widetilde{\otimes}$ in place of $\boxtimes$.

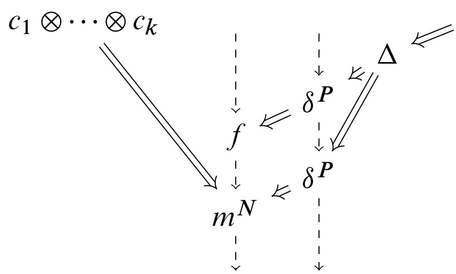
Proof We will discuss the left action by $T^*\mathcal{C}$; the other cases are similar. On one hand,

$[(c_1 \otimes \cdots \otimes c_k) \cdot f] \boxtimes \mathbb{I}_P =$



while on the other hand:

$(c_1 \otimes \cdots \otimes c_k) \cdot (f \boxtimes \mathbb{I}_P) =$



But, by the definition of δ^P, these two diagrams are exactly the same.

The analogue if P is a type *AA* module follows from the definition of $\widetilde{\otimes}$ in terms of $\boxtimes$. $\square$

Lemma 2.3.33 *For any strongly unital $\mathcal{A}_\infty$–algebras $\mathcal{A}$ and $\mathcal{B}$ and $\mathcal{A}_\infty$–bimodule ${}_{\mathcal{B}}M_{\mathcal{A}}$, ${}_{\mathcal{B}}M_{\mathcal{A}}$ is quasi-isomorphic to $\mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, {}_{\mathcal{B}}M_{\mathcal{A}})$ as a $(\mathcal{B}, \mathcal{A})$–bimodule.*

Proof For nonnegative integers i and j, define maps

$$\phi_{i,1,j}\colon (B_+)^{\otimes i} \otimes \mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, {}_{\mathcal{B}}M_{\mathcal{A}}) \otimes (A_+)^{\otimes j} \to {}_{\mathcal{B}}M_{\mathcal{A}}$$

by

$$\phi_{i,1,j}(b_1, \dots, b_i, f, a_1, \dots, a_j) = \begin{cases} f_{j+1}(1, a_1, \dots, a_j) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

One can check that the $\phi_{i,1,j}$ piece together to give a chain map from the complex $T^*(B_+[1]) \otimes \mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, {}_{\mathcal{B}}M_{\mathcal{A}}) \otimes T^*(A_+[1])$ to ${}_{\mathcal{B}}M_{\mathcal{A}}$; ie ϕ gives a bimodule morphism.

We claim that the component $\phi_{0,1,0}\colon \mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, {}_{\mathcal{B}}M_{\mathcal{A}}) \to {}_{\mathcal{B}}M_{\mathcal{A}}$ is an isomorphism in homology. To this end, observe that

$$\mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, {}_{\mathcal{B}}M_{\mathcal{A}}) \cong \mathrm{Hom}_{\boldsymbol{k}}(A \otimes T^*(A_+[1]), M),$$

where the isomorphism is as chain complexes and $\mathrm{Hom}_{\boldsymbol{k}}$ means the full chain complex of morphisms, not just the subset of chain maps. The chain complex structure on $A \otimes T^*(A_+[1])$ that makes this isomorphism true is simply the bar resolution of $\boldsymbol{k}$ (ie it is ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes \boldsymbol{k}$, where here $\boldsymbol{k}$ is thought of as a $\mathcal{A}$–module using the augmentation ϵ). Thus, by Proposition 2.3.22, it follows that $\mathrm{Hom}_{\boldsymbol{k}}(A \otimes T^*(\mathcal{A}_+[1]), M)$ is homotopy equivalent to $\mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{k}, M) \cong M$. Indeed, it is straightforward to verify that the homotopy equivalence is furnished by our map $\phi_{0,1,0}$. □

Lemma 2.3.34 *Let ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}_{\mathcal{A}}N_{\mathcal{C}}$ be bimodules and ${}^{\mathcal{C}}P_{\mathcal{E}}$ a type DA structure. Then*

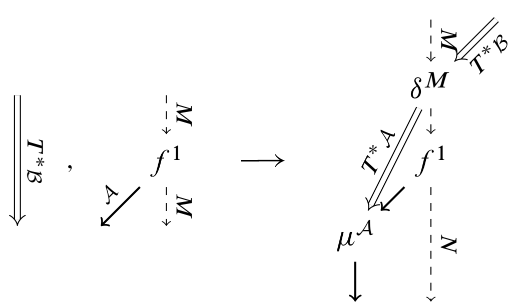
$$\mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}N_{\mathcal{C}} \boxtimes {}^{\mathcal{C}}P_{\mathcal{E}}) \cong \mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}N_{\mathcal{C}}) \boxtimes {}^{\mathcal{C}}P_{\mathcal{E}}$$

as $(\mathcal{B}, \mathcal{E})$–bimodules.

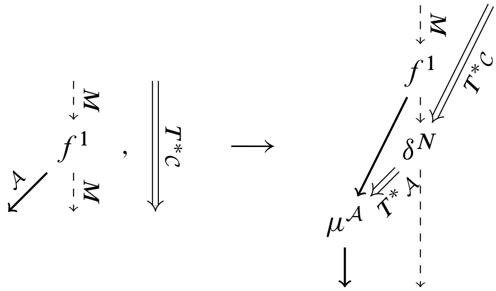
Proof This is immediate from the definitions. □

For computations, it is often more convenient to work with Mor complexes of type D structures, which tend to be much smaller. We will outline how this theory works, leaving the reader to supply most of the proofs.

Definition 2.3.35 Let ${}^{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{A}}N_{\mathcal{C}}$ be type *DA* bimodules. Let $\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}}) = \mathrm{Mor}({}^{\mathcal{A}}\mathcal{F}(M), {}^{\mathcal{A}}\mathcal{F}(N))$ where $\mathcal{F}$ is the functor forgetting the right action. Endow $\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}})$ with an $\mathcal{A}_\infty$ $(\mathcal{C}, \mathcal{B})$–bimodule structure by setting $m^{\mathrm{Mor}}_{i,1,j} = 0$ if i and j are both nonzero, and defining the product $T^*B \otimes \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}}) \to \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}})$ by



and the product $\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}}) \otimes T^*\mathcal{C} \to \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}})$ by:



The obvious analogue of Proposition 2.3.31 holds in this context. Moreover, the following analogue of Proposition 2.3.32 is true:

Proposition 2.3.36 *Let ${}^{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{A}}N_{\mathcal{C}}$ be type DA bimodules and ${}_{\mathcal{E}}P_{\mathcal{A}}$ a type AA bimodule. Then the "tensoring with the identity map"*

$$\mathbb{I}_P \boxtimes \cdot\colon \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}}) \to \mathrm{Mor}_{\mathcal{E}}({}_{\mathcal{E}}P_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{E}}P_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{C}})$$

is a map of $(\mathcal{C},\mathcal{B})$–bimodules.

Corollary 2.3.37 *If $\mathcal{A}$ is a dg algebra then $\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}})$ is canonically isomorphic to the chain complex of maps*

$$\psi\colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{C}}$$

which commute with the $\mathcal{A}$ action, equipped with the differential

$$\partial\psi = \partial_{\mathcal{A}\boxtimes N} \circ \psi + \psi \circ \partial_{\mathcal{A}\boxtimes M}.$$

If ${}^{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{A}}N_{\mathcal{C}}$ are homotopy equivalent to bounded type DA structures then the inclusion of this subcomplex into $\mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{C}})$ is a quasi-isomorphism.

Proof Taking $P = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ in Proposition 2.3.36, we get a map

$$\mathbb{I}_{\mathcal{A}} \boxtimes \cdot\colon \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}}) \to \mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{C}}).$$

It is clear that this map is injective, and so identifies $\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N_{\mathcal{C}})$ with some subcomplex of the $\mathcal{A}_\infty$–maps ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{C}}$. It is straightforward to see that it is the stated subcomplex. The fact that this subcomplex is homotopy equivalent to the entire $\mathrm{Mor}_{\mathcal{A}}$–complex follows from Proposition 2.3.18. □

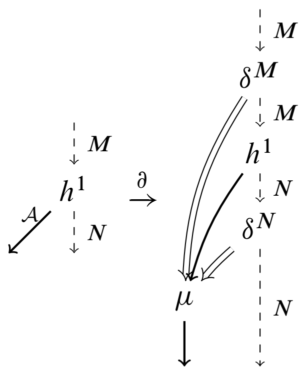
There is one more case that we will consider: that of the type *DA* structure on the type D structure morphisms from a type *DA* module to a type *DD* module. Before discussing this morphism space, we pause to note another interpretation of the chain complex of morphisms between two type D structures:

Lemma 2.3.38 *Let ${}^{\mathcal{A}}M$ and ${}^{\mathcal{A}}N$ be type D structures. Then*

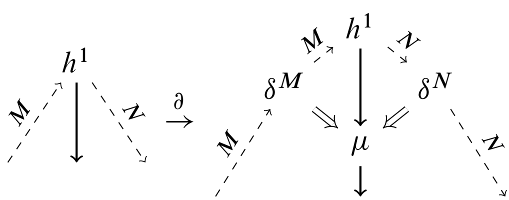
$$\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M, {}^{\mathcal{A}}N) \cong \overline{M}^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N.$$

Here $\overline{M}^{\mathcal{A}}$ denotes the opposite type D structure to ${}^{\mathcal{A}}M$, as in Definition 2.2.31.

Proof This is immediate from the definitions, as follows. The chain complex for the morphism complex isomorphism



which corresponds to the differential on the box complex:



This completes the proof. □

Now, suppose that ${}^{\mathcal{A}}M_{\mathcal{B}}$ is a type *DA* structure and ${}^{\mathcal{A}}N^{\mathcal{C}}$ is a type *DD* module. Then we can give the morphism space $\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N^{\mathcal{C}})$ the structure of a type *DA* module via the isomorphism

$$\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N^{\mathcal{C}}) \cong {}_{\mathcal{B}}\overline{M}^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N^{\mathcal{C}}. \tag{2.3.39}$$

(The opposite type *DA* structure ${}_{\mathcal{B}}\overline{M}^{\mathcal{A}}$ is defined in Definition 2.2.53.) This is related to the module structure on the space of type *AA* morphisms as follows:

Proposition 2.3.40 *For ${}^{\mathcal{A}}M_{\mathcal{B}} \in {}^{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ and ${}^{\mathcal{A}}N^{\mathcal{C}} \in {}^{Alg}\mathsf{Mod}^{\mathcal{C}}$, there is a canonical inclusion*

$$\mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}M_{\mathcal{B}}, {}^{\mathcal{A}}N^{\mathcal{C}}) \to \mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N^{\mathcal{C}})$$

respecting the bimodule structure and inducing an isomorphism on homology.

Proof That there is such an inclusion inducing an isomorphism on homology follows from Proposition 2.3.18. We leave verification that it respects the bimodule structure to the reader. □

2.3.5 Hochschild homology We next turn to the Hochschild homology, or self tensor product, of a bimodule with two actions of the same algebra. We first introduce the classical Hochschild complex of an $\mathcal{A}_\infty$–bimodule and then give a version for type *DA* structures analogous to the $\boxtimes$ tensor product; we also prove that the two definitions are equivalent in the obvious sense (Proposition 2.3.54). The Hochschild complex of a type *DA* structure arises naturally when studying the knot Floer homology of open books; see Section 7.2.

Definition 2.3.41 Let $\mathcal{A}$ be an $\mathcal{A}_\infty$–algebra over a commutative ground ring $\boldsymbol{k}$, and ${}_{\mathcal{A}}M_{\mathcal{A}}$ be an $\mathcal{A}_\infty$–bimodule. The *Hochschild complex $CH({}_{\mathcal{A}}M_{\mathcal{A}})$ of* ${}_{\mathcal{A}}M_{\mathcal{A}}$ is defined as follows. Let $CH_n(M)$ be the $\mathbb{F}_2$ vector space which is the quotient of

$$M \otimes_{\boldsymbol{k}} \overbrace{A[1] \otimes_{\boldsymbol{k}} \cdots \otimes_{\boldsymbol{k}} A[1]}^{n}$$

by the relations

$$e \cdot x \otimes a_1 \otimes \cdots \otimes a_n = x \otimes a_1 \otimes \cdots \otimes a_n \cdot e,$$

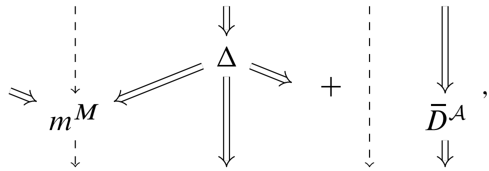
where e ranges over $\boldsymbol{k}$. As a vector space,

$$CH({}_{\mathcal{A}}M_{\mathcal{A}}) = \bigoplus_{n=0}^{\infty} CH_n(M).$$

The differential on $CH({}_{\mathcal{A}}M_{\mathcal{A}})$ is given by

$$\begin{aligned} &D(x \otimes a_1 \otimes \cdots \otimes a_\ell) \\ &\qquad = \sum_{m+n \le \ell} m_{m,1,n}(a_{\ell-m+1}, \dots, a_\ell, x, a_1, \dots, a_n) \otimes a_{n+1} \otimes \cdots \otimes a_{\ell-m} \\ &\qquad\qquad + \sum_{1 \le m < n \le \ell} x \otimes a_1 \cdots \otimes \mu_{n-m}(a_m, \dots, a_{n-1}) \otimes a_n \otimes \cdots \otimes a_\ell. \end{aligned}$$

It is perhaps more instructive to think of $CH_n(M)$ as generated by equivalence classes of collections of n elements arranged on a circle (the equivalence relation coming from a circular tensor product). From this description, then D is a sum of maps which take any collection of $i \leq n$ consecutive terms and apply whichever higher multiplication map is available to this collection. The result of this component of D lies in $CH_{n-i+1}(M)$. Graphically, D is



where in the first diagram a bundle of strands in $\otimes^* \mathcal{A}$ runs off the right edge and comes back on the left.

Note that the subscript n on CH_n is not a grading (though it is a filtration). The grading on CH_n is given by

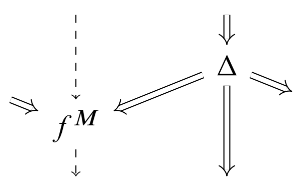
$$\mathrm{gr}(x \otimes a_1 \otimes \cdots \otimes a_n) = \mathrm{gr}(x) + \mathrm{gr}(a_1) + \cdots + \mathrm{gr}(a_n) + n.$$

Lemma 2.3.42 *The endomorphism D of $CH({}_{\mathcal{A}}M_{\mathcal{A}})$ is a differential. If ${}_{\mathcal{A}}M_{\mathcal{A}}$ and ${}_{\mathcal{A}}N_{\mathcal{A}}$ are $\mathcal{A}_\infty$–homotopy equivalent bimodules, the Hochschild complexes $CH({}_{\mathcal{A}}M_{\mathcal{A}})$ and $CH({}_{\mathcal{A}}N_{\mathcal{A}})$ are homotopy equivalent chain complexes.*

Proof The fact that $D^2 = 0$ follows easily from the fact that m and μ satisfy the $\mathcal{A}_\infty$–relations. For the second statement, for $f \in \mathrm{Mor}({}_{\mathcal{A}}M_{\mathcal{A}}, {}_{\mathcal{A}}N_{\mathcal{A}})$ define

$$CH(f)\colon CH({}_{\mathcal{A}}M_{\mathcal{A}}) \to CH({}_{\mathcal{A}}N_{\mathcal{A}})$$

by:

$CH(f) =$ 

(As before, a bundle of strands running off the right edge of a diagram comes back on the right.) It is easy to verify that this definition makes CH into a *dg* functor, and that $CH(\mathbb{I}) = \mathbb{I}$. The result follows. $\square$

Definition 2.3.43 The homology of $CH(M)$, denoted by $HH(M)$, is called the *Hochschild homology* of the bimodule M.

We next show that, in certain cases, the Hochschild homology can be computed from a much smaller complex.

First, some terminology. Given a $\boldsymbol{k}$–bimodule N, let $[N,\boldsymbol{k}]$ be the submodule of N generated by all elements of the form $nk-kn$, where $n\in N$ and $k\in\boldsymbol{k}$, and let $N^\circ = N/[N,\boldsymbol{k}]$ denote the vector space quotient of N by $[N,\boldsymbol{k}]$. We call N° the *cyclicization* of N. Note that for bimodules M and N, any $(\boldsymbol{k},\boldsymbol{k})$–bilinear map $N\to M$ descends to a linear map $N^\circ\to M^\circ$.

Now, let ${}^{\mathcal{A}}N_{\mathcal{A}}$ be a type *DA* structure, and ${}_{\mathcal{A}}M_{\mathcal{A}} = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}\boxtimes {}^{\mathcal{A}}N_{\mathcal{A}}$ the associated type *AA* bimodule. Let

$$\delta^1_j\colon N\otimes A_+^{\otimes j}\to A\otimes N$$

denote the structure maps for ${}^{\mathcal{A}}N_{\mathcal{A}}$. We will assume that ${}^{\mathcal{A}}N_{\mathcal{A}}$ is bounded in the sense of Definition 2.2.46. The vector space N° will be the underlying vector space for the smaller model for the Hochschild complex of ${}_{\mathcal{A}}M_{\mathcal{A}}$.

The maps δ^1_j fit together to give a degree -1 map

$$\widetilde{\delta}\colon N\otimes T^*(A_+[1])\to A[1]\otimes N\otimes T^*(A_+[1]),$$

defined by

$$\widetilde{\delta}(x\otimes a_1\otimes\cdots\otimes a_m)=\sum_{0\le j\le m}\delta^1_j(x\otimes\cdots a_1\otimes\cdots\otimes a_j)\otimes a_{j+1}\otimes\cdots\otimes a_m.$$

There is an $\mathbb{F}_2$–linear, degree 1 cyclic rotation map

$$R\colon (A[1]\otimes N\otimes T^*(A_+[1]))^\circ\to (N\otimes T^*(A_+[1]))^\circ$$

defined by

$$R(a_0\otimes x\otimes a_1\otimes\cdots\otimes a_m)=x\otimes a_1\otimes\cdots\otimes a_m\otimes[(\mathbb{I}-\epsilon)(a_0)],$$

where ϵ denotes the augmentation on $\mathcal{A}$. Note that the map R would not make sense without cyclicizing: if $\iota_1,\iota_2\in\boldsymbol{k}$ are orthogonal idempotents, and $a\in A$ and $n\in N$ are such that $a=\iota_1 a\iota_1$ and $n=\iota_2 n\iota_1$ then $a\otimes n=0$ but $R(a\otimes n)=n\otimes a\neq 0$. We can similarly define a map

$$\overline{R}\colon (T^*(A_+[1])\otimes N\otimes T^*(A_+[1]))\to (N\otimes T^*(A_+[1]))^\circ$$

by

$$\overline{R}(a_0\otimes\cdots\otimes a_n\otimes x\otimes a_{n+1}\otimes\cdots\otimes a_m)=x\otimes a_{n+1}\otimes\cdots\otimes a_m\otimes a_0\otimes\cdots\otimes a_n.$$

The map $\mathbb{I}-\epsilon$ and its tensor powers will come up frequently, so we let $\pi\colon A[1]^{\otimes k}\to A_+[1]^{\otimes k}$ (respectively $\pi\colon T^*(A[1])\to T^*(A_+[1])$) denote $(\mathbb{I}-\epsilon)^{\otimes k}$ (respectively

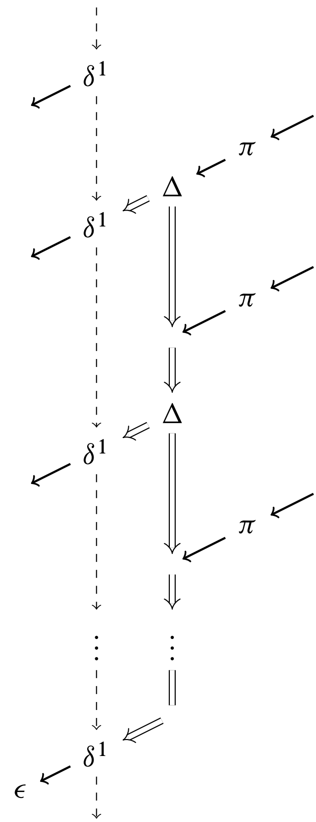
$\bigoplus_k (\mathbb{I}-\epsilon)^{\otimes k}$). We will need also the obvious inclusion map $\iota\colon N \to N \otimes T^*(A_+[1])$, and its cyclicization, which we also denote by ι.

Provided that ${}^{\mathcal{A}}N_{\mathcal{A}}$ is bounded, these ingredients can be assembled to form a linear map $\widetilde{\partial}\colon N^\circ \to N^\circ$ defined by

$$\delta^1 = \sum_{n=1}^{\infty} \epsilon \circ \widetilde{\delta} \circ \overbrace{(R\circ\widetilde{\delta})\circ\cdots\circ(R\circ\widetilde{\delta})}^{n-1} \circ \iota. \tag{2.3.44}$$

We can draw the map $\widetilde{\partial}$ as:

(2.3.45) $\widetilde{\partial} = \sum$

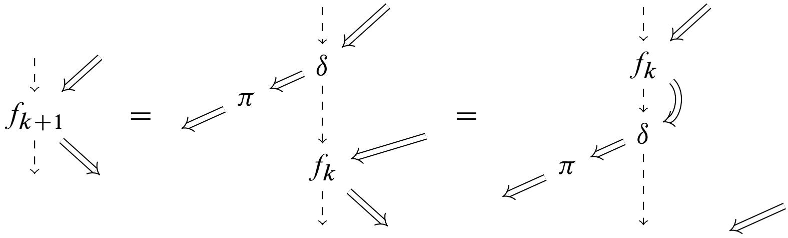


Lemma 2.3.46 *Assume ${}^{\mathcal{A}}N_{\mathcal{A}}$ is bounded. For any $x \in {}^{\mathcal{A}}N_{\mathcal{A}}$ the sum defining $\widetilde{\partial}(x)$ is finite.*

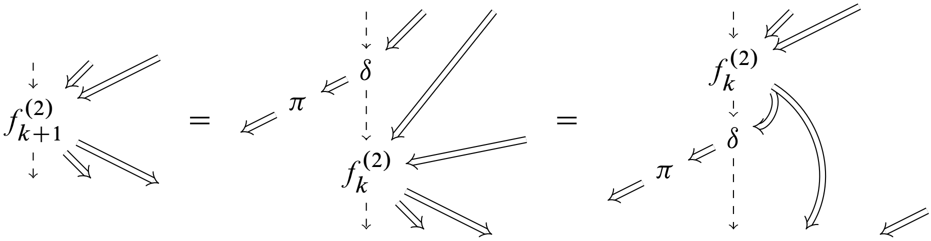
Proof This is immediate from the definitions. $\square$

Our next goal is to show $\widetilde{\partial}^2 = 0$. In order to do this, we develop some more notation.

For N and $\mathcal{A}$ as above, define maps $f_k\colon (N \otimes T^*(A_+[1]))^\circ \to (N \otimes T^*(A_+[1]))^\circ$ by defining f_0 to be the identity map and defining f_k by the diagram:



Similarly, define maps $f_k^{(2)}\colon (N \otimes T^*(A_+[1]) \otimes T^*(A_+[1]))^\circ \to (N \otimes T^*(A_+[1]) \otimes T^*(A_+[1]))^\circ$ by letting $f_0^{(2)}$ be the identity map and:



Lemma 2.3.47 *Assume that ${}^{\mathcal{A}}N_{\mathcal{A}}$ is a bounded type DA structure. Then given any $x \in {}^{\mathcal{A}}N_{\mathcal{A}}$, there is a constant $C = C(x)$ with the property that for all $k \geq C$, $f_k(x, \cdot\,)) = f_{k+1}(x, \cdot\,))$ and moreover $f_k(x \otimes a) = 0$ if $a \in (A_+[1])^{\otimes i}$ for some $i > 0$. Consequently, for k sufficiently large the f_k are induced by a map*

$$f_\infty\colon N^\circ \to (N \otimes T^*(A_+[1]))^\circ$$

in the sense that for $k > C(x)$,

$$f_k(x, a) = \begin{cases} f_\infty^{(2)}(xa) & a \in \boldsymbol{k} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, given $x \in {}^{\mathcal{A}}N_{\mathcal{A}}$, there is a constant $C = C(x)$ such that for $k \geq C$, $f_k^{(2)}(x, \cdot, \cdot) = f_{k+1}^{(2)}(x, \cdot, \cdot)$; $f_k(x \otimes a \otimes b) = 0$ if a or b is in $(A_+[1])^{\otimes i}$ for some $i > 0$; and so the $f_k^{(2)}$ are induced by a map

$$f_\infty^{(2)}\colon N^\circ \to (N \otimes T^*(A_+[1]) \otimes T^*(A_+[1]))^\circ$$

in the same sense.

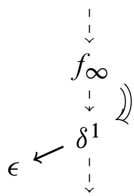
Proof Recall that $\delta^N = \mathbb{I}_N + \delta^1 + \cdots$. By the definition of admissibility there is a C so that $\delta^n = 0$ for $n > C$. In f_k the operator δ occurs k times in a row, and consequently all but C of these terms must be $\mathbb{I}_N$. This implies that $f_k(x,\cdot)) = f_{k+1}(x,\cdot))$ for sufficient large k. To see that f_k is induced from f_∞, note that $\mathbb{I}_N(x \otimes a) = 0$ if $a \notin T^0(A_+[1]) = \boldsymbol{k}$.

The arguments for the corresponding statements about $f_k^{(2)}$ are similar. □

With this notation, we are now able to reinterpret the Hochschild differential:

Lemma 2.3.48 *The operator $\widetilde{\partial}$ is given by:*

(2.3.49) $\widetilde{\partial} =$

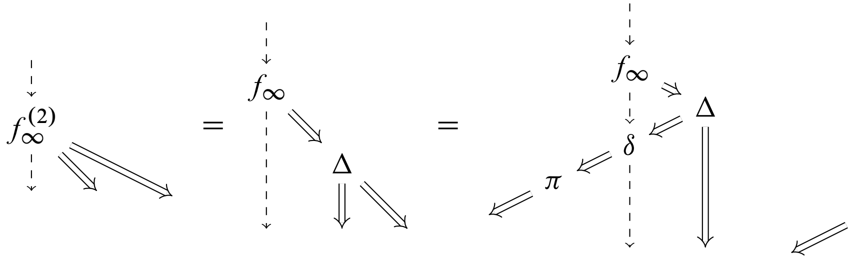


Proof Each term in (2.3.49) can be expanded to give a term in (2.3.45) (by expanding δ into copies of δ^1). We must show that each term in (2.3.45) occurs exactly once this way. The idea is to read the expression in (2.3.45) from bottom to top. More precisely, consider a term in (2.3.45); we want to write this term in the form of (2.3.49). Label the δ^1's occurring in order as $\delta^1_{(1)},\dots,\delta^1_{(k)}$. The operation $\delta^1_{(k)}$ has a sequence of inputs $a_1,\dots,a_\ell$. The input a_1 came from some $\delta^1_{(i)}$ for some $i<k$. Let $\delta_{(1)} = \delta^1_{(k-1)} \circ R \circ \cdots \circ R \circ \delta^1_{(i)}$. Similarly, the operation $\delta^1_{(i-1)}$ has inputs $a'_1,\dots,a'_l$, where a'_1 is produced by $\delta^1_{(j)}$ for some $j<i-1$. Let $\delta_{(2)} = \delta^1_{(i-1)} \circ R \circ \cdots \circ R \circ \delta^1_{(j-1)}$. Repeat, producing operations $\delta_{(3)},\dots,\delta_{(m)}$. Then the term in (2.3.49) with operations $\delta_{(m)},\dots,\delta_{(1)},\delta^1_{(k)}$ corresponds to the given term in (2.3.45), and moreover it is clear from the construction that this is the unique sequence of operations corresponding to the given term. □

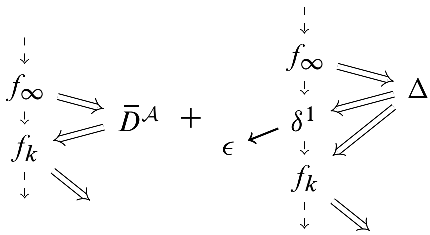
Next, we summarize the properties of the operators we have introduced:

Lemma 2.3.50 *With notation as above:*

(1) *We have $f_\infty^{(2)} = (\mathbb{I} \otimes \Delta) \circ f_\infty = [(\overline{R} \circ \delta) \otimes \mathbb{I}] \circ (\mathbb{I} \otimes \Delta) \circ f_\infty$. That is:*

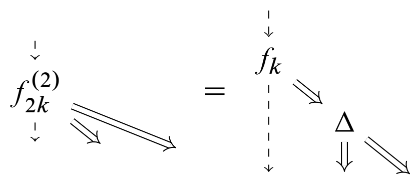


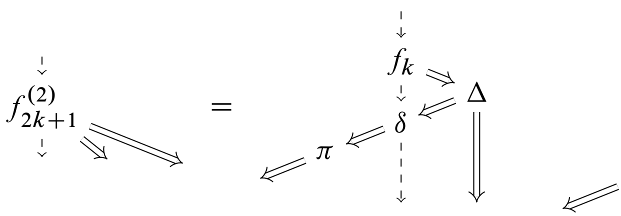
(2) *The operator*



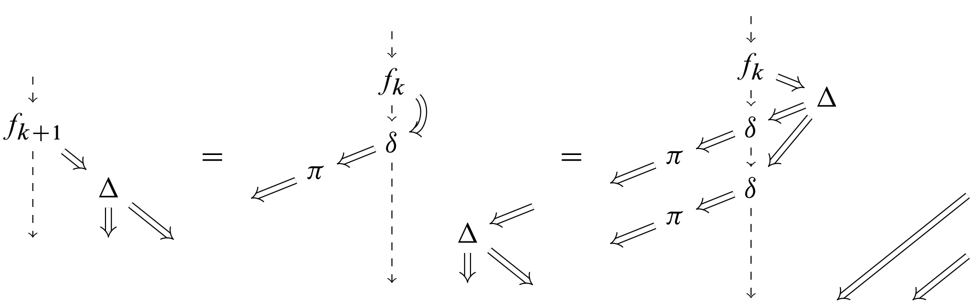
is independent of k.

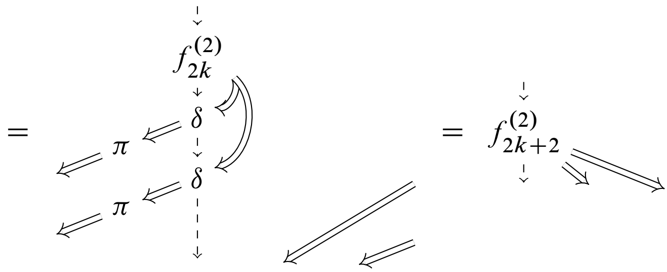
Proof We prove part (1) by induction on k, showing more generally that:





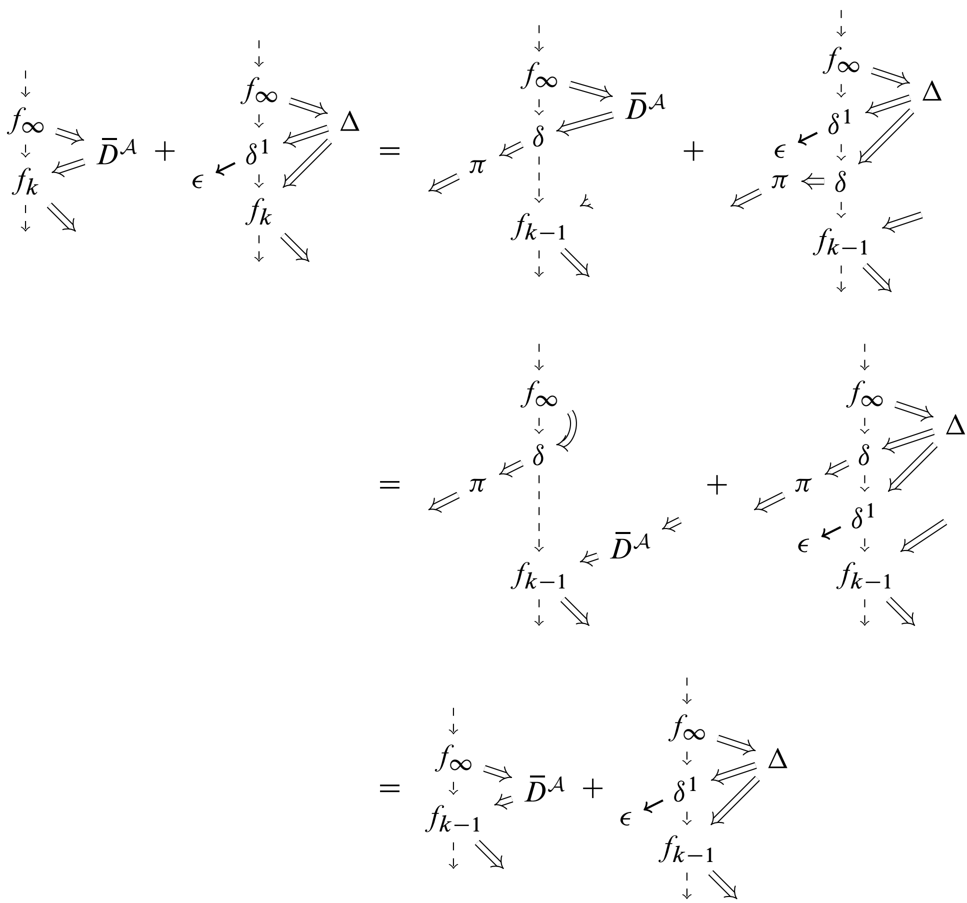
(These identities relate to the portion of f_k etc with no algebra inputs.) Indeed, the second equality follows from the first and the definition of $f_k^{(2)}$. For the first equality, the $k = 0$ case is trivial: both sides reduce to the identity map on N. For the inductive step:





where the first equality uses the inductive definition of f_{k+1}; the second equality uses the fact that δ and π respect the coalgebra structure of T^*A_+; the third equality uses the inductive hypothesis; and the fourth uses the inductive definition of $f^{(2)}_{2k+2}$ (twice).

For part (2), observe that



where the first equality uses the definition of f_k, the second uses the type *DA* structure relation, combined with the fact that

$$\bar{D}^{\mathcal{A}} \circ \pi + \pi \circ \bar{D}^{\mathcal{A}} = \epsilon \otimes \pi + \pi \otimes \epsilon$$

(which follows from the assumption that $\mathcal{A}$ is strictly unital), and the third uses part (1) and the definition of f_∞. □

Proposition 2.3.51 *If $({}^{\mathcal{A}}N_{\mathcal{A}}, \delta)$ is a bounded type DA structure, then $\widetilde{\partial}^2 = 0$.*

Proof To see that $\widetilde{\partial}^2 = 0$, we prove that for any k:

$$\widetilde{\partial}^2 = \begin{tikzcd}[row sep=small, column sep=small]
 & {} \ar[d, dashed] & \\
 & f_\infty \ar[dr, Rightarrow] \ar[dd, dashed] & \\
 & & \bar{D}^{\mathcal{A}} \ar[dl, Rightarrow] \\
 & f_k \ar[dd, dashed] \ar[dd, Rightarrow, bend left] & \\
 & & \\
 & \delta^1 \ar[dl] \ar[d, dashed] & \\
 \epsilon & {} &
\end{tikzcd}
+
\begin{tikzcd}[row sep=small, column sep=small]
 & {} \ar[d, dashed] & \\
 & f_\infty \ar[dr, Rightarrow] \ar[dd, dashed] & \\
 & & \Delta \ar[dl, Rightarrow] \ar[ddl, Rightarrow] \\
 \epsilon & \delta^1 \ar[l] \ar[d, dashed] & \\
 & f_k \ar[d, dashed] \ar[d, Rightarrow, bend left] & \\
 & \delta^1 \ar[dl] \ar[d, dashed] & \\
 \epsilon & {} &
\end{tikzcd} \tag{2.3.52}$$

Indeed, for $k \gg 0$, the first term of (2.3.52) vanishes (because of the $\bar{D}^{\mathcal{A}}$ followed by the f_k; see Lemma 2.3.47), while, in light of Lemma 2.3.47, the second term reduces to the reinterpretation of $\widetilde{\partial}$ from Lemma 2.3.48.

On the other hand, it is immediate from part (2) of Lemma 2.3.50 that the expression on the right of (2.3.52) is independent of k. For $k = 0$, (2.3.52) reduces to

$$\widetilde{\partial}^2 = \begin{tikzcd}[row sep=small, column sep=small]
 & {} \ar[d, dashed] & \\
 & f_\infty \ar[dr, Rightarrow] \ar[dd, dashed] & \\
 & & \bar{D}^{\mathcal{A}} \ar[dl, Rightarrow] \\
 & \delta^1 \ar[dl] \ar[d, dashed] & \\
 \epsilon & {} &
\end{tikzcd}
+
\begin{tikzcd}[row sep=small, column sep=small]
 & {} \ar[d, dashed] & \\
 & f_\infty \ar[dr, Rightarrow] \ar[dd, dashed] & \\
 & & \Delta \ar[dl, Rightarrow] \ar[ddl, Rightarrow] \\
 \epsilon & \delta^1 \ar[l] \ar[d, dashed] & \\
 & \delta^1 \ar[dl] \ar[d, dashed] & \\
 \epsilon & {} &
\end{tikzcd}$$

which is zero by the type *DA* structure relation (with one algebra element output). □

We let $\widetilde{CH}({}^{\mathcal{A}}N_{\mathcal{A}}) = (N/[\boldsymbol{k}, N], \widetilde{\partial})$ denote the chain complex associated as above to the type *DA* structure ${}^{\mathcal{A}}N_{\mathcal{A}}$.

Similarly, given a bounded type *DA* morphism between two bounded type *DA* structures $f\colon {}^{\mathcal{A}}N_{\mathcal{A}} \to {}^{\mathcal{A}}N'_{\mathcal{A}}$, define a map $\widetilde{CH}(f)\colon N/[\boldsymbol{k}, N] \to N'/[\boldsymbol{k}, N']$ by

$$\widetilde{CH}(f) = \sum_{n,m=0}^{\infty} \epsilon \circ \overbrace{(\widetilde{\delta} \circ R) \circ \cdots \circ (\widetilde{\delta} \circ R)}^{n} \circ f \circ \overbrace{(R \circ \widetilde{\delta}) \circ \cdots \circ (R \circ \widetilde{\delta})}^{m} \circ \iota.$$

Proposition 2.3.53 *The assignment $\widetilde{CH}$ is an $\mathcal{A}_\infty$ functor from the category of bounded type DA bimodules to $\mathsf{Mod}_{\boldsymbol{k}}$, the category of chain complexes over $\boldsymbol{k}$. In particular, if $f\colon {}^{\mathcal{A}}N_{\mathcal{A}} \to {}^{\mathcal{A}}N'_{\mathcal{A}}$ is a bounded type DA homomorphism, then $\widetilde{CH}(f)$ is a chain map, and if f and f' are homotopic morphisms, then the maps $\widetilde{CH}(f)$ and $\widetilde{CH}(f')$ are chain homotopic.*

Proof The proof is essentially the same as the proof that $\widetilde{\partial}^2 = 0$ (Proposition 2.3.51), and we leave it to the reader. □

The relation between $\widetilde{CH}({}^{\mathcal{A}}N_{\mathcal{A}})$ and $CH({}_{\mathcal{A}}M_{\mathcal{A}})$ is similar to the relation between $\boxtimes$ and $\widetilde{\otimes}$ (Proposition 2.3.18). In particular:

Proposition 2.3.54 *Let $\mathcal{A}$ be a dg algebra.*

(1) *Suppose that ${}_{\mathcal{A}}M_{\mathcal{A}}$ is a type AA module. Then ${}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}M_{\mathcal{A}}$ is a bounded type DA structure and the complex $CH({}_{\mathcal{A}}M_{\mathcal{A}})$ is isomorphic to the complex*

$$\widetilde{CH}({}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}M_{\mathcal{A}}).$$

(2) *Suppose that ${}^{\mathcal{A}}N_{\mathcal{A}}$ is a bounded type DA structure. Then the complex $\widetilde{CH}({}^{\mathcal{A}}N_{\mathcal{A}})$ is homotopy equivalent to the complex $CH({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{A}})$.*

Proof Part (1) is immediate from the definitions. Part (2) follows from part (1), which gives the first isomorphism in the string

$$CH({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{A}}) \cong \widetilde{CH}({}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{A}}) \simeq \widetilde{CH}({}^{\mathcal{A}}N_{\mathcal{A}});$$

together with the following observation: since ${}^{\mathcal{A}}N_{\mathcal{A}}$ is bounded, the natural map

$$\kappa \otimes \mathbb{I}_N {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{A}} \to {}^{\mathcal{A}}N_{\mathcal{A}}$$

is a *bounded* homotopy equivalence (a fact which can be seen by looking at the maps from Proposition 2.3.22). In view of this fact, Proposition 2.3.53 supplies the second homotopy equivalence. □

Remark 2.3.55 The extension of Proposition 2.3.54 to $\mathcal{A}_\infty$–algebras is straightforward; the reason that we restrict to the *dg* case is that we have not defined the bar resolution more generally.

2.4 Equivalences of categories

In Section 2 we introduced many different categories of modules and bimodules. In Section 2.3.3 we showed that the *dg* categories $\mathsf{Mod}^{\mathcal{A}}$ and $\mathsf{Mod}_{\mathcal{A}}$ are quasiequivalent, and also corresponding statements for bimodules; in particular, their homological categories $\mathsf{H}(\mathsf{Mod}_{\mathcal{A}})$ and $\mathsf{H}(\mathsf{Mod}^{\mathcal{A}})$ are equivalent triangulated categories. In this section we continue to tame the multitude, showing that:

- If $\mathcal{A}$ is a *dg* algebra then $\mathsf{H}(\mathsf{Mod}_{\mathcal{A}})$ is triangle-equivalent to the derived category of *dg* $\mathcal{A}$–modules.
- If $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic *dg* (or, more generally, $\mathcal{A}_\infty$–) algebras then $\mathsf{Mod}_{\mathcal{A}}$ and $\mathsf{Mod}_{\mathcal{B}}$ are quasiequivalent.

The results in this section are not purely of aesthetic interest: various of them will be used in Sections 8 and 9, with consequences that are useful for computation.

2.4.1 Homotopy equivalence and quasi-isomorphism Let $\mathcal{A}$ be a differential graded algebra. Throughout this section, the word *honest* is used to distinguish ordinary ("honest") differential graded modules from $\mathcal{A}_\infty$–modules. We can consider the following different models for the derived category of $\mathcal{A}$–modules:

- The category $\mathcal{D}_{H,\mathrm{qi}}$ with objects honest *dg* $\mathcal{A}$–modules and morphisms obtained by localizing the homotopy category of honest module maps with respect to quasi-isomorphisms. (Recall that a quasi-isomorphism is a chain map inducing an isomorphism on homology.)
- The category $\mathcal{D}_{H,\infty}$ with objects honest *dg* $\mathcal{A}$–modules and morphisms $\mathcal{A}_\infty$–homotopy classes of $\mathcal{A}_\infty$–morphisms.
- The category $\mathcal{D}_{H,\infty\mathrm{qi}}$ obtained from $\mathcal{D}_{H,\infty}$ by localizing with respect to $\mathcal{A}_\infty$–quasi-isomorphisms.
- The category $\mathcal{D}_{\infty,\infty}$ with objects $\mathcal{A}_\infty$–modules and morphisms $\mathcal{A}_\infty$–homotopy classes of $\mathcal{A}_\infty$–morphisms. (This has been denoted $\mathsf{H}(\mathsf{Mod}_{\mathcal{A}})$ elsewhere in this section.)
- The category $\mathcal{D}_{\infty,\infty\mathrm{qi}}$ obtained from $\mathcal{D}_{\infty,\infty}$ by localizing with respect to $\mathcal{A}_\infty$–quasi-isomorphisms.

Proposition 2.4.1 *The categories $\mathcal{D}_{H,\mathrm{qi}}$, $\mathcal{D}_{H,\infty}$, $\mathcal{D}_{\infty,\infty}$ and $\mathcal{D}_{\infty,\infty\mathrm{qi}}$ are all equivalent triangulated categories. Corresponding statements hold for categories of bimodules.*

Proof The proof is standard so we will only sketch it. The main point is that all of these categories are equivalent to the full subcategory of honest, projective modules. To see this, one observes that:

(1) The bar resolution functor $M \mapsto \mathrm{Bar}(M)$ maps the category of ($\mathcal{A}_\infty$) modules into the subcategory of projective modules, and takes $\mathcal{A}_\infty$–module homomorphisms to honest *dg* module homomorphisms.

(2) Any map between projective *dg* modules inducing an isomorphism on homology has a homotopy inverse (see eg [5, Lemma 10.12.2.2]).

(3) The canonical map $\mathrm{Bar}(M) \to M$ is an isomorphism in any of $\mathcal{D}_{H,\mathrm{qi}}$, $\mathcal{D}_{H,\infty}$, $\mathcal{D}_{\infty,\infty}$ and $\mathcal{D}_{\infty,\infty\mathrm{qi}}$, and so the bar resolution functor is naturally isomorphic to the identity functor.

These, together, imply the result. □

Note that, concretely, Proposition 2.4.1 implies that every quasi-isomorphism of $\mathcal{A}_\infty$–modules is a homotopy equivalence. The analogue for (bounded) type D structures, and for bimodules, is given in Corollary 2.4.4.

The honest homotopy category of *dg* $\mathcal{A}$–modules is not in general equivalent to the derived category; it is part (3) of the proof that breaks down.

Proposition 2.4.2 *With respect to the identification from Proposition 2.4.1 the tensor product $\widetilde{\otimes}$ of Definition 2.3.17 is identified with the usual derived tensor product.*

Proof This is clear from the definitions and the fact that the bar resolution of a module is projective. □

Definition 2.4.3 A map $f\colon {}^{\mathcal{A}}M \to {}^{\mathcal{A}}N$ of type D modules is a *quasi-isomorphism* if the induced map $\mathbb{I}_{\mathcal{A}} \boxtimes f\colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M \to {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ is a quasi-isomorphism. Similarly, a map $f\colon {}^{\mathcal{A}}M_{\mathcal{B}} \to {}^{\mathcal{A}}N_{\mathcal{B}}$ of type *DA* modules is called a *quasi-isomorphism* if the induced map $\mathbb{I}_{\mathcal{A}} \boxtimes f\colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M_{\mathcal{B}} \to {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N_{\mathcal{B}}$ is a quasi-isomorphism, and a map $f\colon {}^{\mathcal{A}}M^{\mathcal{B}} \to {}^{\mathcal{A}}N^{\mathcal{B}}$ of type *DD* modules is called a *quasi-isomorphism* if the induced map $\mathbb{I}_{\mathcal{A}} \boxtimes f \boxtimes \mathbb{I}_{\mathcal{B}}\colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \to {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}$ is a quasi-isomorphism.

Corollary 2.4.4 *Let ${}^{\mathcal{A}}M, {}^{\mathcal{A}}N \in {}^{\mathcal{A}}\mathsf{Mod}$. A map $f\colon {}^{\mathcal{A}}N \to {}^{\mathcal{A}}N$ is a quasi-isomorphism if and only if it is a homotopy equivalence. The same holds if M and N are instead in ${}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$, ${}^{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}}$ or ${}^{\mathcal{A}}\mathsf{Mod}^{\mathcal{B}}$.*

Proof For type D structures, this is immediate from Propositions 2.3.18 and 2.4.1. The bimodule analogues follow from these results together with Proposition 2.4.11, below. □

2.4.2 Induction and restriction Consider a map of $\mathcal{A}_\infty$–algebras $\phi\colon \mathcal{A} \to \mathcal{B}$. Associated to ϕ are restriction and induction functors

$$\mathrm{Rest}_\phi\colon \mathsf{Mod}_{\mathcal{B}} \to \mathsf{Mod}_{\mathcal{A}},$$
$${}^{\phi}\mathrm{Induct}\colon {}^{\mathcal{A}}\mathsf{Mod} \to {}^{\mathcal{B}}\mathsf{Mod},$$

which are defined by $\cdot \boxtimes {}^{\mathcal{B}}[\phi]_{\mathcal{A}}$ and ${}^{\mathcal{B}}[\phi]_{\mathcal{A}} \boxtimes \cdot$ respectively, where ${}^{\mathcal{B}}[\phi]_{\mathcal{A}}$ is as defined in Definition 2.2.48.[3] One can define restriction functors of left modules and induction functors of right type D structures similarly, using ${}_{\mathcal{A}}[\phi]^{\mathcal{B}}$ instead of ${}^{\mathcal{B}}[\phi]_{\mathcal{A}}$.

It is obvious from their definitions that these functors behave well with respect to composition of algebra homomorphisms:

Lemma 2.4.5 *If $\phi\colon \mathcal{A} \to \mathcal{B}$ and $\psi\colon \mathcal{B} \to \mathcal{C}$ are $\mathcal{A}_\infty$–algebra homomorphisms then*

$$\mathrm{Rest}_\phi \circ \mathrm{Rest}_\psi = \mathrm{Rest}_{\psi\circ\phi},$$
$${}^{\psi}\mathrm{Induct} \circ {}^{\phi}\mathrm{Induct} = {}^{\psi\circ\phi}\mathrm{Induct}.$$

For most of the rest of this section we restrict to the case that $\mathcal{A}$ and $\mathcal{B}$ are *dg* algebras.

Lemma 2.4.6 *Let $\phi\colon \mathcal{A} \to \mathcal{B}$ be an $\mathcal{A}_\infty$ morphism. Then there is a natural map of $\mathcal{A}$–$\mathcal{A}$ type AA bimodules*

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}_{\mathcal{A}}[\phi]^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}[\phi]_{\mathcal{A}}.$$

Similarly, if $\mathcal{A}$ and $\mathcal{B}$ are dg algebras then there is a natural map of $\mathcal{B}$–$\mathcal{B}$ type DD bimodules

$${}^{\mathcal{B}}[\phi]_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}[\phi]^{\mathcal{B}} \to {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}}.$$

When ϕ is a quasi-isomorphism, then these two natural maps are quasi-isomorphisms.

[3] Since ${}^{\mathcal{A}}\mathsf{Mod}$ and ${}^{\mathcal{B}}\mathsf{Mod}$ are $\mathcal{A}_\infty$–categories, the induction functor is, of course, an $\mathcal{A}_\infty$–functor; if $\mathcal{A}$ and $\mathcal{B}$ are *dg* algebras then this complication disappears.

Proof Define a map $f\colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}_{\mathcal{A}}[\phi]^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}[\phi]_{\mathcal{A}}$ by

$$f_{i,1,j}(a_1,\dots,a_i,a',a''_1,\dots,a''_j) = 1_{\boldsymbol{k}} \otimes \phi_{i+1+j}(a_1,\dots,a_i,a',a''_1,\dots,a''_j) \otimes 1_{\boldsymbol{k}}.$$

Using the fact that

$$\begin{aligned} m_{[\phi]\boxtimes\mathcal{B}\boxtimes[\phi]}&(a_1,\dots,a_i,1_{\boldsymbol{k}}\otimes b\otimes 1_{\boldsymbol{k}},a''_1,\dots,a''_j) \\ &= 1_{\boldsymbol{k}} \otimes \mu_{\mathcal{B}}(F^{\phi}(a_1,\dots,a_i),b,F^{\phi}(a''_1,\dots,a''_j)) \otimes 1_{\boldsymbol{k}} \end{aligned}$$

it is easy to verify that f is an $\mathcal{A}_\infty$–bimodule homomorphism (ie $d(f)=0$). Furthermore, if ϕ_1 induces an isomorphism on homology, $f_{0,1,0}$ induces an isomorphism on homology, ie f is a quasi-isomorphism.

For the second part, define a map $g\colon {}^{\mathcal{B}}[\phi]_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}[\phi]^{\mathcal{B}} \to {}^{\mathcal{B}}\mathrm{Bar}^{\mathcal{B}}$ of separated type *DD* modules by

$$g(1_{\boldsymbol{k}} \otimes [a_1|\cdots|a_k] \otimes 1_{\boldsymbol{k}}) = 1_{\boldsymbol{k}} \otimes [F^{\phi}(a_1\otimes\cdots\otimes a_k)] \otimes 1_{\boldsymbol{k}}.$$

Then it is easy to see that g is a type *DD* homomorphism (ie $d(g)=0$). Moreover, if ϕ is a quasi-isomorphism then F^{ϕ} is a quasi-isomorphism, and hence g is a quasi-isomorphism. □

Definition 2.4.7 Fix $\mathcal{A}_\infty$–algebras $\mathcal{A}$ and $\mathcal{B}$. A *quasi-inverse* to a type *DA* bimodule ${}^{\mathcal{A}}P_{\mathcal{B}}$ is a type *DA* bimodule ${}^{\mathcal{B}}Q_{\mathcal{A}}$ with the property that ${}^{\mathcal{A}}P_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Q_{\mathcal{A}} \simeq {}^{\mathcal{A}}\mathbb{I}_{\mathcal{A}}$ and ${}^{\mathcal{B}}Q_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}P_{\mathcal{B}} \simeq {}^{\mathcal{B}}\mathbb{I}_{\mathcal{B}}$. A type *DA* bimodule is *quasi-invertible* if it has a quasi-inverse.

There are obvious analogous definitions if P is a type *AA* and Q is a type *DD* module, and if P and Q are both type *AA* modules (using $\widetilde{\otimes}$ instead of $\boxtimes$).

Although a quasi-isomorphism between $\mathcal{A}_\infty$–algebras need not be invertible, it does have a quasi-inverse:

Proposition 2.4.8 *If $\phi\colon \mathcal{A}\to\mathcal{B}$ is a quasi-isomorphism, then ${}^{\mathcal{A}}[\phi]_{\mathcal{B}}$ is quasi-invertible.*

Proof We will see the quasi-inverse to $M = {}^{\mathcal{A}}[\phi]_{\mathcal{B}}$ is $N = {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}[\phi]^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$. Indeed,

$$N\boxtimes M \cong {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}[\phi]^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}[\phi]_{\mathcal{B}} \simeq {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \simeq {}^{\mathcal{B}}[\mathbb{I}]_{\mathcal{B}},$$

where the first quasi-isomorphism uses Lemma 2.4.6 and the second uses Lemma 2.3.19.

Similarly,

$$M\boxtimes N \cong {}^{\mathcal{A}}[\phi]_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathrm{Bar}(\mathcal{B})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}[\phi]^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \simeq {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \simeq {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}},$$

where again the first quasi-isomorphism uses Lemma 2.4.6 and the second uses Lemma 2.3.19. □

One reason to be interested in quasi-invertible bimodules is the following:

Lemma 2.4.9 *If ${}^{\mathcal{B}}M_{\mathcal{A}}$ is quasi-invertible then the functors ${}^{\mathcal{B}}M_{\mathcal{A}} \boxtimes \cdot$ and $\cdot \boxtimes {}^{\mathcal{B}}M_{\mathcal{A}}$ are quasiequivalences of categories. Analogous statements hold for type DD and AA modules. An analogous statement holds for type AA modules with $\widetilde{\otimes}$ in place of $\boxtimes$.*

The proof is straightforward.

We have now proved the *dg* case of the following proposition, which we state merely for the reader's edification (compare [5, Theorem 10.12.5.1]):

Proposition 2.4.10 *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{A}_\infty$–algebras and $\phi\colon \mathcal{A} \to \mathcal{B}$ a quasi-isomorphism. Let* $\mathrm{Induct}_\phi\colon \mathsf{Mod}_{\mathcal{A}} \to \mathsf{Mod}_{\mathcal{B}}$ *denote the functor*

$$M_{\mathcal{A}} \mapsto \mathrm{Induct}^{\phi}(M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}\mathrm{Bar}(\mathcal{A})^{\mathcal{A}})^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}.$$

Then Rest_ϕ *and* Induct_ϕ *are inverse quasiequivalences of dg categories. In particular, the derived categories* $\mathsf{H}(\mathsf{Mod}_{\mathcal{A}})$ *and* $\mathsf{H}(\mathsf{Mod}_{\mathcal{B}})$ *are equivalent. Corresponding statements apply to categories of type D structures and bimodules of all kinds.*

The argument we have given works in general, with the only obstruction being the fact that we have not defined type *DD* modules over $\mathcal{A}_\infty$–algebras. This is not a serious difficulty for the purpose of this result; cf Remark 2.3.28.

2.4.3 Bimodules and $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$–modules Let $\mathcal{A}$ and $\mathcal{B}$ be differential graded algebras. An honest $(\mathcal{A}, \mathcal{B})$–bimodule is exactly the same as a right $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$–module. For $\mathcal{A}_\infty$–bimodules, this is not quite true: a $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$–module has distinct operations $m_3(x, \overline{\gamma} \otimes 1, 1 \otimes \eta)$ and $m_3(x, 1 \otimes \eta, \overline{\gamma} \otimes 1)$, both of which intuitively correspond to the operation $m_3(\eta, x, \gamma)$ on an $\mathcal{A}_\infty$–bimodule.

Nevertheless, the two categories are equivalent. More precisely, let M be a right $\mathcal{A}_\infty$ $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$–module. We define an $\mathcal{A}_\infty$ $(\mathcal{A}, \mathcal{B})$–bimodule structure on M as follows. Given sequences $S_{\mathcal{A}} = (a_1, \dots, a_m)$ of elements of $\mathcal{A}$ and $S_{\mathcal{B}} = (b_1, \dots, b_n)$ of elements of $\mathcal{B}$, we say that a sequence $S_{\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}} = (c_1, \dots, c_{m+n})$ of elements of $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}$ *interleaves* $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ if:

- Each c_k is either $\overline{a}_i \otimes 1$ or $1 \otimes b_i$.
- The sequence obtained from $\{c_i\}$ by forgetting the $1 \otimes b_i$ is exactly $(\overline{a}_1 \otimes 1, \dots, \overline{a}_m \otimes 1)$.
- The sequence obtained from $\{c_i\}$ by forgetting the $\overline{a}_i \otimes 1$ is exactly $(1 \otimes b_1, \dots, 1 \otimes b_n)$.

The $(\mathcal{A},\mathcal{B})$–bimodule structure on M is defined by

$$m_{m,1,n}(a_1,\dots,a_m,x,b_1,\dots,b_n)=\sum_{\substack{(c_1,\dots,c_{m+n})\text{ interleaves}\\(a_1,\dots,a_m)\text{ and }(b_1,\dots,b_n)}} m_{m+n+1}(x,c_1,\dots,c_{m+n}).$$

It is routine to verify that these higher products do, in fact, make M into an $(\mathcal{A},\mathcal{B})$–bimodule. Moreover, this construction extends in an obvious way to maps, leading to a functor from $\mathcal{M}_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}}$ to ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$.

Proposition 2.4.11 *The categories* $\mathsf{H}(\mathsf{Mod}_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}})$ *and* $\mathsf{H}({}_{\mathcal{A}}\mathsf{Mod}_{\mathcal{B}})$ *are equivalent triangulated categories.*

Proof This follows from Proposition 2.4.1 for $(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})$ modules and for $(\mathcal{A},\mathcal{B})$ bimodules. □

2.5 Group-valued gradings

The gradings on Floer homology theories differ from gradings in classical homology or homological algebra in (at least) three important ways:

(1) It is often easiest to consider Floer complexes as relatively graded groups, rather than absolutely graded ones.

(2) The relative grading is usually only partially defined. That is, Floer chain complexes break up as direct sums in which there is a relative grading on each summand but no way to compare the gradings across summands.

(3) The relative gradings on Floer complexes are often cyclic, ie by $\mathbb{Z}/n$ rather than by $\mathbb{Z}$.

All three points lead to difficulties with homological algebra. In particular, there is no notion of a degree-0 morphism between two different relatively graded modules, nor is it clear how to define the cone of a morphism between relatively graded modules.

In the literature, points (1) and (2) are usually treated by fixing a lift of the (partially defined) relative grading to an absolute grading, or quantifying over all such lifts; see Seidel [34]. (Alternatively, for Heegaard Floer theory of rational homology spheres there is a natural absolute $\mathbb{Q}$–grading lifting the relative $\mathbb{Z}$–grading; see the second author and Szabó [30].) In papers where point (3) leads to difficulties, authors usually either restrict the generality of their results or work with a periodic $\mathbb{Z}$–graded lift of the $\mathbb{Z}/n$–graded module (although this, too, leads to difficulties with homological algebra).

None of these approaches seem satisfactory for bordered Floer theory. Indeed, we observed in [21] that there is a natural grading on the algebras involved in bordered Floer theory by noncommutative groups G, and on the modules involved by G–sets. This section is devoted to reviewing and expanding the algebraic framework of such gradings. While group-valued gradings have occurred before in the literature (see Năstăsescu and Van Oystaeyen [27] and Khovanov[15]), our perspective and goals seem somewhat different.

Partially-defined relative $\mathbb{Z}$– or $\mathbb{Z}/n$–gradings are a special case of the construction here. More dramatically, cyclic gradings arise naturally when taking tensor products of noncyclically-graded modules in the G–set graded context.

This section is organized as follows. In Section 2.5.1 we review the basic definitions of G–valued gradings from [21], and extend the theory to bimodules. Section 2.5.1 should provide enough background on noncommutative gradings for the reader with a little faith to understand most of the rest of this paper, except for Section 9. Section 2.5.2 extends the notion of *dg* categories as appropriate for categories of G–set graded modules; the generalization is called a $\mathbb{Z}$–set graded *dg* category. Section 2.5.3 then organizes the G–set graded modules into $\mathbb{Z}$–set-graded *dg* categories. This then allows one to extend easily the homological algebra introduced earlier in this section to G–set graded modules.

2.5.1 Basics of group-valued gradings We start by recalling some notions of group-valued gradings of algebras and modules from [21], and then generalize them somewhat. Note that all of our $\mathcal{A}_\infty$–algebras and modules have underlying vector spaces (or $\boldsymbol{k}$–modules), and all of the structure maps of $\mathcal{A}_\infty$–algebras and modules are maps between tensor products of vector spaces. So to grade these modules and speak about the degrees of structure maps it suffices to explain gradings of vector spaces and tensor products of vector spaces. We do so as follows:

Definition 2.5.1 Let (G,λ) be a pair of a group G and a distinguished element λ in the center of G. A *G–graded $\boldsymbol{k}$–bimodule* is a $\boldsymbol{k}$–bimodule V which is decomposed as a direct sum

$$V = \bigoplus_{g\in G} V_g.$$

We say that an element $v \in V_g$ is *homogeneous of degree* g and write $\mathrm{gr}(v) = g$.

If V and W are two G–graded $\boldsymbol{k}$–bimodules then $V \otimes W$ is itself G–graded by $\mathrm{gr}(v\otimes w) = \mathrm{gr}(v)\,\mathrm{gr}(w)$.

For a G–graded $\boldsymbol{k}$–bimodule V, the space $V[n]$ is a G–graded $\boldsymbol{k}$–bimodule with gradings shifted by λ^n: $V[n]_g = V_{\lambda^{-n}g}$.

A homomorphism $f\colon V \to W$ between G–graded $\boldsymbol{k}$–bimodules is *homogeneous of degree* $k \in \mathbb{Z}$ if for all $g \in G$, $f(V_g) \subset W_{g\lambda^k}$.

Using Definition 2.5.1, the definitions of $\mathcal{A}_\infty$–algebras and $\mathcal{A}_\infty$–algebra homomorphisms (Sections 2.1.1 and 2.1.2) carry over to the G–graded case without change. For example:

Definition 2.5.2 Let (G,λ) be a group with a distinguished central element. An *$\mathcal{A}_\infty$–algebra graded by* (G,λ) is an $\mathcal{A}_\infty$–algebra $\mathcal{A}$ with a grading gr by G (as a $\boldsymbol{k}$–bimodule), ie a decomposition $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, satisfying the following condition: For homogeneous elements a_i, we require that

$$\operatorname{gr}(\mu_j(a_1,\dots,a_j)) = \operatorname{gr}(a_1)\cdots\operatorname{gr}(a_j)\lambda^{j-2}. \tag{2.5.3}$$

Example 2.5.4 In the case that $G=\mathbb{Z}$ and $\lambda = 1$, Definition 2.5.2 reduces to a classical ($\mathbb{Z}$–graded) $\mathcal{A}_\infty$–algebra.

If $\mathcal{A}$ is G–graded, we could consider modules that are also G–graded. However, we prefer to consider modules which are graded by G–sets. Again, it suffices to explain the notion of $\boldsymbol{k}$–modules graded by G–sets:

Definition 2.5.5 Let (G,λ) be a group with a distinguished central element, and let S be a right G–set. An *S–graded $\boldsymbol{k}$–module* is a $\boldsymbol{k}$–module V which is decomposed as a direct sum

$$M = \bigoplus_{s\in S} V_s.$$

If V is an S–graded $\boldsymbol{k}$–module and W is a G–graded $\boldsymbol{k}$–bimodule, then $V\otimes W$ is itself S–graded by $\operatorname{gr}(v\otimes w) = \operatorname{gr}(v)\operatorname{gr}(w)$. Similarly if T is a left G–set and V is a T–graded $\boldsymbol{k}$–module, then $W\otimes V$ is T–graded by $\operatorname{gr}(w\otimes v) = \operatorname{gr}(w)\operatorname{gr}(v)$.

For any module V graded by a set with an action of λ, the space $V[n]$ is V with shifted grading: $V[n]_x = V_{\lambda^{-n}x}$. Note that we need not distinguish between left and right actions of λ since λ is central.

Via Definition 2.5.5 the definitions of $\mathcal{A}_\infty$–modules and type D structures (Sections 2.2.2 and 2.2.3) carry over to the G–set graded case without change. For example:

Definition 2.5.6 For (G,λ) a group with a distinguished central element, $\mathcal{A}$ a G–graded $\mathcal{A}_\infty$–algebra, and S a right G–set, a *right S–graded $\mathcal{A}_\infty$–module* is an $\mathcal{A}_\infty$–module $M_{\mathcal{A}}$ whose underlying $\boldsymbol{k}$–module is graded by S, such that for homogeneous elements $x\in M$ and $a_i \in A$,

$$\operatorname{gr}(m_{j+1}(x,a_1,\dots,a_j)) = \operatorname{gr}(x)\cdot\lambda^{j-1}\operatorname{gr}(a_1)\cdots\operatorname{gr}(a_j).$$

Example 2.5.7 The case that $G = \mathbb{Z}$, $\lambda = 1$, and S is a freely transitive G–set is equivalent to relatively $\mathbb{Z}$–graded modules. In particular, given a transitive $\mathbb{Z}$–set graded module M one can define a relative grading on M by $\mathrm{gr}(x, y) = n$ if x and y are homogeneous and $\mathrm{gr}(x)\lambda^n = \mathrm{gr}(y)$. Conversely, given a relatively $\mathbb{Z}$–graded module M it is not hard to construct a canonical $\mathbb{Z}$–set S and an associated S–graded module.

Definition 2.5.8 For S a G–set and M an S–graded module, we say that a G–invariant subset $T \subset S$ is *essential* if each G–orbit in T contains an element s so that M_s is nontrivial.

The theory of G–set gradings has a simple extension to bimodules:

Definition 2.5.9 If (G_1, λ_1) and (G_2, λ_2) are two groups with distinguished central elements, define the group

$$G_1 \times_\lambda G_2 := G_1 \times G_2/(\lambda_1 = \lambda_2)$$

with the distinguished central element $\lambda = [\lambda_1] = [\lambda_2]$. So, a $(G_1 \times_\lambda G_2)$–set is a set with commuting actions of G_1 and G_2, where the actions of λ_1 and λ_2 agree.

A *left-right* (G_1, G_2)*–set graded* $\boldsymbol{k}$*–module* is a left $(G_1 \times_\lambda G_2^{\mathrm{op}})$–set graded $\boldsymbol{k}$–module. (Left-left and right-right graded modules are defined similarly.)

Via Definition 2.5.9 the definitions of bimodules of various types (Section 2.2.4) carry over to the G–set graded case without change: if $\mathcal{A}$ is G_1–graded and $\mathcal{B}$ is G_2–graded, then we consider bimodules that are left-right (G_1, G_2)–set graded.

We next turn to tensor products of G–set graded modules.

Definition 2.5.10 If (G, λ) is a group with a distinguished central element, S is a right G–set, and T is a left G–set, define their *twisted product*

$$S \times_G T := (S \times T)/(s \times gt) \sim (sg \times t).$$

The set $S \times_G T$ has an action of λ defined by $\lambda \cdot [s \times t] := [s\lambda \times t] = [s \times \lambda t]$, but in general has no further structure. For V and W two $\boldsymbol{k}$–modules graded by S and T, respectively, $V \otimes W$ is graded by the $\mathbb{Z}$–set $S \times_G T$, where if $v \in V_s$ and $w \in W_t$, then $v \otimes w \in (V \otimes W)_{[s \times t]}$.

Thus, for instance, if $M_{\mathcal{A}}$ is graded by S and ${}^{\mathcal{A}}N$ is graded by T, $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ is graded by $S \times_G T$.

More generally, for the tensor product of bimodules, if G_1, G_2, G_3 are all groups with distinguished central elements, S is a left-right (G_1, G_2)–set, and T is a left-right (G_2, G_3)–set, then $S \times_{G_2} T$ is a left-right (G_1, G_3)–set in an obvious way. As before, if V and W are two spaces graded by S and T, respectively, $V \otimes W$ is graded by $S \times_{G_2} T$. In particular, if ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}^{\mathcal{B}}N_{\mathcal{C}}$ are two left-right set-graded $\mathcal{A}_\infty$–bimodules, then ${}_{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}N_{\mathcal{C}}$ is also a left-right set-graded $\mathcal{A}_\infty$–bimodule.

Note that the complex $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ is not in general $\mathbb{Z}$–graded; rather, the action of λ on $S \times_G T$ breaks up the morphism space into a sum of chain complexes according to the orbits of the action, with possibly cyclic grading on each orbit. (See [21, Example 2.46] for an example where the grading ends up being a finite cyclic group, even though the action of λ on each side has infinite order.)

The grading on the Hochschild complex of a G–set graded bimodule behaves as one would expect:

Definition 2.5.11 For S a left-right (G, G)–set, define

$$S^\circ := S/(s \cdot g) \sim (g \cdot s),$$

where $s \in S$, $g \in G$. S° has a $\mathbb{Z}$–action given by multiplication by λ on the left or right.

There is a quotient map from S to S°, which we denote $s \mapsto s^\circ$.

Lemma 2.5.12 *Let ${}^{\mathcal{A}}M_{\mathcal{A}}$ be a type DA bimodule graded by a left-right (G, G)–set S. Then $\widetilde{CH}(M)$ is graded by S°. The analogous statement holds for the Hochschild complex of a type AA bimodule.*

There are two ways to form G–set graded modules into categories. The easier of the two, which will suffice for most of the paper, is to define a category for each transitive G–set S, and to restrict to morphisms which only shift the grading by a power of λ:

Definition 2.5.13 Fix (G, λ) a group with a distinguished central element and S a right G–set where the action of G is transitive. Let n be the order of λ in S. (This order is well defined since the G action is transitive.) For V and W two S–graded $\boldsymbol{k}$–modules, let $\widetilde{\mathrm{Hom}}_S(V, W)$ be the $(\mathbb{Z}/n\mathbb{Z})$–graded $\mathbb{F}_2$ vector space with

$$\widetilde{\mathrm{Hom}}_S(V, W)_i := \bigoplus_{s \in S} \mathrm{Hom}(V_s, W_{s\lambda^i}).$$

These can be organized into a category $\widetilde{\mathsf{Mod}}_{\boldsymbol{k},S}$, whose objects are S–graded $\boldsymbol{k}$–modules and morphism spaces are $\widetilde{\mathrm{Hom}}_S(V, W)$.

For $\mathcal{A}$ a G–graded $\mathcal{A}_\infty$–algebra, the category $\widetilde{\mathsf{Mod}}_{\mathcal{A},S}$ of right S–graded $\mathcal{A}_\infty$–modules over $\mathcal{A}$ is the *dg* category whose objects are right S–graded $\mathcal{A}_\infty$–modules $M_{\mathcal{A}}$ and whose morphism spaces are

$$\widetilde{\mathrm{Mor}}_{\mathcal{A},S}(M_{\mathcal{A}}, N_{\mathcal{A}}) = \widetilde{\mathrm{Hom}}_S(M \otimes T^*(A_+[1]), N),$$

with the differential as in Definition 2.2.20. (Here we have extended the notion of *dg* categories in the obvious way to allow cyclic gradings.)

The definitions in Section 2.2.1 of homotopic morphisms and homotopy equivalences carry over unchanged. We extend the definitions of $\mathsf{Z}(\widetilde{\mathsf{Mod}}_{\mathcal{A},S})$ and $\mathsf{H}(\widetilde{\mathsf{Mod}}_{\mathcal{A},S})$ in the obvious ways, and $\mathsf{H}(\widetilde{\mathsf{Mod}}_{\mathcal{A},S})$ is still a triangulated category.

We define categories of other kinds of modules in the G–set graded case analogously. For instance we have a category ${}^{\mathcal{A}}\widetilde{\mathsf{Mod}}_S$ of left S–graded type D structures over $\mathcal{A}$, as well as categories of bimodules of various types.

The invariants $\widehat{CFDD}(Y,\mathfrak{s})$, $\widehat{CFAA}(Y,\mathfrak{s})$ and $\widehat{CFDA}(Y,\mathfrak{s})$ defined in Section 6 will be well defined up to isomorphism in $\mathsf{H}(\widetilde{\mathsf{Mod}}_S)$ (for appropriate grading sets S).

2.5.2 Set-graded *dg* categories It is natural to talk about morphisms between modules graded by different G–sets, or in other words to collect the S–graded modules for all G–sets S into a single category. In doing so, one encounters the following issues:

- There is no natural way to define a degree-0 morphism from an S–graded module to a T–graded module if S is different from T.
- More generally, if M and N are graded by G–sets S and T respectively then, like $M \otimes N$, the collection of morphisms $\mathrm{Mor}(M,N)$ is graded by a $\mathbb{Z}$–set constructed from S and T rather than simply by $\mathbb{Z}$.
- Further, with this construction, there are natural morphisms, like the identity morphism, which are not homogeneous (not supported in a single grading).

We will formalize these notions as follows. We will define a category of $\mathbb{Z}$–set graded chain complexes. If M and N are graded by G–sets S and T respectively then the morphisms from M to N will be a $\mathbb{Z}$–set graded chain complex. (That is, the categories of G–set graded modules of various kinds are enriched over the category of $\mathbb{Z}$–set graded chain complexes.)

Definition 2.5.14 Let S and T be two sets, each endowed with a $\mathbb{Z}$–action (written as multiplication by λ). A *relation* between S and T is a subset $R \subset S \times T$. We write sRt to mean that $(s,t) \in R$. We say that R is *λ–invariant* if it satisfies the property that sRt if and only if $(\lambda s)R(\lambda t)$. The *composite* of two relations $R_1 \subset S \times T$ and $R_2 \subset T \times U$, written $R_1 \circ R_2 \subset S \times U$, is the relation

$$R_1 \circ R_2 = \{(s,u) \mid s \in S, u \in U, \exists t \in T : sR_1t \text{ and } tR_2u\}.$$

Definition 2.5.15 A *$\mathbb{Z}$–set graded chain complex* over $\boldsymbol{k}$ is a triple (S, C, ∂), where S is a set with a $\mathbb{Z}$ action (written as multiplication by λ), C is a $\boldsymbol{k}$–module graded by S, and ∂ is an operator on C satisfying $\partial^2 = 0$ and such that for $x \in C_s$, $\partial x \in C_{\lambda^{-1}s}$. We form $\mathbb{Z}$–set graded chain complexes into a category as follows. A *morphism* from (S, C, ∂_1) to (T, C', ∂_2) is

- a λ–invariant relation R between S and T, and
- a $\boldsymbol{k}$–module map $\phi\colon C \to C'$ that is compatible with R, in the sense that

$$\phi(C_s) \subset \bigoplus_{\substack{t \in T \\ sRt}} C'_t.$$

Let $\mathrm{Mor}((S, C, \partial_1), (T, C', \partial_2))$ denote this space of morphisms, with its natural differential.

If in addition ϕ intertwines the actions of ∂_1 and ∂_2, we say that ϕ is a *homomorphism*. To compose two morphisms, we compose the relations in the sense of Definition 2.5.14 and compose the $\boldsymbol{k}$–module maps.

The category of $\mathbb{Z}$–set graded chain complexes has objects triples (S, C, ∂) as above and $\mathrm{Hom}((S, C, \partial), (T, D, \partial'))$ the set of homomorphisms (not just morphisms) from (S, C, ∂) to (T, D, ∂').

Given two $\mathbb{Z}$–set graded complexes (S, C, ∂_1) and (T, D, ∂_2), we can form their *tensor product*

$$(S, C, \partial_1) \otimes (T, D, \partial_2) := (S \times_{\mathbb{Z}} T, C \otimes_{\boldsymbol{k}} D, \partial_1 \otimes \mathbb{I}_D + \mathbb{I}_C \otimes \partial_2),$$

where $\mathrm{gr}(x \otimes y) = [\mathrm{gr}(x) \times \mathrm{gr}(y)]$ for homogeneous x, y. This tensor product extends in an obvious way to (homo)morphisms, giving a monoidal structure on this category.

Example 2.5.16 Consider a 3–manifold Y. The Heegaard Floer complex $\widehat{CF}(Y)$ is a set-graded chain complex where the grading set S is (noncanonically) given by

$$\bigcup_{\mathfrak{s} \in \mathrm{spin}^c(Y)} \mathbb{Z}/\operatorname{div}(c_1(\mathfrak{s})).$$

(In fact, the grading set can be defined canonically. First one fixes a Heegaard diagram and base generator and uses these to define a G–set grading as in Section 6.5. Then one observes that different base generator or Heegaard diagram lead to canonically isomorphic G–sets.)

The most useful notion of an element of a $\mathbb{Z}$–set graded chain complex (S, C, ∂) is not the naive notion of an element of C: elements carry with them some additional grading information, as follows.

Definition 2.5.17 Let $\underline{\boldsymbol{k}}$ denote the $\mathbb{Z}$–set graded chain complex $(\mathbb{Z}, \boldsymbol{k}_0, 0)$, where $\boldsymbol{k}_0$ denotes a copy of $\boldsymbol{k}$ lying in grading 0. An *element* of a $\mathbb{Z}$–set graded chain complex (S, C, ∂) is a morphism of $\mathbb{Z}$–set graded chain complexes $\underline{\boldsymbol{k}} \to (S, C, \partial)$. (Note that $\underline{\boldsymbol{k}}$ is the identity for the tensor product, and so this is natural from the point of view of category theory.)

Lemma 2.5.18 *An element of (S, C, ∂) (in the sense of Definition 2.5.17) is equivalent to a pair (T, x) where $T \subset S$ and $x \in \bigoplus_{s \in T} C_s$.*

Proof A morphism $\underline{\boldsymbol{k}} \to (S, C, \partial)$ is given by a λ–invariant relation R between $\mathbb{Z}$ and S, and a map ϕ. Let T be $\{t \in S \mid 0 R t\}$, the set of elements that are related to 0, and let $x = \phi(1)$. In the opposite direction, for any subset T of S, there is a unique λ–invariant relation R between $\mathbb{Z}$ and S with the property that T is the set of elements related to 0; and similarly, any element $x \in C$ can be viewed as $\phi(1)$ for a uniquely determined $\phi\colon \underline{\boldsymbol{k}} \to C$. □

For $T \subset S$ we let C_T denote the set of elements of C with grading set T, ie the elements of the form (T, x). For $s \in S$, then, $C_{\{s\}}$ is isomorphic to C_s as defined in Definition 2.5.15.

We can use the tensor product to define an internal Mor for (appropriately finite) $\mathbb{Z}$–set graded chain complexes, just as for ordinary chain complexes. We start by explaining G–set gradings of dual spaces:

Definition 2.5.19 For G any group and S a right G–set, let S^* be a set with elements s^* in bijection with elements s of S, and with a left G–action defined by $g \cdot s^* := (s \cdot g^{-1})^*$. If V is an S–graded $\boldsymbol{k}$–module, the *graded dual* of V, which we will denote $V^{\mathrm{gr}\,*}$, is an S^*–graded module with

$$(V^{\mathrm{gr}\,*})_{s^*} := (V_s)^*.$$

With these definitions, if $\phi : V \to V$ is homogeneous with respect to the right action of g on S, then $\phi^*\colon V^* \to V^*$ is homogeneous with respect to the left action of g on S^*.

Definition 2.5.20 Suppose that (S, C, ∂_1) and (T, D, ∂_2) are set-graded chain complexes such that C is supported on finitely many orbits of the action of $\mathbb{Z}$ on S. Then we define a $\mathbb{Z}$–set graded chain complex $\mathrm{Mor}((S, C, \partial_1), (T, D, \partial_2))$ to be a chain complex graded by $T \times_\lambda S^*$ with

$$\mathrm{Mor}((S, C, \partial_1), (T, D, \partial_2))_u := \prod_{s^* \in S^*} \bigoplus_{\substack{t \in T \\ u = [t \times s^*]}} \mathrm{Hom}(C_s, D_t).$$

The differential on $\mathrm{Mor}((S, C, \partial_1), (T, D, \partial_2))$ is given as usual by

$$(\partial\phi)(x) = \partial(\phi(x)) + \phi(\partial x).$$

Note that if C is finite-dimensional, then

$$\mathrm{Mor}((S, C, \partial_1), (T, D, \partial_2)) \simeq (T, D, \partial_2) \otimes (S^*, C^{\mathrm{gr}\,*}, \partial_1^*).$$

We have defined two different notions of "a morphism" between $\mathbb{Z}$–set graded chain complexes. Fortunately, they agree:

Lemma 2.5.21 *The set of elements (Definition 2.5.17) of the internal morphism complex* $\mathrm{Mor}((S, C, \partial_1), (T, D, \partial_2))$ *(Definition 2.5.20) are in natural bijection with the morphisms of* $\mathbb{Z}$*–set graded complexes defined in Definition 2.5.15.*

Proof An element of the internal morphism complex gives a morphism as follows. Thanks to Lemma 2.5.18, we can think of an element of the internal morphism complex as a subset R' of $T \times_\lambda S^*$, together with a choice of

$$\phi \in \bigoplus_{u \in R' \subset T \times_\lambda S^*} \mathrm{Mor}((S, C, \partial_1), (T, D, \partial_2))_u = \bigoplus_{u \in R' \subset T \times_\lambda S^*} \prod_{s^* \in S^*} \bigoplus_{\substack{t \in T \\ u = [t \times s^*]}} \mathrm{Hom}(C_s, D_t).$$

This ϕ, of course, can be thought of as a $\boldsymbol{k}$–module map from C to D. Now, subsets of $T \times_\lambda S^*$ are in one-to-one correspondence with subsets of $(T \times S)/\mathbb{Z}$, which in turn are in one-to-one correspondence with λ–invariant relations R between S and T. Under this correspondence, we think of ϕ as a $\boldsymbol{k}$–module map from C to D which is compatible with R (in the sense of Definition 2.5.15).

Conversely, given a morphism, an element of the morphism complex can be constructed by reversing the above process. □

We next turn to issues of injectivity. The notion of injectivity for morphisms of $\mathbb{Z}$–set graded chain complexes takes into account also grading information, as follows:

Definition 2.5.22 A relation $R \subset S \times T$ is said to be *injective* if for every $s \in S$, there is a $t \in T$ so that sRt and $s'Rt$ does not hold for any other $s' \in S$.

A morphism $(R, \phi) \in \mathrm{Hom}((S, C, \partial), (T, D, \partial'))$ (in the sense of Definition 2.5.15) is said to be *homology injective* if R is injective in the sense above and the element ϕ, thought of as a homomorphism from C to D, induces an injective map on homology.

If R is a relation between S and T, and $\Sigma \subset S$, let $\Sigma R \subset T$ denote the subset of all $t \in T$ with the property that sRt holds for some $s \in \Sigma$.

Lemma 2.5.23 *A relation R is injective in the sense of Definition 2.5.22 if and only if for any two subsets $\Sigma, \Sigma' \subset S$, $\Sigma R = \Sigma' R$ implies that $\Sigma = \Sigma'$ (ie R induces an injective function from subsets of S to subsets of T).*

Proof Suppose first that R is injective in the sense of Definition 2.5.22. For $s \in S$, let $f(s) \in T$ be the element promised by that definition. Then R induces a bijection between S and $f(S)$, and if $\Sigma R = \Sigma' R$, then $f(\Sigma) = \Sigma R \cap f(S) = \Sigma' R \cap f(S) = f(\Sigma')$, so $\Sigma = \Sigma'$.

Conversely, suppose that R induces an injective function on subsets of S. Then in particular, for each element $s \in S$, $SR \supsetneq (S \setminus \{s\})R$. Let $f(s) \in T$ be an element of SR that is not in $(S \setminus \{s\})R$. Then $f(s)$ satisfies the conditions of Definition 2.5.22. $\square$

Lemma 2.5.24 *A homomorphism (R, ϕ) of $\mathbb{Z}$–set graded complexes is injective in the sense of Definition 2.5.22 if and only if composition with ϕ induces an injection on the set of elements of (S, C) (in the sense of Definition 2.5.17) to the set of elements of (T, D).*

Proof Let (R, ϕ) be a morphism from (S, C) to (T, D). For an element (Σ, x) of (S, C) (where $\Sigma \subset S$ and $x \in C_\Sigma$), the image (element of (T, D)) under (R, ϕ) is the pair $(\Sigma R, \phi(x))$.

From Lemma 2.5.23, we see (R, ϕ) is injective on elements of (S, C) of the form $(\Sigma, 0)$ (with $\Sigma \subset S$) if and only if R is injective; and it is injective on more general elements if and only if ϕ, thought of as a homomorphism from C to D, is injective. $\square$

Since the set of morphisms between $\mathbb{Z}$–set graded chain complex (S, C, ∂_1) and (T, D, ∂_2) (with C supported on finitely many $\mathbb{Z}$–orbits) is itself a chain complex, there is an obvious notion of when two morphisms are homotopic. In fact, the definition of homotopic morphisms can be extended without difficulty to the general case without the finiteness restriction on C.

Definition 2.5.25 A *$\mathbb{Z}$–set graded dg category* is a category $\mathscr{C}$ where the morphism spaces are $\mathbb{Z}$–set graded chain complexes over $\mathbb{F}_2$ and composition of morphisms gives a $\mathbb{Z}$–set graded chain maps $\circ\colon \mathrm{Mor}(y,z)\otimes\mathrm{Mor}(x,y)\to\mathrm{Mor}(x,z)$.

Example 2.5.26 An ordinary *dg* category $\mathscr{C}$ gives a $\mathbb{Z}$–set graded *dg* category $\widetilde{\mathscr{C}}$ as follows. Let $\mathrm{Ob}(\widetilde{\mathscr{C}})=\mathrm{Ob}(\mathscr{C})$. For objects $M,N\in\mathrm{Ob}(\widetilde{\mathscr{C}})$ define $\mathrm{Mor}_{\widetilde{\mathscr{C}}}(M,N)=(\mathbb{Z},\mathrm{Mor}_{\mathscr{C}}(M,N),\partial)$, where ∂ is the differential on $\mathrm{Mor}_{\mathscr{C}}(M,N)$.

Example 2.5.27 As a special case of Example 2.5.26, consider a $\mathbb{Z}$–graded $\mathcal{A}_\infty$–algebra $\mathcal{A}$ and its $\mathbb{Z}$–set graded *dg* category of right modules $\mathcal{M}_{\mathcal{A}}$. Given modules $M_{\mathcal{A}}$ and $N_{\mathcal{A}}$, the space $\mathrm{Mor}(M_{\mathcal{A}},N_{\mathcal{A}})$ is a $\mathbb{Z}$–set graded chain complex. As in Definition 2.5.17, an element of $\mathrm{Mor}(M_{\mathcal{A}},N_{\mathcal{A}})$ — ie a morphism from $M_{\mathcal{A}}$ to $N_{\mathcal{A}}$ — has some grading information built into it.

More precisely, an "element" of the morphism space $\mathrm{Mor}(M_{\mathcal{A}},N_{\mathcal{A}})$ consists of a pair (S,ϕ), where $\phi\colon M\otimes T^*\mathcal{A}\to N$ is a map of $\mathcal{A}_\infty$–modules and $S\subset\mathbb{Z}$ is such that if $x\in M_i$ then $\phi_1(x)\subset\bigoplus_{j\in S+i}N_j$, and similarly for higher products.

The definitions in Section 2.2.1 of homotopic morphisms and homotopy equivalences carry over unchanged to an arbitrary $\mathbb{Z}$–set graded *dg* category $\mathscr{C}$. The definitions of $\mathsf{Z}_*(\mathscr{C})$ and $\mathsf{H}_*(\mathscr{C})$ also carry over. However, it no longer makes sense to talk about morphisms of degree 0, so we no longer have $\mathsf{Z}(\mathscr{C})$ or the triangulated category $\mathsf{H}(\mathscr{C})$.

Definition 2.5.28 A *$\mathbb{Z}$–set graded dg functor* $F\colon\mathscr{C}\to\mathscr{D}$ is a functor enriched so that for $x,y\in\mathscr{C}$, the map $F\colon\mathrm{Mor}(x,y)\to\mathrm{Mor}(F(x),F(y))$ is a $\mathbb{Z}$–set graded chain complex homomorphism (Definition 2.5.15).

Lemma 2.2.9, Definitions 2.2.10 and 2.2.11, and Proposition 2.2.12 carry over almost unchanged to the context of $\mathbb{Z}$–set graded *dg* functors. (In Definition 2.2.11, consider $\mathsf{H}_*(F)$ instead of $\mathsf{H}(F)$.) The basic example is provided by tensor products: see Example 2.5.35.

2.5.3 Categories of G–set graded modules

We now define a category of G–set graded modules where the sets may vary.

We start by defining the grading set for the Hom–space:

Definition 2.5.29 If (G,λ) is a group with a distinguished central element, S and T are two right G–sets, and V and W two $\boldsymbol{k}$–modules graded by S and T, respectively, then $\mathrm{Hom}_{\boldsymbol{k}}(V,W)$ is defined to be $V^{\mathrm{gr}\,*}\otimes W$ (as a $\mathbb{F}_2$ vector space) with its $T\times_G S^*$ grading.

Remark 2.5.30 One might imagine the grading set for the Hom space between S– and T–graded modules should be the set of G–equivariant maps $S \to T$. However, in many situations (eg, if $G = \mathbb{Z}$, $S = \mathbb{Z}/2$ and $T = \mathbb{Z}$) this prevents there being any homogeneous homomorphisms at all. Our philosophy is that any module map (between finite-dimensional modules, say) should be decomposable as a sum of homogeneous maps.

Definition 2.5.31 For $\mathcal{A}$ a (G, λ)–graded $\mathcal{A}_\infty$–algebra over $\boldsymbol{k}$, let $\mathsf{Mod}_{\mathcal{A}}$ be the set-graded *dg* category whose objects are strictly unital, set-graded $\mathcal{A}_\infty$–modules (ie pairs $(S, M_{\mathcal{A}})$ where $M_{\mathcal{A}}$ is graded by S), and whose morphism spaces are

$$\mathrm{Mor}_{\mathcal{A}}((S, M_{\mathcal{A}}), (T, N_{\mathcal{A}})) := \mathrm{Hom}_{\boldsymbol{k}}(M \otimes T^*(A_+[1]), N), \tag{2.5.32}$$

with a grading by $T \times_G S^*$ and differential as in Definition 2.2.20. (The right-hand side of (2.5.32) is to be interpreted as a $\mathbb{Z}$–set graded chain complex.) To define composition, we need to give $\mathbb{Z}$–set graded chain maps

$$\mathrm{Mor}_{\mathcal{A}}((S, M_{\mathcal{A}}), (T, N_{\mathcal{A}})) \otimes \mathrm{Mor}_{\mathcal{A}}((T, N_{\mathcal{A}}), (U, P_{\mathcal{A}})) \to \mathrm{Mor}_{\mathcal{A}}((S, M_{\mathcal{A}}), (U, P_{\mathcal{A}})).$$

On the chain level, this map is defined as in Section 2.2.2. It preserves the grading relation R generated by

$$((s^* \times_G t) \times_\lambda (t^* \times_G u)) \; R \; (s^* \times_G u)$$

for $s \in S$, $t \in T$, and $u \in U$.

Example 2.5.33 Specializing Definition 2.5.31 to the case where $\mathcal{A} = \boldsymbol{k} = \mathbb{F}_2$, we see that $\mathsf{Mod}_{\mathbb{F}_2}$ is the category of $\mathbb{Z}$–set graded chain complexes. Indeed, the above definition allows us to consider the category of set-graded chain complexes as a set-graded *dg* category (compare Example 2.2.2).

For a fixed transitive G–set S, the category $\widetilde{\mathsf{Mod}}_{\mathcal{A},S}$ with grading set S (compare Definition 2.5.13) is a subcategory of $\mathsf{Mod}_{\mathcal{A}}$, but not a full subcategory: morphisms in $\widetilde{\mathsf{Mod}}_{\mathcal{A},S}$ are only allowed to shift the grading by a power of λ, while in $\mathsf{Mod}_{\mathcal{A}}$ there is no such restriction.

We define $\mathsf{Mod}^{\mathcal{A}}$ similarly, using type D structures instead of $\mathcal{A}_\infty$–modules. When $\mathcal{A}$ is an $\mathcal{A}_\infty$–algebra rather than a *dg* algebra, $\mathcal{M}^{\mathcal{A}}$ is a $\mathbb{Z}$–set graded $\mathcal{A}_\infty$–category, which is defined analogously to $\mathbb{Z}$–set graded *dg* categories. The variants ${}_{\mathcal{A}}\mathsf{Mod}$ and ${}^{\mathcal{A}}\mathsf{Mod}$ are defined symmetrically, using left actions instead of right actions. Categories of bimodules are defined similarly; cf Definition 2.5.9.

We can consider $\mathbb{Z}$–set graded functors from the set-graded category $\mathcal{M}_{\mathcal{A}}$ (from Definition 2.5.31) to the category of $\mathbb{Z}$–set graded chain complexes. Although the definition of this notion can be pieced together by what has been written so far, we spell it out for the reader's convenience:

Example 2.5.34 Let $\mathcal{A}$ be a (G,λ)–graded $\mathcal{A}_\infty$–algebra, let $\mathcal{M}_\mathcal{A}$ (whose objects are pairs $(S, M_\mathcal{A})$, where S is a G–set and $M_\mathcal{A}$ is graded by S) be its category of G–set graded $\mathcal{A}_\infty$–modules, and let $\mathsf{Mod}_{\mathbb{F}_2}$ denote the category of $\mathbb{Z}$–set graded chain complexes. A *differential set-graded functor* (or, less precisely, a *dg functor*) $\mathcal{F}\colon \mathcal{M}_\mathcal{A} \to \mathsf{Mod}_{\mathbb{F}_2}$ consists of the following data:

- For each $(S, M_\mathcal{A}) \in \mathcal{M}_\mathcal{A}$, a pair $\mathcal{F}(S, M_\mathcal{A}) = (S', M')$, where S' is a $\mathbb{Z}$–set and M' is a chain complex graded by S'.
- For each pair $(S, M_\mathcal{A}), (T, N_\mathcal{A}) \in \mathcal{M}_\mathcal{A}$, an element $\mathcal{F}_{(S,M),(T,N)}$ (in the sense of Definition 2.5.17) of the internal hom set
$$\mathrm{Hom}_\mathcal{C}(\mathrm{Mor}_\mathcal{A}((S,M),(T,N)), \mathrm{Mor}_\mathcal{C}((S',M'),(T',N'))),$$
which in turn consists of a pair $(R_{(S,M),(T,N)}, \Phi_{(S,M),(T,N)})$, where R is a λ–invariant relation
$$R_{(S,M),(T,N)} \subset (T \times_G S^*)^* \times (T' \times_\lambda (S')^*)$$
and $\Phi_{(S,M),(T,N)}$ is a cycle in
$$\mathrm{Mor}_\mathcal{C}(\mathrm{Mor}_\mathcal{A}((S,M),(T,N)), \mathrm{Mor}_\mathcal{C}((S',M'),(T',N')))$$
which is supported in $R_{(S,M),(T,N)}$.

Moreover, we demand that the $\mathcal{F}_{(S,M),(T,N)}$ respect composition, in the sense that the following two conditions are satisfied:

(1) $R_{(S,M),(U,L)} = R_{(S,M),(T,N)} \circ R_{(T,N),(U,L)}$.

(2) The following diagram commutes:
$$\begin{tikzcd}
\begin{array}{c}\mathrm{Mor}_\mathcal{A}((S,M),(T,N))\\ \otimes \mathrm{Mor}_\mathcal{A}((T,N),(U,L))\end{array} \arrow[r, "\circ_\mathcal{A}"] \arrow[d, "\begin{array}{c}\Phi_{(S,M),(T,N)}\\ \otimes\Phi_{(T,N),(U,L)}\end{array}"'] & \mathrm{Mor}_\mathcal{A}((S,M),(U,L)) \arrow[d, "\Phi_{(S,M),(U,L)}"] \\
\begin{array}{c}\mathrm{Mor}_\mathcal{C}(\mathcal{F}(S,M),\mathcal{F}(T,N))\\ \otimes \mathrm{Mor}_\mathcal{C}(\mathcal{F}(T,N),\mathcal{F}(U,L))\end{array} \arrow[r, "\circ_\mathcal{C}"] & \mathrm{Mor}_\mathcal{C}(\mathcal{F}(S,M),\mathcal{F}(U,L))
\end{tikzcd}$$

The following example (and its generalizations to bimodules) will be of importance to us:

Example 2.5.35 Let $\mathcal{A}$ be a G–graded $\mathcal{A}_\infty$–algebra, and fix a G–set graded type D structure $(U, {}^{\mathcal{A}}P)$. The operation $(S, M_{\mathcal{A}}) \mapsto (S \times_G U, M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}P)$ induces a *dg* functor $\mathcal{F}$ from the category of G–set graded $\mathcal{A}_\infty$–modules, $\mathsf{Mod}_{\mathcal{A}}$, to the category of $\mathbb{Z}$–set graded chain complexes. In particular, given two G–set graded $\mathcal{A}_\infty$–modules $(S, M_{\mathcal{A}})$ and $(T, N_{\mathcal{A}})$, we define a homomorphism

$$\mathcal{F}_{(S,M),(T,N)} = (R, \Phi) \in \mathrm{Hom}(\mathrm{Mor}(M, N), \mathrm{Mor}(M \boxtimes P, N \boxtimes P))$$

as follows: R is the tautological relation in

$$(T \times_G S^*) \times ((T \times_G U) \times_\lambda (S \times_G U)^*)) = (T \times_G S^*) \times ((T \times_G U) \times_\lambda (U^* \times_G S^*))$$

given by

$$[t \times_G s^*] R [[t \times_G u] \times_\lambda [u^* \times_G s^*]],$$

and Φ is the map $\cdot \boxtimes \mathbb{I}_N$ from Section 2.3.2.

Definition 2.5.36 A *dg* functor $\mathcal{F}\colon \mathcal{M}_{\mathcal{A}} \to \mathsf{Mod}_{\mathbb{F}_2}$ (as in Example 2.5.34) is said to be *homology faithful* if the homomorphisms $\mathcal{F}_{(S,M),(T,N)}$ are homology injective in the sense of Definition 2.5.22 (that is, $\mathcal{F}$ induces an injective functor $\mathsf{H}(\mathcal{M}_{\mathcal{A}}) \to \mathsf{H}(\mathsf{Mod}_{\mathbb{F}_2})$).

It is necessary to take morphisms of bimodules over one of the two actions, and to consider the bimodule structure on the result (see especially Sections 2.3.4 and 9). This operation interacts with the gradings as follows:

Definition 2.5.37 If S is left $(G_1 \times_\lambda G_2^{\mathrm{op}})$–set and T is a left $(G_3 \times_\lambda G_2^{\mathrm{op}})$–set, define

$$\begin{aligned}\mathrm{Hom}_{G_2}(S, T) &:= \{\langle s, t\rangle \mid s \in S, t \in T\}/\{\langle sg, tg\rangle \sim \langle s, t\rangle \mid g \in G_2\}\\ &= T \times_{G_2} S^*.\end{aligned}$$

We view the result as a left $(G_3 \times_\lambda G_1^{\mathrm{op}})$–set, as in the description as a product over G_2.

If V and W are graded by S and T, respectively, then $\mathrm{Hom}_{\boldsymbol{k}}(V, W)$ may be graded by $\mathrm{Hom}_{G_2}(S, T)$. In particular, all of the one-sided Mor–spaces between various bimodules are graded by sets like this. For instance, if $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are $\mathcal{A}_\infty$–algebras graded by G_1, G_2, G_3, respectively, and ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}_{\mathcal{C}}N_{\mathcal{B}}$ are graded by S and T as above, then $\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}})$ has underlying space $\mathrm{Hom}_{\boldsymbol{k}}(M \otimes T^*(\mathcal{B}[1]), N)$, which is graded by $\mathrm{Hom}_{G_2}(S, T)$. With this grading on $\mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{C}}N_{\mathcal{B}})$, the differential is a graded map.

3 Pointed matched circles

3.1 The algebra of a pointed matched circle

We start by recalling the following definition [21]:

Definition 3.1 A *pointed matched circle* is an oriented circle Z, equipped with a basepoint z and additional $4k$ points $\boldsymbol{a} = \{a_1, \dots, a_{4k}\}$ (all distinct from z) which are partitioned into pairs in such a manner that the one-manifold obtained by performing surgeries on the $2k$ pairs of points gives a circle.

The pairing of points can be thought of as map M taking an element of $\boldsymbol{a}$ to its equivalence class (consisting of two elements). Alternately, we may think of the pairing as an involution $x \mapsto x'$ on $\boldsymbol{a}$, taking a point to the other element of its pair. We often abbreviate the data $(Z, \{a_1, \dots, a_{4k}\}, M, z)$ by $\mathcal{Z}$.

Construction 3.2 Given a pointed matched circle $\mathcal{Z}$, we can associate a surface whose boundary is a circle, containing a marked point z. We denote this surface $F^\circ(\mathcal{Z})$, and let $F(\mathcal{Z})$ denote the result of filling in the boundary component of $F^\circ(\mathcal{Z})$ with a disk D. Note that any two surfaces specified by the same pointed matched circle are homeomorphic, via a homeomorphism which is uniquely determined up to isotopy.

Any surface of genus $k > 1$ can be represented by more than one pointed matched circle. There are two convenient families of pointed matched circles which can be used to describe an arbitrary oriented surface. One is the *split* pointed matched circle, which is obtained as the k–fold connected sum of pointed matched circles representing genus-one surfaces (and dropping extra basepoints). The other we call the *antipodal* pointed matched circle, where $x \mapsto x'$ is the map exchanging antipodal points. See Figure 7. One can also reverse the orientation of a pointed matched circle $\mathcal{Z}$ to get a new pointed matched circle $-\mathcal{Z}$.

Recall from [21] the algebra $\mathcal{A}(n,k)$. An $\mathbb{F}_2$–basis for $\mathcal{A}(n,k)$ is given by triples $(\boldsymbol{S}, \boldsymbol{T}, \phi)$, where $\boldsymbol{S}$ and $\boldsymbol{T}$ are k–element subsets of $[n] = \{1, \dots, n\}$ and $\phi\colon \boldsymbol{S} \to \boldsymbol{T}$ is a bijection with $\phi(x) \geq x$ for each $x \in \boldsymbol{S}$. Associated to any such triple is a number $\mathrm{inv}(\phi)$, the number of inversions of ϕ. We define

$$(\boldsymbol{S}_1, \boldsymbol{T}_1, \phi_1) \cdot (\boldsymbol{S}_2, \boldsymbol{T}_2, \phi_2) \\ = \begin{cases} (\boldsymbol{S}_1, \boldsymbol{T}_2, \phi_2 \circ \phi_1) & \boldsymbol{T}_1 = \boldsymbol{S}_2,\ \mathrm{inv}(\phi_2 \circ \phi_1) = \mathrm{inv}(\phi_1) + \mathrm{inv}(\phi_2) \\ 0 & \text{otherwise.} \end{cases}$$

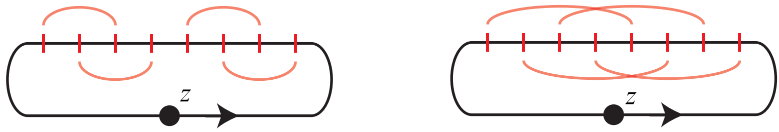
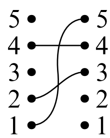


Figure 7: Two convenient pointed matched circles for the genus-2 surface: on the left, we have the split pointed matched circle; on the right, we have the antipodal pointed matched circle.

Let $\mathrm{Inv}(\phi)$ denote the set of inversions of ϕ, so $\mathrm{inv}(\phi) = \#\mathrm{Inv}(\phi)$. For $\sigma = (i, j) \in \mathrm{Inv}(\phi)$ define ϕ_σ by $\phi_\sigma(i) = \phi(j)$, $\phi_\sigma(j) = \phi(i)$, and $\phi_\sigma(k) = \phi(k)$ for $k \neq i, j$. There is a differential on $\mathcal{A}(n,k)$ given by

$$\partial(\boldsymbol{S}, \boldsymbol{T}, \phi) = \sum_{\substack{\sigma \in \mathrm{Inv}(\phi) \\ \mathrm{inv}(\phi_\sigma) = \mathrm{inv}(\phi) - 1}} (\boldsymbol{S}, \boldsymbol{T}, \phi_\sigma).$$

We draw basis elements of $\mathcal{A}(n,k)$ as strand diagrams with upward-veering strands, like this:



The product becomes concatenation (with the convention that double crossings are set to zero) and the differential corresponds to smoothing crossings.

Now, let $\mathcal{Z}$ be a pointed matched circle. For each $i = -k, \dots, k$ we define a subalgebra $\mathcal{A}(\mathcal{Z}, i) \subset \mathcal{A}(4k, k+i)$ as follows. Cutting Z at the basepoint z, the orientation of Z identifies $\boldsymbol{a}$ with $[4k]$, so we can view the matching M as inducing an involution $x \mapsto x'$ of $[4k]$, which we extend to an involution on the set of subsets of $[4k]$. For $a = (\boldsymbol{S}, \boldsymbol{T}, \phi)$ a basis element of $\mathcal{A}(4k, k+i)$ and $\boldsymbol{i} \subset \mathrm{Fix}(\phi)$ a set of fixed points of ϕ define a map

$$\phi_{\boldsymbol{i}} \colon (\boldsymbol{S} \setminus \boldsymbol{i}) \cup (\boldsymbol{i}') \to (\boldsymbol{T} \setminus \boldsymbol{i}) \cup (\boldsymbol{i}')$$

by

$$\phi_{\boldsymbol{i}}(j) = \begin{cases} \phi(j) & \text{if } j \notin \boldsymbol{i}', \\ j & \text{if } j \in \boldsymbol{i}'. \end{cases}$$

Define

$$E(\boldsymbol{S}, \boldsymbol{T}, \phi) = \sum_{\boldsymbol{i} \subset \mathrm{Fix}(\phi)} ((\boldsymbol{S} \setminus \boldsymbol{i}) \cup \boldsymbol{i}', (\boldsymbol{T} \setminus \boldsymbol{i}) \cup \boldsymbol{i}', \phi_{\boldsymbol{i}}).$$

(Loosely speaking, $E(\boldsymbol{S}, \boldsymbol{T}, \phi)$ is obtained by "smearing" the locations of the horizontal strands in ϕ according to the matching.) Then, $\mathcal{A}(\mathcal{Z}, i)$ is the subalgebra of $\mathcal{A}(4k, k+i)$ generated by

$$\{E(\boldsymbol{S}, \boldsymbol{T}, \phi) \mid \boldsymbol{S} \cap \boldsymbol{S}' = \boldsymbol{T} \cap \boldsymbol{T}' = \varnothing\}.$$

(In fact, these elements span $\mathcal{A}(\mathcal{Z}, i)$ as an $\mathbb{F}_2$–vector space.)

Note in particular that the primitive idempotents in $\mathcal{A}(\mathcal{Z}, i)$ correspond naturally to $(k+i)$–element subsets of $[2k] = [4k]/M$. Given a $(k+i)$–element subset $\boldsymbol{s}$ of $[2k]$ we let $I(\boldsymbol{s})$ denote the corresponding idempotent.

The subalgebra $\mathcal{A}(\mathcal{Z}, 0) \subset \mathcal{A}(\mathcal{Z})$ is the summand directly relevant to the case of 3–manifolds with connected boundary. The other summands become necessary when considering three-manifolds with disconnected boundary (as in, for instance, Theorem 7). Consider the extreme cases $\mathcal{A}(\mathcal{Z}, -k)$ and $\mathcal{A}(\mathcal{Z}, k)$. The algebra $\mathcal{A}(\mathcal{Z}, -k)$ is isomorphic to $\mathbb{F}_2$; the generator is the identity permutation of the empty set. By contrast, the algebra $\mathcal{A}(\mathcal{Z}, k)$ is quite large, but it follows from Theorem 9 in Section 4 that $H_*(\mathcal{A}(\mathcal{Z}, k)) \cong \mathbb{F}_2$.

We can put the algebras together to form

$$\mathcal{A}(\mathcal{Z}) = \bigoplus_{i=-k}^{k} \mathcal{A}(\mathcal{Z}, i),$$

which we think of as an algebra with an extra integer grading (represented by the integer i), so that elements with different grading multiply to 0. We call this grading the *strands grading* on $\mathcal{A}(\mathcal{Z})$.

Let $\mathcal{I}(\mathcal{Z}, i)$ denote the ring of idempotents in $\mathcal{A}(\mathcal{Z}, i)$ and $\mathcal{I}(\mathcal{Z})$ the ring of idempotents in $\mathcal{A}(\mathcal{Z})$. Of course, we can think of $\mathcal{A}(\mathcal{Z}, i) = \mathcal{A}(\mathcal{Z}) \cdot \mathcal{I}(\mathcal{Z}, i) = \mathcal{I}(\mathcal{Z}, i) \cdot \mathcal{A}(\mathcal{Z})$. We think of the ring of idempotents as the ground ring. The algebra $\mathcal{A}(\mathcal{Z})$ has a natural augmentation $\epsilon\colon \mathcal{A}(\mathcal{Z}) \to \mathcal{I}(\mathcal{Z})$, sending any strand diagram with a nonhorizontal strand to 0.

The algebra $\mathcal{A}(\mathcal{Z})$ can also be defined in terms of Reeb chords in $(Z, \boldsymbol{a})$; see [21, Section 3.1.3]. In particular, given a set of Reeb chords $\boldsymbol{\rho}$ there is an associated algebra element $a(\boldsymbol{\rho})$ (which may be zero if $\boldsymbol{\rho}$ does not satisfy some compatibility conditions).

Remark 3.3 It is not hard to show that if $\mathcal{Z}$ represents a surface F of genus k then the Grothendieck group of projective $\mathcal{A}(\mathcal{Z}, i)$–modules has rank $\binom{2k}{k+i}$. It is interesting to compare with the homology of $\mathrm{Sym}^{k+i}(F^\circ)$.

3.2 Gradings

As discussed in [21, Section 3.3], the algebra $\mathcal{A}(\mathcal{Z})$ is not graded by $\mathbb{Z}$, but rather by a noncommutative group G, a $\mathbb{Z}$–central extension of $H_1(F(\mathcal{Z}))$. In fact, it is more naturally graded by a larger group G'. We recall that construction first (and discuss some of its key properties) and then we turn to the refinement to gradings in G (which depends on some auxiliary choices).

Let $Z' = Z \setminus z$. Fix $p \in \boldsymbol{a}$ and $\alpha \in H_1(Z', \boldsymbol{a})$. We define the *multiplicity* $m(\alpha, p)$ of α at p to be the average of the local multiplicity of α just above p and just below p. Extend this to a bilinear pairing

$$m\colon H_1(Z', \boldsymbol{a}) \times H_0(\boldsymbol{a}) \to \tfrac{1}{2}\mathbb{Z}.$$

Let $G'(\mathcal{Z})$ denote the $\mathbb{Z}$–central extension of $H_1(Z', \boldsymbol{a})$ with the commutation relation

$$\widetilde{\alpha} \cdot \widetilde{\beta} = \widetilde{\beta} \cdot \widetilde{\alpha} \cdot \lambda^{2m(\beta, \partial\alpha)},$$

where $\alpha, \beta \in H_1(Z', \boldsymbol{a})$, $\widetilde{\alpha}$ and $\widetilde{\beta}$ are lifts of α and β to G', λ is a generator for the center, and $\partial\colon H_1(Z', \boldsymbol{a}) \to H_0(\boldsymbol{a})$ denotes the connecting homomorphism. The group $G'(\mathcal{Z})$ can be realized explicitly as an index two subgroup of the group $\frac{1}{2}\mathbb{Z} \times H_1(Z', \boldsymbol{a})$ endowed with the multiplication map

$$(\ell_1, \alpha_1) \cdot (\ell_2, \alpha_2) = (\ell_1 + \ell_2 + m(\alpha_2, \partial\alpha_1), \alpha_1 + \alpha_2),$$

This index-two subgroup is generated by elements $\lambda = (1, 0)$ and $(-\frac{1}{2}, [i, i+1])$, where $i = 1, \dots, 4k-1$. We call ℓ (respectively α) the *Maslov component* (respectively spin^c *component*) of (ℓ, α). Concretely, $G'(\mathcal{Z})$ consists of pairs (ℓ, α) where

$$\ell \equiv \tfrac{1}{4}\#(\text{parity changes in } \alpha) \pmod 1.$$

A basis element a of the algebra $\mathcal{A}(\mathcal{Z})$ has an associated one-chain in $H_1(Z, \boldsymbol{a})$, $[a]$. Recall also that a is a linear combination of basic elements of $\mathcal{A}(4k, k+i)$; let a_0 be any of the terms appearing in this linear combination. Using a_0 and $[a]$ we can construct the $G'(\mathcal{Z})$–grading by the formulas

$$\begin{aligned} \iota(a) &= \operatorname{inv}(a_0) - m([a_0], S), \\ \operatorname{gr}'(a) &= (\iota(a), [a]), \end{aligned} \tag{3.1}$$

where S is the initial idempotent of a_0 (ie $I(S) \cdot a_0 = a_0$). A short argument shows that the quantity $\iota(a)$ is independent of the choice of a_0; for more details, see [21, Proposition 3.40]. It is also not hard to show that for any a, $\operatorname{gr}'(a)$ is an element of $G'(\mathcal{Z})$, and the map gr' gives $\mathcal{A}(\mathcal{Z})$ a $G'(\mathcal{Z})$–grading, in the sense that $\operatorname{gr}'(a \cdot b) = \operatorname{gr}'(a) \cdot \operatorname{gr}'(b)$ and $\operatorname{gr}'(\partial a) = \lambda^{-1} \operatorname{gr}'(a)$; see [21, Proposition 3.39].

As an immediate application of the grading, our algebras satisfy the condition of Definition 2.1.9.

Lemma 3.2 *For any pointed matched circle $\mathcal{Z}$, the algebra $\mathcal{A}(\mathcal{Z})$ has nilpotent augmentation ideal.*

Proof This is immediate from the facts that $\mathcal{A}(\mathcal{Z})$ is finite-dimensional and elements of the augmentation ideal have positive $H_1(Z', \boldsymbol{a})$ gradings. □

In Section 8.1, we will use a stronger grading-positivity result for our algebras:

Lemma 3.3 *For any pointed matched circle $\mathcal{Z}$, $\mathcal{A}(\mathcal{Z})$ is gr'–graded in nonpositive Maslov degrees. More precisely, if a is a generator of $\mathcal{A}(\mathcal{Z})$, then the Maslov degree of a is less than or equal to zero, with equality if and only if a is an idempotent.*

Proof We will show that $\iota(a)$ is less than or equal to $-\frac{1}{2}$ times the number of moving (nonhorizontal) strands in a. Letting Σ denote the set of moving strands in a, observe that

$$\iota(a) = -\frac{\#|\Sigma|}{2} + \sum_{\substack{s_i, s_j \in \Sigma \\ s_i \neq s_j}} (\#(s_i \cap s_j) - m(I(s_i), [s_j]) - m(I(s_j), [s_i])),$$

where $I(s_i)$ denotes the initial point of the strand s_i, and $[s_i]$ is its associated interval. (Horizontal strands do not contribute to $\iota(a)$.) Evidently, the only positive contributions here come from crossings of s_i and s_j. For each, the contribution is at most 1 for each pair $\{s_i, s_j\}$. However, if s_i and s_j cross, then either the initial point of s_i is contained in the interior of $[s_j]$ or the initial point of s_j is contained in the interior of $[s_i]$. Thus, we can cancel off each positive contribution with a corresponding -1. □

3.2.1 Refined gradings The *Heisenberg group $G(\mathcal{Z})$ of $H_1(F)$* is the central extension of $H_1(F; \mathbb{Z})$ by a subgroup $\mathbb{Z}$ generated by λ, with commutation relation

$$g \cdot h = h \cdot g \cdot \lambda^{2\#([g] \cap [h])}$$

for any $g, h \in G(\mathcal{Z})$. (Here, $[g]$ and $[h]$ are the images of g and h in $H_1(F)$.)

As in [21, Section 3.3.2], there is a natural inclusion $i_*\colon H_1(F) \to H_1(Z', \boldsymbol{a})$, with image the kernel of $M_* \circ \partial$. Since $[g] \cap [h] = m(i_*[h], \partial i_*[g])$, it follows that this inclusion $G(\mathcal{Z}) \hookrightarrow G'(\mathcal{Z})$ is a group homomorphism. As explained in [21, Section 3.3.2], the G' grading on the algebra can be refined to a G–valued grading. (The refined grading leads to cleaner statements of the pairing theorems; see, for instance, Theorem 13 in Section 7.) That construction involves certain choices, as we now elaborate.

Definition 3.4 Fix a pointed matched circle. Given an element $\alpha \in H_1(Z, \boldsymbol{a})$ and subsets $\boldsymbol{s}$ and $\boldsymbol{t}$ of $[2k]$ we say α is *compatible with the idempotents $I(\boldsymbol{s})$ and $I(\boldsymbol{t})$* if

$$M_*(\partial\alpha) = \boldsymbol{t} - \boldsymbol{s}.$$

In particular, for a generator a of $\mathcal{A}(\mathcal{Z})$ with $I(\boldsymbol{s}) \cdot a \cdot I(\boldsymbol{t}) = a$, the homology class $[a]$ is compatible with $\boldsymbol{s}$ and $\boldsymbol{t}$.

Definition 3.5 *Grading refinement data* for the algebra $\mathcal{A}(\mathcal{Z})$ consists of a function

$$\psi\colon \{\boldsymbol{s} \mid \boldsymbol{s} \subset [2k]\} \to G'(\mathcal{Z})$$

satisfying the condition that if $g' \in G'(\mathcal{Z})$ is a group element so that $[g']$ is compatible with $I(\boldsymbol{s})$ and $I(\boldsymbol{t})$, then $\psi(\boldsymbol{s}) \cdot g' \cdot \psi(\boldsymbol{t})^{-1}$ lies in $G(\mathcal{Z}) \subset G'(\mathcal{Z})$.

Grading refinement data for $\mathcal{A}(\mathcal{Z}, i)$ can be specified by choosing, for each $i = 0, \dots, 2k$, a base idempotent $\boldsymbol{t}_i \subset [2k]$ with $|\boldsymbol{t}_i| = i$ and maps

$$\psi_i\colon \{\boldsymbol{s} \mid \boldsymbol{s} \subset [2k],\ |\boldsymbol{s}| = i\} \to G'(\mathcal{Z})$$

satisfying

$$M_*\partial[\psi_i(\boldsymbol{s})] = \boldsymbol{s} - \boldsymbol{t}_i.$$

Definition 3.6 Given grading refinement data ψ as above, we can define a corresponding $G(\mathcal{Z})$–valued grading gr_ψ on $\mathcal{A}(\mathcal{Z})$ as follows. For any generator a of $\mathcal{A}(\mathcal{Z}, i) \subset \mathcal{A}(\mathcal{Z})$ with idempotents $\boldsymbol{s}$ and $\boldsymbol{t}$, define

$$\mathrm{gr}_\psi(a) = \psi(\boldsymbol{s}) \cdot \mathrm{gr}'(a) \cdot \psi(\boldsymbol{t})^{-1}.$$

It is straightforward to verify that this is indeed a grading with values in $G(\mathcal{Z}) \subset G'(\mathcal{Z})$.

For fixed refinement data ψ, we can consider the categories of $G(\mathcal{Z})$–graded $\mathcal{A}_\infty$–modules and type D structures (and corresponding bimodules). For the few times we wish to call attention to all this information, we will decorate the relevant categories, writing, for example, $\mathsf{Mod}_{\mathcal{A},\psi}^{G(\mathcal{Z})}$ for the category of $G(\mathcal{Z})$–graded $\mathcal{A}_\infty$–modules over $\mathcal{A}$ with its $G(\mathcal{Z})$–grading induced by the refinement data ψ.

The reader is warned that if ψ and ψ' are two different refinement data for $\mathcal{Z}$ then the induced $G(\mathcal{Z})$–graded algebras are typically not graded quasi-isomorphic. They do, however, have isomorphic module categories, according to the following:

Proposition 3.7 *If ψ and ψ' are two different refinement data for $\mathcal{A}(\mathcal{Z})$, the identity type DA bimodule ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ can be given a grading, ${}^{\mathcal{A},\psi}[\mathbb{I}]_{\mathcal{A},\psi'}$ so that the functors*

$$(\cdot \boxtimes {}^{\mathcal{A},\psi}[\mathbb{I}]_{\mathcal{A},\psi'})\colon \mathsf{Mod}^{G(\mathcal{Z})}_{\mathcal{A},\psi} \to \mathsf{Mod}^{G(\mathcal{Z})}_{\mathcal{A},\psi'},$$
$$({}^{\mathcal{A},\psi}[\mathbb{I}]_{\mathcal{A},\psi'} \boxtimes \cdot)\colon \mathsf{Mod}_{G(\mathcal{Z})}^{\mathcal{A},\psi'} \to \mathsf{Mod}_{G(\mathcal{Z})}^{\mathcal{A},\psi},$$

induce equivalences of categories.

(See Definition 2.2.48 for the definition of ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$.)

Proof Consider the left-right $(G(\mathcal{Z}), G(\mathcal{Z}))$–set $T = G(\mathcal{Z})$, with action given by $(g_1 \times g_2) \cdot g = g_1 \cdot g \cdot g_2$. We endow ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ with a grading by T, as follows. Recall that generators of ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ are of the form $I(\boldsymbol{s})$ where $\boldsymbol{s}$ is any subset of $[2k]$. We endow the generator $I(\boldsymbol{s})$ of ${}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ with the grading

$$\mathrm{gr}^{\psi}_{\psi'}(I(\boldsymbol{s})) = \psi(\boldsymbol{s}) \cdot \psi'(\boldsymbol{s})^{-1} \in T. \tag{3.8}$$

We denote the resulting T–graded module by ${}^{\mathcal{A},\psi}[\mathbb{I}]_{\mathcal{A},\psi'}$. We must check that, for any homogeneous algebra element $a = I(\boldsymbol{s}) \cdot a \cdot I(\boldsymbol{t})$, the gradings of $I(s) \cdot a$ and $a \cdot I(t)$ are consistent. We compute (using Definition 3.6):

$$\begin{aligned} \mathrm{gr}^{\psi}_{\psi'}(I(\boldsymbol{s})) \cdot \mathrm{gr}_{\psi'}(a) &= (\psi(\boldsymbol{s})\psi'(\boldsymbol{s})^{-1})(\psi'(\boldsymbol{s})\,\mathrm{gr}'(a)\psi'(\boldsymbol{t})^{-1}) \\ &= \psi(\boldsymbol{s})\,\mathrm{gr}'(a)\psi'(\boldsymbol{t})^{-1} \\ &= \mathrm{gr}_{\psi}(a) \cdot \mathrm{gr}^{\psi}_{\psi'}(I(\boldsymbol{t})), \end{aligned} \tag{3.9}$$

showing that ${}^{\mathcal{A},\psi}[\mathbb{I}]_{\mathcal{A},\psi'}$ is indeed a T–graded $\mathcal{A}(\mathcal{Z},\psi)$–$\mathcal{A}(\mathcal{Z},\psi')$ bimodule. An inverse to this bimodule is supplied by the analogously-defined bimodule ${}^{\mathcal{A},\psi'}[\mathbb{I}]_{\mathcal{A},\psi}$, where we use the canonical identification $T \times_{G(\mathcal{Z})} T \to T$ induced by multiplication to grade the isomorphism from the tensor product to the identity. □

Definition 3.10 Let S be a right G'–space and $M_{\mathcal{A}}$ an $\mathcal{A}_\infty$–module over $\mathcal{A}$ graded by S. Consider choices of $\boldsymbol{x}, \boldsymbol{y} \in M_{\mathcal{A}}$ and $\boldsymbol{s}, \boldsymbol{t} \subset [2k]$ with the following properties:

- $\boldsymbol{x} \cdot I(\boldsymbol{s}) = \boldsymbol{x}$.
- $\boldsymbol{y} \cdot I(\boldsymbol{t}) = \boldsymbol{y}$.
- $\boldsymbol{x}$ and $\boldsymbol{y}$ are nonzero elements which are homogeneous with respect to the S–grading.

We say $(S, M_{\mathcal{A}})$ is *G–refinable* if for all choices of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{s}$, and $\boldsymbol{t}$ as above, any group element $g \in G'$ such that $\mathrm{gr}'(\boldsymbol{x}) \cdot g = \mathrm{gr}'(\boldsymbol{y})$ is compatible with the idempotents $I(\boldsymbol{s})$ and $I(\boldsymbol{t})$.

Remark 3.11 Let $(S, M_{\mathcal{A}})$ be G–refinable. If $\boldsymbol{x} \in M_{\mathcal{A}}$ is a homogeneous element, then there is an elementary idempotent $I(\boldsymbol{s})$ with the property that $\boldsymbol{x} \cdot I(\boldsymbol{s}) = \boldsymbol{x}$. To see this, note that if $\boldsymbol{x}$ is homogeneous of some degree (in S), then $\boldsymbol{x} \cdot I(\boldsymbol{s})$ and $\boldsymbol{x} \cdot I(\boldsymbol{t})$ are homogeneous of the same degree; thus, if both elements are nonzero, then the identity element $e \in G'$ is compatible with the idempotents $I(\boldsymbol{s})$ and $I(\boldsymbol{t})$, which in turn forces $\boldsymbol{s} = \boldsymbol{t}$.

Lemma 3.12 *For any grading refinement data ψ, there is a natural functor*

$$\mathcal{F}^{\psi}\colon \mathsf{Mod}^{G(\mathcal{Z})}_{\mathcal{A},\psi} \to \mathsf{Mod}^{G'(\mathcal{Z})}_{\mathcal{A}}$$

which is homology faithful (ie the induced functor on homology is injective on morphisms). These functors are compatible with changing refinement data, in the sense that there is a natural isomorphism of functors

$$\mathcal{F}^{\psi} \cong \mathcal{F}^{\psi'} \circ (\cdot \boxtimes^{\mathcal{A},\psi} \mathbb{I}_{\mathcal{A},\psi'}).$$

Let $M_{\mathcal{A}} \in \mathsf{Mod}^{G'(\mathcal{Z})}_{\mathcal{A}}$ be an object with grading set S that is essential in the sense of Definition 2.5.8. Then, $M_{\mathcal{A}}$ is isomorphic to $\mathcal{F}^{\psi}(N_{\mathcal{A},\psi})$ for some $N_{\mathcal{A},\psi} \in \mathsf{Mod}^{G(\mathcal{Z})}_{\mathcal{A},\psi}$ if and only if $M_{\mathcal{A}}$ is G–refinable.

Proof First we define the functor $\mathcal{F}^{\psi}$. Recall that objects in $\mathsf{Mod}^{G(\mathcal{Z})}_{\mathcal{A},\psi}$ consist of pairs $(S, M_{\mathcal{A}})$, where

- S is a right $G(\mathcal{Z})$–set and
- $M_{\mathcal{A}}$ is a right $\mathcal{A}$ $\mathcal{A}_{\infty}$–infinity module graded by S

(satisfying the necessary compatibility conditions). The functor is defined on objects by

$$\mathcal{F}^{\psi}(S, M_{\mathcal{A}}) = (S \times_{G(\mathcal{Z})} G'(\mathcal{Z}), N_{\mathcal{A}}),$$

where on the right-hand side, the module $M_{\mathcal{A}}$ is given the grading $\mathrm{gr}(\boldsymbol{x}) = \mathrm{gr}'(\boldsymbol{x}) \cdot \psi^{-1}(\boldsymbol{s})$, where $\boldsymbol{x} \cdot I(\boldsymbol{s}) = \boldsymbol{x}$.

We now must define the functor on morphisms. (See Example 2.5.34.) Given two objects $(M_{\mathcal{A}}, S)$ and $(N_{\mathcal{A}}, T)$ in $\mathsf{Mod}^{G(\mathcal{Z})}_{\mathcal{A},\psi}$, we must define a homomorphism of set-graded complexes

$$\begin{aligned}\mathrm{Mor}^{G(\mathcal{Z})}_{\mathcal{A},\psi}&((S, M_{\mathcal{A}}), (T, N_{\mathcal{A}}))\\ &\to \mathrm{Mor}^{G'(\mathcal{Z})}_{\mathcal{A}}((S \times_{G(\mathcal{Z})} G'(\mathcal{Z}), M_{\mathcal{A}}), (T \times_{G(\mathcal{Z})} G'(\mathcal{Z}), N_{\mathcal{A}})).\end{aligned}$$

This consists of two pieces of data: a chain map Φ of morphism spaces, which in this case we take to be the identity map (as a graded linear map), and a λ–invariant relation (on which it is supported)

$$R \subset (T \times_{G(\mathcal{Z})} S^*) \times_\lambda ((T \times_{G(\mathcal{Z})} G'(\mathcal{Z})) \times_{G'(\mathcal{Z})} (S \times_{G(\mathcal{Z})} G'(\mathcal{Z}))^*)$$

which in this case we take to be the relation

$$(t \times_G s^*) R (t \times_G g') \times_{G'} (g' \times_G s^*),$$

for arbitrary $s \in S$, $t \in T$, $g' \in G'$.

The identity map Φ is clearly injective, both as a chain map and on homology. To check that R is injective (as in Definition 2.5.22), note that

$$(t \times_G s^*) R (t \times_G 1) \times_{G'} (1 \times_G s^*)$$

and that

$$(t_1 \times_G 1) \times_{G'} (1 \times_G s_1^*) = (t_2 \times_G g') \times_{G'} (g' \times_G s_2^*)$$

if and only if there is an element $k \in G'$ so that

$$(t_1 \times_G 1) \times (1 \times_G s_1^*) = (t_2 \times_G g'k) \times k^{-1} \cdot (g' \times_G s_2^*) = (t_2 \times_G g'k) \times (g'k \times_G s_2^*);$$

ie there are $g, h \in G$ so that the following equations hold:

$$\begin{aligned} t_1 \times 1 &= t_2 \cdot g \times g^{-1} g'k, \\ s_1 \times 1 &= s_2 \cdot h \times h^{-1} g'k. \end{aligned}$$

We conclude that $g = h$, and hence $t_1 = t_2 \cdot g$ and $s_1 = s_2 \cdot g$; ie $(s_1 \times_G t_1^*) = (s_2 \times_G t_2^*)$, as needed.

One can think of the functor $\mathcal{F}^\psi$ as given by $(\cdot \boxtimes {}^{\mathcal{A},\psi}\mathbb{I}_{\mathcal{A}})$, where we think of ${}^{\mathcal{A},\psi}\mathbb{I}_{\mathcal{A}}$ as the identity bimodule with its $G(\mathcal{Z})$–$G'(\mathcal{Z})$ grading by the set $G'(\mathcal{Z})$, endowed with the left $G(\mathcal{Z})$ action by left translation, and right $G'(\mathcal{Z})$–grading by $G'(\mathcal{Z})$. The stated isomorphism of functors (gotten by varying ψ) follows from the fact that

$${}^{\mathcal{A}'}_{\psi}\mathbb{I}_{\mathcal{A},\psi} \boxtimes {}^{\mathcal{A},\psi}\mathbb{I}_{\mathcal{A}} \cong {}^{\mathcal{A},\psi'}\mathbb{I}_{\mathcal{A}}.$$

Finally, we prove the claim about the image of $\mathcal{F}^\psi$. Given a $G(\mathcal{Z})$–refinable module $(M_{\mathcal{A}}, S)$, we proceed as follows. For each nonzero, gr'–homogeneous element $\boldsymbol{x}$ we define

$$\mathrm{gr}(\boldsymbol{x}) = \mathrm{gr}'(\boldsymbol{x}) \cdot \psi(\boldsymbol{s})^{-1}, \tag{3.13}$$

where $I(\boldsymbol{s})$ is the idempotent with $\boldsymbol{x} \cdot I(\boldsymbol{s}) = \boldsymbol{x}$ (see Remark 3.11). Let $\mathrm{gr}(M_{\mathcal{A}})$ denote the image in S of all homogeneous elements. Let $T \subset S$ denote the $G(\mathcal{Z})$–orbits of $\mathrm{gr}(M_{\mathcal{A}})$. The grading gr can be viewed as giving a T–grading on $M_{\mathcal{A}}$, for the $G(\mathcal{Z})$–set T. Moreover, applying $\mathcal{F}^{\psi}$ to this object, we obtain an object which is isomorphic to $M_{\mathcal{A}}$ with its original S–grading, via the $G'(\mathcal{Z})$–space isomorphism $T \times_{G(\mathcal{Z})} G'(\mathcal{Z}) \cong S$ (given by $(t, g') \mapsto t \cdot g'$). (Note that we are using here the fact that the grading set S is essential; otherwise the canonical map $T \times_{G(\mathcal{Z})} G'(\mathcal{Z}) \to S$ would fail to surject.) $\square$

Remark 3.14 We have phrased Lemma 3.12 in terms of right $\mathcal{A}_\infty$–modules over $\mathcal{A}$; but the same arguments work for type D structures, and indeed bimodules of various types. Note, however, when refining left, rather than right, modules, (3.13) gets replaced by $\mathrm{gr}(\boldsymbol{x}) = \psi(\boldsymbol{s}) \cdot \mathrm{gr}'(\boldsymbol{x})$. Similarly, if M is a left-right $\mathcal{A}$–bimodule, we define $\mathrm{gr}(\boldsymbol{x}) = \psi(\boldsymbol{s}) \cdot \mathrm{gr}'(\boldsymbol{x}) \cdot \psi(\boldsymbol{t})^{-1}$ when $I(\boldsymbol{s}) \cdot \boldsymbol{x} \cdot I(\boldsymbol{t}) = \boldsymbol{x}$.

When working with type D modules, we also need to relate the grading on $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$. Recall from [21, Equation (10.19)] that, if $r\colon Z \to -Z$ is the (orientation-reversing) identity map, then

$$R(j, \alpha) = (j, r_*(\alpha)) \tag{3.15}$$

defines an group antihomomorphism from $G(\mathcal{Z})$ to $G(-\mathcal{Z})$. For $\boldsymbol{s} \subset [2k]$ corresponding to an idempotent in $\mathcal{A}(\mathcal{Z})$, let $\boldsymbol{s}' = [2k] \setminus \boldsymbol{s}$ correspond to the complementary idempotent in $\mathcal{A}(-\mathcal{Z})$.

Definition 3.16 Given grading refinement data ψ for $\mathcal{A}(\mathcal{Z})$, the *reverse* of ψ is defined on the idempotents of $\mathcal{A}(-\mathcal{Z})$ by

$$\psi'(\boldsymbol{s}') = R(\psi(\boldsymbol{s}))^{-1}.$$

Lemma 3.17 *The reverse grading refinement data ψ' of Definition 3.16 is grading refinement data for $\mathcal{A}(-\mathcal{Z})$ (ie satisfies Definition 3.5).*

(Compare [21, Equation (10.35)].)

Proof For $g' \in G'(-\mathcal{Z})$ compatible with $I(\boldsymbol{s}')$ and $I(\boldsymbol{t}')$, we have

$$\psi'(\boldsymbol{s}') \cdot g' \cdot \psi'(\boldsymbol{t}')^{-1} = R(\psi(\boldsymbol{s}))^{-1} \cdot g' \cdot R(\psi(\boldsymbol{t})) = R(\psi(\boldsymbol{t}) \cdot R(g') \cdot \psi(\boldsymbol{s})^{-1}).$$

Now observe that $R(g')$ is compatible with $I(\boldsymbol{t})$ and $I(\boldsymbol{s})$, so $\psi(\boldsymbol{t}) \cdot R(g') \cdot \psi(\boldsymbol{s})^{-1}$ is in $G(\mathcal{Z})$ as ψ is grading refinement data for $\mathcal{A}(\mathcal{Z})$. Thus the last line in the displayed equation is in $G(-\mathcal{Z})$, as desired. $\square$

3.3 An example: The algebra of the surface of genus-one

We recall the algebra for a genus-one surface, as described in [21, Section 11.1].

Consider the surface F of genus 1. This can be represented by a unique pointed matched circle $\mathcal{Z}$. The corresponding algebra $\mathcal{A}(\mathcal{Z})$ has three nontrivial summands

$$\mathcal{A}(\mathcal{Z}) = \mathcal{A}(\mathcal{Z},-1) \oplus \mathcal{A}(\mathcal{Z},0) \oplus \mathcal{A}(\mathcal{Z},1).$$

The two outermost summands are not very interesting. The summand $\mathcal{A}(\mathcal{Z},-1)$ is isomorphic to $\mathbb{F}_2$ (there are zero moving strands). Although $\mathcal{A}(\mathcal{Z},1)$ is not one-dimensional, its homology is, and indeed $\mathcal{A}(\mathcal{Z},1)$ is quasi-isomorphic to $\mathbb{F}_2$.

Thus, in the genus-one case, the interesting algebra is the summand $\mathcal{A} = \mathcal{A}(\mathcal{Z},0)$.

That algebra is generated (over $\mathbb{F}_2$) by two idempotents ι_0 and ι_1, and six nontrivial elements ρ_1, ρ_2, ρ_3, ρ_{12}, ρ_{23}, and ρ_{123}.

The differential is zero, and the nonzero products are

$$\rho_1\rho_2 = \rho_{12}, \quad \rho_2\rho_3 = \rho_{23}, \quad \rho_1\rho_{23} = \rho_{123}, \quad \rho_{12}\rho_3 = \rho_{123}.$$

(All other products of two ρ's vanish.) There are also compatibility conditions with the idempotents:

$$\begin{aligned} \rho_1 &= \iota_0\rho_1\iota_1, & \rho_2 &= \iota_1\rho_2\iota_0, & \rho_3 &= \iota_0\rho_3\iota_1, \\ \rho_{12} &= \iota_0\rho_{12}\iota_0, & \rho_{23} &= \iota_1\rho_{23}\iota_1, & \rho_{123} &= \iota_0\rho_{123}\iota_1. \end{aligned}$$

The unrefined grading takes values in the group G' generated by quadruples $(m;a,b,c)$ where $j \in \frac{1}{2}\mathbb{Z}$, $a,b,c \in \mathbb{Z}$ and j is an integer if $a,b,c \in 2\mathbb{Z}$ or if $b \in 2\mathbb{Z}$ and $a,c \notin 2\mathbb{Z}$, and a half-integer otherwise. The multiplication on G' is given by

$$(m_1;a_1,b_1,c_1)\cdot(m_2;a_2,b_2,c_2) \\ = \left(m_1+m_2+\tfrac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}; a_1+a_2, b_1+b_2, c_1+c_2\right).$$

Here λ is the element $(1;0,0,0)$. Gradings on the algebra are specified by

$$\mathrm{gr}'(\rho_1) = (-\tfrac{1}{2};1,0,0), \quad \mathrm{gr}'(\rho_2) = (-\tfrac{1}{2};0,1,0), \quad \mathrm{gr}'(\rho_3) = (-\tfrac{1}{2};0,0,1).$$

The group $G \subset G'$ is generated by elements $(1;0,0,0)$, $(\frac{1}{2};0,1,1)$ and $(\frac{1}{2};1,1,0)$.

One choice of grading refinement data $\{\psi\}$ is given by the function $\psi\colon \{\iota_0,\iota_1\} \to G'$

$$\psi(\iota_0) = (0;0,0,0), \quad \psi(\iota_1) = (-\tfrac{1}{2};1,0,0). \tag{3.1}$$

With respect to this refinement, then, the induced G–grading on the algebra is specified by

$$\mathrm{gr}(\rho_1) = (0; 0, 0, 0), \quad \mathrm{gr}(\rho_2) = (-\tfrac{1}{2}; 1, 1, 0), \quad \mathrm{gr}(\rho_3) = (0; -1, 0, 1).$$

3.4 Induction and restriction functors

Consider two pointed matched circles $\mathcal{Z} = (Z, \boldsymbol{a}, M, z)$ and $\mathcal{Z}' = (Z', \boldsymbol{a}', M', z')$. Taking the connect sum of Z and Z' at the points z and z' we obtain a new matched circle $(Z \# Z', \boldsymbol{a} \cup \boldsymbol{a}', M \cup M')$. There are two natural choices of where to put the basepoint for $Z \# Z'$; for definiteness, we will put the basepoint in $Z \cap (Z \# Z')$, slightly counterclockwise of where the original z was. Thus, we obtain a new pointed matched circle $\mathcal{Z} \# \mathcal{Z}'$. See Figure 8. If $\mathcal{Z}$ specifies a surface F and $\mathcal{Z}'$ specifies F' then $\mathcal{Z} \# \mathcal{Z}'$ specifies $F \# F'$.

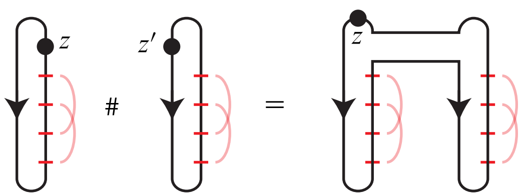


Figure 8: The connect sum of pointed matched circles

There is an obvious inclusion map

$$i_{\mathcal{Z},\mathcal{Z}'}\colon \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}') \to \mathcal{A}(\mathcal{Z} \# \mathcal{Z}').$$

This is a morphism of differential algebras. If we fix grading refinement data for ψ_1 for $\mathcal{A}(\mathcal{Z})$, ψ_2 for $\mathcal{A}(\mathcal{Z}')$, and ψ_{12} for $\mathcal{A}(\mathcal{Z} \# \mathcal{Z}')$ independently, then the bimodule ${}^{\mathcal{A}(\mathcal{Z}\#\mathcal{Z}')}[i_{\mathcal{Z},\mathcal{Z}'}]_{\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(\mathcal{Z}')}$ is graded by $G(\mathcal{Z}) \times_{\mathbb{Z}} G(\mathcal{Z}') \cong G(\mathcal{Z} \# \mathcal{Z}')$ as follows. For $\iota_{\boldsymbol{s}_1 \times \boldsymbol{s}_2}$ an idempotent in $\mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}')$, set the grading of the generator $I(\boldsymbol{s}_1 \times \boldsymbol{s}_2)$ to be

$$\mathrm{gr}(I(\boldsymbol{s}_1 \times \boldsymbol{s}_2)) = \psi_{12}(\boldsymbol{s}_1 \times \boldsymbol{s}_2)\psi_1(\boldsymbol{s}_1)^{-1}\psi_2(\boldsymbol{s}_2)^{-1}. \tag{3.1}$$

(Compare Equation (3.8).) We must check that, for $a \in \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}')$ from idempotent $\boldsymbol{s}_1 \times \boldsymbol{s}_2$ to $\boldsymbol{t}_1 \times \boldsymbol{t}_2$, that $\iota_{\boldsymbol{s}_1 \times \boldsymbol{s}_2} \cdot a$ and $a \cdot \iota_{\boldsymbol{t}_1 \times \boldsymbol{t}_2}$ have consistent grading, ie

$$\mathrm{gr}(I(\boldsymbol{s}_1 \times \boldsymbol{s}_2)) \cdot \mathrm{gr}_{\psi_1 \times \psi_2}(a) = \mathrm{gr}_{\psi_{12}}(a) \cdot \mathrm{gr}(I(\boldsymbol{t}_1 \times \boldsymbol{t}_2)).$$

This straightforward computation is analogous to Equation (3.9). Thus, these inclusion maps induce (graded) restriction functors of module categories

$$\mathrm{Rest}_{\mathcal{Z},\mathcal{Z}'}\colon \mathsf{Mod}_{\mathcal{A}(\mathcal{Z}\#\mathcal{Z}')} \to \mathsf{Mod}_{\mathcal{A}(\mathcal{Z}),\mathcal{A}(\mathcal{Z}')}; \tag{3.2}$$

see Section 2.4.2.

There are also projection maps

$$p_{\mathcal{Z},\mathcal{Z}'}\colon \mathcal{A}(\mathcal{Z} \# \mathcal{Z}') \to \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}')$$

obtained by setting to zero any basis element which crosses between Z and Z'. (This can also be thought of as adding a second basepoint to $\mathcal{Z} \# \mathcal{Z}'$.) For arbitrary grading refinement data for $\mathcal{A}(\mathcal{Z})$, $\mathcal{A}(\mathcal{Z}')$, and $\mathcal{A}(\mathcal{Z} \# \mathcal{Z}')$, the bimodule ${}^{\mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(\mathcal{Z}')}[p_{\mathcal{Z},\mathcal{Z}'}]_{\mathcal{A}(\mathcal{Z}\#\mathcal{Z}')}$ can be graded by Equation (3.1) as before. Thus, we get an induction functor

$$\mathrm{Induct}^{\mathcal{Z},\mathcal{Z}'}\colon {}^{\mathcal{A}(\mathcal{Z}\#\mathcal{Z}')}\mathsf{Mod} \to {}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(\mathcal{Z}')}\mathsf{Mod};$$

again, see Section 2.4.2.

These restriction and induction functors will be used in defining the type *AA* and *DD* modules, respectively, in Section 6.

3.5 Mapping class groupoid

Fix an integer k. Let $\mathcal{Z}$ be a pointed matched circle on $4k$ points, and let $F^\circ(\mathcal{Z})$ be the associated surface-with-boundary.

Given pointed matched circles $\mathcal{Z}_1$ and $\mathcal{Z}_2$ let

$$MCG_0(\mathcal{Z}_1,\mathcal{Z}_2) = \{\phi\colon F^\circ(\mathcal{Z}_1) \xrightarrow{\cong} F^\circ(\mathcal{Z}_2) \mid \phi(z_1) = z_2\}/\text{isotopy}$$

denote the set of basepoint-respecting isotopy class of homeomorphisms $\phi\colon F^\circ(\mathcal{Z}_1) \to F^\circ(\mathcal{Z}_2)$ carrying z_1 to z_2, where $z_i \in \partial F^\circ(\mathcal{Z}_i)$ is the basepoint. (The subscript 0 is to indicate that the maps respect the boundary and the basepoint.)

The *genus-k bordered mapping class groupoid $MCG_0(k)$* is the category whose objects are pointed matched circles with $4k$ points and with morphism set between $\mathcal{Z}_1$ and $\mathcal{Z}_2$ given by $MCG_0(\mathcal{Z}_1,\mathcal{Z}_2)$. This is clearly a groupoid: each morphism is invertible, and any two objects can be connected by a morphism. Moreover, the endomorphisms of a given pointed matched circle is identified with the strongly based mapping class group of the surface of genus k.

Remark 3.1 Our mapping class groupoid is closely related to the *Ptolemy groupoid* studied by Penner [31]. In particular, our pointed matched circles are called *chord diagrams* in the fat graph literature; see Andersen, Bene and Penner [1] and Bene [4].

4 Homology of the algebra

In this section we compute the homology $H(\mathcal{A}(\mathcal{Z}))$ of the algebras $\mathcal{A}(\mathcal{Z})$. We will denote $H(\mathcal{A}(\mathcal{Z}))$ by $\mathcal{H}(\mathcal{Z})$. Like $\mathcal{A}(\mathcal{Z})$, $\mathcal{H}(\mathcal{Z})$ is graded by $G'(\mathcal{Z})$. Specifically, the degree (ℓ, h) part of $\mathcal{H}(\mathcal{Z})$ is represented by cycles whose spin^c part is h, and whose number of crossings is determined, up to an additive constant depending on the one-chain and initial idempotent, by ℓ; see Equation (3.1).

This calculation will be used later in proving functorial properties of bordered Heegaard Floer homology (especially in showing that the identity diffeomorphism induces the identity bimodule, Theorem 4). For the purpose of the following statement, recall that if $p \in [4k]$, we let $M(p)$ denote its corresponding equivalence class with respect to the matching.

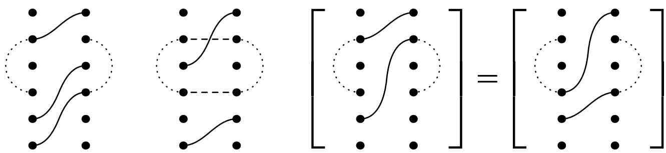


Figure 9: Illustration of the statement of Theorem 9: from left to right, we have an algebra element whose homology class is ruled out by condition (2), an algebra element whose homology class is ruled out by condition (3), and two algebra elements representing the same homology class. In each case only a portion of the matched circle is drawn.

Theorem 9 *Let s, t be a pair of subsets of $[2k]$. The degree (ℓ, h) part of $I(s) \cdot \mathcal{H}(\mathcal{Z}) \cdot I(t)$ is trivial unless all of the following conditions hold:*

(1) *The homology class h is compatible with $I(s)$ and $I(t)$, in the sense of Definition 3.4.*

(2) *The local multiplicities of $h \in H_1(Z, \boldsymbol{a})$ are 0 or 1.*

(3) *If $p_1, p_2 \in \boldsymbol{a}$ are matched points (so $M(p_1) = M(p_2)$), $p_1 \in \mathrm{Int}(\mathrm{supp}(h))$, and $p_2 \notin \mathrm{Int}(\mathrm{supp}(h))$, then $M(p_1) \notin s \cap t$.*

(4) *The Maslov degree ℓ is minimal among all algebra elements with associated one-chain h.*

Furthermore, for every (ℓ, h), $I(s)$, and $I(t)$ satisfying the conditions above, the degree (ℓ, h) part of $I(s) \cdot \mathcal{H} \cdot I(t)$ is 1–dimensional, and is represented by any crossingless diagram of the correct grading.

The above result can be used to calculate the ranks of the homology. For example, by counting the spin^c parts allowed in the above theorem, one finds that for the split pointed matched circle for the surface of genus-two $\mathcal{Z}_{\mathrm{spl}}$, $\mathcal{H}(\mathcal{Z}_{\mathrm{spl}})$ has total rank 164, divided up according to the number of strands as follows:

$$\sum_i \dim(\mathcal{H}(\mathcal{Z}_{\mathrm{spl}}, i)) \cdot T^i = T^{-2} + 32 \cdot T^{-1} + 98 + 32 \cdot T^1 + T^2.$$

On the other hand, if $\mathcal{Z}$ is the antipodal pointed matched circle for the surface of genus-two then $\mathcal{H}(\mathcal{Z}, 0)$ has rank 70, though the other ranks are the same; ie

$$\sum_i \dim(\mathcal{H}(\mathcal{Z}, i)) \cdot T^i = T^{-2} + 32 \cdot T^{-1} + 70 + 32 \cdot T + T^2$$

(so the total rank is 136). The different ranks here underscore the fact that the algebra is really associated to a pointed matched circle, rather than its underlying surface. We shall, however, see later that the derived categories of modules for different representatives of the same surface are equivalent; cf Theorem 1.

4.1 Computing the homology

Before proving Theorem 9, we first study the (rather simple) homology of $\mathcal{A}(n,k)$.

Lemma 4.1 *For $\boldsymbol{S}$ and $\boldsymbol{T}$ two subsets of $\{1, \dots, n\}$, if the unique homology class $h \in H_1(Z, \boldsymbol{a})$ satisfying $\partial h = \boldsymbol{T} - \boldsymbol{S}$ has all nonnegative local multiplicities then $I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$ is nonempty. If some local multiplicity is at least two, then $I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$ has dimension at least 2 (ie there is some element with a crossing in it).*

Proof If $h = 0$, then $\boldsymbol{S} = \boldsymbol{T}$ and we can take the element $I(\boldsymbol{S})$. If h has nonnegative local multiplicities, pick some maximal interval $[i, j] \subset Z$ with positive multiplicity, connect i to j, and subtract 1 from all multiplicities in $[i, j]$. The result still has nonnegative multiplicity and by induction we can find an element of $\mathcal{A}(n, k-1)$ with these multiplicities. When we add the first strand we get our desired element. If h has multiplicity at least 2 anywhere, we will consider nested intervals in this procedure and therefore construct an element of the strands algebra with a crossing. □

Proposition 4.2 *For $\boldsymbol{S}$ and $\boldsymbol{T}$ two subsets of $\{1, \dots, n\}$, if $I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$ is 1–dimensional, so is its homology; otherwise the homology is 0.*

Proof Each summand $I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$ has a well-defined induced one-chain h with $\partial h = \boldsymbol{T} - \boldsymbol{S}$. If $I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$ is more than one-dimensional, it is easy to see that at least one of following two conditions is satisfied:

(1) Some element of $\boldsymbol{S} \cap \boldsymbol{T}$ is in the interior of the support of h.

(2) The chain h has local multiplicity at least 2 at some point.

Thus, our goal is to show that if h satisfies either of the above two conditions then the corresponding summand of $\mathcal{A}(n,k)$ is acyclic.

We start with case (1): suppose that $i \in \boldsymbol{S} \cap \boldsymbol{T}$ is contained in a part of the support of h where the local multiplicity (both just above and just below i) is 1. (The other possibilities in case (1) will be handled in case (2) below.) Then we define a map

$$H\colon I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T}) \to I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$$

as follows. Recall that the part of the algebra $I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$ is spanned by upward-veering maps $\phi\colon \boldsymbol{S} \to \boldsymbol{T}$. Given such a map ϕ, define $H(\phi)$ by

$$H(\phi) = \begin{cases} 0 & \phi(i) = i, \\ \phi_{(i,j)} & \phi(i) \neq i, \phi(j) = i. \end{cases}$$

(Recall that for σ a transposition, ϕ_σ was defined in Section 3 and switches the roles of the inputs i and j.)

We claim that this map H is a nullhomotopy of the identity on this part of the algebra:

$$\partial \circ H + H \circ \partial = \mathbb{I}. \tag{4.3}$$

It suffices to verify this for generators $\phi\colon \boldsymbol{S} \to \boldsymbol{T}$, with the same two cases as in the definition of H:

- Suppose $\phi(i) = i$. Then, $H(\phi) = 0$. Moreover, there is exactly one term in $\partial\phi$ for which H does not vanish: it is the term corresponding to the resolution of the horizontal strand at i with a unique strand corresponding to $\phi(j) = k$, where $j < i < k$. (Note that this term is nonzero: since the local multiplicity of h at i is one, the resolution of this crossing cannot introduce a new double-crossing.) Of course, H applied to this resolution is ϕ; thus we have verified (4.3) in this case.
- Suppose there are $k < i < j$ with $\phi(i) = j$ and $\phi(k) = i$. Now, terms in $\partial H(\phi)$ and those in $H\partial(\phi)$ pair off except for the single term in $\partial H(\phi)$ where we resolve the crossing in $H(\phi)$ which did not already appear in ϕ. This term, of course, gives ϕ.

We have verified (4.3), and hence the homology of $I(\boldsymbol{S}) \cdot \mathcal{A}(n,k) \cdot I(\boldsymbol{T})$ is trivial in this part of case (1).

For case (2), we proceed similarly. Now, choose i so that the local multiplicity of h just below i is 1 and the local multiplicity just above i is 2. Indeed, we can choose i so that the local multiplicity of h at any point less than i is strictly less than 2. We then let j denote the element of $\boldsymbol{S}$ just below i. Since we have already considered case (1), we can also assume without loss of generality that the local multiplicity of h just below j is zero. Once again, we define a map H satisfying (4.3), by defining $H(\phi)$ for $\phi\colon \boldsymbol{S} \to \boldsymbol{T}$; only now, the cases are slightly different:

$$H(\phi) = \begin{cases} 0 & \phi(i) < \phi(j), \\ \phi_{(i,j)} & \phi(j) < \phi(i). \end{cases}$$

We verify (4.3) for generators $\phi\colon \boldsymbol{S} \to \boldsymbol{T}$. In either case, there is a strand s_i starting at i and a strand s_j starting at j.

- Suppose that $\phi(i) < \phi(j)$. Then $H(\phi) = 0$, and there is exactly one term in $\partial\phi$ for which $H(\phi)$ is nonzero, corresponding to the resolution of the crossing of s_i and s_j. (Again, the assumptions on the multiplicities guarantee that this term is nonzero.)
- Suppose that $\phi(i) > \phi(j)$. We divide the crossings in ϕ into two kinds: a *special* crossing is a crossing of a strand s with s_i, which has the property that s also crosses s_j. Any other crossing is said to be *generic*. If ψ is a resolution of ϕ at a generic crossing, then it is easy to see that $\psi(i) > \psi(j)$, and hence that the terms in $H(\partial\phi)$ which come from generic resolutions of ϕ cancel corresponding terms in $\partial H(\phi)$. If ψ is a resolution of ϕ at a special crossing, however, $\psi(i) < \psi(j)$ and hence $H(\psi) = 0$. Likewise, the corresponding terms in $\partial H(\phi)$ which are gotten by resolving the special crossings are also zero, as these resolutions all introduce double-crossings in $H(\phi)$. Finally, there is one leftover term in $\partial H(\phi)$ which does not appear in $H(\partial\phi)$, namely the term gotten by resolving the crossing between s_i and s_j; this gives ϕ back, as desired.

This completes the proof of the proposition. □

Remark 4.4 We could alternately prove Proposition 4.2 by identifying $I(\boldsymbol{S})\cdot\mathcal{A}(n,k)\cdot I(\boldsymbol{T})$ with an interval in Hasse diagram of the Bruhat order of the symmetric group on k letters, and applying the results of Björner and Wachs [6], which imply that such an interval is acyclic if it has more than one element. The argument above can be viewed as an explicit proof of this fact.

Proof of Theorem 9 We first prove the necessary hypotheses as stated.

(1) Without the compatibility condition, the degree (ℓ, h) part of $I(\boldsymbol{s}) \cdot \mathcal{A}(\mathcal{Z}) \cdot I(\boldsymbol{t})$ is trivial.

(2) If the local multiplicity of h is negative anywhere, then the degree $(*, h)$ part of $I(\boldsymbol{s}) \cdot \mathcal{A}(\mathcal{Z}) \cdot I(\boldsymbol{t})$ is empty. Otherwise, suppose that the multiplicity of h is bigger than 1 somewhere. Consider the filtration on $I(\boldsymbol{s}) \cdot \mathcal{A}(\mathcal{Z}) \cdot I(\boldsymbol{t})$ given by the number of horizontal strands. The differential on $\mathcal{A}(\mathcal{Z})$ can decrease the number of horizontal strands by at most one, in the case where we smooth a crossing with a horizontal strand. We will show that the homology of the associated graded complex $C(\ell, h, *) = \bigoplus_{m \in \mathbb{Z}} C(\ell, h, m)$ is zero. Here ℓ and h are the gradings inherited from before, and m is the newly-introduced grading, which we think of as the number of *nonhorizontal* strands. We claim that $C(*, h, m)$ is nearly identified with the corresponding subcomplex of $\mathcal{A}(4k, m)$. The complex is not quite the same since we need to forbid horizontal strands; however, we can shift down the endpoints on the right by one half unit, to obtain an embedding of $C(*, h, m)$ into $I(\boldsymbol{S}) \cdot \mathcal{A}(8k, m) \cdot I(\boldsymbol{T}') \subset \mathcal{A}(8k, m)$ (where here $\boldsymbol{T}'$ is gotten from $\boldsymbol{T}$ by shifting down by one half). If h has local multiplicity greater than or equal to 2 somewhere, then so do the generators of $I(\boldsymbol{S}) \cdot \mathcal{A}(8k, m) \cdot I(\boldsymbol{T}')$. It follows from Lemma 4.1 that the dimension of $I(\boldsymbol{S}) \cdot \mathcal{A}(8k, m) \cdot I(\boldsymbol{T}')$ is at least two, and hence, by Proposition 4.2, that its homology is trivial.

(3) Suppose not; then $M(p_1)$ is in both the initial and final idempotent. Modify the filtration from case (2) by considering the filtration on $I(\boldsymbol{s}) \cdot \mathcal{A}(\mathcal{Z}) \cdot I(\boldsymbol{t})$ given by the number of horizontal strands *other than the one at* p_1 (if there is one). Again we can identify the associated graded pieces with an appropriate part of $\mathcal{A}(8k, m)$, where we shift all right endpoints other than p_1 down by half a unit. (To see this identification, note that since $p_2 \notin \mathrm{Int}(\mathrm{supp}(h))$, there are no crossings with the horizontal strand at p_2.) If we delete p_1 from the initial and final idempotent, by Lemma 4.1 we can construct an algebra element with these endpoints in the corresponding summand of $\mathcal{A}(8k, m-1)$. If we then add a horizontal strand at p_1, we introduce an element with a crossing (since $p_1 \in \mathrm{Int}(\mathrm{supp}(h))$). It follows from Proposition 4.2 that the homology of this piece is trivial.

If none of the other conditions apply, our homology class h consists of a disjoint union of intervals with multiplicity 1, so that for every matched pair $\{p_1, p_2\}$ that is contained in both $\boldsymbol{s}$ and $\boldsymbol{t}$, either both of p_1 and p_2 are in $\mathrm{Int}(\mathrm{supp}(h))$ or neither is. Let m be the number where both are. As before, consider the filtration by the number of horizontal strands. In this case the filtration is actually a grading (which the differential drops by one), since the only possible crossings are between nonhorizontal strands and horizontal strands. Then the complex $I(\boldsymbol{s})\cdot\mathcal{A}(\mathcal{Z})\cdot I(\boldsymbol{t})$ is isomorphic to the standard complex for the m–dimensional hypercube, with the gradings matching (up to an overall shift). In particular, there is a unique element in H_0 (which is the lowest Maslov grading), represented by any generator in this grading. Such a generator corresponds to a crossingless matching. □

4.2 Massey products

Recall (Corollary 2.1.18) that one can endow $\mathcal{H}(\mathcal{Z})$ with an $\mathcal{A}_\infty$–structure so that $\mathcal{A}(\mathcal{Z})$ is quasi-isomorphic to $\mathcal{H}(\mathcal{Z})$. As discussed in Section 2.1.3, while many of the induced higher products on $\mathcal{H}(\mathcal{Z})$ are not entirely canonical, some of them are. The aim of the present section is to show that the homology $\mathcal{H}(\mathcal{Z})$ is generated by canonically determined higher products of chords of length one, in a suitable sense. This will have as a corollary a certain rigidity of the algebra (Proposition 4.7), which will be useful for us in Section 8.1.

Proposition 4.1 *Let $\zeta\in\mathcal{H}(\mathcal{Z})$ be a nontrivial, homogeneous homology class whose support has length greater than one. Then there is some $m>1$ and a Massey admissible sequence (in the sense of Definition 2.1.21) of homogeneous elements $\alpha_1,\dots,\alpha_m\in\mathcal{H}(\mathcal{Z})$ with nonzero support with the property that $\zeta=\overline{\mu}_m(\alpha_1,\dots,\alpha_m)$.*

We will use the following technical lemma to ensure Massey admissibility.

Lemma 4.2 *Let $\mathcal{A}=\mathcal{A}(\mathcal{Z})$. Let $\alpha_1,\dots,\alpha_m$ be a collection of homogeneous homology classes in $\mathcal{H}$, and choose homogeneous representing cycles $a_1,\dots,a_m$. Suppose that there are elements $\xi_{i,j}\in\mathcal{A}$ defined for $1\le i<j\le m$ so that:*

- *For $i=2,\dots,m$, $\xi_{i-1,i}=a_i$.*
- *$d\xi_{i,j}=\sum_{i<k<j}\xi_{i,k}\cdot\xi_{k,j}$.*
- *For each $j>i+1$, we have $d\xi_{i,j}\neq 0$.*
- *For $1<i<m$, there is no algebra element whose support is $[\alpha_1]+\cdots+[\alpha_i]$ and whose initial idempotent agrees with the initial idempotent of α_1.*

Then the associated homology classes $\alpha_1,\dots,\alpha_m$ form a Massey admissible sequence, and moreover $a_1\cdot\xi_{1,m}$ represents $\overline{\mu}_m(\alpha_1,\dots,\alpha_m)$ (for any choice of compatible $\overline{\mu}_i$).

Proof We wish to apply Lemma 2.1.22. When $1\le i<j\le m$, $\xi_{i,j}$ is a chain whose grading is $\lambda\cdot g(i+1,j)$, where $g(i,j)$ is the grading of $\overline{\mu}_{j-i+1}(\alpha_i,\dots,\alpha_j)$. (We write now $\lambda\cdot g(i,j)$ in place of the $g(i,j)+1$ appearing in Definition 2.1.21, where the

grading set was $\mathbb{Z}$.) Since for $1 < i < i+1 < j \leq m$ the differential of $d\xi_{i,j}$ is nonzero (by hypothesis), it follows from Theorem 9 that the homology group $\mathcal{H}_{\lambda\cdot g(i+1,j)}$ is 0 for all $1 \leq i < i+1 < j \leq m$. Moreover, the final condition ensures that for $1 < j < m$, $\overline{\mu}_i(\alpha_1,\dots,\alpha_j) = 0$ since in fact there is no algebra element in the appropriate grading. Similarly, $\mathcal{H}_{\lambda\cdot g(1,j)}$ is 0. Thus, $\alpha_1,\dots,\alpha_m$ is a Massey admissible sequence, and in fact Lemma 2.1.22 applies (after we extend the $\xi_{i,j}$'s above by setting $\xi_{0,1} = a_1$ and $\xi_{0,j} = 0$ for $1 < j < m$). $\square$

Before proving Proposition 4.1, we introduce some more terminology, and then some further lemmas.

Let ζ be an element of $\mathcal{H}(\mathcal{Z})$ supported in grading (ℓ, h). A point p in $\boldsymbol{a}$ is called *fully unoccupied* if $M(p)$ does not appear in either the initial or the final idempotent of ζ. A point p_1 in $\boldsymbol{a}$ is called *fully internal* if both p_1 and its mate p_2 are contained in $\mathrm{Int}(\mathrm{supp}(h))$. It is called *fully internal and unoccupied* if p_1 is fully internal and $M(p_1)$ is not contained in the initial (and hence also the terminal) idempotent.

Lemma 4.3 *Suppose that ζ is a nontrivial homology class in $\mathcal{H}(\mathcal{Z})$ supported in degrees (ℓ, h), and suppose that there is a point p which is fully internal and unoccupied. Then, we have a Massey admissible sequence of $m > 1$ homology classes $\alpha_1,\dots,\alpha_m$, each of which has nontrivial support, with*

$$\zeta = \overline{\mu}_m(\alpha_1,\dots,\alpha_m).$$

Proof According to Theorem 9, ζ can be represented by a single diagram (rather than a formal sum of diagrams). Now, any representative for ζ has some strand which crosses p. In fact, we can find a representative for ζ with the property that on the strand s_1 through p, there are no points which are fully internal and occupied. We find a representative $a \in \mathcal{A}$ for ζ with the property that all the other strands of a become stationary after the moment when s_1 hits p_1, after which point only s_1 moves. See Figure 10 for an illustration.

This provides a factorization (in $\mathcal{A}$) $a = b_1 \cdot b_2$, where b_1 is the part of ζ before s_1 crosses p, and b_2 corresponds to the portion of s_1 starting at p (and whose initial idempotent coincides with the terminal idempotent of b_1). Our condition on the strand s_1 ensures that b_1 is a cycle. By contrast, b_2 need not be a cycle: we can break s_1 into $m-1$ segments by positions $q_2,\dots,q_{m-1}$, where the q_i are those points in the interior of the support of s_1 but above p which are occupied but only partially internal in b_2. In order for ζ to be homologically nontrivial, each q_i must be matched with some corresponding q_i' contained in the support of b_1. The q_i' are necessarily in the terminal but not initial idempotent of b_1, so that there is a strand in b_1 entering q_i'.

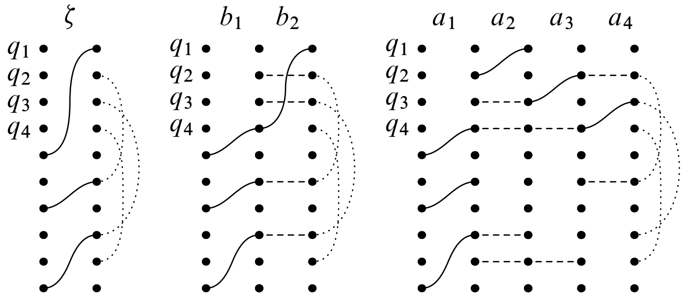


Figure 10: Illustration of the proof of Lemma 4.3: we start with the cycle ζ, which has a strand s_1 which crosses the position q_4, which is fully internal and unoccupied. The strand s_1 then crosses two other positions (q_3 and q_2) which are matched with terminal points in ζ. Thus, factoring the portion of the strand starting at q_4 to the right, we obtain a factorization of ζ into $b_1 \cdot b_2$, where b_2 is not a cycle. The Massey factorization into $a_1, \dots, a_4$ is shown on the right.

(The q_i' are not in the interior of h, since the q_i were not fully internal and occupied; we will use this observation towards the end of the proof.) We label the q_i with the following conventions:

- q_1 is the terminal point of the strand s_1.
- $\{q_i\}$ are ordered in the opposite order to the order they are encountered along s_1.
- $q_m = p$.

We construct cycles $\{a_i\}_{i=1}^m$ as follows. Let $a_1 = b_1$. To define the other a_i, note that the terminal idempotent of b_1 has the form $I(\boldsymbol{s})$, where $\boldsymbol{s} = \boldsymbol{s}_1 \setminus \{M(q_1)\}$ and $\boldsymbol{s}_1$ includes all the $M(q_i)$. Now, for $i = 2, \dots, m$, the initial idempotent of a_i is $\boldsymbol{s}_1 \setminus \{M(q_{i-1})\}$ and a_i consists of a single moving strand, which is the portion of s_1 from q_i to q_{i-1}. Clearly, each a_i is a cycle.

Now, define $\xi_{i,j}$ for $1 \le i < j \le m$ to be the algebra element obtained from the substrand of s_1 which goes from q_j to q_i, and with initial idempotent $I(\boldsymbol{s}_1 \setminus \{M(q_j)\})$. In particular for $2 \le i \le m$, we have $\xi_{i-1,i} = a_i$. Any algebra element with support $[a_1] + \cdots + [a_i]$ with $i < m$ has some strand which starts at q_i; but $M(q_i)$ is not in the initial idempotent of α_1, so there are no such elements. (Here we are using the fact that none of the q_i was fully internal and occupied, as promised). Lemma 4.2 applies, showing $\alpha_1, \dots, \alpha_m$ is Massey admissible, and further that $\bar{\mu}_m(\alpha_1, \dots, \alpha_m) = a_1 \cdot \xi_{1,m} = b_1 \cdot b_2 = [\zeta]$. □

Lemma 4.4 *Suppose that ζ is a nontrivial homology class in $\mathcal{H}(\mathcal{Z})$ supported in degrees (ℓ, h), and suppose that there is a point p in the interior of the support of h which is matched with another point p' on the boundary of the support. Then we have a Massey admissible sequence of $m > 1$ homology classes $\alpha_1, \dots, \alpha_m$, each with nontrivial support, with*

$$\zeta = \overline{\mu}_m(\alpha_1, \dots, \alpha_m).$$

Proof By hypothesis, $M(p)$ occurs in either the initial or the terminal idempotent of ζ, but not both. We focus first on the case where $M(p)$ is in the initial idempotent.

As in the proof of Lemma 4.3, we find a strand s_1 which moves across p, and in fact, we can find a representative of ζ so that no points which are fully internal and occupied are encountered on that strand. We then find a representative a for ζ so that all the other strands of a are stationary after the moment where s_1 hits p.

As before, this gives a factorization $a = b_1 \cdot b_2$, where b_2 corresponds to the substrand of s_1 starting at p. Once again, b_1 is a cycle. The strands picture for b_2 might have some crossings, which are in one-to-one correspondence with points in the support of s_1 above p and which are matched with terminal points of b_1. (Figure 10 can be modified to give a picture of the present case: simply cut off the bottom-most length one interval in the pointed matched circle, so that now q_4 is matched with an initial point of b_1.) Our sequence $a_1, \dots, a_m$ is now obtained by the same mechanism as in the proof of Lemma 4.3.

The case where $M(p)$ is in the terminal idempotent of ζ follows similarly. In this case, we find a representative for our strands so that all the strands of z are stationary until the moment where s_1 hits p. This now gives a factorization $z = b_1 \cdot b_2$, where b_1 corresponds to the substrand of s_1 starting at its initial point and going until p, and b_2 consists of the rest of s_1 and all other strands. Now, b_2 is a cycle. The strands picture for b_1 might have some crossings, which are in one-to-one correspondence with the points in the support of s_1 after its initial point but before p. A slight modification to Lemma 4.2 then applies. □

Lemma 4.5 *Suppose that ζ is a nontrivial homology class in $\mathcal{H}(\mathcal{Z})$ supported in degrees (ℓ, h), and suppose that there is no point in the interior of the support of h which is matched with either an initial or terminal point of ζ, or which is fully unoccupied. Then, there is some terminal point which is not matched with any other point in the support of h.*

Proof Suppose on the contrary that every terminal point is matched with another point in h. Since no point in the interior is matched with an initial or terminal point, it follows that each terminal point is matched with an initial point. Moreover, since there are no fully unoccupied points, and ζ is nontrivial, criterion (3) in Theorem 9 ensures every point in the interior of h is matched with another point in the interior of h. It is now easy to see that the support of h disconnects the one-manifold obtained by doing surgery on the matched pairs in $\mathcal{Z}$, contrary to the definition of a pointed match circle. □

Proof of Proposition 4.1 Suppose $\zeta \in I(\boldsymbol{s}) \cdot \mathcal{H}(\mathcal{Z}) \cdot I(\boldsymbol{t})$ is a nontrivial homology class whose support has length greater than one. Let p_1 be any point in the interior of the support of ζ, and let p_2 be the other point with $M(p_1) = M(p_2)$. We have the following cases:

- If $M(p_1) \notin \boldsymbol{s} \cup \boldsymbol{t}$, then Lemma 4.3 provides the desired factorization.
- If p_2 is in the boundary of the support of ζ, then Lemma 4.4 provides the desired factorization.
- If p_2 is not contained in the closure of the support of ζ, then but $M(p_1) \in \boldsymbol{s} \cup \boldsymbol{t}$, then this also forces $M(p_1) \in \boldsymbol{s} \cap \boldsymbol{t}$ (since p_1 is in the interior of the support of ζ), which, in view of Theorem 9(3), contradicts the assumption that ζ is homologically nontrivial.

Thus, we can assume that every interior point is occupied in both initial and terminal idempotents and is equivalent to some other point in the interior (ie it is fully internal and occupied). Moreover, Lemma 4.5 ensures that there is a point p on the boundary of the support which is not equivalent to any other point in the support of h. We assume that p is an initial point; the case when it is a terminal point is similar.

Consider the point q just above p. We have shown that it is in the terminal idempotent, whether or not it is in the interior of the support; in any case, there is a representing cycle a which contains a length one strand from p to q. We can factor this strand off the right, so as to factor a as a product of two cycles. □

As a digression, the following is a corollary of the above proofs, showing that for the *split* genus-k matched circle, Proposition 4.1 can be strengthened.

Corollary 4.6 *For the split genus k matched circle $\mathcal{Z}_{spl}$, $\mathcal{H}(\mathcal{Z}_{spl})$ is generated as an algebra by homology classes of Reeb chords of length 1.*

Proof Suppose first there is a point p_1 on the boundary of the support which is matched with a point p_2 in the interior. Then, since the since the distance between p_1

and p_2 is two, there is a unique point q in between them. We can factor off the chord between p_1 and q on one side or the other (depending on whether the idempotent of q is occupied or not), so that the two factors are both cycles.

Otherwise, if Lemma 4.3 applies, the factorization of a into algebra elements $b_1 \cdot b_2$ is a factorization into cycles.

In the remaining case of Proposition 4.1 (when neither Lemma 4.3 nor Lemma 4.4 applies), we obtain a factorization into cycles as in the proof of Proposition 4.1. $\square$

Aspects of Proposition 4.1 are illustrated in Figure 11. In particular, the second example shows that $\mathcal{A}(\mathcal{Z}_{\mathrm{spl}})$ is not formal when the genus is bigger than one, and the last example shows that Corollary 4.6 is not true for the antipodal matched circle of genus 2.

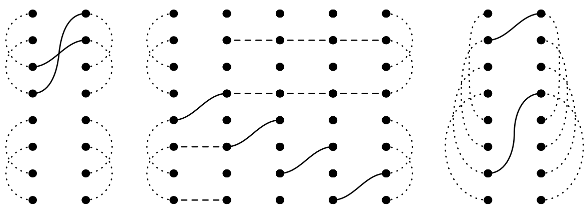


Figure 11: Factorization in the algebra: on the left, we have exhibited an element of the algebra for the split genus-two matched circle which cannot be factored into length one chords; however, this is not a cycle. (Note also that its differential cannot be written as a product of length one chords.) In the middle, there are 4 algebra elements in the split genus-two case with a well defined, nonzero Massey product, showing that the algebra is not formal. On the right, we have exhibited a homology class for the antipodal pointed matched circle which can be written as a Massey product, but not an ordinary product, of Reeb chords of length 1.

Proposition 4.1 has the following consequence which will be used in the proof of Theorem 4:

Proposition 4.7 *Suppose that $\phi\colon \mathcal{A}(\mathcal{Z}) \to \mathcal{A}(\mathcal{Z})$ is a G–graded $\mathcal{A}_\infty$ morphism such that $\phi_1(\xi) = \xi$ where ξ is any Reeb chord of length one. Then ϕ_1 induces the identity map on homology.*

Proof Precomposing ϕ with the standard quasi-isomorphism $f\colon \mathcal{H} \to \mathcal{A}$, we reduce to the following statement: Let $\psi\colon \mathcal{H} \to \mathcal{A}$ be a morphism of $\mathcal{A}_\infty$–algebras with the property that for each length 1 chord ξ, $\psi(\xi)$ is a cycle representing the homology class of ξ, then ψ_1 induces the identity map on homology; ie if ζ is any homology class, then $\psi_1(\zeta)$ is a cycle representing ζ.

We prove this by induction on the total size of the support of ζ. Let ζ be a nontrivial homology class with support of size bigger than 1, and find a corresponding Massey admissible sequence $\alpha_1,\dots,\alpha_m$ so that $\zeta = \bar{\mu}_m(\alpha_1,\dots,\alpha_m)$, where the support of each α_i is smaller than the support of ζ. Define $\xi_{i,j} = \psi_{j-i}(\alpha_{i+1},\dots,\alpha_j)$, so that $\xi_{i-1,i} = \psi_1(\alpha_i)$. By the inductive hypothesis, $\xi_{i-1,i}$ is a cycle representing the homology class α_i. The $\mathcal{A}_\infty$ relation for ψ, together with Massey admissibility, ensures that for $(i,j)\neq(0,m)$,

$$d\xi_{i,j} = \sum_{i<k<j} \xi_{i,k}\cdot\xi_{k,j}.$$

Lemma 2.1.22 ensures $\sum_{0<k<m}\xi_{0,k}\cdot\xi_{k,m}$ is a cycle representing $\bar{\mu}_m(\alpha_1,\dots,\alpha_m)$. On the other hand, another application of the $\mathcal{A}_\infty$ relation (and Massey admissibility) gives

$$d\xi_{0,m} = \sum_{0<k<m} \xi_{0,k}\cdot\xi_{k,m} + \psi_1(\bar{\mu}_m(\alpha_1,\dots,\alpha_m));$$

ie $\psi_1(\zeta)=\psi_1(\bar{\mu}_m(\alpha_1,\dots,\alpha_m))$ represents the homology class $\bar{\mu}_m(\alpha_1,\dots,\alpha_m)=\zeta$. □

Remark 4.8 The above proof works, provided ϕ preserves the relevant notions of homogeneity: ie it works whether ϕ is G– or G'–graded. In the application (see Section 8.1), we are interested in the case where ϕ is G–graded.

Remark 4.9 The $\mathcal{A}_\infty$–structure on $\mathcal{H}(\mathcal{Z})$ is nilpotent (Definition 2.1.9), by the argument of Lemma 3.2.

4.3 A smaller model for $\mathcal{A}(\mathcal{Z})$

Let $\mathcal{Z}$ be a pointed matched circle. Of course, $\mathcal{H}(\mathcal{Z})$ is derived equivalent to $\mathcal{A}(\mathcal{Z})$. Thus, for our purposes, we could always work with $\mathcal{A}_\infty$–modules over this homology. This has the advantage that the underlying algebra has smaller rank, but the disadvantage that now one must always keep track of $\mathcal{A}_\infty$ operations. There is, however, a natural intermediate level: there is a differential graded algebra $\mathcal{A}'(\mathcal{Z})$ which is a quotient of $\mathcal{A}$, but which is quasi-isomorphic to $\mathcal{A}$.

Definition 4.1 Let $\mathcal{I}\subset\mathcal{A}(\mathcal{Z})$ denote the differential ideal generated by all algebra elements which have local multiplicity greater than 1 somewhere.

Proposition 4.2 *The quotient map*

$$\mathcal{A}(\mathcal{Z}) \to \mathcal{A}(\mathcal{Z})/\mathcal{I} = \mathcal{A}'(\mathcal{Z})$$

is a quasi-isomorphism. Moreover, the map sending $M \mapsto M \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})} \mathcal{A}'(\mathcal{Z})$ induces an equivalence of derived categories.

Proof The quasi-isomorphism statement is a direct consequence of Theorem 9(2). The equivalence of derived categories statement follows; see Proposition 2.4.10 (or indeed [5, Theorem 10.12.5.1]). □

In view of Proposition 4.2, we could with use $\mathcal{A}'(\mathcal{Z})$ in place of $\mathcal{A}(\mathcal{Z})$ throughout the present paper. We chose not to do this for aesthetic reasons; but note that, for practical calculations, it is indeed preferable to work in $\mathcal{A}'(\mathcal{Z})$.

5 Bordered Heegaard diagrams

In this section, we extend the notion of a bordered Heegaard diagram of [21, Chapter 4] to 3–manifolds with two boundary components. (The generalization to manifolds with more than two boundary components is straightforward, and we mostly leave it to the interested reader; see also Remark 5.7.) This generalization was first sketched in the appendix to [21], which the reader may want to consult for a condensed treatment.

First, we recall the notion of strongly bordered three-manifolds with two boundary components, introduced in Definition 1.3. We adapt the definition slightly, so that borderings are specified by pointed matched circles, and give notation which will be used later.

Definition 5.1 A *strongly bordered three-manifold with two boundary components* $\mathcal{Y}$ is a tuple

$$\mathcal{Y} = (Y, \mathcal{Z}_L, \Delta_L, z_L, \phi_L, \mathcal{Z}_R, \Delta_R, z_R, \phi_R, \gamma_z, \nu_z),$$

where:

- Y is a compact, oriented three-manifold with two boundary components $\partial_L Y$ and $\partial_R Y$.
- $\Delta_L \subset \partial_L Y$ and $\Delta_R \subset \partial_R Y$ are preferred disks.
- $z_L \in \partial\Delta_L$ and $z_R \in \partial\Delta_R$ are basepoints on the boundaries of the disks.

- ϕ_L and ϕ_R are homeomorphisms
$$\phi_L\colon (F(\mathcal{Z}_L), D_L, z_L) \to (\partial_L Y, \Delta_L, z_L),$$
$$\phi_R\colon (F(\mathcal{Z}_R), D_R, z_R) \to (\partial_R Y, \Delta_R, z_R).$$
(Here, D_L and D_R are the preferred disks in $F(\mathcal{Z}_L)$ and $F(\mathcal{Z}_R)$, equipped with the basepoints z_L and z_R coming from the pointed matched circles; see Construction 3.2. In the interest of notational simplicity, we do not distinguish the notation for the basepoints in the model surface from the preferred basepoints on ∂Y.)
- γ_z is a path in Y connecting z_L to z_R.
- ν_z is an isotopy class of nowhere vanishing normal vector fields to γ_z pointing into Δ_L at z_L and into Δ_R at z_R.

We wish to describe Heegaard diagrams which specify the above data. Before doing this, we recall how we specify bordered three-manifolds (with one boundary component) by diagrams.

Definition 5.2 A *pointed bordered Heegaard diagram with one boundary component* is a quadruple $\mathcal{H} = (\bar{\Sigma}, \bar{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$, where $\bar{\Sigma}$ is a compact surface of genus g with one boundary component; $\boldsymbol{\beta}$ is a g–tuple of pairwise disjoint curves in the interior Σ of $\bar{\Sigma}$;
$$\bar{\boldsymbol{\alpha}} = \{\overbrace{\bar{\alpha}_1^a, \dots, \bar{\alpha}_{2k}^a}^{\bar{\boldsymbol{\alpha}}^a}, \overbrace{\alpha_1^c, \dots, \alpha_{g-k}^c}^{\boldsymbol{\alpha}^c}\}$$
is a collection of pairwise disjoint embedded arcs (the $\bar{\alpha}_i^a$) with boundary on $\partial\bar{\Sigma}$ and circles (the α_i^c) in the interior Σ of $\bar{\Sigma}$; and z is a basepoint in $\partial\bar{\Sigma} \setminus \boldsymbol{\alpha}^a$. We require that $\bar{\Sigma} \setminus \bar{\boldsymbol{\alpha}}$ and $\bar{\Sigma} \setminus \boldsymbol{\beta}$ both be connected; this translates to the condition that the α–(respectively β–) curves be linearly independent in $H_1(\bar{\Sigma}, \partial\bar{\Sigma})$.

Given a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, the boundary $\partial\bar{\Sigma}$ is a pointed matched circle in a natural way, with $\boldsymbol{a} = \bar{\boldsymbol{\alpha}}^a \cap \partial\bar{\Sigma}$, M the matching pairing up the endpoints of each $\bar{\alpha}_i^a$, and z the basepoint z. We call this pointed matched circle $\mathcal{Z}(\mathcal{H})$.

Construction 5.3 A pointed bordered Heegaard diagram $\mathcal{H}$ with one boundary component specifies a 3–manifold $Y(\mathcal{H})$ with one boundary component as follows:

(1) Thicken $\bar{\Sigma}$ to $\bar{\Sigma} \times [0,1]$.
(2) Attach three-dimensional two-handles along the α–circles in $\bar{\Sigma} \times \{0\}$.
(3) Attach three-dimensional two-handles along the β–circles in $\bar{\Sigma} \times \{1\}$.

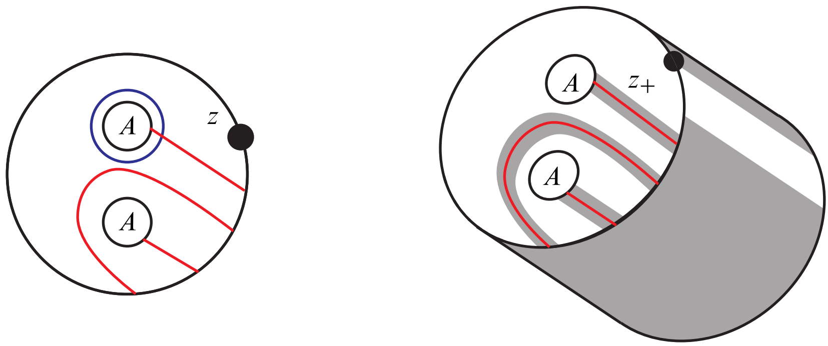


Figure 12: A pointed bordered Heegaard diagram and the associated 3–manifold: the picture on the left is a Heegaard diagram for the bordered solid torus shown on the right. The shaded part of the boundary is $F^\circ(\mathcal{Z}(\mathcal{H}))$.

A parameterization of the boundary is specified as follows. Consider the graph

$$(\bar{\boldsymbol{\alpha}}^a \cup (\partial\bar{\Sigma} \setminus \mathrm{nbd}(z))) \times \{0\} \subset \bar{\Sigma} \times \{0\},$$

thought of as a subset of ∂Y. The closure F° of a neighborhood of this graph is naturally identified with the surface-with-boundary $F^\circ(\mathcal{Z}(\mathcal{H}))$ associated to the pointed matched circle $\mathcal{Z}(\mathcal{H})$. (This identification *reverses* orientation with our orientation conventions. Note that this neighborhood of the α–arcs is orientation-reversing homeomorphic to a portion of $\partial Y(\mathcal{H})$, so that the orientation of $F^\circ(\mathcal{Z}(\mathcal{H}))$ agrees with the orientation of $\partial Y(\mathcal{H})$.) The deleted neighborhood of z is an interval with endpoints z_- and z_+, and we can take either of them (say, z_+) as corresponding to the basepoint on the boundary. The complement of F° in ∂Y is a disk. See Figure 12.

Thus fortified, we turn to the two boundary component case.

Definition 5.4 An *arced bordered Heegaard diagram with two boundary components* is a quadruple $(\bar{\Sigma}, \bar{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{z})$ where:

- $\bar{\Sigma}_g$ is a compact surface of genus g with two boundary components, $\partial_L\bar{\Sigma}$ and $\partial_R\bar{\Sigma}$.
- $\boldsymbol{\beta}$ is a g–tuple of pairwise disjoint curves in the interior Σ of $\bar{\Sigma}$.
- $\bar{\boldsymbol{\alpha}}$ is a collection

$$\bar{\boldsymbol{\alpha}} = \{\overbrace{\bar{\alpha}_1^{a,L}, \dots, \bar{\alpha}_{2\ell}^{a,L}}^{\bar{\boldsymbol{\alpha}}^{a,L}}, \overbrace{\bar{\alpha}_1^{a,R}, \dots, \bar{\alpha}_{2r}^{a,R}}^{\bar{\boldsymbol{\alpha}}^{a,R}}, \overbrace{\alpha_1^c, \dots, \alpha_{g-\ell-r}^c}^{\boldsymbol{\alpha}^c}\}$$

is a collection of pairwise disjoint embedded arcs with boundary on $\partial_L \bar{\Sigma}$ (the $\bar{\alpha}_i^{a,L}$), arcs with boundary on $\partial_R \bar{\Sigma}$ (the $\bar{\alpha}_i^{a,R}$), and circles (the α_i^c) in the interior Σ of $\bar{\Sigma}$.

- z is a path in $\bar{\Sigma} \setminus (\bar{\boldsymbol{\alpha}} \cup \boldsymbol{\beta})$ between $\partial_L \bar{\Sigma}$ and $\partial_R \bar{\Sigma}$.

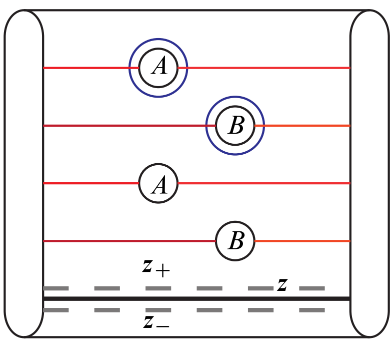
These are required to satisfy:

- $\bar{\Sigma} \setminus \bar{\boldsymbol{\alpha}}$ and $\bar{\Sigma} \setminus \boldsymbol{\beta}$ are connected.
- $\bar{\boldsymbol{\alpha}}$ intersects $\boldsymbol{\beta}$ transversely.

An arced bordered Heegaard diagram $\mathcal{H}$ with two boundary components specifies two pointed matched circles

$$\mathcal{Z}_L(\mathcal{H}) = (\partial_L \bar{\Sigma}, \bar{\boldsymbol{\alpha}}^{a,L} \cap \partial_L \bar{\Sigma}, M_L, z \cap \partial_L \bar{\Sigma}),$$
$$\mathcal{Z}_R(\mathcal{H}) = (\partial_R \bar{\Sigma}, \bar{\boldsymbol{\alpha}}^{a,R} \cap \partial_R \bar{\Sigma}, M_R, z \cap \partial_R \bar{\Sigma}),$$

where M_L (respectively M_R) is the pairing matching the endpoints of each arc in $\bar{\alpha}_i^{a,L}$ (respectively $\bar{\alpha}_i^{a,R}$).



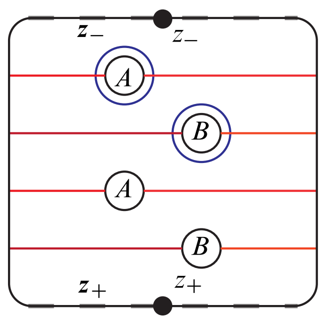


Figure 13: Drilling Heegaard diagrams: on the left, we have an arced bordered Heegaard diagram $\mathcal{H}$ for $T^2 \times [0,1]$; on the right, we have the result $\mathcal{H}_{\mathrm{dr}}$ of drilling the tunnel z from $\mathcal{H}$, a bordered Heegaard diagram with a single boundary component.

Definition 5.5 Given an arced bordered Heegaard diagram $\mathcal{H} = (\bar{\Sigma}, \bar{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z)$ with two boundary components, there is a bordered Heegaard diagram $\mathcal{H}_{\mathrm{dr}}$ with one boundary component, obtained by deleting a neighborhood of the arc z from $\bar{\Sigma}$. The boundary of the deleted neighborhood of z consists of two disjoint pushoffs z_+ and z_- of z, and $\partial \bar{\Sigma}_{dr}$ is

$$(\partial_L \bar{\Sigma} \setminus \mathrm{nbd}(z_L)) \cup (\partial_R \bar{\Sigma} \setminus \mathrm{nbd}(z_R)) \cup z_+ \cup z_-.$$

We can equally well choose to put the basepoint of $\mathcal{H}_{\mathrm{dr}}$ on $\boldsymbol{z}_+$ or on $\boldsymbol{z}_-$. We will call these two choices of basepoint z_+ and z_- respectively. For either of these choices, we call the pointed bordered Heegaard diagram $\mathcal{H}_{\mathrm{dr}}$ a diagram obtained from $\mathcal{H}$ by *drilling*. Note that $\mathcal{Z}(\mathcal{H}_{\mathrm{dr}})$ is the pointed matched circle $\mathcal{Z}_L(\mathcal{H})\,\#\,\mathcal{Z}_R(\mathcal{H})$.

We will use the notation $\mathcal{H}_{\mathrm{dr2}}$ to denote the doubly pointed Heegaard diagram obtained by viewing both z_+ and z_- as basepoints of the drilled diagram.

See Figure 13 for an illustration of the drilling construction on Heegaard diagrams.

Construction 5.6 An arced bordered Heegaard diagram with two boundary components in the sense of Definition 5.4 gives rise to a strongly bordered three-manifold in the sense of Definition 5.1, as follows. Let $\mathcal{H}_{\mathrm{dr}}$ be the Heegaard diagram obtained from $\mathcal{H}$ by drilling (Definition 5.5). The boundary of $Y(\mathcal{H}_{\mathrm{dr}})$ is decomposed as a connect sum $F(\mathcal{Z}_L)\,\#\,F(\mathcal{Z}_R)$. We attach a three-dimensional two-handle along the connect sum annulus to obtain Y. To see the other structure, we perform this construction with more care. The boundary of $Y(\mathcal{H}_{\mathrm{dr}})$ consists of three pieces:

- A neighborhood of the graph
$$(\overline{\boldsymbol{\alpha}}^a_L \cup (\partial_L\overline{\Sigma}\setminus \mathrm{nbd}(z_L)))\times\{0\}\subset\overline{\Sigma}\times\{0\},$$
whose closure is identified in an orientation-reversing way with $F^\circ(\mathcal{Z}_L)$, which we denote F°_L. (Note that F°_L contains the basepoint z^+_L on its boundary.)
- A neighborhood of the graph
$$(\overline{\boldsymbol{\alpha}}^a_R \cup (\partial_R\overline{\Sigma}\setminus \mathrm{nbd}(z_R)))\times\{0\}\subset\overline{\Sigma}\times\{0\},$$
whose closure is identified in an orientation reversing way with $F^\circ_R := F^\circ(\mathcal{Z}_R)$. (Note that F°_L contains the basepoint z^+_R on its boundary.)
- An annulus A, equipped with a path $\boldsymbol{z}^+$ connecting z^+_L to z^+_R (this is the path $\boldsymbol{z}_+$ on the boundary of $\overline{\Sigma}_{dr}=\overline{\Sigma}_{dr}\times\{0\}$, thought of as a subset of the boundary of $Y(\mathcal{H}_{\mathrm{dr}})$.)

Now, we attach a three-dimensional two-handle to $Y(\mathcal{H}_{\mathrm{dr}})$ along the annulus A. More precisely, let Δ be a two-dimensional disk. We glue $\Delta\times[0,1]$ to $Y(\mathcal{H}_{\mathrm{dr}})$, identifying $(\partial\Delta)\times[0,1]$ with the annulus A, so that $(\partial\Delta)\times\{0\}$ is glued to the boundary of F°_L, while $\Delta\times\{1\}$ is glued to the boundary of F°_R. Let $\Delta_L=\Delta\times\{0\}$ and $\Delta_R=\Delta\times\{1\}$. It is easy to see that this gives a three-manifold homeomorphic to Y. Moreover, in this model, the boundary of Y consists of the disjoint union of $F^\circ_L\cup\Delta_L$ and $F^\circ_R\cup\Delta_R$, and Δ_L and Δ_R respectively contain z^+_L and z^+_R on their boundary. Our preferred disks are Δ_L and Δ_R, and their basepoints are z_L^+ and z_R^+. The homeomorphisms ϕ_L and ϕ_R are supplied by the pointed matched circles. The arc γ_z is supplied by $\mathbf{z}_+$, which in turn gives a path on $(\partial\Delta)\times[0,1]$. The framing is specified so as to point into $\Delta\times\{t\}$ for each $t\in[0,1]$. See Figure 14.

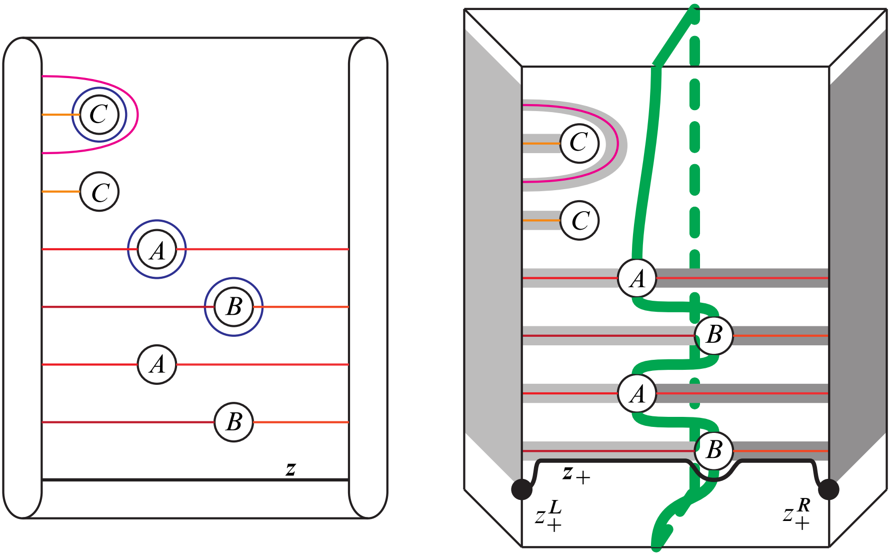


Figure 14: Constructing a bordered 3–manifold with two boundary components from an arced bordered Heegaard diagram: the Heegaard diagram on the left represents an elementary cobordism from the genus-two surface to the genus-one surface. The lightly shaded region on the right picture is $F^\circ(\partial_L\mathcal{H})$ (a surface of genus-two), while the darkly shaded one is $F^\circ(\partial_R\mathcal{H})$ (a surface of genus-one). There is a 2–handle attached along the thick (green) curve.

We call the data $(Y,\Delta_L,z_L,\Delta_R,z_R,\phi_L,\phi_R,\gamma_z)$ from Construction 5.6 the strongly bordered 3–manifold associated to the arced bordered diagram $\mathcal{H}$. We will often abuse notation and use Y or ${}_{\phi_L}Y_{\phi_R}$ or ${}_{z_L}Y_{z_R}$ to denote all the data of a strongly bordered 3–manifold (depending on which pieces we want to emphasize).

Remark 5.7 In the case of strongly bordered Heegaard diagrams with more than two boundary components, one replaces the arc $\mathbf{z}$ with a tree (or, as a special case, a sequence of arcs) connecting the various boundary components.

There is an inverse to the drilling construction, *filling*, defined as follows.

Definition 5.8 Let $\mathcal{H}_{\mathrm{dr}} = (\bar{\Sigma}, \bar{\boldsymbol{\alpha}}, \boldsymbol{\beta}, z_+)$ be a pointed bordered Heegaard diagram with one boundary component. Let z_- be another point in $\partial\bar{\Sigma}$, and suppose that, writing $\partial\bar{\Sigma} \setminus \{z_+, z_-\} = I_L \amalg I_R$, there are no α–arcs running between I_L and I_R. Then $\{z_+, z_-\}$ decomposes $\mathcal{Z}(\mathcal{H}_{\mathrm{dr}})$ as a connect sum, $\mathcal{Z}(\mathcal{H}_{\mathrm{dr}}) = \mathcal{Z}_L \# \mathcal{Z}_R$. Attaching a band (two-dimensional one-handle) to $\mathcal{H}_{\mathrm{dr}}$ between z_+ and z_- gives a Heegaard diagram $\mathcal{H}_f$ with two boundary components, with $\partial_L \mathcal{H}_f = \mathcal{Z}_L$ and $\partial_R \mathcal{H}_f = \mathcal{Z}_R$.

The three-manifold $Y(\mathcal{H}_f)$ is obtained from $Y(\mathcal{H}_{\mathrm{dr}})$ by attaching a 3–dimensional 2–handle to $\partial Y(\mathcal{H}_{\mathrm{dr}})$ along the connect sum curve.

5.1 Arced bordered Heegaard diagrams

We can use the drilling construction to rephrase questions about arced bordered Heegaard diagrams with two boundary components in terms of ordinary (one boundary component) bordered three-manifolds. For example, we have the following:

Definition 5.1 Let $\mathcal{H}$ be an arced bordered Heegaard diagram.

- A *generator* of $\mathcal{H}$ is a generator of $\mathcal{H}_{\mathrm{dr}}$. We let $\mathfrak{S}(\mathcal{H})$ denote the set of generators of $\mathcal{H}$.
- Given generators $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{S}(\mathcal{H})$, the set of *domains connecting* $\boldsymbol{x}$ *and* $\boldsymbol{y}$, $\pi_2(\boldsymbol{x}, \boldsymbol{y})$, is the set of domains in $\mathcal{H}_{\mathrm{dr}}$ connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ that do not cross either z_+ or z_-. We view domains as linear combinations of components of $\bar{\Sigma} \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. Recall that $\partial^\partial(B)$ denotes the intersection of ∂B with $\partial\bar{\Sigma}$.
- Let $\pi_2^\partial(\boldsymbol{x}, \boldsymbol{y}) = \{B \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \mid \partial^\partial B = 0\}$. These are the *provincial* domains from $\boldsymbol{x}$ to $\boldsymbol{y}$.
  There are natural isomorphisms
  $$\pi_2(\boldsymbol{x}, \boldsymbol{x}) \cong H_2(Y(\mathcal{H}), \partial Y(\mathcal{H})),$$
  $$\pi_2^\partial(\boldsymbol{x}, \boldsymbol{x}) \cong H_2(Y_{\mathrm{dr}}(\mathcal{H})),$$
  corresponding to [21, Lemmas 4.18 and 4.20].
- We call elements of $\pi_2(\boldsymbol{x}, \boldsymbol{x})$ *periodic domains.*
- The arced bordered Heegaard diagram with two boundary components $\mathcal{H}$ is called *admissible* (respectively *provincially admissible*) if the associated drilled Heegaard diagram $\mathcal{H}_{\mathrm{dr2}}$ is admissible (respectively provincially admissible) in the sense of [21, Definition 4.24] (respectively [21, Definition 4.23]), ie if every nontrivial periodic domain (respectively provincial periodic domain) of $\mathcal{H}$ has both positive and negative coefficients.

Proposition 5.2 *Any strongly bordered three-manifold with two boundary components comes from an admissible arced bordered Heegaard diagram with two boundary components.*

Proof Choose a bordered Heegaard diagram for the bordered three-manifold (with one boundary component) $Y \setminus \mathrm{nbd}(\gamma_z)$. This can be done according to [21, Lemma 4.9]; moreover, it can be made admissible by [21, Proposition 4.10]. (Note that admissibility with only one basepoint which is gotten from [21, Proposition 4.10] is slightly stronger than the admissibility we require here.) The filling construction of Definition 5.8 then produces the desired diagram for Y. □

Proposition 5.3 *If $\mathcal{H}$ and $\mathcal{H}'$ specify the same strongly bordered 3–manifold then $\mathcal{H}$ and $\mathcal{H}'$ are related by a sequence of the following moves:*

- *Isotopies of the α– and β–curves.*
- *Handleslides among the α–circles and among the β–circles.*
- *Handleslides of an α–arc over an α–circle.*
- *Stabilizations of the diagram.*

Moreover, if $\mathcal{H}$ and $\mathcal{H}'$ are admissible (respectively provincially admissible) then the moves can be chosen so that all intermediate diagrams are admissible (respectively provincially admissible).

Proof Suppose $\mathcal{H}$ and $\mathcal{H}'$ are provincially admissible and specify the same strongly bordered 3–manifold. Then $\mathcal{H}_{\mathrm{dr}}$ and $\mathcal{H}'_{\mathrm{dr}}$ specify the same bordered three-manifold (with one boundary component). Thus, $\mathcal{H}_{\mathrm{dr}}$ and $\mathcal{H}'_{\mathrm{dr}}$ can be connected by a sequence of provincially admissible Heegaard moves which do not cross either of the basepoints, as in [21, Propositions 4.10 and 4.25]. Filling all the diagrams, we obtain the desired sequence connecting $\mathcal{H}$ to $\mathcal{H}'$.

The case when $\mathcal{H}$ and $\mathcal{H}'$ are admissible is similar, except that $\mathcal{H}_{\mathrm{dr}}$ and $\mathcal{H}'_{\mathrm{dr}}$ are not necessarily admissible, as there may be periodic domains with positive coefficients crossing the extra basepoint (say z_-, if z_+ was the basepoint for $\mathcal{H}$). The doubly-pointed diagrams $\mathcal{H}_{\mathrm{dr}2}$ and $\mathcal{H}'_{\mathrm{dr}2}$ are admissible (in the obvious sense), and [21, Propositions 4.10 and 4.25] adapt easily to the doubly-pointed case. □

We call two diagrams which are related by the moves of Proposition 5.3 *equivalent.*

In the case of 3–manifolds with two boundary components, we can refine some of the notions related to domains. Given a domain B, $\partial^\partial_L B$ (respectively $\partial^\partial_R B$)

denote the intersection of ∂B with $\partial_L \overline{\Sigma}$ (respectively $\partial_R \overline{\Sigma}$). Let $\pi_2^{\partial_L}(\boldsymbol{x}, \boldsymbol{y}) = \{A \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \mid \partial_L^{\partial} A = 0\}$ denote the set of *left-provincial* domains connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\pi_2^{\partial_R}(\boldsymbol{x}, \boldsymbol{y}) = \{A \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \mid \partial_R^{\partial} A = 0\}$ the set of *right-provincial* domains connecting $\boldsymbol{x}$ and $\boldsymbol{y}$. It is easy to show that

$$\pi_2^{\partial_L}(\boldsymbol{x}, \boldsymbol{x}) \cong H_2(Y(\mathcal{H}), \partial_R Y(\mathcal{H})),$$
$$\pi_2^{\partial_R}(\boldsymbol{x}, \boldsymbol{x}) \cong H_2(Y(\mathcal{H}), \partial_L Y(\mathcal{H})).$$

Definition 5.4 An arced bordered Heegaard diagram $\mathcal{H}$ with two boundary components is called *left* (respectively *right*) *admissible* if every nontrivial right-provincial (respectively left-provincial) periodic domain has both positive and negative coefficients.

It is easy to show that Proposition 5.3 still holds if one replaces “admissible” by “left admissible” or “right admissible”.

Lemma 5.5 *The Heegaard diagram $\mathcal{H}$ is left (respectively right) admissible if and only if there is an area form on $\overline{\Sigma}$ with respect to which every right-provincial (respectively left-provincial) periodic domain has signed area 0. The diagram $\mathcal{H}$ is admissible if and only if there is an area form on $\overline{\Sigma}$ with respect to which every periodic domain has signed area 0.*

Proof The proof is exactly the same as the proof of [21, Lemma 4.26], which in turn is the same as the proof of the second author and Szabó [29, Lemma 4.12]. □

Note that

admissible $\Longrightarrow$ left or right admissible

$\Longrightarrow$ left and right admissible $\Longrightarrow$ provincially admissible.

All of these implications are strict.

Finally we discuss how spin^c–structures on manifolds with two boundary components relate to arced, bordered Heegaard diagrams.

Before doing this, we recall some generalities (see [28]). Suppose that M is a three-manifold, equipped with an oriented, nullhomologous knot C. Then, there is a notion of *relative* spin^c *structures*, denoted $\underline{\mathrm{spin}}^c(M, C)$. These are defined to be spin^c structures on the zero-surgery manifold $M_0(C)$. If C is equipped with a Seifert surface F, there is an identification

$$\underline{\mathrm{spin}}^c(M, C) \cong \mathrm{spin}^c(M) \oplus \mathbb{Z}.$$

The projection onto $\mathbb{Z}$ is gotten by

$$\underline{\mathfrak{s}} \mapsto \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}), \widehat{F}\rangle,$$

where $\widehat{F}$ is gotten by closing off F in $M_0(C)$. This projection to $\mathbb{Z}$ is called the *Alexander grading* on relative spin^c structures.

Let $\mathcal{Y}$ be a strongly bordered 3–manifold with two boundary components specified by an arced bordered Heegaard diagram Y. A meridian C of $\gamma_z \subset Y$ specifies a knot K in $Y_{\mathrm{dr}} = Y(\mathcal{H}_{\mathrm{dr}})$. This knot is nullhomologous. Indeed, the surface $F^\circ(\mathcal{Z}_R Y)$ provides a natural choice of Seifert surface for K (and in particular an orientation for K).

Lemma 5.6 *There is a natural identification* $\underline{\mathrm{spin}}^c(Y_{\mathrm{dr}}, K) \cong \mathrm{spin}^c(Y)$. *Under this identification, the Alexander grading of* $\underline{\mathfrak{s}}$ *corresponds to the evaluation of the corresponding* spin^c *structure on* $\partial_R Y$.

Proof Let Y' denote zero-surgery on Y_{dr} along K. It is easy to see that Y' is naturally identified with the three-manifold obtained from Y by attaching a one-handle to its boundary connecting the left and right basepoints. The identification $\underline{\mathrm{spin}}^c(Y_{\mathrm{dr}}, K) \cong \mathrm{spin}^c(Y)$ follows at once. Under this identification, $\partial_R Y$ is clearly homologous to the capped-off Seifert surface. $\square$

As in [28], an oriented knot is specified by a Heegaard diagram with two (ordered) basepoints z_+ and z_- (denoted w and z in [28]). The oriented knot in the three-manifold is specified as follows: draw an arc from z_- to z_+ which crosses only β–circles, and then close this up by drawing an arc from z_+ to z_- which crosses only α–circles. (It might be necessary to push the two arcs into the two handlebodies to make the knot be embedded.)

Lemma 5.7 *The Heegaard diagram* $\mathcal{H}_{\mathrm{dr2}}$ *is a doubly-pointed bordered Heegaard diagram for* (Y_{dr}, K), *where* K *is oriented as the boundary of* $F^\circ(\mathcal{Z}_R Y)$.

Proof Let γ be a path in Σ_{dr} connecting z_+ to z_- in the complement of the α–curves and η a path in Σ_{dr} connecting z_- to z_+ in the complement of the β–circles. Then the pushoff of $\gamma \cup \eta$ is the knot specified by $\mathcal{H}_{\mathrm{dr2}}$. We can choose γ to lie in a neighborhood of the α^R–arcs and η to lie near $\partial_R \overline{\Sigma_{dr}}$ and cross only α^R–arcs. Then it is clear that $\gamma \cup \eta$ is isotopic to $\pm K$. With our orientation conventions, if we wish for K to be oriented as the boundary of $F^\circ(\mathcal{Z}_R)$, then we order z_+ and z_- so that the arc from z_+ to z_- in $\mathcal{Z}(Y_{\mathrm{dr}})$ (with its induced orientation) contains the matched pairs for $\mathcal{Z}_R$. (Equivalently, if we think of the arc $\boldsymbol{z}$ in $\mathcal{H}$ connecting $\partial_L \mathcal{H}$ to $\partial_R \mathcal{H}$ as running left to right, then $\boldsymbol{z}_+$ is gotten by translating $\boldsymbol{z}$ upwards in $\mathcal{H}$, and $\boldsymbol{z}_-$ is gotten by pushing it down.) $\square$

Definition 5.8 We can define a map $\mathfrak{s}\colon \mathfrak{S}(\mathcal{H}) \to \mathrm{spin}^c(Y)$ as follows. View $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$ as a generator for $\mathfrak{S}(\mathcal{H}_{\mathrm{dr}})$, let $\underline{\mathfrak{s}}(\boldsymbol{x})$ denote its corresponding relative spin^c structure, and then let $\mathfrak{s}(\boldsymbol{x})$ be the corresponding spin^c structure in $\mathrm{spin}^c(Y)$, according to the equivalence of Lemma 5.6. Let

$$\mathfrak{S}(\mathcal{H}, \mathfrak{s}) = \{\boldsymbol{x} \in \mathfrak{S}(\mathcal{H}) \mid \mathfrak{s}(\boldsymbol{x}) = \mathfrak{s}\}.$$

Lemma 5.9 *Given $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{S}(\mathcal{H})$, we have $\pi_2(\boldsymbol{x}, \boldsymbol{y}) \neq \varnothing$ if and only if $\mathfrak{s}(\boldsymbol{x}) = \mathfrak{s}(\boldsymbol{y})$.*

Proof In view of Lemma 5.6, this statement is equivalent to the corresponding statement for knot Floer homology (see [28, Section 2.3; 29, Sections 2.4 and 2.6]). Recall that this is proved first by constructing a difference element $\epsilon(\boldsymbol{x}, \boldsymbol{y}) \in H_1(Y_{\mathrm{dr}} \setminus C, \partial Y_{\mathrm{dr}})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{S}(\mathcal{H})$ (which vanishes if and only if $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ is nonempty), and then showing that $\underline{\mathfrak{s}}(\boldsymbol{y}) = \underline{\mathfrak{s}}(\boldsymbol{x}) + \mathrm{PD}(\epsilon(\boldsymbol{x}, \boldsymbol{y}))$. □

As noted in the discussion of the Alexander grading above, there is a restriction map $\mathrm{spin}^c(Y) \to \mathrm{spin}^c(\partial_R Y) \cong \mathbb{Z}$. We will see in the proof of Theorem 14 in Section 7 that this restriction map is closely related to the strands grading on the algebra.

5.2 Gluing Heegaard diagrams

In Section 7, we will see how bordered Floer homology groups transform under three gluing operations one can perform on bordered three-manifolds.

The first of these gluing operations glues a bordered three-manifold with one boundary component to one with two boundary components.

Construction 5.1 Suppose Y is a bordered three-manifold with one boundary component whose parameterization is specified by a homeomorphism $\phi\colon F(\mathcal{Z}) \to \partial Y$, and let Y' be a strongly bordered three-manifold with two boundary components $\partial_L Y'$ and $\partial_R Y'$ with parameterizations specified by

$$\phi'_L\colon F(\mathcal{Z}'_L) \to \partial_L Y', \quad \phi'_R\colon F(\mathcal{Z}'_R) \to \partial_R Y'.$$

Suppose moreover that $\mathcal{Z} = -\mathcal{Z}'_L$. Then we can form the bordered three-manifold

$$Y\ {}_{\partial Y}\!\cup_{\partial_L Y'} Y',$$

which is obtained by gluing Y to Y' via the identification of ∂Y with $-\partial_L Y'$ given by $\phi'_L \circ \phi^{-1}$. (Note that for the purpose of this definition, we do not need Y' to be strongly bordered, just bordered; however, in our applications Y' will be equipped with this extra data.)

The second of these gluing operations glues two strongly bordered three-manifolds with two boundary components.

Construction 5.2 Fix two strongly bordered three-manifolds with two boundary components

$$(Y', \mathcal{Z}'_L, \Delta'_L, z'_L, \phi'_L, \mathcal{Z}'_R, \Delta'_R, z'_R, \phi'_R, \gamma'_z, \nu'_z),$$
$$(Y'', \mathcal{Z}''_L, \Delta''_L, z''_L, \phi''_L, \mathcal{Z}''_R, \Delta''_R, z''_R, \phi''_R, \gamma''_z, \nu''_z).$$

Suppose moreover that $\mathcal{Z}'_R = -\mathcal{Z}''_L$. Then we can form a new strongly bordered three-manifold with two boundary components. The underlying three-manifold is

$$Y''' = Y' {}_{\partial_R Y'} \cup_{\partial_L Y''} Y'',$$

gotten by gluing $\partial_R Y'$ to $\partial_L Y''$ via $\phi''_L \circ (\phi'_R)^{-1}$. The path γ''' is gotten by connecting γ'_z to γ''_z. Framings are obtained similarly.

The third gluing operation is a kind of self-gluing.

Construction 5.3 Suppose that

$$(Y, \mathcal{Z}_L, \Delta_L, z_L, \phi_L, \mathcal{Z}_R, \Delta_R, z_R, \phi_R, \gamma_z, \nu_z)$$

is a strongly bordered three-manifold with two boundary components; and suppose moreover that $\mathcal{Z}_L = -\mathcal{Z}_R$. Then identifying the two boundary components of Y together, we obtain a new three-manifold which is equipped with a framed knot, gotten by gluing up the framed arc γ_z. Performing surgery on this framed knot in the self-glued three-manifold, we obtain a new three-manifold denoted (Y°, K), equipped with a knot K gotten as the core of the surgery torus. We call (Y°, K) the *generalized open book* associated to the strongly bordered three-manifold Y.

The justification for this terminology is the following. If we consider a strongly based mapping class $\phi\colon F^\circ(\mathcal{Z}) \to F^\circ(\mathcal{Z})$, there is an associated strongly bordered three-manifold M_ϕ whose underlying topological space is $[0,1] \times F(\mathcal{Z})$, parameterized by the identity on one boundary and (the map on the closed surface induced by) ϕ on the other; see Construction 5.2 and Lemma 5.6. The associated three-manifold (Y°, K) gotten as above is classically known as the *open book* associated to ϕ.

On the level of Heegaard diagrams, the three gluing operations can be described as follows.

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed bordered Heegaard diagram with one boundary component, and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \mathbf{z}')$ an arced bordered Heegaard diagram with two boundary components. Suppose that the pointed matched circle $\mathcal{Z}(\mathcal{H})$ associated to $\mathcal{H}$ is the orientation reverse $-\mathcal{Z}_L(\mathcal{H}')$ of the left pointed matched circle associated to $\mathcal{H}'$. Then gluing the boundary of $\mathcal{H}$ to the left boundary of $\mathcal{H}'$ we obtain a new pointed bordered Heegaard diagram

$$\mathcal{H}\ {}_{\partial}\!\cup_{\partial_L} \mathcal{H}' = (\Sigma\ {}_{\partial}\!\cup_{\partial_L} \Sigma', \boldsymbol{\alpha}\ {}_{\partial}\!\cup_{\partial_L} \boldsymbol{\alpha}', \boldsymbol{\beta} \cup \boldsymbol{\beta}', \partial_R \boldsymbol{z}').$$

Similarly, if $\mathcal{H}''$ is another arced bordered Heegaard diagram with two boundary components, such that $\mathcal{Z}_L(\mathcal{H}'') = -\mathcal{Z}_R(\mathcal{H}')$ then we can glue $\mathcal{H}''$ to $\mathcal{H}'$ to get

$$\mathcal{H}'\ {}_{\partial_R}\!\cup_{\partial_L} \mathcal{H}'' = (\Sigma'\ {}_{\partial_R}\!\cup_{\partial_L} \Sigma'', \boldsymbol{\alpha}'{}_{\partial_R} \cup_{\partial_L} \boldsymbol{\alpha}'', \boldsymbol{\beta}' \cup \boldsymbol{\beta}'', \boldsymbol{z}'\ {}_{\partial_R}\!\cup_{\partial_L} \boldsymbol{z}'').$$

Lemma 5.4 *With notation from above, gluing bordered Heegaard diagrams corresponds to gluing bordered three-manifolds as follows:*

$$Y(\mathcal{H}\ {}_{\partial}\!\cup_{\partial_L} \mathcal{H}') = Y(\mathcal{H}) \cup_{\phi'_L \circ \phi^{-1}} Y(\mathcal{H}'),$$
$$Y(\mathcal{H}'\ {}_{\partial}\!\cup_{\partial_L} \mathcal{H}'') = Y(\mathcal{H}') \cup_{\phi''_L \circ (\phi'_R)^{-1}} Y(\mathcal{H}'').$$

Proof This is straightforward. □

Finally, we have the following construction mirroring Construction 5.3 on the level of Heegaard diagrams:

Construction 5.5 Suppose that $\mathcal{H}$ is an arced bordered Heegaard diagram with two boundary components such that $\mathcal{Z}_L\mathcal{H} = -\mathcal{Z}_R\mathcal{H}$. Then we can glue $\partial_L\mathcal{H}$ to $\partial_R\mathcal{H}$. The result is a closed surface $\Sigma^{\square}$ of genus $g+1$ with g α–circles and g β–circles, as well as a closed curve $\boldsymbol{z}^{\circ}$ corresponding to the arc $\boldsymbol{z}$. Place basepoints z_+ and z_- on the two sides of $\boldsymbol{z}^{\circ}$, and then surger out the arc $\boldsymbol{z}^{\circ}$ from $\Sigma^{\square}$. The result is a doubly-pointed Heegaard diagram $\mathcal{H}^{\circ} = (\Sigma^{\circ}, \boldsymbol{\alpha}^{\circ}, \boldsymbol{\beta}^{\circ}, z_+, z_-)$, which we call the *self-glued diagram* associated to $\mathcal{H}$.

Lemma 5.6 *The self-glued diagram $\mathcal{H}^{\circ}$ of Construction 5.5 represents the generalized open book associated to the strongly bordered three-manifold (Y°, K) of Construction 5.3.*

Proof This is straightforward; see also Lemma 5.7. □

Next, we discuss how the gluing constructions interact with the admissibility hypotheses.

Lemma 5.7 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be arced bordered Heegaard diagrams with two boundary components, such that $\mathcal{Z}_R(\mathcal{H}_1) = -\mathcal{Z}_L(\mathcal{H}_2)$. Let $\mathcal{H} = \mathcal{H}_1\ {}_{\partial_R}\!\cup_{\partial_L} \mathcal{H}_2$. If $\mathcal{H}_1$ is right admissible and $\mathcal{H}_2$ is provincially admissible, or if $\mathcal{H}_1$ is provincially admissible and $\mathcal{H}_2$ is left admissible, then $\mathcal{H}$ is provincially admissible. Moreover:*

(1) *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are both left admissible (respectively right admissible) then $\mathcal{H}$ is left admissible (respectively right admissible).*

(2) *If $\mathcal{H}_1$ (respectively $\mathcal{H}_2$) is admissible then $\mathcal{H}$ is left admissible (respectively right admissible).*

(3) *If $\mathcal{H}_1$ (respectively $\mathcal{H}_2$) is admissible and $\mathcal{H}_2$ (respectively $\mathcal{H}_1$) is right admissible (respectively left admissible) then $\mathcal{H}$ is admissible.*

The obvious analogues hold in the case that $\mathcal{H}_1$ has only one boundary component. (Many of the statements become the same in this case.)

Proof We will discuss the case that $\mathcal{H}_1$ is right admissible and $\mathcal{H}_2$ is provincially admissible, and the case that both $\mathcal{H}_1$ and $\mathcal{H}_2$ are left admissible; the other cases are similar.

Suppose $\mathcal{H}_1$ is right admissible. If P is a nontrivial provincial periodic domain in $\mathcal{H}$ then $P \cap \mathcal{H}_1$ is a left-provincial periodic domain in $\mathcal{H}_1$. Hence either $P \cap \mathcal{H}_1$ has both positive and negative coefficients or $P \cap \mathcal{H}_1$ is the trivial domain. In the latter case, $P \cap \mathcal{H}_2$ is a provincial periodic domain, and hence has both positive and negative coefficients. In either case, P has both positive and negative coefficients.

Similarly, suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are both left admissible. If P is a nontrivial right-provincial periodic domain in $\mathcal{H}$ then $P \cap \mathcal{H}_2$ is a right-provincial periodic domain in $\mathcal{H}_2$, and hence either has both positive and negative coefficients or is trivial. In the latter case, $P \cap \mathcal{H}_1$ is a nontrivial right-provincial periodic domain in $\mathcal{H}_1$, and hence has both positive and negative coefficients. □

We say a doubly-pointed Heegaard diagram is *admissible* if all periodic domains (ie domains which miss both basepoints) have both positive and negative local multiplicities. This is the condition required to define knot Floer homology using the given Heegaard diagram. (It corresponds to weakly admissibility for all spin^c structures for singly-pointed Heegaard diagrams, in the sense of [29, Definition 4.10].)

Lemma 5.8 *Let $\mathcal{H}$ denote an arced bordered Heegaard diagram with two boundary components and such that $\mathcal{Z}_L(\mathcal{H}) = -\mathcal{Z}_R(\mathcal{H})$. Suppose that $\mathcal{H}$ is admissible. Then the doubly-pointed Heegaard diagram $\mathcal{H}^\circ$ is admissible.*

Proof Note that the set of periodic domains in $\mathcal{H}^\circ$ is a subset of the set of periodic domains in $\mathcal{H}$. By Lemma 5.5, we can find an area form on $\widehat{\Sigma}$ with respect to which any periodic domain has signed area zero. This induces an area form on Σ° with the corresponding property. □

5.3 Bordered Heegaard diagrams for surface diffeomorphisms

Definition 5.1 Given a bordered 3–manifold (Y, ψ) (where $\psi\colon F(\mathcal{Z}) \to \partial Y$) and $\phi \in MCG_0(\mathcal{Z}, \mathcal{Z}')$, let $\phi(Y, \psi)$ denote the bordered 3–manifold $(Y, \psi \circ \phi^{-1})$, the result of twisting the parameterization of the boundary by ϕ. This has a natural extension to strongly bordered three-manifolds as well: given a strongly bordered three-manifold ${}_{\psi_L}Y_{\psi_R}$, where $\psi_R\colon F(\mathcal{Z}_R) \to \partial_R Y$, and $\phi \in MCG_0(\mathcal{Z}_R, \mathcal{Z}'_R)$, let $\phi({}_{\psi_L}Y_{\psi_R})$ be the strongly bordered three-manifold $({}_{\psi_L}Y_{\psi_R \circ \phi^{-1}})$.

When considering the above action of the mapping class group on bordered three-manifolds, the following strongly bordered 3–manifolds arise naturally:

Construction 5.2 Fix pointed matched circles $\mathcal{Z}_L$ and $\mathcal{Z}_R$, and a strongly based mapping class $\phi\colon (F(\mathcal{Z}_L), D_L, z_L) \to (F(\mathcal{Z}_R), D_R, z_R)$. We can form a corresponding strongly bordered three-manifold with $Y = [0,1] \times F(\mathcal{Z}_R)$, $\partial_L Y = \{0\} \times -F(\mathcal{Z}_R)$, $\partial_R Y = \{1\} \times F(\mathcal{Z}_R)$, $\Delta_L = \{0\} \times D_R$, $\psi_L = \{0\} \times -\phi$, $\Delta_R = \{1\} \times D_R$, $\psi_R = \{1\} \times \mathbb{I}$, $\gamma_z = [0,1] \times \{z_R\}$. We call it the *mapping cylinder of* ϕ, and denote the resulting arced bordered three-manifold by ${}_{\phi}([0,1] \times F(\mathcal{Z}_R))_{\mathbb{I}}$ or simply M_ϕ. Note that $\partial_L M_\phi = -F(\mathcal{Z}_L)$.

Remark 5.3 In general, for two topological spaces X and Y and a map $\phi\colon X \to Y$, the mapping cylinder of ϕ is the quotient space

$$M_\phi = ([0,1] \times X \amalg Y)/((1,x) \sim \phi(x)),$$

equipped with maps

$$\begin{aligned} \psi_L\colon X &\hookrightarrow M_\phi, & \psi_R\colon Y &\hookrightarrow M_\phi, \\ \psi_L(x) &= 0 \times x, & \psi_R(y) &= y. \end{aligned}$$

This definition works for arbitrary maps ϕ and, for instance, gives a CW complex if X and Y are CW complexes and ϕ is a cellular map. In the case when ϕ is a homeomorphism, the above space is equivalent to $[0,1] \times X$ with $\psi_L = \{0\} \times \mathbb{I}_X$ and $\psi_R = \{1\} \times \phi^{-1}$, which in turn is equivalent to $[0,1] \times Y$ with $\psi_L = \{0\} \times \phi$ and $\psi_R = \{1\} \times \mathbb{I}_Y$.

Lemma 5.4 *Any strongly bordered 3–manifold Y whose underlying space can be identified with a product of a surface with an interval (so that γ_z is identified with the product of a point with the interval, respecting the framing) is of the form M_ϕ for some choice of strongly based mapping class ϕ. Moreover, two such strongly bordered three-manifolds are isomorphic if and only if they represent the same strongly based mapping class.*

Proof Suppose that ${}_{\psi_L}Y_{\psi_R}$ is a strongly bordered three-manifold whose underlying space is homeomorphic to the product of a surface with an interval. To keep orientations consistent, let $F(\mathcal{Z}_L) = -\partial_L(Y)$ and $F(\mathcal{Z}_R) = \partial_R(Y)$. We extract a strongly based mapping class $\phi \in MCG_0(F(\mathcal{Z}_L), F(\mathcal{Z}_R))$ as follows. Fix a diffeomorphism $\Phi\colon [0,1]\times F(\mathcal{Z}_R) \to Y$ so that the following hold:

- $\Phi|_{\{1\}\times F(\mathcal{Z}_R)} = \psi_R$.
- $\Phi([0,1]\times z_R) = \gamma_z$.
- $\Phi(\{0\}\times D_R) = \Delta_L$.
- $\Phi(\{1\}\times D_R) = \Delta_R$.
- The normal vector ν_z is never tangent to $\Phi([0,1]\times \partial D_R)$.

We then define

$$\phi = (\Phi|_{\{0\}\times F(\mathcal{Z}_R)})^{-1} \circ (-\psi_L)\colon F(\mathcal{Z}_L(\mathcal{H})) \to F(\mathcal{Z}_R(\mathcal{H})).$$

Then Φ provides an isomorphism between Y and the strongly bordered three-manifold ${}_{\phi}([0,1]\times F(\mathcal{Z}_R))_{\mathbb{I}}$.

Next, we claim that the strongly based mapping class of ϕ is independent of the choices made. Indeed, if $\Phi'\colon [0,1]\times F(\mathcal{Z}_R) \to Y$ is an alternate choice of Φ, then $\Phi^{-1}\circ\Phi'$ is a pseudoisotopy from $(\Phi|_{\{0\}\times F(\mathcal{Z}_R)})^{-1}\circ \Phi'|_{\{0\}\times F(\mathcal{Z}_R)}$ to the identity map. (A *pseudoisotopy* between two self-diffeomorphisms f_0 and f_1 of a closed manifold M is a self-diffeomorphism of $[0,1]\times M$ that restricts to f_0 on $\{0\}\times M$ and f_1 on $\{1\}\times M$.) Since the equivalence relations induced by pseudoisotopy and isotopy agree in dimension 2,[4] it follows that

$$(\Phi|_{\{0\}\times F(\mathcal{Z}_R)})^{-1} \circ (-\psi_L) \quad \text{and} \quad (\Phi'|_{\{0\}\times F(\mathcal{Z}_R)})^{-1} \circ (-\psi_L)$$

are isotopic, as desired. □

These three-manifolds encode the action of the mapping class group on bordered three-manifolds, in the following sense.

Lemma 5.5 *Fix a strongly based diffeomorphism $\phi\colon F(\mathcal{Z}) \to F(\mathcal{Z}')$. The associated strongly bordered three-manifold M_ϕ has the following properties:*

[4]Proof: if a map ϕ is pseudoisotopic (concordant) to the identity then, in particular, ϕ is homotopic to the identity. But homotopic homeomorphisms of surfaces are isotopic.

(1) *If (Y,ψ) is a bordered three-manifold, where $\psi\colon F(\mathcal{Z})\to\partial Y$ is a strongly based mapping class, then $\phi(Y,\psi)$ is obtained by gluing Y and M_ϕ,*

$$\phi(Y,\psi)\cong Y\ {}_{\partial Y}\!\cup_{\partial_L} M_\phi,$$

canonically.

(2) *If ${}_{\psi_L}Y_{\psi_R}$ is a strongly bordered three-manifold with two boundary components, where $\psi_R\colon F(\mathcal{Z}_R)\to\partial_R Y$ and $\mathcal{Z}\cong\mathcal{Z}_R$, then $\phi({}_{\psi_L}Y_{\psi_R})$ is obtained by gluing Y and M_ϕ along $\partial_R Y$,*

$$\phi({}_{\psi_L}Y_{\psi_R})\cong({}_{\psi_L}Y_{\psi_R})\ {}_{\partial_R}\!\cup_{\partial_L} M_\phi,$$

canonically.

(3) *Given another strongly based diffeomorphism $\phi'\colon F(\mathcal{Z}')\to F(\mathcal{Z}'')$, we have that $M_{\phi'\circ\phi}$ is obtained from gluing*

$$M_{\phi'\circ\phi}\cong M_\phi\ {}_{\partial_R}\!\cup_{\partial_L} M_{\phi'}$$

canonically.

Proof It is straightforward to construct the isomorphism realizing properties (1) and (2). Property (3) follows from property (2), as

$$\begin{aligned}\phi'(M_\phi)&=\phi'({}_\phi([0,1]\times F(\mathcal{Z}'))_{\mathbb{I}})\\&={}_\phi([0,1]\times F(\mathcal{Z}'))_{(\phi')^{-1}}\\&={}_{\phi'\circ\phi}([0,1]\times F(\mathcal{Z}''))_{\mathbb{I}}\\&=M_{\phi'\circ\phi}.\end{aligned}$$

This concludes the proof. □

Similarly, for self gluing, we have:

Lemma 5.6 *Let $\phi\colon F(\mathcal{Z})\to F(\mathcal{Z})$ be a strongly based diffeomorphism. Then the generalized open book associated to M_ϕ (Construction 5.3) agrees with the open book associated to ϕ (with the orientation conventions from Geiges [10, Section 4.4.2] or Etnyre [8, Section 2], say).*

Proof The open book associated to ϕ is given by

$$([0,1]\times(F(\mathcal{Z})\setminus\Delta))\Big/\left(\begin{matrix}(1,x)\sim(0,\phi(x))\\(t,x)\sim(t',x)\text{ for }x\in\partial\Delta\end{matrix}\right),$$

which agrees with the conventions from Construction 5.3. □

In the sequel, we will find it convenient to reformulate the above properties in terms of Heegaard diagrams.

Definition 5.7 Fix a strongly based mapping class ϕ. We say that a Heegaard diagram $\mathcal{H}$ *represents* ϕ if its underlying three-manifold $Y(\mathcal{H})$ is homeomorphic (respecting the marking) to the mapping cylinder M_ϕ of Construction 5.2.

Lemma 5.8 *If $\mathcal{H}$ and $\mathcal{H}'$ represent the same element $\phi \in MCG_0(F(\mathcal{Z}_L), F(\mathcal{Z}_R))$ then $\mathcal{H}$ and $\mathcal{H}'$ are equivalent.*

Proof By Proposition 5.3, it suffices to show that $Y(\mathcal{H})$ and $Y(\mathcal{H}')$ are isomorphic strongly bordered 3–manifolds. But this follows from Lemma 5.4. □

Lemma 5.9 *Fix strongly bordered mapping classes $\phi \in MCG_0(F(\mathcal{Z}), F(\mathcal{Z}'))$ and $\psi \in MCG_0(F(\mathcal{Z}'), F(\mathcal{Z}''))$, and let $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ be Heegaard diagrams representing ϕ and ψ respectively. Then the union $(\mathcal{H}_\phi)\ {}_{\partial_R}\!\cup_{\partial_L} (\mathcal{H}_\psi)$ is a Heegaard diagram which represents the composite $\psi \circ \phi$.*

Proof This follows from Lemma 5.4 and part (3) of Lemma 5.5. □

It will be useful to have an explicit construction of a Heegaard diagram associated to a strongly based mapping class ϕ.

Let $\mathcal{Z}$ be a pointed matched circle. Consider the product of the circle and an interval, $[0,1] \times Z$. Attach one-handles to $\{1\} \times Z$ as specified by the matching. For each pair p_i and q_i on the pointed matched circle which are matched (ie with $M(p_i) = q_i$) we run an arc a_i through the one-handle and extend a_i as $\{p_i, q_i\} \times [0,1]$ through the annulus $[0,1] \times Z$, so that its boundary lies on $\{0\} \times \mathcal{Z}$. This gives a surface-with-boundary F_0 which is homeomorphic to the surface $F(\mathcal{Z})$ with two disks removed, and which is equipped with $2k$ arcs $\{a_1, \dots, a_{2k}\}$. Equip F_0 with an additional $2k$ arcs $\{b_1, \dots, b_{2k}\}$, which are chosen so that the arc b_i is contained in the i^{th} one-handle attached to the original annulus, and is dual to a_i. (That is, b_i meets a_i in a single, transverse intersection point, and is disjoint from all the a_j with $i \neq j$.) The boundary of F_0 has two components, one of which contains all the endpoints of the b_i, which we denote $\partial_b F_0$, and the other which contains all the endpoints of the a_i. The basepoint in the pointed matched circle equips F_0 with an arc ζ which connects the two boundary components of F_0. See Figure 15 on the left.

Now, let $\overline{F}_0$ be another copy of F_0 with orientation reversed, equipped with curves $\{\overline{a}_i\}_{i=1}^{2k}$ and $\{\overline{b}_i\}_{i=1}^{2k}$. Let Σ be the surface with two boundary components, obtained from $F_0 \amalg \overline{F}_0$, by identifying $\partial_b F_0$ with $\partial_b \overline{F}_0$ in such a manner that the boundary of b_i is identified with the boundary of $\overline{b}_i$, and the $\partial_b F_0$ boundary point of ζ is identified with the corresponding boundary point of $\overline{\zeta}$. The surface Σ comes with $4k$ arcs $\alpha_i^{a,L}$ and $\alpha_i^{a,R}$ given by $\alpha_i^{a,L} = a_i$ and $\alpha_i^{a,R} = \overline{a}_i$. The surface Σ also comes with $2k$ circles $\beta_i = b_i \cup \overline{b}_i$ and one more arc $z = \zeta \cup \overline{\zeta}$. Then $(\Sigma, \alpha_1^{i,L}, \dots, \alpha_{2k}^{i,R}, \beta_1, \dots, \beta_{2k}, z)$ is a diagram for the identity map. Again, see Figure 15, in the middle.

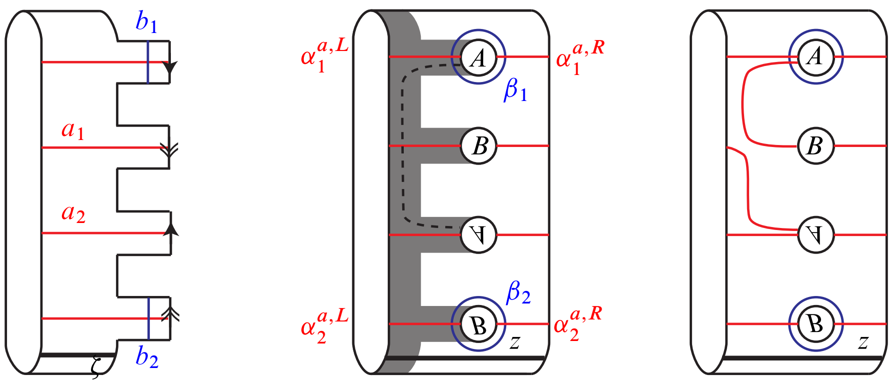


Figure 15: Constructing a Heegaard diagram for a surface automorphism: on the left, we have the surface F_0 and the arcs a_i and b_i in it; in the center, we have the surface Σ, and α– and β–curves giving a diagram for the identity map. The subsurface F_0 of Σ is shaded; note that $F_0 \setminus \zeta$ is (orientation-preserving) homeomorphic to F°. Right: the resulting diagram for a Dehn twist along the dashed curve in Σ.

Definition 5.10 For ϕ a strongly based diffeomorphism $F(\mathcal{Z}) \to F(\mathcal{Z}')$, let Σ be the surface obtained by gluing $F_0(\mathcal{Z}')$ and $\overline{F}_0(\mathcal{Z})$ along ∂_b, $\alpha_i^{a,L} = \phi(a_i) \subset F_0(\mathcal{Z}')$, $\alpha_i^{a,R} = \overline{a}_i \subset \overline{F}_0(\mathcal{Z})$, $\beta_i = b_i \cup \overline{b}_i$, and $z = \zeta \cup \overline{\zeta}$. We call $\mathcal{H}(\phi) = (\Sigma, \alpha_1^{a,L}, \dots, \alpha_{2k}^{a,L}, \alpha_1^{a,R}, \dots, \alpha_{2k}^{a,R}, \beta_1, \dots, \beta_{2k}, z)$ the *canonical bordered Heegaard diagram associated to* ϕ.

(Again, see Figure 15.)

Lemma 5.11 *The canonical bordered Heegaard diagram* $\mathcal{H}(\phi)$ *associated to* ϕ *represents the map* ϕ *in the sense of Definition 5.7.*

Proof We first verify the statement in the case where ψ is the identity mapping class. Let $\mathcal{H}$ be the corresponding Heegaard diagram. Clearly, the three-manifold $Y(\mathcal{H})$ is a surface times an interval. Thus, according to Lemma 5.4, $Y(\mathcal{H})$ represents some mapping class ψ. Moreover, after performing some handleslides and cancelations, one can see that $\mathcal{H}\,{}_{\partial_R\mathcal{H}}\cup_{\partial_L\mathcal{H}}\mathcal{H}$ is equivalent to $\mathcal{H}$. Thus, by Lemams 5.9 and 5.4, $\psi\circ\psi$ represents the same mapping class as ψ. Thus, ψ represents the identity mapping class.

In general, the mapping cylinder M_ϕ is ${}_\phi([0,1]\times F(\mathcal{Z}'))_{\mathbb{I}}$, which we can think of as obtained from the mapping cylinder for the identity by twisting the parametrization by ϕ on the left, which is just what we did above by changing the $\alpha^{a,R}$ arcs. □

6 Bimodules for bordered manifolds

Recall from [21] that bordered Floer homology associates to a bordered 3–manifold Y with one boundary component modules $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$. The module $\widehat{CFD}(Y)$ is, more precisely, a type D structure in the sense of Definition 2.2.23, and encodes all the holomorphic curve counts in its differential. The module $\widehat{CFA}(Y)$ is an $\mathcal{A}_\infty$–module, which encodes holomorphic curve counts in all its actions.

For a 3–manifold Y with two boundary components, we can treat each boundary component in either a type A or a type D manner. Treating both as type A boundaries leads to $\widehat{CFAA}(Y)$. Treating one as type A and the other as type D leads to $\widehat{CFDA}(Y)$. Treating both as type D leads to $\widehat{CFDD}(Y)$.

It turns out that both $\widehat{CFAA}(Y)$ and $\widehat{CFDD}(Y)$ can be obtained from the modules for 3–manifolds with a single boundary component via the drilling construction of Definition 5.5 and the restriction / induction functors of Section 3.4. Defining the module $\widehat{CFDA}(Y)$ seems to require some new work, though the ideas (and analytic machinery) are all present in the single boundary component cases.

The reader is encouraged to consult Section 10 for examples of the bimodules. The reader may also want to refer to [21, Appendix A] for an abbreviated (and, in the case of $\widehat{CFAA}$ and $\widehat{CFDD}$, slightly different) account of this material.

6.1 The type *AA* bimodule

Definition 6.1 Let Y be a strongly bordered 3–manifold with $\partial_L Y = F(\mathcal{Z}_L)$ and $\partial_R Y = F(\mathcal{Z}_R)$. Fix an arced bordered Heegaard diagram $\mathcal{H}$ for Y, and assume $\mathcal{H}$ is provincially admissible. Then, define

$$\widehat{CFAA}(\mathcal{H}) = \mathrm{Rest}_{\mathcal{Z},\mathcal{Z}'}(\widehat{CFA}(\mathcal{H}_{\mathrm{dr}}))$$

which, in light of Section 2.4.3, we can view as a bimodule with $\mathcal{A}_\infty$–commuting right actions of $\mathcal{A}(\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$.

The module $\widehat{CFAA}(\mathcal{H})$ decomposes as a direct sum

$$\widehat{CFAA}(\mathcal{H}) = \bigoplus_{\mathfrak{s}\in\mathrm{spin}^c(Y)} \widehat{CFAA}(Y,\mathfrak{s}).$$

Geometrically, Definition 6.1 means that $\widehat{CFAA}(\mathcal{H})$ is generated over $\mathbb{F}_2$ by g–tuples of points in $\boldsymbol{\alpha}\cap\boldsymbol{\beta}$, one on each α– and β–circle and no two on the same α–arc. The differential counts provincial holomorphic curves, ie curves not approaching $\partial\overline{\overline{\Sigma}}$. The right bimodule structure come from counting curves with asymptotics at $\partial_L\overline{\overline{\Sigma}}$ and $\partial_R\overline{\overline{\Sigma}}$, with appropriate height constraints on the asymptotics.

Proposition 6.2 *If $\mathcal{H}$ and $\mathcal{H}'$ are Heegaard diagrams for the same strongly bordered 3–manifold Y then $\widehat{CFAA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_L),\mathcal{A}(\mathcal{Z}_R)}$ and $\widehat{CFAA}(\mathcal{H}')_{\mathcal{A}(\mathcal{Z}_L),\mathcal{A}(\mathcal{Z}_R)}$ are $\mathcal{A}_\infty$–homotopy equivalent bimodules.*

Proof As in the proof of Proposition 5.3, the drilled Heegaard diagrams $\mathcal{H}_{\mathrm{dr}}$ and $\mathcal{H}'_{\mathrm{dr}}$ are equivalent, so the result follows from invariance of $\widehat{CFA}$ [21, Theorem 7.17]. □

Because of Proposition 6.2, we are justified in writing $\widehat{CFAA}(Y)$ to denote the (homotopy equivalence class of) $\widehat{CFAA}(\mathcal{H})$ for some (any) diagram $\mathcal{H}$ for Y.

Lemma 6.3 *If $\mathcal{H}$ is admissible (respectively left admissible, right admissible) in the sense of Definition 5.1 (respectively Definition 5.4) then $\widehat{CFAA}(\mathcal{H})$ is bounded (respectively left bounded, right bounded) in the sense of Definition 2.2.41.*

Proof This follows easily from Lemma 5.5, similarly to [21, Lemma 7.7]. □

6.2 The type *DD* bimodule

Definition 6.1 Let Y be a strongly bordered 3–manifold with $\partial_L Y = F(\mathcal{Z}_L)$ and $\partial_R Y = F(\mathcal{Z}_R)$. Fix an arced bordered Heegaard diagram $\mathcal{H}$ for Y, and assume $\mathcal{H}$ is provincially admissible. Then, define

$$\widehat{CFDD}(\mathcal{H}) = \mathrm{Induct}^{-\mathcal{Z}_L,-\mathcal{Z}_R}(\widehat{CFD}(\mathcal{H}_{\mathrm{dr}})),$$

a type *DD* structure over $\mathcal{A}(-\mathcal{Z}_L)$ and $\mathcal{A}(-\mathcal{Z}_R)$ (where $-\mathcal{Z}$ denotes $\mathcal{Z}$ with its orientation reversed).

The module $\widehat{CFDD}(\mathcal{H})$ decomposes as a direct sum

$$\widehat{CFDD}(\mathcal{H}) = \bigoplus_{\mathfrak{s}\in\mathrm{spin}^c(Y)} \widehat{CFDD}(Y,\mathfrak{s}).$$

Geometrically, Definition 6.1 means that $\widehat{CFDD}(\mathcal{H})$ is generated as a type *DD* structure over $\mathcal{A}(\mathcal{Z}_L)\otimes\mathcal{A}(\mathcal{Z}_R)$ by g–tuples of points in $\boldsymbol{\alpha}\cap\boldsymbol{\beta}$, one on each α– and β–circle and no two on the same α–arc. The differential counts holomorphic curves with asymptotics at $\partial\overline{\Sigma}$, without height constraints. These curves contribute coefficients corresponding to their asymptotics.

Proposition 6.2 *If $\mathcal{H}$ and $\mathcal{H}'$ are Heegaard diagrams for the same strongly bordered 3–manifold Y then*

$$^{\mathcal{A}(-\mathcal{Z}_L),\mathcal{A}(-\mathcal{Z}_R)}\widehat{CFDD}(\mathcal{H}) \quad \text{and} \quad ^{\mathcal{A}(-\mathcal{Z}_L),\mathcal{A}(-\mathcal{Z}_R)}\widehat{CFDD}(\mathcal{H}')$$

are homotopy equivalent bimodules.

Proof As in invariance of $\widehat{CFAA}$, the proof of Proposition 5.3 implies that the drilled Heegaard diagrams $\mathcal{H}_{\mathrm{dr}}$ and $\mathcal{H}'_{\mathrm{dr}}$ are equivalent, so the result follows from invariance of $\widehat{CFD}$ [21, Theorem 6.16]. □

Because of Proposition 6.2, we are justified in writing $\widehat{CFDD}(Y)$ to denote the (homotopy equivalence class of) $\widehat{CFDD}(\mathcal{H})$ for some (any) diagram $\mathcal{H}$ for Y.

Lemma 6.3 *If $\mathcal{H}$ is admissible (respectively left admissible, right admissible) in the sense of Definition 5.1 (respectively Definition 5.4) then $\widehat{CFDD}(\mathcal{H})$ is bounded (respectively left bounded, right bounded) in the sense of Definition 2.2.56.*

Proof As for $\widehat{CFAA}$, this follows easily from Lemma 5.5. □

6.3 The type *DA* bimodule

Fix a provincially admissible arced bordered Heegaard diagram $\mathcal{H}=(\overline{\Sigma}_g,\overline{\boldsymbol{\alpha}},\boldsymbol{\beta},\boldsymbol{z})$, with boundaries $\mathcal{Z}_L$ and $\mathcal{Z}_R$ representing surfaces of genus k_L and k_R. We will associate to $\mathcal{H}$ a bimodule over $\mathcal{A}(-\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$,

$$^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_R)}.$$

(As usual, $-\mathcal{Z}_L$ denotes the orientation reverse of $\mathcal{Z}_L$.)

Recall that $\S(\mathcal{H})$ is the set of g–tuples of points $\boldsymbol{x}$ in Σ so that

- exactly one x_i lies on each α– and each β–circle, and
- no two x_i lie on the same α–arc.

Let $X(\mathcal{H})$ be the $\mathbb{F}_2$–vector space spanned by $\S(\mathcal{H})$.

Given $\boldsymbol{x} \in \S(\mathcal{H})$ let $o_L(\boldsymbol{x})$ denote the indices of the α^L–arcs occupied by $\boldsymbol{x}$ and $o_R(\boldsymbol{x})$ the indices of the α^R–arcs occupied by $\boldsymbol{x}$. We let $\S(\mathcal{H}, i) \subset \S(\mathcal{H})$ denote the subset of generators with $\# o_R(\boldsymbol{x}) = i$. Note that $\# o_L(\boldsymbol{x}) + \# o_R(\boldsymbol{x}) = k_L + k_R$. Let $I_{L,D}(\boldsymbol{x}) = I([2k_L] \setminus o^L(\boldsymbol{x}))$ and $I_{R,A} = I(o^R(\boldsymbol{x}))$. We define a left (respectively right) action of $\mathcal{I}(\mathcal{Z}_L)$ (respectively $\mathcal{I}(\mathcal{Z}_R)$) on $\S(\mathcal{H})$ by

$$I(\boldsymbol{s}) \cdot \boldsymbol{x} \cdot I(\boldsymbol{t}) := \begin{cases} \boldsymbol{x} & I(\boldsymbol{s}) = I_{L,D}(\boldsymbol{x}) \text{ and } I(\boldsymbol{t}) = I_{R,A}(\boldsymbol{x}), \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{s}$ and $\boldsymbol{t}$ are subsets of $[2k]$.

As an $(\mathcal{I}(\mathcal{Z}_L), \mathcal{I}(\mathcal{Z}_R))$–bimodule, ${}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_R)}$ is $\S(\mathcal{H})$, with the above action.

Our next task is to define the type *DA* structure maps on $\widehat{CFDA}(\mathcal{H})$, for which we resort to holomorphic curves.

As in [21], we will count holomorphic curves in

$$((\Sigma \setminus \boldsymbol{z}) \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})).$$

To avoid repeating the seemingly innumerable definitions and propositions of [21, Chapter 5], we will use the drilling construction of Section 5 and simply use moduli spaces defined in [21]. (Since we are considering only curves missing the region containing $\boldsymbol{z}$, moduli spaces in the tunneled diagram contain the moduli spaces in the original diagram.)

So, let $\mathcal{H}_{\mathrm{dr}} = (\overline{\Sigma}_{\mathrm{dr}}, \overline{\boldsymbol{\alpha}}_{\mathrm{dr}}, \boldsymbol{\beta}_{\mathrm{dr}}, z_+)$ denote the bordered Heegaard diagram with one boundary component obtained by drilling a tunnel from $\mathcal{H}$. Reeb chords in $(\partial\overline{\Sigma}_{\mathrm{dr}} \setminus z_+, \overline{\boldsymbol{\alpha}}_{\mathrm{dr}} \cap \overline{\Sigma}_{\mathrm{dr}})$ come in three kinds:

- Reeb chords connecting points in $\partial\overline{\boldsymbol{\alpha}}^{a,L}_{\mathrm{dr}}$; we refer to these as *left Reeb chords*, and decorate them with an "L".
- Reeb chords connecting points in $\partial\overline{\boldsymbol{\alpha}}^{a,R}_{\mathrm{dr}}$; we refer to these as *right Reeb chords*, and decorate them with an "R".
- Reeb chords connecting points in $\partial\overline{\boldsymbol{\alpha}}^{a,L}_{\mathrm{dr}}$ to points in $\partial\overline{\boldsymbol{\alpha}}^{a,R}_{\mathrm{dr}}$; we refer to these as *mixed Reeb chords* and shall have no use for them in the present discussion.

Note that there is a one-to-one correspondence between $\S(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H}_{\mathrm{dr}})$.

Recall that a decorated source $S^{\triangleright}$ is a Riemann surface S with boundary and boundary punctures, where each puncture is either labeled $+\infty$, $-\infty$ or $e\infty$, and the $e\infty$

punctures are further labeled by Reeb chords at east infinity. Given generators $\boldsymbol{x}$ and $\boldsymbol{y}$, a homology class $B \in \pi_2(\boldsymbol{x},\boldsymbol{y})$ connecting $\boldsymbol{x}$ to $\boldsymbol{y}$, a decorated source $S^{\triangleright}$, and an ordered partition $\vec{\boldsymbol{\rho}}$ of the Reeb chords labeling punctures of $S^{\triangleright}$ we have a moduli space

$$\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};S^{\triangleright};\vec{\boldsymbol{\rho}})$$

of holomorphic curves $u\colon S \to \Sigma_{\mathrm{dr}} \times [0,1] \times \mathbb{R}$ in the homology class B with asymptotics specified by $\boldsymbol{x}$, $\boldsymbol{y}$ and $\vec{\boldsymbol{\rho}}$ [21, Definition 5.11]. The expected dimension of $\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};S^{\triangleright};\vec{\boldsymbol{\rho}})$ is given by

$$g - \chi(S) + 2e(B) + |\vec{\boldsymbol{\rho}}| - 1 =: \mathrm{ind}(B, S^{\triangleright}, \vec{\boldsymbol{\rho}}) - 1; \tag{6.1}$$

see [21, Proposition 5.8]. If the curve u is an embedding then the Euler characteristic of S is determined by

$$\chi(S) = \chi_{\mathrm{emb}}(B,\vec{\boldsymbol{\rho}}) := g + e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B) - \iota(\vec{\boldsymbol{\rho}}); \tag{6.2}$$

see [21, Proposition 5.62]. In particular, this leads us to define

$$\mathrm{ind}(B,\vec{\boldsymbol{\rho}}) := e(B) + n_{\boldsymbol{x}}(B) + n_{\boldsymbol{y}}(B) + \iota(\vec{\boldsymbol{\rho}}) + |\vec{\boldsymbol{\rho}}|. \tag{6.3}$$

If the asymptotic data $(\boldsymbol{x},\boldsymbol{\rho})$ is such that $u^{-1}(\hat{\alpha}_i^{a,L} \times (1,t))$ (respectively $u^{-1}(\hat{\alpha}_i^{a,R} \times (1,t))$) consists of at most one point for any given t (ie $(\boldsymbol{x},\boldsymbol{\rho})$ is *strongly boundary monotonic*) then the moduli space $\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};S^{\triangleright};\vec{\boldsymbol{\rho}})$ is well behaved, and we can understand its codimension-one boundary:

Proposition 6.4 [21, Theorem 5.55] *Suppose that $(\boldsymbol{x},\vec{\boldsymbol{\rho}})$ is strongly boundary monotonic. Fix $\boldsymbol{y}$, $B \in \pi_2(\boldsymbol{x},\boldsymbol{y})$, and $S^{\triangleright}$ with $e\infty$ punctures labeled by $\vec{\boldsymbol{\rho}}$, such that $\mathrm{ind}(B,S^{\triangleright},\vec{\boldsymbol{\rho}}) = 2$. Let $\mathcal{M} = \mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};S^{\triangleright};\vec{\boldsymbol{\rho}})$. Then the total number of all*

(1) *two-story ends of $\mathcal{M}$,*

(2) *join curve ends of $\mathcal{M}$,*

(3) *odd shuffle curve ends of $\mathcal{M}$, and*

(4) *collision of levels i and $i+1$ in $\mathcal{M}$, where $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_{i+1}$ are weakly composable*

is even.

Examples of the four types of degenerations are shown in Figure 16.

To define the multiplications on $\widehat{CFDA}(\mathcal{H})$ we collect certain of the $\mathcal{M}(\boldsymbol{x},\boldsymbol{y};S^{\triangleright};\vec{\boldsymbol{\rho}})$. Specifically, given a sequence $\vec{\rho}^L$ of left Reeb chords and a sequence of sets of right Reeb chords $(\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$, we say that a sequence $\vec{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_m)$ *interleaves*

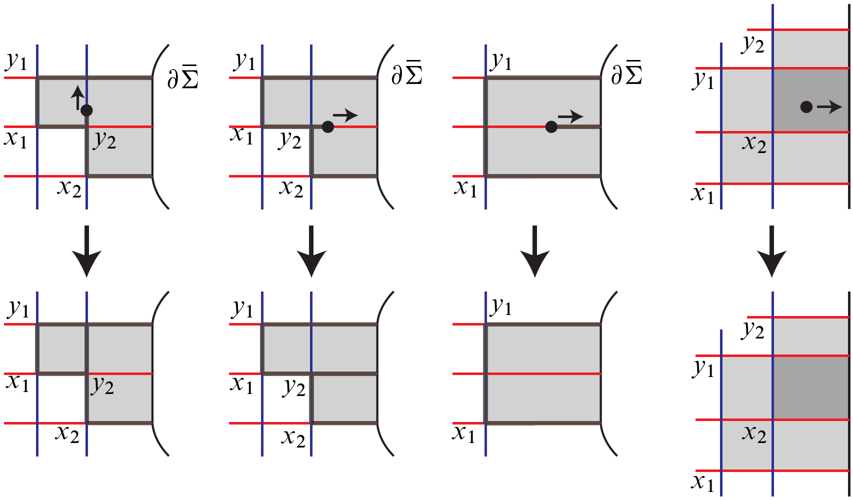


Figure 16: Examples of the four codimension-one degenerations: on the far left, we have a two-level splitting; in the center left, degenerating a join curve; in the center right, degenerating a split curve (a collision of levels); on the far right, degenerating a shuffle curve. The dark dots indicate branch points. This figure is adapted from [21, Figures 5.1 and 5.5].

$(\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$ if, as a multiset, $\{\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_m\}=\vec{\rho}^L\amalg\{\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R\}$, and the orderings of $\vec{\rho}^L$ and $(\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_n)$ agree with the orderings induced by $\vec{\boldsymbol{\rho}}$.

Now, let

$$\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)=\bigcup_{\substack{\vec{\boldsymbol{\rho}}\text{ interleaves }(\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)\\ \chi(S)=\chi_{\mathrm{emb}}(B,\vec{\boldsymbol{\rho}})}}\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};S^{\triangleright};\vec{\boldsymbol{\rho}}).$$

Lemma 6.5 *If $\vec{\boldsymbol{\rho}}$ and $\vec{\boldsymbol{\rho}}'$ both interleave $(\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$ then $(\boldsymbol{x},\vec{\boldsymbol{\rho}})$ is strongly boundary monotonic if and only if $(\boldsymbol{x},\vec{\boldsymbol{\rho}}')$ is strongly boundary monotonic. Moreover, for any homology class B,* $\mathrm{ind}(B,\vec{\boldsymbol{\rho}})=\mathrm{ind}(B,\vec{\boldsymbol{\rho}}')$.

Proof The boundary monotonicity statement is immediate from the definition. It is also immediate from the definitions that if $\vec{\boldsymbol{\rho}}$ interleaves $(\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$ then

$$\iota(\vec{\boldsymbol{\rho}})=\iota(\vec{\rho}^L)+\iota(\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R).$$

So the statement about ind follows from the definition, Equation (6.3). □

Consequently, it makes sense to talk about a triple $(\boldsymbol{x};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$ being strongly boundary monotonic: such a triple is strongly boundary monotonic if for some (equivalently, any) sequence $\vec{\boldsymbol{\rho}}$ interleaving $(\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$, $(\boldsymbol{x},\vec{\boldsymbol{\rho}})$ is strongly boundary monotonic. Similarly, define $\operatorname{ind}(B;\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$ to be $\operatorname{ind}(B,\vec{\boldsymbol{\rho}})$ for any $\vec{\boldsymbol{\rho}}$ which interleaves $(\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$.

With these moduli spaces in hand, define a type *DA* structure (Definition 2.2.43) on $\widehat{CFDA}$ by

$$\begin{aligned}(6.6)\quad &\delta^1_{n+1}(\boldsymbol{x},a(\boldsymbol{\rho}_1^R),\dots,a(\boldsymbol{\rho}_n^R))\\ &:=\sum_{\boldsymbol{y}\in\S(\mathcal{H})}\sum_{\substack{B\in\pi_2(\boldsymbol{x},\boldsymbol{y})\\ \operatorname{ind}(B;\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)=1}}\#(\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R))\cdot a(-\rho_1^L)\cdots a(-\rho_m^L)\boldsymbol{y}\end{aligned}$$

where $\vec{\rho}^L=(\rho_1^L,\dots,\rho_m^L)$ and $-\rho_i$ denotes ρ_i with its orientation reversed. The reader may find it helpful to compare this definition with [21, Chapter 7]. Also, note that the case $n=0$ is essentially the differential on $\widehat{CFD}$ from [21, Chapter 6]. Finally, notice that if $(\boldsymbol{x};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$ is not strongly boundary monotonic then either the left-hand side of (6.6) is nonsensical or the element $a(-\rho_1^L)\cdots a(-\rho_m^L)\boldsymbol{y}$ is automatically 0.

Lemma 6.7 *Under the provincial admissibility hypothesis, the sum defining δ_n is finite.*

Proof This is a trivial adaptation of [21, Lemma 7.7]. □

This completes the definition of $\widehat{CFDA}(\mathcal{H})$. It remains to check that:

- The maps δ_n satisfy the compatibility conditions of Definition 2.2.43 (see (2.2.44)).
- If $\mathcal{H}$ and $\mathcal{H}'$ define the same strongly bordered 3–manifold then $\widehat{CFDA}(\mathcal{H})$ is $\mathcal{A}_\infty$–homotopy-equivalent to $\widehat{CFDA}(\mathcal{H}')$.

We start by refining Proposition 6.4.

Proposition 6.8 *Fix generators $\boldsymbol{x}$ and $\boldsymbol{y}$, $B\in\pi_2(\boldsymbol{x},\boldsymbol{y})$, a sequence of left Reeb chords $\vec{\rho}^L$ and a sequence of sets of right Reeb chords $(\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$, such that $(\boldsymbol{x};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$ is strongly boundary monotonic. Assume that*

$$\operatorname{ind}(B;\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)=2.$$

Then the sum of the following numbers is even:

(1) *The number of two-story ends of* $\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R)$, *ie*

$$\sum_{\substack{\mathrm{ind}(B_1;\vec{\rho}_1^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_i^R)=1\\ \mathrm{ind}(B_2;\vec{\rho}_2^L;\boldsymbol{\rho}_{i+1}^R,\dots,\boldsymbol{\rho}_n^R)=1}} \#(\mathcal{M}^{B_1}(\boldsymbol{x},\boldsymbol{w};\vec{\rho}_1^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_i^R)\times\mathcal{M}^{B_2}(\boldsymbol{w},\boldsymbol{y};\vec{\rho}_2^R;\boldsymbol{\rho}_{i+1}^R,\dots,\boldsymbol{\rho}_n^R)),$$

where the sum is over $\boldsymbol{w}\in\S(\mathcal{H})$, $B_1\in\pi_2(\boldsymbol{x},\boldsymbol{w})$, $B_2\in\pi_2(\boldsymbol{w},\boldsymbol{y})$, $B=B_1*B_2$, $i=0,\dots,n$ *and* $(\vec{\rho}_1^L,\vec{\rho}_2^L)=\vec{\rho}^L$.

(2) *The number of join curve ends among right Reeb chords, ie*

$$\sum_{\substack{i=1,\dots,n\\ \rho_{i,j}=\rho_a\uplus\rho_b}} \#\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_i^{R,a,b},\dots,\boldsymbol{\rho}_n^R),$$

where $\boldsymbol{\rho}_i^{R,a,b}$ *is obtained from* $\boldsymbol{\rho}_i^R$ *by replacing* $\rho_{i,j}\in\boldsymbol{\rho}_i^R$ *by* ρ_a,ρ_b.

(3) *The number of odd shuffle curve ends among right Reeb chords, ie*

$$\sum_{i=1}^{n}\#\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_i^{R,\prime},\dots,\boldsymbol{\rho}_n^R),$$

where $\boldsymbol{\rho}_i^{R,\prime}$ *is obtained from* $\boldsymbol{\rho}_i^R$ *by performing a weak shuffle.*

(4) *The number of collisions among right levels, ie*

$$\sum_{i=1}^{n}\#\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\rho}^L;\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_i^R\uplus\boldsymbol{\rho}_{i+1}^R,\dots,\boldsymbol{\rho}_n^R),$$

where $\boldsymbol{\rho}_i$ *and* $\boldsymbol{\rho}_{i+1}$ *are weakly composable.*

(5) *The number of join curve ends among left Reeb chords, ie*

$$\sum_{i=1}^{n}\#\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\boldsymbol{\rho}}^{L,\prime};\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R),$$

where $\vec{\boldsymbol{\rho}}^{L,\prime}=(\rho_1^L,\dots,\rho_{i-1}^L,\{\rho_a^L,\rho_b^L\},\dots,\rho_m^L)$ *is obtained by replacing* $\rho_i^L=\rho_a^L\uplus\rho_b^L$ *in* $\vec{\rho}^L$ *with* $\{\rho_a^L,\rho_b^L\}$.

(6) *The number of split curve ends among left Reeb chords, ie*

$$\sum_{\rho_i^{L,+}=\rho_{i+1}^{L,-}}\#\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};(\rho_1^L,\dots,\rho_{i-1}^L,\rho_i^L\uplus\rho_{i+1}^L,\dots,\rho_m^L);\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R).$$

(7) *The number of other collisions of left levels* ρ_i,ρ_{i+1}, *ie*

$$\sum\#\mathcal{M}^B(\boldsymbol{x},\boldsymbol{y};\vec{\boldsymbol{\rho}}^{L,\prime};\boldsymbol{\rho}_1^R,\dots,\boldsymbol{\rho}_n^R),$$

where

$$\vec{\boldsymbol{\rho}}^{L,\prime} = (\rho_1^L, \dots, \rho_{i-1}^L, \{\rho_i^L, \rho_{i+1}^L\}, \dots, \rho_n^L) \quad \text{and} \quad \rho_i^{L,+} \neq \rho_{i+1}^{L,-}.$$

Moreover, ρ_i^L *and* ρ_{i+1}^L *must satisfy*

- $\rho_i^{L,+}$ *and* $\rho_{i+1}^{L,-}$ *do not lie on the same* α*–arc, and*
- (ρ_i^L, ρ_{i+1}^L) *are not interleaved (in that order).*

Proof Recall that embedded curves have maximal index and, in codimension one, families of embedded curves converge to embedded curves [21, Proposition 5.62, Lemmas 5.69 and 5.70]. So summing Proposition 6.4 over all $S^\triangleright$ with embedded Euler characteristic, there are four kinds of ends not accounted for:

- The first is collisions of levels between right and left Reeb chords; these cancel in pairs.
- The second is collisions of right levels which are not composable; these are prohibited by [21, Lemma 5.70].
- The third is collisions of left levels not satisfying the conditions set out; the first condition comes from the fact that boundary degenerations are prohibited (see [21, Lemma 5.54]). (Note that collisions where $\rho_i^{L,+} = \rho_{i+1}^{L,-}$ are included in sum (6).) The second comes from the fact that $\{\rho_i^L\}$ and $\{\rho_{i+1}^L\}$ must be composable [21, Lemma 5.70].
- The last possibility is shuffle curve ends among left Reeb chords. These are prohibited because each part of $\vec{\rho}^L$ has only a single Reeb chord.

The result follows. $\square$

Proposition 6.9 *The maps* δ_n *satisfy the compatibility conditions of a type DA structure.*

Proof The proof is a combination of the proofs of [21, Propositions 6.7 and 7.12], and we shall be somewhat terse. We must show that for any $\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n$,

$$\begin{aligned} 0 &= (\partial \otimes \mathbb{I}_N)(\delta_{n+1}(\boldsymbol{x} \otimes a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_n))) \\ &\quad + \sum_{i+j=n+2} (\mu_2 \otimes \mathbb{I}_N) \circ (\mathbb{I}_A \otimes \delta_i)(\delta_j(\boldsymbol{x} \otimes a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_{j-1})) \otimes a(\boldsymbol{\rho}_j) \otimes \cdots \otimes a(\boldsymbol{\rho}_n)) \\ &\quad + \sum_{i=1}^{n} \delta_{n+1}(\boldsymbol{x} \otimes a(\boldsymbol{\rho}_1) \otimes \cdots \otimes \partial a(\boldsymbol{\rho}_i) \otimes \cdots \otimes \boldsymbol{\rho}_n) \\ &\quad + \sum_{i=1}^{n} \delta_n(\boldsymbol{x} \otimes a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_i) a(\boldsymbol{\rho}_{i+1}) \otimes \cdots \otimes \boldsymbol{\rho}_n); \end{aligned}$$

cf Definition 2.2.43.

The second term corresponds to two level splittings, sum (1) of Proposition 6.8. The third term corresponds to the right join and shuffle ends, sums (2) and (3) of Proposition 6.8; see also [21, proof of Proposition 7.12]. The fourth term corresponds to the collisions of right levels, sum (4) of Proposition 6.8; again, see also [21, proof of Proposition 7.12]. The first term corresponds to sum (6) of Proposition 6.8; compare [21, Lemma 6.11].

It remains to see that the sums (5) and (7) of Proposition 6.8 cancel in pairs as long as $a(-\rho_1^L)\cdots a(-\rho_m^L) \neq 0$. (See also [21, Proof of Proposition 6.7] for this part of the proof.) This product being nonzero imposes the following additional conditions on collisions of left levels ρ_i and ρ_{i+1}:

- $\rho_i^{L,-}$ and $\rho_{i+1}^{L,-}$ lie on different α–arcs. Similarly, $\rho_i^{L,+}$ and $\rho_{i+1}^{L,+}$ lie on different α–arcs. See [21, Lemma 6.9].
- If $\rho_i^{L,-}$ and $\rho_{i+1}^{L,+}$ lie on the same α–arc then $\rho_i^{L,-} = \rho_{i+1}^{L,+}$. This is immediate from $a(-\rho_i^L)a(-\rho_{i+1}^L) \neq 0$.
- (ρ_{i+1}^L, ρ_i^L) are not interleaved (in that order). Again, this is immediate from the fact that $a(-\rho_i^L)a(-\rho_{i+1}^L) \neq 0$.

Thus, the two allowed kinds of left collisions which are not algebraically 0 are:

- Collisions with $\rho_i^{L,-} = \rho_{i+1}^{L,+}$; these moduli spaces cancel with the join curve ends of the factorization $a(-\rho_1^L)\cdots a(-(\rho_i^L \uplus \rho_{i+1}^L))\cdots a(-\rho_m^L)$.
- Collisions with the endpoints of ρ_i^L and ρ_{i+1}^L lying on four different α–arcs, and with ρ_i^L and ρ_{i+1}^L either nested or disjoint; in this case, the same degeneration also occurs for the factorization with $a(\rho_i^L)$ and $a(\rho_{i+1}^L)$ switched.

This concludes the proof. □

We next turn to the issue of invariance.

Proposition 6.10 *If $\mathcal{H}$ and $\mathcal{H}'$ are provincially admissible arced bordered Heegaard diagrams defining the same strongly bordered 3–manifold ${}_{\mathcal{Z}_L}Y_{\mathcal{Z}_R}$, then the corresponding bimodules*

$$ {}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_R)} \quad \text{and} \quad {}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H}')_{\mathcal{A}(\mathcal{Z}_R)} $$

are $\mathcal{A}_\infty$–homotopy equivalent.

Proof As in the case of a single boundary component, the invariance is proved by constructing homotopy equivalences corresponding to each of the Heegaard moves of Proposition 5.3; the reader should have no difficulty adapting the proof from [21, Section 7.1] to the present situation. □

Because of Proposition 6.10, we are justified in writing $\widehat{CFDA}(Y)$ to denote the (homotopy equivalence class of) $\widehat{CFDA}(\mathcal{H})$ for some (any) diagram $\mathcal{H}$ for Y.

We conclude this section with a lemma about admissibility.

Lemma 6.11 *If $\mathcal{H}$ is admissible (respectively left admissible, right admissible) in the sense of Definition 5.1 (respectively Definition 5.4) then $\widehat{CFDA}(\mathcal{H})$ is bounded (respectively left bounded, right bounded) in the sense of Definition 2.2.46.*

Proof As for $\widehat{CFAA}$ and $\widehat{CFDD}$, this follows easily from Lemma 5.5. □

Remark 6.12 It follows from the pairing theorems of Section 7 that the module $\widehat{CFDA}(Y)$ is determined by $\widehat{CFDD}(Y)$ (or, equally well, $\widehat{CFAA}(Y)$). Consequently, the bimodules associated to 3–manifolds with two boundary components are completely determined by the invariants of 3–manifolds with connected boundaries, via the induction/restriction functors.

6.4 Modules associated to surface automorphisms

Given a strongly based diffeomorphism $\psi\colon (F, D, z) \to (F, D, z)$, define

$$\begin{aligned}
\widehat{CFAA}(\psi)_{\mathcal{A}(F),\mathcal{A}(F)} &:= \widehat{CFAA}(\mathcal{H})_{\mathcal{A}(F),\mathcal{A}(F)},\\
{}^{\mathcal{A}(-F),\mathcal{A}(-F)}\widehat{CFDD}(\psi) &:= {}^{\mathcal{A}(-F),\mathcal{A}(-F)}\widehat{CFDD}(\mathcal{H}),\\
{}^{\mathcal{A}(-F)}\widehat{CFDA}(\psi)_{\mathcal{A}(F)} &:= {}^{\mathcal{A}(-F)}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(F)},
\end{aligned}$$

where $\mathcal{H}$ is any Heegaard diagram representing ψ (in the sense of Definition 5.7).

Proof of Theorem 3 This is immediate from Lemma 5.8 and invariance of $\widehat{CFAA}$, $\widehat{CFDD}$ and $\widehat{CFDA}$, Propositions 6.2, 6.2 and 6.10 respectively. □

6.5 Gradings

Suppose Y is a strongly bordered three-manifold with boundary parameterized by $\mathcal{Z}_L$ and $\mathcal{Z}_R$, and choose a compatible provincially admissible arced bordered Heegaard diagram $\mathcal{H}$.

The gradings on $\widehat{CFDD}(Y)$ and $\widehat{CFAA}(Y)$ are induced by the induction and restriction functors, so we will focus on the grading of $\widehat{CFDA}(Y) = {}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z}_R)}$. (An alternate approach to the gradings on $\widehat{CFDD}(Y)$ and $\widehat{CFAA}(Y)$ is in Remark 6.16.)

The bimodule $\widehat{CFDA}(Y)$ interacts with the strands grading in the following way. The condition that $\#o_L(\boldsymbol{x}) + \#o_R(\boldsymbol{x}) = k_L + k_R$ for all generators $\boldsymbol{x}$ gives

$$\mathcal{I}(-\mathcal{Z}_L, i) \cdot \widehat{CFDA}(Y) = \widehat{CFDA}(Y) \cdot \mathcal{I}(\mathcal{Z}_R, i).$$

Thus, defining $\widehat{CFDA}(Y, i) := \widehat{CFDA}(Y) \cdot \mathcal{I}(\mathcal{Z}_R, i)$, we have that $\widehat{CFDA}(Y, i)$ is a type *DA* structure over $\mathcal{A}(-\mathcal{Z}_L, i)$ and $\mathcal{A}(\mathcal{Z}_R, i)$, and

$$(6.1)\qquad {}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_R)} = \bigoplus_{i \in \mathbb{Z}} {}^{\mathcal{A}(-\mathcal{Z}_L, i)}\widehat{CFDA}(\mathcal{H}, i)_{\mathcal{A}(\mathcal{Z}_R, i)}.$$

Moreover, by Lemma 5.9, there is a natural splitting of $\widehat{CFDA}(Y)$ according to spin^c structures: $\widehat{CFDA}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{spin}^c(Y)} \widehat{CFDA}(Y, \mathfrak{s})$. In particular, to define the summand $\widehat{CFDA}(Y, \mathfrak{s})$, we repeat the construction from Section 6.3, using only the subset $\S(\mathcal{H}, \mathfrak{s}) \subset \S(\mathcal{H})$ of generators representing $\mathfrak{s}$.

We would like to endow $\widehat{CFDA}(Y)$ with the structure of a left-right $(G'(-\mathcal{Z}_L), G'(\mathcal{Z}_R))$–set graded bimodule (in the sense of Definition 2.5.9); ie we would like to grade $\widehat{CFDA}(Y)$ by a set with a compatible right action by

$$G'_{DA}(\partial\mathcal{H}) := G'(-\mathcal{Z}_L)^{\mathrm{op}} \times_\lambda G'(\mathcal{Z}_R).$$

(When the Heegaard diagram is clear from the context, we will sometimes write G'_{DA} to mean $G'_{DA}(\partial\mathcal{H})$.) This is done by a suitable adaptation of the grading on $\widehat{CFD}$ and $\widehat{CFA}$ from [21].

We will often work in the isomorphic group

$$G'_{AA}(\partial\mathcal{H}) = G'(\mathcal{Z}_L) \times_\lambda G'(\mathcal{Z}_R).$$

Recall from (3.15) that, if $r\colon Z \to -Z$ is the (orientation-reversing) identity map, then

$$R(j, \alpha) = (j, r_*(\alpha))$$

defines a group antihomomorphism from $G(\mathcal{Z})$ to $G(-\mathcal{Z})$, and so an isomorphism $G(\mathcal{Z}) \cong G(-\mathcal{Z})^{\mathrm{op}}$. Then

$$R \times_\lambda \mathbb{I}\colon G'_{AA}(\partial\mathcal{H}) \to G'_{DA}(\partial\mathcal{H})$$

is a canonical isomorphism, which we denote $\widetilde{R}$.

We will construct the grading one spin^c structure at a time. Specifically, fix a Heegaard diagram $\mathcal{H}$ for Y and a spin^c structure $\mathfrak{s}$ over Y. Suppose that there is at least one generator for $\mathfrak{S}(\mathcal{H})$ which represents $\mathfrak{s}$; we will return to the case that $\mathcal{H}$ has no generator representing $\mathfrak{s}$ at the end of this subsection.

There is a map

$$g'\colon \pi_2(\boldsymbol{x},\boldsymbol{y}) \to G'_{AA}(\partial\mathcal{H})$$

defined by

$$g'(B) = (-e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B), \partial^{\partial_L} B, \partial^{\partial_R} B), \tag{6.2}$$

where $e(B)$ is the Euler measure of B and $n_{\boldsymbol{x}}(B)$ is sum of the average local multiplicities of B at each coordinate of $\boldsymbol{x}$. (Compare [21, Section 10.2].)

Recall from Section 3.2 that $G'(\mathcal{Z})$ is an index two subgroup of the group $\frac{1}{2}\mathbb{Z} \times H_1(Z', \boldsymbol{a})$ (with a twisted multiplication), so we must show that $g'(B) \in G'_{AA}(\partial\mathcal{H})$. To this end, we have the following:

Lemma 6.3 *The tuple $g'(B)$ defined in (6.2) is an element of $G'_{AA}(\partial\mathcal{H})$.*

Proof This follows from [21, Proposition 10.3] by drilling. □

Lemma 6.4 *If $B_1 \in \pi_2(\boldsymbol{x},\boldsymbol{y})$ and $B_2 \in \pi_2(\boldsymbol{y},\boldsymbol{w})$, then*

$$g'(B_1 * B_2) = g'(B_1)\cdot g'(B_2). \tag{6.5}$$

Proof This follows from [21, Lemma 10.4] by drilling. □

For $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$, let $P'_{\boldsymbol{x}} \subset G'_{AA}(\partial\mathcal{H})$ be $g'(\pi_2(\boldsymbol{x},\boldsymbol{x}))$.

Corollary 6.6 *For $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$, $P'_{\boldsymbol{x}}$ is a subgroup of $G'_{AA}(\partial\mathcal{H})$. Also, if $\boldsymbol{y} \in \mathfrak{S}(\mathcal{H})$ is another generator and $C \in \pi_2(\boldsymbol{x},\boldsymbol{y})$, then $P'_{\boldsymbol{x}} = g'(C)\cdot P'_{\boldsymbol{y}} \cdot g'(C)^{-1}$.*

Proof Both parts follow immediately from Lemma 6.4. □

Definition 6.7 Fix $\boldsymbol{x}_0 \in \mathfrak{S}(\mathcal{H},\mathfrak{s})$. Let $S'_{DA}(\mathcal{H},\boldsymbol{x}_0)$ denote the quotient $\widetilde{R}(P'_{\boldsymbol{x}_0})\backslash G'_{DA}(\partial\mathcal{H})$ as a set with a right action of $G'_{DA}(\partial\mathcal{H})$, or equivalently as a left-right $(G'(-\mathcal{Z}_L), G'(\mathcal{Z}_R))$–set. There is a grading on $\widehat{CFDA}(\mathcal{H},\mathfrak{s})$ with values in $S'_{DA}(\mathcal{H},\boldsymbol{x}_0)$, defined by $\mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{x}) = [\widetilde{R}(g'(B))]$ for any $B \in \pi_2(\boldsymbol{x}_0,\boldsymbol{x})$.

With the above definition, Lemma 6.4 ensures that if $B \in \pi_2(\boldsymbol{x},\boldsymbol{y})$, then

$$\mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{y}) = \mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{x})\cdot \widetilde{R}(g'(B)), \tag{6.8}$$

where the multiplication on the right is right translation in $G'_{DA}(\partial\mathcal{H})$ of the right coset $\mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{x})$.

Lemma 6.9 *If $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, and $(\vec{\rho}^L, \vec{\boldsymbol{\rho}}^R)$ is compatible with B, then*

$$\lambda^{|\vec{\boldsymbol{\rho}}^R| - \mathrm{ind}(B; \vec{\rho}^L; \vec{\boldsymbol{\rho}}^R)} \cdot \mathrm{gr}'(\vec{\boldsymbol{\rho}}^R) = \widetilde{R}(g'(B)) \cdot \mathrm{gr}'(-\vec{\rho}^L)$$

inside $G'_{DA}(\partial\mathcal{H})$.

Proof This is a combination of the following facts:

$$\begin{aligned}
\mathrm{gr}'(\vec{\boldsymbol{\rho}}^R) &= (\iota(\vec{\boldsymbol{\rho}}^R), 0, \partial^\partial_R B), \\
\mathrm{gr}'(-\vec{\rho}^L) &= (-|\vec{\rho}^L| - \iota(\vec{\rho}^L), -\partial^\partial_L B, 0), \\
\mathrm{ind}(B; \vec{\rho}^L; \vec{\boldsymbol{\rho}}^R) &= e(B) + n_{\boldsymbol{x}}(B) + n_{\boldsymbol{y}}(B) + \iota(\vec{\boldsymbol{\rho}}^R) + \iota(\vec{\rho}^L) + |\vec{\rho}^L| + |\vec{\boldsymbol{\rho}}^R|.
\end{aligned}$$

The first of these equations is verified in [21, Lemma 5.60]; the second is verified in the proof of [21, Lemma 10.20]; and the third is the definition, (6.3). Note that the two terms on the right-hand side, $\mathrm{gr}'(-\vec{\rho}^L)$ and $\widetilde{R}(g'(B))$, commute with each other, as the spin^c component of $\mathrm{gr}'(-\vec{\rho}^L)$ is the negative of the portion of the spin^c component of $\widetilde{R}(g'(B))$ that lies on the left boundary. □

Proposition 6.10 *The map $\mathrm{gr}'_{\boldsymbol{x}_0}$ defines a grading of $\widehat{CFDA}$ as a DA structure with values in the right $G'_{DA}(\partial\mathcal{H})$–set $S'_{DA}(\mathcal{H}, \boldsymbol{x}_0)$. Different choices of $\boldsymbol{x}_0 \in \mathfrak{S}(\mathcal{H}, \mathfrak{s})$ lead to canonically isomorphic $G'_{DA}(\partial\mathcal{H})$–set graded modules.*

Proof For the first part, suppose that $a(-\vec{\rho}^L) \otimes \boldsymbol{y}$ appears with nonzero multiplicity in $\delta^1_{n+1}(\boldsymbol{x}, a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_n))$. Then there is a domain $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ so that $(\vec{\rho}^L, \vec{\boldsymbol{\rho}}^R)$ is compatible with B, and indeed $\mathrm{ind}(B, \vec{\rho}^L; \vec{\boldsymbol{\rho}}^R) = 1$. We need to know

$$\lambda^{n-1} \cdot \mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{x}) \cdot \prod_{i=1}^{n} \mathrm{gr}'(\boldsymbol{\rho}_i^R) = \mathrm{gr}'(-\vec{\rho}^L) \star \mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{y})$$

(where here $\star$ refers to the left action of $G'(-\mathcal{Z}_L)$, which in turn can be viewed as right translation by an element of $G'(-\mathcal{Z}_L)^{\mathrm{op}} \subset G'_{DA}(\partial\mathcal{H})$). But this follows from Lemma 6.9 and (6.8).

Suppose now $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are two different choices of generator both of which represent $\mathfrak{s}$. This means that there is a domain $C \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x}_1)$. We define now an identification

$$\Phi'^{\boldsymbol{x}_1}_{\boldsymbol{x}_0} \colon \widetilde{R}(P'_{\boldsymbol{x}_1}) \backslash G'_{DA}(\partial\mathcal{H}) \to \widetilde{R}(P'_{\boldsymbol{x}_0}) \backslash G'_{DA}(\partial\mathcal{H})$$

by

$$\Phi'^{\boldsymbol{x}_1}_{\boldsymbol{x}_0}(\widetilde{R}(P'_{\boldsymbol{x}_1}) \cdot h) = \widetilde{R}(P'_{\boldsymbol{x}_0}) \cdot \widetilde{R}(g'(C)) \cdot h.$$

This gives a well-defined map on coset spaces since $P'_{\boldsymbol{x}_0} = g'(C) \cdot P'_{\boldsymbol{x}_1} \cdot g'(C)^{-1}$, by Corollary 6.6. Now, for any other $\boldsymbol{y}$ representing $\mathfrak{s}$ and any $B \in \pi_2(\boldsymbol{x}_1, \boldsymbol{y})$,

$$\begin{aligned}\Phi'^{\boldsymbol{x}_1}_{\boldsymbol{x}_0}(\mathrm{gr}'_{\boldsymbol{x}_1}(\boldsymbol{y})) = \Phi'^{\boldsymbol{x}_1}_{\boldsymbol{x}_0}(\widetilde{R}(P'_{\boldsymbol{x}_1}) \cdot \widetilde{R}(g'(B))) &= \widetilde{R}(P'_{\boldsymbol{x}_0}) \cdot \widetilde{R}(g'(C)) \cdot \widetilde{R}(g'(B)) \\ &= \widetilde{R}(P'_{\boldsymbol{x}_0}) \cdot \widetilde{R}(g'(C * B)) = \mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{y}),\end{aligned}$$

by the various definitions and another application of Corollary 6.6. Thus, the desired isomorphism

$$(\widehat{CFDA}(\mathcal{H}, \mathfrak{s}), \mathrm{gr}'_{\boldsymbol{x}_1}, S'_{DA}(\mathcal{H}, \boldsymbol{x}_1)) \mapsto (\widehat{CFDA}(\mathcal{H}, \mathfrak{s}), \mathrm{gr}'_{\boldsymbol{x}_0}, S'_{DA}(\mathcal{H}, \boldsymbol{x}_0))$$

is supplied by the identity map on the modules, combined with the map $\Phi'^{\boldsymbol{x}_1}_{\boldsymbol{x}_0}$ on the $G'_{DA}(\partial\mathcal{H})$–sets. □

Finally, we comment briefly on the case that $\mathcal{H}$ has no generators representing the spin^c–structure $\mathfrak{s}$. In this case, $\widehat{CFDA}(\mathcal{H}, \mathfrak{s})$ is the trivial module, but (perhaps) we should still specify its grading set. Choose another diagram $\mathcal{H}'$ so that there is a generator $\boldsymbol{x}_0 \in \mathfrak{S}(\mathcal{H}')$ with $\mathfrak{s}(\boldsymbol{x}_0) = \mathfrak{s}$, and define the grading set for $\widehat{CFDA}(\mathcal{H}, \mathfrak{s})$ to be $S'_{DA}(\mathcal{H}', \boldsymbol{x}_0)$ and the grading on $\widehat{CFDA}(\mathcal{H}, \mathfrak{s})$ to be the unique map from $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) = \varnothing$ to $S'_{DA}(\mathcal{H}', \boldsymbol{x}_0)$. A simplified version of the invariance proof (keeping track only of the G'–set gradings, and not the modules themselves) shows that, up to isomorphism (in the category of G'–set graded bimodules), this is independent of the choice of $\mathcal{H}'$; see the proof of Proposition 6.14 for more details.

6.5.1 Refined gradings We now give the bimodule $\widehat{CFDA}(Y, \mathfrak{s})$ a grading by a left-right $(G(-\mathcal{Z}_L), G(\mathcal{Z}_R))$–set, using the smaller grading group from Section 3.2.1. Let $G_{AA}(\partial\mathcal{H}) = G(\mathcal{Z}_L) \times_\lambda G(\mathcal{Z}_R)$ and $G_{DA}(\partial\mathcal{H}) = G(-\mathcal{Z}_L)^{\mathrm{op}} \times_\lambda G(\mathcal{Z}_R)$.

The existence of a refinement is a formal consequence of the following:

Lemma 6.11 *The image $P'_{\boldsymbol{x}}$ of $\pi_2(\boldsymbol{x}, \boldsymbol{x})$ in $G'_{AA}(\partial\mathcal{H})$ is in fact contained in $G_{AA}(\partial\mathcal{H})$. Moreover, given two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ representing $\mathfrak{s}$, if $\mathrm{gr}'(\boldsymbol{x}) \cdot \widetilde{R}(g) = \mathrm{gr}'(\boldsymbol{y})$, with $g = g_L \times_\lambda g_R$, then $R(g_L)$ is compatible with the idempotents $I_{L,D}(\boldsymbol{x})$ and $I_{L,D}(\boldsymbol{y})$, and g_R is compatible with the idempotents $I_{R,A}(\boldsymbol{x})$ and $I_{R,A}(\boldsymbol{y})$, in the sense of Definition 3.4.*

Proof Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ represent $\mathfrak{s}$, and let $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$. It is clear that the homology class $\partial^\partial_L(B)$ is compatible with the idempotents $I_{L,A}(\boldsymbol{x})$ and $I_{L,A}(\boldsymbol{y})$, or equivalently that $r_*(\partial^\partial_L(B))$ is compatible with $I_{L,D}(\boldsymbol{x})$ and $I_{L,D}(\boldsymbol{y})$. Similarly, $\partial^\partial_R(B)$ is compatible with $I_{R,A}(\boldsymbol{x})$ and $I_{R,A}(\boldsymbol{y})$.

Specializing to the case where $\boldsymbol{x} = \boldsymbol{y}$, $P'_{\boldsymbol{x}}$ is contained in $G_{AA}(\partial\mathcal{H}) \subset G'_{AA}(\partial\mathcal{H})$.

We turn to the second condition. For notational simplicity, let $I_{AA}(\boldsymbol{x})$ denote the pair of idempotents $(I_{L,A}(\boldsymbol{x}), I_{R,A}(\boldsymbol{x}))$. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ both represent $\mathfrak{s}$, there is some $B \in \pi_2(\boldsymbol{x},\boldsymbol{y})$. Moreover, $\mathrm{gr}'(\boldsymbol{y}) = \mathrm{gr}'(\boldsymbol{x}) \cdot \widetilde{R}(g'(B))$, which ensures that $g = h \cdot g'(B)$ for some $h \in P'_{\boldsymbol{x}}$ (since $\mathrm{gr}'(\boldsymbol{x})$ and $\mathrm{gr}'(\boldsymbol{y})$ are cosets of $P'_{\boldsymbol{x}}$). Since $h \in G_{AA}(\partial\mathcal{H})$ and $g'(B)$ is compatible with $I(\boldsymbol{x})$ and $I(\boldsymbol{y})$, it follows readily that g is also compatible with the idempotents $I_{AA}(\boldsymbol{x})$ and $I_{AA}(\boldsymbol{y})$, or equivalently that $R(g_L)$ is compatible with $I_{L,D}(\boldsymbol{x})$ and $I_{L,D}(\boldsymbol{y})$ and g_R is compatible with $I_{R,A}(\boldsymbol{x})$ and $I_{R,A}(\boldsymbol{y})$. $\square$

Lemma 6.11 ensures the type DA bimodule $\widehat{CFDA}(\mathcal{H},\mathfrak{s})$ is refinable, in the sense of Definition 3.10, provided our Heegaard diagram has a generator which represents $\mathfrak{s}$ (except now we are using left-right $(G(-\mathcal{Z}_L), G(\mathcal{Z}_R))$–sets, rather than just right G–sets). Thus, the analogue of Lemma 3.12 (adapted to bimodules) applies, allowing us to think of $\widehat{CFDA}(\mathcal{H},\mathfrak{s})$ as a left-right $(G(-\mathcal{Z}_L), G(\mathcal{Z}_R))$–graded type DA structure.

More concretely, fix a reference point $\boldsymbol{x}_0 \in \mathfrak{S}(\mathcal{H},\mathfrak{s})$, and fix refinement data $\psi_{L,A}$ and $\psi_{R,A}$ for $\mathcal{A}(\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$, respectively. Let $\psi_{L,D}$ be the reverse of $\psi_{L,A}$ (see Definition 3.16), which is grading refinement data for $\mathcal{A}(-\mathcal{Z}_L)$ by Lemma 3.17. Define $\psi_{AA}(\boldsymbol{x}) \in G'_{AA}(\partial\mathcal{H})$ and $\psi_{DA}(\boldsymbol{x}) \in G'_{DA}(\partial\mathcal{H})$ by

$$\psi_{AA}(\boldsymbol{x}) = (\psi_{L,A}(I_{L,A}(\boldsymbol{x})), \psi_{R,A}(I_{R,A}(\boldsymbol{x}))),$$
$$\psi_{DA}(\boldsymbol{x}) = (\psi_{L,D}(I_{L,D}(\boldsymbol{x}))^{-1}, \psi_{R,A}(I_{R,A}(\boldsymbol{x}))) = \widetilde{R}(\psi_{AA}(\boldsymbol{x})).$$

For $B \in \pi_2(\boldsymbol{x},\boldsymbol{y})$, let $g(B) = \psi_{AA}(\boldsymbol{x}) \cdot g'(B) \cdot \psi_{AA}(\boldsymbol{y})$; by Lemma 6.11, $g(B) \in G_{AA}(\partial\mathcal{H})$. Let $P_{\boldsymbol{x}} = g(\pi_2(\boldsymbol{x},\boldsymbol{x}))$. Let $S_{DA}(\mathcal{H},\boldsymbol{x}_0)$ be the quotient $\widetilde{R}(P_{\boldsymbol{x}_0})\backslash G_{DA}(\partial\mathcal{H})$ as a right $G_{DA}(\partial\mathcal{H})$–set. For any $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H},\mathfrak{s})$, define

$$\begin{aligned} \mathrm{gr}_{\boldsymbol{x}_0}(\boldsymbol{x}) = \mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{x}) \cdot \widetilde{R}(\psi_{AA}(\boldsymbol{x})^{-1}) &= \mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{x}) \cdot \psi_{DA}(\boldsymbol{x})^{-1} \\ &= \psi_{L,D}(\boldsymbol{x}) \cdot \mathrm{gr}'_{\boldsymbol{x}_0}(\boldsymbol{x}) \cdot \psi_{R,A}(\boldsymbol{x})^{-1} \end{aligned} \tag{6.12}$$

as an element of $S_{DA}(\mathcal{H},\boldsymbol{x}_0)$. (Compare Equation (3.13).)

For a different choice of initial point $\boldsymbol{x}_1$ representing $\mathfrak{s}$, we define a canonical identification of grading sets as in the proof of Proposition 6.10:

$$\Phi^{\boldsymbol{x}_1}_{\boldsymbol{x}_0}(\widetilde{R}(P_{\boldsymbol{x}_1}) \cdot h) = \widetilde{R}(P_{\boldsymbol{x}_0}) \cdot \widetilde{R}(g(C)) \cdot h, \tag{6.13}$$

where $C \in \pi_2(\boldsymbol{x}_0,\boldsymbol{x}_1)$.

Proposition 6.14 *Fix a* spin^c *structure $\mathfrak{s}$ on some strongly bordered 3–manifold ${}_{\mathcal{Z}_L}Y_{\mathcal{Z}_R}$, and let $\mathcal{H}$ and $\mathcal{H}'$ be Heegaard diagrams for Y in each of which there is at* *least one generator representing $\mathfrak{s}$. Fix refinement data ψ_L and ψ_R for $\mathcal{A}(-\mathcal{Z}_L)$ and $\mathcal{A}(\mathcal{Z}_R)$ respectively. Then the induced bimodules*

$$^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H},\mathfrak{s})_{\mathcal{A}(\mathcal{Z}_R)}, \quad ^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H}',\mathfrak{s})_{\mathcal{A}(\mathcal{Z}_R)}$$

are $\mathcal{A}_\infty$–homotopy equivalent as left-right $(G(-\mathcal{Z}_L), G(\mathcal{Z}_R))$–graded bimodules.

Proof Invariance in the ungraded sense was verified in Proposition 6.10. We will first produce a map of $G'_{DA}(\partial\mathcal{H})$–sets $S'_{DA}(\mathcal{H},\mathfrak{s}) \to S'_{DA}(\mathcal{H}',\mathfrak{s})$ compatible with the continuation isomorphism of Proposition 6.10. The isomorphism of Proposition 6.10 is a sequence of elementary isomorphisms corresponding to stabilizations, handleslides, and isotopies and changes of almost complex structure, and we will treat these separately. Note that, by Proposition 6.10, in each case we are free to choose base generators for $\mathcal{H}$ and $\mathcal{H}'$ that are convenient.

Suppose $\mathcal{H}'$ is obtained from $\mathcal{H}$ by stabilizing in the region containing the basepoint z. Then there is an obvious identification of generators in $\mathfrak{S}(\mathcal{H},\mathfrak{s})$ and $\mathfrak{S}(\mathcal{H}',\mathfrak{s})$. Fix a base generator $\boldsymbol{x}_0 \in \mathfrak{S}(\mathcal{H},\mathfrak{s})$ and let $\boldsymbol{x}'_0$ denote the corresponding generator of $\mathfrak{S}(\mathcal{H}',\mathfrak{s})$. Then the obvious identification between $P'_{\boldsymbol{x}_0}$ and $P'_{\boldsymbol{x}'_0}$ induces an identification of grading sets, which commutes with the stabilization isomorphism on $\widehat{CFDA}$.

Next, suppose $\mathcal{H}'$ is obtained from $\mathcal{H}$ by a handleslide. There are several cases: a handleslide among the β–circles, among the α–circles, or of an α–arc over an α–circle. The case of sliding an arc over a circle is the most complicated, so we will restrict to that one; for definiteness, suppose that $\alpha_1^{a,L}$ is slid over α_1^c. As in [21, Section 6.3.2], arrange that each α–arc of $\mathcal{H}'$ is close to the corresponding α–arc of $\mathcal{H}$ and intersects it in a single point, denoted θ_i^L or θ_i^R, and that for each θ_i^L (respectively θ_i^R) there is a bigon in $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\alpha}')$ originating at θ_i^L (respectively θ_i^R) (ie the θ_i^L would correspond to the top graded generator in a closed diagram). Let $f_{\mathcal{H},\mathcal{H}'}$ denote the triangle map giving the isomorphism of Proposition 6.10.

Then, fix a base generator $\boldsymbol{x}_0 \in \mathfrak{S}(\mathcal{H},\mathfrak{s})$. There is a corresponding generator $\boldsymbol{x}'_0 \in \mathfrak{S}(\mathcal{H}',\mathfrak{s})$ so that there is a provincial domain in $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\alpha}',\boldsymbol{\beta})$ connecting $\boldsymbol{x}_0$, $\boldsymbol{x}'_0$ and a generator Θ_0 composed of θ_i^L and θ_i^R; and so that this domain consists of a disjoint union of triangles supported in the isotopy region and possibly an annulus with boundary on $\alpha_1^{a,L}$, $\alpha_1^{a,L,\prime}$ and α_1^c. Then there is an obvious identification between $P'_{\boldsymbol{x}_0}$ and $P'_{\boldsymbol{x}'_0}$, which gives an identification of grading sets $S'_{DA}(\mathcal{H},\mathfrak{s}) \cong S'_{DA}(\mathcal{H}',\mathfrak{s})$.

To see that this identification is compatible with the isomorphism of Proposition 6.10, recall that to each domain B' in $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\alpha}',\boldsymbol{\beta})$ counted in the triangle map there is an associated domain B in $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$. Moreover, if $(B',\vec{\boldsymbol{\rho}})$ contributes to $f_{\mathcal{H},\mathcal{H}'}$ then $\operatorname{ind}(B,\vec{\boldsymbol{\rho}}) = 0$. It follows that from this and the fact that the map $f_{\mathcal{H},\mathcal{H}'}$ treats the $\vec{\boldsymbol{\rho}}$

in the same way that δ^1 does that the map $f_{\mathcal{H},\mathcal{H}'}$ preserves the relative $G'_{DA}(\partial\mathcal{H})$ gradings.

The case that $\mathcal{H}'$ differs from $\mathcal{H}$ by an isotopy or a change of complex structure is similar to, but easier than, the case of a handleslide, so we leave it to the reader.

Finally, we turn to the $G_{DA}(\partial\mathcal{H})$–set gradings. In view of Lemmas 3.12 and 6.11, the G'–set gradings can be lifted to G–set gradings. Moreover, since we choose the same refinement data for the two Heegaard diagrams, compatibility of the map of G'–sets with the isomorphism of Proposition 6.10 implies compatibility of the map of G–sets with the isomorphism of Proposition 6.10. □

The above proposition allows us to use Heegaard-diagram-free notation for $\widehat{CFDA}$: we write ${}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(Y,\mathfrak{s})_{\mathcal{A}(\mathcal{Z}_R)}$ for ${}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H},\mathfrak{s})_{\mathcal{A}(\mathcal{Z}_R)}$, where $\mathcal{H}$ is any Heegaard diagram which represents Y. We also let $S_{DA}(Y,\mathfrak{s})$ (or just $S(Y,\mathfrak{s})$) denote the corresponding grading set.

Moreover, we can let

$$S(Y) = \bigcup_{\mathfrak{s}\in\mathrm{spin}^c(Y)} S(Y,\mathfrak{s}), \tag{6.15}$$

and define

$${}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}_R)} = \bigoplus_{\mathfrak{s}\in\mathrm{spin}^c(Y)} {}^{\mathcal{A}(-\mathcal{Z}_L)}\widehat{CFDA}(Y,\mathfrak{s})_{\mathcal{A}(\mathcal{Z}_R)},$$

and think of it as graded by $S(Y)$.

Remark 6.16 Instead of using induction and restriction, the gradings on the bimodules $\widehat{CFDD}(Y)$ and $\widehat{CFAA}(Y)$ can be treated similarly to the discussion above. The invariant $\widehat{CFAA}(\mathcal{H})$ is graded by the right $G'_{AA}(\partial\mathcal{H})$–set $S'_{AA}(\partial\mathcal{H}) = P'_{\boldsymbol{x}}\backslash G'_{AA}(\partial\mathcal{H})$. The refined grading on $\widehat{CFAA}(\mathcal{H})$ is by the right $G_{AA} = G(\mathcal{Z}_L)\times_{\mathbb{Z}} G(\mathcal{Z}_R)$–set $S_{AA}(\partial\mathcal{H}) = P_{\boldsymbol{x}}\backslash G_{AA}$. Tracing through the definitions shows that this grading set (and the corresponding grading) is the same as that given by the restriction functor.

Similarly, the invariant $\widehat{CFDD}(\mathcal{H})$ is graded by the left $G'_{DD}(\partial\mathcal{H}) = G'(-\mathcal{Z}_L)\times_{\mathbb{Z}} G'(-\mathcal{Z}_R)$–set $S'_{DD}(\partial\mathcal{H}) = G'_{AA}(\partial\mathcal{H})/RR(P'_{\boldsymbol{x}})$ (where RR denotes the map $G'(\mathcal{Z}_L)^{\mathrm{op}}\times_{\mathbb{Z}} G'(\mathcal{Z}_R)^{\mathrm{op}} \to G'(-\mathcal{Z}_L)\times_{\mathbb{Z}} G'(-\mathcal{Z}_R)$ gotten by applying R to each factor). The refined grading on $\widehat{CFDD}(\mathcal{H})$ is by the left $G_{DD}(\partial\mathcal{H}) = G(-\mathcal{Z}_L)\times_{\mathbb{Z}} G(-\mathcal{Z}_R)$–set $S_{DD}(\partial\mathcal{H}) = G_{AA}(\partial\mathcal{H})/RR(P_{\boldsymbol{x}})$. Again, it follows from the definitions that this agrees with the grading given by the induction functor.

6.6 Invariance

We collect the results from this section into the following:

Theorem 10 *Let Y_{12} be a three-manifold with boundary, strongly bordered by $\mathcal{Z}_1$ and $\mathcal{Z}_2$, and fix grading refinement data for $\mathcal{A}(\mathcal{Z}_1)$ and $\mathcal{A}(\mathcal{Z}_2)$. Then we can associate the following G–set graded bimodules to Y_{12}:*

$$\widehat{CFAA}(Y_{12})_{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_2)}, \quad {}_{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDA}(Y_{12})_{\mathcal{A}(\mathcal{Z}_2)}, \quad {}_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(Y_{12}).$$

The quasi-isomorphism types of these G–set graded bimodules are diffeomorphism invariants of the bordered three-manifold Y_{12} with its strong boundary framing.

Proof Without the gradings, this is immediate from Propositions 6.2, 6.2 and 6.10. The fact that the isomorphisms respect the grading on $\widehat{CFDA}$ is proved in Proposition 6.14; the proofs that the isomorphisms respect the gradings on $\widehat{CFAA}$ and $\widehat{CFDD}$ are analogous. □

7 Pairing theorems

Theorems 2, 5 and 7 can all be seen as pairing theorems, which express how the bordered Floer homology groups transform as bordered three manifolds are glued in the three situations discussed in Section 5.2. The aim of the present section is to study how the bordered invariants change under these three gluing operations, to obtain proofs of the aforementioned three theorems. (Indeed, we obtain three generalizations, Theorems 11, 12 and 14 below.)

7.1 Pairing along a connected surface

Here is the promised generalization of Theorem 2:

Theorem 11 *Let Y_{12} be a strongly bordered three-manifold with boundary parameterized by $-\mathcal{Z}_1$ and $\mathcal{Z}_2$. Let Y_1 be a three-manifold with boundary parameterized by $\mathcal{Z}_1$. Then there are $\mathcal{A}_\infty$–homotopy equivalences:*

$$\begin{aligned}
\widehat{CFA}(Y_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} \widehat{CFDA}(Y_{12}) &\simeq \widehat{CFA}(Y_1 \cup_{F_1} Y_{12}),\\
\widehat{CFAA}(Y_{12}) \boxtimes_{\mathcal{A}(-\mathcal{Z}_1)} \widehat{CFD}(Y_1) &\simeq \widehat{CFA}(Y_1 \cup_{F_1} Y_{12}),\\
\widehat{CFA}(Y_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_1)} \widehat{CFDD}(Y_{12}) &\simeq \widehat{CFD}(Y_1 \cup_{F_1} Y_{12}),\\
\widehat{CFDA}(Y_{12}) \boxtimes_{\mathcal{A}(-\mathcal{Z}_1)} \widehat{CFD}(Y_1) &\simeq \widehat{CFD}(Y_1 \cup_{F_1} Y_{12}).
\end{aligned}$$

The first two are equivalences of type A structures over $\mathcal{A}(\mathcal{Z}_2)$, while the second two are equivalences of type D structures over $\mathcal{A}(\mathcal{Z}_1)$.

(Here and later, the $\mathcal{A}_\infty$–(bi)modules like $\widehat{CFDA}(Y_{12})$ are required to be appropriately bounded for the tensor product to exist. This is always possible, as we just choose the Heegaard diagram to be the appropriate variant of admissible.)

In a similar spirit, we have the following generalization of Theorem 5:

Theorem 12 *Let Y_{12} be a strongly bordered three-manifold with boundary parameterized by $-\mathcal{Z}_1$ and $\mathcal{Z}_2$. Let Y_{23} be a strongly bordered three-manifold with boundary parameterized by $-\mathcal{Z}_2$ and $\mathcal{Z}_3$. Then there are $\mathcal{A}_\infty$–quasi-isomorphisms:*

$$\widehat{CFDA}(Y_{12}) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(Y_{23}) \simeq {}^{\mathcal{A}(\mathcal{Z}_1)}\widehat{CFDA}(Y_{12} \cup_{F_2} Y_{23})_{\mathcal{A}(\mathcal{Z}_3)},$$
$$\widehat{CFAA}(Y_{12}) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(Y_{23}) \simeq \widehat{CFAA}(Y_{12} \cup_{F_2} Y_{23})_{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(\mathcal{Z}_3)},$$
$$\widehat{CFDA}(Y_{12}) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDD}(Y_{23}) \simeq {}^{\mathcal{A}(\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_3)}\widehat{CFDD}(Y_{12} \cup_{F_2} Y_{23}),$$
$$\widehat{CFAA}(Y_{12}) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDD}(Y_{23}) \simeq {}^{\mathcal{A}(-\mathcal{Z}_3)}\widehat{CFDA}(Y_{12} \cup_{F_2} Y_{23})_{\mathcal{A}(-\mathcal{Z}_1)}.$$

Proof of Theorems 11 and 12 Both of the proofs of the pairing theorem in [21] extend easily to these cases. To belabor the point, we will prove in detail the first equivalences of Theorem 11 via nice diagrams; the proofs of the other parts of the theorems proceed similarly.

So, let $\mathcal{H}_1$ be a nice diagram for Y_1; existence of such is guaranteed by [21, Proposition 8.2]. Let $\mathcal{H}'$ be a Heegaard diagram for Y_{12}. Apply the algorithm from [21, Proposition 8.2] to $\mathcal{H}'_{\mathrm{dr}}$ and then fill in the tunnel; the result is a Heegaard diagram $\mathcal{H}_{12}$ for Y_{12} so that $\mathcal{H}_{\mathrm{dr}12}$ is nice. (We will simply call $\mathcal{H}_{12}$ nice in this case.)

We turn to the first isomorphism of Theorem 11. Note that the fact that $\mathcal{H}_1$ and $\mathcal{H}_{12}$ are nice implies (by [21, Lemma 8.3]) that they are admissible; Proposition 2.3.10 and Lemma 6.11 imply that the box product $\widehat{CFA}(\mathcal{H}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(\mathcal{H}_{12})$ is well defined. On the other side, Lemma 5.7 implies that the glued diagram $\mathcal{H} = \mathcal{H}_1 \,{}_{\partial}\cup_{\partial_L} \mathcal{H}_{12}$ is admissible. (In fact, $\mathcal{H}$ is nice, and hence admissible by [21, Lemma 8.3].)

We claim that $\widehat{CFA}(\mathcal{H}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(\mathcal{H}_{12})$ is exactly equal to $\widehat{CFA}(\mathcal{H})$. As a first step, note that there is an obvious correspondence between generators. (In particular, only $\widehat{CFDA}(\mathcal{H}_{12}, 0) \subset \widehat{CFDA}(\mathcal{H}_{12})$ contributes to the tensor product.) So, we need to check that this identification respects the differentials and right module structures.

On the one hand, the diagram $\mathcal{H}$ is obviously nice. So, by [21, Proposition 8.4], the differential on $\widehat{CFA}(\mathcal{H})$ counts (provincial) rectangles and bigons, while the only nontrivial (right) algebra actions correspond to unions of half-strips through $\partial_R\mathcal{H}$. Note that these half strips are entirely contained in $\mathcal{H}_{12} \subset \mathcal{H}$.

On the other hand, in $\widehat{CFA}(\mathcal{H}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(\mathcal{H}_{12})$, the differential comes from three different contributions:

- Provincial curves in $\mathcal{H}_1$, which correspond to bigons and rectangles by [21, Proposition 8.4].
- Provincial curves in $\mathcal{H}_{12}$, which again correspond to bigons and rectangles by [21, Proposition 8.4].
- Contributions of the form
$$\boldsymbol{x}_1 \otimes \boldsymbol{x}_{12} \xrightarrow{\mathbb{I}\otimes\delta^1} \boldsymbol{x}_1 \otimes (\rho \otimes \boldsymbol{y}_{12}) = (\boldsymbol{x}_1 \otimes \rho) \otimes \boldsymbol{y}_{12} \xrightarrow{m_2\otimes\mathbb{I}} \boldsymbol{y}_1 \otimes \boldsymbol{y}_2.$$

The third kind of contributions correspond exactly to rectangles crossing $\partial\mathcal{H}_1 = \partial_L\mathcal{H}_{12}$. The other two correspond to bigons and rectangles in $\mathcal{H}$ contained entirely in one of $\mathcal{H}_1$ or $\mathcal{H}_{12}$. Consequently, the differentials on $\widehat{CFA}(\mathcal{H}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(\mathcal{H}_{12})$ and $\widehat{CFA}(\mathcal{H})$ agree.

Since $\mathcal{H}_{12}$ is nice, the right module structure comes entirely from juxtapositions of half-strips crossing $\partial_R\mathcal{H}_{12}$. These are exactly the same curves which define the module structure on $\widehat{CFA}(\mathcal{H})$, and they contribute in the same way.

Thus, we have an isomorphism of right differential modules
$$\widehat{CFA}(\mathcal{H}_1) \boxtimes_{\mathcal{A}(\mathcal{Z}_2)} \widehat{CFDA}(\mathcal{H}_{12}) \cong \widehat{CFA}(\mathcal{H}). \qquad \square$$

Proof of Theorems 2 and 5 Theorem 2 (respectively Theorem 5) is an immediate consequence of Theorem 11 (respectively Theorem 12), together with the definition of the bimodule of a surface diffeomorphism, and the interpretation of the derived tensor product in terms of $\boxtimes$, Propositions 2.3.18 and 2.4.2. (A schematic, illustrating one way to keep the compositions straight, is given in Figure 17.) $\square$

Similarly, we have the following:

Proof of Corollary 1.1 This is a special case of Theorem 2, using the identity map for ψ. $\square$

7.1.1 Gradings We discuss now how the pairing theorem intertwines the gradings on the two sides. For definiteness, we will consider the $\boxtimes$ product of two type *DA* modules; the other cases are similar.

Fix strongly bordered 3–manifolds Y_1 and Y_2 with two boundary components, where Y_i is parameterized by $\mathcal{Z}_L(Y_i)$ and $\mathcal{Z}_R(Y_i)$, with $\mathcal{Z}_R(Y_1) = -\mathcal{Z}_L(Y_2)$, so that we can form the manifold $Y = Y_1\,{}_{\partial_R Y_1}\!\cup_{\partial_L Y_2} Y_2$. For brevity, let $\mathcal{Z}_{\mathrm{mid}}$ be $\mathcal{Z}_R(Y_1) = -\mathcal{Z}_L(Y_2)$. As in (6.15), we let $S_{DA}(Y_i)$ and $S_{DA}(Y)$ denote the various grading sets for the bordered Floer homology bimodules. Since $\mathcal{Z}_L Y = \mathcal{Z}_L Y_1$ and $\mathcal{Z}_R Y = \mathcal{Z}_R Y_2$, $S_{DA}(Y_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(Y_2)$ is naturally a left-right $(G(-\mathcal{Z}_L(Y)), G(\mathcal{Z}_R(Y)))$–set.

As before, we will write, for instance, $G_{DA}(\partial Y)$ for $G(-\mathcal{Z}_L(Y))^{\mathrm{op}} \times_\lambda G(\mathcal{Z}_R(Y))$.

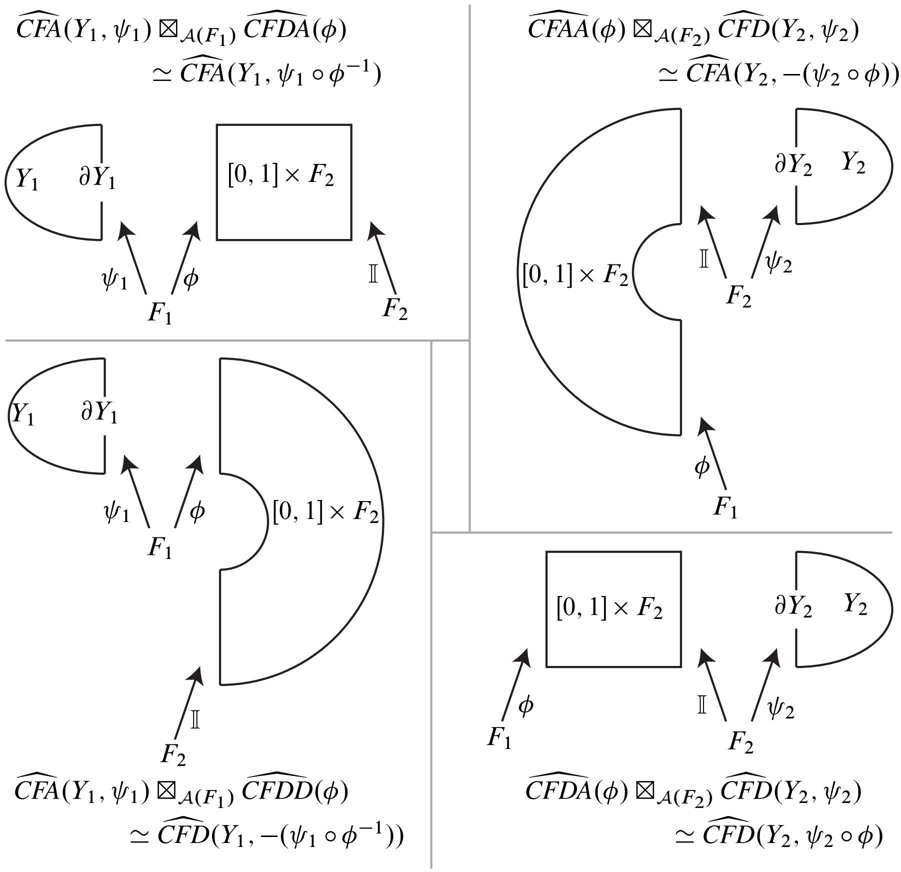


Figure 17: Schematic illustration of Theorem 2: the four cases are shown in order left-to-right, top-to-bottom. To compute the parametrization of the boundary after gluing, start at the unglued boundary component and follow the arrows until you reach ∂Y_i, composing the maps labeling the arrows (or their inverses).

Theorem 13 *If $\mathcal{Z}_R(Y_1) = -\mathcal{Z}_L(Y_2) = \mathcal{Z}_{\mathrm{mid}}$ and $Y = Y_1\ {}_{\partial_L Y_1}\!\cup_{\partial_R Y_2} Y_2$ as above, there is an identification of $G_{DA}(\partial Y)$–sets*

$$S_{DA}(Y_1)\times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(Y_2) \cong S_{DA}(Y)$$

so that the isomorphism in Theorem 11 is a $G_{DA}(\partial Y)$–set graded isomorphism.

In fact, we can refine this statement slightly: there is a natural identification between spin^c–structures on Y and $G(\partial Y)$–orbits in $S_{DA}(Y_1)\times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(Y_2) \cong S_{DA}(Y)$, which is refined by the identification in Theorem 13; the identification is given in the proof.

Proof of Theorem 13 The identification of $G_{DA}(\partial Y)$–sets

$$S_{DA}(Y_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(Y_2) \cong S_{DA}(Y)$$

is given one spin^c structure (over Y) at a time. More precisely, fix $\mathfrak{s} \in \mathrm{spin}^c(Y)$, and let $\mathfrak{s}_i$ denote its restriction to Y_i. We will exhibit an identification

$$S_{DA}(Y_1,\mathfrak{s}_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(Y_2,\mathfrak{s}_2) \cong \bigcup_{h \in H_1(\partial_R Y_1;\mathbb{Z})} S_{DA}(Y,\mathfrak{s}+\mathrm{PD}[h]).$$

We work with a Heegaard diagram $\mathcal{H} = \mathcal{H}_1 {}_{\mathcal{Z}_R}\cup_{\mathcal{Z}_L} \mathcal{H}_2$ for Y which has a generator $\boldsymbol{x}$ representing $\mathfrak{s}$, so that the restrictions $\boldsymbol{x}_i$ of $\boldsymbol{x}$ to $\mathcal{H}_i$ represent $\mathfrak{s}_i$. Thus, our goal is to construct a map

$$S_{DA}(\mathcal{H}_1,\boldsymbol{x}_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(\mathcal{H}_2,\boldsymbol{x}_2) \cong \bigcup_{h \in H_1(\partial_R Y_1;\mathbb{Z})} S_{DA}(Y,\mathfrak{s}_z(\boldsymbol{x})+\mathrm{PD}[h]).$$

First, however, we construct the identification on the level of orbit spaces, ie spin^c structures. And indeed, before this, we construct a map

$$p_{\boldsymbol{x}}\colon S_{DA}(\mathcal{H}_1,\boldsymbol{x}_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(\mathcal{H}_2,\boldsymbol{x}_2) \to H_1(Y,\partial Y),$$

as follows. Recall from Section 6.5 that $S_{DA}(\mathcal{H}_1,\boldsymbol{x}_1)$ and $S_{DA}(\mathcal{H}_2,\boldsymbol{x}_2)$ are the coset spaces $\widetilde{R}(P_{\boldsymbol{x}_1})\backslash G_{DA}(\partial\mathcal{H}_1)$ and $\widetilde{R}(P_{\boldsymbol{x}_2})\backslash G_{DA}(\partial\mathcal{H}_2)$ respectively. For brevity, we will write $\widetilde{P}_{\boldsymbol{x}}$ for $\widetilde{R}(P_{\boldsymbol{x}})$. Thus we can write elements of $S_{DA}(\mathcal{H}_1,\boldsymbol{x}_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(\mathcal{H}_2,\boldsymbol{x}_2)$ as $(\widetilde{P}_{\boldsymbol{x}_1}\cdot(n_1,\alpha_1,\beta_1)) \times (\widetilde{P}_{\boldsymbol{x}_2}\cdot(n_2,\alpha_2,\beta_2))$, where we have $n_i \in \mathbb{Z}$, $\alpha_i \in H_1(F(-\mathcal{Z}_L(Y_i)))$ and $\beta_i \in H_1(F(\mathcal{Z}_R(Y_i)))$. We then define

$$p_{\boldsymbol{x}}((\widetilde{P}_{\boldsymbol{x}_1}\cdot(n_1,\alpha_1,\beta_1)) \times (\widetilde{P}_{\boldsymbol{x}_2}\cdot(n_2,\alpha_2,\beta_2))) = i_*(\beta_1+\alpha_2),$$

where here i_* is the inclusion map from $H_1(\partial_R Y_1) = H_1(-\partial_L Y_2)$ to $H_1(Y;\mathbb{Z})$. With this definition, elements

$$s,t \in S_{DA}(\mathcal{H}_1,\boldsymbol{x}_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(\mathcal{H}_2,\boldsymbol{x}_2)$$

lie in the same $G_{DA}(\partial Y)$–orbit if and only if $p_{\boldsymbol{x}}(s) = p_{\boldsymbol{x}}(t)$.

The map $p_{\boldsymbol{x}}$ depends on the base generator $\boldsymbol{x}$. However, the map

$$q = q_{\boldsymbol{x}}\colon S_{DA}(\mathcal{H}_1,\boldsymbol{x}_1) \times_{G(\mathcal{Z}_{\mathrm{mid}})} S_{DA}(\mathcal{H}_2,\boldsymbol{x}_2) \to \mathrm{spin}^c(Y),$$
$$q_{\boldsymbol{x}}(s_1 \times s_2) = \mathfrak{s}(\boldsymbol{x}) + p_{\boldsymbol{x}}(s_1 \times s_2)$$

is independent of $\boldsymbol{x}$, in the following sense. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two choices of base generators, then $q_{\boldsymbol{x}} \circ \Phi_{\boldsymbol{x}}^{\boldsymbol{y}} = q_{\boldsymbol{y}}$, where here

$$\Phi_{\boldsymbol{x}}^{\boldsymbol{y}} \colon S_{DA}(\mathcal{H}_1, \boldsymbol{x}_1) \times_{G(\mathcal{Z}_{\text{mid}})} S_{DA}(\mathcal{H}_1, \boldsymbol{x}_2) \to S_{DA}(\mathcal{H}_1, \boldsymbol{y}_1) \times_{G(\mathcal{Z}_{\text{mid}})} S_{DA}(\mathcal{H}_1, \boldsymbol{y}_2),$$

given by $\Phi_{\boldsymbol{x}}^{\boldsymbol{y}} = (\Phi_{\boldsymbol{x}_1}^{\boldsymbol{y}_1} \times \Phi_{\boldsymbol{x}_2}^{\boldsymbol{y}_2})$, is the map gotten by putting together the two canonical identifications of grading sets (see (6.13); see also Proposition 6.10). More explicitly, if there are $C_i \in \pi_2(\boldsymbol{x}_i, \boldsymbol{y}_i)$, then

$$\Phi_{\boldsymbol{x}}^{\boldsymbol{y}}(\widetilde{P}_{\boldsymbol{y}_1} \cdot g_1 \times \widetilde{P}_{\boldsymbol{y}_2} \cdot g_2) = (\widetilde{P}_{\boldsymbol{x}_1} \cdot \widetilde{R}(g(C_1)) \cdot g_1 \times \widetilde{P}_{\boldsymbol{x}_2} \cdot \widetilde{R}(g(C_2)) \cdot g_2).$$

Now, $i_*(\partial_R^\partial[C_1] + r_*(\partial_L^\partial[C_2])) = \epsilon(\boldsymbol{x}, \boldsymbol{y})$, where $\epsilon(\boldsymbol{x}, \boldsymbol{y})$ is the map giving the difference in spin^c structures between $\boldsymbol{x}$ and $\boldsymbol{y}$ as in the proof of Lemma 5.9. Thus,

$$\begin{aligned}
q_{\boldsymbol{x}} \circ \Phi_{\boldsymbol{x}}^{\boldsymbol{y}}(\widetilde{P}_{\boldsymbol{x}_1} \cdot (n_1, \alpha_1, \beta_1) \times \widetilde{P}_{\boldsymbol{x}_2} \cdot (n_2, \alpha_2, \beta_2)) \\
&= q_{\boldsymbol{x}}(\widetilde{P}_{\boldsymbol{x}_1} \cdot \widetilde{R}(g(C_1)) \cdot (n_1, \alpha_1, \beta_1) \times \widetilde{P}_{\boldsymbol{x}_2} \cdot \widetilde{R}(g(C_2)) \cdot (n_2, \alpha_2, \beta_2)) \\
&= \mathfrak{s}(\boldsymbol{x}) + i_*(\beta_1 + \partial_R^\partial[C_1] + r_*(\partial_L^\partial[C_2]) + \alpha_2) \\
&= \mathfrak{s}(\boldsymbol{x}) + \epsilon(\boldsymbol{x}, \boldsymbol{y}) + i_*(\beta_1 + \alpha_2) \\
&= \mathfrak{s}(\boldsymbol{y}) + i_*(\beta_1 + \alpha_2) \\
&= q_{\boldsymbol{y}}(\widetilde{P}_{\boldsymbol{y}_1} \cdot (n_1, \alpha_1, \beta_1) \times \widetilde{P}_{\boldsymbol{y}_2} \cdot (n_2, \alpha_2, \beta_2)),
\end{aligned}$$

as claimed.

Thus, if we write $\mathfrak{s}_i = \mathfrak{s}|_{Y_i}$, the map q defines an identification of $G_{DA}(\partial Y)$–orbits in

$$S_{DA}(Y_1, \mathfrak{s}_1) \times_{G(\mathcal{Z}_{\text{mid}})} S_{DA}(Y_2, \mathfrak{s}_2)$$

and those spin^c structures on Y which are of the form $\mathfrak{s} + \mathrm{PD}[i_*(h)]$ for some $h \in H_1(\partial_R Y_1)$.

We refine this to a map of grading sets, as follows. Given orbits O_1 and O_2 of $G_{DA}(\partial Y_1)$ and $G_{DA}(\partial Y_2)$ respectively, fix a $G_{DA}(\partial Y)$–orbit O_{12} in $O_1 \times O_2$, and suppose that there is a generator $\boldsymbol{x}$ for $\mathcal{H}$ which represents the corresponding spin^c structure. Without loss of generality, we can think of $S_{DA}(Y, \mathfrak{s})$ as the orbit of $\mathrm{gr}(\boldsymbol{x})$, and its components $\boldsymbol{x}_i$ as determining the grading sets $S_{DA}(\mathcal{H}_i, \boldsymbol{x}_i)$, so that O_{12} is contained in the orbit of $\mathrm{gr}(\boldsymbol{x}_1) \times \mathrm{gr}(\boldsymbol{x}_2)$. Then define a map $O_{12} \to S_{DA}(Y, \mathfrak{s})$ as follows. Any element of O_{12} can be represented as

$$\widetilde{P}_{\boldsymbol{x}_1} \cdot (n_1, \alpha_1, \beta_1) \times \widetilde{P}_{\boldsymbol{x}_2} \cdot (n_2, \alpha_2, \beta_2),$$

where $\beta_1 + \alpha_2 = 0$. Then define a map $\phi \colon O_{12} \to S_{DA}(Y, \mathfrak{s})$ by

$$\phi(\widetilde{P}_{\boldsymbol{x}_1} \cdot (n_1, \alpha_1, \beta_1) \times \widetilde{P}_{\boldsymbol{x}_2} \cdot (n_2, \alpha_2, \beta_2)) = \widetilde{P}_{\boldsymbol{x}} \cdot (n_1 + n_2, \alpha_1, \beta_2).$$

It is clear that this defines a map of $G_{DA}(\partial\mathcal{H}) = G(-\mathcal{Z}_L)^{\mathrm{op}} \times_\lambda G(\mathcal{Z}_R)$ sets.

To verify this map respects the gradings on $\widehat{CFDA}(\mathcal{H}_1) \boxtimes \widehat{CFDA}(\mathcal{H}_2)$ and $\widehat{CFDA}(\mathcal{H})$, suppose $\boldsymbol{y} = \boldsymbol{y}_1 \otimes \boldsymbol{y}_2$ is another generator of $\widehat{CFDA}(\mathcal{H})$ and $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$. We can decompose B as $B_1 \times B_2$ where $B_i \in \pi_2(\boldsymbol{x}_i, \boldsymbol{y}_i)$ and $\partial_R^\partial B_1 = \partial_L^\partial B_2$. Then

$$\begin{aligned}
\mathrm{gr}_{\boldsymbol{x}}(\boldsymbol{y}) &= \widetilde{P}_{\boldsymbol{x}} \cdot (-e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B), r_*(\partial_L^\partial B_1), \partial_R^\partial B_2) \cdot \psi_{DA}(\boldsymbol{y})^{-1}, \\
\mathrm{gr}_{\boldsymbol{x}_1}(\boldsymbol{y}_1) &= \widetilde{P}_{\boldsymbol{x}_1} \cdot (-e(B_1) - n_{\boldsymbol{x}}(B_1) - n_{\boldsymbol{y}}(B_1), r_*(\partial_L^\partial B_1), \partial_R^\partial B_1) \cdot \psi_{DA}(\boldsymbol{y}_1)^{-1}, \\
\mathrm{gr}_{\boldsymbol{x}_2}(\boldsymbol{y}_2) &= \widetilde{P}_{\boldsymbol{x}_2} \cdot (-e(B_2) - n_{\boldsymbol{x}}(B_2) - n_{\boldsymbol{y}}(B_2), r_*(\partial_L^\partial B_2), \partial_R^\partial B_2) \cdot \psi_{DA}(\boldsymbol{y}_2)^{-1}.
\end{aligned}$$

(By Section 6.5.1, $\psi_{DA}(\boldsymbol{y}) = (\psi_{L,D}(I_{L,D}(\boldsymbol{y}))^{-1}, \psi_{R,A}(I_{R,A}(\boldsymbol{y})))$.) Thus

$$\begin{aligned}
&\phi(\mathrm{gr}(\boldsymbol{y}_1) \times \mathrm{gr}(\boldsymbol{y}_2)) \\
&\quad= \phi(\widetilde{P}_{\boldsymbol{x}_1} \cdot (-e(B_1) - n_{\boldsymbol{x}_1}(B_1) - n_{\boldsymbol{y}_1}(B_1), r_*(\partial_L^\partial B_1), \partial_R^\partial B_1) \cdot \psi_{DA}(\boldsymbol{y}_1)^{-1} \\
&\qquad\qquad \times \widetilde{P}_{\boldsymbol{x}_2}(-e(B_2) - n_{\boldsymbol{x}_2}(B_2) - n_{\boldsymbol{y}_2}(B_2), r_*(\partial_L^\partial B_2), \partial_R^\partial B_2) \cdot \psi_{DA}(\boldsymbol{y}_2)^{-1}) \\
&\quad= \phi(\widetilde{P}_{\boldsymbol{x}_1}(-e(B_1) - n_{\boldsymbol{x}_1}(B_1) - n_{\boldsymbol{y}_1}(B_1), r_*(\partial_L^\partial B_1), \partial_R^\partial B_1)(\psi_{L,D}(I_{L,D}(\boldsymbol{y}_1)), 0) \\
&\qquad \times \widetilde{P}_{\boldsymbol{x}_2}(-e(B_2) - n_{\boldsymbol{x}_2}(B_2) - n_{\boldsymbol{y}_2}(B_2), r_*(\partial_L^\partial B_2), \partial_R^\partial B_2)(0, \psi_{R,A}(I_{R,A}(\boldsymbol{y}_2))^{-1})) \\
&\quad= \widetilde{P}_{\boldsymbol{x}} \cdot (-e(B_1) - n_{\boldsymbol{x}_1}(B_1) - n_{\boldsymbol{y}_1}(B_1) - e(B_2) - n_{\boldsymbol{x}_2}(B_2) - n_{\boldsymbol{y}_2}(B_2), r_*(\partial_L^\partial B_1), \partial_R^\partial B_2) \\
&\qquad\qquad \cdot (\psi_{L,D}(I_{L,D}(\boldsymbol{y}_1)), \psi_{R,A}(I_{R,A}(\boldsymbol{y}_2))^{-1}) \\
&\quad= (-e(B) - n_{\boldsymbol{x}}(B) - n_{\boldsymbol{y}}(B), r_*(\partial_L^\partial B), \partial_R B) \cdot \psi_{DA}(\boldsymbol{y})^{-1} \\
&\quad= \mathrm{gr}(\boldsymbol{y}_1 \otimes \boldsymbol{y}_2),
\end{aligned}$$

as desired. □

7.2 Hochschild homology and knot Floer homology

To give a precise statement of the self-pairing theorem, we will need to discuss the relevant Alexander grading on knot Floer homology for generalized open books.

Let Y be a strongly bordered three-manifold with two boundary components specified by $-\mathcal{Z}$ and $\mathcal{Z}$, and let (Y°, K) be its associated generalized open book, as in Construction 5.3.

Recall that $F^\circ(\mathcal{Z}) = F(\mathcal{Z}) \setminus \mathbb{D}^2$. Then we can think of $F^\circ \subset Y^\circ$ as an embedded Seifert surface for K. As such, it induces an integral grading on the knot Floer homology $\widehat{CFK}(Y^\circ, K)$. Specifically, thinking of knot Floer homology as graded by relative spin^c structures $\mathrm{spin}^c(Y^\circ, K)$, the summand of $\widehat{HFK}(Y^\circ, K)$ in Alexander grading i is the sum of knot Floer homology groups over all relative spin^c structures $\mathfrak{s}$ with

$\frac{1}{2}\langle c_1(\widehat{\mathfrak{s}}), [\widehat{F}]\rangle = i$, where here $\widehat{F}$ is the surface gotten by capping off F° in the zero-surgery of $Y^\circ(F)$, and $\widehat{\mathfrak{s}}$ is the extension of the relative spin^c structure $\mathfrak{s}$ over the zero-surgery.

On the bordered side, the bimodule $\widehat{CFDA}(Y)$ splits according to the strands grading $\widehat{CFDA}(Y) = \bigoplus_{i\in\mathbb{Z}} \widehat{CFDA}(Y,i)$ (see (6.1)); and hence, so does its Hochschild homology.

The following is a generalization of Theorem 7:

Theorem 14 *Let Y be a strongly bordered three-manifold with two boundary components parameterized by $-\mathcal{Z}$ and $\mathcal{Z}$. Let (Y°, K) be the open book obtained by gluing the boundary components of Y together and performing 0–surgery on γ. Then there is an identification between the knot Floer homology of the generalized open book and the Hochschild homology of the bimodule of Y*

$$\widehat{HFK}(Y^\circ, K) \cong HH({}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z})}),$$

which identifies the Alexander grading on knot Floer homology with the strands grading on the bimodule; ie

$$\widehat{HFK}(Y^\circ, K, i) \cong HH({}^{\mathcal{A}(\mathcal{Z},i)}\widehat{CFDA}(Y,i)_{\mathcal{A}(\mathcal{Z},i)}).$$

Moreover, this isomorphism intertwines the $\mathbb{Z}$–set gradings on $HH({}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z})})$ (from Lemma 2.5.12) and on $\widehat{HFK}(Y^\circ, K)$.

(For the statement about gradings, we have chosen the same grading refinement data ψ (Definition 3.5) for the two sides of $\mathcal{H}$.)

We prove this theorem in two ways, first with nice diagrams and then with deforming the diagonal.

Proof via nice diagrams As in the proof of Theorems 11 and 12, choose a nice diagram $\mathcal{H}$ for Y. By [21, Lemma 8.3], the diagram $\mathcal{H}$ is admissible. Hence, by Lemma 6.11, the bimodule $\widehat{CFDA}(\mathcal{H})$ is bounded. Also, by Lemma 5.8, the doubly-pointed Heegaard diagram $\mathcal{H}^\circ$ is weakly admissible.

Note that, by assumption, $\mathcal{Z}_R(\mathcal{H}) \cong -\mathcal{Z}_L(\mathcal{H})$; denote $\mathcal{Z}_R(\mathcal{H})$ simply by $\mathcal{Z}$.

View $\widehat{CFDA}(\mathcal{H})$ as a type *DA* structure with structure maps

$$\delta_{n+1}\colon X(\mathcal{H}) \otimes \mathcal{A}(\mathcal{Z})^{\otimes n} \to \mathcal{A}(\mathcal{Z}) \otimes X(\mathcal{H}).$$

By Proposition 2.3.54, the Hochschild homology of $\widehat{CFDA}(\mathcal{H})$ is computed as the homology of $(X(\mathcal{H})^\circ, \widetilde{\partial})$ where $X(\mathcal{H})^\circ = X(\mathcal{H})/[\mathcal{I}(\mathcal{Z}), X(\mathcal{H})]$ is the cyclicization of $X(\mathcal{H})$ and $\widetilde{\partial}$ is as in Equation (2.3.44).

Let $\mathcal{H}^\circ$ be the doubly-pointed Heegaard diagram for (Y°, K) gotten by self-gluing $\mathcal{H}$, as in Construction 5.5. We will show the complex $(X(\mathcal{H})^\circ, \widetilde{\partial})$ is exactly $\widehat{CFK}(\mathcal{H}^\circ)$. First, as an $\mathbb{F}_2$–vector space, $X(\mathcal{H})^\circ$ is isomorphic to $\widehat{CFK}(\mathcal{H}^\circ)$: the $\mathbb{F}_2$–vector space $X(\mathcal{H})^\circ$ has basis the generators $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$ such that $I_{L,D}(\boldsymbol{x}) = I_{R,A}(\boldsymbol{x})$: there is a natural one-to-one correspondence between such generators in $\S(\mathcal{H})$ and the generators of $\S(\mathcal{H}^\circ)$.

Since $\mathcal{H}$ is nice, the definition of $\widetilde{\partial}$, Equation (2.3.44), simplifies considerably. Indeed, for $n > 1$,

$$\pi \circ (R \circ \delta)^n \circ \iota = 0.$$

Consequently, the differential $\widetilde{\partial}$ has two contributions, corresponding to the cases $n = 0$ and $n = 1$. The $n = 0$ part of $\widetilde{\partial}$ corresponds to provincial domains in the differential on $\widehat{CFDA}(\mathcal{H})$, ie rectangles and bigons in $\mathcal{H}$. These also contribute in exactly the same way to the differential on $\widehat{CFK}(\mathcal{H}^\circ)$.

The $n = 1$ part of $\widetilde{\partial}$ corresponds to chains of the form

$$\boldsymbol{x} \xrightarrow{\partial} \rho \boldsymbol{y} \xrightarrow{R} \boldsymbol{y} \rho \xrightarrow{m_2} \boldsymbol{w}.$$

These correspond exactly to rectangles in $\mathcal{H}^\circ$ which cross $\partial_L \mathcal{H} = \partial_R \mathcal{H}$: the first arrow comes from one half of the rectangle, which crosses the left boundary in a chord ρ, while the third arrow comes from the other half, crossing the boundary in the same ρ. In total, this rectangle contributes exactly as it would for $\widehat{CFK}(\mathcal{H})$.

Certainly no bigons in $\mathcal{H}^\circ$ cross through $\partial_L \mathcal{H}$, and no rectangle can cross $\partial_L \mathcal{H}$ twice. So, the differential on $\widehat{CFK}(\mathcal{H}^\circ)$ is exactly the same as the differential $\widetilde{\partial}$ on $X(\mathcal{H})^\circ$, proving the isomorphism.

We turn next to the strands grading. Let $\S(\mathcal{H})^\circ$ be the generators in $\S(\mathcal{H})$ that survive in $X(\mathcal{H})^\circ$. This set is naturally identified with $\S(\mathcal{H}^\circ)$. To verify the statement about the Alexander and strand gradings, it suffices to show that generators $\boldsymbol{x} \in \S(\mathcal{H})^\circ$ with $\# o_R(\boldsymbol{x}) = k + i$ are mapped under the natural one-to-one correspondence to generators $\boldsymbol{x}^\circ \in \S(\mathcal{H}^\circ)$ with Alexander grading equal to i. (Recall that $o_R(\boldsymbol{x})$ denotes the set of α^R–arcs which are occupied by the generator $\boldsymbol{x}$.)

To verify the this assertion, observe that the surface F° is isotopic to the union of

- a regular neighborhood N of $\bigcup_{i=1}^{2k} \alpha_i^{a,R} \cup \partial_R \overline{\Sigma}$ and
- the descending disk of $\partial N \setminus \partial_R \overline{\Sigma}$.

So, it follows from the description of $\mathfrak{s}(\boldsymbol{x}^\circ)$ from [29, Section 2.6; 28, Section 2.3] that

$$\langle c_1(\widehat{\mathfrak{s}(\boldsymbol{x}^\circ)}), \widehat{F^\circ} \rangle = -2k + 2\#(\boldsymbol{x}^\circ \cap N) = 2i.$$

Finally we turn to the $\mathbb{Z}$–set gradings. Fix a generator $\boldsymbol{x}^\circ \in \mathfrak{S}(\mathcal{H}^\circ)$ for the self-glued Heegaard diagram, and let $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$ denote the corresponding generator for the bordered Heegaard diagram. Regard $\mathrm{gr}(\boldsymbol{x})$ as an element of $S_{DA}(\mathcal{H})/\sim = \widetilde{P}_{\boldsymbol{x}_0}\backslash G_{DA}(\partial\mathcal{H})/\sim$, where $\sim$ is the equivalence relation

$$(n,\alpha,\beta) \sim (n,\alpha+\beta,0),$$

and otherwise the notation is as in Section 6.5. (This is the same as the equivalence relation from Lemma 2.5.12. For brevity, we denote $G_{DA}(\partial\mathcal{H})$ by G.)

We first show that the divisibility of $\mathrm{gr}(\boldsymbol{x})$ and $\mathrm{gr}(\boldsymbol{x}^\circ)$ are the same. Indeed, $n\cdot\mathrm{gr}(\boldsymbol{x}) = \mathrm{gr}(\boldsymbol{x})$ means there is a periodic domain $P \in \pi_2(\boldsymbol{x},\boldsymbol{x})$ with $\widetilde{R}(g(P)) = (n,\alpha,-\alpha)$. But then P closes up to give a periodic domain in $\pi_2(\boldsymbol{x}^\circ,\boldsymbol{x}^\circ)$ with $\mathrm{ind}(P) = n$.

Next we identify the $\mathbb{Z}$–orbits in $S_{DA}(\mathcal{H})/\sim$ with the $\mathbb{Z}$–orbits of the grading set of $\mathcal{H}^\circ$, ie the relative spin^c–structures on $Y^\circ \setminus K$, as follows. Fix a spin^c–structure $\mathfrak{s}$ on $\mathcal{H}$ and let $\boldsymbol{x}$ be a generator representing $\mathfrak{s}$. Recall that the spin^c–structure $\mathfrak{s}$ corresponds to the G–orbit of $\mathrm{gr}(\boldsymbol{x})$; we will use $\boldsymbol{x}$ as the base generator for this orbit. The sum map $G \to H_1(\partial_L(Y(\mathcal{H})))$ given by

$$(n,\alpha,\beta) \mapsto \alpha+\beta$$

does not descend to the G–orbit of $\mathrm{gr}(\boldsymbol{x})$, as there may be elements of $\widetilde{P}_{\boldsymbol{x}}$ with nontrivial image under this map. However, the inclusion $H_1(\partial_L(Y(\mathcal{H}))) \to H_1(Y^\circ\setminus K)$ kills the image of $\widetilde{P}_{\boldsymbol{x}}$. If we further compose with the Poincaré duality isomorphism $H_1(Y^\circ \setminus K) \cong H^2(Y^\circ, K)$ we get a map

$$p_{\boldsymbol{x}}\colon \mathrm{gr}(\boldsymbol{x})\cdot G \to H^2(Y^\circ, K).$$

This map depends on the choice of $\boldsymbol{x}$; however, the map $q\colon \mathrm{gr}(\boldsymbol{x})\cdot G \to \mathrm{spin}^c(Y,K)$ defined by $q(\mathrm{gr}(\boldsymbol{x})\cdot g) = \mathfrak{s}(\boldsymbol{x}^\circ) + p_{\boldsymbol{x}}(g)$ is independent of the choice of $\boldsymbol{x}$. In fact, q descends to an identification of $\mathbb{Z}$–orbits in $S_{DA}(\mathcal{H})/\sim$ with relative spin^c–structures.

Now, focus on the $\mathbb{Z}$–orbit in $S_{DA}(\mathcal{H})/\sim$ which contains the generator $\boldsymbol{x}$. The map of grading sets (on this $\mathbb{Z}$–orbit) is completely determined by the requirement that $\mathrm{gr}(\boldsymbol{x})$ map to $\mathrm{gr}(\boldsymbol{x}^\circ)$. It remains to check that this map is compatible with the isomorphism

$$HH({}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(Y)_{\mathcal{A}(\mathcal{Z})}) \cong \widehat{HFK}(Y^\circ, K).$$

Let $\boldsymbol{y}$ be in the $\mathbb{Z}$–orbit of $\boldsymbol{x}$. Then there is a domain $B \in \pi_2(\boldsymbol{x},\boldsymbol{y})$ so that $r_*(\partial^\partial_L B) = -\partial^\partial_R B$. Taking $\boldsymbol{x}$ as our base generator (ie setting $\mathrm{gr}(\boldsymbol{x}) = \widetilde{P}_{\boldsymbol{x}} \subset G_{DA}(\partial\mathcal{H})$), we have

$$\begin{aligned}\mathrm{gr}(\boldsymbol{y}) &= \widetilde{P}_{\boldsymbol{x}}\cdot(-e(B)-n_{\boldsymbol{x}}(B)-n_{\boldsymbol{y}}(B), r_*(\partial^\partial_L B), \partial^\partial_R B)\\ &\qquad\qquad\cdot(\psi(I_{L,D}(\boldsymbol{y})), \psi(I_{R,A}(\boldsymbol{y}))^{-1}),\\ \mathrm{gr}(\boldsymbol{y})^\circ &= [(-e(B)-n_{\boldsymbol{x}}(B)-n_{\boldsymbol{y}}(B),0,0)] = \lambda^{\mathrm{gr}(\boldsymbol{x}^\circ,\boldsymbol{y}^\circ)},\end{aligned}$$

where $\mathrm{gr}(\boldsymbol{y})^\circ$ is defined in Definition 2.5.11; the factors $\psi(I_{L,D}(\boldsymbol{y}))$ and $\psi(I_{R,A}(\boldsymbol{y}))^{-1}$ cancel in $S(\mathcal{H})^\circ$ because $I_{L,D}(\boldsymbol{y})=I_{R,A}(\boldsymbol{y})$; $\mathrm{gr}(\boldsymbol{x}^\circ,\boldsymbol{y}^\circ)$ denotes the $(\mathbb{Z}/n)$–grading difference between $\boldsymbol{x}^\circ$ and $\boldsymbol{y}^\circ$; and the last equality follows from the fact that B gives a domain B° in $\pi_2(\boldsymbol{x}^\circ,\boldsymbol{y}^\circ)$ with the same Euler measure and point measures as B. □

Proof via deforming the diagonal (sketch) Fix an admissible Heegaard diagram $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z)$ for Y. Given a holomorphic map $u\colon S\to\Sigma\times[0,1]\times\mathbb{R}$ with right punctures $p_1^R,\dots,p_i^R$ and left punctures $p_1^L,\dots,p_j^L$ we have points

$$\mathrm{ev}_R(u)=(t\circ u(p_1^R),\dots,t\circ u(p_i^R))\in\mathbb{R}^i,$$
$$\mathrm{ev}_L(u)=(t\circ u(p_1^L),\dots,t\circ u(p_i^L))\in\mathbb{R}^j.$$

By a *self-matched curve* we mean a holomorphic map $u\colon S\to\Sigma\times[0,1]\times\mathbb{R}$ with the same number of right punctures as left punctures, labeled by the same Reeb chords in the same order, and such that $\mathrm{ev}_R(u)=\mathrm{ev}_L(u)$. Let $\mathcal{M}^B_{SM}$ denote the set of embedded, self-matched curves in the homology class B. One can show that, generically, $\mathcal{M}^B_{SM}$ is a manifold, transversely cut out and of dimension $e(B)+n_{\boldsymbol{x}}(B)+n_{\boldsymbol{y}}(B)-1$. Also, for appropriate almost complex structures, the differential on $\widehat{CFK}(\mathcal{H}^\circ)$ is given by

$$\partial\boldsymbol{x}=\sum_{\boldsymbol{y}}\sum_{\substack{B\in\pi_2(\boldsymbol{x},\boldsymbol{y})\\ e(B)+n_{\boldsymbol{x}}(B)+n_{\boldsymbol{y}}(B)=1}}(\#\mathcal{M}^B_{SM})\boldsymbol{y}.$$

Next, we deform the condition of being self-matched in two stages. First, for $t\in[0,\infty)$, a *T–shifted self-matched curve* is a curve u with right punctures $p_1^R,\dots,p_j^R$ and left punctures $p_1^L,\dots,p_j^L$ so that for each $i=1,\dots,j$,

$$t\circ u(p_i^R)+T=t\circ u(p_i^L)$$

(and u is asymptotic to the same Reeb chords at p_i^R and p_i^L). A 0–shifted self-matched curve is just a self-matched curve as previously defined. Let $\mathcal{M}^B_{T-S,SM}$ denote the moduli space of T–shifted self-matched curves. Defining ∂ instead using $\mathcal{M}^B_{T-S,SM}$, we get a new chain complex which is homotopy equivalent to $\widehat{CFK}(\mathcal{H}^\circ)$.

Now, take $T\to\infty$. Sequences of T_i–shifted self-matched curves with $T_i\to\infty$ converge to many-story holomorphic combs $(u_1,u_2,\dots,u_l)$ where each $u_i\in\pi_2(\boldsymbol{x}_i,\boldsymbol{x}_{i+1})$ (with $\boldsymbol{x}_1=\boldsymbol{x}$ and $\boldsymbol{x}_{l+1}=\boldsymbol{y}$), subject to the following condition. Let $p^R_{i,1},\dots,p^R_{i,j_i^R}$ denote the right punctures of u_i and $p^L_{i,1},\dots,p^L_{i,j_i^L}$ denote the left punctures of u_i. Then:

- For each $i=1,\dots,l-1$, $j_i^R=j_{i+1}^L$, and $j_l^R=j_1^L=0$.
- For each $i=1,\dots,l-1$ and $j=1,\dots,j_i$, u is asymptotic to the same Reeb chords at $p^R_{i,j}$ and $p^L_{i+1,j}$.

- For each $i = 1, \dots, l-1$ and $j = 1, \dots, j_i - 1$,

$$t \circ u_i(p^R_{i,l+1}) - t \circ u_i(p^R_{i,l}) = t \circ u_{i+1}(p^L_{i+1,l+1}) - t \circ u_{i+1}(p^L_{i+1,l}). \tag{7.1}$$

We call such combs *∞–shifted self-matched combs*, and let $\mathcal{M}^B_{\infty\text{–}S,SM}$ denote the moduli space of ∞–shifted self-matched combs. Using $\mathcal{M}^B_{\infty\text{–}S,SM}$ instead of $\mathcal{M}^B_{SM}$, we again obtain a new chain complex, homotopy equivalent to $\widehat{CFK}(\mathcal{H}^\circ)$. (See Figure 18 for an example of this shifting and the further steps in the proof.)

Next, we further deform the diagonal as follows. An *∞–shifted, T–self-matched holomorphic comb* is a holomorphic comb $(u_1, u_2, \dots, u_i)$ satisfying the same conditions as a ∞–shifted self-matched comb, except that (7.1) is replaced with the formula

$$T \cdot (t \circ u_i(p^R_{i,l+1}) - t \circ u_i(p^R_{i,l})) = t \circ u_{i+1}(p^L_{i+1,l+1}) - t \circ u_{i+1}(p^L_{i+1,l}).$$

An ∞–shifted, 1–self-matched comb is the same as an ∞–shifted self-matched comb.

Let $\mathcal{M}^B_{\infty\text{–}S,T\text{–}SM}$ denote the moduli space of ∞–shifted, T–self-matched combs. By replacing $\mathcal{M}^B_{SM}$ with $\mathcal{M}^B_{\infty\text{–}S,T\text{–}SM}$, we get another chain complex homotopy equivalent to $\widehat{CFK}(\mathcal{H}^\circ)$.

Now, send $T \to \infty$. One can show that sequences of ∞–shifted, T_i–self-matched combs converge to holomorphic combs $(u_1, u_2, \dots, u_l)$ such that:

- Each u_i is asymptotic to a sequence of sets of Reeb chords $\vec{\boldsymbol{\rho}}^R_i = (\boldsymbol{\rho}^R_{i,1}, \dots, \boldsymbol{\rho}^R_{i,j^R_i})$ at $\partial_R \Sigma$, and to a sequence of Reeb chords $\vec{\rho}^L_i = (\rho^L_{i,1}, \dots, \rho^L_{i,j^L_i})$ at $\partial_L \Sigma$.
- The sequence of algebra elements $a(\boldsymbol{\rho}^R_{1,1}), \dots, a(\boldsymbol{\rho}^R_{1,j^R_1}), a(\boldsymbol{\rho}^R_{2,1}), \dots$ and the sequence of algebra elements $a(-\vec{\rho}^L_1), a(-\vec{\rho}^L_2), \dots$ are the same.
- Each u_i is rigid, as a one-story comb with the specified asymptotics.

We call such a comb a *∞–shifted, ∞–self-matched holomorphic comb*, and denote the moduli space of such combs by $\mathcal{M}^B_{\infty\text{–}S,\infty\text{–}SM}$ Replacing $\mathcal{M}^B_{SM}$ by $\mathcal{M}^B_{\infty\text{–}S,\infty\text{–}SM}$, we obtain another chain complex homotopy equivalent to $\widehat{CFK}(\mathcal{H}^\circ)$. But this chain complex also has an alternate description: it is the Hochschild complex for $\widehat{CFDA}(\mathcal{H})$ of Proposition 2.3.54. This implies the result.

The statements about gradings follows exactly as in the "nice diagrams" version of the proof. $\square$

Proof of Theorem 7 This is immediate from Lemma 5.6, Theorem 14 and the definition of $\widehat{CFDA}(\psi)$ in Section 6.4. $\square$

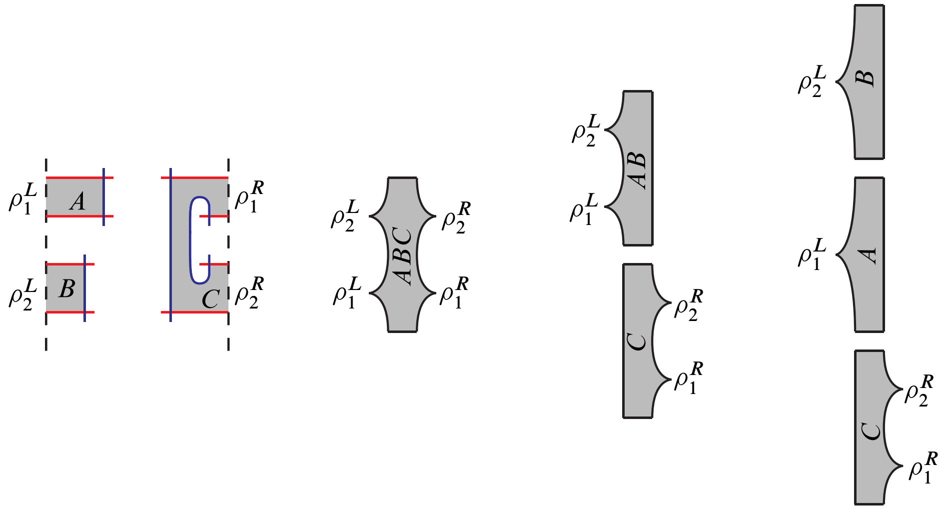


Figure 18: Hochschild homology via deforming the diagonal: on the far left, we have a region in a Heegaard diagram $\mathcal{H}$, contributing to the differential on $\widehat{CFK}(\mathcal{H}^\circ)$. All of the α–arcs shown are parts of different α–curves; center left, we have a schematic of the corresponding self-matched curve; center right, we have a schematic of the corresponding ∞–shifted self-matched curve; far right, we have a schematic of the corresponding ∞–shifted ∞–self-matched curve. Another interesting example can be obtained by reflecting the diagram horizontally.

8 The mapping class group action

In this section, we show that the bimodules $\widehat{CFDA}(\phi)$ associated to surface diffeomorphisms ϕ induce an action of the bordered mapping class group on the derived category of $\mathcal{A}(\mathcal{Z})$–modules. A key step towards establishing this result is that the bimodule associated to the identity surface diffeomorphism is the identity map, ie $\widehat{CFDA}(\mathbb{I}_{F(\mathcal{Z})})$ is homotopy equivalent to ${}^{\mathcal{A}(\mathcal{Z})}[\mathbb{I}]_{\mathcal{A}(\mathcal{Z})}$ (Definition 2.2.48), verifying Theorem 4. This is done in Section 8.1. The mapping class group action on the derived categories is stated precisely in Section 8.2, and verified in Section 8.3.

8.1 Identity bimodules

We first prove that the identity map on $\mathcal{Z}$ induces a bimodule which is quasi-isomorphic to the identity bimodule on $\mathcal{A}(\mathcal{Z})$, as stated in Theorem 4.

We make our notation slightly more precise than in the original statement of the theorem, writing $\mathcal{A}(\mathcal{Z})$ for $\mathcal{A}(F)$, so the desired result is $\widehat{CFDA}(\mathbb{I}_{F(\mathcal{Z})}) \simeq {}^{\mathcal{A}(\mathcal{Z})}[\mathbb{I}_{\mathcal{A}(\mathcal{Z})}]_{\mathcal{A}(\mathcal{Z})}$

During the proof we also make our notation also less precise, dropping the subscript from the identity $\mathbb{I}$ (which could be either the algebra of $\mathcal{Z}$, or the surface associated to $\mathcal{Z}$): it should be clear from the context.

Proof of Theorem 4 We start by arguing that ${}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})}$ is quasi-invertible in the sense of Definition 2.4.7. Consider the canonical bordered Heegaard diagram for the identity diffeomorphism (Definition 5.10), illustrated in Figure 19 (left). It is clear from inspection that ${}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})}$ is isomorphic to $\mathcal{A}(F)$ as a left $\mathcal{A}(F)$–module. Moreover, $\delta_1^1 = 0$: any nontrivial domain meets the type A boundary in the canonical diagram. Thus, Lemma 2.2.50 applies, showing that $\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})}$ is isomorphic to ${}^{\mathcal{A}(F)}[\phi]_{\mathcal{A}(F)}$ for some $\mathcal{A}_\infty$–endomorphism ϕ of $\mathcal{A}(F)$.

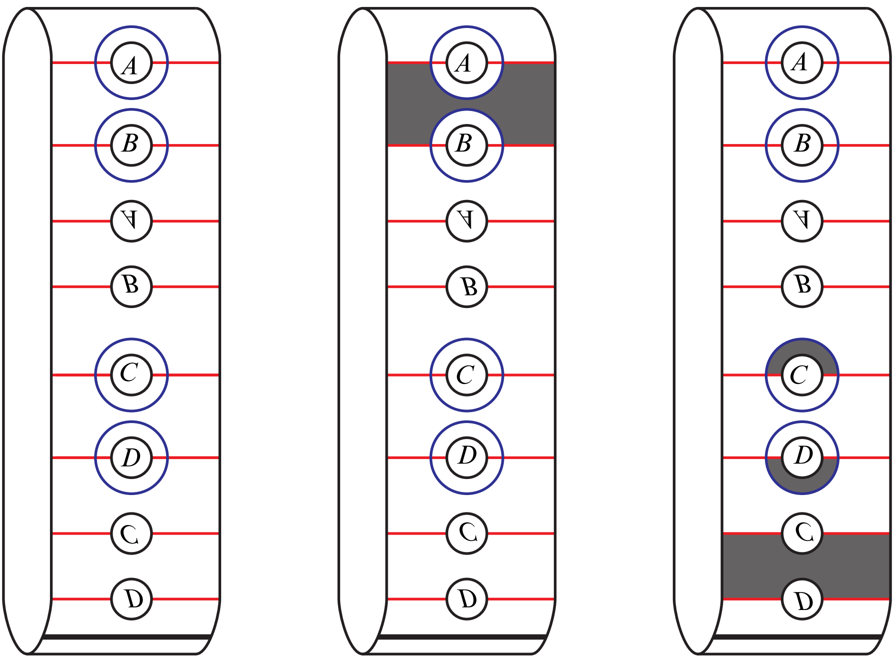


Figure 19: The canonical Heegaard diagram for the identity map: on the left, we have the identity map of the split genus-2 surface; center and right, we have two of the holomorphic disks implying $\phi_*(\rho) = \rho$ for any length 1 chord ρ.

For any Reeb chord ρ of length 1, $\phi_*(\rho) = \rho$, as there is an obvious holomorphic disk; see Figure 19 (center, right). By Proposition 4.7, ϕ_1 induces the identity map on the homology of $\mathcal{A}(F)$; in particular, ϕ is a quasi-isomorphism. By Proposition 2.4.8 ${}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})}$ is quasi-invertible.

Next, according to Theorem 5,

$$^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})} \simeq {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})}.$$

Hence, applying $\boxtimes$ with the quasi-inverse to $^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})}$ to both sides of the above quasi-isomorphism, we obtain the desired quasi-isomorphism

$$^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathbb{I})_{\mathcal{A}(\mathcal{Z})} \simeq {}^{\mathcal{A}(\mathcal{Z})}[\mathbb{I}]_{\mathcal{A}(\mathcal{Z})}. \qquad \square$$

Corollary 8.1 *Let $\phi \in MCG_0(F(\mathcal{Z}))$. Then the functors*

$$\cdot \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z})} \colon \mathsf{Mod}_{\mathcal{A}(\mathcal{Z})} \to \mathsf{Mod}_{\mathcal{A}(\mathcal{Z})},$$
$$^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z})} \boxtimes \cdot \colon {}^{\mathcal{A}(\mathcal{Z})}\mathsf{Mod} \to {}^{\mathcal{A}(\mathcal{Z})}\mathsf{Mod},$$

are auto quasiequivalences of $\mathbb{Z}$–set graded differential categories.

More generally, if $\phi \in MCG_0(F(\mathcal{Z}), F(\mathcal{Z}'))$ is in the mapping class groupoid, then the functors

$$\cdot \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}')} \colon \mathsf{Mod}_{\mathcal{A}(\mathcal{Z})} \to \mathsf{Mod}_{\mathcal{A}(\mathcal{Z}')},$$
$$\cdot \boxtimes {}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z}')}\widehat{CFDD}(\phi) \colon \mathsf{Mod}_{\mathcal{A}(\mathcal{Z})} \to {}^{\mathcal{A}(-\mathcal{Z}')}\mathsf{Mod},$$
$$^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}')} \boxtimes \cdot \colon {}^{\mathcal{A}(\mathcal{Z}')}\mathsf{Mod} \to {}^{\mathcal{A}(\mathcal{Z}')}\mathsf{Mod},$$
$$\widehat{CFAA}(\phi)_{\mathcal{A}(-\mathcal{Z}),\mathcal{A}(\mathcal{Z}')} \boxtimes \cdot \colon {}^{\mathcal{A}(\mathcal{Z}')}\mathsf{Mod} \to \mathsf{Mod}_{\mathcal{A}(-\mathcal{Z}')},$$

are quasiequivalences of $\mathbb{Z}$–set graded differential categories.

Proof We will prove that

$$\cdot \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}')} \colon \mathsf{Mod}_{\mathcal{A}(\mathcal{Z})} \to \mathsf{Mod}_{\mathcal{A}(\mathcal{Z}')}$$

is a quasiequivalence; the other cases are similar. By Lemma 2.4.9 it suffices to show that $^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\phi)_{\mathcal{A}(\mathcal{Z}')}$ is quasi-invertible.

Fix any Heegaard diagrams $\mathcal{H}$ for ϕ and $\mathcal{H}'$ for ϕ^{-1}. It follows from Lemma 5.9 that $\mathcal{H}_{\partial_R} \cup_{\partial_L} \mathcal{H}'$ is a Heegaard diagram for the identity map $\mathbb{I}_{F(\mathcal{Z})} \in MCG_0(F(\mathcal{Z}), F(\mathcal{Z}))$ while $\mathcal{H}'_{\partial_R} \cup_{\partial_L} \mathcal{H}$ is a Heegaard diagram for $\mathbb{I}_{F(\mathcal{Z}')} \in MCG_0(F(\mathcal{Z}'), F(\mathcal{Z}'))$. By Theorem 12,

$$^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}')} \boxtimes {}^{\mathcal{A}(\mathcal{Z}')}\widehat{CFDA}(\mathcal{H}')_{\mathcal{A}(\mathcal{Z})} \simeq {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathcal{H}_{\partial_R} \cup_{\partial_L} \mathcal{H}')_{\mathcal{A}(\mathcal{Z}')},$$
$$^{\mathcal{A}(\mathcal{Z}')}\widehat{CFDA}(\mathcal{H}')_{\mathcal{A}(\mathcal{Z})} \boxtimes {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}')} \simeq {}^{\mathcal{A}(\mathcal{Z}')}\widehat{CFDA}(\mathcal{H}'_{\partial_R} \cup_{\partial_L} \mathcal{H})_{\mathcal{A}(\mathcal{Z}')}.$$

By Lemma 5.8 and Proposition 6.10,

$$
\begin{aligned}
{}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathcal{H}\ {}_{\partial_R}\cup_{\partial_L}\mathcal{H}')_{\mathcal{A}(\mathcal{Z})} &\simeq {}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathcal{H}(\mathbb{I}_{F(\mathcal{Z})}))_{\mathcal{A}(\mathcal{Z})},\\
{}^{\mathcal{A}(\mathcal{Z}')}\widehat{CFDA}(\mathcal{H}'\ {}_{\partial_R}\cup_{\partial_L}\mathcal{H})_{\mathcal{A}(\mathcal{Z}')} &\simeq {}^{\mathcal{A}(\mathcal{Z}')}\widehat{CFDA}(\mathcal{H}(\mathbb{I}_{F(\mathcal{Z}')}))_{\mathcal{A}(\mathcal{Z}')}.
\end{aligned}
$$

So, by Theorem 4,

$$
\begin{aligned}
{}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}')}\boxtimes{}^{\mathcal{A}(\mathcal{Z}')}\widehat{CFDA}(\mathcal{H}')_{\mathcal{A}(\mathcal{Z})} &\simeq {}^{\mathcal{A}(\mathcal{Z})}[\mathbb{I}]_{\mathcal{A}(\mathcal{Z})},\\
{}^{\mathcal{A}(\mathcal{Z}')}\widehat{CFDA}(\mathcal{H}')_{\mathcal{A}(\mathcal{Z})}\boxtimes{}^{\mathcal{A}(\mathcal{Z})}\widehat{CFDA}(\mathcal{H})_{\mathcal{A}(\mathcal{Z}')} &\simeq {}^{\mathcal{A}(\mathcal{Z}')}[\mathbb{I}]_{\mathcal{A}(\mathcal{Z}')}.
\end{aligned}
$$

This proves the claim. □

Finally, as another corollary, we have Theorem 1, the statement that different pointed matched circles for a given surface have equivalent derived categories.

Proof of Theorem 1 This follows from Corollary 8.1 by choosing any mapping class $\phi\in MCG_0(\mathcal{Z}_1,\mathcal{Z}_2)$. □

8.2 Group actions on categories

To state the mapping class group(oid) action precisely requires a little categorical algebra, which we review in this subsection.

Definition 8.1 Let G be a group and $\mathscr{C}$ a category. Let $\mathrm{End}(\mathscr{C})$ denote the class of functors $F\colon \mathscr{C}\to\mathscr{C}$.

- A *strict action* of G on $\mathscr{C}$ is a map $A\colon G\to\mathrm{End}(\mathscr{C})$ such that $A(\mathbb{I})$ is the identity functor and if $g,h\in G$ then $A(gh)=A(g)\circ A(h)$.
- A *weak action* of G on $\mathscr{C}$ is a map $A\colon G\to\mathrm{End}(\mathscr{C})$ together with a natural isomorphism A_0 of the identity functor $\mathbb{I}_{\mathscr{C}}$ to $A(\mathbb{I})$; and for each $g,h\in G$, an isomorphism $A_2(g,h)$ from $A(g)\circ A(h)$ to $A(gh)$, so that the following diagrams commute:

$$
\begin{tikzcd}
A(g)\circ A(h)\circ A(k) \arrow[r, "A_2(g,h)\circ\mathbb{I}_{A(k)}"] \arrow[d, "\mathbb{I}_{A(g)}\circ A_2(h,k)"'] & A(gh)\circ A(k) \arrow[d, "A_2(gh,k)"] \\
A(g)\circ A(hk) \arrow[r, "A_2(g,hk)"'] & A(ghk)
\end{tikzcd}
$$

$$
\begin{tikzcd}
A(g)\circ \mathbb{I}_{\mathscr{C}} \arrow[r, "="] \arrow[d, "A(g)\circ A_0"'] & A(g) \\
A(g)\circ A(\mathbb{I}) \arrow[r, "A_2(g,\mathbb{I})"'] & A(g\mathbb{I}) \arrow[u, "="']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}_{\mathscr{C}}\circ A(g) \arrow[r, "="] \arrow[d, "A_0\circ A(g)"'] & A(g) \\
A(\mathbb{I})\circ A(g) \arrow[r, "A_2(\mathbb{I},g)"'] & A(\mathbb{I}g) \arrow[u, "="']
\end{tikzcd}
$$

(Compare Mac Lane [25, Section XI.2].)

If $\mathscr{C}$ has some extra structure (for example, if it is triangulated) then we replace $\mathrm{End}(\mathscr{C})$ by the class of endofunctors preserving that structure.

Remark 8.2 This terminology is not entirely standard. In particular, the reader is cautioned that some sources call our weak action a strong action.

We want to extend the notion of weak group actions to actions of groupoids, so first we reinterpret it. Recall:

Definition 8.3 If $\mathscr{D}$ and $\mathscr{E}$ are 2–categories then a *weak 2–functor* from $\mathscr{D}$ to $\mathscr{E}$ consists of:

- A map $A\colon \mathrm{Ob}_{\mathscr{D}} \to \mathrm{Ob}_{\mathscr{E}}$.
- For each $a, b \in \mathrm{Ob}_{\mathscr{D}}$ a functor $A_{a,b}\colon \mathrm{Mor}_{\mathscr{D}}(a,b) \to \mathrm{Mor}_{\mathscr{E}}(A(a), A(b))$.
- For each $a \in \mathrm{Ob}_{\mathscr{D}}$ a 2–morphism $A_a \in 2\mathrm{Mor}(\mathbb{I}_{A(a)}, A_{a,a}(\mathbb{I}_a))$.
- For each $a, b, c \in \mathrm{Ob}_{\mathscr{D}}$, $f \in \mathrm{Mor}_{\mathscr{D}}(b,c)$, and $g \in \mathrm{Mor}_{\mathscr{D}}(a,b)$, a 2–morphism

  $$A_{a,b,c}(f,g) \in 2\mathrm{Mor}(A_{b,c}(f)\circ A_{a,b}(g), A_{a,c}(f\circ g)),$$

  forming a natural transformation of functors; more precisely, as f and g vary, both $A_{b,c}(f)\circ A_{a,b}(g)$ and $A_{a,c}(f\circ g)$ give functors from $\mathrm{Mor}_{\mathscr{D}}(b,c)\times \mathrm{Mor}_{\mathscr{D}}(a,b)$ to $\mathrm{Mor}_{\mathscr{E}}(A(a), A(c))$; then $A_{a,b,c}$ is required to be a natural transformation between these two functors.

These data must satisfy:

- For any objects $a, b, c, d \in \mathrm{Ob}_{\mathscr{D}}$ and morphisms $f \in \mathrm{Mor}_{\mathscr{D}}(c, d)$, $g \in \mathrm{Mor}_{\mathscr{D}}(b, c)$ and $h \in \mathrm{Mor}_{\mathscr{D}}(a, b)$ the diagram

$$\begin{tikzcd}
A_{c,d}(f) \circ (A_{b,c}(g) \circ A_{a,b}(h)) \arrow[r, "="] \arrow[d, "\mathbb{I}_{A_{c,d}(f)} \circ A_{a,b,c}(g,h)"'] & (A_{c,d}(f) \circ A_{b,c}(g)) \circ A_{a,b}(h) \arrow[d, "A_{b,c,d}(f,g) \circ \mathbb{I}_{A_{a,b}(h)}"] \\
A_{c,d}(f) \circ A_{a,c}(g \circ h) \arrow[d, "A_{a,c,d}(f, g\circ h)"'] & A_{b,d}(f \circ g) \circ A_{a,b}(h) \arrow[d, "A_{a,b,d}(f \circ g, h)"] \\
A_{a,d}(f \circ (g \circ h)) \arrow[r, "="'] & A_{a,d}((f \circ g) \circ h)
\end{tikzcd}$$

  commutes.

- For any morphism $f \in \mathrm{Mor}(a, b)$, the diagrams

$$\begin{tikzcd}
A_{a,b}(f) \circ \mathbb{I}_{A(b)} \arrow[r, "="] \arrow[d, "A_{a,b}(f) \circ A_b"'] & A_{a,b}(f) \\
A_{a,b}(f) \circ A(\mathbb{I}_b) \arrow[r, "A_{a,b,b}(f,\mathbb{I})"'] & A_{a,b}(f \circ \mathbb{I}_b) \arrow[u, "="']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{I}_{A(a)} \circ A_{a,b}(f) \arrow[r, "="] \arrow[d, "A_a \circ A_{a,b}(f)"'] & A_{a,b}(f) \\
A(\mathbb{I}_a) \circ A_{a,b}(f) \arrow[r, "A_{a,a,b}(\mathbb{I}, f)"'] & A_{a,b}(\mathbb{I}_a \circ f) \arrow[u, "="']
\end{tikzcd}$$

  commute.

See Bénabou [3, Definition 4.1], which defines the notion more generally for weak 2–categories (or bicategories), although we choose to keep the standard convention for order of composition.

We may view a group G as a 2–category Γ with a single object $\bullet$, $\mathrm{Mor}(\bullet, \bullet) = G$, and $2\mathrm{Mor}(g, h)$ empty if $g \neq h$ and consisting of the identity map if $g = h$. A category $\mathscr{C}$ specifies a 2–category $\mathscr{E}nd(\mathscr{C})$ with a single object $*$, $\mathrm{Mor}(*, *) = \mathrm{End}(\mathscr{C})$, and $2\mathrm{Mor}(F_1, F_2)$ the set of natural transformations from F_1 to F_2.

Lemma 8.4 *With the above setup, a weak action of G on $\mathscr{C}$ is a weak 2–functor from Γ to $\mathscr{E}nd(\mathscr{C})$.*

Proof This is largely immediate from the definitions: Given a weak action A, we define a weak 2–functor B by:

- $B(\bullet) = *$.
- $B_{\bullet,\bullet}(g) = A(g)$; this function on the objects of $\mathrm{Mor}_\Gamma(\bullet,\bullet)$ extends trivially to the morphisms of $\mathrm{Mor}_\Gamma(\bullet,\bullet)$ (which are the 2–morphisms of Γ).
- $B_{\bullet,\bullet,\bullet}(g,h) = A_2(g,h)$; this map automatically defines a natural transformation, since $\mathrm{Mor}_\Gamma(\bullet,\bullet)\times\mathrm{Mor}_\Gamma(\bullet,\bullet)$ has only identity morphisms.

The diagrams that are required to commute are precisely the same in the two cases. □

This leads easily to the notion of a groupoid action.

Definition 8.5 Let $\mathscr{C}_1,\dots,\mathscr{C}_n$ be categories and Γ a groupoid. Make Γ into a 2–category with only identity 2–morphisms. Let $\mathscr{E}nd(\{\mathscr{C}_1,\dots,\mathscr{C}_n\})$ denote the full 2–subcategory of $\mathscr{C}at$ generated by $\mathscr{C}_1,\dots,\mathscr{C}_n$. That is,

$$\begin{aligned}\mathrm{Ob}(\mathscr{C}) &= \{\mathscr{C}_1,\dots,\mathscr{C}_n\},\\ \mathrm{Mor}(\mathscr{C}_i,\mathscr{C}_j) &= \{\text{functors } F\colon \mathscr{C}_i\to\mathscr{C}_j\},\\ 2\mathrm{Mor}(F_1,F_2) &= \{\text{natural transformations from } F_1 \text{ to } F_2\}.\end{aligned}$$

Then a *weak action of* Γ *on* $\{\mathscr{C}_1,\dots,\mathscr{C}_n\}$ is a weak 2–functor from Γ to $\mathscr{E}nd(\mathscr{C})$. (Again, if the categories $\mathscr{C}_1,\dots,\mathscr{C}_n$ are triangulated, say, then we restrict to triangulated functors.)

(There is an obvious analogue when $\mathscr{C}$ has infinitely many elements, but we shall not need this.)

Recall from Definition 2.5.31 that $\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})}$ denotes the category of G–set graded right $\mathcal{A}_\infty$–modules over $\mathcal{A}(\mathcal{Z})$, and $\mathsf{H}_*(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})})$ the category whose objects are set-graded $\mathcal{A}_\infty$–modules over $\mathcal{A}(\mathcal{Z})$ and morphisms are $\mathcal{A}_\infty$–homotopy classes of $\mathcal{A}_\infty$–module maps. There is no natural notion of degree-0 morphisms in $\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})}$, and so $\mathsf{H}_*(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})})$ is not a triangulated category in the usual sense. However, $\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})}$ does have a (nonfull) subcategory $\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),G(\mathcal{Z})}$ of modules graded by the grading group $G(\mathcal{Z})$ (considered as a right $G(\mathcal{Z})$–set); see Definition 2.5.13. On this subcategory there is a notion of degree-0 morphisms and so one obtains a triangulated category $\mathsf{H}(\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),G(\mathcal{Z})})$.

The goal of the rest of this section is to prove:

Theorem 15 *The bimodules $\widehat{CFDA}(\phi)$ induce a weak action of the genus-k bordered mapping class groupoid $MCG_0(k)$ on $\{\mathsf{H}_*(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})}) \mid \text{genus}(F(\mathcal{Z})) = k\}$. This action preserves the subcategories $\{\mathsf{H}(\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),G(\mathcal{Z})})\}$ and acts by triangulated functors on $\{\mathsf{H}(\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),G(\mathcal{Z})})\}$.*

In particular, for any pointed matched circle $\mathcal{Z}$, the bimodules $\widehat{CFDA}(\phi)$ for the maps $\phi\colon F(\mathcal{Z}) \to F(\mathcal{Z})$ induce a weak action of the genus-k bordered mapping class group on $\mathsf{H}_*(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})})$, and for different choices of $\mathcal{Z}$ these actions are conjugate (in the obvious sense). One easily digestible piece of Theorem 15 is that tensoring with the bimodules $\widehat{CFDA}(\phi)$ induces equivalences of categories; this fact is Corollary 8.1.

Remark 8.6 It is important to note that it is the *bordered* mapping class group which acts on the category, rather than the ordinary one. For example, if $\mathcal{Z}$ represents a surface of genus one, and ϕ denotes the mapping class which is gotten by Dehn twist around the boundary of $\mathcal{Z}$, then the tensor product with ϕ induces a nontrivial action on the category of $\mathcal{A}(\mathcal{Z})$–modules. For instance, there is a module with rank one over $\mathbb{F}_2$ (and trivial differential) whose tensor product with $\widehat{CFDA}(\phi)$ has homology with rank 9; see Figure 20.

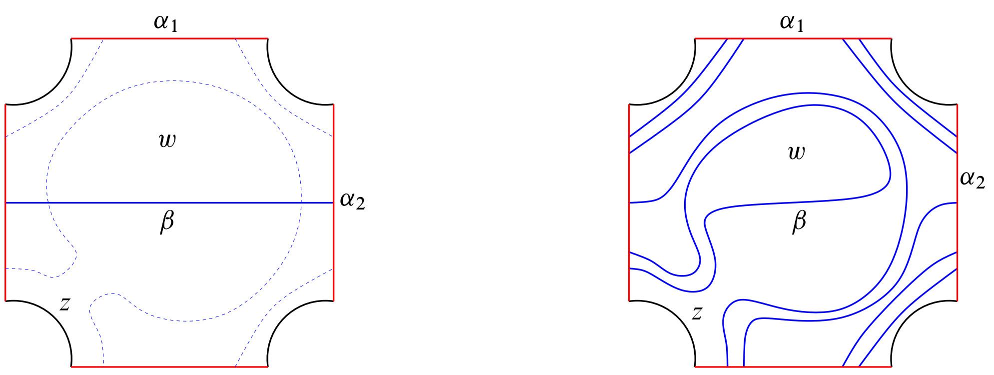


Figure 20: Action of the unbased mapping class group: the differential module corresponding to one idempotent can be thought of as the type A module associated to the doubly-pointed Heegaard diagram illustrated on the left. The Dehn twist along the dotted curve (which can be thought of as Dehn twist around the disk in the torus minus a disk) acts to give the diagram on the right. Since there are no differentials, the rank of the homology of the resulting module is 9. (The diagram depicts a left-handed Dehn twist applied to the β–circle, which corresponds to the action of a right-handed Dehn twist on the bordered three-manifold.)

Remark 8.7 It is possible to refine Theorem 15 by allowing triangulated actions for different grading sets beyond just $G(\mathcal{Z})$. For instance, consider the category $\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),*}$ defined to be the disjoint union, over all right $G(\mathcal{Z})$–sets S on which λ acts freely, of $\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),S}$. Here the "disjoint union" of categories means the category whose objects are the union of the objects of the summands, with no morphisms between objects from different summands, and with the inherited morphisms between objects from the same summand. Then each $\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),S}$ and therefore $\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),*}$ are *dg* categories in the usual sense, and $MCG_0(k)$ acts by triangulated functors on $\{\mathsf{H}(\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),*})\}$.

Remark 8.8 Group actions on algebraic categories have seen considerable interest recently; see for instance Ganter and Kapranov [9], as well as the references in Khovanov and Thomas [16]. The reader might also wonder about groupish structures with more interesting 2–morphisms; such objects are called 2*–groups* and are studied in Baez and Lauda [2].

Remark 8.9 One could try to strengthen Theorem 15 as follows. Consider the 2–category with objects pointed matched circles, $\mathrm{Mor}(\mathcal{Z},\mathcal{Z}')$ the set of diffeomorphisms $F^\circ(\mathcal{Z})\to F^\circ(\mathcal{Z}')$, and 2–morphisms $2\mathrm{Mor}(\phi,\psi)$ the set of isotopy classes of isotopies from ϕ to ψ. Then one could try to associate a weak 2–functor from this category to the weak 2–category of algebras, bimodules and homotopy classes of $\mathcal{A}_\infty$–bimodule maps. That is, one would associate a well-defined bimodule to each diffeomorphism and an $\mathcal{A}_\infty$–homotopy equivalence of bimodules (well defined up to homotopy) to each (isotopy class of) isotopy, satisfying appropriate coherence axioms. (One could of course imagine going farther and look for a functor between the ∞–category of surfaces, diffeomorphisms, paths of diffeomorphisms, … and the ∞–category of differential algebras, differential bimodules, differential bimodule homomorphisms, ….) Although this approach would lead to a slightly stronger result, it would require additional technicalities (for example, consistent choices of perturbation data).

8.3 Construction of the mapping class group action

Before constructing the mapping class group action, we will need to study the DD–identity bimodule.

We will need to consider simultaneously $\mathcal{Z}$ and $-\mathcal{Z}$. To this end, recall from Section 3.2 that there is an orientation-reversing map

$$r\colon \mathcal{Z}\to -\mathcal{Z},$$

which induces the map $a_i\mapsto a_{4k+1-i}$ on the points in $\mathcal{Z}$ (with respect to their orderings induced by the orientations on Z and $-Z$). This induces a map from $[4k]/M$ to $[4k]/r(M)$.

Definition 8.1 Let $\mathcal{Z}$ be a pointed matched circle. If $\boldsymbol{s}$ and $r(\boldsymbol{t})$ form a partition of $[4k]/M$, we say that the idempotents $I_{\mathcal{A}(\mathcal{Z})}(\boldsymbol{s})$ and $I_{\mathcal{A}(-\mathcal{Z})}(\boldsymbol{t})$ for $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$ are *complementary idempotents*.

Complementary idempotents show up in the generating set for $\widehat{CFDD}$ for the standard Heegaard diagram for the identity map, as follows:

Lemma 8.2 *Consider the standard Heegaard diagram $\mathcal{H}$ for the identity map pictured in Figure 19. The generating set $\mathfrak{S}(\mathcal{H})$ is in one-to-one correspondence with the set of idempotents $\mathcal{A}(\mathcal{Z})$: indeed, for each pair (I, I') of complementary idempotents, there is a unique generator $\boldsymbol{x} = \boldsymbol{x}(I)$ satisfying $(I \otimes I') \cdot \boldsymbol{x} = \boldsymbol{x}$.*

Proof This follows from a straightforward inspection of the diagram. $\square$

We turn now to gradings on the identity type DD bimodule. The map

$$R\colon G(-\mathcal{Z}) \to G(\mathcal{Z})^{\mathrm{op}}$$

defined by $R(s, \eta) = (s, r_*(\eta))$ is a group isomorphism. Using this, we give $G(\mathcal{Z})$ the structure of a left $G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z})$–set by the rule

$$(g_1 \times_{\mathbb{Z}} g_2) * h := g_1 \cdot h \cdot R(g_2),$$

where the operation $\cdot$ on the right-hand-side refers to multiplication in $G(\mathcal{Z})$. When referring to $G(\mathcal{Z})$ as a $G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z})$–set in this way, we denote it by T.

Lemma 8.3 *For $\mathcal{H}$ the standard Heegaard diagram for the identity map, there is a natural identification of the grading set $S_{DD}(\mathcal{H})$ of $\widehat{CFDD}(\mathcal{H})$ with the $G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z})$–set T.*

Proof Recall that $P_{\boldsymbol{x}_0}$ is the image of the space of periodic domains under the map from (6.2). $S_{DD}(\mathcal{H})$ can be defined as the quotient of $G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z})$ by $(R \times R)(P_{\boldsymbol{x}_0})$. Another glance at the standard Heegaard diagram shows that $P_{\boldsymbol{x}_0}$ is generated by $(0, -r_*(m)) \times_{\mathbb{Z}} (0, m)$, as m runs over intervals in $\mathcal{Z}$ connecting matched pairs. The map

$$F\colon G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z}) \to T$$

defined by $F(g_1 \times g_2) = g_1 \cdot R(g_2)$ induces an isomorphism of $G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z})$–spaces

$$f\colon (G(\mathcal{Z}) \times_{\mathbb{Z}} G(-\mathcal{Z}))/((R \times R)(P_{\boldsymbol{x}_0})) \to T. \qquad \square$$

Lemma 8.4 *Let $\mathcal{Z}$ be a pointed matched circle. Any grading-preserving homotopy auto-equivalence of ${}^{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{CFDD}(\mathbb{I})$ as a type DD bimodule is homotopic to the identity.*

Proof It suffices to verify that for the standard Heegaard diagram $\mathcal{H}$ of the identity map, pictured in Figure 19, the identity map is the only grading-preserving automorphism ϕ from $\widehat{CFDD}(\mathcal{H})$ to $\widehat{CFDD}(\mathcal{H})$.

It follows from Lemma 8.2 that our automorphism ϕ is specified by algebra elements $a(I,J) \in \mathcal{A}(\mathcal{Z})$ and $b(I,J) \in \mathcal{A}(-\mathcal{Z})$, indexed by pairs I and J of primitive idempotents for $\mathcal{A}(\mathcal{Z})$, whose compatibility with the idempotents is given by

$$I \cdot a(I,J) \cdot J = a(I,J), \quad I' \cdot b(I,J) \cdot J' = b(I,J),$$

(where I' is complementary to I and J' is complementary to J), so that

$$\phi(\boldsymbol{x}(I)) = \sum_J (a(I,J) \otimes b(I,J)) \otimes \boldsymbol{x}(J).$$

To draw conclusions, we must turn to gradings. To this end, note that any two generators $\boldsymbol{x}(I)$ and $\boldsymbol{x}(J)$ can be connected by a domain B with $e(B) + n_{\boldsymbol{x}(I)}(B) + n_{\boldsymbol{x}(J)}(B) = 0$, and $\partial^{\partial_R}(B) = r_*(\partial^{\partial_L}(B))$. Thus, if we fix I, we can find one-chains α_J for each idempotent J, with the property that

$$\mathrm{gr}'(\boldsymbol{x}(J)) = ((0,\alpha_J) \times_{\mathbb{Z}} (0, r_*(\alpha_J))) * \mathrm{gr}'(\boldsymbol{x}(I)).$$

Let ψ and ψ' be the grading refinement data for $\mathcal{A}(\mathcal{Z})$ and $\mathcal{A}(-\mathcal{Z})$, respectively. Recall that

$$\mathrm{gr}((a(I,J)\otimes b(I,J))\otimes \boldsymbol{x}(J)) = (\psi(I)\times_{\mathbb{Z}}\psi'(I'))\cdot \mathrm{gr}'(a(I,J)\otimes b(I,J)) * \mathrm{gr}'(\boldsymbol{x}(J)).$$

Thus, if $\mathrm{gr}((a(I,J) \otimes b(I,J)) \otimes \boldsymbol{x}(J)) = \mathrm{gr}(\boldsymbol{x}(I))$, then it follows that

$$\mathrm{gr}'(a(I,J) \otimes b(I,J)) \cdot \mathrm{gr}'(\boldsymbol{x}(J)) = \mathrm{gr}'(\boldsymbol{x}(I)),$$

so that

$$\mathrm{gr}'(a(I,J) \otimes b(I,J)) * ((0,\alpha_J) \times_{\mathbb{Z}} (0, r_*(\alpha_J))) * \mathrm{gr}'(\boldsymbol{x}(I)) = \mathrm{gr}'(\boldsymbol{x}(I)).$$

Hence (according to Lemma 8.3) there is another one-chain β with the property that

$$\mathrm{gr}'(a(I,J) \otimes b(I,J)) \cdot ((0,\alpha_J) \times_{\mathbb{Z}} (0, r_*(\alpha_J))) = (0,\beta) \times_{\mathbb{Z}} (0, r_*(\beta)).$$

Using $m_{\mathcal{Z}}(\alpha_J, \partial\beta) = -m_{-\mathcal{Z}}(r_*(\alpha_J), \partial r_*(\beta))$, it follows that $\mathrm{gr}'(a(I,J) \otimes b(I,J)) = (0, \beta - \alpha_J) \times_{\mathbb{Z}} (0, r_*(\beta - \alpha_J))$. From Lemma 3.3, it follows that each of $a(I,J)$ and $b(I,J)$ is an idempotent.

Thus, ϕ has the form

$$\phi(\boldsymbol{x}(I)) = c(I, I) \otimes \boldsymbol{x}(I),$$

where here $c(I, I)$ can be either 0 or 1. The fact that ϕ is an automorphism implies that each of these terms is 1; ie ϕ is the identity map. $\square$

Lemma 8.5 *Let $\mathcal{H}$ and $\mathcal{H}'$ be Heegaard diagrams representing $\phi \in MCG_0(F(\mathcal{Z}), F(\mathcal{Z}'))$. Let*

$$f, g\colon \widehat{CFDA}(\mathcal{H}) \to \widehat{CFDA}(\mathcal{H}')$$

be graded $\mathcal{A}_\infty$–homotopy equivalences of bimodules. Then f and g are $\mathcal{A}_\infty$–homotopic.

Proof We start by considering the identity map $\mathbb{I}_{F(\mathcal{Z})} \in MCG_0(F(\mathcal{Z}), F(\mathcal{Z}))$, with the Heegaard diagrams $\mathcal{H} = \mathcal{H}'$. By Lemma 8.4, $\widehat{CFDD}(\mathcal{H})$ has only one homotopy class of auto-equivalences. By Corollary 8.1, $\widehat{CFAA}(\mathbb{I}) \boxtimes \cdot$ is an quasiequivalence of categories from the category of type *DD* structures to the category of type DA structures, and by Theorem 12,

$$\widehat{CFAA}(\mathbb{I}) \boxtimes \widehat{CFDD}(\mathcal{H}) \simeq \widehat{CFDA}(\mathcal{H}).$$

Thus $\widehat{CFDA}(\mathcal{H})$ also has only one homotopy class of auto-equivalences.

The case where $\mathcal{H} \neq \mathcal{H}'$ follows readily: $\widehat{CFDA}(\mathcal{H})$ and $\widehat{CFDA}(\mathcal{H}')$ are homotopy equivalent, so the set of equivalences from $\widehat{CFDA}(\mathcal{H})$ to $\widehat{CFDA}(\mathcal{H}')$ is identified with the set of homotopy auto-equivalences of $\widehat{CFDA}(\mathcal{H})$. Finally, the case where $\phi \neq \mathbb{I}$ follows, since $\widehat{CFDA}(\phi)$ is carried by the equivalence of categories given by $\widehat{CFDA}(\phi^{-1}) \boxtimes \cdot$ to $\widehat{CFDA}(\mathbb{I})$ (again, by Theorem 12). $\square$

Proof of Theorem 15 For each mapping class $[\phi] \in MCG_0(F(\mathcal{Z}), F(\mathcal{Z}'))$ choose a diffeomorphism $\phi\colon F(\mathcal{Z}) \to F(\mathcal{Z}')$ representing $[\phi]$. Recall from Section 6.4 that there is a canonical bordered Heegaard diagram $\mathcal{H}(\phi)$ associated to ϕ. For each ϕ choose also a generic almost complex structure J_ϕ on $\mathcal{H}(\phi) \times [0, 1] \times \mathbb{R}$. Associated to these choices is a well-defined bimodule $\widehat{CFDA}(\phi) = \widehat{CFDA}(\mathcal{H}(\phi), J_\phi)$. Let $\mathcal{F}_\phi$ denote the functor

$$\cdot \boxtimes \widehat{CFDA}(\phi)\colon \mathsf{H}(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})}) \to \mathsf{H}(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z}')}).$$

Now we define an action A of $MCG_0(k)$ on $\{\mathsf{H}(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})}) \mid \mathrm{genus}(F(\mathcal{Z})) = k\}$, as in Definition 8.5, as follows. First, let

$$A(\mathcal{Z}) = \mathsf{H}(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})}), \quad A_{\mathcal{Z},\mathcal{Z}'}([\phi]) = \mathcal{F}_\phi.$$

It remains to define the correction terms A_a and $A_{a,b,c}$ of Definition 8.3.

By Theorem 4 and Lemma 8.5, given a pointed matched circle $\mathcal{Z}$ there is a unique graded isomorphism $\widehat{CFDA}(\mathbb{I}_{F(\mathcal{Z})}) \to {}^{\mathcal{A}(\mathcal{Z})}[\mathbb{I}]_{\mathcal{A}(\mathcal{Z})}$. Together with the canonical natural isomorphism between the identity functor and $\cdot\,) \boxtimes {}^{\mathcal{A}(\mathcal{Z})}[\mathbb{I}]_{\mathcal{A}(\mathcal{Z})}$ (see Lemma 2.3.20) this defines a natural transformation

$$A_{\mathcal{Z}}\colon \mathbb{I}_{\mathsf{H}(\mathsf{Mod}_{\mathcal{A}(\mathcal{Z})})} \to \mathcal{F}_{\mathbb{I}_{F(\mathcal{Z})}}.$$

Similarly, given $[\phi_{12}] \in MCG_0(\mathcal{A}(\mathcal{Z}_1), \mathcal{A}(\mathcal{Z}_2))$ and $[\phi_{23}] \in MCG_0(\mathcal{A}(\mathcal{Z}_2), \mathcal{A}(\mathcal{Z}_3))$, let ϕ_{13} denote the chosen representative of $[\phi_{23} \circ \phi_{12}] \in MCG(\mathcal{A}(\mathcal{Z}_1), \mathcal{A}(\mathcal{Z}_3))$. Then, by the pairing theorem (Theorem 12) and Lemma 8.5 there is a unique isomorphism (in the derived category)

$$\Phi_{123}\colon \widehat{CFDA}(\phi_{12}) \boxtimes \widehat{CFDA}(\phi_{23}) \to \widehat{CFDA}(\phi_{13}).$$

This isomorphism induces a natural transformation

$$A_{\mathcal{Z}_1,\mathcal{Z}_2,\mathcal{Z}_3}([\phi_{23}],[\phi_{12}])\colon \mathcal{F}_{\phi_{23}} \circ \mathcal{F}_{\phi_{12}} \to \mathcal{F}_{\phi_{13}}.$$

This completes the definition of the weak 2–functor A. It remains to check that A satisfies the three commutative diagrams of Definition 8.3. But these follow trivially from Lemma 8.5: both paths around the diagram correspond to graded isomorphisms between a bimodule M and a bimodule of the form $\widehat{CFDA}(\phi)$, and that lemma guarantees that there is a unique such isomorphism, so the diagrams must commute.

Finally, it is immediate from the definitions that these functors preserve the subcategories $\mathsf{H}(\widetilde{\mathsf{Mod}}_{\mathcal{A}(\mathcal{Z}),G(\mathcal{Z})})$, and act by triangulated functors on them. □

Remark 8.6 We have constructed an action of the mapping class group action on the module category. The reader might be interested in its behavior on grading sets. Specifically, if $\phi \in MCG_0(F(\mathcal{Z}), F(\mathcal{Z}))$ is a strongly based mapping class, then one can show that the grading set for $\widehat{CFDA}(\phi, \mathfrak{s})$ is a left-right $G(\mathcal{Z})$–$G(\mathcal{Z})$–set, for which both the left and the right actions are simply transitive. In turn, it is easy to see that isomorphism classes of such $G(\mathcal{Z})$–$G(\mathcal{Z})$–sets are in one-to-one correspondence with outer automorphisms of $G(\mathcal{Z})$ (fixing λ). Thus, we obtain a representation $MCG_0(F(\mathcal{Z}), F(\mathcal{Z})) \to \mathrm{Out}(G(\mathcal{Z}))$. Projecting onto $H_1(F(\mathcal{Z}))$, of course, we get the induced representation of the mapping class group on homology. But in fact, the entire representation is also determined by its behavior on homology. For more on this, see [20]; see also Lemma 10.3 for an explicit action in the case where $g = 1$.

9 Duality

In this section we will deduce Theorem 6 from the fact that $\widehat{CFAA}(\mathbb{I})$ and $\widehat{CFDD}(\mathbb{I})$ are quasi-inverses to each other (Definition 2.4.7) (which in turn follows from the pairing theorem (Theorem 5) and the fact that $\widehat{CFDA}(\mathbb{I}) \simeq {}^{\mathcal{A}}[\mathbb{I}]_{\mathcal{A}}$ (Theorem 4)) and the algebraic results from Section 2 (particularly Section 2.3.4). We give a genus-1 illustration of Theorem 6 in Section 10.3.

To keep notation simple, fix a pointed matched circle $\mathcal{Z}$ and let $\mathcal{A} = \mathcal{A}(\mathcal{Z})$ and $\mathcal{B} = \mathcal{A}(-\mathcal{Z})$.

Recall from Section 2.3.3 that given a type D structure ${}^{\mathcal{A}}M$ we let ${}_{\mathcal{A}}M = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}M$ denote the corresponding type A module; and similarly for bimodules.

Lemma 9.1 *Let $\mathcal{C}$ and $\mathcal{E}$ be $\mathcal{A}_\infty$–algebras. For any bimodules ${}_{\mathcal{E}}^{\mathcal{B}}M$ and ${}_{\mathcal{C}}^{\mathcal{B}}N$, the "tensoring with the identity map" morphism (see Proposition 2.3.32)*

$$\mathrm{Mor}^{\mathcal{B}}({}_{\mathcal{E}}^{\mathcal{B}}M, {}_{\mathcal{C}}^{\mathcal{B}}N) \to \mathrm{Mor}_{\mathcal{A}}(\widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}} \boxtimes {}_{\mathcal{E}}^{\mathcal{B}}M, \widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}} \boxtimes {}_{\mathcal{C}}^{\mathcal{B}}N)$$

is a quasi-isomorphism of $(\mathcal{C},\mathcal{E})$–bimodules.

Proof The fact that this map respects the bimodule structure is immediate from Proposition 2.3.36.

As noted above, by Theorems 4 and 12, it is immediate that the bimodules ${}^{\mathcal{B},\mathcal{A}}\widehat{CFDD}(\mathbb{I})$ and $\widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}}$ are quasi-inverses to each other, as in Definition 2.4.7. It follows from Lemma 2.4.9 that the functors

$$\widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}} \boxtimes {}_{\mathcal{B}} \cdot \colon \mathsf{H}_*({}^{\mathcal{B}}\mathsf{Mod}) \to \mathsf{H}_*(\mathsf{Mod}_{\mathcal{A}}),$$
$$\cdot_{\mathcal{A}} \boxtimes {}^{\mathcal{A},\mathcal{B}}\widehat{CFDD}(\mathbb{I})\colon \mathsf{H}_*(\mathsf{Mod}_{\mathcal{A}}) \to \mathsf{H}_*({}^{\mathcal{B}}\mathsf{Mod}),$$

are inverse equivalences of categories. But, on the level of morphisms, this says exactly that the "tensoring with the identity" map

$$\mathrm{Mor}^{\mathcal{B}}({}^{\mathcal{B}}M, {}^{\mathcal{B}}N) \to \mathrm{Mor}_{\mathcal{A}}(\widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}} \boxtimes {}^{\mathcal{B}}M, \widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}} \boxtimes {}^{\mathcal{B}}N)$$

is a quasi-isomorphism. □

Proposition 9.2 *There is a quasi-isomorphism of bimodules*

$$\mathrm{Mor}^{\mathcal{B}}({}_{\mathcal{A}}^{\mathcal{B}}\widehat{CFDD}(\mathbb{I}), {}^{\mathcal{B}}[\mathbb{I}]_{\mathcal{B}}) \simeq \widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}}.$$

Proof There are quasi-isomorphisms of G–set graded $(\mathcal{B},\mathcal{A})$–bimodules

$$\begin{aligned}\mathrm{Mor}^{\mathcal{B}}({}_{\mathcal{A}}^{\mathcal{B}}\widehat{CFDD}(\mathbb{I}), {}^{\mathcal{B}}[\mathbb{I}]_{\mathcal{B}}) &\to \mathrm{Mor}_{\mathcal{A}}(\widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}} \boxtimes_{\mathcal{B}} {}_{\mathcal{A}}^{\mathcal{B}}\widehat{CFDD}(\mathbb{I}), \widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}})\\ &\to \mathrm{Mor}_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, \widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}})\\ &\to \widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}}.\end{aligned}$$

Here, the first map is induced by tensoring with the identity, and is a quasi-isomorphism by Lemma 9.1. The second map is induced by naturality of Mor and is a quasi-isomorphism because $\widehat{CFDD}$ and $\widehat{CFAA}$ are quasi-inverses. The third map comes from Lemma 2.3.33. Note that all of these maps respect the G–set grading. □

We are now in a position to prove Theorem 6, which we restate and generalize as follows:

Theorem 16 *Let Y be a bordered three-manifold with boundary parameterized by $F(\mathcal{Z})$. Then*

$$\begin{aligned}\widehat{CFA}(Y)_{\mathcal{A}(\mathcal{Z})} &\simeq \mathrm{Mor}_{\mathcal{A}(-\mathcal{Z})}({}_{\mathcal{A}(\mathcal{Z}),\mathcal{A}(-\mathcal{Z})}\widehat{CFDD}(\mathbb{I}), {}_{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y)) \\ &\simeq \mathrm{Mor}^{\mathcal{A}(-\mathcal{Z})}({}^{\mathcal{A}(-\mathcal{Z})}_{\mathcal{A}(\mathcal{Z})}\widehat{CFDD}(\mathbb{I}), {}^{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y)).\end{aligned}$$

Similarly, suppose Y_{12} is a strongly bordered three-manifold with boundary parameterized by $F(\mathcal{Z}_1)$ and $F(\mathcal{Z}_2)$. Then

$$\begin{aligned}&{}^{\mathcal{A}(-\mathcal{Z}_1)}\widehat{CFDA}(Y_{12})_{\mathcal{A}(\mathcal{Z}_2)} \\ &\qquad\simeq \mathrm{Mor}^{\mathcal{A}(-\mathcal{Z}_2)}\big({}^{\mathcal{A}(-\mathcal{Z}_2)}_{\mathcal{A}(\mathcal{Z}_2)}\widehat{CFDD}(\mathbb{I}_{\mathcal{Z}_2}), {}^{\mathcal{A}(-\mathcal{Z}_1),\mathcal{A}(-\mathcal{Z}_2)}\widehat{CFDD}(Y_{12})\big).\end{aligned}$$

Proof As before, let $\mathcal{A}=\mathcal{A}(\mathcal{Z})$ and $\mathcal{B}=\mathcal{A}(-\mathcal{Z})$. By Proposition 2.3.40,

$$\mathrm{Mor}^{\mathcal{B}}({}^{\mathcal{B}}_{\mathcal{A}}\widehat{CFDD}(\mathbb{I}), {}^{\mathcal{B}}\widehat{CFD}(Y)) \simeq \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{A},\mathcal{B}}\widehat{CFDD}(\mathbb{I}), {}_{\mathcal{B}}\widehat{CFD}(Y)).$$

By definition (Equation (2.3.39)), Proposition 9.2, and Theorem 11 respectively,

$$\begin{aligned}\mathrm{Mor}^{\mathcal{B}}({}^{\mathcal{B}}_{\mathcal{A}}\widehat{CFDD}(\mathbb{I}), {}^{\mathcal{B}}\widehat{CFD}(Y)) &\simeq \mathrm{Mor}^{\mathcal{B}}({}^{\mathcal{B}}_{\mathcal{A}}\widehat{CFDD}(\mathbb{I}), {}^{\mathcal{B}}[\mathbb{I}]_{\mathcal{B}}) \boxtimes {}^{\mathcal{B}}\widehat{CFD}(Y) \\ &\simeq \widehat{CFAA}(\mathbb{I})_{\mathcal{A},\mathcal{B}} \boxtimes {}^{\mathcal{B}}\widehat{CFD}(Y) \\ &\simeq \widehat{CFA}(Y)_{\mathcal{A}}.\end{aligned}$$

These isomorphisms are all maps of G–set graded modules.

The bimodule case is similar. □

Remark 9.3 One might imagine that

$$\widehat{CFA}(Y)_{\mathcal{A}} \simeq \mathrm{Mor}^{\mathcal{B}}({}^{\mathcal{A},\mathcal{B}}\widehat{CFDD}(\mathbb{I}), {}^{\mathcal{B}}\widehat{CFD}(Y)) \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}.$$

This is in fact the case. However, the most obvious analogue for bimodules is false. See the authors [22].

Remark 9.4 Like tensoring with $\widehat{CFDD}(\mathbb{I})$ or $\widehat{CFAA}(\mathbb{I})$, the duality exchanges the actions of $\mathcal{A}(F,i)$ and $\mathcal{A}(-F,-i)$. (Since modules associated to manifolds with connected boundary are supported in $i=0$, this reversal of spin^c–structures is invisible in Theorem 6.)

Of course, the above discussion is much more useful once one calculates $\widehat{CFDD}(\mathbb{I})$.

Remark 9.5 Consider the type *DD* structure B which is generated by elements of the form $I(r)\otimes I(s)$, where here $r,s\subset[2k]$ are complementary sets of subsets of $[2k]=[4k]/M$, for a given pointed matched circle. Let R be the set of Reeb chords for $\mathcal{Z}$. For $\rho\in R$, let $a(\rho)\in\mathcal{A}(\mathcal{Z})$ denote the algebra element associated to ρ, and let $a(-\rho)\in\mathcal{A}(-\mathcal{Z})$ denote the corresponding algebra element for the pointed matched circle with the opposite orientation. We endow B with a differential which is given by

$$\delta^1(I(r)\otimes I(s))=\sum_{\substack{\rho\in R\\ I(r)a(\rho)=a(\rho)I(r')\\ (r',s')\text{ complementary}}}(a(\rho)\otimes a(-\rho))\otimes(I(r')\otimes I(s')).$$

It is shown in [20] that the above explicitly-defined bimodule B is in fact quasi-isomorphic to the bimodule $\widehat{CFDD}(\mathbb{I})$. (See also Proposition 10.1 for a verification of this in the genus-one case.)

It is interesting to compare the bimodule of Remark 9.5 with the dualizing modules in the theory of Koszul algebras. See for example [32].

10 Bimodules for the torus

In [21, Appendix A], we stated various bimodules for the torus. These included the bimodules $\widehat{CFDD}$ and $\widehat{CFAA}$ for the identity cobordism, and also the type *DA* bimodules for Dehn twists along generators for the mapping class group. In this section we verify those claims. Bimodules for generators of the mapping class groupoid of a surface with arbitrary genus, given using a different mechanism, are given in [20].

We use the notation for the torus algebra from Section 3.3 (mostly from [21]). In Section 10.1, we calculate the *AA* and *DD* bimodules for the identity map. (The same techniques can be used to calculate the *DA* bimodule of the identity map: we do not bother with this, in view of Theorem 4, which ensures that it is simply the identity bimodule. It is worth noting, though, that a direct calculation along the lines of Section 10.1 would result in a different, though quasi-isomorphic, bimodule.) In Section 10.2 we calculate the type *DA* bimodules which represent the mapping class group generators in genus-one. We conclude by giving an illustration of the duality theorem (Theorem 6) in Section 10.3.

Note that we focus here on the case $\mathcal{A} = \mathcal{A}(F, 0)$. This is because $\mathcal{A}(F, -1) \cong \mathbb{F}_2$, and $\mathcal{A}(F, 1)$ is quasi-isomorphic to $\mathbb{F}_2$, so all quasi-invertible bimodules over these algebras are quasi-isomorphic to $\mathbb{F}_2$.

10.1 Bimodules for the identity map of the torus

Consider the unique pointed matched circle $\mathcal{Z}$ for a surface of genus 1. In the present section, we describe the bimodule $\widehat{CFAA}(\mathbb{I}, 0)$, but first, we must set up some notation. Recall that $\widehat{CFAA}(\mathbb{I}, 0)$ is a right-right $\mathcal{A}(\mathcal{Z}, 0)$–$\mathcal{A}(-\mathcal{Z}, 0)$–bimodule. We write $\mathcal{A} = \mathcal{A}(\mathcal{Z}, 0)$, and $\mathcal{B} = \mathcal{A}(-\mathcal{Z}, 0)$. Of course $\mathcal{A} \cong \mathcal{B}$, but we still find it convenient to distinguish them, to help record which actions we are using. Specifically, we think of $\mathcal{B}$ as having idempotents which we denote j_0 and j_1 (corresponding to ι_0 and ι_1 in $\mathcal{A}$), and generators σ_i (corresponding to the ρ_i in $\mathcal{A}$). With these conventions, then, we claim that the *AA* bimodule for the identity diffeomorphism of the torus (in the $i = 0$ summand) is the *AA* bimodule illustrated in Figure 21. The idempotent actions on the generators can be easily determined from the actions on the algebra (or indeed they can be read immediately off the Heegaard diagram in Figure 22). We include only two of them here and leave the others to the reader:

$$\boldsymbol{x} \cdot (\iota_0 \otimes j_0) = \boldsymbol{x}, \quad \boldsymbol{y} \cdot (\iota_1 \otimes j_1) = \boldsymbol{y}.$$

Note that one can contract arrows to reduce to a quasi-isomorphic bimodule with only two generators; however, in that model, the $\mathcal{A}_\infty$ operations look rather complicated. (In particular, there are infinitely many different ones, ie the module is not operationally bounded in the sense of Definition 2.2.41.)

We verify our description of the *AA* bimodule for the identity by drawing a suitable Heegaard diagram and analyzing the holomorphic curves. Unfortunately, the canonical Heegaard diagram for the identity map given by Definition 5.10 is not admissible. This means that, although there are relatively few generators, we could have infinitely many nontrivial $\mathcal{A}_\infty$–products (and hence infinitely many domains to consider). To simplify matters, then, we apply some finger moves to get an admissible diagram $\mathcal{H}$, as illustrated in Figure 22.

The diagram $\mathcal{H}$ has six generators: $\boldsymbol{y} = ae$, $\boldsymbol{x} = bd$, $\boldsymbol{w}_1 = bf_1$, $\boldsymbol{w}_2 = bf_2$, $\boldsymbol{z}_1 = c_1e$, and $\boldsymbol{z}_2 = c_2e$. These can be connected by the domains labeled in Figure 22 to form the graph shown in Figure 23.

This graph has the property that any homotopy class connecting two generators can be realized as a path in the graph. Indeed, any positive domain connecting two generators

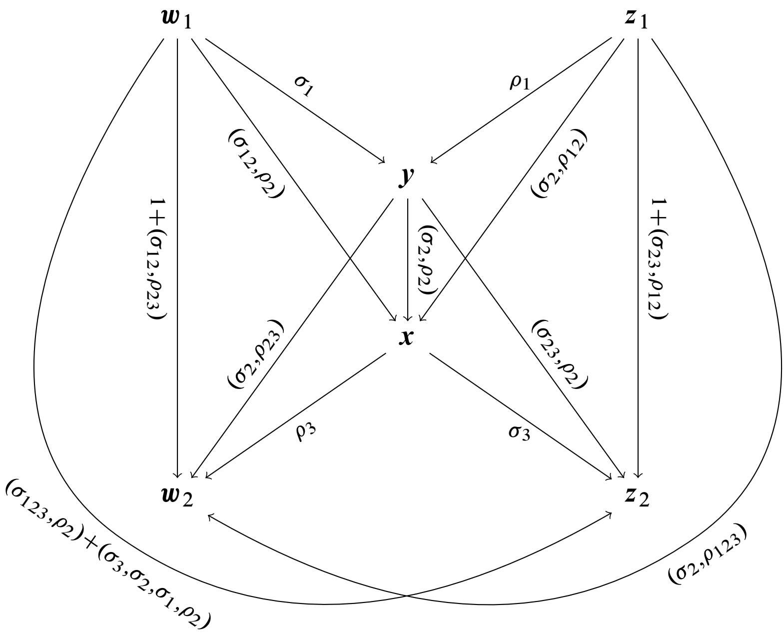


Figure 21: The type *AA* bimodule $\widehat{CFAA}(\mathbb{I},0)$: the labels on the arrows indicate the $\mathcal{A}_\infty$ operations. For instance, the label (σ_{23},ρ_2) on the arrow from $\boldsymbol{y}$ to $\boldsymbol{z}_2$ means that $m(\boldsymbol{y},\sigma_{23},\rho_2)$ contains a term $\boldsymbol{z}_2$.

can be represented by a connected path of downward-pointing edges. For example, there are two positive domains from $\boldsymbol{w}_1$ to $\boldsymbol{w}_2$, and these are B_2 and the composite domain S_1MR_3.

The regions corresponding to the various domains from Figure 23 represent polygons. It follows that all the $\mathcal{A}_\infty$ operations from Figure 21 which drop height by one are given by the corresponding labels; ie

$$m(\boldsymbol{w}_1,\sigma_1)=\boldsymbol{y},\quad m(\boldsymbol{z}_1,\rho_1)=\boldsymbol{y},\quad m(\boldsymbol{y},\sigma_2,\rho_2)=\boldsymbol{x},\quad m(\boldsymbol{x},\rho_3)=\boldsymbol{w}_2,$$
$$m(\boldsymbol{x},\sigma_3)=\boldsymbol{z}_2,\qquad m(\boldsymbol{w}_1)=\boldsymbol{w}_2,\qquad m(\boldsymbol{z}_1)=\boldsymbol{z}_2.$$

The $\mathcal{A}_\infty$ relation for a type *AA* bimodule applied to the $\boldsymbol{x}$ components of $m^2(\boldsymbol{w}_1\otimes\sigma_1\otimes\sigma_2\otimes\rho_2)$ and $m^2(\boldsymbol{z}_1\otimes\sigma_2\otimes\rho_1\otimes\rho_2)$ forces the arrows from $\boldsymbol{w}_1$ and $\boldsymbol{z}_1$ to $\boldsymbol{x}$:

$$m(\boldsymbol{w}_1,\sigma_{12},\rho_2)=\boldsymbol{x},$$
$$m(\boldsymbol{z}_1,\sigma_2,\rho_{12})=\boldsymbol{x}.$$

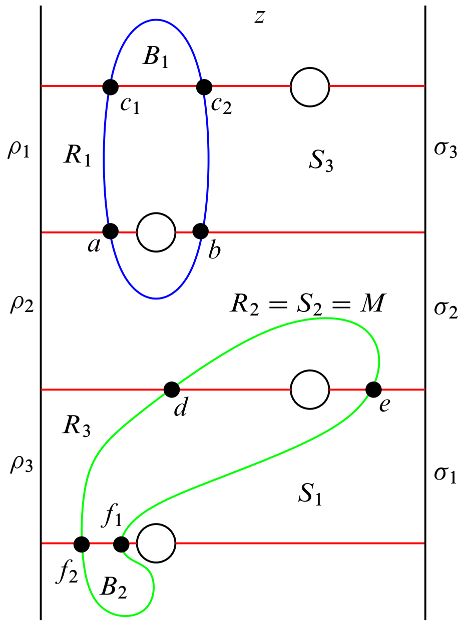


Figure 22: Admissible Heegaard diagram for the identity map of the solid torus: there are two handles attached to the surface, connecting the circles that are vertically aligned with each other.

$$\begin{tikzcd}
\boldsymbol{w}_1 \arrow[dr, "S_1"] \arrow[ddd, "B_2"] & & \boldsymbol{z}_1 \arrow[dl, "R_1"'] \arrow[ddd, "B_1"] \\
& \boldsymbol{y} \arrow[d, "M"] & \\
& \boldsymbol{x} \arrow[dl, "R_3"'] \arrow[dr, "S_3"] & \\
\boldsymbol{w}_2 & & \boldsymbol{z}_2
\end{tikzcd}$$

Figure 23: Graph of domains

Similar considerations force the arrows from $\boldsymbol{y}$ to $\boldsymbol{w}_2$ and $\boldsymbol{z}_2$, from $\boldsymbol{z}_1$ to $\boldsymbol{z}_2$, and from $\boldsymbol{w}_1$ to $\boldsymbol{w}_2$:

$$m(\boldsymbol{y}, \sigma_2, \rho_{23}) = \boldsymbol{w}_2, \qquad m(\boldsymbol{y}, \sigma_{23}, \rho_2) = \boldsymbol{z}_2,$$

$$m(\boldsymbol{w}_1, \sigma_{12}, \rho_{23}) = \boldsymbol{w}_2, \quad m(\boldsymbol{z}_1, \sigma_{23}, \rho_{12}) = \boldsymbol{z}_2.$$

Considering the $\boldsymbol{w}_2$ and $\boldsymbol{z}_2$ components of $m^2(\boldsymbol{z}_1 \otimes \sigma_2 \otimes \rho_1 \otimes \rho_{23})$ and $m^2(\boldsymbol{w}_1 \otimes \sigma_1 \otimes \sigma_{23} \otimes \rho_2)$ respectively, we see that

$$m(\boldsymbol{z}_1, \sigma_2, \rho_{123}) = \boldsymbol{w}_2,$$
$$m(\boldsymbol{w}_1, \sigma_{123}, \rho_2) = \boldsymbol{z}_2.$$

We must finally consider the possibility of alternative higher multiplications supported by the positive domains we have found. For example, the domain $S_1 M S_3$, as we have already seen, represents $m(\boldsymbol{w}_1, \sigma_2, \rho_{123}) = \boldsymbol{z}_2$, but has an alternative decomposition, where we cut to the right wherever possible. This gives rise to a polygon representing

$$m(\boldsymbol{w}_1, \sigma_3, \sigma_2, \sigma_1, \rho_2) = \boldsymbol{z}_2.$$

Inspecting the existing positive domains, it is clear that no other one can give rise to an alternate $\mathcal{A}_\infty$ operation, as corresponding cuts are not possible.

This concludes the verification that the identity map gives rise to the bimodule pictured in Figure 21. Having found all the holomorphic curves for the *AA* bimodule, it is easy to write down also the type *DD* bimodule. To do this, we proceed as follows: we relabel the algebra elements on the regions to be compatible with type D labelings (ie swap the order of the generators on the boundary), then throw out some curves which cannot contribute for idempotent reasons, and finally add up other curve counts.

For instance, a curve which used to count as $m(\boldsymbol{w}_1, \sigma_1) = \boldsymbol{y}$ now counts as giving a term of $\sigma_3 \otimes \boldsymbol{y}$ in $\partial \boldsymbol{w}_1$. Also, the curve which used to count as $m(\boldsymbol{w}_1, \sigma_{12}, \rho_2) = \boldsymbol{x}$ now does not count. Finally, the contributions

$$m(\boldsymbol{w}_1, \sigma_{123}, \rho_2) = \boldsymbol{z}_2 \quad \text{and} \quad m(\boldsymbol{w}_1, \sigma_3, \sigma_2, \sigma_1, \rho_2) = \boldsymbol{z}_2$$

both contribute to the coefficient of $(\sigma_{123}\rho_2) \otimes \boldsymbol{z}_2$ in $\partial \boldsymbol{w}_1$ (and hence they cancel). The results are summarized in Figure 24.

We can simplify this further, to obtain the following:

Proposition 10.1 *There is a type DD identity bimodule for the torus with two generators $\boldsymbol{p}$ and $\boldsymbol{q}$, satisfying*

$$(\iota_0 \otimes j_0) \cdot \boldsymbol{p} = \boldsymbol{p}, \quad (\iota_1 \otimes j_1) \cdot \boldsymbol{q} = \boldsymbol{q},$$

and differential given by

$$\partial \boldsymbol{p} = (\rho_1\sigma_3 + \rho_3\sigma_1 + \rho_{123}\sigma_{123}) \otimes \boldsymbol{q}, \quad \partial \boldsymbol{q} = (\rho_2\sigma_2) \otimes \boldsymbol{p}.$$

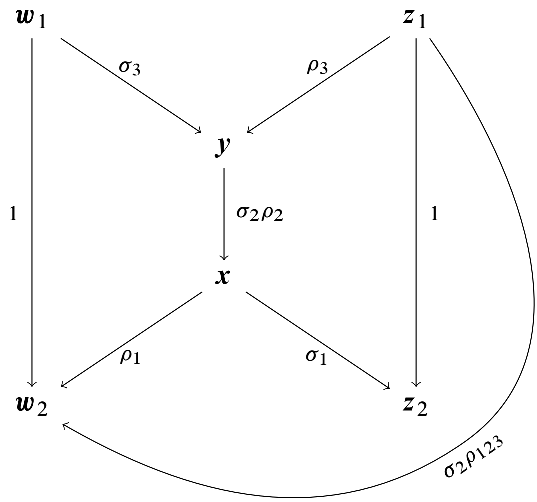


Figure 24: The type *DD* bimodule $\mathit{CFDD}^\wedge(\mathbb{I},0)$: the labels on the arrows indicate differentials. For instance, the arrow from $\boldsymbol{y}$ to $\boldsymbol{x}$ signifies a term of $(\rho_2\sigma_2)\otimes\boldsymbol{x}$ in $\partial\boldsymbol{y}$.

Proof Substitute

$$\boldsymbol{p} = \boldsymbol{x} + \rho_1\otimes\boldsymbol{w}_1 + \sigma_1\otimes\boldsymbol{z}_1 + \sigma_{12}\rho_{123}\otimes\boldsymbol{w}_1,$$
$$\boldsymbol{q} = \boldsymbol{y},$$

in the description from Figure 24 to get a quasi-isomorphic submodule with the stated differential. The idempotent actions follow immediately from the diagram. (Note that the idempotents are the same as those arising from the interpretation of the generators as generators for a type *AA* bimodule.) □

The type *DD* module given in Figure 24 is bounded in the sense of Definition 2.2.56, while the module given in Proposition 10.1 is merely left and right bounded.

Determining relative gradings is also a straightforward matter, as in Lemma 8.3.

10.2 The *DA* bimodules for the mapping class group of the torus

The mapping class group is generated by Dehn twists τ_m and τ_ℓ along meridian and longitude respectively (ie τ_m takes an n–framed knot complement to an $(n+1)$–framed knot complement). We describe the summand $\widehat{\mathit{CFDA}}(\,\cdot\,,0)$ of the type *DA* modules for Dehn twists about these two curves and their inverses.

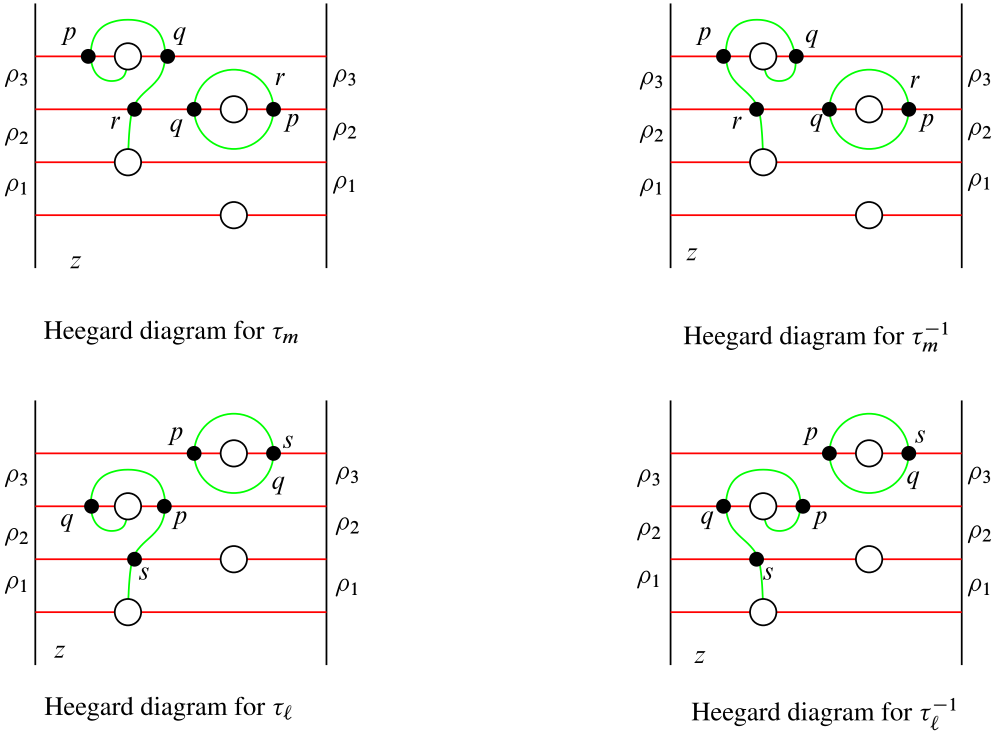


Figure 25: Heegaard diagrams for mapping class group elements: we have illustrated here genus-two bordered diagrams for the generators of the genus-one mapping class group and their inverses, as indicated. In each of the four diagrams, there are three generators in the $i=0$ summand.

Heegaard diagrams for τ_m, τ_m^{-1}, τ_ℓ and τ_ℓ^{-1} are illustrated in Figure 25. Each of the four type *DA* structures $\widehat{CFDA}(\tau_m, 0)$, $\widehat{CFDA}(\tau_m^{-1}, 0)$, $\widehat{CFDA}(\tau_\ell, 0)$, and $\widehat{CFDA}(\tau_\ell^{-1}, 0)$ has three generators, which are labeled by four possible letters $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{s}$. ($\widehat{CFDA}(\tau_m^{\pm 1})$ does not have $\boldsymbol{s}$ and $\widehat{CFDA}(\tau_\ell^{\pm 1})$ does not have $\boldsymbol{r}$.) The compatibility with the idempotents is given as follows:

$$\iota_0 \cdot \boldsymbol{p} \cdot \iota_0 = \boldsymbol{p}, \quad \iota_1 \cdot \boldsymbol{q} \cdot \iota_1 = \boldsymbol{q}, \quad \iota_1 \cdot \boldsymbol{r} \cdot \iota_0 = \boldsymbol{r}, \quad \iota_0 \cdot \boldsymbol{s} \cdot \iota_1 = \boldsymbol{s}.$$

Next, we study the grading set for the modules. We will adopt the notation for elements of G' from [21, Chapter 11]; that is, elements of $G'(T^2)$ are written as tuples $g = (m; i, j, k)$ where $m \in \frac{1}{2}\mathbb{Z}$ is the Maslov component of g and i, j, k are the local multiplicities of $[g] \in H_1(\mathcal{Z} \setminus z, \boldsymbol{a})$ at the three relevant components of $\mathcal{Z} \setminus \boldsymbol{a}$; see Section 3.3. Since G is a subgroup of G', this gives us notation for elements of G as well.

Definition 10.1 Let $f\colon G\to G$ be a group homomorphism. Let G_f be the associated left-right G–G–space whose underlying set is G, and whose action is given by $g_1\star s\ast g_2 = g_1\cdot s\cdot f(g_2)$, where $\star$ denotes the left action on G_f, $\ast$ denotes the right action on G_f, and $\cdot$ denotes multiplication in G.

With this definition, if $G_f\cong G_g$ as left-right G–G–spaces, then f and g are conjugate to one another.

Recall that the actions of the generating Dehn twists on homology is given by

$$\begin{aligned}
(\tau_m)_*(m) &= m, & (\tau_m)_*(\ell) &= m+\ell,\\
(\tau_m^{-1})_*(m) &= m, & (\tau_m^{-1})_*(\ell) &= -m+\ell,\\
(\tau_\ell)_*(m) &= m-\ell, & (\tau_\ell)_*(\ell) &= \ell,\\
(\tau_\ell^{-1})_*(m) &= m+\ell, & (\tau_\ell^{-1})_*(\ell) &= \ell.
\end{aligned}$$

Here, we are thinking of the homology classes m and ℓ as represented by local multiplicities

$$m=(0,1,1),\quad \ell=(1,1,0).$$

These have canonical lifts to elements in G, gotten by $\widetilde{m}=\mathrm{gr}(\rho_{23})$ and $\widetilde{\ell}=\mathrm{gr}(\rho_{12})$; ie (using the grading refinement data from (3.1)), we get

$$\widetilde{m}=(\tfrac{1}{2};0,1,1),\quad \widetilde{\ell}=(-\tfrac{1}{2};1,1,0)$$

We define the following lifts of the action of homology to automorphisms of $G(\mathcal{Z})$: f_m, f_m^{-1}, f_ℓ, and f_ℓ^{-1}. These are determined by the property that they fix λ, and transform the other generators according to the following:

$$\begin{aligned}
f^{\tau_m}(\widetilde{m}) &= \widetilde{m}, & f^{\tau_m}(\widetilde{\ell}) &= \lambda\cdot\widetilde{m}\cdot\widetilde{\ell},\\
f^{\tau_m^{-1}}(\widetilde{m}) &= \widetilde{m}, & f^{\tau_m^{-1}}(\widetilde{\ell}) &= \lambda^{-1}\cdot\widetilde{m}^{-1}\cdot\widetilde{\ell},\\
f^{\tau_\ell}(\widetilde{m}) &= \lambda^{-1}\cdot\widetilde{m}\cdot\widetilde{\ell}^{-1}, & f^{\tau_\ell}(\widetilde{\ell}) &= \widetilde{\ell},\\
f^{\tau_\ell^{-1}}(\widetilde{m}) &= \lambda\cdot\widetilde{m}\cdot\widetilde{\ell}, & f^{\tau_\ell^{-1}}(\widetilde{\ell}) &= \widetilde{\ell}.
\end{aligned}\tag{10.2}$$

Lemma 10.3 *Let $\phi\in\{\tau_m,\tau_m^{-1},\tau_\ell,\tau_\ell^{-1}\}$. For the grading refinement data of (3.1), if we base our grading sets around $\boldsymbol{p}$, the intersection point in the ι_0 idempotent on both the left and the right, then the grading set for $\widehat{CFDA}(\phi)$ is identified with G_{f^ϕ}, where f^ϕ is the homomorphism corresponding to ϕ as described in (10.2).*

With respect to these identifications of grading sets, we find

$$\begin{aligned}
\mathrm{gr}_m(\boldsymbol{p}) &= (0;0,0,0), & \mathrm{gr}_m(\boldsymbol{q}) &= (0;0,0,0), & \mathrm{gr}_m(\boldsymbol{r}) &= (\tfrac{1}{2};1,1,0),\\
\mathrm{gr}_{m^{-1}}(\boldsymbol{p}) &= (0;0,0,0), & \mathrm{gr}_{m^{-1}}(\boldsymbol{q}) &= (0;0,0,0), & \mathrm{gr}_{m^{-1}}(\boldsymbol{r}) &= (-\tfrac{1}{2};-1,-1,0),\\
\mathrm{gr}_{\ell}(\boldsymbol{p}) &= (0;0,0,0), & \mathrm{gr}_{\ell}(\boldsymbol{q}) &= (0;0,0,0), & \mathrm{gr}_{\ell}(\boldsymbol{s}) &= (\tfrac{1}{2};1,1,0),\\
\mathrm{gr}_{\ell^{-1}}(\boldsymbol{p}) &= (0;0,0,0), & \mathrm{gr}_{\ell^{-1}}(\boldsymbol{q}) &= (0;0,0,0), & \mathrm{gr}_{\ell^{-1}}(\boldsymbol{s}) &= (-\tfrac{1}{2};-1,-1,0),
\end{aligned}$$

where the subscript on gr *indicates which diagram we are considering.*

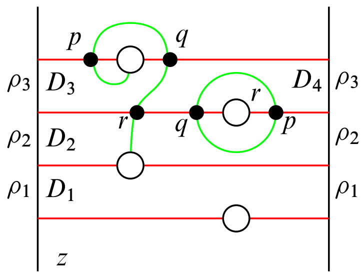


Figure 26: Heegaard diagram for τ_m, with labeled domains

Proof We describe in detail the calculations in the case where $\phi = \tau_m$ (and consider $\mathrm{gr} = \mathrm{gr}_m$). (The other cases are entirely parallel.) A basis for the space of periodic domains $\pi_2(\boldsymbol{r},\boldsymbol{r})$ is given by the domains $D_2 + D_3 + D_4$ and $D_1 + D_2 - D_4$, with diagrams labeled as in Figure 26.

Let $\mathcal{Z}_L \amalg \mathcal{Z}_R$ be the boundary of the Heegaard diagram for ϕ. The grading set for the larger (G') grading takes its values in the quotient of $G'(-\mathcal{Z}_L) \times_\lambda G'(\mathcal{Z}_R)$ (viewed as a left-right $G'(-\mathcal{Z}_L)$–$G'(\mathcal{Z}_R)$–set) by relations coming from the periodic domains; specifically,

$$\begin{aligned}
\mathrm{gr}'(\boldsymbol{p}) &= (\tfrac{1}{2};0,-1,-1)\cdot \mathrm{gr}'(\boldsymbol{p})\cdot(-\tfrac{1}{2};0,1,1),\\
\mathrm{gr}'(\boldsymbol{p}) &= (\tfrac{1}{2};-1,-1,0)\cdot \mathrm{gr}'(\boldsymbol{p})\cdot(-1;1,0,-1).
\end{aligned}$$

The first of these equations comes from $D_2+D_3+D_4$, which has $e(D_2+D_3+D_4) = -2$ and $2n_{\boldsymbol{p}}(D_2+D_3+D_4) = 2$, $r_*(\partial^{\partial_L}(D_2+D_3+D_4)) = (0,-1,-1)$, $\partial^{\partial_R}(D_2+D_3+D_4) = (0,1,1)$. The second equation comes from $D_1+D_2-D_4$, which has $e(D_1+D_2-D_4) = -\frac{1}{2}$ and $2n_{\boldsymbol{p}}(D_1+D_2-D_4) = 1$.) According to (6.12) (see also Remark 3.14), we have $\mathrm{gr}(\boldsymbol{p}) = \psi(\iota_0)\cdot \mathrm{gr}'(\boldsymbol{p})\cdot \psi(\iota_0)^{-1}$; but since $\psi(\iota_0)$ is the identity, the above relations can be restated as

$$\mathrm{gr}(\boldsymbol{p}) = (\tfrac{1}{2}; 0, -1, -1) \cdot \mathrm{gr}(\boldsymbol{p}) \cdot (-\tfrac{1}{2}; 0, 1, 1),$$
$$\mathrm{gr}(\boldsymbol{p}) = (\tfrac{1}{2}; -1, -1, 0) \cdot \mathrm{gr}(\boldsymbol{p}) \cdot (-1; 1, 0, -1).$$

It is now easy to see that the map from the grading set to $G_{f^{\tau_m}}$ given by

$$g_1 \cdot \mathrm{gr}(\boldsymbol{p}) \cdot g_2 \mapsto g_1 \cdot f^{\tau_m}(g_2) \tag{10.4}$$

(which evidently sends $\mathrm{gr}(\boldsymbol{p})$ to $(0;0,0,0)$) is an isomorphism of left-right G'–G'–sets.

To calculate the grading of $\boldsymbol{r}$, for example, consider the domain D_2 from $\boldsymbol{r}$ to $\boldsymbol{p}$. This domain has $e(D_2) = 0$ and $n_{\boldsymbol{r}} + n_{\boldsymbol{p}} = 1/2$, and hence it gives the relation $\mathrm{gr}'(\boldsymbol{p}) = (-\tfrac{1}{2}; 0, -1, 0) \star \mathrm{gr}'(\boldsymbol{r})$. Since $\mathrm{gr}(\boldsymbol{r}) = \psi(\iota_1) \cdot \mathrm{gr}'(\boldsymbol{r}) \cdot \psi(\iota_0)^{-1}$, it follows that

$$\mathrm{gr}(\boldsymbol{r}) = (\tfrac{1}{2}; 1, 1, 0).$$

A similar calculation (only now using the domain D_1) shows $\mathrm{gr}(\boldsymbol{p}) = \mathrm{gr}(\boldsymbol{q})$. □

Next we turn to computing the explicit bimodules.

Proposition 10.5 *The type DA bimodules for τ_m, τ_m^{-1}, τ_ℓ and τ_ℓ^{-1} are as given in Figure 27.*

Proof We give the proof in detail for $\widehat{CFDA}(\tau_m)$. We enumerate domains which contribute to the type *DA* actions, organizing them by the algebra elements they contribute on the type *D* side. These algebra elements in turn are determined by how the domain meets $\partial_L \mathcal{H}$.

Algebra element 1 The only domain which is disjoint from $\partial_L \mathcal{H}$ is D_4. That represents a rectangle, which therefore contributes to δ^1. We denote this by writing

$$(\boldsymbol{r}, \rho_3) \xrightarrow{D_4} \boldsymbol{q},$$

to mean that $\boldsymbol{q}$ occurs in $\delta^1(\boldsymbol{r}, \rho_3)$.

Algebra element $\boldsymbol{\rho_1}$ There is only one valid domain which could contribute ρ_1, namely the domain D_1 itself. Indeed, it is a rectangle, starting at $\boldsymbol{p}$ and ending at $\boldsymbol{q}$. Therefore, $\delta^1(\boldsymbol{p}, \rho_1)$ contains $\rho_1 \boldsymbol{q}$, or graphically

$$(\boldsymbol{p}, \rho_1) \xrightarrow{D_1} \rho_1 \otimes \boldsymbol{q}.$$

Algebra element $\boldsymbol{\rho_2}$ The only domain which could contribute in this case is D_2; and that in turn is a bigon, giving

$$(\boldsymbol{r}) \xrightarrow{D_2} \rho_2 \otimes \boldsymbol{p}.$$

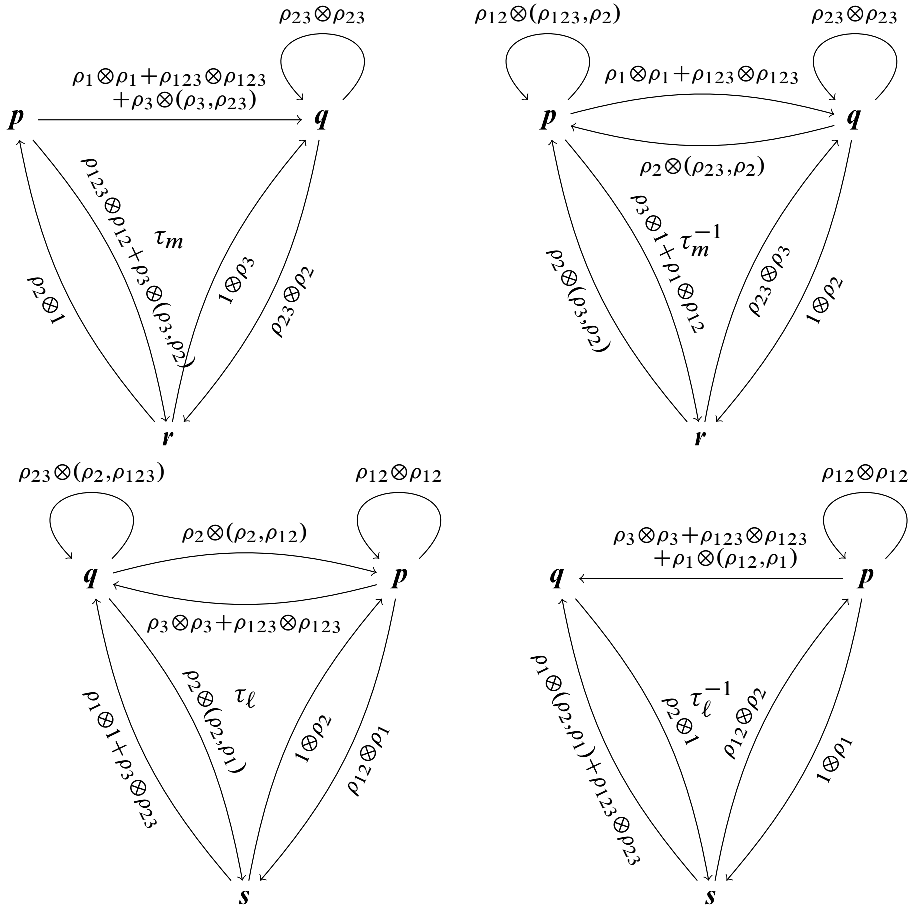


Figure 27: Type *DA* bimodules for torus mapping class group action. These are the module associated to τ_m, τ_m^{-1}, τ_ℓ, τ_ℓ^1 respectively. The notation is as follows. Consider the module for τ_m. The label $\rho_1\otimes\rho_1$ on the horizontal arrow indicates that $m(\boldsymbol{p},\rho_1)$ contains a term of the form $\rho_1\boldsymbol{q}$. Similarly, the label $\rho_3\otimes(\rho_3,\rho_{23})$ on that arrow indicates that $m(\boldsymbol{p},\rho_3,\rho_{23})$ contains a term of the form $\rho_3\boldsymbol{q}$.

Algebra element $\boldsymbol{\rho_3}$ There are only two domains which can contribute ρ_3 on the type D side, and these are D_3+D_4 and D_3+2D_4, which we consider in turn.

$\boldsymbol{D_3+D_4}$ This domain has two possible interpretations: either its input consists of $(\boldsymbol{p},\rho_2\rho_3)$ or $(\boldsymbol{p},\rho_3,\rho_2)$. But the first interpretation is not valid: bilinearity over the idempotents forces such a term to vanish. The second interpretation leads to considering the domain $D_3 D_4$, thought of as having a cut form $\boldsymbol{p}$ out to $\partial_R \mathcal{H}$. As such, it represents an annulus with a cut parameter at $\boldsymbol{r}$ (whose other endpoints go out to the other boundary component). Such an annulus is always represented by a holomorphic curve. Hence we have

$$(\boldsymbol{p}, \rho_3, \rho_2) \xrightarrow{D_3+D_4} \rho_3 \otimes \boldsymbol{r}.$$

$\boldsymbol{D_3 + 2D_4}$ There are three possible interpretations of this domain: $(\boldsymbol{p}, \rho_3, \rho_{23})$, $(\boldsymbol{p}, \rho_{23}, \rho_3)$ or $(\boldsymbol{p}, \rho_3, \rho_2, \rho_3)$. The second interpretation is impossible because of idempotents.

The third interpretation gives a moduli space whose expected dimension is nonzero; ie the Maslov index of the moduli space is wrong. This is neatly formulated in terms of the gradings calculated in Lemma 10.3. Specifically, substituting gradings calculated from that lemma, we see that

$$\begin{aligned}\lambda^{-1} \cdot \mathrm{gr}(\boldsymbol{p} \otimes \rho_3[1] \otimes \rho_2[1] \otimes \rho_3[1]) &= \lambda^2 \cdot \mathrm{gr}(\boldsymbol{p}) \cdot f^{\phi}(\mathrm{gr}(\rho_3) \cdot \mathrm{gr}(\rho_2) \cdot \mathrm{gr}(\rho_3)) \\ &= (1; -1, 0, 1).\end{aligned}$$

(Note that the notation $\rho_i[1]$ means the element $\rho_i \in \mathcal{A}$ with a shift in its grading.) This is different from

$$\mathrm{gr}(\rho_3 \otimes \boldsymbol{q}) = \mathrm{gr}(\rho_3) \cdot \mathrm{gr}(\boldsymbol{q}) = (0; -1, 0, 1).$$

The reader might be concerned that this calculation depends on several auxiliary choices, such as the choice of refinement data ψ and the identification of the grading set given in Lemma 10.3. However, the conclusion that

$$\lambda^{-1}\, \mathrm{gr}(\boldsymbol{p} \otimes \rho_3[1] \otimes \rho_2[1] \otimes \rho_3[1]) = \lambda \cdot \mathrm{gr}(\rho_3 \otimes \boldsymbol{q}),$$

which rules out the possibility that $\rho_3 \otimes \boldsymbol{q}$ appears in $m(\boldsymbol{p}, \rho_3, \rho_2, \rho_3)$, is independent of this choice (and indeed the exponent of λ appearing on the right-hand side here gives the dimension of the relevant moduli space.)

This leaves only the first possible interpretation

$$Q_1\colon (\boldsymbol{p}, \rho_3, \rho_{23}) \xrightarrow{D_3+2D_4} \rho_3 \otimes \boldsymbol{q}.$$

(The label Q here signifies that we have not (yet) determined that this contribution is indeed 1 (mod 2).) Consider the $\mathcal{A}_\infty$ relation with inputs $(\boldsymbol{p}, \rho_3, \rho_2, \rho_3)$ and output $\rho_3 \otimes \boldsymbol{q}$. The composite of

$$(\boldsymbol{p}, \rho_3, \rho_2) \xrightarrow{D_3+D_4} \rho_3 \otimes \boldsymbol{r}$$

with

$$(\boldsymbol{r}, \rho_3) \xrightarrow{D_4} \boldsymbol{q}$$

gives one nontrivial term in this relation; the only possible alternate contribution is gotten from the arrow Q_1 under consideration. Thus, we have verified the existence of

$$(\boldsymbol{p}, \rho_3, \rho_{23}) \xrightarrow{D_3+2D_4} \rho_3 \otimes \boldsymbol{q}.$$

Algebra element $\boldsymbol{\rho_{12}}$ The possible domain is $D_1 + D_2$, thought of as a map

$$Q_2\colon (\boldsymbol{r}, \rho_1) \xrightarrow{D_1+D_2} \rho_{12} \cdot \boldsymbol{q}.$$

But this is incompatible with the idempotents of $\boldsymbol{q}$.

Algebra element $\boldsymbol{\rho_{23}}$ The domains are D_2+D_3, $D_2+D_3+D_4$ and $D_2+D_3+2D_4$.

$\boldsymbol{D_2+D_3}$ This domain must have a cut out to $\partial_L \mathcal{H}$. After this cut is made, the domain is a rectangle, giving a contribution

$$(\boldsymbol{q}, \rho_2) \xrightarrow{D_2+D_3} \rho_{23} \otimes \boldsymbol{r}.$$

$\boldsymbol{D_2+D_3+D_4}$ This is a periodic domain, so could be interpreted to have initial and terminal generator either $\boldsymbol{p}$, $\boldsymbol{q}$, or $\boldsymbol{r}$. However, $\rho_{23} \otimes \boldsymbol{p}$ cannot appear as the target of a type *DA* action on $\boldsymbol{p}$: the left idempotent of $\boldsymbol{p}$ is ι_0, while the left idempotent of ρ_{23} is ι_1. Thus, we need consider only cases where the initial and terminal point are $\boldsymbol{q}$ or $\boldsymbol{r}$. We exclude first the latter case. Idempotents ensure that the only possible interpretation of $D_2 + D_3 + D_4$ as a nontrivial contribution to the type *DA* module with initial generator $\boldsymbol{r}$ is to think of it as a domain from $(\boldsymbol{r}, \rho_3, \rho_2)$ to $\rho_{23} \otimes \boldsymbol{r}$. But this is ruled out by gradings because

$$\begin{aligned}\lambda^{-1} \cdot \mathrm{gr}(\boldsymbol{r} \otimes \rho_3[1] \otimes \rho_2[1]) &= \lambda \cdot \mathrm{gr}(\boldsymbol{r}) \cdot f^{\phi}(\mathrm{gr}(\rho_3) \cdot \mathrm{gr}(\rho_2)) \\ &= (1; 1, 2, 1)),\end{aligned}$$

which is different from

$$\mathrm{gr}(\rho_{23}) \cdot \mathrm{gr}(\boldsymbol{r}) = (0; 1, 2, 1).$$

Thus the initial and terminal generator must be $\boldsymbol{q}$. Examining idempotents once again, we see that there are only two possible interpretations of this domain. One is as a domain from $(\boldsymbol{q}, \rho_2, \rho_3)$ to $\rho_{23} \otimes \boldsymbol{q}$. But this is ruled out by looking at gradings. The only remaining interpretation of the domain is as a map

$$Q_3\colon (\boldsymbol{q}, \rho_{23}) \xrightarrow{D_2+D_3+D_4} \rho_{23} \otimes \boldsymbol{q}.$$

We will now establish the existence of the contribution by Q_3, using the $\mathcal{A}_\infty$ relation, together with the information about the type *DA* bimodule which we have so far collected. Specifically, consider the $(\rho_{23} \otimes \boldsymbol{q})$–coefficient of the $\mathcal{A}_\infty$ relation with inputs $(\boldsymbol{q}, \rho_2, \rho_3)$. We know that $D_2 + D_3$ and D_4 give a nonzero contribution of $m(m(\boldsymbol{q}, \rho_2), \rho_3)$. Other terms in this $\mathcal{A}_\infty$ relation include $m(\boldsymbol{q}, \rho_{23})$, which is counted by Q_3. Possible other terms are $m(m(\boldsymbol{q}), \rho_2, \rho_3)$ and $m(m(\boldsymbol{q}, \rho_2, \rho_3))$. Looking back at those terms which contribute algebra elements 1, ρ_2, ρ_3 and ρ_{23}, we see there are no possible such terms. This forces the existence of

$$(\boldsymbol{q}, \rho_{23}) \xrightarrow{D_2+D_3+D_4} \rho_{23} \otimes \boldsymbol{q}.$$

$D_2 + D_3 + 2D_4$ This domain starts at $\boldsymbol{r}$ and terminates at $\boldsymbol{q}$. There are two possible interpretations of the domain to $\rho_{23} \otimes \boldsymbol{q}$: one starts at $(\boldsymbol{r}, \rho_3, \rho_{23})$ while the other starts at $(\boldsymbol{r}, \rho_3, \rho_2, \rho_3)$. Both possibilities are excluded by considering gradings:

$$\lambda^{-1}\,\mathrm{gr}(\boldsymbol{r} \otimes \rho_3[1] \otimes \rho_{23}[1]) = \lambda \cdot \mathrm{gr}(\rho_{23} \otimes \boldsymbol{q}),$$
$$\lambda^{-1}\,\mathrm{gr}(\boldsymbol{r} \otimes \rho_3[1] \otimes \rho_2[1] \otimes \rho_3[1]) = \lambda^2 \cdot \mathrm{gr}(\rho_{23} \otimes \boldsymbol{q}).$$

Algebra element $\boldsymbol{\rho_{123}}$ The domains now are $D_1+D_2+D_3$ and $D_1+D_2+D_3+D_4$.

$D_1 + D_2 + D_3 + D_4$, first visit There are two conceivable interpretations of this domain compatible with the idempotents: one with input $(\boldsymbol{p}, \rho_{123})$, and the other with input $(\boldsymbol{p}, \rho_3, \rho_2, \rho_1)$. The first is forced to exist by the $(\rho_{123} \otimes \boldsymbol{q})$–coefficient of the $\mathcal{A}_\infty$ relation with inputs $(\boldsymbol{p}, \rho_1, \rho_{23})$, since we have already verified that $m_2(m_2(\boldsymbol{p}, \rho_1), \rho_{23}) = \rho_{123} \otimes \boldsymbol{q}$. We will return to the second interpretation of $D_1 + D_2 + D_3 + D_4$; but first we turn to the easier analysis of $D_1 + D_2 + D_3$.

$D_1 + D_2 + D_3$ The initial point is $\boldsymbol{p}$ and the terminal point $\boldsymbol{r}$, and hence it follows readily that the only interpretation of this domain is as a map

$$Q_3\colon m(\boldsymbol{p}, \rho_{12}) \xrightarrow{D_1+D_2+D_3} \rho_{123} \otimes \boldsymbol{r}.$$

We verify the existence of this map, by considering the $\rho_{123} \otimes \boldsymbol{r}$–component of the $\mathcal{A}_\infty$ relation with inputs $(\boldsymbol{p}, \rho_{12}, \rho_3)$. One contribution to this is furnished by the domain $D_1+D_2+D_3+D_4$, interpreted as giving the operation $m_2(\boldsymbol{p}, \rho_{123}) = \rho_{123} \otimes \boldsymbol{r}$. The only conceivable alternate contribution is provided by the composite of the presently considered domain (Q_3) with

$$(\boldsymbol{r}, \rho_3) \xrightarrow{D_4} \rho_1 \otimes \boldsymbol{q},$$

hence forcing the existence of

$$m(\boldsymbol{p}, \rho_{12}) \xrightarrow{D_1+D_2+D_3} \rho_{123} \otimes \boldsymbol{r}.$$

$D_1 + D_2 + D_3 + D_4$ revisited Recall that this domain had two interpretations; one with one with input $(\boldsymbol{p}, \rho_{123})$, and the other with input $(\boldsymbol{p}, \rho_3, \rho_2, \rho_1)$. We already verified the existence of the curve with the first interpretation.

We will see that $m_4(\boldsymbol{p}, \rho_3, \rho_2, \rho_1)$ vanishes (ie the contribution of $D_1 + D_2 + D_3 + D_4$ under this second interpretation is zero), but this is surprisingly subtle. The $\mathcal{A}_\infty$ relations on the type DA structure are not rich enough to give information in this case: the inputs cannot be factored, and the product of any nonidempotent element with coefficient ρ_{123} vanishes. Moreover, there are two types of curves which can contribute to this m_4: in one, there are two cuts going out to $\partial_L \mathcal{H}$, and in the other, there are no such cuts. To calculate the m_4, we use some information which can be extracted from the type *AA* bimodule, which has the advantage that the two kinds of curves are counted differently. To this end, we label chords on the left of the diagram in a type A manner, ie placing algebra elements σ_1, σ_2 and σ_3 along $\partial_L \mathcal{H}$ in the regions D_3, D_2 and D_1 respectively.

Interpret the domain $D_1 + D_2 + D_3 + D_4$ as contributing in the type *AA* bimodule. Cutting all the way out to the boundary at each corner, we obtain a rectangle. Traversing its boundary, we find a contribution

$$X\colon (\boldsymbol{p}, \sigma_3, \sigma_2, \sigma_1, \rho_3, \rho_2, \rho_1) \xrightarrow{D_1+D_2+D_3+D_4} \boldsymbol{q}.$$

We wish to argue that there is also a curve contributing

$$Y\colon (\boldsymbol{p}, \sigma_{123}, \rho_3, \rho_2, \rho_1) \xrightarrow{D_1+D_2+D_3+D_4} \boldsymbol{q}.$$

To see this, we consider the $\mathcal{A}_\infty$ relation with inputs $(\boldsymbol{p}, \rho_3, \rho_2, \rho_1, \sigma_1, \sigma_{23})$ (recall that the positions of the σ_i relative to the ρ_i have no meaning). One contribution to this $\mathcal{A}_\infty$ relation is the juxtaposition

$$(\boldsymbol{p}, \rho_3, \rho_2, \sigma_1) \xrightarrow{D_3+D_4} \boldsymbol{r}$$

(an annulus, which is easily seen to have a representative) with

$$(\boldsymbol{r}, \rho_1, \sigma_{23}) \xrightarrow{D_1+D_2} \boldsymbol{q}$$

(a rectangle). The only term that can cancel this juxtaposition is $m_4(\boldsymbol{p}, \sigma_{123}, \rho_3, \rho_2, \rho_1)$, forcing the existence of the curve Y.

Turning attention back to the type *DA* bimodule, observe that both the curves X and Y contribute to the $(\rho_{123} \otimes \boldsymbol{q})$–coefficient of $m_4(\boldsymbol{r}, \rho_3, \rho_2, \rho_1)$. Thus, taken together we see that the contribution of $D_1 + D_2 + D_3 + D_4$ to $m_4(\boldsymbol{p}, \rho_3, \rho_2, \rho_1)$ vanishes. (Recall that in the calculation of the type *DD* bimodule for the identity a similar mechanism applied: the module was deduced from the *AA* identity bimodule, and there were two distinct $\mathcal{A}_\infty$ operations in the AA bimodules from $\boldsymbol{w}_1$ to $\boldsymbol{z}_2$ which canceled when interpreted as contributions in the type DD identity bimodule.)

This completes the verification that $\widehat{CFDA}(\tau_m)$ has the form stated in Figure 27. The other mapping class group generators can be calculated in an entirely parallel way. $\square$

10.3 Duality for the genus-one handlebody

We illustrate now the duality theorem, Theorem 6, by explicitly verifying that the description of the type A module for the genus-one handlebody gotten by inspecting a Heegaard diagram with one generator is quasi-isomorphic to the one gotten from the duality theorem. This calculation will use the type DD identity bimodule in the genus-one case.

Continuing notation from Section 10.1, we let $\mathcal{A} = \mathcal{A}(\mathcal{Z}, 0)$ and $\mathcal{B} = \mathcal{A}(-\mathcal{Z}, 0)$. We abbreviate ${}^{\mathcal{A},\mathcal{B}}D$ for ${}^{\mathcal{A},\mathcal{B}}\widehat{CFDD}(\mathbb{I})$.

Our goal is to calculate the $\mathcal{B}$–module $\mathrm{Mor}^{\mathcal{A}}({}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \boxtimes {}^{\mathcal{A},\mathcal{B}}D, {}^{\mathcal{A}}N)$, where here ${}^{\mathcal{A}}N$ is the type D module associated to a 0–framed solid torus from [21, Section 11.2]. We will use the interpretation of this morphism space provided by Corollary 2.3.37. ${}^{\mathcal{A}}N$ has a single generator as a type D module; and hence ${}_{\mathcal{A}}N = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$ is spanned by three elements: $\boldsymbol{x}$, $\rho_2 \otimes \boldsymbol{x}$ and $\rho_{12} \otimes \boldsymbol{x}$. Note that $\rho_2 \otimes \boldsymbol{x}$ is in the left ι_1–idempotent, while the two other elements are in the left ι_0–idempotent.

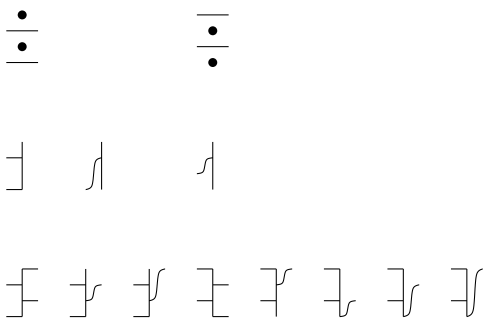


Figure 28: Module generators: in the top row, we have displayed the two primitive idempotents (ι_0 and ι_1 respectively) in the torus algebra. In the second row, we exhibit the three generators ($\boldsymbol{x}$, $\rho_{12} \otimes \boldsymbol{x}$, and $\rho_2 \otimes \boldsymbol{x}$ respectively) of the module for a framed handlebody. In the third row, we have displayed the eight generators of ${}^{\mathcal{A}}_{\mathcal{B}}D$. They are $\iota_0 \otimes j_0$, $\iota_0 \otimes \sigma_2$, $\iota_0 \otimes \sigma_{12}$, $\iota_1 \otimes j_1$, $\iota_1 \otimes \sigma_1$, $\iota_1 \otimes \sigma_3$, $\iota_1 \otimes \sigma_{23}$, and $\iota_1 \otimes \sigma_{123}$ respectively.

The differential on ${}^{\mathcal{A}}N$ is given by $\delta^1(\boldsymbol{x}) = \rho_{12} \otimes \boldsymbol{x}$, so the differential on ${}_{\mathcal{A}}N$ is given by $\partial(\boldsymbol{x}) = \rho_{12} \otimes \boldsymbol{x}$ and $\partial(\rho_2 \otimes \boldsymbol{x}) = \partial(\rho_{12} \otimes \boldsymbol{x}) = 0$.

The module ${}_{\mathcal{B}}^{\mathcal{A}}D = {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \boxtimes {}^{\mathcal{A}\otimes\mathcal{B}}D$ has eight generators, which are naturally labeled $\iota_0 \otimes j_0$, $\iota_0 \otimes \sigma_2$, $\iota_0 \otimes \sigma_{12}$, $\iota_1 \otimes j_1$, $\iota_1 \otimes \sigma_1$, $\iota_1 \otimes \sigma_3$, $\iota_1 \otimes \sigma_{23}$, and $\iota_1 \otimes \sigma_{123}$. These are all enumerated in Figure 28. A homomorphism from ${}_{\mathcal{B}}^{\mathcal{A}}D$ to ${}^{\mathcal{A}}N$ is uniquely specified by where it takes each of those eight generators. Of those eight generators, three are in the ι_0–idempotent, whereas five are in the ι_1–idempotent. Thus, the vector space $\operatorname{Mor}^{\mathcal{A}}({}_{\mathcal{B}}^{\mathcal{A}}D, {}^{\mathcal{A}}N)$ has 11 basis vectors, which are gotten by the six possible maps sending any of the three elements $\{\iota_0 \otimes j_0,\ \iota_0 \otimes \sigma_2\ \ \iota_0 \otimes \sigma_{12}\}$ to any of the two elements $\{\boldsymbol{x}, \rho_{12} \otimes \boldsymbol{x}\}$, or the five maps gotten by sending any of the five elements $\{\iota_1 \otimes j_1, \iota_1 \otimes \sigma_1, \iota_1 \otimes \sigma_3, \iota_1 \otimes \sigma_{23}, \iota_1 \otimes \sigma_{123}\}$ to $\rho_2 \otimes \boldsymbol{x}$.

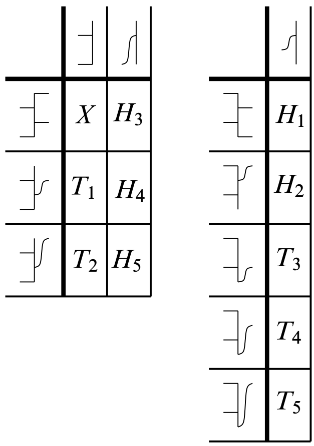


Figure 29: Differentials in $\operatorname{Mor}^{\mathcal{A}}({}_{\mathcal{B}}^{\mathcal{A}}D, {}^{\mathcal{A}}N)$: we have two charts, indicating the two types of elements (corresponding to the two left-idempotent), wherein we relabel the eleven basis vectors $\operatorname{Mor}^{\mathcal{A}}({}_{\mathcal{B}}^{\mathcal{A}}D, {}^{\mathcal{A}}N)$ by X, $\{T_i\}_{i=1}^{5}$ and $\{H_i\}_{i=1}^{S}$. For instance, T_1 is the element which carries $\iota_0 \otimes \sigma_2$ to the element $\boldsymbol{x}$, while H_4 takes $\iota_0 \otimes \sigma_2$ to $\rho_{12} \otimes \boldsymbol{x}$. There are three additional components to the differential not captured by the convention that $\partial T_i = H_i + \cdots$: an H_3 term in ∂X, an H_4 term in ∂T_1, and an H_5 term in ∂T_2.

These basis vectors are relabeled as indicated in Figure 29. The relabeling has the convenient property that for all $i = 1, \dots, 5$, the differential of T_i contains a nontrivial component in H_i; indeed, these are all the components of the differential coming from ${}_{\mathcal{B}}^{\mathcal{A}}D$. There are three additional components in the differential, taking X to H_3, T_1 to H_4, and T_2 to H_5. (These are all components coming from the differential in ${}^{\mathcal{A}}N$).

The homology of this complex is carried by $X + T_3$. It is straightforward to verify that

$$(X + T_3) \cdot \sigma_3 = H_1, \qquad \partial(T_1 + T_4) = H_1,$$
$$(T_1 + T_4) \cdot \sigma_2 = X + T_3, \quad (T_1 + T_4) \cdot \sigma_{23} = H_1.$$

Setting $y = X + T_3$, we see readily that this module is $\mathcal{A}_\infty$–homotopy equivalent to the module generated by y, with $\mathcal{A}_\infty$ relations indexed by integers $i \geq 0$

$$m_{3+i}(y, \sigma_3, \overbrace{\sigma_{23}, \dots, \sigma_{23}}^{i}, \sigma_2) = y,$$

which is in fact the module which is obtained from the inspection of holomorphic disks (see [21, Lemma 11.22]).

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Department of Mathematics, Columbia University
MC 4425, 2990 Broadway, New York, NY 10027, USA

Department of Mathematics, Princeton University
Fine Hall, Washington Road, Princeton, NY 08540, USA

Department of Mathematics, Indiana University
831 E Third St, Bloomington, NY 47405, USA

lipshitz@math.columbia.edu, petero@math.princeton.edu, dpthurst@indiana.edu

Proposed: Yasha Eliashberg Received: 1 July 2011
Seconded: Ciprian Manolescu, Tomasz Mrowka Revised: 23 April 2014