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Brauer groups and étale cohomology in derived algebraic geometry

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In this paper, we study Azumaya algebras and Brauer groups in derived algebraic geometry. We establish various fundamental facts about Brauer groups in this setting, and we provide a computational tool, which we use to compute the Brauer group in several examples. In particular, we show that the Brauer group of the sphere spectrum vanishes, which solves a conjecture of Baker and Richter, and we use this to prove two uniqueness theorems for the stable homotopy category. Our key technical results include the local geometricity, in the sense of Artin n–stacks, of the moduli space of perfect modules over a smooth and proper algebra, the étale local triviality of Azumaya algebras over connective derived schemes and a local to global principle for the algebraicity of stacks of stable categories.

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1 Introduction

1.1 Setting

Derived algebraic geometry is a generalization of classical Grothendieck-style algebraic geometry aimed at bringing techniques from geometry to bear on problems in homotopy theory, and used to unify and explain many disparate results about categories of sheaves on schemes. It has been used by Arinkin and Gaitsgory [3] to formulate a precise version of the geometric Langlands conjecture, by Ben-Zvi, Francis and Nadler [9] to study integral transforms and Hochschild homology of coherent sheaves, by Lurie and others to study topological modular forms, by Toën and Vaquié [58] to study moduli spaces of complexes of vector bundles and by Toën [57] to study derived Azumaya algebras. Moreover, the philosophy of derived algebraic geometry is closely related to noncommutative geometry and to the idea of hidden smoothness of Kontsevich.

The basic objects in derived algebraic geometry are "derived" versions of commutative rings. There are various things this might mean. For instance, it could mean simply a graded commutative ring, or a commutative differential-graded ring, such as the

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de Rham complex $\Omega^*(M)$ of a manifold M. Or, it could mean a commutative ring spectrum, which is to say a spectrum equipped with a coherently homotopy commutative and associative multiplication. The basic example of such a commutative ring spectrum is the sphere spectrum $\mathbb{S}$, which is the initial commutative ring spectrum, and hence plays the role of the integers $\mathbb{Z}$ in derived algebraic geometry. Commutative ring spectra are in a precise sense the universal class of derived commutative rings. We work throughout this paper with connective commutative ring spectra, their module categories and their associated schemes.

While a substantial portion of the theory we develop in this paper has been studied previously for simplicial commutative rings, it is important for applications to homotopy theory and differential geometry to have results applicable to the much broader class of commutative (or $\mathbb{E}_\infty$) ring spectra, as the vast majority of the rings which arise in these contexts are only of this more general form. Simplicial commutative rings are special cases of commutative differential graded rings, and an $\mathbb{E}_\infty$–ring spectrum admits an $\mathbb{E}_\infty$–dg model if and only if it is a commutative algebra over the Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{Z}$. To give an idea of how specialized a class this is, note that an arbitrary spectrum M is an $\mathrm{H}\mathbb{Z}$–module precisely when all of its k–invariants are trivial, meaning that it decomposes as a product of spectra $\Sigma^n \mathrm{H}\pi_n M$, or that it has no nontrivial extensions in its "composition series" (Postnikov tower). Rather, the basic $\mathbb{E}_\infty$–ring is the sphere spectrum $\mathbb{S}$, which is the group completion of the symmetric monoidal category of finite sets and *automorphisms* (as opposed to only *identities*, which yields $\mathbb{Z}$) and captures substantial information from differential and $\mathbb{F}_1$–geometry and contains the homological complexity of the symmetric groups. Similarly, the algebraic K–theory spectra, as well as other important spectra such as those arising from bordism theories of manifolds, in the study of the mapping class group and the Mumford conjecture, or in topological Hochschild or cyclic homology, tend not to exist in the differential graded world.

Nevertheless, an $\mathbb{E}_\infty$–ring spectrum R should be regarded as a nilpotent thickening of its underlying commutative ring $\pi_0 R$, in much the same way as the Grothendieck school successfully incorporated nilpotent elements of ordinary rings into algebraic geometry via scheme theory. Of course, this relies upon the "local" theory of homotopical commutative algebra, which, thanks to the efforts of many mathematicians, is now well established. In particular, there is a good notion of étale map of commutative ring spectra, and so the basic geometric objects in our paper will be glued together, in this topology, from commutative ring spectra. We adopt Grothendieck's "functor of points" perspective; specifically, we fix a base $\mathbb{E}_\infty$–ring R and consider the category of connective commutative R–algebras, $\mathrm{CAlg}_R^{\mathrm{cn}}$. A sheaf is then a space-valued functor on $\mathrm{CAlg}_R^{\mathrm{cn}}$ which satisfies descent for the étale topology in the appropriate homotopical sense. For instance, if S is a commutative R–algebra, there is the representable sheaf $\operatorname{Spec} S$ whose space of T–points is the mapping space $\operatorname{map}(S,T)$ in the ∞–category of connective commutative R–algebras.

Just as in ordinary algebraic geometry, one is really only interested in a subclass of sheaves which are geometric in some sense. An important feature of derived algebraic geometry is the presence of higher versions of Artin stacks, an idea due to Simpson [53]; roughly, this is the smallest class of sheaves which contain the representables $\operatorname{Spec} S$ and is closed under formation of quotients by smooth groupoid actions. By restricting attention to these sheaves, it is possible to prove many EGA-style statements. The situation is entirely analogous to the classes of schemes or algebraic spaces in ordinary algebraic geometry, which can be similarly expressed as the closure of the affines under formation of Zariski or étale quotients, respectively. The difference is that we allow our sheaves to take values in spaces, a model for the theory of higher groupoids, and that we require the larger class which is closed under smooth actions, so that it contains objects such as the deloopings $\mathrm{B}^n A$ of a smooth abelian group scheme A. These are familiar objects: the Artin 1–stack $\mathrm{B}A$ is the moduli space of A–torsors, and the Artin 2–stack $\mathrm{B}^2 A$ is the moduli space of gerbes with band A.

One of the main goals of this paper is to study Azumaya algebras over these derived geometric objects. Historically, the notion of Azumaya algebra, due to Auslander and Goldman [5], arose from an attempt to generalize the Brauer group of a field. It was then globalized by Grothendieck [31], who defined an Azumaya algebra $\mathcal{A}$ over a scheme X as a sheaf of coherent $\mathcal{O}_X$–algebras that is étale locally a matrix algebra. In other words, there is a surjective étale map $p\colon U \to X$ such that $p^*\mathcal{A} \cong \mathrm{M}_n(\mathcal{O}_U)$. The Brauer group of a scheme classifies Azumaya algebras up to Morita equivalence, that is, up to equivalence of their stacks of modules. The original examples of Azumaya algebras are central simple algebras over a field k; by Wedderburn's Theorem, these are precisely the algebras $\mathrm{M}_n(D)$, where D is a division algebra of finite dimension over its center k. The algebra of quaternion numbers over $\mathbb{R}$ is thus an example of an Azumaya $\mathbb{R}$–algebra, and represents the generator of $\operatorname{Br}(\mathbb{R}) \cong \mathbb{Z}/2$.

In more geometric settings, the first example of an Azumaya algebra is the endomorphism algebra of a vector bundle, though these have trivial Brauer class. Locally, any Azumaya algebra is the endomorphism algebra of a vector bundle, but the vector bundles do not generally glue to a vector bundle on the total space. However, every Azumaya algebra is the endomorphism algebra of a twisted vector bundle, a perspective that has recently gained a great deal of importance. For instance, in the theory of moduli spaces of vector bundles, there is always a twisted universal vector bundle, and the class of its endomorphism algebra in the Brauer group is precisely the obstruction to the existence of a universal (nontwisted) vector bundle on the moduli space. Brauer groups and Azumaya algebras play an important role in many areas of mathematics, but especially in arithmetic geometry, algebraic geometry and applications to mathematical physics. In arithmetic geometry, they are closely related to Tate's conjecture on l–adic cohomology of schemes over finite fields, and they play a critical role in studying rational points of varieties through, for example, the Brauer–Manin obstructions to the Hasse principle. In algebraic geometry, Azumaya algebras arise naturally when studying moduli spaces of vector bundles, and Brauer classes appear when considering certain constructions motivated from physics in homological mirror symmetry. The Brauer group was also used by Artin and Mumford [4] to construct one of the first examples of a nonrational unirational complex variety.

As an abstract group, defined via the above equivalence relation, the Brauer group is difficult to compute directly. Instead, one introduces the cohomological Brauer group of a scheme, $\mathrm{Br}'(X) = \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)_{\mathrm{tors}}$. There is an inclusion $\mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$. A first critical problem, posed by Grothendieck, is whether this inclusion is an equality. Unfortunately, the answer is "no" in general, although de Jong has written a proof [36] of a theorem of O Gabber that equality holds if X is quasiprojective, or more generally has an ample line bundle. However, by expanding the notion of Azumaya algebra to derived Azumaya algebra, as done in Lieblich [39, Chapter 3] and Toën [57], the answer to the corresponding question is "yes," at least for quasicompact and quasiseparated schemes. This was shown by Toën, who also shows that the result holds for quasicompact and quasiseparated derived schemes built from simplicial commutative rings. One of the purposes of the present paper is to generalize this theorem to quasicompact and quasiseparated derived schemes based on connective commutative ring spectra, which is necessary for our applications to homotopy theory. To any class $\alpha \in \mathrm{Br}'(X)$ there is an associated category Mod_X^{α} of complexes of quasicoherent α–twisted sheaves. When this derived category is equivalent to $\mathrm{Mod}_{\mathscr{A}}$ for an ordinary Azumaya algebra $\mathscr{A}$, then $\alpha \in \mathrm{Br}(X)$. However, even when this fails, as long as X is quasicompact and quasiseparated, there is a derived Azumaya algebra $\mathscr{A}$ such that $\mathrm{Mod}_X^{\alpha} \simeq \mathrm{Mod}_{\mathscr{A}}$. Hence derived Azumaya algebras are locally endomorphisms algebras of complexes of vector bundles, and not just vector bundles, and therefore the appropriate notion of Morita equivalence is based on tilting complexes instead of bimodules.

One of the main features of this category $\mathrm{Mod}_X^{\alpha} \simeq \mathrm{Mod}_{\mathscr{A}}$ of quasicoherent α–twisted sheaves is that it allows us to define the α–twisted K–theory spectrum $K^{\alpha}(X)$ of X as the K–theory of the subcategory of perfect objects (see Definition 6.5). The reason this is sensible is that, given an Azumaya $\mathcal{O}_X$–algebra $\mathscr{A}$, there is an Azumaya $\mathcal{O}_X$–algebra $\mathscr{B}$ such that $\mathrm{Mod}_{\mathscr{A}} \otimes \mathrm{Mod}_{\mathscr{B}} \simeq \mathrm{Mod}_X$; moreover, $\mathscr{B}$ can be taken to be the opposite $\mathcal{O}_X$–algebra $\mathscr{A}^{\mathrm{op}}$ and $\mathrm{Mod}_{\mathscr{A}^{\mathrm{op}}} \simeq \mathrm{Mod}_X^{-\alpha}$. Note that because $\mathrm{Br}'(X) = \mathrm{H}^2_{\mathrm{et}}(X; \mathbb{G}_m)$, this is entirely analogous to what happens topologically, where the twists are typically given by elements of the cohomology group $\mathrm{H}^2(X;\mathbb{C}^\times)\cong\mathrm{H}^3(X;\mathbb{Z})$ in the complex case and elements of $\mathrm{H}^2(X;\mathbb{R}^\times)$ in the real case; see the authors and Gómez [2]. While we do not study the twisted K–theory of derived schemes in this paper, the basic structural features (such as additivity and localization) follow from the untwisted case as in [57], using the fact that our categories of α–twisted sheaves $\mathrm{Mod}_X^\alpha\simeq\mathrm{Mod}_{\mathscr{A}}$ admit global generators with endomorphism algebra $\mathscr{A}$.

1.2 Summary

We now give a detailed summary of the paper. By definition, an R–algebra A is Azumaya if it is a compact generator of the ∞–category of R–modules and if the multiplication action

$$A\otimes_R A^{\mathrm{op}}\longrightarrow \mathrm{End}_R(A)$$

of $A\otimes_R A^{\mathrm{op}}$ on A is an equivalence. This definition is due to Auslander and Goldman [5] in the case of discrete commutative rings, and it has been studied in the settings of schemes by Grothendieck [31], $\mathbb{E}_\infty$–ring spectra by Baker, Richter and Szymik [8], and derived algebraic geometry over simplicial commutative rings by Toën [57]. In a slightly different direction, it has also been studied in the setting of higher categories by Borceux and Vitale [15] and Johnson [35]. All of these variations ultimately rely on the idea of an Azumaya algebra as an algebra whose module category is invertible with respect to a certain "Morita" symmetric monoidal structure.

Although we restrict to Azumaya algebras over commutative ring spectra, we note that the notion of Azumaya algebra makes sense over any $\mathbb{E}_3$–ring spectrum. The reason for this is that if R is an $\mathbb{E}_3$–ring, then Mod_R is naturally a $\mathbb{E}_2$–monoidal ∞–category, and so its ∞–category of modules is naturally $\mathbb{E}_1$–monoidal. The theory of Azumaya algebras is closely related to the notions of smoothness and properness in noncommutative geometry, which have been studied extensively starting from Kapranov [37]. These and related ideas have been used to great success to prove theorems in algebraic geometry. For instance, van den Bergh [10] uses noncommutative algebras to give a proof of the Bondal–Orlov conjecture, showing that birational smooth projective 3–folds are derived equivalent.

One of main points of the paper is to establish the following theorem, which says that all Azumaya algebras over the sphere spectrum are Morita equivalent. This proves a conjecture of Baker and Richter.

Corollary 7.17 *The Brauer group of the sphere spectrum is zero.*

The proof of the theorem highlights the differences between our approach and the approaches of Baker, Richter and Szymik [8] and Toën [57]. While Brauer groups of commutative ring spectra were introduced in [8], they are impossible to compute without identifying them with cohomological objects; this is what we do for connective commutative ring spectra. This transition is similar to the move from the algebraic Brauer group of Auslander and Goldman [5] to the cohomological Brauer group of Grothendieck [31]. On the other hand, Toën has a similar cohomological philosophy in [57], but a key point in his proof fails dramatically for connective ring spectra in general, and hence requires a radically different proof; see Section 6.

This theorem follows from several other important results, which we now outline.

Theorem 3.15 *Let $\mathscr{C}$ be a compactly generated R–linear category (a stable presentable ∞–category enriched in R–modules). Then*

(1) *$\mathscr{C}$ is dualizable in $\mathrm{Cat}_{R,\omega}$ if and only if $\mathscr{C}$ is equivalent to Mod_A for a smooth and proper R–algebra A,*

(2) *$\mathscr{C}$ is invertible in $\mathrm{Cat}_{R,\omega}$ if and only if $\mathscr{C}$ is equivalent to Mod_A for an Azumaya R–algebra A.*

The analogous results were proved for simplicial commutative rings in [57]. A final algebraic ingredient is the fact that smooth and proper R–algebras are compact. In particular, Azumaya algebras are compact algebras. This is a key point later in our analysis of the geometricity of the sheaf of perfect modules for an Azumaya algebra. To establish it requires showing that the ∞–category of spectra Sp is compact in the ∞–category of all compactly generated $\mathbb{S}$–linear categories, which does not follow immediately from the fact that it is the unit object in this symmetric monoidal ∞–category. The theory of smooth and proper algebras is also fundamental in the theory of noncommutative motives, and has been studied in that setting by Cisinski and Tabuada [18] and Blumberg, Gepner and Tabuada [12].

Suppose that R is an $\mathbb{E}_k$–ring spectrum for $3 \le k \le \infty$. Then, the characterization of Azumaya algebras above lets us define the Brauer space of R as the Picard space

$$\mathrm{Br}_{\mathrm{alg}}(R) = \mathrm{Pic}(\mathrm{Cat}_{R,\omega})$$

of the $\mathbb{E}_{k-2}$–monoidal ∞–category of compactly generated R–linear categories. This space is a grouplike $\mathbb{E}_{k-2}$–space, and so is, in particular, a $(k-2)$–fold loop space. The Brauer group is the abelian group

$$\pi_0 \,\mathrm{Br}_{\mathrm{alg}}(R).$$

When $k = \infty$, it follows that there is a Brauer spectrum $\mathrm{br}_{\mathrm{alg}}(R)$. One strength of this definition is that it generalizes well to other settings, such as arbitrary compactly generated $\mathbb{E}_{k-2}$–monoidal stable ∞–categories. We do not develop this theory in our paper, instead working only with $\mathbb{E}_\infty$–ring spectra, but it is closely related to ideas about Brauer groups of 2–categories.

Let us take a moment to place this idea in context. We can describe the space $\mathrm{Br}_{\mathrm{alg}}(R)$ as follows. The 0–simplices are ∞–categories Mod_A where A is an Azumaya R–algebra. A 1–cell from A to B is an equivalence $\mathrm{Mod}_A \simeq \mathrm{Mod}_B$; these may be identified with certain right $A^{\mathrm{op}} \otimes_R B$–modules. A 2–cell is the data of an equivalence between bimodules, and so forth. When R is an ordinary ring, there is no interesting data in degree higher than 2. However, when R is a derived ring, the higher homotopy groups appear in the homotopy of $\mathrm{Br}_{\mathrm{alg}}(R)$; see (1) below. Thus, our Brauer space can be viewed as a generalization of the Brauer 3–group of Gordon, Power and Street [28] and Duskin [21], and as a generalization of the approach to Brauer groups by Vitale in [60] and [15].

The subject of derived algebraic geometry is increasingly important due to its utility in proving theorems in homotopy theory and algebraic geometry. As we will see in this paper, even to derive purely homotopy-theoretic results about modules over the sphere spectrum, we will need to employ derived algebraic geometry in an essentially nontrivial way. Such methods are essential even in ordinary algebra. For instance, the classical proof (see Grothendieck [33]) that the Brauer group of the integers vanishes employs geometric methods and cohomology.

In order to utilize cohomological methods to compute Brauer groups of derived schemes, it is necessary to show that Azumaya algebras are locally Morita equivalent to the base. This local triviality holds in the étale topology, but not in the Zariski topology (as is shown by the quaternions over $\mathbb{R}$). This is not easy to prove and uses the geometry of smooth higher Artin sheaves. Higher Artin sheaves are built inductively out of affine schemes by taking iterated quotients by smooth equivalence relations. We study these sheaves in Section 4, and we prove the following theorem.

Theorem 4.47 *If $p\colon X \to Y$ is a smooth locally geometric morphism that is surjective on geometric points, then for every S–point $\mathrm{Spec}\, S \to Y$ there exists an étale cover $\mathrm{Spec}\, T \to \mathrm{Spec}\, S$ and a T–point $\mathrm{Spec}\, T \to X$ such that*

$$\begin{tikzcd} \mathrm{Spec}\, T \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\, S \arrow[r] & Y \end{tikzcd}$$

commutes.

Briefly, the theorem says that if $f\colon X \to \operatorname{Spec} R$ is a smooth surjection, where R is a connective commutative ring spectrum and X is filtered by higher Artin stacks, then f has étale local sections. This extends the classical result about smooth morphisms of schemes to derived algebraic geometry, and has been established in other contexts by Toën and Vezzosi [59]. To use this result on sections of smooth morphisms, we first need to establish the following theorem, showing that a certain moduli sheaf is sufficiently geometric; it is due to [58] in the simplicial commutative setting.

Theorem 5.8 *Let A be a compact R–algebra. Then, the stack $\mathbf{M}_A$ is locally geometric and locally of finite presentation, and the functor $\pi\colon \mathbf{M}_A \to \mathbf{M}_R$ is locally geometric and locally of finite presentation.*

Compact R–algebras are those R–algebras that admit a finite presentation in the ∞–category Alg_R. This class includes the smooth and proper R–algebras, but is much bigger. After we finished our paper, we were informed that Pandit had established this result when A is smooth and proper in his thesis [49].

This is the case in particular for $\mathscr{C} = \mathrm{Mod}_A$ when A is an Azumaya algebra, in which case the subsheaf $\mathbf{Mor}_A \subseteq \mathbf{M}_A$ that classifies Morita equivalences from A to R is smooth and surjective over $\operatorname{Spec} R$. This is used to prove the following theorem.

Theorem 5.11 *Let R be a connective $\mathbb{E}_\infty$ ring spectrum, and let A be an Azumaya R–algebra. Then, there is a faithfully flat étale R–algebra S such that $A \otimes_R S$ is Morita equivalent to S.*

For nonconnective commutative ring spectra the question of étale-local triviality is more subtle: there are examples where it fails. One possibility is to use Galois descent instead of étale descent. This is the subject of current work by the second author and Lawson [26].

In Section 6, we study families of linear categories over sheaves in order to establish the following key result regarding the existence of compact generators.

Theorem 6.1 (Local-global principle) *Let $\mathscr{C}$ be an R–linear category with descent, and suppose that $R \to S$ is an étale cover such that $\mathscr{C} \otimes_R S$ has a compact generator. Then, $\mathscr{C}$ has a compact generator.*

This local-global principle is proved by establishing analogous statements for Zariski covers, for finite flat covers and for Nisnevich covers. The method for showing the Zariski local-global result follows work of Thomason and Trobaugh [56], Bökstedt and

Neeman [13], Neeman [47] and Bondal and van den Bergh [14] on derived categories of schemes. The local-global principle for finite flat covers is straightforward. The real work is in establishing the principle for étale covers. Lurie proves in [43, Theorem 2.9] that the Morel–Voevodsky Theorem, which reduces Nisnevich descent to affine Nisnevich excision, holds for connective $\mathbb{E}_\infty$–ring spectra. Thus, we show a local-global principle for affine Nisnevich squares. This idea parallels work of Lurie on a local-global principle for the compact generation of linear categories (as opposed, in our work, for compact generation by a single object). Toën [57] proves a similar local-global principle for fppf covers in the setting of simplicial commutative rings, but his proofs both of the étale and the fppf local-global principles do not obviously generalize to $\mathbb{E}_\infty$–ring spectra because it is typically not the case that there are algebra structures on module-theoretic quotients of ring spectra.

The local-global principle shows that if $\mathscr{C}$ is a linear category with descent over a quasicompact and quasiseparated derived scheme such that $\mathscr{C}$ is étale locally equivalent to modules over an Azumaya algebra, then $\mathscr{C}$ is globally equivalent to modules over an Azumaya algebra. This solves the $\mathrm{Br} = \mathrm{Br}'$ problem of Grothendieck for derived schemes.

Theorem 7.2 *For any quasicompact and quasiseparated derived scheme X, we have* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

The local-global principle has another interesting application: if X is a quasicompact and quasiseparated derived scheme over the p–local sphere, then the ∞–category $L_{K(n)}\mathrm{Mod}_X$ of $K(n)$–local objects is compactly generated.

In Section 7, we define a Brauer sheaf $\mathbf{Br}$. If X is an étale sheaf, the Brauer space $\mathbf{Br}(X)$ of X is the space of maps from X to $\mathbf{Br}$ in the ∞–topos $\mathrm{Shv}_R^{\text{ét}}$. In the case of an affine scheme $\mathrm{Spec}\, R$, combining the étale-triviality of Azumaya algebras and the étale local-global principle, we find that $\mathrm{Br}_{\mathrm{alg}}(R) \simeq \mathbf{Br}(\mathrm{Spec}\, R)$. One advantage of using the Brauer sheaf $\mathbf{Br}$ is that it is a delooping of the Picard sheaf: $\Omega\mathbf{Br} \simeq \mathbf{Pic}$. This allows us to compute the homotopy sheaves of $\mathbf{Br}$:

$$\pi_k \mathbf{Br} \simeq \begin{cases} 0 & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k = 1, \\ \pi_0 \mathbb{O}^\times & \text{if } k = 2, \\ \pi_{k-2}\mathbb{O} & \text{if } k \geq 3, \end{cases}$$

where $\mathbb{O}$ denotes the structure sheaf of $\mathrm{Shv}_R^{\text{ét}}$. We introduce a computation tool, a descent spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \pi_q \mathbf{Br}) \Rightarrow \pi_{q-p}\mathbf{Br}(X),$$

which converges if X is affine or has finite étale cohomological dimension. When $X = \mathrm{Spec}\, R$, the spectral sequence collapses, and we find that

$$\pi_k \mathbf{Br}(R) \cong \begin{cases} \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, \pi_0 R, \mathbb{Z}) \times \mathrm{H}^2_{\text{ét}}(\mathrm{Spec}\, \pi_0 R, \mathbb{G}_m) & \text{if } k = 0, \\ \mathrm{H}^0_{\text{ét}}(\mathrm{Spec}\, \pi_0 R, \mathbb{Z}) \times \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\, \pi_0 R, \mathbb{G}_m) & \text{if } k = 1, \\ \pi_0 R^\times & \text{if } k = 2, \\ \pi_{k-2} R & \text{if } k \geq 3. \end{cases} \tag{1}$$

In particular, we recover [8, Corollary 6.2], one of the main results of that paper, which establishes the existence of many Azumaya algebras over commutative ring spectra. That is, the splitting when $k = 0$, establishes a map $\mathrm{Br}(\pi_0 R) \to \pi_0 \mathbf{Br}(R)$.

It follows that the Brauer group vanishes in many interesting cases; for example

$$\pi_0 \mathbf{Br}(ko) = 0, \quad \pi_0 \mathbf{Br}(ku) = 0, \quad \pi_0 \mathbf{Br}(MU) = 0, \quad \pi_0 \mathbf{Br}(tmf) = 0.$$

For examples of nonzero Brauer groups, we find that

$$\pi_0 \mathbf{Br}(\mathbb{S}[\tfrac{1}{p}]) \simeq \mathbb{Z}/2,$$

and for the p–local sphere spectrum, the Brauer group fits into an exact sequence

$$0 \to \pi_0 \mathbf{Br}(\mathbb{S}_{(p)}) \to \mathbb{Z}/2 \oplus \bigoplus_p \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Note that the p–inverted sphere and the p–local sphere give examples of non-Eilenberg–Mac Lane $\mathbb{E}_\infty$–ring spectra with nonzero Brauer groups. By (1), if R is a connective $\mathbb{E}_\infty$–ring spectrum, we can compute the homotopy groups of $\mathbf{Br}(R)$ whenever we can compute the relevant étale cohomology groups of $\mathrm{Spec}\, \pi_0 R$. For example, $\pi_0 \mathbf{Br}(R) = 0$ if R is any connective $\mathbb{E}_\infty$–ring spectrum such that $\pi_0 R \cong \mathbb{Z}$ or $\mathbb{W}_k$, the ring of Witt vectors over $\mathbb{F}_{p^k}$.

We state as theorems two consequences of the vanishing of the Brauer group of the sphere spectrum. These theorems follow immediately from the fact that $\mathrm{Br}(\mathbb{S}) = 0$ and Theorems 3.15 and 7.2.

Theorem 1.1 *Let $\mathscr{C}$ be a compactly generated stable presentable ∞–category, and suppose that there exists a stable presentable ∞–category $\mathscr{D}$ such that $\mathscr{C} \otimes \mathscr{D} \simeq \mathrm{Mod}_{\mathbb{S}}$, the ∞–category of spectra. Then, $\mathscr{C} \simeq \mathrm{Mod}_{\mathbb{S}}$.*

Theorem 1.2 *Let $\mathscr{C}$ be a stable presentable ∞–category such that there exists a faithfully flat étale $\mathbb{S}$–algebra T such that $\mathscr{C} \otimes \mathrm{Mod}_T \simeq \mathrm{Mod}_T$. Then, $\mathscr{C} \simeq \mathrm{Mod}_{\mathbb{S}}$.*

The first theorem says that if $\mathscr{C}$ is compactly generated and invertible as a $\mathrm{Mod}_{\mathbb{S}}$–module, then $\mathscr{C}$ is already equivalent to $\mathrm{Mod}_{\mathbb{S}}$. The second theorem says that if $\mathscr{C}$ is étale locally equivalent to spectra, then $\mathscr{C}$ is already equivalent to the ∞–category of spectra. These give strong uniqueness, or rigidity, results for $\mathbb{S}$–modules. Such statements have a long history, and are related to the conjecture of Margolis, which gives conditions for a triangulated category to be equivalent to the stable homotopy category. The conjecture was proven for triangulated categories with models by Schwede and Shipley [51]. Our results extend theirs and also those of Schwede [50].

We briefly outline the contents of our paper. We start in Section 2 by giving some background on rings, modules, and the étale topology in the context of derived algebraic geometry. In Section 3, we consider the module categories of R–algebras A under various conditions, including compactness, properness, and smoothness. We prove there the characterization that A is Azumaya (resp. smooth and proper) if and only if Mod_A is invertible (resp. dualizable) in a certain symmetric monoidal ∞–category of R–linear categories. We develop the theory of higher Artin sheaves in derived algebraic geometry in Section 4. In Section 5, we harness the notion of geometric sheaves to study the moduli space of A–modules for nice R–algebras A. Specializing to the case of Azumaya algebras, we prove that the sheaf of Morita equivalences from A to R is smooth and surjective over $\operatorname{Spec} R$, and hence has étale-local sections. It follows that Azumaya R–algebras are étale locally trivial. We consider the problem of when a stack of linear categories over a stack admits a perfect generator in Section 6. In the final section, Section 7, we study the Brauer group, define the Brauer spectral sequence, and give the computations, including the important theorem that the Brauer group of the sphere spectrum vanishes.

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2 Ring and module theory

In this section of the paper, we give some background on ring spectra and their module categories, compactness, Grothendieck topologies on commutative ring spectra, and Tor-amplitude.

2.1 Rings and modules

Lurie [45] gives good notions of module categories for ring objects in symmetric monoidal ∞–categories. We refer to that book for details on the construction of the objects introduced in the rest of this section. If $\mathscr{C}$ is a symmetric monoidal ∞–category, and if A is an algebra object, by which we mean an $\mathbb{E}_1$–algebra in $\mathscr{C}$, then there is an ∞–category $\mathrm{Mod}_A(\mathscr{C})$ of right A–modules in $\mathscr{C}$; similarly there is an ∞–category of left A–modules ${}_A\mathrm{Mod}(\mathscr{C}) \simeq \mathrm{Mod}_{A^{\mathrm{op}}}(\mathscr{C})$. Given two algebras A and B, there is an ∞–category ${}_A\mathrm{Mod}_B(\mathscr{C})$ of (A,B)–bimodules in $\mathscr{C}$, which is equivalent to $\mathrm{Mod}_{A^{\mathrm{op}}\otimes B}(\mathscr{C})$. The $\mathbb{E}_k$–algebras in $\mathscr{C}$ form an ∞–category $\mathrm{Alg}_{\mathbb{E}_k}(\mathscr{C})$. When $k=1$, we write $\mathrm{Alg}(\mathscr{C})$ for this ∞–category, and when $k=\infty$, we write $\mathrm{CAlg}(\mathscr{C})$. When $\mathscr{C}=\mathrm{Sp}$, the ∞–category of spectra with the smash product tensor structure, we will write more simply $\mathrm{Alg}_{\mathbb{E}_k}$, Alg, CAlg, Mod_A, ${}_A\mathrm{Mod}$, ${}_A\mathrm{Mod}_B$ and so forth for the ∞–categories of $\mathbb{E}_k$–ring spectra, associative ring spectra, commutative ring spectra, etc. When A is a discrete associative ring, then the ∞–category of right modules $\mathrm{Mod}_{\mathrm{H}A}$ over the Eilenberg–Mac Lane spectrum of $\mathrm{H}A$ is equivalent to the ∞–category of chain complexes on A.

2.2 Compact objects and generators

We introduce the notion of compactness, which will play a crucial role in everything that follows.

Definition 2.1 Let $\mathscr{C}$ denote an ∞–category which is closed under κ–filtered colimits [41, Section 5.3.1]. A functor $f\colon \mathscr{C}\to\mathscr{D}$ is said to be *κ–continuous* if f preserves κ–filtered colimits. In the special case that f preserves ω–filtered colimits, we simply say that f is continuous.

Definition 2.2 Let $\mathscr{C}$ denote an ∞–category which is closed under κ–filtered colimits. An object x of $\mathscr{C}$ is said to be κ–compact if the mapping space functor

$$\mathrm{map}_{\mathscr{C}}(x,-)\colon \mathscr{C}\longrightarrow\mathscr{S}$$

is κ–continuous, where $\mathscr{S}$ is the ∞–category of spaces. We say that x is compact if it is ω–compact.

Definition 2.3 Let $\mathscr{C}$ be an ∞–category which is closed under geometric realizations (in other words, colimits of simplicial diagrams). An object x of $\mathscr{C}$ is said to be *projective* if the mapping space functor

$$\mathrm{map}_{\mathscr{C}}(x,-)\colon \mathscr{C}\longrightarrow\mathscr{S}$$

preserves geometric realizations.

A compact projective object x of an ∞–category $\mathscr{C}$ corepresents a functor which preserves both filtered colimits and geometric realizations. Both filtered colimits and the simplicial indexing category Δ^{op} are examples of *sifted colimits* (that is, colimits indexed by simplicial sets K such that the diagram $K \to K \times K$ is cofinal), and x is compact projective if and only if $\mathrm{map}_{\mathscr{C}}(x, -)$ preserves sifted colimits (see [41, Corollary 5.5.8.17]).

An ∞–category with all small colimits is said to be κ–compactly generated if the natural map $\mathrm{Ind}_{\kappa}(\mathscr{C}^{\kappa}) \to \mathscr{C}$ is an equivalence, where Ind_{κ} denotes the κ–filtered cocompletion [41, Section 5.3.5]. By definition, a presentable ∞–category is an ∞–category that is κ–compactly generated for some infinite regular cardinal κ. When $\mathscr{C}$ is ω–compactly generated, we say simply that it is compactly generated. If $\mathscr{C}$ is compactly generated, and if $\mathscr{D}$ is a full subcategory of $\mathscr{C}^{\omega}$ such that the closure of $\mathscr{D}$ in $\mathscr{C}$ under finite colimits and retracts is equivalent to $\mathscr{C}^{\omega}$, then we say that $\mathscr{C}$ is compactly generated by $\mathscr{D}$.

Lemma 2.4 *A stable presentable ∞–category $\mathscr{C}$ is compactly generated by a set X of compact objects if for any object $y \in \mathscr{C}$, $\mathrm{Map}_{\mathscr{C}}(x, y) \simeq 0$ for all $x \in X$ if and only if $y \simeq *$.*

Proof See the proof of [52, Lemma 2.2.1]. □

Returning to the algebraic situation of the previous section, if A is an $\mathbb{E}_1$–ring spectrum, then Mod_A is a stable presentable ∞–category. Presentability follows from [45, Corollary 4.2.3.7.(1)] and stability is straightforward. In particular, Mod_A admits all small colimits. Moreover, Mod_A is compactly generated by the single object A. We will often refer to compact objects of Mod_A as perfect modules, in keeping with the usual terminology of algebraic geometry.

Definition 2.5 A connective A–module P is projective if it is projective as an object of $\mathrm{Mod}_A^{\mathrm{cn}}$, the ∞–category of connective A–modules.

The following argument shows why there are no nonzero projective objects of Mod_A in general. Suppose that M is a projective object of Mod_R. For any R–module N, we can write the suspension of N as the geometric realization

$$\Sigma N \simeq \left| 0 \leftleftarrows N \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, N \oplus N \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots \right|.$$

Then, by stability,

$$\mathrm{map}_R(\Sigma^{-1} M, N) \simeq \mathrm{map}_R(M, \Sigma N) \simeq |\mathrm{map}(M, N^{\oplus n})| \simeq \mathrm{Bmap}_R(M, N).$$

In particular, $\pi_0 \mathrm{map}_R(\Sigma^{-1}M, N) = 0$ for all R–modules N, so that $\mathrm{id}_M \simeq 0$. Thus, $M \simeq 0$.

We record here a few facts about projective and compact modules.

Proposition 2.6 *Let A be a connective $\mathbb{E}_1$–ring spectrum.*

(1) *A connective right A–module P is projective if and only if it is a retract of a free right A–module.*

(2) *A connective right A–module P is projective if and only if for every surjective map $M \to N$ of right A–modules the map*

$$\mathrm{map}(P, M) \to \mathrm{map}(P, N)$$

is surjective.

(3) *A right A–module P is compact if and only if it is dualizable: there exists a left A–module $P^\vee$ such that the composition*

$$\mathrm{Mod}_A \xrightarrow{\otimes_A P^\vee} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$$

is equivalent to the functor corepresented by P. In this case, $P^\vee$ is a compact left A–module.

(4) *If P is a nonzero compact right A–module, then P has a bottom nonzero homotopy group; that is, there exists some integer N such that*

$$\pi_n P = 0$$

for $n \leq N$ and $\pi_{N+1} P \neq 0$. Moreover, $\pi_{N+1} P$ is finitely presented as a $\pi_0 A$–module.

Proof Part (1) is [45, Proposition 8.2.2.7]. The proof of part (2) is the same as in the discrete case. Part (3) is [45, Proposition 8.2.5.4]. Part (4) is [45, Corollary 8.2.5.5]. □

The following lemma will be used later in the paper.

Lemma 2.7 *Let R be a commutative ring spectrum, A an R–algebra, and P and Q compact right A–modules. Then, for any commutative R–algebra S, the natural map*

$$\mathrm{Map}_A(P, Q) \otimes_R S \to \mathrm{Map}_{A \otimes_R S}(P \otimes_R S, Q \otimes_R S) \tag{2}$$

is an equivalence of S–modules.

Proof The statement is clear when P is a suspension of the free A–module A. We prove the lemma by induction on the cells of P. So, suppose that we have a cofiber sequence of compact modules

$$\Sigma^n A \to N \to P$$

such that

$$\mathrm{Map}_A(N, Q) \otimes_R S \to \mathrm{Map}_{A \otimes_R S}(N \otimes_R S, Q \otimes_R S) \tag{3}$$

is an equivalence. We show that (2) holds. We obtain a morphism of cofiber sequences

$$\begin{array}{ccc}
\mathrm{Map}_A(\Sigma^n A, Q) \otimes_R S & \longrightarrow & \mathrm{Map}_{A\otimes_R S}(\Sigma^n A \otimes_R S, Q \otimes_R S) \\
\downarrow & & \downarrow \\
\mathrm{Map}_A(N, Q) \otimes_R S & \longrightarrow & \mathrm{Map}_{A\otimes_R S}(N \otimes_R S, Q \otimes_R S) \\
\downarrow & & \downarrow \\
\mathrm{Map}_A(P, Q) \otimes_R S & \longrightarrow & \mathrm{Map}_{A\otimes_R S}(P \otimes_R S, Q \otimes_R S).
\end{array}$$

Since the left two vertical arrows are equivalences, the right arrow is an equivalence. To finish the proof, we show that if (3) holds for N and if P is a retract of N, then (2) holds. If $N \simeq P \oplus M$, then there is a commutative square of equivalences

$$\begin{array}{ccc}
\mathrm{Map}_A(N, Q) \otimes_R S & \longrightarrow & \mathrm{Map}_A(P, Q) \otimes_R S \oplus \mathrm{Map}_A(M, Q) \otimes_R S \\
\downarrow & & \downarrow \\
\mathrm{Map}_{A\otimes_R S}(N \otimes_R S, Q \otimes_R S) & \longrightarrow & \mathrm{Map}_{A\otimes_R S}(P \otimes_R S, Q \otimes_R S) \\
 & & \oplus \mathrm{Map}_{A\otimes_R S}(M \otimes_R S, Q \otimes_R S).
\end{array}$$

It follows (by looking for instance at cofibers of the vertical maps) that (2) holds. □

The following form of the Morita Theorem is used frequently to show that certain ∞–categories are categories of modules over some $\mathbb{E}_1$–ring spectrum.

Theorem 2.8 (Morita theory) *Let $\mathcal{C}$ be a stable presentable ∞–category, and let P be an object of $\mathcal{C}$. Then, $\mathcal{C}$ is compactly generated by P if and only if*

$$\mathcal{C} \xrightarrow{\mathrm{Map}_{\mathcal{C}}(P,-)} \mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(P)^{\mathrm{op}}} \tag{4}$$

is an equivalence.

Proof One direction is the theorem of Schwede and Shipley, in the form found in Lurie [45, Theorem 8.1.2.1]. So, suppose that (4) is an equivalence. The functor $\mathrm{Map}_{\mathscr{C}}(P,-)$ automatically preserves filtered colimits because it is an equivalence, so we see that P is compact in $\mathscr{C}$. Since $\mathrm{Map}_{\mathscr{C}}(P,-)$ is conservative, it follows from Lemma 2.4 that $\mathscr{C}$ is compactly generated by P. $\square$

2.3 Topologies on affine connective derived schemes

Fix an $\mathbb{E}_\infty$–ring R, and denote by $\mathrm{CAlg}_R^{\mathrm{cn}}$ the ∞–category of connective commutative R–algebras. Set $\mathrm{Aff}_R^{\mathrm{cn}} = (\mathrm{CAlg}_R^{\mathrm{cn}})^{\mathrm{op}}$, the ∞–category of affine connective derived schemes over $\mathrm{Spec}\, R$. We make extensive use of ∞–topoi arising from Grothendieck topologies on $\mathrm{Aff}_R^{\mathrm{cn}}$. For details on the construction of these ∞–topoi see [41, Chapter 6] and [42, Section 5]. All of these topologies arise from pretopologies consisting of special classes of flat morphisms, a notion we now define.

Definition 2.9 A morphism $f\colon S \to T$ of commutative ring spectra is called flat if

$$\pi_0(f)\colon \pi_0 S \to \pi_0 T$$

is a flat morphism of discrete rings and if f induces isomorphisms

$$\pi_k S \otimes_{\pi_0 S} \pi_0 T \xrightarrow{\simeq} \pi_k T$$

for all integers k.

It is useful to use flatness to give a definition of many other properties of morphisms of $\mathbb{E}_\infty$–ring spectra.

Definition 2.10 If P is a property of flat morphisms of discrete commutative rings, such as being faithful or étale, then a morphism $f\colon R \to T$ of commutative rings is said to be P if f is flat in the sense of Definition 2.9 and if $\pi_0(f)$ is P.

The Zariski topology on $\mathrm{Aff}_R^{\mathrm{cn}}$ is the Grothendieck topology generated by Zariski open covers. Here, a map $\mathrm{Spec}\, T \to \mathrm{Spec}\, S$ is a Zariski open cover if the associated map on ring spectra $S \to T$ is flat and induces a Zariski open cover $\mathrm{Spec}\, \pi_0 T \to \mathrm{Spec}\, \pi_0 S$. The associated ∞–topos of Zariski sheaves is denoted by $\mathrm{Shv}_R^{\mathrm{Zar}}$. Similarly, there is an étale topology on $\mathrm{Aff}_R^{\mathrm{cn}}$ and an associated étale ∞–topos $\mathrm{Shv}_R^{\text{ét}}$. We say a map $\mathrm{Spec}\, T \to \mathrm{Spec}\, S$ is étale if $S \to T$ is flat and étale. Both of these Grothendieck topologies are constructed, explicitly, via the method of [42, Proposition 5.1]; see [42, Proposition 5.4] for how to do this for the flat topology.

2.4 Tor-amplitude

Most of the material below on Tor-amplitude and perfect modules was developed in [11, Exposé I]. We refer also to the exposition in [56]. In the simplicial commutative setting, this is treated in Toën and Vaquié [58]. Throughout this section, R is a connective commutative ring spectrum. We refer to compact R–modules as perfect R–modules. This is to agree with the terminology in the references. Over a scheme X, a complex of quasicoherent $\mathcal{O}_X$–modules is called perfect if its restriction to any affine subscheme is perfect, or, equivalently, compact. While the perfect and compact modules agree for affine schemes, on a general scheme X not every perfect module is compact.

Definition 2.11 An R–module P has Tor-amplitude contained in the interval $[a,b]$ if for any $\pi_0 R$–module M (any module, not any complex of modules),

$$\mathrm{H}_i(P \otimes_R M) = 0$$

for $i \notin [a,b]$. If such integers a,b exist, then P is said to have finite Tor-amplitude.

If P is an R–module, then P has Tor-amplitude contained in $[a,b]$ if and only if $P \otimes_R \pi_0 R$ is a complex of $\pi_0 R$–modules with Tor-amplitude contained in $[a,b]$ in the ordinary sense. Note, however, that our definition differs from that in [11, I 5.2] simply in that we work with homology instead of cohomology.

The next proposition is used in the proof of the proposition that follows, but it seems interesting in its own right.

Proposition 2.12 *The functor*

$$\pi_0\colon \mathrm{Ho}(\mathrm{Mod}_R^{\mathrm{proj}}) \to \mathrm{Mod}_{\pi_0 R}^{\mathrm{proj}}$$

is an equivalence, where the decoration proj *denotes the full subcategory of projective modules.*

Proof This is a special case of [45, Corollary 8.2.2.19]. The analogous map on free modules is an equivalence. Since projectives are summands of free modules, we deduce that the functor π_0 above is fully faithful.

Let P be a projective $\pi_0 R$–module. Then, there exists a free $\pi_0 R$–module F and an idempotent homomorphism $e\colon F \to F$ such that P is the image of e. By definition, P is also the filtered colimit of

$$F \xrightarrow{e} F \xrightarrow{e} F \to \cdots$$

in $\mathrm{Mod}_{\pi_0 R}$. Lift the diagram to a diagram of free R–modules F', and let P' be the filtered colimit. The R–module P' is projective because we can construct a splitting of $F' \to P'$ by mapping F' to each F' in the diagram via the idempotent e'. Then, $\pi_0(P')$ is isomorphic to P. So, the functor is essentially surjective, and hence an equivalence of categories. □

The following proposition provides the technical results needed on perfect modules over connective $\mathbb{E}_\infty$–ring spectra. In particular, parts (4)–(7) will be the key to giving certain inductive proofs about the moduli of objects in module categories in Section 5. We emphasize again that it was the insight of Toën and Vaquié [58] that suggests this approach to studying perfect objects in the context of simplicial commutative rings.

Proposition 2.13 *Let P and Q be R–modules.*

(1) *If P is perfect, then P has finite Tor-amplitude.*

(2) *If R' is a connective commutative R–algebra, and if P is an R–module with Tor-amplitude contained in $[a,b]$, then $P \otimes_R R'$ is an R'–module with Tor-amplitude contained in $[a,b]$.*

(3) *If P has Tor-amplitude contained in $[a,b]$ and Q has Tor-amplitude contained in $[c,d]$, then $P \otimes_R Q$ has Tor-amplitude contained in $[a+c, b+d]$.*

(4) *If P and Q have Tor-amplitude contained in $[a,b]$, then for any morphism $P \to Q$, the cofiber has Tor-amplitude contained in $[a, b+1]$. Dually, the fiber has Tor-amplitude contained in $[a-1, b]$.*

(5) *If P is a perfect R–module with Tor-amplitude contained in $[0,b]$, with $0 \le b$, then P is connective, and $\pi_0 P = \mathrm{H}_0(P \otimes_R \pi_0 R)$.*

(6) *If P is perfect and has Tor-amplitude contained in $[a,a]$, then P is equivalent to $\Sigma^a M$ for a finitely generated projective R–module M.*

(7) *If P is perfect and has Tor-amplitude contained in $[a,b]$, then there exists a morphism*
$$\Sigma^a M \to P$$
such that M is a finitely generated projective R–module and the cofiber is perfect and has Tor-amplitude contained in $[a+1, b]$.

Proof Part (1) follows from [56, Propositions 2.2.12, 2.3.1.(d)]. That the notions of perfection in Thomason, Trobaugh and Lurie agree is explained by [56, Theorem 2.4.4], which is applicable here as the modules which appear in Mod_R are all quasicoherent, and so have quasicoherent homology. Parts (2) and (3) are [11, Proposition 5.6]. If C is the cofiber of $P \to Q$, and if M is a $\pi_0 R$–module, then

$$P \otimes_R M \to Q \otimes_R M \to C \otimes_R M$$

is a cofiber sequence in $\mathrm{Mod}_{\pi_0 R}$. The case of a fiber is dual. Thus, part (4) follows immediately from the long exact sequence in homology.

Consider the Tor spectral sequence

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_* P, \pi_0 R)_q \Rightarrow \pi_{p+q}(P \otimes_R \pi_0 R) = \mathrm{H}_{p+q}(P \otimes_R \pi_0 R)$$

with differentials $d^r_{p,q}$ of degree $(-r, r-1)$ constructed by Elmendorf, Kriz, Mandell and May in [23]. If P is a nontrivial perfect R–module with Tor-amplitude contained in $[0, b]$, then the abutment of the spectral sequence is 0 when $p+q<0$. We know that P has a bottom homotopy group, say π_k. That is, $\pi_k P$ is nonzero, and $\pi_j P = 0$ for $j<k$. Calculating the graded tensor product, we see that $\mathrm{E}^2_{0,k}$ is the coequalizer of

$$\bigoplus_{i+j=k} \pi_i P \otimes_{\pi_0 R} \pi_j R \rightrightarrows \pi_k P$$

in the category of graded $\pi_0 R$–modules. So, $\mathrm{E}^2_{0,k} = \pi_k P$ as a $\pi_0 R$–module. But, by our hypothesis on k, no nonzero differential may hit $E^2_{0,k}$. All differentials out are zero for degree reasons. It follows that $\pi_k(P \otimes_R \pi_0 R) \neq 0$. Therefore, $k \geq 0$, and P is connective. This proves the first statement of part (5), and the second statement follows easily from the same argument.

To prove part (6), we may assume that $a=0$. By [56, Proposition 2.3.1.(d)], we may assume that $P \otimes_R \pi_0 R$ is a bounded complex of finitely generated projective $\pi_0 R$–modules. Because the kernel of a surjective map of finitely generated projective modules is finitely generated projective, by induction, the good truncation $\tau_{\geq 0} P \otimes_R \pi_0 R \xrightarrow{\sim} P \otimes_R \pi_0 R$ is a bounded complex of finitely generated projective $\pi_0 R$–modules that is concentrated in nonnegative degrees. We show now that $\pi_0 P$ is a projective $\pi_0 R$–module. By part (5), $\pi_0 P \cong \mathrm{H}_0(P \otimes_R \pi_0 R)$. Since the homology is zero above degree 0, the good truncation $\tau_{\geq 0} P \otimes_R \pi_0 R$ is a resolution of the finitely presented $\pi_0 R$–module $\mathrm{H}_0(P \otimes_R \pi_0 R)$ by finitely generated projective $\pi_0 R$–modules. It suffices to show that $\mathrm{H}_0(P \otimes_R \pi_0 R)$ is flat by Matsumura [46, Theorem 7.12]. If M is a $\pi_0 R$–module, the Tor spectral sequence computing $\mathrm{H}_*(P \otimes_R \pi_0 R \otimes_{\pi_0 R} M)$ is

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}_p^{\pi_0 R}(\mathrm{H}_*(P \otimes_R \pi_0 R), M)_q \Rightarrow \mathrm{H}_{p+q}(P \otimes_R \pi_0 R \otimes_{\pi_0 R} M).$$

But, for $q>0$, $\mathrm{E}^2_{p,q} = 0$, so that for $p>0$

$$\mathrm{Tor}_p^{\pi_0 R}(\mathrm{H}_0(P \otimes_R \pi_0 R), M) \cong \mathrm{H}_p(P \otimes_R M) = 0,$$

by the Tor-amplitude of P. Thus, $\mathrm{H}_0(P \otimes_R \pi_0 R)$ is flat.

Thus, by the previous theorem and the connectivity of P, there is a natural map

$$Q \to P,$$

where Q is a finitely generated projective R–module and $\pi_0 Q \cong \pi_0 P$. It suffices to show that the cofiber C of this map is equivalent to zero. The R–module C is perfect and has the property that $C \otimes_R \pi_0 R$ is zero. Let $\pi_k C$ be the first nonzero homotopy group of C. Then, $\Sigma^{-k} C$ is connective with Tor-amplitude contained in $[0,0]$. By part (5), $\pi_k C = \mathrm{H}_0(\Sigma^{-k} C \otimes_R \pi_0 R) = 0$, a contradiction. Thus, $C \simeq 0$.

To prove part (7), we assume that $a = 0$. If $b = 0$, the statement follows from part (6). Thus, assume that $b > 0$, and consider $P \otimes_R \pi_0 R$, which is a perfect complex over $\pi_0 R$ with bounded homology. As above, we may assume that $P \otimes_R \pi_0 R$ is in fact a bounded complex of finitely generated projective $\pi_0 R$–modules concentrated in nonnegative degrees. Thus, there is a natural morphism of complexes $Z_0 \to P \otimes_R \pi_0 R$ which induces a surjection in degree 0 homology. Lift Z_0 to a finitely generated projective R–module M, by Proposition 2.12. We can write Z_0 as a split summand of $\pi_0 R^n$, and hence M as a split summand of R^n. Since P is connective by part (5), the composition

$$\pi_0 R^n \to Z_0 \to P \otimes_R \pi_0 R$$

lifts to a map $R^n \to P$. Composing with $M \to R^n$, we obtain a map $M \to P$ which is a surjection on H_0. By the long exact sequence in homology, the cofiber has Tor-amplitude contained in $[1,b]$ (remembering that $b > 0$). Moreover, the cofiber is perfect by the two out of three property for perfect modules [56, Proposition 2.2.13.(b)]. □

2.5 Vanishing loci

We show that the complement of the support of a perfect complex on an affine derived scheme is a quasicompact open subscheme. Recall that a morphism of schemes $X \to Y$ is quasicompact if for every open affine $\operatorname{Spec} R$ of Y, the pullback $X \times_Y \operatorname{Spec} R$ is quasicompact; see Grothendieck [29, Definition I 6.1.1]. The following result is due to Thomason [55] in the ordinary setting of discrete rings, and to Toën and Vaquié [58] for simplicial commutative rings.

Proposition 2.14 *Let R be a connective commutative ring spectrum, and let P be a perfect R–module. The subfunctor $V_P \subseteq \operatorname{Spec} R$ of points $R \to S$ such that $P \otimes_R S$ is quasi-isomorphic to zero is a quasicompact Zariski open immersion.*

Proof If R is discrete, the proposition is [55, Lemma 3.3.c]. To prove the proposition when R is a connective commutative ring spectrum, let $Q = P \otimes_R \pi_0 R$, which is a perfect complex of $\pi_0 R$–modules. Let V_Q be the quasicompact Zariski open subscheme of $\operatorname{Spec} \pi_0 R$ specified by the vanishing of Q by the discrete case. Choose elements $f_1, \dots, f_n \in \pi_0 R$ such that V_Q is the union of the $\operatorname{Spec} \pi_0 R[1/f_i]$. We claim that V_P is the union V of the $\operatorname{Spec} R[1/f_i]$ in $\operatorname{Spec} R$. But, because P is a perfect R–module, given an S–point $\operatorname{Spec} S \to V \subseteq \operatorname{Spec} R$ of V, then

$$(P \otimes_R S) \otimes_S \pi_0 S \simeq 0$$

if and only if

$$P \otimes_R S \simeq 0.$$

Indeed, $P \otimes_R S$ has a bottom homotopy group, say of degree k, and it follows from the proof of Proposition 2.13(5) that

$$\pi_k P \otimes_R S \cong \mathrm{H}_k((P \otimes_R S) \otimes_S \pi_0 S). \qquad \square$$

3 Module categories and their module categories

In this section, we examine the algebra of module categories of $\mathbb{E}_\infty$–ring spectra, viewed as $\mathbb{E}_\infty$–monoids in the ∞–category of stable presentable ∞–categories. This leads to an important module-theoretic characterization of Azumaya R–algebras for an $\mathbb{E}_\infty$–ring spectrum R: an R–algebra A is Azumaya if and only if Mod_A is an invertible Mod_R–module.

3.1 R–linear categories

In [41, Chapter 5], Lurie constructs the ∞–category Pr^{L} of presentable ∞–categories and colimit preserving functors. We refer to Lurie's book for the precise definition and properties of presentable ∞–categories. For us, the main points are that a presentable ∞–category is closed under small limits and colimits and is κ–compactly generated for some infinite regular cardinal κ. Moreover, the ∞–category Pr^{L} is also closed under small limits and colimits, and there is a symmetric monoidal structure on Pr^{L} with unit object the ∞–category of pointed spaces [45, Section 6.3].

A critical fact about Pr^{L} is that if R is an $\mathbb{E}_k$–ring spectrum ($1 \le k \le \infty$), then the ∞–category of right R–modules Mod_R is an $\mathbb{E}_{k-1}$–monoidal stable presentable ∞–category with unit R (where $\infty - 1 = \infty$). We can equivalently view Mod_R as an $\mathbb{E}_{k-1}$–algebra in Pr^{L} by [45, Proposition 8.1.2.6] in this case. This decrease in coherent commutativity is the analogue of the usual fact that there is no tensor product of right A–modules when A is an associative ring. Thus, by [45, Corollary 6.3.5.17], when $2 \leq k \leq \infty$ we may build an ∞–category Cat_R of (right) Mod_R–modules in Pr^{L}. In the notation of [45],

$$\mathrm{Cat}_R = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}}).$$

This ∞–category is $\mathbb{E}_{k-2}$–monoidal and is closed under small limits and colimits. Moreover, the $\mathbb{E}_{k-2}$–monoidal structure is closed; see [45, Remark 6.3.1.17] and the beginning of the next section. The dual of $\mathscr{C}$ is

$$\mathrm{D}_R\mathscr{C} = \mathrm{Fun}^{\mathrm{L}}_R(\mathscr{C}, \mathrm{Mod}_R),$$

the functor category of left adjoint R–linear functors from $\mathscr{C}$ to Mod_R. When R is the sphere spectrum, Cat_R is also denoted by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$; it is the ∞–category of *stable* presentable ∞–categories and colimit preserving functors. Since Mod_R is stable, we could also define Cat_R as $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$. We will refer to the objects of Cat_R as R–linear categories. An R–linear category is thus a stable ∞–category with an enrichment in Mod_R: there are functorial R–module mapping spectra $\mathrm{Map}_{\mathscr{C}}(x, y)$ for x, y in $\mathscr{C}$.

We may also consider the ∞–category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ of compactly generated stable presentable ∞–categories with morphisms the colimit preserving functors that preserve compact objects. Then, $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ inherits a symmetric monoidal structure from $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, as one can check by using the proof of [45, 6.3.1.14] in the ω–compactly generated situation. The ∞–category Mod_R is again an $\mathbb{E}_{k-1}$–monoid in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$, and so we can consider the ∞–category $\mathrm{Cat}_{R,\omega}$ of compactly generated R–linear categories and colimit preserving functors that preserve compact objects. The natural map $\mathrm{Cat}_{R,\omega} \to \mathrm{Cat}_R$ is an $\mathbb{E}_{k-2}$–monoidal map of ∞–categories.

There is a natural equivalence

$$\mathrm{Ind}\colon \mathrm{Cat}^{\mathrm{perf}}_{\infty} \leftrightarrows \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} :(-)^{\omega}$$

of symmetric monoidal ∞–categories, where $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$ is the symmetric monoidal ∞–category of small idempotent complete stable ∞–categories and exact functors. One may also view ∞–category $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$ as the localization of the ∞–category of spectrally enriched categories $\mathrm{Cat}_{\mathrm{Sp}}$ given by inverting the maps $\mathscr{A} \to \mathrm{Mod}^{\omega}_{\mathscr{A}}$ for all (compact) spectral categories $\mathscr{A}$. For details, see [12]. If R is an $\mathbb{E}_k$–ring, this equivalence sends the $\mathbb{E}_{k-1}$–algebra Mod_R to Mod^{ω}_R in $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$. Thus, it induces an equivalence between $\mathrm{Cat}_{R,\omega}$ and $\mathrm{Mod}_{\mathrm{Mod}^{\omega}_R}(\mathrm{Cat}^{\mathrm{perf}}_{\infty})$.

In the rest of this section, we prove some technical results relating algebras and their module categories, which we will need later in the paper. While the statements are true for $\mathbb{E}_k$–ring spectra with $3 \leq k \leq \infty$, for simplicity we treat only $\mathbb{E}_\infty$–ring spectra. Fix an $\mathbb{E}_\infty$–ring R. Let

$$\mathrm{Mod}_*\colon \mathrm{Alg}_R \to (\mathrm{Cat}_R)_{\mathrm{Mod}_R/}$$

be the symmetric monoidal functor which sends an R–algebra A to the R–linear category of right A–modules Mod_A with basepoint A. We abuse notation and write Mod_A for the object (Mod_A, A) of $(\mathrm{Cat}_R)_{\mathrm{Mod}_R/}$. There are analogous functors $\mathrm{Mod}_{*,\omega}\colon \mathrm{Alg}_R \to (\mathrm{Cat}_{R,\omega})_{\mathrm{Mod}_R/}$, and we can forget the basepoint to obtain $\mathrm{Mod}\colon \mathrm{Alg}_R \to \mathrm{Cat}_R$ and $\mathrm{Mod}_\omega\colon \mathrm{Alg}_R \to \mathrm{Cat}_{R,\omega}$.

There is an adjunction

$$\mathrm{Mod}_*\colon \mathrm{Alg}_R \rightleftarrows (\mathrm{Cat}_R)_{\mathrm{Mod}_R/} :\!\mathrm{End}$$

where the right adjoint End takes a pointed R–linear category and sends it to the R–algebra of endomorphisms of the distinguished object.

Proposition 3.1 *For an $\mathbb{E}_\infty$–ring R, the functors* $\mathrm{Mod}_*\colon \mathrm{Alg}_R \to (\mathrm{Cat}_R)_{\mathrm{Mod}_R/}$ *and* $\mathrm{Mod}_{*,\omega}\colon \mathrm{Alg}_R \to (\mathrm{Cat}_{R,\omega})_{\mathrm{Mod}_R/}$ *are fully faithful.*

Proof To check the first statement, for R–algebras A and B, consider the fiber sequence

$$\mathrm{map}_{\mathrm{Mod}_R/}(\mathrm{Mod}_A, \mathrm{Mod}_B) \to \mathrm{map}_R(\mathrm{Mod}_A, \mathrm{Mod}_B) \to \mathrm{map}_R(\mathrm{Mod}_R, \mathrm{Mod}_B).$$

Since Mod_A is dualizable with dual $\mathrm{Mod}_{A^{\mathrm{op}}}$ and using that the symmetric monoidal structure on Cat_R is closed, we can rewrite the fiber sequence as

$$\mathrm{map}_{\mathrm{Mod}_R/}(\mathrm{Mod}_A, \mathrm{Mod}_B) \to \mathrm{Mod}^{\mathrm{eq}}_{A^{\mathrm{op}} \otimes_R B} \to \mathrm{Mod}^{\mathrm{eq}}_B.$$

The fiber of the map over B is equivalent to the space of $A^{\mathrm{op}} \otimes_R B$–module structures compatible with the B–module structure on B, which is simply

$$\mathrm{map}_{\mathrm{Alg}_R}(A, \mathrm{End}_B(B)) \simeq \mathrm{map}_{\mathrm{Alg}_R}(A, B).$$

So, the functor is fully faithful.

To check the second statement, simply note that there is a pullback square

$$\begin{tikzcd}
\mathrm{map}^\omega_R(\mathrm{Mod}_A, \mathrm{Mod}_B) \arrow[r] \arrow[d] & \mathrm{map}^\omega_R(\mathrm{Mod}_R, \mathrm{Mod}_B) \arrow[d] \\
\mathrm{map}_R(\mathrm{Mod}_A, \mathrm{Mod}_B) \arrow[r] & \mathrm{map}_R(\mathrm{Mod}_R, \mathrm{Mod}_B).
\end{tikzcd}$$

of mapping spaces, so the fibers are equivalent. $\square$

Corollary 3.2 *If A is an R–algebra, then the fiber over a compact R–module P of the forgetful map*
$$\mathrm{map}_R^{\omega}(\mathrm{Mod}_A, \mathrm{Mod}_R) \to \mathrm{Mod}_R^{\mathrm{eq}}$$
is naturally equivalent to $\mathrm{map}_{\mathrm{Alg}_R}(A, \mathrm{End}_R(P))$.

Despite the fact that Mod_R is the unit of the symmetric monoidal structure on $\mathrm{Cat}_{R,\omega}$, it is not formal that Mod_R is a compact object in $\mathrm{Cat}_{R,\omega}$. The fact that it is compact is essential in deducing that Azumaya algebras are compact R–algebras (and not just compact as R–modules).

Theorem 3.3 *The unit* Mod_R *is a compact object of* $\mathrm{Cat}_{R,\omega}$.

Proof We begin by showing that the ∞–category of spectra is compact in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$. Equivalently, we must show that the functor $\mathrm{map}(\mathrm{Sp}, -)\colon \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \to \mathcal{S}$ preserves filtered colimits. Since Δ^0 is a compact object of Cat_∞, the underlying space functor $\mathrm{Cat}_\infty \to \mathcal{S}$ preserves filtered colimits, and we see that it is enough to show that
$$\mathrm{Fun}^{\mathrm{L},\omega}(\mathcal{S}, -)\colon \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \to \mathrm{Cat}_\infty$$
preserves filtered colimits. By [41, Proposition 5.5.7.11], we have that the forgetful functor $\mathrm{Cat}_\infty^{\mathrm{Rex}(\omega)} \to \mathrm{Cat}_\infty$ preserves filtered colimits where $\mathrm{Cat}_\infty^{\mathrm{Rex}(\omega)}$ denotes the ∞–category of finitely cocomplete ∞–categories and finite colimit-preserving functors. Recall that taking compact objects $(-)^\omega$ identifies $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ with the full subcategory $\mathrm{Cat}_\infty^{\mathrm{perf}} \subseteq \mathrm{Cat}_\infty^{\mathrm{Rex}(\omega)}$ consisting of the stable and idempotent-complete objects. Moreover, this inclusion admits a left adjoint
$$\mathrm{Stab}(\mathrm{Ind}(-))^\omega\colon \mathrm{Cat}_\infty^{\mathrm{Rex}(\omega)} \to \mathrm{Cat}_\infty^{\mathrm{perf}},$$
and the functor $(-)^\omega\colon \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$ admits a left-adjoint Ind given by ind-completion.

Let $\mathrm{colim}_i\, \mathcal{C}_i \simeq \mathcal{C}$ be a filtered colimit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$. It follows that the canonical map $\mathrm{colim}_i\, \mathcal{C}_i^\omega \to \mathcal{C}^\omega$ is an idempotent completion, so it is fully faithful and any object P in $\mathcal{C}^\omega$ is a retract of an object Q in $\mathrm{colim}_i\, \mathcal{C}_i^\omega$. In particular, there is an idempotent $e \in \pi_0\, \mathrm{end}(Q)$ such that P is the cofiber of

$$\bigoplus_{k=0}^{\infty} Q \xrightarrow{1-s(e)} \bigoplus_{k=0}^{\infty} Q, \tag{5}$$

where $s(e)$ is the map which on the k^{th} component maps Q to the $(k+1)^{\mathrm{st}}$ component via e. Since the colimit $\mathrm{colim}_i\, \mathcal{C}_i^\omega$ is computed in Cat_∞, it follows that Q is the image of an object Q_i in $\mathcal{C}_i^\omega$ for some i. Write Q_j for the image of Q_i in $\mathcal{C}_j$. Because mapping spaces in filtered colimits of ∞–categories are given as the filtered colimit of the mapping spaces, there is a natural equivalence

$$\operatorname*{colim}_{j\geq i} \operatorname{end}(Q_j) \simeq \operatorname{end}(Q).$$

It follows that we may lift e to an idempotent e_j of Q_j for some $j \geq i$. Define P_j to be the summand of Q_j split off by this idempotent as in (5). Then, P_j is compact object of $\mathscr{C}_j$ which maps to P in the colimit. It follows that $\operatorname{colim}_i \mathscr{C}_i^{\omega} \to \mathscr{C}^{\omega}$ is essentially surjective and hence an equivalence.

To deduce that, in general, Mod_R is a compact object of $\mathrm{Cat}_{R,\omega}$, it suffices to note that the forgetful functor $\mathrm{Cat}_{R,\omega} \simeq \mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}) \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ preserves filtered colimits. This follows from [45, Corollary 3.4.4.6], which is applicable because $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \simeq \mathrm{Cat}_{\infty}^{\mathrm{perf}}$, as a symmetric monoidal ∞–categories and the symmetric monoidal structure is closed by [12, Theorem 2.14]. □

From the theorem, we deduce an important fact about the endomorphism functor.

Lemma 3.4 *The right adjoint* $\mathrm{End}\colon (\mathrm{Cat}_{R,\omega})_{\mathrm{Mod}_R/} \to \mathrm{Alg}_R$ *of* Mod_* *preserves filtered colimits.*

Proof A map $\mathrm{Mod}_R \to \mathscr{C}$ in $\mathrm{Cat}_{R,\omega}$ classifies a compact object of $\mathscr{C}$, ie, a pointed R–linear category. Let $\operatorname{colim}_i \mathscr{C}_i \simeq \mathscr{C}$ be a colimit of pointed compactly generated R–linear categories. Let X_i be the image of R in $\mathscr{C}_i$, and let X be the image of R in $\mathscr{C}$. Consider the map of R–algebras

$$\operatorname*{colim}_i \mathrm{End}_{\mathscr{C}_i}(X_i) \longrightarrow \mathrm{End}_{\mathscr{C}}(X).$$

Since the forgetful functors

$$\mathrm{Alg}_R \to \mathrm{Mod}_R \to \mathrm{Sp} \xrightarrow{\Omega^{\infty}\Sigma^n} \mathrm{Spaces}$$

preserve filtered colimits and taken together they detect filtered colimits in Alg_R, it is enough to show that

$$\operatorname*{colim}_i \operatorname{end}_{\mathscr{C}_i^{\omega}}(X_i) \to \operatorname{end}_{\mathscr{C}^{\omega}}(X)$$

is an equivalence. This follows because we know that the filtered colimit of pointed compactly generated R–linear categories agrees with the filtered colimit as compactly generated R–linear categories with the obvious basepoint and, by the theorem, $\operatorname{colim}_i \mathscr{C}_i^{\omega} \simeq \mathscr{C}^{\omega}$ in Cat_{∞}. □

We now prove the important fact that compactness of an R–algebra A is detected purely through the module category of A.

Proposition 3.5 *Let A be an R–algebra. Then, A is compact in Alg_R if and only if Mod_A is compact in $\mathrm{Cat}_{R,\omega}$.*

Proof Assume first that A is compact in Alg_R, and let $\mathscr{C}$ be a filtered colimit of a diagram $\{\mathscr{C}_i\}_{i\in I}$ in $\mathrm{Cat}_{R,\omega}$. Because End preserves filtered colimits by the previous lemma, it is clear that $\mathrm{Mod}_*\colon \mathrm{Alg}_R \to (\mathrm{Cat}_{R,\omega})_{\mathrm{Mod}_R/}$ preserves compact objects. Every object $M \in \operatorname{colim}_i \operatorname{map}(\mathrm{Mod}_R, \mathscr{C}_i) \simeq \operatorname{map}(\mathrm{Mod}_R, \mathscr{C})$ comes from a collection of objects M_i of $\mathscr{C}_i$ for i sufficiently large. For any such M there is a map of fiber sequences

$$
\begin{array}{ccc}
\operatorname{colim}_i \operatorname{map}_{\mathrm{Alg}_R}(A, \mathrm{End}_{\mathscr{C}_i}(M_i)) & \longrightarrow & \operatorname{map}_{\mathrm{Alg}_R}(A, \mathrm{End}_{\mathscr{C}}(M)) \\
\downarrow & & \downarrow \\
\operatorname{colim}_i \operatorname{map}(\mathrm{Mod}_A, \mathscr{C}_i) & \longrightarrow & \operatorname{map}(\mathrm{Mod}_A, \mathscr{C}) \\
\downarrow & & \downarrow \\
\operatorname{colim}_i \operatorname{map}(\mathrm{Mod}_R, \mathscr{C}_i) & \longrightarrow & \operatorname{map}(\mathrm{Mod}_R, \mathscr{C}),
\end{array}
\tag{6}
$$

where the top sequence is a fiber sequence because filtered colimits commute with finite colimits by [41, Proposition 5.3.3.3]. Since Mod_R is compact in $\mathrm{Cat}_{R,\omega}$ and A is compact in Alg_R, the left and right vertical arrows are equivalences. Since this is true for every point of $\operatorname{map}(\mathrm{Mod}_R, \mathscr{C})$, the middle arrow is an equivalence. Thus, Mod_A is compact in $\mathrm{Cat}_{R,\omega}$.

Now, assume that Mod_A is compact in $\mathrm{Cat}_{R,\omega}$. Using (6) and the adjunction

$$\operatorname{map}_{\mathrm{Mod}_R/}(\mathrm{Mod}_A, \mathscr{C}) \simeq \operatorname{map}_{\mathrm{Alg}_R}(A, \mathrm{End}_{\mathscr{C}}(M)),$$

it is easy to see that Mod_A, with basepoint A, is also compact in $(\mathrm{Cat}_{R,\omega})_{\mathrm{Mod}_R/}$. Let $B = \operatorname{colim} B_i$ be a filtered colimit of R–algebras. Then, there are equivalences,

$$
\begin{aligned}
\operatorname{colim} \operatorname{map}_{\mathrm{Alg}_R}(A, B_i) &\simeq \operatorname{colim} \operatorname{map}_{\mathrm{Alg}_R}(A, \mathrm{End}_{\mathrm{Mod}_{B_i}}(B_i)) \\
&\simeq \operatorname{colim} \operatorname{map}_{\mathrm{Mod}_R/}(\mathrm{Mod}_A, \mathrm{Mod}_{B_i}) \\
&\simeq \operatorname{map}_{\mathrm{Mod}_R/}(\mathrm{Mod}_A, \mathrm{Mod}_B) \\
&\simeq \operatorname{map}_{\mathrm{Alg}_R}(A, B).
\end{aligned}
$$

That is, A is a compact object in Alg_R. $\square$

Corollary 3.6 *Compactness is a Morita-invariant property of R–algebras.*

3.2 Smooth and proper algebras

If $\mathscr{C}$ is an object of Cat_R, then the dual of $\mathscr{C}$ is the functor category

$$\mathrm{D}_R\mathscr{C} = \mathrm{Fun}^{\mathrm{L}}_R(\mathscr{C}, \mathrm{Mod}_R)$$

in Mod_R. There is a functorial evaluation map

$$\mathscr{C} \otimes_R \mathrm{D}_R\mathscr{C} \xrightarrow{\mathrm{ev}} \mathrm{Mod}_R.$$

The R–linear category $\mathscr{C}$ is dualizable if there exists a coevaluation map

$$\mathrm{Mod}_R \xrightarrow{\mathrm{coev}} \mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C},$$

which classifies $\mathscr{C}$ as a $\mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C}$–module, such that both maps

$$\mathscr{C} \xrightarrow{\mathscr{C}\otimes_R \mathrm{coev}} \mathscr{C} \otimes_R \mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C} \xrightarrow{\mathrm{ev}\otimes_R \mathscr{C}} \mathscr{C},$$

$$\mathrm{D}_R\mathscr{C} \xrightarrow{\mathrm{coev}\otimes \mathrm{D}_R\mathscr{C}} \mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C} \otimes_R \mathrm{D}_R\mathscr{C} \xrightarrow{\mathrm{D}_R\mathscr{C}\otimes \mathrm{ev}} \mathrm{D}_R\mathscr{C},$$

are equivalent to the identity.

Lemma 3.7 *An object $\mathscr{C}$ is dualizable in $\mathrm{Cat}_{R,\omega}$ if and only if it is dualizable in Cat_R and the evaluation and coevaluation morphisms of its duality data in Cat_R are morphisms in $\mathrm{Cat}_{R,\omega}$.*

Proof Dualizability is detected on the monoidal homotopy category, and the duality data for $\mathscr{C}$ in $\mathrm{Ho}(\mathrm{Cat}_{R,\omega})$ must coincide with the duality data in $\mathrm{Ho}(\mathrm{Cat}_R)$ by uniqueness. □

Definition 3.8 A compactly generated R–linear category $\mathscr{C}$ is proper if its evaluation map is in $\mathrm{Cat}_{R,\omega}$; it is smooth if it is dualizable and its coevaluation map is in $\mathrm{Cat}_{R,\omega}$.

If A is an R–algebra, then Mod_A is proper if and only if A is a perfect R–module. Indeed, in this case, the evaluation map is the map

$$\mathrm{Mod}_{A\otimes_R A^{\mathrm{op}}} \simeq \mathrm{Mod}_A \otimes_R \mathrm{Mod}_{A^{\mathrm{op}}} \to \mathrm{Mod}_R$$

that sends $A\otimes_R A^{\mathrm{op}}$ to A. We say in this case that A is a proper R–algebra. Similarly, Mod_A is smooth if and only if the coevaluation map $\mathrm{Mod}_R \to \mathrm{Mod}_{A^{\mathrm{op}}\otimes_R A}$, which sends R to A, considered as an $A^{\mathrm{op}} \otimes_R A$–module, exists and is in $\mathrm{Cat}_{R,\omega}$. So we see that Mod_A is smooth if and only if A is perfect as an $A^{\mathrm{op}} \otimes_R A$–module. Again, we say in this case that A is a smooth R–algebra. In fact, every smooth R–linear category is equivalent to a module category.

Lemma 3.9 *Suppose that $\mathscr{C}$ is a smooth R–linear category. Then, $\mathscr{C} \simeq \mathrm{Mod}_A$ for some R–algebra A.*

Proof See [57, Lemma 2.6]. The morphism

$$\mathrm{Mod}_R \xrightarrow{\mathrm{coev}} \mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C}$$

is in $\mathrm{Cat}_{R,\omega}$ by hypothesis. Thus, R is sent by coev to a compact object of $\mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C}$. The compact objects of this category are the smallest idempotent complete stable subcategory of $\mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C}$ containing the objects of the form $\mathrm{Map}_{\mathscr{C}}(a,-) \otimes_R b$, where a and b are compact objects of $\mathscr{C}$. This is because $\mathscr{C}$ is compactly generated, so the compact objects of $\mathrm{D}_R\mathscr{C}$ are precisely the duals of the compact objects of $\mathscr{C}$. We can thus write $\mathrm{coev}(R)$ as the result of taking finitely many shifts, cones, and summands of $\mathrm{Map}_{\mathscr{C}}(a_i,-) \otimes_R b_i$, for $i = 1, \ldots, n$. The identity map

$$\mathscr{C} \xrightarrow{\mathscr{C} \otimes_R \mathrm{coev}} \mathscr{C} \otimes_R \mathrm{D}_R\mathscr{C} \otimes_R \mathscr{C} \xrightarrow{\mathrm{ev} \otimes_R \mathscr{C}} \mathscr{C}$$

sends $c \in \mathscr{C}$ to the same diagram built out of $\mathrm{Map}_{\mathscr{C}}(a_i, c) \otimes_R b_i$. It follows that if $\mathrm{Map}_{\mathscr{C}}(a_i, c) \simeq 0$ for $i = 1, \ldots, n$, then $c \simeq 0$. Thus, the a_i form a set of compact generators for $\mathscr{C}$. Letting $A = \mathrm{End}_{\mathscr{C}}(\bigoplus_i a_i)^{\mathrm{op}}$, we get $\mathscr{C} \simeq \mathrm{Mod}_A$ as desired. □

Definition 3.10 An R–linear category is of finite type if there exists a compact R–algebra A such that $\mathscr{C}$ is equivalent to Mod_A.

The condition of being smooth and proper is a strong one for R–algebras: it implies compactness in the ∞–category of R–algebras; see [58, Corollary 2.13] for the dg–statement.

Proposition 3.11 *If $\mathscr{C}$ is a smooth and proper R–linear category, then $\mathscr{C}$ is of finite type.*

Proof Let A be an R–algebra such that $\mathscr{C} \simeq \mathrm{Mod}_A$. To show that A is compact as an R–algebra, it suffices by Proposition 3.5 to show that $\mathscr{C}$ is compact in $\mathrm{Cat}_{R,\omega}$. To this end, fix a filtered colimit $\mathscr{D} = \mathrm{colim}_{i \in I} \mathscr{D}_i$ in $\mathrm{Cat}_{R,\omega}$; we must show that the natural map

$$\operatorname*{colim}_{i \in I} \mathrm{map}_{\mathrm{Cat}_{R,\omega}}(\mathscr{C}, \mathscr{D}_i) \to \mathrm{map}_{\mathrm{Cat}_{R,\omega}}(\mathscr{C}, \mathscr{D})$$

is an equivalence. The dualizability of $\mathscr{C}$ in $\mathrm{Cat}_{R,\omega}$ gives natural equivalences

$$\mathrm{map}_{\mathrm{Cat}_{R,\omega}}(\mathscr{C}, \mathscr{D}) \simeq \mathrm{map}_{\mathrm{Cat}_{R,\omega}}(\mathrm{Mod}_R, D_R\mathscr{C} \otimes_R \mathscr{D}), \tag{7}$$

and as Mod_R is compact as a compactly generated R–linear category, the result follows from the equivalences

$$\operatorname*{colim}_i D_R\mathscr{C}\otimes_R\mathscr{D}_i \simeq D_R\mathscr{C}\otimes_R \operatorname*{colim}_i \mathscr{D}_i \simeq D_R\mathscr{C}\otimes_R\mathscr{D}. \qquad \square$$

The following result is due to Toën and Vaquié [58] in the dg–setting, and the arguments are essentially the same. The result is part of the philosophy of hidden smoothness due to Kontsevich.

Theorem 3.12 *An R–linear category of finite type is smooth.*

Proof It suffices to show that if A is a compact R–algebra, then it is perfect as a right $A^{\mathrm{op}}\otimes_R A$–module. There is a fiber sequence

$$\Omega_{A/R}\to A^{\mathrm{op}}\otimes_R A\to A,$$

where $\Omega_{A/R}$ is the $A^{\mathrm{op}}\otimes_R A$–module of differentials (see Lazarev [38]). So, it is enough to show that $\Omega_{A/R}$ is a perfect $A^{\mathrm{op}}\otimes_R A$–module when A is a compact R–algebra. This follows from the adjunction

$$\mathrm{map}_{A^{\mathrm{op}}\otimes_R A}(\Omega_{A/R}, M)\simeq \mathrm{map}_{(\mathrm{Alg}_R)_{/A}}(A, A\oplus M),$$

together with the fact that, since A is a compact R–algebra, then A is compact in $(\mathrm{Alg}_R)_{/A}$. $\square$

3.3 Azumaya algebras

Let R be an $\mathbb{E}_\infty$–ring spectrum. The following definition is due to Auslander and Goldman [5]. In the derived setting, it and variations on it have been considered by Lieblich [39], Baker and Lazarev [7], Toën [57], Johnson [35] and Baker, Richter and Szymik [8]. Our definition is the same as that of [8].

Definition 3.13 An R–algebra A is an Azumaya R–algebra if A is a compact generator of Mod_R and if the natural R–algebra map giving the bimodule structure on A

$$A\otimes_R A^{\mathrm{op}}\to \mathrm{End}_R(A)$$

is an equivalence of R–algebras.

Note that if A is an Azumaya R–algebra, then, by definition, $A\otimes_R A^{\mathrm{op}}$ is Morita equivalent to R. The standard example of an Azumaya algebra is the endomorphism algebra $\mathrm{End}_R(P)$ of a compact generator of Mod_R. These algebras are not so interesting as they are already Morita equivalent to R. The Brauer group will be the group of Morita equivalence classes of Azumaya algebras, so these endomorphism algebras will represent the trivial class. For more examples and various properties, we refer to [8]. In particular, we will use the fact that if S is an $\mathbb{E}_\infty$–R–algebra, then $A \otimes_R S$ is Azumaya if A is [8, Proposition 1.5]. One main goal of this paper is to show that if R is a connective commutative ring spectrum, then Azumaya algebras are étale locally Morita equivalent to R, which Toën established in the connective commutative dg–setting [57]. The first fact we need is the following theorem.

Theorem 3.14 (Toën [57]) *If $R = \mathrm{H}k$, where k is an algebraically closed field, then every Azumaya R–algebra is Morita equivalent to R.*

We prove now a characterization of Azumaya algebras and smooth and proper algebras. The corresponding statement for dg–algebras is [57, Proposition 2.5].

Theorem 3.15 *Let $\mathscr{C}$ be a compactly generated R–linear category. Then*

(1) *$\mathscr{C}$ is dualizable in $\mathrm{Cat}_{R,\omega}$ if and only if $\mathscr{C}$ is equivalent to Mod_A for a smooth and proper R–algebra A,*

(2) *$\mathscr{C}$ is invertible in $\mathrm{Cat}_{R,\omega}$ if and only if $\mathscr{C}$ is equivalent to Mod_A for an Azumaya R–algebra A.*

Proof If A is smooth and proper, then Mod_A is dualizable in $\mathrm{Cat}_{R,\omega}$ since the evaluation and coevaluation maps are in $\mathrm{Cat}_{R,\omega}$ by hypothesis. If $\mathscr{C}$ is smooth and proper, then $\mathscr{C} \simeq \mathrm{Mod}_A$ for an R–algebra A which is, by definition, smooth and proper.

Suppose that $\mathscr{C}$ is invertible. Then, it follows that it is dualizable in $\mathrm{Cat}_{R,\omega}$, and thus that it is equivalent to Mod_A where A is a smooth and proper R–algebra. So, it suffices to show that Mod_A is invertible if and only if A is Azumaya. The evaluation map

$$\mathrm{Mod}_{A\otimes_R A^{\mathrm{op}}} \to \mathrm{Mod}_R$$

is an equivalence if and only if A is invertible. This map sends $A \otimes_R A^{\mathrm{op}}$ to A, and it is contained in $\mathrm{Cat}_{R,\omega}$ if and only if A is a compact R–module. The evaluation map is essentially surjective if and only if A is a generator of Mod_R. Finally, it is fully faithful if and only if

$$A \otimes_R A^{\mathrm{op}} \simeq \mathrm{End}_{A\otimes_R A^{\mathrm{op}}}(A \otimes_R A^{\mathrm{op}}) \to \mathrm{End}_R(A)$$

is an equivalence. □

We see that we might define the Brauer space of an $\mathbb{E}_\infty$–ring spectrum R to be the grouplike $\mathbb{E}_\infty$–space $\mathrm{Cat}^{\times}_{R,\omega}$. Instead, we will later give an equivalent definition that generalizes more readily to derived schemes.

4 Sheaves

We give in this section preliminaries we will need about sheaves of spaces and ∞–categories. In particular, we study smoothness for morphisms of sheaves of spaces, and we show that under mild hypotheses smooth surjective morphisms admit étale local sections.

4.1 Stacks of algebra and module categories

Roughly speaking, if $\mathcal{X}$ is an ∞–topos and $\mathcal{C}$ is a complete ∞–category, then a $\mathcal{C}$–valued sheaf on $\mathcal{X}$ is a functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ which satisfies descent.

Definition 4.1 Let $\mathcal{C}$ be a complete ∞–category. A $\mathcal{C}$–valued sheaf on $\mathcal{X}$ is a limit-preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}$. The ∞–category $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ is the full subcategory of $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{C})$ consisting of the $\mathcal{C}$–valued sheaves on $\mathcal{X}$.

In the cases we care about, $\mathcal{X}$ will be the ∞–topos associated to a Grothendieck topology on an ∞–category $\mathcal{A}$. In this case a $\mathcal{C}$–valued sheaf on $\mathcal{X}$ is determined by its values on $\mathcal{A}$, because every object in $\mathcal{X}$ is a colimit of representable functors. Moreover, we will typically be in an even more special situation, where the Grothendieck topology is given by a pretopology satisfying the conditions of [42, Propositions 5.1, 5.7]. In this case, a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ is a sheaf if and only if for every covering morphism $X \to Y$ in $\mathcal{A}$, the map

$$F(Y) \to \lim_{\Delta} F(X_\bullet)$$

is an equivalence in $\mathcal{D}$, where $X_\bullet$ is the simplicial object associated to the cover. Similarly, F is a hypercomplete sheaf, or hypersheaf, if for every hypercovering $V_\bullet \to Y$ in $\mathcal{A}$, the map

$$F(Y) \to \lim_{\Delta} F(V_\bullet)$$

is an equivalence; see [42, Section 5] for details. In particular, Lurie proves that the collection of faithfully flat morphisms in $(\mathrm{CAlg}_R)^{\mathrm{op}}$ satisfies the necessary conditions. Thus, the collection of faithfully flat étale morphisms (Section 2.3) in $(\mathrm{CAlg}_R^{\mathrm{cn}})^{\mathrm{op}}$ does as well.

In practice, our sheaves will be one of the following three types: sheaves of ∞–groupoids (spaces), which we call sheaves; sheaves of spectra; or, sheaves of (not necessarily small) ∞–categories, which we call stacks. Thus, for instance, a stack on an ∞–topos $\mathcal{X}$ is a limit-preserving functor $\mathcal{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. We will also consider sheaves of ring spectra and stacks of symmetric monoidal ∞–categories. A presheaf of symmetric monoidal ∞–categories is a stack if and only if the underlying presheaf of ∞–categories is a stack. Indeed, the forgetful functor $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty) \to \widehat{\mathrm{Cat}}_\infty$ preserves and detects limits [45, Corollary 3.2.2.5].

The conventions spelled out in the previous paragraph might cause some confusion. We have chosen to emphasize the ∞–categorical notion that groupoids are spaces in our definitions. As a result, we end up saying "sheaf of Morita equivalences," "classifying sheaf," or "Deligne–Mumford sheaf," instead of the more comfortable "stack of Morita equivalences," "classifying stack," and "Deligne–Mumford stack." Our stacks will be sheaves of ∞–categories. This approach is justified by the fact that the three examples just given are actually objects of the underlying ∞–topoi. Since the objects of the ∞–topos themselves are sheaves of spaces, there is no longer any need to have a separate notion of a sheaf of groupoids.

From a stack, we can produce a sheaf of (not necessarily small) spaces as follows. There is a pair of adjoint functors

$$i\colon \widehat{\mathrm{Gpd}}_\infty \leftrightarrows \widehat{\mathrm{Cat}}_\infty : (-)^{\mathrm{eq}},$$

where the left adjoint i is the natural inclusion, and $(-)^{\mathrm{eq}}$ sends an ∞–category $\mathscr{C}$ to its maximal subgroupoid $\mathscr{C}^{\mathrm{eq}}$. If $\mathscr{M}\colon \mathscr{X}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ is a stack, then the associated sheaf $\mathscr{M}^{\mathrm{eq}}$ is the composition of $\mathscr{M}$ with $(-)^{\mathrm{eq}}$, which is a sheaf because $(-)^{\mathrm{eq}}$ preserves limits.

In the remainder of the section, we will recall some facts about étale (hyper)descent. Let R be a connective $\mathbb{E}_\infty$–ring, and let $\mathrm{Shv}_R^{\text{ét}}$ denote the big étale ∞–topos on R. Given any commutative R–algebra U, connective or not, there is a presheaf $X = \operatorname{Spec} U$ whose values on an R–algebra S are given by

$$X(S) = \mathrm{map}_{\mathrm{CAlg}_R}(U, S).$$

This presheaf is in fact a sheaf, which says that the étale topology on $\mathrm{Aff}_R^{\mathrm{cn}}$ is subcanonical, though much more is true [42, Theorem 5.14].

Proposition 4.2 *For any commutative R–algebra U, the presheaf $\operatorname{Spec} U$ is an étale hypersheaf.*

Proof Indeed, let $S \to T^\bullet$ be an étale hypercovering. This determines a map $\mathrm{N}(\Delta_+) \to \mathrm{CAlg}_R$, which is a limit diagram by [42, Lemma 5.13]. $\square$

Let $\mathscr{M}od\colon (\mathrm{Aff}_R^{\mathrm{cn}})^{\mathrm{op}} = \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be the presheaf of symmetric monoidal ∞–categories that sends S to Mod_S. By [42, Theorem 6.1], this presheaf satisfies descent for étale hypercovers. It follows that we may uniquely extend $\mathcal{M}\mathrm{od}$ to a hyperstack on all of $\mathrm{Shv}_R^{\text{ét}}$. Concretely, when X is an object of $\mathrm{Shv}_R^{\text{ét}}$, we let

$$\mathrm{Mod}_X = \lim_{\mathrm{Spec}\, S \to X} \mathrm{Mod}_S$$

be the stable presentable symmetric monoidal ∞–category of modules over X. We are actually keeping track of the symmetric monoidal structure on Mod_S, and hence on Mod_X by forming the limit in the ∞–category $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. However, the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ preserves limits, so we choose to ignore the intricacies of symmetric monoidal ∞–categories and suppress the symmetric monoidal structure from the notation.

By composing $\mathcal{M}\mathrm{od}\colon \mathrm{Shv}_R^{\text{ét}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ with the limit-preserving functor

$$\mathrm{Alg}\colon \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

that sends a presentable symmetric monoidal ∞–category to the ∞–category of algebra objects (which is also presentable by [45, Corollary 3.2.3.5] and symmetric monoidal by [45, Proposition 3.2.4.3 and Example 3.2.4.4]), we obtain the hyperstack of algebras $\mathcal{A}\mathrm{lg}$ on $\mathrm{Shv}_R^{\text{ét}}$. There is a substack $\mathcal{A}\mathrm{z}$ of Azumaya algebras: an algebra $\mathcal{A}$ over X is Azumaya if its restriction to any affine scheme is Azumaya.

Recall that if $\mathcal{C}$ is a symmetric monoidal ∞–category, then its space of units $\mathrm{Pic}(\mathcal{C})$ is the grouplike $\mathbb{E}_\infty$–space consisting of invertible elements of $\mathcal{C}$ and equivalences. When $\mathcal{C}$ is presentable, then $\mathrm{Pic}(\mathcal{C})$ is a small space, as proven in [1, Theorem 8.9]. Thus, there is a functor

$$\mathrm{Pic}\colon \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S}),$$

where $\mathrm{CAlg}^{\mathrm{gp}}(\mathcal{S})$ denotes the full subcategory of $\mathrm{CAlg}(\mathcal{S})$ of grouplike $\mathbb{E}_\infty$–spaces.

Proposition 4.3 *If we have that $\mathcal{M}$ is a hyperstack of presentable symmetric monoidal ∞–categories, then the presheaf* $\mathrm{Pic}(\mathcal{M})$ *is a hypersheaf.*

Proof By [1, Theorem 8.10], Pic is a right adjoint, so it preserves limits. □

Applying the lemma to the particular stack $\mathcal{M}\mathrm{od}$ on $\mathrm{Shv}_R^{\text{ét}}$, we obtain the Picard sheaf **Pic**, and we let **pic** be the associated sheaf of spectra.

Now, we introduce a stack of R–linear categories, $\mathcal{C}\mathrm{at}_R^{\mathrm{desc}}$, which classifies R–linear categories satisfying étale hyperdescent. Let $\mathcal{C}\mathrm{at}_R\colon (\mathrm{Aff}_R^{\mathrm{cn}})^{\mathrm{op}} = \mathrm{CAlg}_R^{\mathrm{cn}} \to \widehat{\mathrm{Cat}}_\infty$ be the composite functor

$$\mathcal{C}\mathrm{at}_R\colon (\mathrm{Aff}_R^{\mathrm{cn}})^{\mathrm{op}} = \mathrm{CAlg}_R^{\mathrm{cn}} \xrightarrow{\mathrm{Mod}} \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \xrightarrow{\mathrm{Mod}} \widehat{\mathrm{Cat}}_\infty$$

whose value at S is the ∞–category Cat_S of S–linear ∞–categories (equivalently, Mod_S–modules in the symmetric monoidal ∞–category Pr^{L}).

Say that an R–linear category $\mathscr{C}$ satisfies étale hyperdescent if for each connective commutative R–algebra S and each étale hypercover $S \to T^\bullet$, the canonical map

$$\mathscr{C} \otimes_R S \to \lim_{\Delta} \mathscr{C} \otimes_R T^\bullet$$

is an equivalence. We write $\mathrm{Cat}_S^{\mathrm{desc}} \subseteq \mathrm{Cat}_S$ for the full subcategory of Cat_S consisting of the S–linear ∞–categories with étale hyperdescent and $\mathscr{C}\mathrm{at}_R^{\mathrm{desc}} \subseteq \mathscr{C}\mathrm{at}_R$ for the full subfunctor of R–linear categories with étale hyperdescent.

Example 4.4 If A is an R–algebra, then Mod_A is an R–linear category that satisfies étale hyperdescent. Indeed, in this case Mod_A is dualizable in Cat_R with dual $\mathrm{Mod}_{A^{\mathrm{op}}}$. Therefore, if S is a connective $\mathbb{E}_\infty$–R–algebra, then

$$\mathrm{Mod}_A \otimes_R S \simeq \mathrm{D}_R\mathrm{D}_R\mathrm{Mod}_A \otimes_R S \simeq \mathrm{Fun}_R(\mathrm{D}_R\mathrm{Mod}_A, \mathrm{Mod}_S)$$

in Cat_R. Because functors out of $\mathrm{D}_R\mathrm{Mod}_A$ commutes with limits, Mod_A is an R–linear category with hyperdescent. More generally, every compactly generated R–linear category satisfies étale hyperdescent by [43, Corollary 6.11].

The important fact about $\mathscr{C}\mathrm{at}_R^{\mathrm{desc}}$ that we need is that it satisfies étale hyperdescent itself.

Proposition 4.5 *The functor $\mathscr{C}\mathrm{at}_R^{\mathrm{desc}}$ is an étale hyperstack on $\mathrm{Aff}_R^{\mathrm{cn}}$.*

Proof Lurie proves in [43, Theorem 7.5] that the prestack of R–linear categories satisfying flat hyperdescent is a flat hyperstack. The same proof works here. □

4.2 The cotangent complex and formal smoothness

We consider notions of smoothness for maps $p\colon X \to Y$ of sheaves in $\mathrm{Shv}_R^{\text{ét}}$. References for this material include [58] and Lurie [40; 44].

Definition 4.6 Let $p\colon X \to Y$ be a map of sheaves. Then, for any point $x \in X(S)$ and any connective S–module M, the space of derivations $\mathrm{der}_p(x, M)$ is the fiber of the canonical map

$$X(S \oplus M) \to X(S) \times_{Y(S)} Y(S \oplus M)$$

over the point corresponding to x and the map $\mathrm{Spec}\, S \oplus M \to \mathrm{Spec}\, S \xrightarrow{x} X \to Y$, where the first map is induced by the map $(\mathrm{id}, 0)\colon S \to S \oplus M$. If $Y \simeq \mathrm{Spec}\, R$ is a terminal object, write $\mathrm{der}_X(-,-)$ for $\mathrm{der}_p(-,-)$.

Definition 4.7 Let $p\colon X \to Y$ be a map of sheaves. An object L of Mod_X is a relative cotangent complex for p if there exist equivalences

$$\mathrm{map}_S(x^*\mathrm{L}, M) \simeq \mathrm{der}_p(x, M)$$

which are natural in x and connective modules M. When L exists and is unique up to equivalence then we write L_p and refer to this as *the* cotangent complex of p. We will often abuse notation and write $\mathrm{L}_{X/Y}$ for L_p when no confusion will result. When $Y \simeq \mathrm{Spec}\, R$ is a terminal object, we write L_X in place of $\mathrm{L}_{X/Y}$.

Note that if L is a cotangent complex for p, then the space of derivations $\mathrm{der}_p(x, M)$ is never empty. If S is a ring spectrum, an S–module M is almost connective if it is k–connective for some integer k. If X is a sheaf, an object M of Mod_X is almost connective, if its restriction to any $x\colon \mathrm{Spec}\, S \to X$ is almost connective.

Lemma 4.8 *If $p\colon X \to Y$ has at least one cotangent complex* L *that is almost connective, then all cotangent complexes are equivalent, so L_p exists.*

Proof Suppose that L and L' are two cotangent complexes for p, and suppose that L is almost connective. Let $x\colon \mathrm{Spec}\, S \to X$ be an S–point. We show that there is an equivalence $x^*\mathrm{L}' \to x^*\mathrm{L}$, natural in x. Suppose that $\Sigma^n x^*\mathrm{L}$ is connective. Then, using the chain of equivalences

$$\begin{aligned}\mathrm{map}_S(x^*\mathrm{L}, x^*\mathrm{L}') &\simeq \Omega^n \mathrm{map}_S(x^*\mathrm{L}, \Sigma^n x^*\mathrm{L})\\ &\simeq \Omega^n \mathrm{der}_p(x, \Sigma^n x^*\mathrm{L})\\ &\simeq \Omega^n \mathrm{map}_S(x^*\mathrm{L}', \Sigma^n x^*\mathrm{L})\\ &\simeq \mathrm{map}_S(\Sigma^n x^*\mathrm{L}', \Sigma^n x^*\mathrm{L}),\end{aligned}$$

we see there is a unique map $x^*\mathrm{L}' \to x^*\mathrm{L}$ corresponding to the identity on $x^*\mathrm{L}$, which does not depend on n, and so is natural in x. If there exists an integer k such that $\pi_k x^*\mathrm{L}' \to \pi_k x^*\mathrm{L}$ is not an isomorphism, then $\Sigma^k x^*\mathrm{L}' \to \Sigma^k x^*\mathrm{L}$ is not an equivalence. So,

$$\Omega^k \mathrm{map}_S(x^*\mathrm{L}, M) \to \Omega^k \mathrm{map}_S(x^*\mathrm{L}', M)$$

is not an equivalence, which is a contradiction. Thus, $\mathrm{L}' \to \mathrm{L}$ is an equivalence. □

Monomorphisms of sheaves always have cotangent complexes, which vanish.

Lemma 4.9 *Let $f\colon X \to Y$ be a monomorphism of sheaves. Then, $\mathrm{L}_f \simeq 0$.*

Proof Suppose that $x\colon \operatorname{Spec} S \to X$ is a point and that M is an S–module, and consider the diagram

$$\begin{tikzcd} X(S \oplus M) \arrow[r] & X(S) \times_{Y(S)} Y(S \oplus M) \arrow[r] \arrow[d] & Y(S \oplus M) \arrow[d] \\ & X(S) \arrow[r] & Y(S) \end{tikzcd}$$

in which the square is a pullback. The bottom horizontal arrow is a monomorphism. So, the map $X(S) \times_{Y(S)} Y(S \oplus M) \to Y(S \oplus M)$ is a monomorphism. The composite $X(S \oplus M) \to Y(S \oplus M)$ is also a monomorphism. Therefore, the map $X(S \oplus M) \to X(S) \times_{Y(S)} Y(S \oplus M)$ is a monomorphism, and hence the fibers are either empty or contractible. But, the space of derivations

$$\mathrm{der}_f(x, M)$$

is the fiber over $x\colon \operatorname{Spec} S \to X$ and $\operatorname{Spec} S \oplus M \to Y$, with the latter induced by the composition

$$\operatorname{Spec} S \oplus M \to \operatorname{Spec} S \xrightarrow{x} X \xrightarrow{f} Y.$$

It follows that the composite $\operatorname{Spec} S \oplus M \to X$ is in the fiber, so it is contractible. Hence, 0 corepresents derivations. □

The following two lemmas can be proved with straightforward arguments using only the definition of the space of derivations.

Lemma 4.10 *If $f\colon X \to Y$ is a map of sheaves, and if L_X and L_Y exist, then there is a cofiber sequence*

$$f^*\mathrm{L}_Y \to \mathrm{L}_X \to \mathrm{L}_f$$

in Mod_X. In particular the cotangent complex of f exists.

Lemma 4.11 *Let $\{X_i\}$ be a diagram of sheaves in Shv_R indexed by a simplicial set I, and let X be the limit. Suppose that the cotangent complex L_{X_i} exists for each i in I, and write L_X for the colimit of the diagram $\{\mathrm{L}_{X_i}|_X\}$ in Mod_X. If L_X is almost connective, then L_X is a cotangent complex for X.*

The inclusion functor $\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}_R^{\mathrm{cn}}$ induces a functor

$$\tau_{\leq n}^*\colon \mathrm{Shv}_R^{\text{ét}} \simeq \mathrm{Shv}^{\text{ét}}(\mathrm{CAlg}_R^{\mathrm{cn}}) \to \mathrm{Shv}^{\text{ét}}(\tau_{\leq n}\mathrm{CAlg}_R^{\mathrm{cn}}).$$

If S is a connective commutative R–algebra, then $\tau^*_{\leq n}\operatorname{Spec} S \simeq \operatorname{Spec} \tau_{\leq n} S$. Indeed, if T is any n–truncated connective commutative R–algebra, then the natural map

$$\operatorname{map}(\tau_{\leq n} S, T) \to \operatorname{map}(S, T)$$

is an equivalence.

Lemma 4.12 *If $f\colon X \to Y$ is a morphism of sheaves with a cotangent complex L_f such that $\tau^*_{\leq n} f$ is an equivalence, then $\tau_{\leq n}\mathrm{L}_f \simeq 0$.*

Proof We sketch the proof. The proof is the same as for the affine case, the details of which can be found in [45, Lemma 8.4.3.17]. In fact, the natural map

$$\tau_{\leq n}(f^*\mathrm{L}_Y) \to \tau_{\leq n}\mathrm{L}_X$$

is an equivalence. To check this, it is enough to map into an n–truncated $\mathcal{O}_X$–module M. By the universal property of the cotangent complex, we check that the morphism

$$\operatorname{map}_{\mathrm{CAlg}_{R/\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{O}_X \oplus M) \to \operatorname{map}_{\mathrm{CAlg}_{R/f_*\mathcal{O}_X}}(\mathcal{O}_Y, f_*\mathcal{O}_X \oplus f_*M)$$

is an equivalence, which follows from the fact that

$$\operatorname{map}_{\mathrm{CAlg}_{R/\tau_{\leq n}\mathcal{O}_X}}(\mathcal{O}_X, \tau_{\leq n}\mathcal{O}_X \oplus M) \to \operatorname{map}_{\mathrm{CAlg}_{R/\tau_{\leq n} f_*\mathcal{O}_X}}(\mathcal{O}_Y, \tau_{\leq n} f_*\mathcal{O}_X \oplus f_*M)$$

is an equivalence, since $\tau_{\leq n}\mathcal{O}_Y \to \tau_{\leq n} f_*\mathcal{O}_X$ is an equivalence by hypothesis. □

Let R be a commutative ring spectrum. Then, the forgetful functor $\mathrm{CAlg}_R \to \mathrm{Mod}_R$ has a left adjoint

$$\mathrm{Sym}_R\colon \mathrm{Mod}_R \to \mathrm{CAlg}_R.$$

If M is an R–algebra, then $\mathrm{Sym}_R(M)$ is called the symmetric R–algebra on M. For the existence of the functor Sym_R, see [45, Section 3.1.3]. We can compute the cotangent complexes of the affine schemes of these symmetric algebras, which provides the essential step in showing that all maps between connective affine schemes have cotangent complexes.

Lemma 4.13 *Let M be an almost connective R–module, and let $S = \mathrm{Sym}_R(M)$. Then the cotangent complex $\mathrm{L}_{\operatorname{Spec} S}$ of $\operatorname{Spec} S \to \operatorname{Spec} R$ exists and is equivalent to the S–module $M \otimes_R S$.*

Proof See [45, Proposition 8.4.3.14]. For any S–module N, there is a sequence of equivalences:

$$\operatorname{map}_S(M \otimes_R S, N) \simeq \operatorname{map}_R(M, N) \simeq \operatorname{map}_{R/S}(M, S \oplus N) \simeq \operatorname{map}_{(\mathrm{CAlg}_R)/S}(S, S \oplus N)$$

Thus, $M \otimes_R S$ is an almost connective cotangent complex for $\operatorname{Spec} S$, and therefore the unique cotangent complex. □

Proposition 4.14 *If $S \to T$ is a map of connective commutative R–algebras, then $\mathrm{L}_{\operatorname{Spec} T/\operatorname{Spec} S}$ exists and is connective.*

Proof We can write T as a colimit $\operatorname{colim}_i T_i$ of symmetric algebras $T_i = \operatorname{Sym}_S S^{\oplus n_i}$ so that $\operatorname{Spec} T$ is a limit of $\operatorname{Spec} T_i$. By Lemma 4.11, the cotangent complex L_T is the colimit of the restrictions of L_{T_i} to $\operatorname{Spec} T$. By Lemma 4.13, each L_{T_i} is connective. Since colimits of connective T–modules are connective, L_T is connective. □

A map of connective commutative R–algebras $\phi\colon \widetilde{S} \to S$ is a nilpotent thickening if $\pi_0(\phi)\colon \pi_0\widetilde{S} \to \pi_0 S$ is surjective and if the kernel of $\pi_0(\phi)$ is a nilpotent ideal. Note that if S is a connective commutative R–algebra, then the maps $\tau_{\leq m} S \to \tau_{\leq n} S$ for $m \geq n$ in the Postnikov tower of S are nilpotent thickenings.

Definition 4.15 A map of sheaves $p\colon X \to Y$ is formally smooth if for every nilpotent thickening $\widetilde{S} \to S$ the induced map

$$X(\widetilde{S}) \to X(S) \times_{Y(S)} Y(\widetilde{S}) \tag{8}$$

is surjective (that is, surjective on π_0). We say $p\colon X \to Y$ is formally étale if the maps in (8) are isomorphisms on π_0.

We need the following nontrivial proposition from [42].

Proposition 4.16 [42, Proposition 7.26] *If S and T are connective commutative R–algebras, then a map $\operatorname{Spec} T \to \operatorname{Spec} S$ is formally smooth if and only if $\mathrm{L}_{\operatorname{Spec} T/\operatorname{Spec} S}$ is a projective T–module.*

To consider the stronger notion of smoothness, we need to consider the notion of compactness for commutative algebras, and we will need to know later that this notion agrees with the usual notion of finite presentation for ordinary commutative rings.

Definition 4.17 A map $S \to T$ of connective commutative ring spectra is locally of finite presentation if T is a compact object of CAlg_S [45, Definition 8.2.5.26].

If T is a connective commutative S–algebra that is compact in CAlg_S, then it is compact in $\mathrm{CAlg}_S^{\mathrm{cn}}$. Since truncation preserves compact objects by [41, Corollary 5.5.7.4(iii)], it follows that $\tau_{\leq 0} T$ is compact in $\tau_{\leq 0}\mathrm{CAlg}_S^{\mathrm{cn}}$.

Lemma 4.18 *Let R be a discrete commutative ring, and let S be a discrete commutative R–algebra. Then, S is compact as a discrete commutative R–algebra if and only if S is finitely presented.*

Proof Suppose that S is finitely presented, so it can be written as a quotient $R[X]/I$, where X is a finite set, and I is a finitely generated ideal. We can write S as the pushout

$$\begin{tikzcd} \mathrm{Sym}_R I \arrow[r] \arrow[d] & R[X] \arrow[d] \\ R \arrow[r] & S \end{tikzcd}$$

of R–algebras. But, this means that if $R[Y] \to \mathrm{Sym}_R I$ exhibits $\mathrm{Sym}_R I$ as a finitely generated R–algebra, the following square is also a pushout square:

$$\begin{tikzcd} R[Y] \arrow[r] \arrow[d] & R[X] \arrow[d] \\ R \arrow[r] & S \end{tikzcd}$$

Since $R[Y]$, $R[X]$ and R are compact, it follows that S is compact as well.

Now, suppose that S is a compact (discrete) commutative R–algebra. Then, S is a retract of a finitely presented commutative R–algebra $R[X]/I$. Indeed, we can write S as a filtered colimit of finitely presented commutative R–algebras; by compactness, the identity map on S factors through a finite stage. It suffices to show that the kernel of $R[X]/I \to S$ is finitely generated. We proceed by Noetherian induction. Let ϕ be the composition

$$R[X]/I \to S \to R[X]/I.$$

We may write $I = (p_1(X), \dots, p_k(X))$, an ideal generated by k polynomials in X. Let R_0 be the subring of R generated over $\mathbb{Z}$ by the coefficients appearing in the p_i's and in the polynomials $\phi(x_i)$ for $x_i \in X$. This is a finitely generated commutative $\mathbb{Z}$–algebra, so it is in particular Noetherian. By our choice of R_0, we can define an ideal I_0 of $R_0[X]$ generated by the same polynomials. Moreover, ϕ defines a morphism $\phi_0\colon R_0[X]/I_0 \to R_0[X]/I_0$. Let S_0 be the image of ϕ_0, which is a subring of $R_0[X]/I_0$. There is an exact sequence of R_0–modules

$$0 \to J_0 \to R_0[X]/I_0 \to S_0 \to 0.$$

Since S_0 is finitely generated and $R_0[X]/I_0$ is Noetherian, it follows that S_0 is finitely presented, and that J_0 is finitely generated. Tensoring with R over R_0, we obtain an exact sequence

$$J_0 \otimes_{R_0} R \to R[X]/I \to S \to 0.$$

The kernel J_0 is finitely generated, and it surjects onto the kernel J of $R[X]/I \to S$. It follows that J is finitely generated, and hence that S is finitely presented. □

The previous lemma implies that the next definition agrees with the usual definition of smooth maps between ordinary affine schemes.

Definition 4.19 A map $\operatorname{Spec} T \to \operatorname{Spec} S$ is smooth if it is formally smooth and $S \to T$ is locally of finite presentation.

At first glance, the condition that $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective in the next lemma might seem strange. But, we will show in Theorem 4.47 that this is satisfied if $\operatorname{Spec} T \to \operatorname{Spec} S$ is smooth, and if the map $\operatorname{Spec} \pi_0 T \to \operatorname{Spec} \pi_0 S$ is a surjective map of ordinary schemes.

Lemma 4.20 *Let $R \to S \to T$ be maps of connective commutative algebras. If T is locally of finite presentation over R and over S, and if $\operatorname{Spec} T \to \operatorname{Spec} S$ is a surjective map in $\mathrm{Shv}_R^{\text{ét}}$, then S is locally of finite presentation over R.*

Proof First, note that the maps $\pi_0 R \to \pi_0 S \to \pi_0 T$ satisfy the same hypotheses by Lemma 4.18. Thus, by Grothendieck [30, Proposition I.1.4.3(v)], $\pi_0 S$ is a $\pi_0 R$–algebra that is locally of finite presentation. Now, by [45, Theorem 8.4.3.18], it is enough to show that $\mathrm{L}_S = \mathrm{L}_{R/S}$ is a perfect S–module. There is a fiber sequence

$$\mathrm{L}_S \otimes_S T \to \mathrm{L}_T \to \mathrm{L}_{S/T}$$

of cotangent complexes. Again, by [45, Theorem 8.4.3.18], L_T and $\mathrm{L}_{S/T}$ are perfect since $R \to T$ and $S \to T$ are locally of finite presentation. It follows that $\mathrm{L}_S \otimes_S T$ is perfect. Since $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective in $\mathrm{Shv}_R^{\text{ét}}$, there are étale local sections. Thus, there is a faithfully flat étale S–algebra P and maps $S \to T \to P$. Since $\mathrm{L}_S \otimes_S T$ is perfect, the U–module $\mathrm{L}_S \otimes_S U$ is perfect. But, by faithfully flat descent, it follows that L_S is perfect (to see this, one can either refer forward to Lemma 5.4, or use the fact that an S–module is perfect if and only if it is dualizable and the fact that dualizability data can be constructed étale locally by Example 4.4). □

The following example will be used later in the paper.

Example 4.21 The sheaf of R–module endomorphisms of $R^{\oplus n}$ is representable by an affine monoid scheme M_n, where

$$\mathrm{M}_n = \operatorname{Spec}\operatorname{Sym}_R \operatorname{End}_R(R^{\oplus n}).$$

Given a commutative R–algebra S, an element of $\mathrm{M}_n(S)$ is a commutative R–algebra map

$$\operatorname{Sym}_R \operatorname{End}_R(R^{\oplus n}) \to S.$$

But these are equivalent to the R–module maps

$$R^{\oplus n^2} \simeq \operatorname{End}_R(R^{\oplus n}) \to S,$$

which by adjunction is the S–module

$$S^{\oplus n^2} \simeq \operatorname{End}_S(S^{\oplus n}).$$

Since the space of S–module endomorphisms of $S^{\oplus n}$ has a natural monoid structure, this shows that M_n is a monoid scheme. We can invert the determinant element of $\pi_0\mathrm{M}_n$, so the sheaf of S–module automorphisms of $S^{\oplus n}$ is representable by an affine group scheme GL_n. Because the cotangent complex of M_n at an S–point is $S^{\oplus n^2}$, which is a projective S–module, the affine schemes M_n and GL_n are smooth over R.

4.3 Geometric sheaves

Let R be a connective commutative ring spectrum. The goal of this section is to study certain geometric classes of sheaves in $\mathrm{Shv}_R^{\text{ét}}$ built inductively from the representable sheaves by forming smooth quotients. The notions of n–stack here have been studied extensively by Simpson [53], Toën and Vezzosi [59] and Lurie [40], and we base our approach on theirs.

We define n–geometric morphisms and smooth n–geometric morphisms inductively as follows.

- A morphism $f\colon X \to Y$ in $\mathrm{Shv}_R^{\text{ét}}$ is 0–geometric if for any $\operatorname{Spec} S \to Y$, the fiber product $X \times_Y \operatorname{Spec} S$ is equivalent to $\coprod_i \operatorname{Spec} T_i$ for some connective commutative R–algebras T_i.
- A 0–geometric morphism f is smooth if $X \times_Y \operatorname{Spec} S \to \operatorname{Spec} S$ is smooth for all $\operatorname{Spec} S \to Y$. To be clear, if $X \times_Y \operatorname{Spec} S \simeq \coprod_i \operatorname{Spec} T_i$, then this means that each morphism $\operatorname{Spec} T_i \to \operatorname{Spec} S$ is smooth in the sense of Definition 4.19.
- A morphism $X \to Y$ in $\mathrm{Shv}_R^{\text{ét}}$ is n–geometric if for any $\operatorname{Spec} S \to Y$, there is a smooth surjective $(n-1)$–geometric morphism $U \to X \times_Y \operatorname{Spec} S$, where U is a disjoint union of affines.

- An n–geometric morphism $f\colon X \to Y$ is smooth if for every $\operatorname{Spec} S \to Y$ we may take the above map $U \to X \times_Y \operatorname{Spec} S$ such that the composition $U \to X \times_Y \operatorname{Spec} S \to \operatorname{Spec} S$ is a smooth 0–geometric morphism.

We say that an n–geometric morphism $X \to Y$ is an n–submersion if it is smooth and surjective. If, moreover, X is a disjoint sum of representables, then we call such a morphism an n–atlas. A 1–geometric sheaf with a Zariski atlas is a derived scheme. What this means is that a 1–geometric sheaf X has an atlas $\coprod_i \operatorname{Spec} T_i \to X$ which is a 0–geometric morphism, and, for every point $\operatorname{Spec} S \to X$, the pullback $\operatorname{Spec} T_i \times_X \operatorname{Spec} S \to \operatorname{Spec} S$ is Zariski open. Similarly, a 1–geometric sheaf with an étale atlas is a Deligne–Mumford stack.

Any 0–geometric sheaf X is a disjoint union of sheaves $\coprod_{i\in I} \operatorname{Spec} S_i$, where the S_i are connective commutative R–algebras. If I is *finite*, then we call the sheaf representable. In this case $X = \coprod_{i=1}^n \operatorname{Spec} S_i \simeq \operatorname{Spec}(S_1 \times \cdots \times S_n)$.

A 0–geometric sheaf is quasicompact if it is representable, and a 0–geometric morphism $f\colon X \to Y$ is quasicompact if, for all $\operatorname{Spec} S \to Y$, the pullback $X \times_Y \operatorname{Spec} S$ is representable. Inductively, an n–geometric sheaf X if quasicompact if there exists an $(n-1)$–geometric quasicompact submersion of the form $\operatorname{Spec} S \to X$, and an n–geometric morphism $f\colon X \to Y$ is quasicompact if for each map $\operatorname{Spec} S \to Y$, the fiber $X \times_Y \operatorname{Spec} S$ is a quasicompact n–geometric sheaf. Finally, an n–geometric morphism $f\colon X \to Y$ is quasiseparated if the diagonal $X \to X \times_Y X$ is quasicompact.

Definition 4.22 An n–geometric sheaf X is locally of finite presentation over $\operatorname{Spec} R$ if it has an $(n-1)$–atlas
$$\coprod_i \operatorname{Spec} S_i \to X$$
such that each S_i is a connective commutative R–algebra that is locally of finite presentation. An n–geometric morphism $X \to Y$ is locally of finite presentation if for every S–point of Y, $X \times_Y \operatorname{Spec} S$ is locally of finite presentation over $\operatorname{Spec} S$. By definition, a smooth n–geometric morphism is locally of finite presentation.

It is important to have a theory of sheaves that are only locally geometric. A sheaf $X \to \operatorname{Spec} S$ is locally geometric if it can be written as a filtered colimit
$$X \simeq \operatorname*{colim}_i X_i,$$
where each sheaf X_i is n_i–geometric for some n_i and where the maps $X_i \to X$ are monomorphisms. If we can furthermore take the X_i to be locally of finite presentation, we say that X is locally geometric and locally of finite presentation. A morphism

$f\colon X \to Y$ is locally geometric (locally of finite presentation) if for every $\operatorname{Spec} S \to Y$, the pullback $X \times_Y \operatorname{Spec} S$ is locally geometric (locally of finite presentation) over $\operatorname{Spec} S$.

We say that a locally geometric morphism $X \to Y$ which is locally of finite presentation is smooth if for every $\operatorname{Spec} S \to Y$, the pullback $X \times_Y \operatorname{Spec} S \to \operatorname{Spec} S$ has a cotangent complex of Tor-amplitude contained in $[-n, 0]$ for some nonnegative integer n (depending on S).

Note the following easy but important facts.

Lemma 4.23 *If $f\colon X \to Z$ is an n–geometric morphism (resp. smooth n–geometric morphism), and if $Y \to Z$ is any morphism, then the pullback $f_Y\colon X \times_Z Y \to Y$ is n–geometric (resp. n–geometric and smooth).*

Lemma 4.24 *A morphism $f\colon X \to Y$ is n–geometric if and only if for every map $\operatorname{Spec} S \to Y$, the morphism $X \times_Y \operatorname{Spec} S \to \operatorname{Spec} S$ is n–geometric.*

The following lemma can be found in [40]. We include a proof for the reader's convenience.

Lemma 4.25 *Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are composable morphisms of sheaves.*

(1) *If f and g are n–geometric (resp. smooth and n–geometric), then $g \circ f$ is n–geometric (resp. smooth and n–geometric).*

(2) *If f is an n–submersion and $g \circ f$ is $(n+1)$–geometric, then g is $(n+1)$–geometric.*

(3) *If $g \circ f$ is n–geometric and g is $(n+1)$–geometric, then f is n–geometric.*

Proof We prove (1) by induction on n. Using Lemma 4.24, it suffices to suppose that Z is representable. Assume that $n = 0$. Then, the fact that g is 0–geometric implies that Y is representable, and the fact that f is 0–geometric then implies that Z is representable. Evidently any morphism of representables is 0–representable, and compositions of smooth morphisms of representables are smooth. Now assume the statement (1) for $(n-1)$–geometric morphisms. Since we assume that Z is representable, it suffices to find an $(n-1)$–submersion $U \to X$ where U is a disjoint union of representables. Since g is n–representable, there is an $(n-1)$–submersion $V \to Y$ where V is a sum of representables. Constructing the pullback $X \times_Y V$, we know by the n–geometricity of f and the formal representability of V that there is an $(n-1)$–submersion $U \to X \times_Y V$ with U a sum of representables. This is summarized in the following diagram:

$$\begin{tikzcd}
U \arrow[r, "(n-1)\text{–sub}"] & X \times_Y V \arrow[r] \arrow[d, "(n-1)\text{–sub}"'] & V \arrow[d, "(n-1)\text{–sub}"] & \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Since, by the induction hypothesis, composition of $(n-1)$–submersions are $(n-1)$–submersions, the inductive step follows.

To prove (2), it is again enough to assume that Z is representable. Then, there is an n–atlas $u\colon U \to X$ since $g \circ f$ is $(n+1)$–geometric. Since f is an n–submersion, the composition $f \circ u$ is an n–atlas by part (1). Hence, g is $(n+1)$–geometric.

To prove (3), suppose that $p : \operatorname{Spec} S \to Y$ is a point of Y. Consider the diagram

$$\begin{tikzcd}
& X \times_Y \operatorname{Spec} S \arrow[r] \arrow[d] & \operatorname{Spec} S \arrow[d] & V \arrow[dl, "n\text{–atlas}"] \\
U \arrow[r, "(n-1)\text{–atlas}"'] \arrow[urrr, dashed] & X \times_Z \operatorname{Spec} S \arrow[r] \arrow[d] & Y \times_Z \operatorname{Spec} S \arrow[r] \arrow[d] & \operatorname{Spec} S \arrow[d, "g \circ p"] \arrow[dl, "p"] \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z,
\end{tikzcd}$$

where the squares are all pullback squares, $U \to X \times_Z \operatorname{Spec} S$ is an $(n-1)$–atlas (or the identity map if $n = 0$) and $V \to Y \times_Z \operatorname{Spec} S$ is an n–atlas. Since $V \to Y \times_Z \operatorname{Spec} S$ is surjective, up to refining U, we may assume that the composite $U \to Y \times_Z \operatorname{Spec} S$ factors through V. The map $U \to V$ is thus 0–geometric. By part (1), the map $U \to Y \times_Z \operatorname{Spec} S$ is n–geometric. By part (2), $X \times_Z \operatorname{Spec} S \to Y \times_Z \operatorname{Spec} S$ is n–geometric. Therefore, $X \times_Y \operatorname{Spec} S \to \operatorname{Spec} S$ is n–geometric, and we conclude by Lemma 4.24 that f is n–geometric. □

Remark 4.26 The previous lemma goes through as stated with the additional assumptions and conclusions of quasicompactness.

Lemma 4.27 *Suppose that X is an n–geometric sheaf that is locally of finite presentation. If $U = \coprod_i \operatorname{Spec} T_i \xrightarrow{p} X$ is any atlas, then each T_i is locally of finitely presentation over R.*

Proof Let $V = \coprod_i \operatorname{Spec} S_i \xrightarrow{q} X$ be an atlas where each S_i is locally of finite presentation over R. Since $V \to X$ is a surjection of sheaves, we may assume, possibly by refining U, that there is a factorization of p as $U \to V \xrightarrow{q} X$. Now, consider the fiber product $U \times_X V$, which is a smooth $(n-1)$–geometric sheaf over either U or V. Let $W = \coprod_i \operatorname{Spec} P_i \to U \times_X V$ be an atlas. Since $U \times_X V \to U$ is surjective, we may arrange indices so that the composition $W \to U$ is a coproduct of smooth surjections of the form $\operatorname{Spec} P_i \to \operatorname{Spec} T_i$. Assume also that in the map $U \to V$ we have $\operatorname{Spec} T_i \to \operatorname{Spec} S_i$. Then, there is a composition of commutative ring maps $S_i \to T_i \to P_i$. The composite is locally of finite presentation since it is smooth, the map $T_i \to P_i$ is locally of finite presentation for the same reason, and by construction the map $\operatorname{Spec} P_i \to \operatorname{Spec} T_i$ is surjective. Thus, the conditions of Lemma 4.20 are satisfied. It follows that T_i is locally of finite presentation over S_i. Since S_i is locally of finite presentation over R, it follows that T_i is locally of finite presentation over R. $\square$

Now we prove an analogue of [40, Principle 5.3.5].

Lemma 4.28 *Suppose that* P *is a property of sheaves. Suppose that every disjoint union of affines has property* P*, and suppose that whenever $U \to X$ is a surjective morphism of sheaves such that $U_X^k = U \times_X \cdots \times_X U$ has property* P *for all $k \geq 0$, then X has property* P*. Then, all n–geometric sheaves have property* P*.*

Proof Let X be an n–geometric sheaf. Then, there exists a smooth $(n-1)$–geometric surjection $U \to X$, where U is a disjoint union of affines. Each product U_X^k is $(n-1)$–geometric. So, it suffices to observe that the statement follows by induction. $\square$

Lemma 4.29 *Let $X \to Y$ be a surjection of sheaves. Suppose that X and $X \times_Y X$ are n–geometric stacks and that the projections $X \times_Y X \to X$ are n–geometric and smooth. Then, Y is an $(n+1)$–geometric stack. If, in addition, X is quasicompact and $X \to Y$ is a quasicompact morphism, then Y is quasicompact. Finally, if X is locally of finite presentation, then so is Y.*

Proof Let

$$\coprod_i \operatorname{Spec} S_i \to X$$

be an atlas for X. For any i and j, we have a diagram of pullbacks

$$\begin{tikzcd}
\operatorname{Spec} S_i \times_Y \operatorname{Spec} S_j \arrow[r] \arrow[d] & \operatorname{Spec} S_i \times_Y X \arrow[r] \arrow[d] & \operatorname{Spec} S_i \arrow[d] \\
X \times_Y \operatorname{Spec} S_j \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} S_j \arrow[r] & X \arrow[r] & Y.
\end{tikzcd}$$

We will show that the composite $\coprod_i \operatorname{Spec} S_i \to X \to Y$ is an n–submersion. The surjectivity follows by hypothesis. To check that $\operatorname{Spec} S_i \to X$ is smooth and n–geometric, it is enough to check on the fiber of $\operatorname{Spec} S_j$ for all j. Since $X \times_Y X \to X$ and $\operatorname{Spec} S_i \to X$ are smooth and n–geometric, $\operatorname{Spec} S_i \times_Y \operatorname{Spec} S_j \to X \times_Y \operatorname{Spec} S_j$ and $X \times_Y \operatorname{Spec} S_j \to S_j$ are smooth and n–geometric. Therefore, the composite is as well, which completes the proof of the first statement. To prove the second statement, note that we can take $\coprod_i \operatorname{Spec} S_i$ to be a finite disjoint union, since X is quasicompact. Then, since $\coprod_i \operatorname{Spec} S \to X$ and $X \to Y$ are quasicompact, it follows that the composition is quasicompact by Remark 4.26. The third statement is immediate, since $\coprod_i \operatorname{Spec} S_i$ can be chosen so that each S_i is locally of finite presentation. □

Lemma 4.30 *Suppose that $f\colon X \to Y$ is a morphism of sheaves where X is an n–geometric sheaf and the diagonal $Y \to Y \times_{\operatorname{Spec} R} Y$ is n–geometric. Then, f is n–geometric. Moreover, if f is smooth and surjective, then Y is $(n+1)$–geometric.*

Proof Since X is n–geometric, there is an $(n-1)$–submersion $\coprod_i \operatorname{Spec} T_i \to X$. Suppose that $\operatorname{Spec} S \to Y$ is arbitrary. Form the fiber products $X \times_Y \operatorname{Spec} S$ and $\coprod_i \operatorname{Spec} T_i \times_Y \operatorname{Spec} S$, and note that the map

$$\coprod_i \operatorname{Spec} T_i \times_Y \operatorname{Spec} S \to X \times_Y \operatorname{Spec} S$$

is an $(n-1)$–submersion. The map

$$\coprod_i \operatorname{Spec} T_i \times_Y \operatorname{Spec} S \to \coprod_i \operatorname{Spec} T_i \otimes_R S$$

is n–geometric because it is the pullback of $\coprod_i \operatorname{Spec} T_i \times_{\operatorname{Spec} R} \operatorname{Spec} S \to Y \times_{\operatorname{Spec} R} Y$ along the diagonal map $Y \to Y \times \operatorname{Spec} R$. Therefore $\coprod_i \operatorname{Spec} T_i \times_Y \operatorname{Spec} S$ admits an $(n-1)$–submersion from a disjoint union of affines $\coprod_{i,j} \operatorname{Spec} U_{ij}$. We obtain the following diagram:

$$
\begin{tikzcd}
\coprod \operatorname{Spec} U_{i,j} \arrow[r, "(n-1)\text{-sub}"] & \coprod \operatorname{Spec} T_i \times_Y \operatorname{Spec} S \arrow[r, "(n-1)\text{-sub}"] \arrow[dl, "n\text{-geometric}"'] \arrow[d] & X \times_Y \operatorname{Spec} S \arrow[r] \arrow[d] & \operatorname{Spec} S \arrow[d] \\
\coprod \operatorname{Spec} T_i \otimes_R S & \coprod \operatorname{Spec} T_i \arrow[r, "(n-1)\text{-sub}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}
$$

By Lemma 4.25, the composition of the top two horizontal maps is also an $(n-1)$–submersion from a disjoint union of affines, establishing that f is n–geometric. The second claim is clear. □

Lemma 4.31 *If X is n–geometric, and if $p\colon \operatorname{Spec} S \to X$ is a point of X, then $\Omega_p X = \operatorname{Spec} S \times_X \operatorname{Spec} S \to \operatorname{Spec} S$ is an $(n-1)$–geometric morphism. The projection $X^{S^m} \to X$ induced by choosing a point in the m–sphere is an $(n-m)$–geometric map.*

Proof We use the equivalent description of $\Omega_p X$ as the pullback in the diagram

$$
\begin{tikzcd}
\Omega_p X \arrow[r] \arrow[d] & \operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} S \arrow[d, "(p,p)"] \\
X \arrow[r] & X \times_{\operatorname{Spec} R} X.
\end{tikzcd}
$$

Since the diagonal of X is $(n-1)$–geometric, it follows that the composite

$$\Omega_p X \to \operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} S \to \operatorname{Spec} S$$

is also $(n-1)$–geometric. To prove the second statement, it suffices to note that the fiber of the projection map over a point $p\colon \operatorname{Spec} S \to X$ is the m–fold iterated loop space $\Omega_p^m X$. So, this follows from the first part of the lemma. □

Example 4.32 If G is a smooth n–geometric stack of groups (ie, grouplike A_∞–spaces), then $\mathrm{B}G$ is a pointed smooth $(n+1)$–geometric stack. Indeed, the loop space of $\mathrm{B}G$ at the canonical point is just G. Therefore, the point $\operatorname{Spec} R \to \mathrm{B}G$ is an n–submersion. Using Lemma 4.25(2), the claim follows.

For the next two lemmas, fix a base sheaf Z in $\mathrm{Shv}_R^{\text{ét}}$, and consider the ∞–topos $\mathrm{Shv}_{/Z}^{\text{ét}}$ of objects over Z. Let $\mathrm{Shv}_{/Z}^{n}$ be the full subcategory of $\mathrm{Shv}_{/Z}^{\text{ét}}$ consisting of the n–geometric morphisms $Y \to Z$.

Lemma 4.33 *The full subcategory $\mathrm{Shv}_{/Z}^{n}$ of $\mathrm{Shv}_{/Z}^{\text{ét}}$ is closed under finite limits in $\mathrm{Shv}_{/Z}^{\text{ét}}$.*

Proof As $\mathrm{Shv}^n_{/Z}$ has a terminal object which agrees with the terminal object of $\mathrm{Shv}^{\text{ét}}_{/Z}$, it is enough to check the case of pullbacks. Suppose that $X \to Y$ and $W \to Y$ are two morphisms in $\mathrm{Shv}^n_{/Z}$. In order to check that $X \times_Y W$ is in $\mathrm{Shv}^n_{/Z}$ it suffices to check that $X \times_Y W$ is in $\mathrm{Shv}^n_{/Y}$ since $Y \to Z$ is n–geometric. Moreover, we can obviously reduce to the case that $Y \simeq \operatorname{Spec} S$ is representable. Then, X and W are n–geometric stacks over S. Taking an n–atlas $U \to X$ and an n–atlas $V \to W$, the fiber product $U \times_Y V$ is an n–atlas for $X \times_Y W$ by using the stability of geometricity and smoothness under pullbacks. □

Lemma 4.34 *The full subcategory of* $\mathrm{Shv}^{\text{ét}}_{/Z}$ *consisting of quasicompact* 0*–geometric sheaves over* Z *is closed under* all *limits in* $\mathrm{Shv}^{\text{ét}}_{/Z}$.

Proof It suffices to note that arbitrary limits of representables are representable, since the ∞–category of connective commutative R–algebras has all colimits. □

Lemma 4.35 *A finite limit of* n*–geometric morphisms (locally of finite presentation) is* n*–geometric (and locally of finite presentation).*

Proof The proof is by induction on n. The base case $n = 0$ simply follows because finite limits of representable sheaves are representable, and finite limits distribute over coproducts. Suppose the lemma is true for k–geometric sheaves for all $k < n$, and let $f_i\colon X_i \to Y_i$ be a finite diagram of n–geometric morphisms (locally of finite presentation). Let $f\colon X \to Y$ be the limit. Let $\operatorname{Spec} S \to Y$ be an S–point. Then, we may construct an atlas for the pullback $X \times_Y \operatorname{Spec} S$ as the (finite) limit of a compatible family of atlases for the pullbacks $X_i \times_{Y_i} \operatorname{Spec} S$. The morphism from this atlas to $X \times_Y \operatorname{Spec} S$ is $(n-1)$–geometric by the inductive hypothesis. It is also clear that it is a submersion. If the maps are locally of finite presentation, then the atlases over each $X_i \times_{Y_i} \operatorname{Spec} S$ may be chosen to be locally of finite presentation, and hence their (finite) limit is again locally of finite presentation. □

Lemma 4.36 *Let* $X^\bullet$ *be a cosimplicial diagram of quasiseparated* n*–geometric sheaves over* Z*. Then, the limit* $X = \lim_\Delta X^\bullet$ *is* n*–geometric over* Z.

Proof By Goerss and Jardine [27, Proposition VII.1.7], there is pushout diagram for any m,

$$\begin{array}{ccc}
\Delta^\bullet_m \times \partial\Delta^m & \longrightarrow & sk_{m-1}\Delta^\bullet \\
\downarrow & & \downarrow \\
\Delta^\bullet_m \times \Delta^m & \longrightarrow & sk_m\Delta^\bullet,
\end{array}$$

of cosimplicial spaces. Given $X^\bullet$ we obtain a pullback diagram of sheaves

$$\begin{tikzcd}
\mathrm{map}(sk_m\Delta^\bullet, X^\bullet) \arrow[r] \arrow[d] & \mathrm{map}(\Delta^\bullet_m \times \Delta^m, X^\bullet) \simeq X^m \arrow[d] \\
\mathrm{map}(sk_{m-1}\Delta^\bullet, X^\bullet) \arrow[r] & \mathrm{map}(\Delta^\bullet_m \times \partial\Delta^m, X^\bullet) \simeq (X^m)^{S^{m-1}}.
\end{tikzcd}$$

By Lemma 4.31, the map $(X^m)^{S^{m-1}} \to X^m$ is $(n-m-1)$–geometric. Since X^m is n–geometric, Lemma 4.25(3) implies that the left-hand vertical maps above are $(n-m-2)$–geometric. Thus, if $m \geq n-2$, we see that the left-hand vertical maps above are 0–geometric. Moreover, by hypothesis, each diagonal $X^m \to X^m \times_Z X^m$ is quasicompact, so, pulling back, we see that each of the maps

$$(X^m)^{S^{m-1}} \to (X^m)^{S^{m-2}} \to \cdots \to (X^m)^{S^1} \to X^m$$

is quasicompact, so the composite $(X^m)^{S^{m-1}} \to X^m$ and the section $X^m \to (X^m)^{S^{m-1}}$ are as well by Remark 4.26. We conclude that the left-hand vertical map is 0–geometric and quasicompact. As we have equivalences

$$\lim_\Delta X^\bullet \simeq \lim_m \mathrm{map}(sk_m\Delta^\bullet, X^\bullet) \simeq \lim_{m\geq n-2} \mathrm{map}(sk_m\Delta^\bullet, X^\bullet),$$

$\lim_\Delta X^\bullet$ is a limit of quasicompact 0–geometric morphisms over $\mathrm{map}(sk_{n-2}\Delta^\bullet, X^\bullet)$, which, as a finite limit of n–geometric sheaves over Z, is n–geometric over Z. Hence, by Lemma 4.34, the limit is n–geometric. □

Proposition 4.37 *If Y is a retract of a sheaf X over Z, and if X is quasiseparated and n–geometric over Z, then Y is n–geometric over Z.*

Proof We refer to [41, Section 4.4.5] for details about retracts in ∞–categories . In particular, any retract in $\mathrm{Shv}^{\text{ét}}_{/Z}$ is given as the limit of a diagram $\widetilde{X}\colon \mathrm{Idem} \to \mathrm{Shv}^{\text{ét}}_{/Z}$, where Idem is an ∞–category with only one object $*$ and with finitely many simplices in each degree. Let $X = \widetilde{X}(*)$. It follows that the cosimplicial replacement (see Bousfield and Kan [16, XI 5.1]) of p is a cosimplicial sheaf $X^\bullet$ which in degree k is a finite product of copies of X. Thus, if p takes values in quasiseparated n–geometric sheaves over Z, then each X^k is still quasiseparated and n–geometric. By Lemma 4.36, the retract of X classified by $\widetilde{X}$ is n–geometric. □

Lemma 4.38 *If X is a quasiseparated n–geometric sheaf that is locally of finite presentation, and if Y is a retract of X, then Y is locally of finite presentation.*

Proof By the previous lemma, Y is itself n–geometric. Let $U = \coprod_i \operatorname{Spec} T_i \to Y$ be an atlas, and let $V = \coprod_i \operatorname{Spec} S_i \to X \times_Y U$ be an atlas for the fiber product. Since the composition $V \to X \times_Y U \to X$ is an $(n-1)$–geometric submersion, it follows that V is an atlas for X. By Lemma 4.27, each S_i is locally of finite presentation over R. Taking the pullback of $X \times_Y U \to X$ over $Y \to X$, we get $U \to X \times_Y U$, since $Y \to X \to Y$ is the identity. Possibly by refining U, we can assume that $U \to X \times_Y U$ factors through the surjection $V \to X \times_Y U$. We thus have shown that each T_i is a retract of S_j for some j. Since S_j is locally of finite presentation over R, it follows that T_i is as well. □

We now prove in two lemmas that images of smooth n–geometric morphisms are Zariski open. This is a generalization of the fact that images of smooth maps of schemes are Zariski open. Restricting a sheaf in $\mathrm{Shv}_R^{\text{ét}}$ to discrete connective commutative rings induces a geometric morphism of ∞–topoi $\pi_0^*\colon \mathrm{Shv}_R^{\text{ét}} \to \mathrm{Shv}_{\pi_0 R}^{\text{ét}}$. Note that $\pi_0^* \operatorname{Spec} S$ is equivalent to $\operatorname{Spec} \pi_0 S$.

Lemma 4.39 *Let S be a connective commutative R–algebra. Then, a subobject Z of $\operatorname{Spec} S$ is Zariski open if and only if $\pi_0^* Z$ is Zariski open in $\operatorname{Spec} \pi_0 S$.*

Proof The necessity is trivial. So, suppose that $\pi_0^* Z$ is Zariski open. Because π_0^* admits a left adjoint, we see that π_0^* preserves (-1)–truncated objects and finite limits. Thus, π_0^* preserves subobjects, so $\pi_0^* Z$ is a subobject of $\operatorname{Spec} \pi_0 S$. Let F be the set of $f \in \pi_0 S$ such that $\operatorname{Spec} S[f^{-1}] \to \operatorname{Spec} S$ factors through Z. Note that $f \in F$ if and only if $\operatorname{Spec} \pi_0 S[f^{-1}] \to \operatorname{Spec} \pi_0 S$ factors through $\pi_0^* Z$. By construction, there is a monomorphism over $\operatorname{Spec} S$

$$W := \bigcup_{f \in F} \operatorname{Spec} S[f^{-1}] \to Z.$$

Since $\pi_0^* Z$ is Zariski open, it follows that $\pi_0^* W = \pi_0^* Z$. The counit map of the adjunction

$$\pi_0^*\colon \mathrm{Shv}_R^{\text{ét}} \rightleftarrows \mathrm{Shv}_{\pi_0 R}^{\text{ét}} :\pi_{0*}$$

gives a map $Z \to \pi_{0*}\pi_0^* Z = \pi_{0*}\pi_0^* W$. Now, we can recover W from $\pi_{0*}\pi^* W$ as the pullback

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \operatorname{Spec} S \arrow[d] \\ \pi_{0*}\pi_0^* W \arrow[r] & \pi_{0*}\pi_0^* \operatorname{Spec} S. \end{tikzcd}$$

Indeed, since W is open, it is a union of $\operatorname{Spec} S[f_i^{-1}]$. This is clear when W is a basic open subscheme $\operatorname{Spec} S[f^{-1}]$, and the general case follows from the fact that π_0^* induces an equivalence between the small Zariski site of $\operatorname{Spec} S$ and (the nerve of) the small Zariski site of $\operatorname{Spec} \pi_0 S$. Thus, there are maps $W \to Z$ and

$$Z \to \pi_{0*}\pi_0^* Z \times_{\pi_{0*} \operatorname{Spec} \pi_0 S} \operatorname{Spec} S \xrightarrow{\sim} W$$

over $\operatorname{Spec} S$. Since W and Z are subobjects of $\operatorname{Spec} S$, it follows that they are equivalent. Thus, Z is Zariski open. □

The image of a map $f\colon X \to Y$ of sheaves is defined as the epi-mono factorization $X \twoheadrightarrow \operatorname{im}(f) \rightarrowtail Y$. In particular, the morphism $\operatorname{im}(f) \rightarrowtail Y$ is a monomorphism.

Lemma 4.40 *Let $f\colon X \to Y$ be a smooth n–geometric morphism. Then, the map $\operatorname{im}(f) \to Y$ is a Zariski open immersion.*

Proof We may assume without loss of generality that $Y = \operatorname{Spec} S$ for some connective commutative R–algebra S. Then, by hypothesis, there is a smooth $(n-1)$–geometric chart

$$\coprod_i \operatorname{Spec} T_i \to X$$

such that the compositions $g_i\colon \operatorname{Spec} T_i \to \operatorname{Spec} S$ are smooth, and thus have cotangent complexes L_{g_i} which are projective. By [45, Proposition 8.4.3.9], $\pi_0 \mathrm{L}_{g_i}$ is the cotangent complex of $\pi_0^*(g_i)\colon \operatorname{Spec} \pi_0 T_i \to \operatorname{Spec} \pi_0 S$, and it is projective. Since g_i is locally of finite presentation, by Lemma 4.18, $\pi_0^*(g_i)$ is locally of finite presentation, and hence smooth. Since smooth morphisms of discrete schemes are flat by [30, Theorem 17.5.1], the image of $\pi_0^*(g_i)$ has an open image in $\operatorname{Spec} \pi_0 S$. By the previous lemma, it follows that the image of g_i in $\operatorname{Spec} S$ is open. □

4.4 Cotangent complexes of smooth morphisms

In this section, we show that n–geometric morphisms have cotangent complexes, and we give a criterion for an n–geometric morphism to be smooth in terms of formal smoothness and the cotangent complex.

Let S be a connective commutative R–algebra, and let M be a connective S–module. Then, a small extension of S by M over R is a connective commutative R–algebra $\widetilde{S}$ together with an R–algebra section d of $S \oplus \Sigma M \to S$ such that $\widetilde{S}$ is equivalent to the pullback

$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & S \arrow[d] \\ S \arrow[r, "d"] & S \oplus \Sigma M. \end{tikzcd}$$

The ∞–category of small extensions $\mathrm{CAlg}^{\mathrm{small}}_{R/S}$ is the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}_{R/S}$ spanned by small extensions of S over R.

Lemma 4.41 *There is a natural equivalence $\mathrm{CAlg}^{\mathrm{small}}_{R/S} \simeq (\tau_{>0}\mathrm{Mod}_S)_{\mathrm{L}_{R/S}/}$. The composite*

$$\mathrm{CAlg}^{\mathrm{small}}_{R/S} \simeq (\tau_{>0}\mathrm{Mod}_S)_{\mathrm{L}_{R/S}/} \to \tau_{>0}\mathrm{Mod}_S$$

is given by taking the cofiber ΣM of $\widetilde{S} \to S$.

Proof By adjunction, the space of R–algebra sections of $S \oplus \Sigma M \to S$ is equivalent to the space of S–module maps $\mathrm{L}_{R/S} \to \Sigma M$. □

The previous lemma allows us to compute the cotangent complex of a small extension.

Lemma 4.42 *Let $\widetilde{S} \to S$ be a small extension of S by M. Then, the cotangent complex L_i of $i : \mathrm{Spec}\, S \to \mathrm{Spec}\, \widetilde{S}$ is naturally equivalent to ΣM.*

Proof By the previous lemma, it suffices to show that $\widetilde{S}$ is an initial object of $\mathrm{CAlg}^{\mathrm{small}}_{\widetilde{S}/S}$. It is easy to check that $\widetilde{S}$ is a small extension of S over $\widetilde{S}$. As

$$\widetilde{S} \to S$$

is the initial object of $\mathrm{CAlg}^{\mathrm{cn}}_{\widetilde{S}/S}$, it follows that it is an initial object of $\mathrm{CAlg}^{\mathrm{small}}_{\widetilde{S}/S}$. □

A sheaf X is infinitesimally cohesive if for all R–algebras S and all small extensions $\widetilde{S} \simeq S \times_{S \oplus \Sigma M} S$ of S by an S–module M the natural map

$$X(\widetilde{S}) \to X(S) \times_{X(S \oplus \Sigma M)} X(S)$$

is an equivalence.

Lemma 4.43 *Let X be an infinitesimally cohesive sheaf with a cotangent complex L_X, let u: $\mathrm{Spec}\, S \to X$ be a point and let $\widetilde{S} \to S$ be a small extension of S by M classified by a class $x \in \pi_0\mathrm{map}_S(\mathrm{L}_{\mathrm{Spec}\, S}, \Sigma M)$. Then, u extends to $\widetilde{u}$: $\mathrm{Spec}\, \widetilde{S} \to X$ if and only if the image of x vanishes under the map induced by $u^*\mathrm{L}_X \to \mathrm{L}_{\mathrm{Spec}\, S}$, $\pi_0\mathrm{map}_S(\mathrm{L}_{\mathrm{Spec}\, S}, \Sigma M) \to \pi_0\mathrm{map}_S(u^*\mathrm{L}_X, \Sigma M)$.*

Proof Since X is infinitesimally cohesive, there is a cartesian square

$$\begin{tikzcd}
X_u(\widetilde{S}) \arrow[r] \arrow[d] & * \arrow[d, "{(u,0)}"] \\
* \arrow[r, "\alpha"] & X_u(S \oplus \Sigma M),
\end{tikzcd}$$

where $X_u(\widetilde{S}), X_u(S \oplus \Sigma M)$ are the fibers of $X(\widetilde{S}) \to X(S), X(S \oplus \Sigma M) \to X(S)$ over u, and α is induced by $\operatorname{Spec} d \colon \operatorname{Spec} S \oplus M \to \operatorname{Spec} S$. By definition of the cotangent complex, $X_u(S \oplus \Sigma M) \simeq \operatorname{map}_S(u^*\mathrm{L}_X, \Sigma M)$. So, $X_u(\widetilde{S})$ is nonempty if and only if the point α of $\operatorname{map}_S(u^*\mathrm{L}_X, \Sigma M)$ is 0. But, d is classified by x, so that α is the image of x in $\operatorname{map}_S(u^*\mathrm{L}_X, \Sigma M)$, as claimed. □

A sheaf X is nilcomplete if for any connective commutative R–algebra S the canonical map

$$X(S) \to \lim_n X(\tau_{\leq n} S)$$

is an equivalence. If T is any commutative R–algebra, then $X = \operatorname{Spec} T$ is nilcomplete. Indeed, if S is a connective commutative R–algebra, then

$$X(S) = \operatorname{map}_{\mathrm{CAlg}_R}(T, S) \simeq \lim_n \operatorname{map}_{\mathrm{CAlg}_R}(T, \tau_{\leq n} S) = \lim_n X(\tau_{\leq n} S).$$

A map of sheaves $p \colon X \to Y$ is nilcomplete if for all connective commutative R–algebras S and all S–points of Y the fiber product $X \times_Y \operatorname{Spec} S$ is nilcomplete.

Remark 4.44 Suppose that $S \xrightarrow{\sim} \lim_\alpha S_\alpha$ is a limit diagram of connective commutative R–algebras such that each map $S \to S_\alpha$ induces an isomorphism on π_0. In this case, the underlying small étale ∞–topoi of S and each S_α are equivalent. Given a sheaf X in $\mathrm{Shv}_R^{\text{ét}}$, let X_S (resp. X_{S_α}) denote the restriction of X to the small étale site of S. Thus, for instance, the space of global sections $X_{S_\alpha}(S)$ is equivalent to $X(S_\alpha)$. In order for $X(S) \to \lim_\alpha X(S_\alpha)$ to be an equivalence, it suffices to show that X_S is equivalent to $\lim_\alpha X_{S_\alpha}$.

As we now show, all n–geometric morphisms have cotangent complexes, and we use this to show that the property of smoothness for n–geometric morphisms can be detected via a Tor-amplitude condition on the cotangent complex. The proof of the next proposition is a mix of several proofs in [40], particularly Propositions 5.1.5 and 5.3.7.

Proposition 4.45 *An n–geometric morphism $f \colon X \to Y$ is infinitesimally cohesive, nilcomplete, and has a $(-n)$–connective cotangent complex L_f. If f is smooth, then L_f is perfect of Tor-amplitude contained in* $[-n, 0]$. *Finally, if f is smooth, then it is formally smooth.*

Proof We prove the proposition by induction on n. We may assume moreover that $Y = \operatorname{Spec} S$, and prove the statements for X and L_X. When X is a disjoint union of affines, it is automatically infinitesimally cohesive and nilcomplete, since maps out of a commutative ring commute with limits. The other statements in the base case $n = 0$ follow from Propositions 4.14 and 4.16. Thus, suppose the proposition is true for k–geometric morphisms with $k < n$. Then, since X is n–geometric, we fix an $(n-1)$–submersion $p\colon U \to X$ where U is a disjoint union of affines $\coprod_i \operatorname{Spec} T_i$. Write p_i for the composition $\operatorname{Spec} T_i \to U \to X$.

To prove the statements about infinitesimal cohesiveness and nilcompleteness, we apply Lemma 4.28 and use Remark 4.44. Let X be an n–geometric sheaf, and let $U \to X$ be a surjection of sheaves. Let $S \xrightarrow{\sim} \lim_\alpha S_\alpha$ be a limit diagram of connective commutative R–algebras such that each map $S \to S_\alpha$ induces an isomorphism on π_0. Consider the simplicial object obtained by taking iterated fiber products of the map

$$\lim_\alpha U_{S_\alpha} \to \lim_\alpha X_{S_\alpha}.$$

By using identifications of the form

$$U_{S_\alpha} \times_{X_{S_\alpha}} U_{S_\alpha} \simeq (U \times_X U)_{S_\alpha},$$

the simplicial objected induces a (-1)–truncated map from the geometric realization $|\lim_\alpha U_{S_\alpha,\bullet}|$ to $\lim_\alpha X_{S_\alpha}$. We obtain a commutative diagram

$$\begin{tikzcd} X_S \arrow[r] & \lim_\alpha X_{S_\alpha} \\ {|U_{S,\bullet}|} \arrow[u] \arrow[r] & {|\lim_\alpha U_{S_\alpha,\bullet}|,} \arrow[u] \end{tikzcd}$$

where the bottom map is an equivalence, the left vertical map is an equivalence and the right vertical map is (-1)–truncated. To show the top map is an equivalence, it is enough to show that for any étale S–algebra T the map $\lim_\alpha U(T_\alpha) \to \lim_\alpha X(T_\alpha)$ is surjective, where $T_\alpha = S_\alpha \otimes_S T$.

Infinitesimal cohesiveness We specialize the above considerations to where S is a small extension of S_0 by M:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S_0 \arrow[d] \\ S_0 \arrow[r] & S_0 \oplus \Sigma M \end{tikzcd}$$

Set $S_1 = S_0 \oplus \Sigma M$. We want to show that for any étale S–algebra T, the natural map

$$\lim_i U(T_i) \to \lim_i X(T_i)$$

is surjective, where $T_i = S_i \otimes_S T$. It suffices to prove this when the value of $\mathrm{Spec}(T_1) \to X$ at the terminal object lifts to U. To show that the map on limits is surjective, it suffices to show that for any point x_0 of $X(T_0)$ that maps to x_1 in $X(T_1)$, and for any lift of x_1 to y_1 in $U(T_1)$, there exists y_0 in $U(T_0)$ mapping to y_1 and x_0 (for either of the maps $T_0 \to T_1$). This surjectivity follows from the fact that the cotangent complex of $U \to X$ exists, which is due to the inductive step of the proof. The surjectivity follows from Lemma 4.43 because the cotangent complex of U over X is perfect and its dual is connective.

Nilcompleteness The proof of nilcompleteness is similar to that of infinitesimal cohesiveness, and is left to the reader.

Existence Fix a T–point $x\colon \mathrm{Spec}\, T \to X$. Since $p\colon U \to X$ is surjective, we can assume that x factors through p. Let $y\colon \mathrm{Spec}\, T \to U$ be such a factorization. Then, there is a natural morphism

$$F\colon \mathrm{der}_{f\circ p}(y, M) \to \mathrm{der}_f(x, M).$$

The 0–point of $\mathrm{der}_f(x, M)$ is the point in the fiber of $X(T \oplus M) \to X(T) \times_{Y(T)} Y(T \oplus M)$ corresponding to $\mathrm{Spec}\, T \oplus M \to \mathrm{Spec}\, T \to X$. The fiber over 0 of F is naturally equivalent to $\mathrm{der}_p(y, M)$. Thus, we have a natural fiber sequence

$$\mathrm{map}_T(\mathrm{L}_p, M) \to \mathrm{map}_T(\mathrm{L}_{f\circ p}, M) \to \mathrm{der}_f(x, M).$$

By the formal smoothness of the smooth $(n-1)$–geometric morphism p, the map of spaces F is surjective. It follows that we can identify $\mathrm{der}_f(x, M)$ with the fiber of the delooped map

$$\mathrm{Bmap}_T(\mathrm{L}_p, M) \to \mathrm{Bmap}_T(\mathrm{L}_{f\circ p}, M).$$

Therefore, the fiber of $\mathrm{L}_{f\circ p} \to \mathrm{L}_p$ is a cotangent complex for f. The connectivity statement is immediate.

Tor-amplitude Now, suppose that $X \to \mathrm{Spec}\, S$ is smooth. Then, we may assume that $\mathrm{Spec}\, T_i$ is smooth over $\mathrm{Spec}\, S$; in particular $\mathrm{Spec}\, T_i$ is locally finitely presented and L_{T_i} is finitely generated projective. By Lemma 4.10, there is a cofiber sequence

$$p_i^*\mathrm{L}_X \to \mathrm{L}_{\mathrm{Spec}\, T_i} \to \mathrm{L}_{\mathrm{Spec}\, T_i/X}.$$

By the inductive hypothesis, $\mathrm{L}_{\mathrm{Spec}\, T_i/X}$ is perfect and has Tor-amplitude contained in $[-n+1, 0]$. Therefore, L_X is perfect and has Tor-amplitude contained in $[-n, 0]$.

Formal smoothness Let $\mathcal{K}$ be the class of nilpotent thickenings $u\colon \widetilde{T} \to T$ that satisfy the left lifting property with respect to f. Since f has a cotangent complex, $\mathcal{K}$ contains all trivial square-zero extensions $T \oplus M \to T$. To see that $\mathcal{K}$ contains all small extensions of T by M, we use the fact that

$$X(\widetilde{T}) \simeq X(T) \otimes_{X(T \oplus \Sigma M)} X(T).$$

Therefore, to check that the projection

$$X(\widetilde{T}) \simeq X(T) \otimes_{X(T \oplus \Sigma M)} X(T) \to X(T)$$

is surjective, it suffices to note that

$$\pi_0 X(T) \times_{\pi_0 X(T \oplus \Sigma M)} \pi_0 X(T) \to \pi_0 X(T) \times_{\pi_0 X(T)} \pi_0 X(T) = \pi_0 X(T)$$

is surjective, because the map of pullback diagrams admits a section induced by the inclusion $\pi_0 X(T) \to \pi_0 X(T \oplus \Sigma M)$. Finally, that $\mathcal{K}$ contains all nilpotent thickenings follows from the method of the proof of [42, Proposition 7.26], which decomposes such a thickening as a limit of small extensions. □

The fact that smooth n–geometric morphisms have perfect cotangent complexes with Tor-amplitude contained in $[-n, 0]$ characterizes smooth morphisms, at least if we include the assumption that the morphism be locally of finite presentation.

Proposition 4.46 *An n–geometric morphism $f\colon X \to Y$ is smooth if and only if it is locally of finite presentation and L_f is a perfect complex with Tor-amplitude contained in $[-n, 0]$.*

Proof We may assume that $Y = \operatorname{Spec} S$. Let $U = \coprod_i \operatorname{Spec} T_i \to X$ be a smooth $(n-1)$–submersion onto X. Then, each composition $\operatorname{Spec} T_i \to \operatorname{Spec} S$ is smooth, and hence locally of finite presentation. Therefore, f is locally of finite presentation. The fact that L_f is perfect with Tor-amplitude contained in $[-n, 0]$ follows from the previous proposition. Suppose now that f is n–geometric, locally of finite presentation, and that L_f has Tor-amplitude contained in $[-n, 0]$. Take a chart $p\colon U = \coprod_i \operatorname{Spec} T_i \to X$ where the $\operatorname{Spec} T_i$ are all locally of finite presentation over $\operatorname{Spec} S$. The pullback sequence

$$p^* \mathrm{L}_X \to \mathrm{L}_U \to \mathrm{L}_p$$

of cotangent complexes, together with the facts that $p^* \mathrm{L}_X$ and L_U are perfect complexes with Tor-amplitude contained in $[-n, 0]$ and $[-n+1, 0]$, respectively, shows that L_U is perfect with Tor-amplitude contained in $[-n, 0]$. But, since U is a disjoint union of affines, L_U is connective. Thus, L_U is equivalent to a finitely generated projective module, so that each $\operatorname{Spec} T_i \to \operatorname{Spec} S$ is smooth, as desired. □

4.5 Étale-local sections of smooth geometric morphisms

The theorem in this section says that smooth morphisms that are surjective on geometric points are in fact surjections of étale sheaves.

Theorem 4.47 *If $p\colon X \to Y$ is a smooth locally geometric morphism that is surjective on geometric points, then for every S–point $\operatorname{Spec} S \to Y$ there exists an étale cover $\operatorname{Spec} T \to \operatorname{Spec} S$ and a T–point $\operatorname{Spec} T \to X$ such that*

$$\begin{tikzcd} \operatorname{Spec} T \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} S \arrow[r] & Y \end{tikzcd}$$

commutes.

Proof We may assume without loss of generality that $Y = \operatorname{Spec} S$, and it then suffices to prove that $X \to \operatorname{Spec} S$ has étale local sections. Write X as a filtered colimit

$$X \simeq \operatorname*{colim}_i X_i$$

of n_i–geometric sheaves, such that each $X_i \to X$ is a monomorphism. By Lemma 4.9, the cotangent complex $\mathrm{L}_{X_i/X}$ vanishes. Suppose that the cotangent complex of X has Tor-amplitude contained in $[-n, 0]$. Then, for i sufficiently large, it follows that $X_i \to \operatorname{Spec} S$ is a smooth n_i–geometric morphism. Since $\operatorname{Spec} S$ is quasicompact, and since the image of X_i in $\operatorname{Spec} S$ is open by Lemma 4.40, it follows that for some i, $X_i \to \operatorname{Spec} S$ is a smooth n_i–geometric morphism that is surjective on geometric points. There exists an $(n-1)$–submersion $U \to X_i$ such that U is the disjoint union of smooth affine S–schemes. Let $\pi_0^* U \to \operatorname{Spec} \pi_0 S$ be the associated map of ordinary schemes. By hypothesis, this morphism is smooth and is surjective on geometric points. By [30, Corollaires IV.17.16.2, IV.17.16.3(ii)], there exists an étale cover $\operatorname{Spec} \pi_0 T \to \operatorname{Spec} \pi_0 S$ and a factorization $\operatorname{Spec} \pi_0 T \to \pi_0^* U \to \operatorname{Spec} \pi_0 S$. The étale map $\pi_0 S \to \pi_0 T$ determines a unique connective commutative S–algebra T by [45, Theorem 8.5.0.6]. We would like to lift the $\pi_0 T$–point of $\pi_0^* U$ to a T–point of U. Since U is a disjoint union of affines, it is nilcomplete. Therefore, $U(T) \simeq \lim_n U(\tau_{\leq n} T)$, so it suffices to show that $U(\tau_{\leq n} T) \to U(\tau_{\leq n-1} T)$ is surjective. This is true since U is formally smooth and $\tau_{\leq n} T \to \tau_{\leq n-1} T$ is a nilpotent thickening. □

5 Moduli of objects in linear ∞–categories

We study moduli spaces of objects in R–linear categories. This extends the work of Toën and Vaquié [58] to the setting of commutative ring spectra. We give some results on local moduli, which form the basis of an important geometricity statement for global moduli sheaves. As a corollary, we show in the final section that if A is an Azumaya algebra over R, the sheaf of Morita equivalences from A to R is smooth over $\operatorname{Spec} R$, and hence has étale local sections.

5.1 Local moduli

In this section we prove the geometricity of the sheaf corepresented by a free commutative R–algebra $\operatorname{Sym}_R(P)$ where P is a perfect R–module, and we show that when P has Tor-amplitude contained in $[-n, 0]$, then this sheaf is smooth. These facts are nontrivial precisely because $\operatorname{Sym}_R(P)$ is not necessarily connective. This turns out to be the main step in understanding the geometricity of more general moduli problems.

Let $\mathbf{Proj}_R$ denote the sheaf of finite rank projective modules.

Proposition 5.1 *The sheaf $\mathbf{Proj}_R$ is equivalent to $\coprod_n \mathrm{B}\,\mathrm{GL}_n$. In particular, $\mathbf{Proj}_R$ is locally 1–geometric and locally of finite presentation.*

Proof A projective module is locally free by Proposition 2.12. Hence, the sheaf of projective rank n modules is equivalent to $\mathrm{B}\,\mathrm{GL}_n$. This sheaf has a 0–atlas $\operatorname{Spec} R \to \mathrm{B}\,\mathrm{GL}_n$, which shows it is 1–geometric and locally of finite presentation. $\square$

Theorem 5.2 *Let P be a perfect R–module with Tor-amplitude contained in $[a, b]$ with $a \leq 0$. Then, the sheaf $\operatorname{Spec}\operatorname{Sym}_R(P)$ is a quasicompact and quasiseparated $(-a)$–geometric stack that is locally of finite presentation. Moreover, the cotangent complex of $\operatorname{Spec}\operatorname{Sym}_R(P)$ at an S–point $x\colon \operatorname{Spec} S \to \operatorname{Spec}\operatorname{Sym}_R(P)$ is the S–module*

$$\mathrm{L}_{\operatorname{Spec}\operatorname{Sym}_R(P),x} \simeq P \otimes_R S.$$

Therefore, if $b \leq 0$, $\operatorname{Spec}\operatorname{Sym}_R(P)$ is smooth.

Proof We prove all of the statements except for quasiseparatedness by induction on $-a$. If $a = 0$, then P is connective by Proposition 2.13, so that $\operatorname{Sym}_R(P)$ is connective as well. Thus, $\operatorname{Spec}\operatorname{Sym}_R(P)$ is 0–geometric and quasicompact. It is locally of finite presentation since the R–module P is perfect. Now, assume that P has Tor-amplitude contained in $[a, b]$ where $a < 0$. By Proposition 2.13, we can write P as the fiber of some map $Q \to \Sigma^{a+1} N$, where Q is a perfect R–module with Tor-amplitude contained in $[a+1,b]$ and N is a finitely generated projective R–module. The fiber sequence induces a fiber sequence of sheaves

$$\operatorname{Spec}\operatorname{Sym}_R(\Sigma^{a+1}N) \to \operatorname{Spec}\operatorname{Sym}_R(Q) \to \operatorname{Spec}\operatorname{Sym}_R(P) \to \operatorname{Spec}\operatorname{Sym}_R(\Sigma^{a}N),$$

where, inductively, both $\operatorname{Spec}\operatorname{Sym}_R(\Sigma^{a+1}N)$ and $\operatorname{Spec}\operatorname{Sym}_R(Q)$ are $(-a-1)$–geometric stacks that are locally of finite presentation. The map

$$\operatorname{Spec}\operatorname{Sym}_R(Q) \to \operatorname{Spec}\operatorname{Sym}_R(P) \tag{9}$$

is surjective, because if S is a connective commutative R–algebra, we get a fiber sequence of spaces

$$\begin{aligned}&\operatorname{map}_R(\Sigma^{a+1}N,S) \to \operatorname{map}_R(Q,S) \to \operatorname{map}_R(P,S)\\ &\qquad\qquad \to \operatorname{map}_R(\Sigma^{a}N,S) \simeq \operatorname{Bmap}_R(\Sigma^{a+1}N,S),\end{aligned}$$

which shows that $\operatorname{map}_R(Q,S)$ is the total space of a principal bundle over $\operatorname{map}_R(P,S)$. The map (9) is also quasicompact, since the fiber $\operatorname{Spec}\operatorname{Sym}_R(\Sigma^{a+1}N)$ is quasicompact. Note that

$$\begin{aligned}\operatorname{Spec}\operatorname{Sym}_R(Q) &\times_{\operatorname{Spec}\operatorname{Sym}_R(P)} \operatorname{Spec}\operatorname{Sym}_R(Q)\\ &\simeq \operatorname{Spec}(\operatorname{Sym}_R(Q) \otimes_{\operatorname{Sym}_R(P)} \operatorname{Sym}_R(Q))\\ &\simeq \operatorname{Spec}\operatorname{Sym}_R(Q \oplus_P Q).\end{aligned}$$

Using that the natural map given by an inclusion followed by the codiagonal

$$Q \to Q \oplus_P Q \to Q$$

is the identity, it follows that $Q \oplus_P Q \simeq Q \oplus \Sigma^{a+1}N$. Therefore,

$$\operatorname{Spec}\operatorname{Sym}_R(Q \oplus_P Q) \simeq \operatorname{Spec}\operatorname{Sym}_R(Q) \times_{\operatorname{Spec} R} \operatorname{Spec}\operatorname{Sym}_R(\Sigma^{a+1}N).$$

It follows that the projection $\operatorname{Spec}\operatorname{Sym}_R(Q \oplus_P Q) \to \operatorname{Spec}\operatorname{Sym}_R(Q)$ is the pullback of a $(-a-1)$–geometric morphism, and so is itself $(-a-1)$–geometric. The projection is smooth because $\operatorname{Spec}\operatorname{Sym}_R(\Sigma^{a+1}N)$ is smooth. Therefore, by all of the statements of Lemma 4.29, $\operatorname{Spec}\operatorname{Sym}_R(P)$ is a quasicompact $(-a)$–geometric stack that is locally of finite presentation. Finally, by Lemma 4.13, the cotangent complex of $\operatorname{Spec}\operatorname{Sym}_R(P)$ is $P \otimes_R \operatorname{Sym}_R(P)$, so $\operatorname{Spec}\operatorname{Sym}_R(P)$ is smooth by Proposition 4.45 if $b \leq 0$.

It remains to show that $\operatorname{Spec}\operatorname{Sym}_R(P)$ is quasiseparated. Let Q be the cofiber of the diagonal morphism $P \to P \oplus P$. Then, the fiber of the diagonal morphism

$$\operatorname{Spec}\operatorname{Sym}_R(P) \to \operatorname{Spec}\operatorname{Sym}_R(P) \times_{\operatorname{Spec} R} \operatorname{Spec}\operatorname{Sym}_R(P) \simeq \operatorname{Spec}\operatorname{Sym}_R(P \oplus P)$$

is equivalent to $\operatorname{Spec}\operatorname{Sym}_R(Q)$, which is quasicompact by the first part of the proof. □

Remark 5.3 If P is a perfect R–module with Tor-amplitude contained in $[a,b]$ and $a \geq 0$, then it also has Tor amplitude contained in $[0,b]$, and the proposition implies that $\operatorname{Spec}\operatorname{Sym}_R(P)$ is a 0–geometric stack.

5.2 The moduli sheaf of objects

In this section, we apply the study of local moduli above to global moduli sheaves of objects. The main theorems in this section, Theorems 5.6 and 5.8, are generalizations of results of [58] to connective $\mathbb{E}_\infty$–ring spectra.

Let $\mathscr{C}$ be a compactly generated R–linear ∞–category. Define a functor

$$\mathcal{M}_{\mathscr{C}}\colon (\mathrm{Aff}_R^{\mathrm{cn}})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$$

whose value at $\operatorname{Spec} S$ is the full subcategory of $\mathrm{D}_R\mathscr{C} \otimes_R \mathrm{Mod}_S \simeq \mathrm{Fun}_R^{\mathrm{L}}(\mathscr{C}, \mathrm{Mod}_S)$ consisting of those objects f such that for every compact object x of $\mathscr{C}$, the value $f(x)$ is a compact object of Mod_S. Put another way, we can define $\mathcal{M}_{\mathscr{C}}$ as the pullback

$$\begin{tikzcd}
\mathcal{M}_{\mathscr{C}} \arrow[r] \arrow[d] & \mathrm{Fun}_R^{\mathrm{L}}(\mathscr{C}, \mathcal{M}\mathrm{od}) \arrow[d] \\
\prod_{x\in\pi_0\mathscr{C}^\omega} \mathcal{M}\mathrm{od}^\omega \arrow[r] & \prod_{x\in\pi_0\mathscr{C}^\omega} \mathcal{M}\mathrm{od},
\end{tikzcd}$$

where $\mathcal{M}\mathrm{od}^\omega$ is the functor of compact objects $\mathcal{M}\mathrm{od}^\omega\colon (\mathrm{Aff}_R^{\mathrm{cn}})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ given by

$$\mathcal{M}\mathrm{od}^\omega(\operatorname{Spec} S) = \mathrm{Mod}_S^\omega.$$

Lemma 5.4 *For any compactly generated R–linear ∞–category $\mathscr{C}$, the functor $\mathcal{M}_{\mathscr{C}}$ satisfies étale hyperdescent.*

Proof It is clear that $\mathrm{Fun}_R^{\mathrm{L}}(\mathscr{C}, \mathcal{M}\mathrm{od})$ satisfies étale hyperdescent since $\mathcal{M}\mathrm{od}$ is an étale hyperstack. Moreover, we claim that the functor of compact objects $\mathcal{M}\mathrm{od}^\omega$ also satisfies étale hyperdescent. It suffices to check that $\mathrm{Mod}_S^\omega \to \lim_\Delta \mathrm{Mod}_{T^\bullet}^\omega$ is an equivalence for any étale hypercover $S \to T^\bullet$. But this follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}_S^\omega \arrow[r] \arrow[d] & \lim_\Delta \mathrm{Mod}_{T^\bullet}^\omega \arrow[d] \\
\mathrm{Mod}_S \arrow[r] & \lim_\Delta \mathrm{Mod}_{T^\bullet}.
\end{tikzcd}$$

The vertical arrows are fully faithful, and the bottom arrow is an equivalence. It follows that the top arrow is fully faithful. It is also essentially surjective for the following reason. The compact objects of Mod_S are precisely the dualizable ones. But, the property of being dualizable can be checked étale locally. Thus, $\mathcal{M}\mathrm{od}^{\omega}$ satisfies étale hyperdescent. Now, by the pullback definition of $\mathcal{M}_{\mathscr{C}}$ above, it follows that $\mathcal{M}_{\mathscr{C}}$ satisfies étale hyperdescent. □

Because it satisfies étale hyperdescent, the functor $\mathcal{M}_{\mathscr{C}}$ extends uniquely to a limit preserving functor $\mathrm{Shv}_R^{\text{ét}} \to \widehat{\mathrm{Cat}}_{\infty}$. We abuse notation and write $\mathcal{M}_{\mathscr{C}}$ for the resulting stack. Let $\mathbf{M}_{\mathscr{C}} = \mathcal{M}_{\mathscr{C}}^{\mathrm{eq}}$ be the sheaf of equivalences in $\mathcal{M}_{\mathscr{C}}$. We call this the moduli sheaf (or moduli space) of objects in $\mathscr{C}$. It is a sheaf of small spaces because $\mathscr{C}^{\omega}$ is small. If $\mathscr{C}$ is Mod_A for some R–algebra A, we also write $\mathcal{M}_A$ for $\mathcal{M}_{\mathrm{Mod}_A}$. This sheaf classifies left A–module structures on perfect S–modules.

Definition 5.5 Let $\mathbf{M}_R^{[a,b]}$ be the full subsheaf of $\mathbf{M}_R$ whose S–points consist of perfect S–modules with Tor-amplitude contained in $[a,b]$. Note that this makes sense since Tor-amplitude is stable under base change by Proposition 2.13. By the same proposition, there is an equivalence

$$\operatorname*{colim}_{a \le b} \mathbf{M}_R^{[a,b]} \xrightarrow{\sim} \mathbf{M}_R,$$

and each map $\mathbf{M}_R^{[a,b]} \to \mathbf{M}_R$ is a monomorphism.

Theorem 5.6 *The sheaf $\mathbf{M}_R$ is locally geometric and locally of finite presentation.*

Proof By the definition of local geometricity, it suffices to show that each $\mathbf{M}_R^{[a,b]}$ is $(n+1)$–geometric and locally of finite presentation where $n = b - a$. To begin, we show that each diagonal map

$$\mathbf{M}_R^{[a,b]} \to \mathbf{M}_R^{[a,b]} \times_{\mathrm{Spec}\, R} \mathbf{M}_R^{[a,b]} \tag{10}$$

is $(b-a)$–geometric and locally of finite presentation. Given a map from $\mathrm{Spec}\, S$ to the product classifying two perfect S–modules P and Q, the pullback along the diagonal is equivalent to $\mathrm{Eq}(P,Q)$, the sheaf over $\mathrm{Spec}\, S$ classifying equivalences between P and Q. This is a Zariski open subsheaf of $\mathrm{Spec}\,\mathrm{Sym}_S(P \otimes_S Q^{\vee})$. Since $P \otimes_S Q^{\vee}$ has Tor-amplitude contained in $[a-b, b-a]$, by Theorem 5.2, $\mathrm{Eq}(P,Q) \to \mathrm{Spec}\, S$ is $(b-a)$–geometric. Therefore, the diagonal map (10) is $(b-a)$–geometric, as desired.

We now proceed by induction on $n = b - a$. When $n = 0$, a–fold suspension induces an equivalence

$$\mathbf{Proj}_R \to \mathbf{M}_R^{[a,a]}.$$

By Example 4.21, $\mathcal{P}\mathrm{roj}_R$ is 1–geometric and locally of finite presentation. Now, suppose that $\mathbf{M}_R^{[a+1,b]}$ is $(b-a)$–geometric and locally of finite presentation. The general outline of the induction is as follows. We construct a sheaf U that admits a 0–geometric smooth map to $\mathbf{M}_R^{[a+1,b]} \times_{\operatorname{Spec} R} \mathbf{M}_R^{[a+1,a+1]}$ and use this to show that U is a $(b-a)$–geometric sheaf locally of finite presentation. Then, we show that U submerses onto $\mathbf{M}_R^{[a,b]}$. By Lemma 4.30, this is enough to conclude that $\mathbf{M}_R^{[a,b]}$ is $(b-a+1)$–representable and locally of finite presentation.

Let U be defined as the pullback of sheaves

$$\begin{tikzcd} U \arrow[r] \arrow[d, "p"'] & \operatorname{Fun}(\Delta^1, \mathcal{M}\mathrm{od}_R^{\omega})^{\mathrm{eq}} \arrow[d] \\ \mathbf{M}_R^{[a+1,b]} \times_{\operatorname{Spec} R} \mathbf{M}_R^{[a+1,a+1]} \arrow[r] & \operatorname{Fun}(\partial\Delta^1, \mathcal{M}\mathrm{od}_R^{\omega})^{\mathrm{eq}}. \end{tikzcd}$$

Suppose that $\operatorname{Spec} S \to \mathbf{M}_R^{[a+1,b]} \times_{\operatorname{Spec} R} \mathbf{M}_R^{[a+1,a+1]}$ is a point classifying a perfect S–module Q of Tor-amplitude contained in $[a+1,b]$ and a perfect S–module $\Sigma^{a+1} M$ of Tor-amplitude contained in $[a+1,a+1]$. The fiber of p at this point is simply the local moduli sheaf

$$\operatorname{Spec} \operatorname{Sym}_S(Q \otimes_S \Sigma^{-a-1} M).$$

As $Q \otimes_S \Sigma^{-a-1} M$ has Tor-amplitude contained in $[0, b-a-1]$, it follows that this local moduli sheaf is 0–geometric and locally of finite presentation because $\operatorname{Sym}_S Q \otimes_S \Sigma^{-a-1} M$ is a compact commutative S–algebra (because $Q \otimes_S \Sigma^{-a-1} M$ is compact). Therefore, p is 0–geometric and locally of finite presentation. Moreover, $\mathbf{M}_R^{[a+1,b]} \times_{\operatorname{Spec} R} \mathbf{M}_R^{[a+1,a+1]}$ is a $(b-a)$–geometric sheaf locally of finite presentation by the inductive hypothesis. So, U is $(b-a)$–geometric by Lemma 4.25, and it is locally of finite presentation.

Let $q\colon U \to \mathbf{M}_R^{[a,b]}$ be the map that sends an object of U to the fiber of the map it classifies in $\operatorname{Fun}(\Delta^1, \mathcal{M}\mathrm{od}_R^{\omega})^{\mathrm{eq}}$, which has the asserted Tor-amplitude by part (5) of Proposition 2.13. Since U is $(b-a)$–geometric and because the diagonal of $\mathbf{M}_R^{[a,b]}$ is $(b-a)$–geometric, it follows from Lemma 4.30 that q is $(b-a)$–geometric. If we prove that q is smooth and surjective, it will follow that $\mathbf{M}_R^{[a,b]}$ is $(b-a+1)$–geometric by Lemma 4.30.

The surjectivity of q is immediate from part (7) of Proposition 2.13. To check smoothness, consider a point $\operatorname{Spec} S \to \mathbf{M}^{[a,b]}$, which classifies a compact S–module P of Tor-amplitude contained in $[a,b]$. Let Z be the fiber product of this map with $U \to \mathbf{M}^{[a,b]}$. The T–points of the sheaf Z may be described as ways of writing $P \otimes_S T$ as a fiber of a map $Q \to \Sigma^{a+1} M$, where M is a finitely generated projective T–module, and Q is a T–module with Tor-amplitude contained in $[a+1,b]$. Possibly after passing to a Zariski cover of $\operatorname{Spec} T$, we may assume that $M \simeq T^{\oplus r}$ is finitely generated and free. In other words, the sheaf Z decomposes as

$$Z \simeq \coprod_{r \geq 0} Z_r,$$

where Z_r classifies maps $\Sigma^a S^{\oplus r} \to P$ with cofiber having Tor-amplitude contained in $[a+1, b]$. Since $\operatorname{Spec}\operatorname{Sym}_S(\Sigma^a (P^\vee)^{\oplus r})$ classifies all maps $\Sigma^a S^{\oplus r} \to P$, we see that Z_r consists of the points of $\operatorname{Spec}\operatorname{Sym}_S(\Sigma^a (P^\vee)^{\oplus r})$ classifying maps $\Sigma^a S^{\oplus r} \to P$ that are surjective on π_a. Since by Proposition 2.6, $\pi_a P$ is finitely generated, this surjectivity condition is open, because the vanishing of the cokernel of $\pi_0 S^{\oplus r} \to \pi_a P$ can be detected on fields by Nakayama's Lemma. As the perfect module $(P^\vee)^{\oplus r}$ has Tor-amplitude contained in $[a-b, 0]$, $\operatorname{Spec}\operatorname{Sym}_S(\Sigma^a (P^\vee)^{\oplus r})$ is smooth by Theorem 5.2. Thus, Z_r is smooth, and hence the morphism $U \to \mathbf{M}^{[a,b]}$ is smooth, which completes the proof. □

To analyze $\mathbf{M}_A$ for other rings A, we use subsheaves $\mathbf{M}_A^{[a,b]}$ of $\mathbf{M}_A$ which are induced by the corresponding subsheaves of $\mathbf{M}_R$: specifically, define $\mathbf{M}_A^{[a,b]}$ to be the pullback in

$$\begin{tikzcd}
\mathbf{M}_A^{[a,b]} \arrow[r, "\pi^{[a,b]}"] \arrow[d] & \mathbf{M}_R^{[a,b]} \arrow[d] \\
\mathbf{M}_A \arrow[r, "\pi"] & \mathbf{M}_R.
\end{tikzcd}$$

Since the filtration $\{\mathbf{M}_R^{[-a,a]}\}_{a\geq 0}$ exhausts $\mathbf{M}_R$, it follows that $\{\mathbf{M}_A^{[-a,a]}\}_{a\geq 0}$ exhausts $\mathbf{M}_A$.

Proposition 5.7 *Let Mod_A be an R–linear category of finite type, so that A is a compact R–algebra. Then, the natural morphism $\pi\colon \mathbf{M}_A^{[a,b]} \to \mathbf{M}_R^{[a,b]}$ is $(b-a)$–geometric and locally of finite presentation.*

Proof It is easy to see using Corollary 3.2 that the space of T–points of the fiber of $\pi^{[a,b]}$ at a point $\operatorname{Spec} S \to \mathbf{M}_R^{[a,b]}$ classifying a perfect S–module P is equivalent to

$$\operatorname{map}_{\mathrm{Alg}_T}(A \otimes_R T, \operatorname{End}_T(P \otimes_S T)).$$

We will write $\mathbf{map}(A \otimes_R S, \operatorname{End}_S(P))$ for the resulting sheaf over $\operatorname{Spec} S$. Now, since $A \otimes_R S$ is of finite presentation as an S–algebra, $A \otimes_R S$ is a retract of a finite colimit of the free S–algebra $S\langle t\rangle$. It follows from Lemmas 4.35 and 4.38 and Proposition 4.37 that to prove that $\mathbf{map}(A \otimes_R S, \operatorname{End}_S(P))$ is $(b-a)$–geometric and locally of finite presentation, it suffices to show that $\mathbf{map}(S\langle t\rangle, \mathrm{End}_S(P))$ is a quasiseparated $(b-a)$–geometric sheaf that is locally of finite presentation. But,

$$\mathbf{map}_{\mathrm{Alg}_S}(S\langle t\rangle, \mathrm{End}_S(P)) \simeq \mathbf{map}_{\mathrm{Mod}_S}(S, \mathrm{End}_S(P)) \simeq \mathrm{Spec}\,\mathrm{Sym}_S(\mathrm{End}_S(P)^\vee).$$

Since $\mathrm{End}_S(P)^\vee \simeq P^\vee \otimes_S P$ is perfect and has Tor-amplitude contained in $[a-b, b-a]$, it follows from Theorem 5.2 that the fiber is $(b-a)$–geometric, quasiseparated, and locally of finite presentation. □

Given the proposition, it is now straightforward to prove the following theorem. After we completed the first version of this paper, we were pointed to the thesis of Pandit [49], which establishes the result in the case where A is smooth and proper using other methods, namely the representability theorem of Lurie.

Theorem 5.8 *Let A be a compact R–algebra. Then, the stack $\mathbf{M}_A$ is locally geometric and locally of finite presentation, and the functor $\pi\colon \mathbf{M}_A \to \mathbf{M}_R$ is locally geometric and locally of finite presentation.*

Proof By the previous proposition, $\mathbf{M}_A^{[a,b]} \to \mathbf{M}_R^{[a,b]}$ is $(b-a)$–geometric and locally of finite presentation. Since $\mathbf{M}_R^{[a,b]}$ is $(b-a+1)$–geometric and locally of finite presentation, it follows from Lemma 4.25, that $\mathbf{M}_A^{[a,b]}$ is also $(b-a+1)$–geometric and locally of finite presentation. It follows that $\mathbf{M}_A$ is locally geometric and locally of finite presentation. To prove the second statement, let $\mathrm{Spec}\, S \to \mathbf{M}_R$ classify a perfect S–module P, which has Tor-amplitude contained in some $[a,b]$. The fiber of π over this point is equivalent to the fiber of $\pi^{[a,b]}$ over P, which by the previous proposition is $(b-a)$–geometric and locally of finite presentation. □

Note that, in the proof, the fiber is not only locally geometric, but $(b-a)$–geometric. However, the geometricity of the fibers changes from point to point.

Corollary 5.9 *Let A be a compact R–algebra, and let $\mathrm{Spec}\, S \to \mathbf{M}_A$ classify a perfect S–module P with a left A–module structure. Then, the cotangent complex of $\mathbf{M}_A$ at the point P is the S–module*

$$\mathrm{L}_{\mathbf{M}_A,P} \simeq \Sigma^{-1}\,\mathrm{End}_{A^{\mathrm{op}}\otimes_R S}(P)^\vee.$$

Proof By Lemma 4.9 and Proposition 4.45, the cotangent complex $\mathrm{L}_{\mathbf{M}_A}$ exists. Consider the loop sheaf $\Omega_P \mathbf{M}_A$. By Lemma 4.11, the cotangent complex of this sheaf at the basepoint $*$ is simply

$$\mathrm{L}_{\Omega_P \mathbf{M}_A, *} \simeq \Sigma \mathrm{L}_{\mathbf{M}_A, P}.$$

Thus, it suffices to compute $\mathrm{L}_{\Omega_P \mathbf{M}_A,*}$. The stack $\Omega_P \mathbf{M}_A$ is described by

$$\begin{aligned}\Omega_P \mathbf{M}_A(T) \simeq \mathrm{aut}_{A^{\mathrm{op}}\otimes_R T}(P \otimes_S T) &\subseteq \Omega^\infty \mathrm{End}_{A^{\mathrm{op}}\otimes_R T}(P \otimes_S T)\\ &\simeq \mathrm{map}_S(S, \mathrm{End}_{A^{\mathrm{op}}\otimes_R T}(P \otimes_S T))\\ &\simeq \mathrm{map}_S(S, \mathrm{End}_{A^{\mathrm{op}}\otimes_R S}(P) \otimes_S T)\\ &\simeq \mathrm{map}_S(\mathrm{End}_{A^{\mathrm{op}}\otimes_R S}(P)^\vee, T)\\ &\simeq \mathrm{map}(\mathrm{Spec}\, T, \mathrm{Spec}\, \mathrm{Sym}_S(\mathrm{End}_{A^{\mathrm{op}}\otimes_R S}(P)^\vee)),\end{aligned}$$

where the equivalence between the second and third lines is by Lemma 2.7. The inclusion is Zariski open, and hence the computation of Theorem 5.2 says that

$$\mathrm{L}_{\Omega_P \mathbf{M}_A,*} \simeq \mathrm{End}_{A^{\mathrm{op}}\otimes_R S}(P)^\vee,$$

which completes the proof. □

5.3 Étale local triviality of Azumaya algebras

Let R be a connective $\mathbb{E}_\infty$–ring spectrum, and let A be an Azumaya R–algebra. We prove now that the stack of Morita equivalences from A to R is smooth and surjective over $\mathrm{Spec}\, R$. As a corollary, we obtain one of the major theorems of the paper: the étale local triviality of Azumaya algebras.

Since A is compact as an R–module, the space $\mathbf{M}_A(S)$ of S–points is equivalent to the space

$$\mathbf{M}_A(S) \simeq \mathrm{Fun}_{\mathrm{Cat}_{S,\omega}}(\mathrm{Mod}_{A\otimes_R S}, \mathrm{Mod}_S)^{\mathrm{eq}}$$

of (compact object preserving) functors between compactly generated S–linear categories. We define the full subsheaf $\mathbf{Mor}_A \subseteq \mathbf{M}_A$ of Morita equivalences from A to R by restricting the S–points to the full subspace of $\mathbf{M}_A(S)$ consisting of the equivalences $\mathrm{Mod}_{A\otimes_R S} \simeq \mathrm{Mod}_S$.

Proposition 5.10 *Suppose that R is a connective $\mathbb{E}_\infty$–ring and that A is an Azumaya algebra. The sheaf $\mathbf{Mor}_A \to \mathrm{Spec}\, R$ of Morita equivalences is locally geometric and smooth.*

Proof We show that $\mathbf{Mor}_A \subseteq \mathbf{M}_A$ is quasicompact and Zariski open. Fix an S–valued point of $\mathbf{M}_A$ classifying an $A^{\mathrm{op}} \otimes_R S$–module P which is compact as an S–module. The bimodule P defines an adjoint pair of functors

$$-\otimes_A P\colon \mathrm{Mod}_{A\otimes_R S} \rightleftarrows \mathrm{Mod}_S : \mathrm{Map}_S(P, -).$$

To show that $\mathbf{Mor}_A \subseteq \mathbf{M}_A$ is open, it suffices to check that the subsheaf of points of $\operatorname{Spec} S$ on which the unit

$$\eta(A)\colon A \to \operatorname{Map}_S(P, A \otimes_A P)$$

and counit

$$\epsilon(S)\colon \operatorname{Map}_S(P, S) \otimes_A P \to S$$

morphisms are equivalences is open in $\operatorname{Spec} S$, since the unit and counit transformations are equivalences if and only if they are equivalences on generators. As A is a perfect S–module by the Azumaya hypothesis, Proposition 2.14 implies that the cofibers of these maps vanish on quasicompact Zariski open subschemes of $\operatorname{Spec} S$. Taking the intersection of these two open subschemes yields the desired quasicompact Zariski open subscheme of $\operatorname{Spec} S$ on which P defines a Morita equivalence. It follows that $\mathbf{Mor}_A$ is locally geometric and locally of finite presentation.

Given a point $P\colon \operatorname{Spec} S \to \mathbf{Mor}_A$, the cotangent complex at P is

$$\mathrm{L}_{\mathbf{Mor}_A, P} \simeq \Sigma^{-1} \operatorname{End}_{A \otimes_R S}(P) \simeq \Sigma^{-1} S,$$

a perfect S–module with Tor-amplitude contained in $[-1,-1]$. Thus, by definition, $\mathbf{Mor}_A$ is a smooth locally geometric sheaf. □

The following theorem is a generalization of [57, Proposition 2.14] to connective $\mathbb{E}_\infty$–ring spectra.

Theorem 5.11 *Let R be a connective $\mathbb{E}_\infty$ ring spectrum, and let A be an Azumaya R–algebra. Then, there is a faithfully flat étale R–algebra S such that $A \otimes_R S$ is Morita equivalent to S.*

Proof The theorem follows immediately from the previous proposition, Theorems 3.14 and 4.47. □

Proposition 5.12 *If R is a connective $\mathbb{E}_\infty$–ring spectrum and A is an Azumaya R–algebra, then the sheaf of Morita equivalences $\mathbf{Mor}_A$ is a $\mathbf{Pic}$–torsor. In particular, it is 1–geometric and smooth.*

Proof The action of $\mathbf{Pic}$ on $\mathbf{Mor}_A$ is simply by tensoring A^{op}–modules with line bundles. Étale-locally, $\mathbf{Mor}_A$ is equivalent to the space of equivalences $\mathrm{Mod}_S \simeq \mathrm{Mod}_S$ by the theorem. This is precisely $\mathbf{Pic}$ over $\operatorname{Spec} S$. □

6 Gluing generators

The main result of the previous section shows that Azumaya algebras over connective $\mathbb{E}_\infty$–rings are étale locally trivial. In this section, we want to show that certain étale cohomological information on derived schemes X can be given by Azumaya algebras. In other words we want to prove that "$\mathrm{Br}(X) = \mathrm{Br}'(X)$" in various cases. This is established once we prove the following theorem.

Theorem 6.1 (Local-global principle) *Let $\mathscr{C}$ be an R–linear category with descent, and suppose that $R \to S$ is an étale cover such that $\mathscr{C} \otimes_R S$ has a compact generator. Then, $\mathscr{C}$ has a compact generator.*

In fact, we prove a version of this theorem for quasicompact and quasiseparated derived schemes. The result we prove expands on [43, Theorem 6.1], which is about a similar statement for the property of being compactly generated.

The proof breaks up into several parts. First, we prove a local-global statement for Zariski covers. This is used in two ways: to reduce the problem from schemes to affine schemes and to help prove Nisnevich descent. Second, we prove a local-global statement for étale covers, following [43, Section 6]. The main insight there is to use the fact that a presheaf is an étale sheaf if and only if it is a sheaf for the Nisnevich and finite étale topologies. Then, by using a theorem of Morel and Voevodsky (see [43, Theorem 2.9]), we can reduce the proof of Nisnevich local-global principle to certain special squares, which we analyze directly using techniques that, essentially, go back to Thomason and Trobaugh [56] and Bökstedt and Neeman [13]. Proof of a local-global principle for finite étale covers is subsumed in a more general statement for finite and flat covers, which is purely ∞–categorical.

This theorem has applications to the module theory of perfect stacks, as is developed in [9; 43], and is related to questions about when derived categories are compactly generated, and has been studied by Thomason and Trobaugh [56], Neeman [47; 48], Bökstedt and Neeman [13] and Bondal and van den Bergh [14].

With two major exceptions, the outline of the proof is already contained in [57]. First, our proof differs significantly from Toën's when it comes to the étale local-global principle. Since Toën works with simplicial commutative rings, he is able to use some concrete constructions based on work of Gabber [25] to reduce to the finite étale case. These constructions, which involve algebras of invariants of symmetric groups acting on polynomial rings and quotient algebras, simply fail in the case of $\mathbb{E}_\infty$–ring spectra. Thus, we use Lurie's idea of using the Morel–Voevodsky result to prove the étale local-global principle. Second, we cannot prove the fppf version contained in [57].

Because of the lack of $\mathbb{E}_\infty$–structures on quotient rings, we do not know how to show that the stack of quasisections used in the proof of [57, Proposition 4.13]. Hence, we work everywhere in the étale topology. For the cases of interest to us, this restriction is not a problem.

6.1 Azumaya algebras and Brauer classes over sheaves

Fix a connective $\mathbb{E}_\infty$–ring spectrum R. In Section 4, we introduced the étale hyperstacks $\mathscr{A}\mathrm{lg}$, $\mathscr{A}\mathrm{lg}^{\mathrm{Az}}$ and $\mathscr{C}\mathrm{at}_R^{\mathrm{desc}}$. Let **Alg**, **Az**, and **Pr** be the associated underlying (large) étale hypersheaves. There are natural maps $\mathbf{Alg} \to \mathbf{Pr}$ and $\mathbf{Az} \to \mathbf{Pr}$. Let **Mr** and **Br** be the étale hypersheafifications of the images of these maps. To be precise, for every connective commutative R–algebra S, there is a map $\mathbf{Alg}(S) \to \mathbf{Pr}(S)$, and the image of this map is full subspace of $\mathbf{Pr}(S)$ consisting of those points of $\mathbf{Pr}(S)$ in the image of $\mathbf{Alg}(S)$. As S varies, these images determine a presheaf of spaces, and **Mr** is the étale sheafification of this presheaf. The story for **Br** is similar, but with **Az** in place of **Alg**.

Definition 6.2 Let X be an object of $\mathrm{Shv}_R^{\text{ét}}$.

(1) A quasicoherent algebra over X is a morphism $X \to \mathbf{Alg}$.

(2) An Azumaya algebra over X is a morphism $X \to \mathbf{Az}$.

(3) A Morita class over X is a morphism $X \to \mathbf{Mr}$.

(4) A Brauer class over X is a morphism $X \to \mathbf{Br}$.

(5) A linear category with descent over X is a morphism $X \to \mathbf{Pr}$.

Note that of the above, only **Az** and **Br** are actually sheaves of *small* spaces.

A Brauer class over X is thus a linear category over X which is étale locally equivalent to modules over some Azumaya algebra. The rest of this section will prove that every Brauer class (resp. Morita class) over X lifts to an Azumaya algebra (resp. algebra) when X is a quasicompact and quasiseparated derived scheme.

If $\alpha\colon \operatorname{Spec} S \to \mathbf{Pr}$ is a linear category with descent over $\operatorname{Spec} S$, let Mod_S^α denote the Mod_S–module classified by α by the Yoneda lemma. The following construction is studied extensively in [9; 43].

Definition 6.3 Let $\alpha\colon X \to \mathbf{Pr}$ be a linear category with descent over X. Then, the ∞–category of α–twisted X–modules is

$$\mathrm{Mod}_X^\alpha = \lim_{f\colon \operatorname{Spec} S \to X} \mathrm{Mod}_S^{\alpha\circ f}.$$

This limit exists and Mod_X^α is stable and presentable, because $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ is closed under limits.

We describe this construction as the right Kan extension of $(\mathrm{Aff}^{\mathrm{cn}}_{/X})^{\mathrm{op}} \simeq \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Pr}^{\mathrm{L}}$, the functor which sends $f\colon \operatorname{Spec} S \to X$ to $\mathrm{Mod}_S^{\alpha\circ f}$, evaluated at X. As a particularly important case, let $\alpha\colon X \to \mathbf{Pr}$ be the linear category with descent over X which sends a point $x\colon \operatorname{Spec} S \to X$ to $\mathrm{Mod}_S^{x^*\alpha} = \mathrm{Mod}_S$. Then, by definition, Mod_X^{α} is simply Mod_X, which is an $\mathbb{E}_\infty$–algebra in Pr^{L}. When X is a (nonderived) scheme, then the homotopy category of Mod_X recovers the usual derived category $D_{\mathrm{qc}}(X)$ of complexes of $\mathcal{O}_X$–modules with quasicoherent cohomology sheaves.

The construction of the stable ∞–category of α–twisted modules commutes with colimits.

Lemma 6.4 *Let $I \to \mathrm{Shv}_R^{\text{ét}}$ be a small diagram of sheaves X_i with colimit X, let $\alpha\colon X \to \mathbf{Pr}$ be a linear category with descent over X, and let α_i be the restriction of X to X_i. Then, the canonical map*

$$\mathrm{Mod}_X^{\alpha} \to \lim_I \mathrm{Mod}_{X_i}^{\alpha_i}$$

is an equivalence.

Proof This follows from our definition of Mod_X^{α} as a right Kan extension. □

To attack our main theorem, the local-global principle, we require some additional terminology.

Definition 6.5 Let $\alpha\colon X \to \mathbf{Pr}$ be a linear category with descent over X.

(1) An object P of Mod_X^{α} is called perfect if for every point $x\colon \operatorname{Spec} S \to X$, x^*P is a compact object of $\mathrm{Mod}_S^{x^*\alpha}$.

(2) An object P of Mod_X^{α} is a perfect generator if for every point $x\colon \operatorname{Spec} S \to X$, the pullback x^*P is a compact generator of $\mathrm{Mod}_S^{x^*\alpha}$.

(3) An object P is a global generator of Mod_X^{α} if it is a compact generator and a perfect generator.

Note that while perfect objects are preserved automatically by any pullback induced by a map $\pi\colon X \to Y$ in $\mathrm{Shv}_X^{\text{ét}}$, it is not the case that compact objects are preserved by pullbacks. For instance, if X is not quasicompact over the base $\operatorname{Spec} R$, then $\mathrm{Mod}_R \to \mathrm{Mod}_X$ sends R to $\mathcal{O}_X$, which is perfect but might not be compact. It is for this reason why perfect objects play such an important role. However, in most cases of interest, it is possible to show that the perfect and compact objects do coincide; see, for example, [9, Section 3].

When X is affine, the next lemma shows that there is no difference between the notions of compact generators and perfect generators of Mod_X^{α}. In particular, every perfect generator is automatically a global generator.

Lemma 6.6 *If α: Spec $S \to$ **Pr** is a linear category with descent, then an object P of Mod_S^α is a compact generator if and only if it is a perfect generator.*

Proof If P is a perfect generator, then P is a compact generator of Mod_S^α by definition. So, suppose that P is a compact generator of Mod_S^α. We must show that for any f: Spec $T \to$ Spec S, where T is a connective $\mathbb{E}_\infty$–S–algebra, then $P \otimes_S T$ is a compact generator of Mod_T^α. There is a commutative diagram of equivalences

$$\begin{tikzcd}
\mathrm{Mod}_S^\alpha \otimes_S \mathrm{Mod}_T \arrow[rr] \arrow[d, "\mathrm{Map}(P,-)"'] & & \mathrm{Mod}_T^{f^*\alpha} \arrow[d] \\
\mathrm{Mod}_{\mathrm{End}(P)^{\mathrm{op}}} \otimes_S \mathrm{Mod}_T \arrow[r] & \mathrm{Mod}_{\mathrm{End}(P)^{\mathrm{op}} \otimes_S T},
\end{tikzcd}$$

in which the left-hand equivalence is Morita theory (Theorem 2.8) and the right-hand equivalence is induced from the other three. By commutativity, the object $P \otimes_S T$ in the upper-left corner is sent to $\mathrm{End}(P)^{\mathrm{op}} \otimes_S T$ in the lower-right corner, which is indeed a compact generator. $\square$

The following lemma will be used below to detect when an object is a compact generator of Mod_S^α by passing to Mod_T^α for an étale cover $S \to T$.

Lemma 6.7 *If $S \to T$ is an étale cover, and if α: Spec $S \to$ **Pr** is an linear category with descent, then a* compact *object P of Mod_S^α is a compact generator of Mod_S^α if and only if $P \otimes_S T$ is a compact generator of Mod_T^α.*

Proof One direction is clear: if P is a compact generator of Mod_S^α, then by the lemma above, it is a perfect generator, so that $P \otimes_S T$ is a compact generator of Mod_T^α. So, suppose that P is a compact object of Mod_S^α such that $P \otimes_S T$ is a compact generator of Mod_T^α. Let $A = \mathrm{End}_S(P)^{\mathrm{op}}$, and let Mod_A be the stable ∞–category of A–modules. Write $T^\bullet$ for the cosimplicial commutative S–algebra associated to the cover $S \to T$. Consider commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}_S^\alpha \arrow[r] \arrow[d, "\mathrm{Map}(P,-)"'] & \lim_\Delta \mathrm{Mod}_{T^\bullet}^\alpha \arrow[d, "\mathrm{Map}(P\otimes_S T^\bullet,-)"] \\
\mathrm{Mod}_A \arrow[r] & \lim_\Delta \mathrm{Mod}_{A\otimes_S T^\bullet}.
\end{tikzcd}$$

The horizontal maps are equivalences since both Mod_S^α and Mod_A satisfy étale descent, the latter by Example 4.4. On the other hand, since $P \otimes_S T$ is a compact generator of Mod_T^α, the right vertical map is an equivalence, since it is the limit of a levelwise equivalence of simplicial ∞–categories. It follows that the left vertical map is an equivalence. In particular, if $\mathrm{Map}(P, M) \simeq 0$, then $M \simeq 0$ in Mod_S^α. Thus, P is a compact generator of Mod_S^α. □

As we now see, the linear categories with descent over X that possess perfect generators are exactly those which are ∞–categories of modules for quasicoherent algebras over X. Our strategy in proving the local-global principle is then to construct perfect generators.

Proposition 6.8 *A linear category with descent* $\alpha\colon X \to \mathbf{Pr}$ *factors through* $\mathbf{Alg} \to \mathbf{Pr}$ *if and only if* Mod_X^α *possesses a perfect generator.*

Proof Suppose that $\alpha\colon X \to \mathbf{Pr}$ factors as

$$X \xrightarrow{\mathcal{A}} \mathbf{Alg} \to \mathbf{Pr}.$$

Then, there is an algebra object A in Mod_X^α, which restricts to an S–algebra $A \otimes S$ over each affine $\mathrm{Spec}\, S \to X$, and which is a compact generator of the S–linear category $\mathrm{Mod}_S^\alpha \simeq \mathrm{Mod}_{A\otimes S}$. Hence, A is a perfect generator. Now, suppose that P is a perfect generator of Mod_X^α. By hypothesis, for any point $x\colon \mathrm{Spec}\, S \to X$, the object P of $\mathrm{Mod}_S^{x^*\alpha}$ induces an equivalence

$$\mathrm{Map}(P, -)\colon \mathrm{Mod}_S^{x^*\alpha} \to \mathrm{Mod}_{\mathrm{End}(P)^{\mathrm{op}}\otimes S}.$$

In other words, we obtain a natural equivalence of functors

$$\mathrm{Mod}^\alpha_{\mathrm{Spec}\, -/X} \to \mathrm{Mod}_{\mathrm{End}(P)^{\mathrm{op}}\otimes -}.$$

Therefore, $\mathrm{End}(P)^{\mathrm{op}}$ classifies a lift of α through $\mathbf{Alg} \to \mathbf{Pr}$. □

6.2 The Zariski local-global principle

There is a long history to the arguments in this section. On the one hand, the ideas about lifting compact objects along localizations goes back to Thomason and Trobaugh [56] and Neeman [47, Theorem 2.1]. On the other hand, the arguments about Zariski gluing appeared in Bökstedt and Neeman [13, Section 6], in an argument about derived categories of quasicoherent sheaves. They were further used in [48, Proposition 2.5] and [14, Theorem 3.1.1] before being used by Toën [57, Proposition 4.9] for module categories over quasicoherent sheaves of algebras.

Given a colimit-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ of stable presentable ∞–categories, the kernel of F is full subcategory of $\mathcal{C}$ consisting of those objects which become equivalent to 0 in $\mathcal{D}$. Since the ∞–category of stable presentable ∞–categories has limits which are computed in $\widehat{\mathrm{Cat}}_\infty$, we see that the kernel of F is stable, presentable, and equipped with a colimit-preserving inclusion into $\mathscr{C}$.

In this section, when U is a quasicompact open subscheme of a derived scheme X, we will write $\mathrm{Mod}^\alpha_{X,Z}$ for the kernel of $\mathrm{Mod}^\alpha_X \to \mathrm{Mod}^\alpha_U$, where Z is the complement of U in X. Of course, this complement will usually not exist as a derived scheme, but only as a closed subspace of X.

The following proposition, which appears essentially in [13], is one of the major points of "derived" geometry in our proof that $\mathrm{Br}(X) = \mathrm{Br}'(X)$. The generator K in the proof is truly a derived object, and thus produces, even in the case of ordinary schemes, a derived Azumaya algebra.

Proposition 6.9 [13, Proposition 6.1; 57, Proposition 3.9] *Let $j\colon U \subset X = \mathrm{Spec}\, S$ be a quasicompact open subscheme with complement Z, and let $\alpha\colon X \to \mathbf{Pr}$ be a S–linear category such that Mod^α_X has a compact generator P. Then, the restriction functor $j^*\colon \mathrm{Mod}^\alpha_X \to \mathrm{Mod}^\alpha_U$ is a localization whose kernel $\mathrm{Mod}^\alpha_{X,Z}$ is generated by a single compact object L in Mod^α_X.*

Proof Note that under these hypotheses, it is enough to treat the special case in which α classifies Mod_S. Indeed, in this case, we have a localization sequence

$$\mathrm{Mod}_{X,Z} \to \mathrm{Mod}_X \to \mathrm{Mod}_U.$$

Since Mod^α_X is dualizable (it admits a compact generator), tensoring with Mod^α_X preserves limits, and we obtain the localization sequence

$$\mathrm{Mod}^\alpha_{X,Z} \to \mathrm{Mod}^\alpha_X \to \mathrm{Mod}^\alpha_U.$$

To complete the proof, it suffices to show that $\mathrm{Mod}_{X,Z}$ has a compact generator. Write

$$U = \bigcup_{i=1}^r \mathrm{Spec}\, S[f_i^{-1}],$$

and let K_i be the cone of $S \xrightarrow{f_i} S$. Then, $K = K_1 \otimes_S \cdots \otimes_S K_r$ is a compact object of Mod^α_X, and $j^*L \simeq 0$. We claim that K is a compact generator of the kernel of $\mathrm{Mod}_{X,Z}$. Suppose that $\mathrm{Map}(K, M) \simeq 0$ and $j^*(M) \simeq 0$. Then,

$$\mathrm{Map}(K_1, \mathrm{Map}(K_2 \otimes_S \cdots \otimes_S K_r, M) \simeq \mathrm{Map}(K, M) \simeq 0,$$

where we are using the fact that K_i is self-dual up to a shift. It follows that f_1 acts invertibly on $\mathrm{Map}(K_2 \otimes_S \cdots \otimes_S K_r, M)$, so that

$$\begin{aligned}\mathrm{Map}(K_2 \otimes_S \cdots \otimes_S K_r, M) &\simeq \mathrm{Map}(K_2 \otimes_S \cdots \otimes_S K_r, M) \otimes_S S[f_1^{-1}] \\ &\simeq \mathrm{Map}(K_2 \otimes_S \cdots \otimes_S K_r, M \otimes_S S[f_1^{-1}]) \simeq 0,\end{aligned}$$

where the last equivalence follows from the fact that $j^* M \simeq 0$ and that $\mathrm{Spec}\, S[f_1^{-1}]$ is contained in U. By induction, it follows that

$$\mathrm{Map}(K_r, M) \simeq 0,$$

and thus that

$$M \simeq M \otimes_S S[f_r^{-1}] \simeq 0.$$

Therefore, K is a compact generator of $\mathrm{Mod}_{X,Z}$. □

We also need the following K–theoretic characterization, due to [56] in the case of schemes and [47] more generally, of when an object lifts through a localization. Recall that if $\mathscr{C}$ is a compactly generated stable ∞–category, then $\mathrm{K}_0(\mathscr{C})$ is the Grothendieck group of the compact objects of $\mathscr{C}$. That is, it is the free abelian group on the *set* of compact objects of $\mathscr{C}$, modulo the relation $[M] = [L] + [N]$ whenever there is a cofiber sequence $L \to M \to N$. Note that $\mathrm{K}_0(\mathscr{C})$ depends only on the triangulated homotopy category $\mathrm{Ho}(\mathscr{C})$.

Proposition 6.10 *Let $\alpha\colon X \to \mathbf{Pr}$ be a linear category such that Mod_X^{α} is compactly generated, where X is a derived scheme over R which can be embedded as a quasicompact open subscheme of an affine $\mathrm{Spec}\, S \in \mathrm{Aff}_R^{\mathrm{cn}}$, and let $U \subseteq X$ be a quasicompact open subscheme. Then, a compact object P of Mod_U^{α} lifts to Mod_X^{α} if and only if it is in the image of $\mathrm{K}_0(\mathrm{Mod}_X^{\alpha}) \to \mathrm{K}_0(\mathrm{Mod}_U^{\alpha})$.*

Proof This follows from Neeman's localization theorem [47, Theorem 2.1] and its corollary [47, Corollary 0.9]. The only thing to check is that $\mathrm{Mod}_{X,Z}^{\alpha}$ is compactly generated by a set of objects that are compact in Mod_X^{α}. For this, we refer to the beginning of the proof of Lemma 6.13, which shows that the inclusion $\mathrm{Mod}_{X,Z}^{\alpha} \to \mathrm{Mod}_X^{\alpha}$ preserves compact objects. □

We are now ready to state and prove our Zariski local-global principle, which is a generalization of the arguments of [13, Section 6] and the theorems [14, Theorem 3.1.1] and [57, Proposition 4.9].

Theorem 6.11 *Let X be a quasicompact, quasiseparated derived scheme over R, and let $\alpha\colon X\to\mathbf{Pr}$ be a linear category with descent over X. If there exists Zariski cover $f\colon \operatorname{Spec} S\to X$ such that $\mathrm{Mod}_S^{f^*\alpha}$ has a compact generator, then there exists a global generator of Mod_X^{α}.*

Proof The proof is by induction on n, the number of affines in an open cover of X over which there are compact generators. If Mod_X^{α} has a compact generator when $X=\operatorname{Spec} S$, then it has a global generator by Lemma 6.6. Now, assume that for all quasicompact, quasiseparated derived schemes Y and all $\beta\colon Y\to\mathbf{Pr}$, if

$$\coprod_{i=1}^{n}\operatorname{Spec} S_i\xrightarrow{\coprod f_i} Y$$

is a Zariski cover such that $\mathrm{Mod}_{S_i}^{f_i^*\beta}$ has a compact generator for $i=1,\dots,n$, then Mod_Y^{β} has a global generator. Let

$$\coprod_{i=1}^{n+1}\operatorname{Spec} T_i\xrightarrow{\coprod g_i} X$$

be a Zariski cover such that each $\mathrm{Mod}_{T_i}^{g_i^*\alpha}$ has a compact generator. The proof will be complete if we produce a global generator of Mod_X^{α}.

Let Y be the union of $\operatorname{Spec} T_i$, $i=1,\dots,n$ in X, let $Z=\operatorname{Spec} T_{n+1}$, and let $W=Y\cap Z$. So, there is a pushout square of sheaves

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X. \end{tikzcd}$$

By Lemma 6.4, it follows that

$$\begin{tikzcd} \mathrm{Mod}_X^{\alpha} \arrow[r] \arrow[d] & \mathrm{Mod}_Z^{\alpha} \arrow[d] \\ \mathrm{Mod}_Y^{\alpha} \arrow[r] & \mathrm{Mod}_W^{\alpha} \end{tikzcd} \tag{11}$$

is a pullback square of stable presentable ∞–categories. By the induction hypothesis, there exists a global generator P_Y of Mod_Y^{α}. The restriction of $P_Y\oplus\Sigma P_Y$ to W lifts to a compact object of Mod_Z^{α} by Proposition 6.10. Since $P_Y\oplus\Sigma P_Y$ is also a compact generator of Mod_Y^{α}, we can assume in fact that the restriction P_W of P_Y to W lifts to a compact object P_Z of Mod_Z^{α}. The cartesian square (11) says that there is an object P_X of Mod_X^{α} that restricts to P_W, P_Y, and P_Z over W, Y, and Z, respectively. The object P_X is in fact compact, because for any M_X in Mod_X^{α}, the mapping space $\mathrm{map}(P_X, M_X)$ is computed as the pullback

$$\begin{tikzcd} \mathrm{map}_X(P_X, M_X) \arrow[r] \arrow[d] & \mathrm{map}_Z(P_Z, M_Z) \arrow[d] \\ \mathrm{map}_Y(P_Y, M_Y) \arrow[r] & \mathrm{map}_W(P_W, M_W). \end{tikzcd}$$

Since finite limits commute with filtered colimits, since the restriction functors preserve colimits, and since P_Z, P_Y, and P_W are compact, it follows that P_X is compact.

Because Z is affine and $W \subset Z$ is quasicompact, by Proposition 6.9, the restriction functor

$$\mathrm{Mod}_Z^{\alpha} \to \mathrm{Mod}_W^{\alpha}$$

is a localization, which kills exactly the stable subcategory of Mod_Z^{α} generated by a compact object Q_Z of Mod_Z^{α}. We may lift Q_Z to an object Q_X of Mod_X^{α} that restricts to 0 over Y using (11). The object Q_X is compact for the same reason that P_X is compact. Then, $L_X = P_X \oplus Q_X$ is a compact object of Mod_X^{α}, which we claim is a global generator of Mod_X^{α}.

Suppose that M_X is an object of Mod_X^{α} such that $\mathrm{Map}_X(L_X, M_X) \simeq 0$. Then, $\mathrm{Map}_X(P_X, M_X) \simeq 0$ and $\mathrm{Map}_X(Q_X, M_X) \simeq 0$. For any N in Mod_X^{α}, we have a cartesian square

$$\begin{tikzcd} \mathrm{Map}_X(N_X, M_X) \arrow[r] \arrow[d] & \mathrm{Map}_Z(N_Z, M_Z) \arrow[d] \\ \mathrm{Map}_Y(N_Y, M_Y) \simeq 0 \arrow[r] & \mathrm{Map}_W(N_W, M_W). \end{tikzcd}$$

When we have that $N_X = Q_X$, the bottom mapping spaces are trivial, and so we have $0 \simeq \mathrm{Map}_X(Q_X, M_X) \simeq \mathrm{Map}_Z(Q_Z, M_Z)$. It follows that M_Z is supported on W. On the other hand, $0 \simeq \mathrm{Map}_X(P_X, M_X) \simeq \mathrm{Map}_Y(P_Y, M_Y)$ since in that case, the right-hand vertical map is an equivalence as M_Z is supported on W. As P_Y is a compact generator of Mod_Y^{α}, the restriction of M to U is trivial. But, the support of M_Z is contained in $W \subset U$, so M is trivial. Therefore, L is a compact generator of Mod_X^{α}.

To prove that L_X is a perfect generator of Mod_X^{α}, it suffices to show that L_Y is a perfect generator of Mod_Y^{α} and that L_Z is a compact generator of Mod_Z^{α} (since Z is affine). Indeed, given any affine $V = \mathrm{Spec}\, S$ mapping into X, we can intersect it with the affine hypercover determined by the T_i. Write $S \to T$ for this map. By hypothesis, $L \otimes T$ is a compact generator for Mod_T^{α}. By Lemma 6.7, it follows that $L \otimes S$ is a compact generator of Mod_S^{α}.

That L_Y is a global generator of Mod_Y^{α} is trivial, since $Q_Y \simeq 0$ and so $L_Y \simeq P_Y$ was chosen to be a global generator of Mod_Y^{α}. If M is an object of Mod_Z^{α} such that $\mathrm{Map}_Z(L_Z, M) \simeq 0$, then $\mathrm{Map}_Z(Q_Z, M) \simeq 0$ so that M is supported on W. Thus,

$$0 \simeq \mathrm{Map}_Z(P_Z, M) \simeq \mathrm{Map}_W(P_W, M) \simeq \mathrm{Map}_Y(P_Y, M).$$

But, P_Y is a global generator of Mod_Y^{α}, and $\mathrm{Mod}_W^{\alpha} \to \mathrm{Mod}_Y^{\alpha}$ is fully faithful. Thus, $M \simeq 0$. □

6.3 The étale local-global principle

In this section, we adapt an idea of Lurie to show that for R–linear ∞–categories, the property of having a compact generator is local for the étale topology. The context of this section is slightly different from that of the rest of Section 6: we do not require that our R–linear categories to satisfy étale hyperdescent. As every R–linear ∞–category satisfies étale descent by [43, Theorem 5.4] (not étale hyperdescent), this is a natural hypothesis to drop when considering étale covers. So, instead of studying morphisms $X \to \mathbf{Pr}$, we instead fix an R–linear category $\mathscr{C}$. If S is a commutative R–algebra, we write $\mathrm{Mod}_S(\mathscr{C})$ for the ∞–category of S–modules in $\mathscr{C}$. In particular, $\mathrm{Mod}_R(\mathscr{C}) \simeq \mathscr{C}$, and more generally $\mathrm{Mod}_S(\mathscr{C}) \simeq \mathscr{C} \otimes_R S$. For a general étale sheaf X, we define

$$\mathrm{Mod}_X(\mathscr{C}) = \lim_{\mathrm{Spec}\, S \to X} \mathrm{Mod}_S(\mathscr{C}).$$

If $\mathscr{C}$ is a linear category with étale hyperdescent arising from a map $\alpha\colon \mathrm{Spec}\, R \to \mathbf{Pr}$, then these definitions agree with our definitions of Mod_X^{α} above.

Lemma 6.12 *Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} \colon G$ be a pair of adjoint functors between stable presentable ∞–categories such that the right adjoint G is conservative and preserves filtered colimits. If P is a compact generator of $\mathscr{C}$, then $F(P)$ is a compact generator of $\mathscr{D}$.*

Proof Since G preserves filtered colimits, F preserves compact objects, so that $F(P)$ is compact. Suppose that M is an object of $\mathscr{D}$ such that $\mathrm{Map}_{\mathscr{D}}(F(P), M) \simeq 0$. Then, $\mathrm{Map}_{\mathscr{C}}(P, G(M)) \simeq 0$. Since P is a compact generator of $\mathscr{C}$, this implies that $G(M) \simeq 0$. The conservativity of G implies that $M \simeq 0$, so that $F(P)$ is a compact generator of $\mathscr{D}$. □

Following Lurie, we let $\mathrm{Test}_{\pi_0 R}$ be the category of (nonderived) $\pi_0 R$–schemes X which admit a quasicompact open immersion $X \to \mathrm{Spec}\, \pi_0 S$, where $\pi_0 S$ is an étale

$\pi_0 R$–algebra. There is a Grothendieck topology on $\mathrm{Test}_{\pi_0 R}$ that extends the Nisnevich topology [43, Proposition 2.7]. Lurie proves [43, Theorem 2.9] a version of the theorem of Morel and Voevodsky which says that for a presheaf F on $\mathrm{Test}_{\pi_0 R}$, being a Nisnevich sheaf is equivalent to satisfying affine Nisnevich excision. Recall that F satisfies affine Nisnevich excision if $F(\varnothing)$ is contractible and for all affine morphisms $X' \to X$ and quasicompact open subschemes $U \subseteq X$ such that $X' - U' \to X - U$ is an isomorphism, where $U' = X' \times_X U$, the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F(X') \\ \big\downarrow & & \big\downarrow \\ F(U) & \longrightarrow & F(U') \end{array}$$

is a pullback square of spaces.

Let $\mathrm{CAlg}_R^{\text{ét}}$ denote the ∞–category of étale R–algebras. There is a fully faithful embedding $\mathrm{CAlg}_R^{\text{ét}} \to \mathrm{N}(\mathrm{Test}_{\pi_0 R}^{\mathrm{op}})$ given by sending S to $\operatorname{Spec} \pi_0 S$. Given an R–linear category $\mathscr{C}$, we extend the construction that sends an étale R–algebra S to $\mathrm{Mod}_S(\mathscr{C})$ to $\mathrm{Test}_{\pi_0 R}$ by right Kan extension. In other words, if X is an object of $\mathrm{Test}_{\pi_0 R}$,

$$\mathrm{Mod}_X(\mathscr{C}) = \lim_{\operatorname{Spec} \pi_0 S \to X} \mathrm{Mod}_S(\mathscr{C}),$$

where the limit runs over all étale R–algebras S and all maps $\operatorname{Spec} \pi_0 S \to X$.

If $j\colon U \subseteq X$ is a quasicompact open immersion in $\mathrm{Test}_{\pi_0 R}$ with complement Z, viewed as a $\pi_0 R$–scheme with its reduced scheme structure, then we let $\mathrm{Mod}_{X,Z}(\mathscr{C})$ be the full subcategory of $\mathrm{Mod}_X(\mathscr{C})$ consisting of those objects M such that $j^* M \simeq 0$ in $\mathrm{Mod}_U(\mathscr{C})$. Roughly speaking, these are the quasicoherent $\mathscr{O}_X$–modules in $\mathscr{C}$ with support contained in Z.

Lemma 6.13 *Let X be an object of $\mathrm{Test}_{\pi_0 R}$, and let $j\colon U \to X$ be a quasicompact open immersion with complement Z. If there exists a compact object Q in $\mathrm{Mod}_X(\mathscr{C})$ such that $\mathrm{Mod}_U(\mathscr{C})$ is generated by $j^* Q$ and if $\mathrm{Mod}_{X,Z}(\mathscr{C})$ has a compact generator P, then $i_! P \oplus Q$ is a compact generator of $\mathrm{Mod}_X(\mathscr{C})$, where $i_!$ is the inclusion functor from $\mathrm{Mod}_{X,Z}(\mathscr{C})$ into $\mathrm{Mod}_X(\mathscr{C})$.*

Proof Since Q is compact by hypothesis, to show $i_! P \oplus Q$ is compact, we must show that $i_! P$ is compact. In fact, we show that $i_!$ preserves compact objects. To see this, consider the right adjoint $i^!$ of $i_!$, which is defined as the fiber of the natural unit natural transformation

$$i^! \to \mathrm{id}_{\mathrm{Mod}_X(\mathscr{C})} \to j_* j^*,$$

where j_* is the right adjoint of j^*. The functor j^*, being a left adjoint, preserves small colimits. By [43, Proposition 5.15], the functor j_* preserves small colimits as well (this is where the quasicompact hypothesis is used). Since $i^!$ is defined via a finite limit diagram, it follows that $i^!$ preserves filtered colimits, and hence that $i_!$ preserves compact objects. Hence, $i_! P \oplus Q$ is a compact object of $\mathrm{Mod}_X(\mathscr{C})$. Suppose now that M is an object of $\mathrm{Mod}_X(\mathscr{C})$ such that

$$\mathrm{Map}_X(i_! P \oplus Q, M) \simeq 0.$$

Then, $\mathrm{Map}_Z(P, i^! M) \simeq \mathrm{Map}_X(i_! P, M) \simeq 0$. Since P is a compact generator of $\mathrm{Mod}_{X,Z}(\mathscr{C})$, it follows that $i^! M \simeq 0$. Hence the unit map $M \to j_* j^* M$ is an equivalence. At the same time,

$$\mathrm{Map}_U(j^* Q, j^* M) \simeq \mathrm{Map}_X(Q, j_* j^* M) \simeq 0.$$

Thus, $j^* M \simeq 0$, since $j^* Q$ is a compact generator of $\mathrm{Mod}_U(\mathscr{C})$. Since $M \simeq j_* j^* M$, we see that $M \simeq 0$. Therefore, $i_! P \oplus Q$ is indeed a compact generator of $\mathrm{Mod}_X(\mathscr{C})$. □

Lemma 6.14 *Let $\mathscr{C}$ be an R–linear category. Let $f\colon X' \to X$ be a morphism in $\mathrm{Test}_{\pi_0 R}$ where X' is affine. Suppose that $\mathrm{Mod}_X(\mathscr{C})$ is compactly generated, and suppose that there exists a quasicompact open subset $U \subseteq X$ with complement Z such that $f|_{Z'}\colon Z' \to Z$ is an equivalence, where $Z' = Z \times_X X'$, and such that $\mathrm{Mod}_{X'}(\mathscr{C})$ and $\mathrm{Mod}_U(\mathscr{C})$ possess compact generators P and Q. Then, $\mathrm{Mod}_X(\mathscr{C})$ has a compact generator.*

Proof We verify the conditions of Lemma 6.13. Because $\mathrm{Mod}(\mathscr{C})$ is a Nisnevich sheaf, there is a cartesian square of R–linear ∞–categories

$$\begin{tikzcd} \mathrm{Mod}_X(\mathscr{C}) \arrow[r] \arrow[d] & \mathrm{Mod}_{X'}(\mathscr{C}) \arrow[d] \\ \mathrm{Mod}_U(\mathscr{C}) \arrow[r] & \mathrm{Mod}_{U'}(\mathscr{C}). \end{tikzcd}$$

Taking the fibers of the vertical maps induces an equivalence $\mathrm{Mod}_{X,Z}(\mathscr{C}) \simeq \mathrm{Mod}_{X',Z'}(\mathscr{C})$. By Proposition 6.9, the fact that $\mathrm{Mod}_{X'}(\mathscr{C})$ has a compact generator implies that $\mathrm{Mod}_{X',Z'}(\mathscr{C})$ has a compact generator, and hence $\mathrm{Mod}_{X,Z}(\mathscr{C})$ has a compact generator. To finish the proof, we show that $\mathrm{Mod}_U(\mathscr{C})$ has a compact generator which is the restriction of a compact object over X. But, by Proposition 6.10, $Q \oplus \Sigma Q$ is the restriction of a compact object of X. It clearly generates $\mathrm{Mod}_U(\mathscr{C})$. □

Let $\mathscr{C}$ be an R–linear ∞–category, and let $\chi_{\mathscr{C}}$ be the presheaf on $\mathrm{CAlg}_R^{\text{ét}}$ defined by

$$\chi_{\mathscr{C}}(S) = \begin{cases} * & \text{if } \mathrm{Mod}_S(\mathscr{C}) \text{ has a compact generator,} \\ \varnothing & \text{otherwise.} \end{cases}$$

The presheaf $\chi_{\mathcal{C}}$ extends to a presheaf $\chi'_{\mathcal{C}}$ on $\mathrm{Test}_{\pi_0 R}$ by right Kan extension. By definition, if X is an object of $\mathrm{Test}_{\pi_0 R}$, then $\chi'_{\mathcal{C}}(X)$ is contractible if and only if $\chi_{\mathcal{C}}(S)$ is nonempty for all R–algebras S and all $\mathrm{Spec}\,\pi_0 S \to X$.

Lemma 6.15 *Suppose that $\mathcal{C}$ is an R–linear ∞–category and that $R \to S$ is a finite faithfully flat cover. Then,* $\mathrm{Mod}_S(\mathcal{C})$ *has a compact generator if and only if* $\mathrm{Mod}_R(\mathcal{C})$ *does.*

Proof If $\mathrm{Mod}_R(\mathcal{C})$ has a compact generator, then by Lemma 6.6 it is a perfect generator, so that $\mathrm{Mod}_S(\mathcal{C})$ has a compact generator. Suppose that $\mathrm{Mod}_S(\mathcal{C})$ has a compact generator P. The functor $\pi_*\colon \mathrm{Mod}_S \to \mathrm{Mod}_R$ has a right adjoint, which is given explicitly by $\pi^!(M) = \mathrm{Map}_R(S, M)$. Since S is a finite and flat R–module, it follows that $\pi^!$ preserves filtered colimits. Therefore, $\pi_*\colon \mathrm{Mod}_S(\mathcal{C}) \to \mathrm{Mod}_R(\mathcal{C})$ has a continuous right adjoint given by tensoring $\mathcal{C}$ with $\pi^!\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$. We abuse notation and write $\pi^!$ for this right adjoint as well. It follows immediately that π_* preserves compact objects so that $\pi_*(P)$ is a compact object of $\mathrm{Mod}_R(\mathcal{C})$. To show that $\pi_*(P)$ is a compact generator of $\mathrm{Mod}_R(\mathcal{C})$, suppose that $\mathrm{Map}_R(\pi_*(P), M) \simeq 0$. Using the adjunction, we get that $\mathrm{Map}_S(P, \pi^!(M)) \simeq 0$. Therefore, $\pi^!(M) \simeq 0$. In general, the functor π_* is conservative. But, $\pi_*\pi^!(M) \simeq S^\vee \otimes_R M$, so that $\pi_*\pi^!$ is conservative by the faithful flatness of S. Therefore, $\pi^!$ is conservative. Thus, $M \simeq 0$, so that $\pi_*(P)$ is a compact generator of $\mathrm{Mod}_R(\mathcal{C})$. □

Now, we come to the étale local-global principle. The idea of the proof is due to Lurie [43, Section 6].

Theorem 6.16 *If $\mathcal{C}$ is an R–linear ∞–category, then $\chi_{\mathcal{C}}$ is an étale sheaf.*

Proof By [43, Theorems 2.9, 3.7], it suffices to show that $\chi_{\mathcal{C}}$ satisfies finite étale descent, and that $\chi'_{\mathcal{C}}$ satisfies affine Nisnevich excision. Finite étale descent follows from Lemma 6.15. To show that $\chi'_{\mathcal{C}}$ satisfies affine Nisnevich excision, suppose that $f\colon X' \to X$ is an affine morphism in Test_R, that $U \subseteq X$ is a quasicompact open subset such that $X' - U' \simeq X - U$, where $U' = X' \times_X U$, and that $\chi'_{\mathcal{C}}(X')$ and $\chi'_{\mathcal{C}}(U)$ are nonempty. Note that by [43, Proposition 6.12 and Lemma 6.17] all of the stable presentable ∞–categories that appear in proof are compactly generated. This is important because we will use Lemma 6.14. To show that $\chi'_{\mathcal{C}}(X)$ is nonempty, let $\mathrm{Spec}\, S \to X$ be a point of X. Pull back the affine elementary Nisnevich square via this map, to obtain

$$\begin{tikzcd} X' \times_X U \times_X \operatorname{Spec} S \arrow[r] \arrow[d] & X' \times_X \operatorname{Spec} S \arrow[d] \\ U \times_X \operatorname{Spec} S \arrow[r] & \operatorname{Spec} S. \end{tikzcd}$$

By our hypotheses, $X' \times_X \operatorname{Spec} S$ is affine, so $\chi_{\mathscr{C}}(X' \times_X \operatorname{Spec} S) \simeq \chi'_{\mathscr{C}}(X' \times_X \operatorname{Spec} S)$ is contractible, as we see by using the map $X' \times_X \operatorname{Spec} S \to X'$. By Lemma 6.14, to complete the proof, it suffices to show that $\operatorname{Mod}_{U \times_X \operatorname{Spec} S}(\mathscr{C})$ has a compact generator. By hypothesis, we know that $\chi'_{\mathscr{C}}(U \times_X \operatorname{Spec} S)$ is nonempty. As U is quasicompact in X, we may write $U \times_X \operatorname{Spec} S$ as a union of Zariski open subschemes:

$$U \times_X \operatorname{Spec} S = \bigcup_{i=1}^{n} \operatorname{Spec} S_i$$

Since $\chi'_{\mathscr{C}}(U \times_X \operatorname{Spec} S)$ is nonempty, $\operatorname{Mod}_{\operatorname{Spec} S_i}(\mathscr{C})$ has a compact generator for all i. Write

$$V_k = \bigcup_{i=1}^{k} \operatorname{Spec} S_i,$$

and assume that $\operatorname{Mod}_{V_k}(\mathscr{C})$ has a compact generator for some k in $[1, n)$. Then, $\operatorname{Spec} S_{k+1} \to V_{k+1}$ and the open $V_k \subseteq V_{k+1}$ satisfy the hypotheses of Lemma 6.14 (take $X = V_{k+1}$, $X' = \operatorname{Spec} S_{k+1}$ and $U = V_k$). Therefore, $\operatorname{Mod}_{V_{k+1}}(\mathscr{C})$ has a compact generator. By induction, we see that $\operatorname{Mod}_{U \times_X \operatorname{Spec} S}(\mathscr{C})$ has a compact generator, as desired. □

6.4 Lifting theorems

Now we put together the local-global principles of the previous sections into one of the main theorems of the paper. In the case of schemes built from simplicial commutative rings, this was proved in [57, Theorem 4.7]. Our proof is rather different, as the étale local-global principle requires different methods for connective $\mathbb{E}_\infty$–rings.

Theorem 6.17 *Let X be a quasicompact, quasiseparated derived scheme. Then, every Morita class $\alpha\colon X \to \mathbf{Mr}$ on X lifts to an algebra $X \to \mathbf{Alg}$.*

Proof By definition of sheafification, the Morita class $\alpha\colon X \to \mathbf{Mr}$ lifts étale locally through $\mathbf{Alg} \to \mathbf{Pr}$. It follows that there is an étale cover $\pi\colon \coprod_i \operatorname{Spec} T_i \to X$ such that $\pi^*\alpha\colon \coprod_i \operatorname{Spec} T_i \to \mathbf{Mr}$ factors through $\mathbf{Alg} \to \mathbf{Pr}$; in other words, $\operatorname{Mod}_{T_i}^{\alpha}$ has a compact generator for all i. Since étale maps are open, we can assume, possibly by refining the cover, that the image of $\operatorname{Spec} T_i$ in X is an affine subscheme $\operatorname{Spec} S_i$. By Theorem 6.16, $\mathrm{Mod}_{S_i}^{\alpha}$ has a compact generator. By Theorem 6.11, it follows that Mod_X^{α} has a perfect generator. This completes the proof by Proposition 6.8. □

We now consider several applications, which show the power of this theorem in establishing the compact generation of various stable presentable ∞–categories. These are motivated by the results of [52] in the affine case.

Example 6.18 If X is a quasicompact and quasiseparated derived scheme, and if E is an R–module such that localization with respect to E is smashing, then $L_E\mathrm{Mod}_X$, the full subcategory of E–local objects in Mod_X, is compactly generated by a single compact object.

Example 6.19 If X is a quasicompact and quasiseparated derived scheme over the p–local sphere, consider the localization $L_{K(n)}\mathrm{Mod}_X$, where $K(n)$ is the n^{th} Morava K–theory at the prime p. In this case, the $K(n)$–localization of $\mathcal{O}_X$ need not be compact in $L_{K(n)}\mathrm{Mod}_X$. However, if F is a finite type n complex, then over any affine $\operatorname{Spec} S \to X$, the $K(n)$–localization of $S \otimes F$ *is* a compact generator of $L_{K(n)}\mathrm{Mod}_S$. It follows from Theorem 6.17 that there is a compact generator of $L_{K(n)}\mathrm{Mod}_X$.

Our main application of the theorem is the following statement.

Corollary 6.20 *Let X be a quasicompact, quasiseparated derived scheme. Then, every Brauer class $\alpha\colon X \to \mathbf{Br}$ on X lifts to an Azumaya algebra $X \to \mathbf{Az}$.*

Note that this theorem is false in nonderived algebraic geometry. There is a nonseparated, but quasicompact and quasiseparated, surface X and a nonzero cohomological Brauer class $\alpha \in \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$ that is not represented by an ordinary Azumaya algebra; see Edidin, Hassett, Kresch and Vistoli [22, Corollary 3.11]. In this case, the Brauer class vanishes on a Zariski cover of X. However, there is no global α–twisted vector bundle, so there cannot be a nonderived Azumaya algebra. The corollary shows that, even in this case, there *is* a derived Azumaya algebra with class α.

7 Brauer groups

We prove our main theorems on the Brauer group, which will, in particular, allow us to show that the Brauer group of the sphere spectrum vanishes.

7.1 The Brauer space

Classically, there are two Brauer groups of a commutative ring or a scheme X. The first is the algebraic Brauer group, which is the group of Morita equivalence classes of Azumaya algebras over X. This notion goes back to Azumaya [6] for algebras free over commutative rings, Auslander and Goldman [5] for the general affine case, and Grothendieck [31] for schemes. The second is the cohomological Brauer group $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$ introduced by Grothendieck [31]. There is an inclusion from the algebraic Brauer group into the cohomological Brauer group (under the reasonable assumption that X has only finitely many connected components), but they are not always identical, as noted above. As a result of Corollary 6.20, the natural generalizations of these definitions to quasicompact, quasiseparated schemes *do* agree. Moreover, these generalizations yield not just groups but in fact grouplike $\mathbb{E}_\infty$–spaces; the Brauer groups are the groups of connected components of these spaces. We work again over some fixed connective $\mathbb{E}_\infty$–ring R.

Definition 7.1 Let X be an étale sheaf. Then, the Brauer space of X is $\mathbf{Br}(X)$, the space of maps from X to $\mathbf{Br}$ in $\mathrm{Shv}^{\text{ét}}_R$. The Brauer group of X is $\pi_0\mathbf{Br}(X)$.

When X is an arbitrary étale sheaf, we cannot say much about the algebraic nature of the classes in $\pi_0\mathbf{Br}(X)$. However, write $\mathbf{Br}_{\text{alg}}(X)$ for the full subspace of $\mathbf{Br}(X)$ of classes $\alpha\colon X \to \mathbf{Br}$ that factor through $\mathbf{Az} \to \mathbf{Br}$. In other words, $\pi_0\mathbf{Br}_{\text{alg}}(X)$ is the subgroup of the Brauer group consisting of those classes representable by an Azumaya algebra over X. When $X = \operatorname{Spec} S$, we will write $\mathbf{Br}(S)$ for $\mathbf{Br}(\operatorname{Spec} S)$.

We now can answer the analogue of the $\mathrm{Br} = \mathrm{Br}'$ question of Grothendieck.

Theorem 7.2 *For any quasicompact and quasiseparated derived scheme X, we have $\mathbf{Br}_{\text{alg}}(X) \simeq \mathbf{Br}(X)$.*

Proof This is the content of Corollary 6.20. □

An important fact about the Brauer space of a connective commutative ring spectrum is that it has a purely categorical formulation. Recall that $\mathrm{Cat}_{S,\omega}$ is the symmetric monoidal ∞–category of compactly generated S–linear categories together with colimit preserving functors that preserve compact objects. We saw in Theorem 3.15 that if A is an S–algebra, then Mod_A is invertible in $\mathrm{Cat}_{S,\omega}$ if and only if A is Azumaya. Write $\mathrm{Cat}^{\times}_{S,\omega}$ for the grouplike $\mathbb{E}_\infty$–space of invertible objects in $\mathrm{Cat}_{S,\omega}$.

Proposition 7.3 *If S is a connective commutative R–algebra, then the natural morphism $\mathrm{Cat}^{\times}_{S,\omega} \to \mathbf{Br}(S)$ is an equivalence.*

Proof Consider the diagram

$$\mathrm{Cat}^{\times}_{S,\omega} \xrightarrow{i} \mathbf{Br}(S) \xrightarrow{j} \mathbf{Pr}(S).$$

The composition $j \circ i$ is fully faithful, by definition. The map j is fully faithful by construction of $\mathbf{Br}$. Thus, i is fully faithful. On the other hand, by Corollary 6.20, the map i is essentially surjective. Thus, i is an equivalence. □

This proposition has the following two interesting corollaries, which will not be used in the sequel.

Corollary 7.4 *The presheaf of spaces which sends a connective commutative R–algebra S to $\mathrm{Cat}^{\times}_{S,\omega}$ is an étale sheaf.*

Corollary 7.5 *The space $\mathbf{Br}(X)$ is a grouplike $\mathbb{E}_\infty$–space.*

Proof The space $\mathbf{Br}(S)$ is a grouplike $\mathbb{E}_\infty$–space for every connective commutative R–algebra S, and the grouplike $\mathbb{E}_\infty$–structure is natural in S. Thus, $\mathbf{Br}$ is a grouplike $\mathbb{E}_\infty$–object in $\mathrm{Shv}^{\text{ét}}_R$. The mapping space

$$\mathbf{Br}(X) = \mathrm{Map}_{\mathrm{Shv}^{\text{ét}}_R}(X, \mathbf{Br})$$

thus inherits a grouplike $\mathbb{E}_\infty$–structure from that on $\mathbf{Br}$. □

As a result of the corollary, when X is an étale sheaf, we may construct via delooping a spectrum $\mathbf{br}(X)$, with $\Omega^\infty \mathbf{br}(X) \simeq \mathbf{Br}(X)$. A similar idea has been pursued recently by Szymik [54], but with rather different methods.

We will need the following proposition, as well as the computations in the following section, to tell us the homotopy sheaves of $\mathbf{Br}$. This will be used to give a complete computation of $\mathbf{Br}(X)$ using a descent spectral sequence when X is affine.

Proposition 7.6 *There is a natural equivalence of étale sheaves $\Omega\mathbf{Br} \simeq \mathbf{Pic}$, where $\mathbf{Pic}$ is the sheaf of line bundles.*

Proof By the étale local triviality of Azumaya algebras proven in Theorem 5.11, it follows that $\mathbf{Br}$ is a connected sheaf and that it is equivalent to the classifying space of the trivial Brauer class. But, the sheaf of auto-equivalences of $\mathcal{M}\mathrm{od}$ is precisely the sheaf of line bundles in $\mathcal{M}\mathrm{od}$. □

7.2 Picard groups of connective ring spectra

In the previous section, we showed that $\Omega\mathbf{Br} \simeq \mathbf{Pic}$, and by the étale local triviality of Azumaya algebras, we know that the sheaf $\pi_0\mathbf{Br}$ vanishes. Thus, to compute the homotopy sheaves of $\mathbf{Br}$, it is enough to compute them for $\mathbf{Pic}$, which is what we now do.

If R is a discrete commutative ring, let $\mathrm{Pic}(R)$ be the Picard group of invertible R–modules. This should be distinguished from $\mathbf{Pic}(\mathrm{H}R)$, the grouplike $\mathbb{E}_\infty$–space of invertible $\mathrm{H}R$–modules, and from $\mathrm{Pic}(\mathrm{H}R)$.

Proposition 7.7 (Fausk [24]) *Let R be a discrete commutative ring. Then, there is an exact sequence*

$$0 \to \mathrm{Pic}(R) \to \pi_0\mathbf{Pic}(\mathrm{H}R) \xrightarrow{c} \mathrm{H}^0(\operatorname{Spec} R, \mathbb{Z}) \to 0,$$

where the inclusion comes from the monoidal functor $\mathrm{Mod}_R \to \mathrm{Mod}_{HR}$, and the map c sends an invertible element L to its degree of connectivity on each connected component of $\operatorname{Spec} R$*. Thus, $c(L) = n$ if and only if $\pi_m(L) = 0$ for $m < n$, and $\pi_n(L) \neq 0$.*

The purpose of this section is to extend Proposition 7.7 to all connective commutative rings. The following lemma is essentially found in Hopkins, Mahowald and Sadofsky [34, page 90]. We remark that if L is an invertible R–module, then L is perfect and L^{-1} is the dual of L, $\mathrm{Map}_R(L, R)$. It follows that there is a canonical evaluation map $\mathrm{ev}\colon L \otimes_R L^{-1} \to R$, which is an equivalence.

Lemma 7.8 *Let R be an $\mathbb{E}_\infty$–ring spectrum, and let L be an invertible R–module. Suppose that there are R–module maps $\phi\colon \Sigma^n R \to L$ and $\omega\colon \Sigma^{-n} R \to L^{-1}$ such that*

$$\mathrm{ev} \circ \phi \otimes_R \omega\colon R \simeq \Sigma^n R \otimes_R \Sigma^{-n} R \to L \otimes_R L^{-1} \to R$$

is homotopic to the identity. Then, ϕ and ω are weak equivalences.

Proof The n^{th} suspension of $\mathrm{ev} \circ \phi \otimes_R \omega$ is homotopic to the composition

$$\Sigma^n R \xrightarrow{\phi} L \simeq L \otimes_R R \xrightarrow{1 \otimes \Sigma^n \omega} L \otimes_R \Sigma^n L^{-1} \to \Sigma^n R.$$

Therefore, $\Sigma^n R$ is a retract of L; specifically, there exists a perfect R–module M and an equivalence $L \simeq \Sigma^n R \oplus M$. Similarly, $L^{-1} \simeq \Sigma^{-n} \oplus N$ for some perfect R–module N. But,

$$R \simeq L \otimes_R L^{-1} \simeq (\Sigma^n R \oplus M) \otimes_R (\Sigma^{-n} R \oplus N) \simeq R \oplus \Sigma^{-n} M \oplus \Sigma^n N \oplus (M \otimes_R N),$$

which shows that M and N are zero, and hence that ϕ and ω are equivalences. □

Theorem 7.9 *Let R be a connective local $\mathbb{E}_\infty$–ring spectrum (that is, $\pi_0 R$ is a local ring). Then, $R \to \tau_{\leq 0} R \simeq \mathrm{H}\pi_0 R$ induces an isomorphism $\pi_0 \mathbf{Pic}(R) \to \pi_0 \mathbf{Pic}(\tau_{\leq 0} R) \cong \mathbb{Z}$.*

Proof Since $\pi_0 R$ is local, $\pi_0 \mathbf{Pic}(\tau_{\leq 0} R) = \mathbb{Z}$ by Proposition 7.7. Thus, it suffices to show that if L is an invertible R–module, then $L \simeq \Sigma^n R$ for some n. Fixing L, we first identify the appropriate integer n.

The invertibility of L implies that L is a perfect R–module. By Proposition 2.6, it follows that L has a bottom homotopy group, say $\pi_n L$. This means that for $m < n$, $\pi_m L = 0$, while $\pi_n L \neq 0$. Similarly, let $\pi_m L^{-1}$ be the bottom homotopy group of L^{-1}. We will show that $n = -m$, and that $L \simeq \Sigma^n R$. Consider the Tor spectral sequence for $L \otimes_R L^{-1}$,

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_* L, \pi_* L^{-1})_q \Rightarrow \pi_{p+q} R.$$

The differential d^r is of degree $(-r, r-1)$. Thus, for degree reasons, $\mathrm{E}^2_{0,n+m} = \mathrm{E}^\infty_{0,n+m}$. In this case, we have

$$(\pi_* L \otimes_{\pi_* R} \pi_* L^{-1})_{n+m} \cong \pi_n L \otimes_{\pi_0 R} \pi_m L^{-1}.$$

Since $\pi_n L$ and $\pi_m L^{-1}$ are nonzero and $\pi_0 R$ is local, the term $\mathrm{E}^2_{0,n+m}$ is nonzero. It is the term of smallest total degree that is nonzero. Thus, since it is permanent in the spectral sequence,

$$\pi_n L \otimes_{\pi_0 R} \pi_m L^{-1} \cong \pi_0 R,$$

and $n = -m$. Again, since $\pi_0 R$ is local, $\pi_n L$ and $\pi_m L^{-1}$ are both in fact isomorphic to $\pi_0 R$.

Choose $\phi \in \pi_n L$ and $\omega \in \pi_m L^{-1}$ so the isomorphism above gives $\phi \otimes_R \omega = 1_R \in \pi_0 R$. The homotopy classes ϕ and ω are represented by R–module maps

$$\phi\colon \Sigma^n R \to L,$$
$$\omega\colon \Sigma^m R \to L^{-1}.$$

Then,

$$R \xrightarrow{\phi \otimes_R \omega} L \otimes_R L^{-1} \to R$$

is homotopic to $\phi \otimes_R \sigma \simeq 1_R$. Thus, applying Lemma 7.8, the R–module maps ϕ and ω are in fact equivalences. This completes the proof. $\square$

Consider the étale sheaf $\mathbf{GL}_1$, which sends a connective commutative R–algebra S to the space of units in S. That is, $\mathbf{GL}_1(S)$ is defined as the pullback in the diagram of spaces

$$\begin{tikzcd} \mathbf{GL}_1(S) \arrow[r] \arrow[d] & \Omega^\infty S \arrow[d] \\ \pi_0 S^\times \arrow[r] & \pi_0 S. \end{tikzcd}$$

The classifying space $\mathrm{B}\mathbf{GL}_1(S)$ of this grouplike $\mathbb{E}_\infty$–space naturally includes as the identity component into $\mathbf{Pic}(S)$. Thus, there is a natural map $\mathbf{BGL}_1 \to \mathbf{Pic}$ from the classifying sheaf of $\mathbf{GL}_1$ into $\mathbf{Pic}$. When S is a local connective commutative R–algebra, then $\mathbf{Pic}(S)$ decomposes as the product $\mathbf{BGL}_1(S) \times \mathbb{Z}$, where the map

$$\mathbb{Z} \longrightarrow \mathbf{Pic}(S)$$

sends n to $\Sigma^n S$. Thus, we have the following corollary.

Corollary 7.10 *The sequence $\mathbf{BGL}_1 \to \mathbf{Pic} \to \mathbb{Z}$ is a split fiber sequence of hypersheaves.*

Proof Since $\mathbf{GL}_1$ is a hypersheaf, so is $\mathbf{BGL}_1$. We also know that $\mathbf{Pic}$ is a hypersheaf by Proposition 4.3. Finally, $\mathbb{Z}$ is by definition the hypersheaf associated to the constant presheaf with values $\mathbb{Z}$. Evidently, the sequence is in fact a sequence of sheaves of grouplike $\mathbb{E}_1$–spaces. Since $\mathbb{Z}$ is freely generated as a sheaf of grouplike $\mathbb{E}_1$–spaces by a single object, the splitting is obtained by taking the canonical basepoint of $\mathbf{Pic}$. □

With this corollary, we can give the computation of the homotopy sheaves of $\mathbf{Br}$, which we need in the next section in order to actually compute the Brauer groups of some connective $\mathbb{E}_\infty$–rings.

Corollary 7.11 *The homotopy sheaves of $\mathbf{Br}$ are*

$$\pi_i \mathbf{Br} \cong \begin{cases} 0 & \text{if } i = 0, \\ \mathbb{Z} & \text{if } i = 1, \\ \pi_0 \mathbb{O}^* & \text{if } i = 2, \\ \pi_{i-2}\mathbb{O} & \text{if } i \geq 3, \end{cases} \tag{12}$$

where $\mathbb{O}$ is the structure sheaf on $\mathrm{Shv}_R^{\text{ét}}$.

7.3 The exact sequence of Picard and Brauer groups

Suppose that $X = U \cup V$ is a derived scheme, written as the union of two Zariski open subschemes. Then, because $\mathbf{Br}$ is an étale sheaf, there is a fiber sequence of spaces

$$\mathbf{Br}(X) \to \mathbf{Br}(U) \times \mathbf{Br}(V) \to \mathbf{Br}(U \cap V).$$

Taking long exact sequences, we obtain the following exact sequence:

$$\begin{aligned}\pi_2\mathbf{Br}(U \cap V) \to \pi_1\mathbf{Br}(X) \to \pi_1\mathbf{Br}(U) \oplus \pi_1\mathbf{Br}(V) \to \pi_1\mathbf{Br}(U \cap V)\\ \to \pi_0\mathbf{Br}(X) \to \pi_0\mathbf{Br}(U) \oplus \pi_0\mathbf{Br}(V) \to \pi_0\mathbf{Br}(U \cap V)\end{aligned}$$

which generalizes the classical Picard–Brauer exact sequence

$$\mathrm{Pic}(X) \to \mathrm{Pic}(U) \oplus \mathrm{Pic}(V) \to \mathrm{Pic}(U \cap V) \to \mathrm{Br}(X) \to \mathrm{Br}(U) \oplus \mathrm{Br}(V) \to \mathrm{Br}(U \cap V),$$

when U, V and X are ordinary schemes. The computations in the next section can be used to show that the sequence is not, in general, exact on the right.

The important connecting morphism $\delta\colon \pi_1\mathbf{Br}(U \cap V) \to \pi_0\mathbf{Br}(X)$ can be described in the following Morita-theoretic way. The ∞–category Mod_X of quasicoherent sheaves on X can be glued from Mod_U and Mod_V by taking the natural equivalence $\mathrm{Mod}_U|_{U\cap V} \simeq \mathrm{Mod}_V|_{U\cap V}$. On the other hand, given a line bundle L over $U \cap V$, we can twist the gluing data by tensoring with L. The resulting category is $\mathrm{Mod}_X^{\delta(L)}$, the ∞–category of quasicoherent $\delta(L)$–twisted sheaves.

7.4 The Brauer space spectral sequence

In this section, we obtain a spectral sequence converging conditionally to the homotopy groups of $\mathbf{Br}(X)$. In most cases of interest, for instance when X is affine or has finite étale cohomological dimension, we show that the spectral sequence converges completely (see [16, Section IX.5]). In particular, the graded pieces of the filtration on the abutment of the spectral sequence are in fact computed by the spectral sequence. As an application, in the next section, we give various example computations of Brauer groups. For now, we fix a connective $\mathbb{E}_\infty$–ring spectrum R.

If A is a grouplike $\mathbb{E}_\infty$–object of $\mathrm{Shv}_R^{\text{ét}}$, and if X is any object of $\mathrm{Shv}_R^{\text{ét}}$, then for every $p \geq 0$, there is a cohomology group

$$\mathrm{H}^p_{\text{ét}}(X, A) = \pi_0\mathrm{Map}_{\mathrm{Shv}_R^{\text{ét}}}(X, \mathbf{B}^p A),$$

where $\mathbf{B}^p A$ denotes a p–fold delooping of A. In particular, if A is a sheaf of abelian groups in $\mathrm{Shv}_X^{\text{ét}}$, then we can view A canonically as a grouplike $\mathbb{E}_\infty$–space. An ∞–topos $\mathcal{X}$ has cohomological dimension $\leq n$ if $\mathrm{H}^m(\mathcal{X}, A) = 0$ for all abelian sheaves A in $\mathcal{X}$ and all $m > n$ [41, Definition 7.2.2.18].

Recall that by [45, Theorem 8.5.0.6], the small étale site on $\operatorname{Spec} S$ is equivalent to the nerve of the small étale site on $\operatorname{Spec} \pi_0 S$. Therefore, by [41, Remark 7.2.2.17], for any sheaf of abelian groups A over S, there is a natural isomorphism

$$\mathrm{H}^p_{\text{ét}}(\operatorname{Spec} S, A) \cong \mathrm{H}^p_{\text{ét}}(\operatorname{Spec} \pi_0 S, A),$$

where the right-hand side denotes the classical étale cohomology groups over $\operatorname{Spec} \pi_0 S$.

Theorem 7.12 *Let X be an object of $\mathrm{Shv}^{\text{ét}}_R$. Then, there is a conditionally convergent spectral sequence*

$$\mathrm{E}_2^{p,q} = \begin{cases} \mathrm{H}^p_{\text{ét}}(X, \pi_q \mathbf{Br}) & \text{if } p \le q, \\ 0 & \text{if } p > q, \end{cases} \quad \Rightarrow \quad \pi_{q-p}\mathbf{Br}(X), \tag{13}$$

with differentials d_r of degree $(r, r-1)$. If X is affine, discrete, or if $(\mathrm{Shv}^{\text{ét}}_R)_{/X}$ has finite cohomological dimension, then the spectral sequence converges completely.

Proof Because $\mathbf{Br}$ is hypercomplete, the map from $\mathbf{Br}$ to the limit of its Postnikov tower $\mathbf{Br} \to \lim_n \tau_{\le n}\mathbf{Br}$ is an equivalence; see [41, Section 6.5]. Taking sections preserves limits, so that

$$\mathbf{Br}(X) \to \lim_n ((\tau_{\le n}\mathbf{Br})(X))$$

is also an equivalence. Thus, $\mathbf{Br}(X)$ is the limit of a tower, and to any such tower there is an associated spectral sequence [16, Chapter IX] which converges conditionally to the homotopy groups of the limit. Using the methods of Brown and Gersten [17], one identifies the E_2–page as (13).

If X is affine, discrete, or if $(\mathrm{Shv}^{\text{ét}}_R)_{/X}$ has finite cohomological dimension, then the spectral sequence degenerates at some finite page. This is clear in the latter case, and if X is discrete the spectral sequence collapses entirely at the E_2–page. So, suppose that $X = \operatorname{Spec} S$. Then, $\mathbf{Br}(X)$ can be computed on the small étale site on $\operatorname{Spec} S$. But, as mentioned above, this site is the nerve of a discrete category, the small étale site on $\operatorname{Spec} \pi_0 S$. Therefore,

$$\mathrm{H}^p_{\text{ét}}(\operatorname{Spec} S, \pi_q\mathbf{Br}) \cong \mathrm{H}^p_{\text{ét}}(\operatorname{Spec} \pi_0 S, \pi_q\mathbf{Br}).$$

Since $\pi_q\mathbf{Br} \simeq \pi_{q-2}\mathcal{O}$ for $q \ge 3$, and since these are all quasicoherent $\pi_0\mathcal{O}$–modules, it follows that

$$\mathrm{H}^p_{\text{ét}}(\operatorname{Spec} S, \pi_q\mathbf{Br}) \cong \mathrm{H}^p_{\text{ét}}(\operatorname{Spec} \pi_0 S, \pi_{q-2}\mathcal{O}) = 0$$

for $q \geq 3$ and $p \geq 1$ by Grothendieck's vanishing theorem. Thus, the only possible differentials are

$$d_2\colon \mathrm{H}^p(\operatorname{Spec} S, \mathbb{Z}) \to \mathrm{H}^{p+2}(\operatorname{Spec} S, \pi_0\mathbb{O}^\times).$$

However, these differentials vanish because $\mathrm{B}\mathbb{Z}$ is in fact a split retract of $\mathbf{Br}$. Therefore, if X is affine, the spectral sequences degenerates at the E_2–page. It follows from the degeneration and the complete convergence lemma [16, IX.5.4] that the spectral sequence converges completely to $\pi_*\mathbf{Br}(X)$. This completes the proof. □

Using the theorem and the remarks preceding it, we deduce the following corollary, which completely computes the homotopy groups of the Brauer space of a connective commutative ring R. In particular, in the case of an Eilenberg–Mac Lane spectrum, the corollary determines the image of the map $\operatorname{Br}(R) \to \pi_0\mathbf{Br}(\mathrm{H}R)$ constructed in [8, Proposition 5.2].

Corollary 7.13 *If R is a connective $\mathbb{E}_\infty$–ring spectrum, then the homotopy groups of $\mathbf{Br}(R)$ are described by*

$$\pi_k\mathbf{Br}(R) \cong \begin{cases} \mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\pi_0 R, \mathbb{Z}) \times \mathrm{H}^2_{\text{ét}}(\operatorname{Spec}\pi_0 R, \mathbb{G}_m) & \text{if } k = 0, \\ \mathrm{H}^0_{\text{ét}}(\operatorname{Spec}\pi_0 R, \mathbb{Z}) \times \mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\pi_0 R, \mathbb{G}_m) & \text{if } k = 1, \\ \pi_0 R^\times & \text{if } k = 2, \\ \pi_{k-2} R & \text{if } k \geq 3. \end{cases}$$

Proof This follows immediately from the degeneration of the Brauer spectral sequence for $\operatorname{Spec} R$ together with the fact that $\mathbf{B}\mathbb{Z}$ splits off of $\mathbf{Br}$. □

Note that in the special case where R is a discrete commutative ring, Szymik obtained similar computations for the purely algebraic Brauer spectrum of $\mathrm{H}R$ defined in [54]. The computations also follow from the next corollary.

Corollary 7.14 *If X is a quasicompact and quasiseparated ordinary scheme, then*

$$\pi_k\mathbf{Br}(X) \cong \begin{cases} \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}) \times \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) & \text{if } k = 0, \\ \mathrm{H}^0_{\text{ét}}(X, \mathbb{Z}) \times \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) & \text{if } k = 1, \\ \mathrm{H}^0_{\text{ét}}(X, \mathbb{G}_m) & \text{if } k = 2, \\ 0 & \text{if } k \geq 3. \end{cases}$$

7.5 Computations of Brauer groups of ring spectra

In this section, we give several examples of Brauer groups of ring spectra and of derived schemes. Our convention throughout this section is to write $\mathrm{Br}(R)$ for the Brauer group of Azumaya algebras over a discrete commutative ring R. This injects but is not, in general, the same as $\pi_0\mathbf{Br}(\mathrm{H}R)$, as we will see below. Note that $\mathrm{Br}(R) \cong \mathrm{H}^2_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_m)_{\text{tors}}$, by Gabber [25]. If R is a regular domain, then by Grothendieck [32, Corollaire 1.8], we have $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_m)_{\text{tors}} = \mathrm{H}^2_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_m)$.

Lemma 7.15 *If X is a normal ordinary scheme, then $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}) = 0$.*

Proof Using the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, it is enough to show that $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}) = 0$. This is can be shown as in Deninger [20, 2.1]. □

However, the $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z})$ term does not always vanish, even when X is ordinary and affine, so there are some truly exotic elements in the derived Brauer group, even over discrete rings. Here is an example: let k be an algebraically closed field, and let $R = k[x, y]/(y^2 - x^3 + x^2)$. Then, $\operatorname{Spec} R$ is a nonnormal affine curve with singular point at $(0, 0)$. The normalization of $\operatorname{Spec} R$ is $\mathbb{A}^1_k$. It follows from DeMeyer [19, page 19] that $\mathrm{Br}(R) = 0$. It is also known that $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}) \cong \mathbb{Z}$. Therefore, we have computed that $\pi_0\mathbf{Br}(\mathrm{H}R) \cong \mathbb{Z}$.[1]

We can show that the Brauer group vanishes in many cases.

Theorem 7.16 *Let R be a connective commutative ring spectrum such that $\pi_0 R$ is either $\mathbb{Z}$ or the ring of Witt vectors $\mathbb{W}_q$ of $\mathbb{F}_q$. Then,*

$$\pi_0\mathbf{Br}(R) = 0.$$

Proof Both $\mathbb{Z}$ and $\mathbb{W}_p$ are normal, so that $\mathrm{H}^1_{\text{ét}}(\pi_0 R, \mathbb{Z}) = 0$. The ring of Witt vectors $\mathbb{W}_q$ is a Hensel local ring with residue field $\mathbb{F}_q$. Thus, by a theorem of Azumaya (see [31, Théorème 1]), there is an isomorphism $\mathrm{Br}(\mathbb{W}_q) \cong \mathrm{Br}(\mathbb{F}_q)$. But, $\mathrm{Br}(\mathbb{F}_q) = 0$ by a theorem of Wedderburn. The Albert–Brauer–Hasse–Noether Theorem from class field theory implies that $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} \mathbb{Z}, \mathbb{G}_m) = 0$ [33, Proposition 2.4]. Thus, in both cases, we have established the required vanishing. □

Corollary 7.17 *The Brauer group of the sphere spectrum is zero.*

[1] We thank Angelo Vistoli for pointing this out to us at `mathoverflow.net/questions/84414`.

Of course, it would be nice to have some more examples where the Brauer group does not vanish. We can give several. First, we recall some standard results, all of which can be found in [33, Section 2]. There is a residue isomorphism

$$h_p\colon \mathrm{Br}(\mathbb{Q}_p) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathbb{F}_p, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z},$$

and, for any open subscheme U of $\operatorname{Spec}\mathbb{Z}$, there is an exact sequence

$$0 \to \mathrm{Br}(U) \to \mathrm{Br}(\mathbb{Q}) \to \bigoplus_{p\in U} \mathrm{Br}(\mathbb{Q}_p),$$

where the sum is over all prime integers p in U. We may also identify $h_{\mathbb{R}}\colon \mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2 \subseteq \mathbb{Q}/\mathbb{Z}$; the unique nonzero class is represented by the real quaternions. Finally, there is an exact sequence

$$0 \to \mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(\mathbb{R}) \oplus \bigoplus_{p} \mathrm{Br}(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where the right-hand map is induced by mapping $\mathrm{Br}(\mathbb{R})$ or $\mathrm{Br}(\mathbb{Q}_p)$ to $\mathbb{Q}/\mathbb{Z}$ and summing. These two exact sequences are compatible in the obvious way.

If $\alpha \in \mathrm{Br}(\mathbb{Q})$ write α_p for the image of α in $\mathrm{Br}(\mathbb{Q}_p)$, and write $\alpha_{\mathbb{R}}$ for the image of α in $\mathrm{Br}(\mathbb{R})$. By examining the two exact sequences above, it follows that

$$\mathrm{Br}(\mathbb{Z}[\tfrac{1}{p}]) \cong \mathbb{Z}/2.$$

Indeed, if α is a class of $\mathrm{Br}(\mathbb{Q})$ that lifts to $\mathrm{Br}(\mathbb{Z}[1/p])$, then it follows that $h_q(\alpha_q) = 0$ for all primes $q \neq p$. Therefore, $h_p(\alpha_p) + h_{\mathbb{R}}(\alpha_{\mathbb{R}}) = 0$. Since there is a unique nonzero class in $\mathrm{Br}(\mathbb{R})$, the result follows.

Similarly, if $\alpha \in \mathrm{Br}(\mathbb{Q})$ lifts to $\mathrm{Br}(\mathbb{Z}_{(p)})$, then $h_p(\alpha_p) = 0$. Thus, there is an exact sequence

$$0 \to \mathrm{Br}(\mathbb{Z}_{(p)}) \to \mathbb{Z}/2 \oplus \bigoplus_{q\neq p} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

We have therefore proven the following corollary to Corollary 7.13.

Corollary 7.18 (1) *The Brauer group of the sphere with p inverted is given by $\pi_0\mathbf{Br}(\mathbb{S}[1/p]) \cong \mathbb{Z}/2$.*

(2) *The Brauer group of the p–local sphere fits into the exact sequence*

$$0 \to \pi_0\mathbf{Br}(\mathbb{S}_{(p)}) \to \mathbb{Z}/2 \oplus \bigoplus_{q\neq p} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

(3) *There is an isomorphism $\pi_0\mathbf{Br}(L_{\mathbb{Q}_p}\mathbb{S}) \cong \mathbb{Q}/\mathbb{Z}$, where $L_{\mathbb{Q}_p}\mathbb{S}$ is the rational p–adic sphere.*

Note the important fact that the first two cases in the corollary give examples of non-Eilenberg–Mac Lane commutative ring spectra with nonzero Brauer groups.

Finally, we mention two examples of ordinary schemes, where the derived Brauer group exhibits different behavior than the classical Brauer group. The first is the scheme X used in [22, Corollary 3.11], which is the gluing of two affine quadric cones along the nonsingular locus, viewed as a derived scheme over the complex numbers. This is a normal, quasicompact, nonseparated, quasiseparated scheme, so it satisfies the hypotheses of the theorems. One can check that $\pi_0\mathbf{Br}(X) = \mathbb{Z}/2$ by Corollary 7.14. This example was studied originally because the classical Brauer group of the scheme X viewed as an ordinary geometric object over $\mathbb{C}$ is $\mathrm{Br}(X) = 0$, while the cohomological Brauer group is $\mathrm{Br}'(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) = \mathbb{Z}/2$. In other words, the nonzero class $\alpha \in \mathrm{Br}'(X)$ is represented by an Azumaya algebra, but not by an ordinary Azumaya algebra (an algebra concentrated in degree 0).

The second example is the surface of Mumford [32, Remarques 1.11(b)]. He constructs a normal surface Y such that $\mathrm{H}^2_{\text{ét}}(Y, \mathbb{G}_m)$ has nontorsion elements. Of course, these can never be the classes of ordinary Azumaya algebras over Y. On the other hand, by Corollary 6.20, they are represented by (derived) Azumaya algebras over Y.

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