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Comparison of models for (∞, n)–categories, I

JULIA E BERGNER
CHARLES REZK

While many different models for $(\infty, 1)$–categories are currently being used, it is known that they are Quillen equivalent to one another. Several higher-order analogues of them are being developed as models for (∞, n)–categories. In this paper, we establish model structures for some naturally arising categories of objects which should be thought of as (∞, n)–categories. Furthermore, we establish Quillen equivalences between them.

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1 Introduction

There has been much recent interest in homotopical notions of higher categories. Given a positive integer n, an n–category has a notion of i–morphisms for all $1 \leq i \leq n$, and one can consider ∞–categories, in which there are i–morphisms for arbitrarily large i. When such higher categories are considered as having strict associativity and unit laws on compositions at all levels, then their definitions are straightforward. However, most examples of interest are better expressed as weak n–categories, where these laws are only required to hold up to isomorphism, and one needs to impose various coherence laws. While there have been many proposed models for weak n–categories (often extending to models for weak ∞–categories), the problem of comparing these models has thus far been intractable.

However, in the world of homotopy theory, models for so-called $(\infty, 1)$–categories, or ∞–categories with all i–morphisms invertible for $i > 1$, have been far more manageable. Several different approaches were taken, some originating from the idea of modeling homotopy theories, others with the intent of developing this kind of special case for higher category theory. While these are by no means the only ones, four models for $(\infty, 1)$–categories have been equipped with appropriate model structures: simplicial categories by the first author in [8], Segal categories by Hirschowitz and Simpson [18] and Pelissier [27], quasicategories by Joyal [21] and Lurie [24] and complete Segal spaces by the second author [30]. They have all been shown to be Quillen equivalent

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to one another in work of the first author [10], Dugger and Spivak [14], Joyal and Tierney [20; 22] and Lurie [24]; a survey by the first author can be found in [11].

Simplicial categories, or categories enriched over simplicial sets, are probably the easiest to understand as $(\infty, 1)$–categories, especially if we apply geometric realization and consider topological categories, or categories enriched over topological spaces. Given any objects x and y in a topological category $\mathcal{C}$, the points of the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ can be regarded as 1–morphisms. Paths between these points are 2–morphisms, but since paths can be reversed, these 2–morphisms are invertible up to homotopy. Homotopies between these paths are 3–morphisms, and we can continue to take homotopies between homotopies to see that we have n–morphisms for arbitrarily large n, all of which are invertible up to homotopy.

Segal categories and quasicategories are two different ways of thinking of weakened versions of simplicial categories, in which composition of mapping spaces is only defined up to homotopy. Segal categories are bisimplicial sets with discrete space at level zero which satisfy a Segal condition, guaranteeing an up-to-homotopy composition. Quasi-categories, on the other hand, are just simplicial sets, generally described in terms of a horn-filling condition which essentially gives the same kind of composition up to homotopy.

Like Segal categories, complete Segal spaces are bisimplicial sets satisfying the Segal condition, but instead of being discrete at level zero, they satisfy a "completeness" condition that makes up for it: essentially, the spaces at level zero are weakly equivalent to the subspace of "homotopy equivalences" sitting inside the space of morphisms. The Quillen equivalence between the model structure for Segal categories and the model structure for complete Segal spaces tells us that this completeness condition exactly compensates for the discreteness of the level zero space in a Segal category.

While $(\infty, 1)$–categories have been enormously useful in many ways, Lurie's recent proof of the cobordism hypothesis [25] has brought attention to the fact that they are not always good enough: for some purposes we need higher versions as well. Thus, we can consider more general (∞, n)–categories, or ∞–categories with i–morphisms invertible for $i > n$. A few models for such objects have been proposed, namely the Segal n–categories of Hirschowitz–Simpson and Pelissier [18; 27], the n–fold complete Segal spaces of Barwick and Lurie [25] and the Θ_n–spaces of the second author [31]. The latter model has the advantage that its model structure is cartesian closed.

One feature of all these models is that they are inductive in nature; beginning with a known way to think about $(\infty, 1)$–categories, some way of defining $(\infty, n+1)$–categories from (∞, n)–categories is given. Our purpose is to look at hybrid inductive definitions. More specifically, we begin with Θ_n–spaces as (∞, n)–categories, but rather than going on to Θ_{n+1}–spaces, instead take categories enriched in them as models for $(\infty, n+1)$–categories. Furthermore, we define a weakened version of these enriched categories, which can be regarded as an $(\infty, n+1)$–version of Segal categories (using a different approach than the Hirschowitz–Simpson model) and prove that the two are Quillen equivalent. In fact, we have two different model structures for these higher Segal categories.

The model we propose for a higher-dimensional analogue of Segal categories is described in terms of functors $\Delta^{op} \to \Theta_n Sp$, where $\Theta_n Sp$ denotes the model category for Θ_n–spaces, satisfying the Segal condition and a discreteness condition with respect to their being Δ^{op}–diagrams. We show that there exist two model structures, just as we have for ordinary Segal categories, which are Quillen equivalent to one another, and that they are in turn Quillen equivalent to the model category of categories enriched over $\Theta_n Sp$. This result generalizes the one establishing the Quillen equivalence between simplicial categories and Segal categories, ie, the case where $n = 1$ [10]. While only one of these model structures is necessary for this Quillen equivalence, the other one is the easier one to describe. Furthermore, we anticipate, as in the $(\infty, 1)$–case, that we will need the second one as we eventually seek to continue the zigzag to establish the equivalence with Θ_{n+1}–spaces. These Quillen equivalences will be the subject of another paper.

Just as in the $(\infty, 1)$–category case, there are a number of preliminary results that need to be established. We first show that we have appropriate model categories and Quillen equivalences when we restrict to Segal objects and the corresponding enriched categories which have a fixed set of objects. To do so, we need to show that rigidification results of Badzioch on algebras over algebraic theories [2] continue to hold when we take these algebras in categories other than that of simplicial sets. Many of the arguments used for the case of simplicial sets can still be used in the general setting. In cases where properties of simplicial sets are not used at all, we usually omit proofs; where there are more subtle changes to be noted, we have included a proof.

We also make use of our understanding of sets of generating cofibrations in a Reedy category, as well as the fact, established in a separate manuscript by the authors [13], that for the category of functors $\Theta_n^{op} \to \mathcal{SS}ets$, the Reedy and injective model structures coincide. By modifying these generating cofibrations appropriately, we are able to find a set of generating cofibrations for our more restrictive situation where the objects at level zero are discrete. From there, we can find the more general model structures and prove the Quillen equivalence with the enriched categories much as we proved it in the earlier case.

1.1 Work still to be done

So far we have not extended the chain of Quillen equivalences to $\Theta_{n+1}Sp$, which would be the end goal, but there are a couple of possible approaches to doing so. We expect to show that our model structure for Segal category objects is Quillen equivalent to the model category of complete Segal objects in $\Theta_n Sp$, which is in turn Quillen equivalent to $\Theta_{n+1}Sp$. This last step should use an inductive argument using the characterization of Θ_n as a wreath product of n copies of Δ, defined by Berger in [6], and be the first in a chain of Quillen equivalences between $\Theta_n Sp$ and the model structure for Barwick's n–fold complete Segal spaces. These results will be the subject of a future paper.

The results of this paper hold for more general cartesian presheaf categories other than $\Theta_n Sp$. However, the proofs require a good deal more subtlety, so these results will be given in a separate paper by the authors [12]. This problem has also been addressed by Simpson [32].

1.2 Related work

There are other models for (∞, n)–categories as well as comparisons being established. For example, Barwick has defined quasi-n–categories and compared them with Θ_n–spaces; this model is also cartesian closed and therefore lends itself to defining a model via enrichment over it [3]. In the case where $n = 2$, Lurie has a model using Verity's complicial sets [23; 34]. Generalizing a result of Toën [33], Barwick and Schommer-Pries have developed a set of axioms which any model for (∞, n)–categories must satisfy [4]. Ayala and Rozenblyum have also given a more geometric model for (∞, n)–categories and have shown that it is Quillen equivalent to $\Theta_n Sp$ [1].

1.3 Outline of the paper

In Section 2 we review some basic material on model categories and simplicial objects, and in Section 3 we establish a model structure for categories enriched in $\Theta_n Sp$. In Sections 4 and 5, we generalize comparisons between Segal categories and simplicial categories in the fixed object set case to more general Segal category objects and enriched categories in $\Theta_n Sp$. Section 6 is devoted to establishing model structures for Segal category objects and in Section 7 we prove that they are Quillen equivalent to the model category of enriched categories. In Section 8 we establish a technical result about fibrations in $\Theta_n Sp$.

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2 Background

Let Δ denote the simplicial indexing category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$. Recall that a *simplicial set* is a functor $\Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{ets}$, where $\mathcal{S}\mathrm{ets}$ denotes the category of sets. Denote by $\mathcal{SS}\mathrm{ets}$ the category of simplicial sets.

A *simplicial space* is a functor $\Delta^{\mathrm{op}} \to \mathcal{SS}\mathrm{ets}$. A simplicial set X can be regarded as a simplicial space in two ways. It can be considered a constant simplicial space with the simplicial set X at each level, and in this case we will also denote the constant simplicial set by X. Alternatively, we can take the simplicial space, which we denote X^t, for which $(X^t)_n$ is the discrete simplicial set X_n. The superscript t is meant to suggest that this simplicial space is the "transpose" of the constant simplicial space.

We recall some basics on model categories. A *model category* $\mathcal{M}$ is a category with three distinguished classes of morphisms: weak equivalences, fibrations and cofibrations, satisfying five axioms; see Dwyer and Spaliński [15, 3.3]. Given a model category structure, one can define the *homotopy category* $\mathrm{Ho}(\mathcal{M})$, which is a localization of $\mathcal{M}$ with respect to the class of weak equivalences; see Hovey [19, 1.2.1]. An object x in a model category $\mathcal{M}$ is *fibrant* if the unique map $x \to *$ to the terminal object is a fibration. Dually, an object x in $\mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to x$ from the initial object is a cofibration.

Recall that an *adjoint pair* of functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D} :G$ satisfies the property that, for any objects X of $\mathcal{C}$ and Y of $\mathcal{D}$, there is a natural isomorphism

$$\varphi\colon \mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, GY).$$

The functor F is called the *left adjoint* and the functor G the *right adjoint*; see Mac Lane [26, IV.1].

Definition 2.1 [19, 1.3.1] An adjoint pair of functors $F\colon \mathcal{M} \leftrightarrows \mathcal{N} :G$ between model categories is a *Quillen pair* if F preserves cofibrations and G preserves fibrations. The left adjoint F is called a *left Quillen functor*, and the right adjoint G is called the *right Quillen functor*.

Definition 2.2 [19, 1.3.12] A Quillen pair of model categories is a *Quillen equivalence* if for all cofibrant X in $\mathcal{M}$ and fibrant Y in $\mathcal{N}$, a map $f\colon FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if the map $\varphi f\colon X \to GY$ is a weak equivalence in $\mathcal{M}$.

We will also need the notion of a simplicial model category $\mathcal{M}$. For any objects X and Y in a simplicial category $\mathcal{M}$, the *function complex* is the simplicial set $\mathrm{Map}(X, Y)$.

A *simplicial model category* $\mathcal{M}$ is a model category $\mathcal{M}$ that is also a simplicial category such that two axioms hold; see Hirschhorn [17, 9.1.6].

Definition 2.3 [17, 17.3.1] A *homotopy function complex* $\mathrm{Map}^h(X,Y)$ in a simplicial model category $\mathcal{M}$ is the simplicial set $\mathrm{Map}(\widetilde{X},\widehat{Y})$ where $\widetilde{X}$ is a cofibrant replacement of X in $\mathcal{M}$ and $\widehat{Y}$ is a fibrant replacement for Y.

Several of the model category structures that we use are obtained by localizing a given model category structure with respect to a map or a set of maps. Suppose that $P=\{f\colon A\to B\}$ is a set of maps with respect to which we would like to localize a model category $\mathcal{M}$.

Definition 2.4 A *P–local* object W is a fibrant object of $\mathcal{M}$ such that for any $f\colon A\to B$ in P, the induced map on homotopy function complexes

$$f^*\colon \mathrm{Map}^h(B,W)\to \mathrm{Map}^h(A,W)$$

is a weak equivalence of simplicial sets. A map $g\colon X\to Y$ in $\mathcal{M}$ is a *P–local equivalence* if for every P–local object W, the induced map on homotopy function complexes

$$g^*\colon \mathrm{Map}^h(Y,W)\to \mathrm{Map}^h(X,W)$$

is a weak equivalence of simplicial sets.

If $\mathcal{M}$ is a sufficiently nice model category, then one can obtain a new model structure with the same underlying category as $\mathcal{M}$ but with weak equivalences the P–local equivalences and fibrant objects the P–local objects [17, 4.1.1].

Suppose that $\mathcal{D}$ is a small category and consider the category of functors $\mathcal{D}\to\mathcal{SS}\mathrm{ets}$, or $\mathcal{D}$–diagrams of spaces. We would like to consider model category structures on the category $\mathcal{SS}\mathrm{ets}^{\mathcal{D}}$ of such diagrams. A natural choice for the weak equivalences in $\mathcal{SS}\mathrm{ets}^{\mathcal{D}}$ is the class of levelwise weak equivalences of simplicial sets. Namely, given two $\mathcal{D}$–diagrams X and Y, we define a map $f\colon X\to Y$ to be a weak equivalence if and only if for each object d of $\mathcal{D}$, the map $X(d)\to Y(d)$ is a weak equivalence of simplicial sets.

There is a model category structure $\mathcal{SS}\mathrm{ets}^{\mathcal{D}}_f$ on the category of $\mathcal{D}$–diagrams with these weak equivalences and in which the fibrations are given by levelwise fibrations of simplicial sets. The cofibrations in $\mathcal{SS}\mathrm{ets}^{\mathcal{D}}_f$ are then the maps of $\mathcal{D}$–diagrams which have the left lifting property with respect to the maps which are levelwise acyclic fibrations. This model structure is often called the *projective* model category structure on $\mathcal{D}$–diagrams of spaces; see Goerss and Jardine [16, IX, 1.4]. Dually, there is a model category structure $\mathcal{SS}\mathrm{ets}_c^{\mathcal{D}}$ in which the cofibrations are given by levelwise cofibrations of simplicial sets, and this model structure is often called the *injective* model category structure [16, VIII, 2.4]. In particular, we obtain these model structures for $\mathcal{D} = \Delta^{\mathrm{op}}$, so that the category $\mathcal{SS}\mathrm{ets}^{\Delta^{\mathrm{op}}}$ is just the category of simplicial spaces.

However, Δ^{op} is a Reedy category [17, 15.1.2], and therefore we also have the Reedy model category structure on simplicial spaces [29]. In this structure, the weak equivalences are again the levelwise weak equivalences of simplicial sets. This model structure is cofibrantly generated, where the generating cofibrations are the maps

$$\partial\Delta[m] \times \Delta[n]^t \cup \Delta[m] \times \partial\Delta[n]^t \to \Delta[m] \times \Delta[n]^t$$

for all $n, m \geq 0$, an the generating acyclic cofibrations are the maps

$$V[m,k] \times \Delta[n]^t \cup \Delta[m] \times \partial\Delta[n]^t \to \Delta[m] \times \Delta[n]^t$$

for all $n \geq 0$, $m \geq 1$, and $0 \leq k \leq m$ [30, 2.4].

However, for simplicial spaces, the Reedy model structure coincides with the injective model structure, as follows.

Proposition 2.5 [17, 15.8.7, 15.8.8] *A map $f\colon X \to Y$ of simplicial spaces is a cofibration in the Reedy model category structure if and only if it is a monomorphism. In particular, every simplicial space is Reedy cofibrant.*

In light of this result, we denote the Reedy model structure on simplicial spaces by $\mathcal{SS}\mathrm{ets}_c^{\Delta^{\mathrm{op}}}$. Both $\mathcal{SS}\mathrm{ets}_c^{\Delta^{\mathrm{op}}}$ and $\mathcal{SS}\mathrm{ets}_f^{\Delta^{\mathrm{op}}}$ are simplicial model categories. In each case, given two simplicial spaces X and Y, we can define $\mathrm{Map}(X,Y)$ by

$$\mathrm{Map}(X,Y)_n = \mathrm{Hom}_{\mathcal{SS}\mathrm{ets}^{\Delta^{\mathrm{op}}}}(X \times \Delta[n], Y).$$

The projective model structure $\mathcal{SS}\mathrm{ets}_f^{\Delta^{\mathrm{op}}}$ is also cofibrantly generated, and a set of generating cofibrations consists of the maps

$$\partial\Delta[m] \times \Delta[n]^t \to \Delta[m] \times \Delta[n]^t$$

for all $m, n \geq 0$ [16, IV.3.1].

3 Categories enriched in Θ_n–spaces

In this section, we begin with a summary of basic definitions and results for Θ_n–spaces; a thorough treatment can be found in [30] for $n = 1$ and [31] for the general case. We then establish a model for $(\infty, n+1)$–categories given by categories enriched in

Θ_n–spaces. Since Θ_n–spaces model (∞, n)–categories, the model structure on these enriched categories is thus a higher-order version of the model structure on simplicial categories.

3.1 Complete Segal spaces

We begin with the essential definitions when $n = 1$, where a Θ_1–space is a complete Segal space.

Definition 3.2 [30, 4.1] A Reedy fibrant simplicial space W is a *Segal space* if for each $k \geq 2$ the Segal map

$$\varphi_k \colon W_k \to \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{k}$$

is a weak equivalence of simplicial sets.

Theorem 3.3 [30, 7.1] *There is a cartesian closed model structure $\mathcal{S}e\mathcal{S}p$ on the category of simplicial spaces in which the fibrant objects are precisely the Segal spaces.*

Because Segal spaces satisfy this Segal condition, we can regard them as being weakened versions of simplicial categories and apply appropriate terminology. The *objects* of a Segal space W are the elements of the set $W_{0,0}$. The *mapping space* $\mathrm{map}_W(x, y)$ is given by the fiber of the map

$$(d_1, d_0) \colon W_1 \to W_0 \times W_0$$

over (x, y). Since W is Reedy fibrant, the fiber is in fact a homotopy fiber and therefore the mapping space is homotopy invariant. Two maps $f, g \in \mathrm{map}_W(x, y)_0$ are *homotopic* if they lie in the same component of the mapping space $\mathrm{map}_W(x, y)$. The space of homotopy equivalences $W_{\mathrm{hoequiv}} \subseteq W_1$ is defined to be the union of all the components containing homotopy equivalences. There is a (nonunique) way to compose mapping spaces, as given explicitly by the second author in [30, Section 4].

The *homotopy category* of W, denoted $\mathrm{Ho}(W)$, has objects the elements of the set $W_{0,0}$ and

$$\mathrm{Hom}_{\mathrm{Ho}(W)}(x, y) = \pi_0 \mathrm{map}_W(x, y).$$

The image of a homotopy equivalence of W in $\mathrm{Ho}(W)$ is an isomorphism.

We can consider maps between Segal spaces that are similar in structure to Dwyer–Kan equivalences of simplicial categories; we even give them the same name.

Definition 3.4 [30] A map $f\colon W \to Z$ of Segal spaces is a *Dwyer–Kan equivalence* if

(1) for any objects x and y of W, the induced map $\mathrm{map}_W(x,y) \to \mathrm{map}_Z(fx, fy)$ is a weak equivalence of simplicial sets,

(2) the induced map $\mathrm{Ho}(W) \to \mathrm{Ho}(Z)$ is an equivalence of categories.

For a Segal space W, notice that the degeneracy map $s_0\colon W_0 \to W_1$ factors through the space of homotopy equivalences W_{hoequiv}, since the image of s_0 consists of "identity maps."

Definition 3.5 [30, Section 6] A Segal space W is a *complete Segal space* if the map $W_0 \to W_{\mathrm{hoequiv}}$ given above is a weak equivalence of simplicial sets.

Theorem 3.6 [30, 7.2] *There is a cartesian closed model structure $\mathcal{CSS}$ on the category of simplicial spaces in which the fibrant objects are precisely the complete Segal spaces.*

3.7 More general Θ_n–spaces

We now turn to Θ_n–spaces as higher-order complete Segal spaces. We begin by recalling the definition of the Θ–construction, as first described by Berger [6]. Let $\mathcal{C}$ be a small category, and define $\Theta\mathcal{C}$ to be the category with objects $[m](c_1, \dots, c_m)$ where $[m]$ is an object of Δ and each c_i is an object of $\mathcal{C}$. A morphism

$$[m](c_1, \dots, c_m) \to [q](d_1, \dots, d_q)$$

is given by $(\delta, \{f_{ij}\})$ where $\delta\colon [m] \to [q]$ in Δ and $f_{ij}\colon c_i \to d_j$ are morphisms in $\mathcal{C}$ indexed by $1 \le i \le m$ and $1 \le j \le q$ where $\delta(i-1) < j \le \delta(i)$ [31, 3.2].

Inductively, let Θ_0 be the terminal category with a single object and no nonidentity morphisms, and then define $\Theta_n = \Theta\Theta_n$. Note that $\Theta_1 = \Delta$. The categories Θ_n have also been studied in unpublished work of Joyal, using a more direct definition.

Looking at the case of Θ_2, we can think of objects as objects of Δ whose arrows are labeled by other objects of Δ, for example, $[4]([2],[3],[0],[1])$ can be depicted as

$$0 \xrightarrow{[2]} 1 \xrightarrow{[3]} 2 \xrightarrow{[0]} 3 \xrightarrow{[1]} 4$$

but since these labels can also be interpreted as strings of arrows, we get a diagram such as

$$\begin{tikzcd}
0 \arrow[r] \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & 1 \arrow[r, bend left=80] \arrow[r, bend left=25] \arrow[r, bend right=25] \arrow[r, bend right=80] \arrow[r, phantom, shift left=4ex, "\Downarrow"] \arrow[r, phantom, "\Downarrow"] \arrow[r, phantom, shift right=4ex, "\Downarrow"] & 2 \arrow[r] & 3 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 4.
\end{tikzcd}$$

The elements of this diagram can be regarded as generating a strict 2–category by composing 1–cells and 2–cells whenever possible. In other words, the objects of Θ_2 can be seen as encoding all possible finite compositions that can take place in a 2–category, much as the objects of Δ can be thought of as listing all the finite compositions that can occur in an ordinary category.

We can consider functors $\Theta_n^{op} \to \mathcal{S}ets$, and the most important example is the following. For any object $[m](c_1, \dots, c_m)$, let $\Theta[m](c_1, \dots, c_m)$ be the analogue of $\Delta[m]$ in $\mathcal{SS}ets$, ie, the representable object for maps into $[m](c_1, \dots, c_m)$.

Here, we consider functors $\Theta_n^{op} \to \mathcal{SS}ets$. Notice that any simplicial set can be regarded as a constant functor of this kind, and any functor $\Theta_n^{op} \to \mathcal{S}ets$, in particular the representable one given above, can be regarded as a levelwise discrete functor to $\mathcal{SS}ets$. Since, unlike in the case of simplicial spaces, the indexing diagrams in each direction are different, we can simply use the notation from the original category to denote such an object. Since Θ_n^{op} is a Reedy category [6], we have the Reedy model structure, as well as the projective and injective model structures, on the category $\mathcal{SS}ets^{\Theta_n^{op}}$. However, we prove in [13] that the injective and Reedy model structures agree here, just as in the case of simplicial spaces.

Given $m \geq 2$ and $c_1, \dots, c_m$ objects of Θ_n, define the object

$$G[m](c_1, \dots, c_m) = \operatorname{colim}(\Theta[1](c_1) \leftarrow \Theta[0] \rightarrow \cdots \leftarrow \Theta[0] \rightarrow \Theta[1](c_m)).$$

Referring to the example of the object

$$[4]([2], [3], [0], [1])$$

of Θ_2 above, we have the representable corresponding to it,

$$\Theta[4]([2], [3], [0], [1]).$$

The corresponding

$$G[4]([2], [3], [0], [1])$$

picks out the representables for each piece, $\Theta[1][2]$, $\Theta[1]([3])$, $\Theta[1]([0])$, and $\Theta[1]([1])$, glued together along the representables corresponding to the intersection points, given by $\Theta[0]$. If we localize with respect to the inclusion

$$G[4]([2],[3],[0],[1]) \to \Theta[4]([2],[3],[0],[1]),$$

then an object is local if having these vertical compositions guarantees the existence of all "horizontal" compositions of 1–cells and 2–cells.

Returning to the general case, there is an inclusion map

$$se^{(c_1,\dots,c_m)}\colon G[m](c_1,\dots,c_m) \to \Theta[m](c_1,\dots,c_m).$$

We define the set

$$Se_{\Theta_n} = \{se^{(c_1,\dots,c_m)} \mid m \geq 2, c_1,\dots,c_m \in \mathrm{ob}(\Theta_n)\}.$$

However, being local with respect to these maps is not sufficient for our purposes, as it only gives an up-to-homotopy composition at level n. (Returning the case of Θ_2, we have not guaranteed that vertical composites exist.) Encoding lower levels of composition is achieved inductively, using the Segal object model structure on the category of functors $\Theta_{n-1}^{\mathrm{op}} \to \mathcal{SS}\mathrm{ets}$. This procedure is rather technical, and full details can be found in [31, Section 8]. The main point is that, if the model structure on the category of functors $\Theta_{n-1}^{\mathrm{op}} \to \mathcal{SS}\mathrm{ets}$ is obtained by localizing with respect to a set $\mathcal{S}$ of maps, we can make use of an intertwining functor $V\colon \Theta(\mathcal{SS}\mathrm{ets}^{\Theta_{n-1}^{\mathrm{op}}}) \to \mathcal{SS}\mathrm{ets}^{\Theta_n^{\mathrm{op}}}$ to translate the set $\mathcal{S}$ into a set $V[1](\mathcal{S})$ of maps in $\mathcal{SS}\mathrm{ets}^{\Theta_n^{\mathrm{op}}}$. We need to localize with respect to this set, in addition to those imposing the Segal conditions for level n.

Let $\mathcal{S}_1 = Se_\Delta$, and for $n \geq 2$, inductively define $\mathcal{S}_n = Se_{\Theta_n} \cup V[1](\mathcal{S}_{n-1})$.

Theorem 3.8 [31, 8.5] *Localizing the model structure $\mathcal{SS}\mathrm{ets}_c^{\Theta_n^{\mathrm{op}}}$ with respect to $\mathcal{S}_n$ results in a cartesian model category whose fibrant objects are higher-order analogues of Segal spaces.*

However, we need to incorporate higher-order completeness conditions as well. Consider the functor $T\colon \Delta \to \Theta_n$ defined by

$$T[k]([m](c_1,\dots,c_m)) = \mathrm{Hom}_\Delta([m],[k])$$

which induces a Quillen pair

$$T_\#\colon \mathcal{SS}\mathrm{ets}_c^{\Delta^{\mathrm{op}}} \to \mathcal{SS}\mathrm{ets}_c^{\Theta_n^{\mathrm{op}}} \;:T^*$$

from which we can reduce to known results for simplicial spaces [31, 4.1]. In particular, define

$$Cpt_{\Delta} = \{E \to \Delta[0]\}$$

and, for $n \geq 2$,

$$Cpt_{\Theta_n} = \{T_{\#}E \to T_{\#}\Delta[0]\}.$$

Let $\mathcal{T}_1 = Se_{\Theta_1} \cup Cpt_{\Theta_1}$ and, for $n \geq 2$,

$$\mathcal{T}_n = Se_{\Theta_n} \cup Cpt_{\Theta_n} \cup V[1](\mathcal{T}_{n-1}).$$

Theorem 3.9 [31, 8.1] *Localizing $\mathcal{SS}\mathrm{ets}_c^{\Theta_n^{\mathrm{op}}}$ with respect to the set $\mathcal{T}_n$ gives a cartesian model category, denoted $\Theta_n Sp$.*

We refer to the fibrant objects of $\Theta_n Sp$ simply as *Θ_n–spaces*.

3.10 Categories enriched in Θ_n–spaces

As complete Segal spaces are known to be equivalent to simplicial categories, establishing them as models for $(\infty, 1)$–categories, $\Theta_{n+1}Sp$ should be Quillen equivalent to a model category whose objects are categories enriched in $\Theta_n Sp$, further strengthening the view that they are indeed models for $(\infty, n+1)$–categories.

The existence of the appropriate model structure for enriched categories can be regarded as a special case of a result of Lurie [24, A.3.2.4]; the $n = 0$ case was proved in [8].

Theorem 3.11 *There is a cofibrantly generated model structure on the category $\Theta_n Sp$–Cat of small categories enriched in $\Theta_n Sp$ in which the weak equivalences $f \colon \mathcal{C} \to \mathcal{D}$ are given by*

- (W1) $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(fx, fy)$ *is a weak equivalence in $\Theta_n Sp$ for any objects x, y,*
- (W2) $\pi_0\mathcal{C} \to \pi_0\mathcal{D}$ *is an equivalence of categories, where $\pi_0\mathcal{C}$ has the same objects as $\mathcal{C}$ and* $\mathrm{Hom}_{\pi_0\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathrm{Ho}(\Theta_n Sp)}(1, \mathcal{C}(x, y))$,

and the generating cofibrations are given by

- (I1) $\{UA \to UB\}$ *where $U \colon \Theta_n Sp \to \Theta_n Sp$–$Cat$ is the functor taking an object A of $\Theta_n Sp$ to the category with two objects x and y,* $\mathrm{Hom}_{UA}(x, y) = A$ *and no other nonidentity morphisms, and $A \to B$ is a generating cofibration of V,*
- (I2) $\varnothing \to \{x\}$, *where $\{x\}$ denotes the category with one object and only the identity morphism.*

Proof We need only verify the conditions of [24, A.3.2.4], of which the only non-straightforward one to check is that weak equivalences are stable under filtered colimits. Suppose I is a filtered category and $F, G\colon I \to \Theta_n Sp$ together with a map $F \to G$ such that for each object i of I, the map $F(i) \to G(i)$ is a weak equivalence in $\Theta_n Sp$. Since $\Theta_n Sp$ is a simplicial model category with all objects cofibrant, then it follows from [17, 18.5.1] that

$$\mathrm{hocolim}_I F \to \mathrm{hocolim}_I G$$

is a weak equivalence in $\Theta_n Sp$.

In fact, using the construction of homotopy colimits in a simplicial model category, we get the same result whether we compute homotopy colimits in $\mathcal{SS}\mathrm{ets}^{\Theta_n^{\mathrm{op}}}$ (the original Reedy model category with levelwise weak equivalences) or in $\Theta_n Sp$, which is a localization of it. Working levelwise in $\mathcal{SS}\mathrm{ets}^{\Theta_n^{\mathrm{op}}}$, we conclude $\mathrm{hocolim}_I F \simeq \mathrm{colim}_I F$, and similarly for G, since this fact is true for filtered diagrams of simplicial sets. Therefore, we have that $\mathrm{colim}_I F \to \mathrm{colim}_I G$ is a weak equivalence in $\Theta_n Sp$. □

Establishing that $\Theta_n Sp$–Cat is Quillen equivalent to $\Theta_{n+1} Sp$ should be achieved via a chain of Quillen equivalences, of which the ones shown in this paper are the beginning.

3.12 Segal spaces in $\Theta_n Sp$

We will have need of the following generalizations of the definitions of Segal spaces.

Definition 3.13 A Reedy fibrant functor $W\colon \Delta^{\mathrm{op}} \to \Theta_n Sp$ is a *$\Theta_n Sp$–Segal space* if the Segal maps

$$W_k \to \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{k}$$

are weak equivalences in $\Theta_n Sp$ for all $k \geq 2$.

Theorem 3.14 *There is a cartesian closed model structure $\mathcal{L}_S(\Theta_n Sp)^{\Delta^{\mathrm{op}}}$ on the category of functors $\Delta^{\mathrm{op}} \to \Theta_n Sp$ in which the fibrant objects are precisely the Segal space objects in $\Theta_n Sp$.*

Proof To obtain the model structure, one can localize the Reedy model structure with respect to the analogues of the maps used to obtain the Segal space model structure. To show that this model structure is cartesian, we follow the same line of argument as used by Rezk in [30, Section 10]. First, we establish that any function object W^X in $\Theta_n Sp^{\Delta^{\mathrm{op}}}$ is local, where W is local and W^X is defined by

$$(W^X)_{[q](c_1, \ldots, c_q), k} = \mathrm{Hom}(X \times \Theta[q](c_1, \ldots, c_q) \times \Delta[k], W).$$

Regarding $\Delta[1]$ as a levelwise discrete object of $\Theta_n Sp^{\Delta^{op}}$, consider the function object $W^{\Delta[1]}$ for any local object W. Proving that $W^{\Delta[1]}$ is again local can be proved just as in Rezk's paper, using the notion of covering. Then, for any $k \geq 2$, $W^{\Delta[k]}$ can be shown to be a retract of $W^{(\Delta[1])^k}$, establishing that $W^{\Delta[k]}$ is also local. If Y is any object of $\Theta_n Sp$, regarded as a constant diagram in $\Theta_n Sp^{\Delta^{op}}$, then $(W^{\Delta[k]})^Y = W^{\Delta[k]\times Y}$ is again local. Since any object X of $\Theta_n Sp^{\Delta^{op}}$ can be written as a homotopy colimit of objects of the form $\Delta[k]\times Y$, any object of the form W^X can be written as a homotopy limit of a objects of the form $W^{\Delta[k]\times Y}$, and therefore W^X is local.

To complete the proof that this cartesian structure is compatible with the model structure, we can use the same argument as Rezk, using properties of adjoints. □

Given a $\Theta_n Sp$–Segal space W, and elements $x, y \in W_{0,[0]}$, we can define mapping objects $\mathrm{map}_W(x, y)$ just as in the case of Segal spaces; they will be Θ_n–spaces. As we have seen above, we can obtain a simplicial space from a Θ_n–space via the functor T^*. Applying the diagonal functor to $T^*\mathrm{map}_W(x, y)$, we get mapping spaces for W. As a consequence, it is possible to define the *homotopy category* $\mathrm{Ho}(W)$ of a $\Theta_n Sp$–Segal space W, where the objects are the elements of $W_{0,[0]}$ and the morphisms are the components of the mapping spaces. We can then extend the definition of Dwyer–Kan equivalence to this setting, as follows.

Definition 3.15 A map $f\colon W \to Z$ of $\Theta_n Sp$–Segal spaces is a *Dwyer–Kan equivalence* if

- for any objects x and y of W, the induced map $\mathrm{map}_W(x, y) \to \mathrm{map}_Z(fx, fy)$ is a weak equivalence in $\Theta_n Sp$,
- the induced functor $\mathrm{Ho}(W) \to \mathrm{Ho}(Z)$ is an equivalence of categories.

4 Fixed-object $\Theta_n Sp$–Segal categories and their model structures

In this section, we first recall basic definitions of Segal categories and generalize them to those of $\Theta_n Sp$–Segal categories. We then go on to establish model structures in the restricted case where all $\Theta_n Sp$–Segal categories have the same set of objects which is preserved by all functions.

4.1 Segal categories

Definition 4.2 [18, Section 2] A *Segal precategory* is a simplicial space X such that the simplicial set X_0 in degree zero is discrete, ie a constant simplicial set.

Again, we can consider the Segal maps

$$\varphi_k \colon X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$$

for each $k \geq 2$. Since X_0 is discrete, the right hand side is actually a homotopy limit.

Definition 4.3 [18, Section 2] A *Segal category* X is a Segal precategory such that each Segal map φ_k is a weak equivalence of simplicial sets for $k \geq 2$.

There is a fibrant replacement functor L taking a Segal precategory X to a Segal category LX. We can think of this functor as a "localization," even though it is not actually obtained from localization of a different model structure [10, Section 5].

Weak equivalences in this setting, again called *Dwyer–Kan equivalences*, are the maps $f \colon X \to Y$ such that the induced map $\mathrm{map}_{LX}(x,y) \to \mathrm{map}_{LY}(fx, fy)$ is a weak equivalence of simplicial sets for any $x, y \in X_0$ and the map $\mathrm{Ho}(LX) \to \mathrm{Ho}(LY)$ is an equivalence of categories.

Theorem 4.4 [10, 5.1, 7.1] *There is a model structure $\mathcal{S}e\mathcal{C}at_c$ on the category of Segal precategories in which the fibrant objects are precisely the Reedy fibrant Segal categories. The weak equivalences are the Dwyer–Kan equivalences. There is also a model structure $\mathcal{S}e\mathcal{C}at_f$ with the same weak equivalences in which the fibrant objects are precisely the projective fibrant Segal categories.*

Theorem 4.5 [10, 7.5, 8.6] *There is a chain of Quillen equivalences*

$$\mathcal{SC} \leftrightarrows \mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{S}e\mathcal{C}at_c$$

where $\mathcal{SC}$ denotes the model structure on the category of simplicial categories.

We would like to generalize these definitions and their corresponding model structures to $\Theta_n Sp$–Segal categories; the goal of this paper is to prove the analogue of the previous theorem in this setting.

Definition 4.6 A $\Theta_n Sp$*–Segal precategory* is a functor $X\colon \Delta^{\mathrm{op}} \to \Theta_n Sp$ such that X_0 is a discrete object in $\Theta_n Sp$, ie, a constant Θ_n–diagram of sets. It is a $\Theta_n Sp$*–Segal category* if, additionally, the Segal maps

$$\varphi_k\colon X_k \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$$

are weak equivalences in $\Theta_n Sp$ for all $k \geq 2$.

We denote by $\Theta_n Sp^{\Delta^{\mathrm{op}}}_{\mathrm{disc}}$ the category of $\Theta_n Sp$–Segal precategories. Notice that if the Segal maps for X are isomorphisms in $\Theta_n Sp$, then X is just a $\Theta_n Sp$–category.

In the remainder of this section, we seek to define model structures on the category of functors $X\colon \Delta^{\mathrm{op}} \to \Theta_n Sp$ with the additional requirement that $X_0 = \mathcal{O}$, the discrete object of $\Theta_n Sp$ given by the a fixed set $\mathcal{O}$, and such that all maps between such functors are required to be the identity on this set. We denote this category $\Theta_n Sp^{\Delta^{\mathrm{op}}}_{\mathcal{O}}$.

4.7 The projective model structure on $\Theta_n Sp^{\Delta^{\mathrm{op}}}_{\mathcal{O}}$

Our first goal is to prove the following result.

Proposition 4.8 *There is a model structure on $\Theta_n Sp^{\Delta^{\mathrm{op}}}_{\mathcal{O}}$ with levelwise weak equivalences and fibrations in $\Theta_n Sp$, denoted by $\Theta_n Sp^{\Delta^{\mathrm{op}}}_{\mathcal{O},f}$.*

To prove this theorem, first notice that limits and colimits can be understood in this category just as they are as in the first author's work [9, 3.5,3.6], ie, they are taken in a category in which the object set $\mathcal{O}$ at level [0] is preserved. We then need sets of generating cofibrations and generating acyclic cofibrations for this proposed model structure. The constructions here are generalizations of those for ordinary Segal categories [9, Section 3].

Just as we did in the case for simplicial sets, we begin by finding suitable sets of generating cofibrations and generating acyclic cofibrations for the projective model structure on the category $\Theta_n Sp^{\Delta^{\mathrm{op}}}$ of all functors $X\colon \Delta^{\mathrm{op}} \to \Theta_n Sp$. By definition, a map $f\colon X \to Y$ in our proposed model structure is an acyclic fibration if and only if, for each $p \geq 0$, the map $f_p\colon X_p \to Y_p$ has the right lifting property with respect to every generating cofibration $A \to B$ in $\Theta_n Sp$. This condition is equivalent to the having a lift in the following diagram, for any $A \to B$ as above and $p \geq 0$:

$$\begin{tikzcd} A \times \Delta[p] \arrow[r] \arrow[d] & X \arrow[d, "\simeq"] \\ B \times \Delta[p] \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Thus, we can regard the set of such maps

$$A \times \Delta[p] \to B \times \Delta[p]$$

as a suitable set of generating cofibrations for $\Theta_n Sp$. Similarly, f is a fibration if and only if each f_p has the right lifting property with respect to every generating acyclic cofibration $C \to D$ in $\Theta_n Sp$. It follows by arguments like the ones given above that a set of generating cofibrations consists of the maps

$$C \times \Delta[p] \to D \times \Delta[p].$$

Because the (constant) Θ_n–space at level zero must be preserved, we need a distinct simplex of each dimension corresponding to each tuple of objects of $\mathcal{O}$. Thus, for any $\underline{x} = (x_0, \dots, x_p) \in \mathcal{O}^{p+1}$, we define $\Delta[p]_{\underline{x}}$ to be the p–simplex $\Delta[p]$, regarded as an object of $\Theta_n Sp^{\Delta^{op}}_{\text{disc}}$, with $(\Delta[p]_{\underline{x}})_0 = \mathcal{O}$; the vertices of the p–simplex are given by $\underline{x}$, and the other elements of $\mathcal{O}$ are added in as 0–simplices so that we have the requisite $\mathcal{O}$ in degree 0. Notice here that we assume that $\underline{x}$ is ordered by the usual ordering on iterated face maps. It remains to find an appropriate means of assuring that each object involved in our generating (acyclic) cofibrations is in fact discrete in degree zero.

For any object A in $\Theta_n Sp$, $p \geq 0$, and $\underline{x} \in \mathcal{O}^{p+1}$, define the object $A_{[p],\underline{x}}$ to be the pushout of the diagram

$$\begin{tikzcd}
A \times (\Delta[p]_{\underline{x}})_0 \arrow[r] \arrow[d] & A \times \Delta[p]_{\underline{x}} \arrow[d] \\
(\Delta[p]_{\underline{x}})_0 \arrow[r] & A_{[p],\underline{x}}.
\end{tikzcd}$$

The idea is that $A_{[p],\underline{x}}$ looks as much as possible like $A \times \Delta[p]_{\underline{x}}$, but has the discrete set $\mathcal{O}$ as the simplicial set at level 0, as required to be in the category in which we are working.

Thus, we define sets

$$I_{\mathcal{O},f} = \{A_{[p],\underline{x}} \to B_{[p],\underline{x}} \mid p \geq 0, A \to B \text{ a generating cofibration in } \Theta_n Sp\},$$
$$J_{\mathcal{O},f} = \{C_{[p],\underline{x}} \to D_{[p],\underline{x}} \mid p \geq 0, C \to D \text{ a generating acyclic cofibration in } \Theta_n Sp\}.$$

Given these generating sets, Proposition 4.8 can be proved just as in the simplicial case [9, 3.7].

4.9 The injective model structure on $\Theta_n Sp_{\mathcal{O}}^{\Delta^{op}}$

Now, we turn to the other model structure with levelwise weak equivalences, where we instead have levelwise cofibrations. A useful fact is the following.

Proposition 4.10 *The Reedy and injective model structures on $\Theta_n Sp^{\Delta^{op}}$ coincide.*

Proof The fact that Reedy cofibrations are levelwise cofibrations in $\Theta_n Sp$ follows from a general result about Reedy categories [17, 15.3.11]. Therefore, it remains to prove that if $f\colon X \to Y$ in $\Theta_n Sp^{\Delta^{op}}$ satisfies the condition that $f_n\colon X_n \to Y_n$ is a cofibration in $\Theta_n Sp$, then f is a Reedy cofibration.

We first need to understand what a "codegeneracy" is in Θ_n. For simplicity, we look at Θ_2. Given an object $[k](c_1, \dots, c_k)$ in Θ_2, there are two kinds of codegeneracies. The first is given by a codegeneracy of a c_i, regarding c_i as an object of Δ. Using a "pasting diagram" interpretation of Θ_2, such a codegeneracy amounts to collapsing one of the 2–cells at horizontal position i. Thus, when we take a simplicial presheaf on Θ_2, the corresponding degeneracy gives a degenerate 2–cell in a position specified by the degeneracy map of the c_i in Δ^{op}. We think of such degeneracies as "vertical" degeneracies.

There is also a kind of "horizontal" degeneracy, but we do not want to allow all such. Given an object $[k](c_1, \dots, c_k)$, a horizontal degeneracy would be given by a codegeneracy of $[k]$ in Δ. But, if we took the i^{th} codegeneracy of $[k]$, where $c_i > 0$, then we would, in effect, we collapsing multiple cells. Thus, we only want to consider such codegeneracies when $c_i = 0$, ie, the case where there are no 2–cells in position i.

In either case, however, a degeneracy is given by a degeneracy in Δ^{op}, and therefore our result about degeneracies in Δ^{op} continues to hold in Θ_2^{op}. This argument can be rephrased as an inductive one, so that it is in fact true for all Θ_n^{op}.

Now, we establish an analogue of [17, 15.8.6] in this situation, namely, that, for every $m \geq 0$, the latching object $L_m X$ is isomorphic to the subobject of X_m consisting of lower-order simplices, ie, objects of $\mathrm{Hom}(\Theta[k](c_1, \dots, c_k), X)$, which are in the image of a degeneracy operator. However, this fact follows from [17, 15.8.4] and the existence of a map from $(L_m X)_{[k](c_1,\dots,c_k)}$ to the degenerate elements of $X_{*,[k](c_1,\dots,c_k)}$.

Using this above description of codegeneracies in Θ_n, we have that for any object W of $\Theta_n Sp$, if $k \geq 0$, $\sigma \in W_{[k](c_1,\dots,c_k)}$ is nondegenerate if and only if no two degeneracies of σ are equal, the analogue of [17, 15.8.5]. Therefore, it follows that the intersection of X_m and $L_m Y$ in Y_m is precisely the object $L_m X$. Therefore, the latching map $X_m \amalg_{L_m X} L_m Y \to Y_m$ is an monomorphism in $\Theta_n Sp$, which is precisely the requirement for f to be a Reedy cofibration. □

Thus, we can use the Reedy structure to understand precise sets of generating cofibrations and generating acyclic cofibrations, but we also know that cofibrations are precisely the monomorphisms and in particular that all objects are cofibrant.

Proposition 4.11 *There is a model structure on $\Theta_n Sp_{\mathcal{O}}^{\Delta^{op}}$ with levelwise weak equivalences and cofibrations in $\Theta_n Sp$, denoted by $\Theta_n Sp_{\mathcal{O},c}^{\Delta^{op}}$.*

To define sets $I_{c,\mathcal{O}}$ and $J_{c,\mathcal{O}}$ which will be our candidates for generating cofibrations and generating acyclic cofibrations, respectively, we first recall the generating cofibrations and acyclic cofibrations in the Reedy model structure. The generating cofibrations are the maps

$$A \times \Delta[p] \cup B \times \partial\Delta[p] \to B \times \Delta[p]$$

for all $p \geq 0$ and $A \to B$ generating cofibrations in $\Theta_n Sp$, and similarly the generating acyclic cofibrations are the maps

$$C \times \Delta[p] \cup C \times \partial\Delta[p] \to D \times \Delta[p]$$

for all $p \geq 0$ and $C \to D$ generating acyclic cofibrations in $\Theta_n Sp$ [17, 15.3].

As in the case of the projective model structure, we need to modify these maps so that they are in the category where we have $\mathcal{O}$ in degree zero. However, the modification here is done slightly differently. As before, we begin by considering the category $\Theta_n Sp_{\mathrm{disc}}^{\Delta^{op}}$ of all Segal precategory objects in $\Theta_n Sp$ and the inclusion functor $\Theta_n Sp_{\mathrm{disc}}^{\Delta^{op}} \to \Theta_n Sp^{\Delta^{op}}$. To force the degree zero space to be exactly the set $\mathcal{O}$, we use the fact that this functor has a left adjoint, which we call the reduction functor. Given an object X of $\Theta_n Sp^{\Delta^{op}}$, we denote its reduction by $(X)_r$. Reducing X essentially amounts to collapsing the space X_0 to its set of components and making the appropriate changes to degeneracies in higher degrees. So, we start by reducing the objects defining the Reedy generating cofibrations and generating acyclic cofibrations to obtain maps of the form

$$(A \times \Delta[p] \cup B \times \partial\Delta[p])_r \to (B \times \Delta[p])_r,$$
$$(C \times \Delta[p] \cup D \times \partial\Delta[p])_r \to (D \times \Delta[p])_r.$$

Then, in order to have our maps fix the object set $\mathcal{O}$, we define a separate such map for each choice of vertices $\underline{x}$ in degree zero and adding in the remaining points of $\mathcal{O}$ if necessary. As above, we use $\Delta[p]_{\underline{x}}$ to denote the object $\Delta[p]$ with the $(p+1)$–tuple $\underline{x}$ of vertices. We then define sets

$$I_{\mathcal{O},c} = \{(A \times \Delta[p]_{\underline{x}} \cup B \times \partial\Delta[p]_{\underline{x}})_r \to (B \times \Delta[p]_{\underline{x}})_r\}$$

for all $p \geq 1$ and $A \to B$, and

$$J_{\mathcal{O},c} = \{(C \times \Delta[p]_{\underline{x}} \cup D \times \partial\Delta[p]_{\underline{x}})_r \to (D \times \Delta[p]_{\underline{x}})_r\}$$

for all $p \geq 1$ and $C \to D$, where the notation $(-)_{\underline{x}}$ indicates the specified vertices.

Then, the proof that we do in fact get a model structure can be proved just as in [9, 3.9].

4.12 Localization of these model categories

However, these two model structures are not enough. We need to localize them so that their fibrant objects are Segal category objects, following [30]. Fortunately, this process can be done just as in the $n = 1$ case. Define a map $\alpha^i\colon [1] \to [p]$ in Δ such that $0 \mapsto i$ and $1 \mapsto i+1$ for each $0 \leq i \leq p-1$. Then for each p defines the object

$$G(p) = \bigcup_{i=0}^{p-1} \alpha^i \Delta[1]$$

and the inclusion map $\varphi^p\colon G(p) \to \Delta[p]$. To obtain the Segal model structure from the Reedy model structure on the category of functors $\Delta^{\mathrm{op}} \to \Theta_n Sp$, the localization is with respect to the coproduct of inclusion maps

$$\varphi = \coprod_{p \geq 0} (G(p) \to \Delta[p]).$$

However, in our case, the objects $G(p)$ and $\Delta[p]$ do not preserve the object set. As before, we can replace $\Delta[p]$ with the objects $\Delta[p]_{\underline{x}}$, where $\underline{x} = (x_0, \ldots, x_p)$ and define

$$G(p)_{\underline{x}} = \bigcup_{i=0}^{p-1} \alpha^i \Delta[1]_{x_i, x_{i+1}}.$$

Now, we need to take coproducts not only over all values of p, but also over all p–tuples of vertices. Here, we can regard these objects as giving a diagram of constant Θ_n–spaces.

Thus, we localize with respect to the set of maps

$$\{G[p]_{\underline{x}} \to \Delta[p]_{\underline{x}} \mid p \geq 0, \underline{x} \in \mathcal{O}^{p+1}\}.$$

Applying this localization to the model structure $\Theta_n Sp^{\Delta^{\mathrm{op}}}_{\mathcal{O},f}$ gives a model structure denoted $\mathcal{L}(\Theta_n Sp)^{\Delta^{\mathrm{op}}}_{\mathcal{O},f}$, and similarly from the model structure $\Theta_n Sp^{\Delta^{\mathrm{op}}}_{\mathcal{O},c}$ we obtain the localized model structure $\mathcal{L}(\Theta_n Sp)^{\Delta^{\mathrm{op}}}_{\mathcal{O},c}$.

5 Rigidification of algebras over algebraic theories

In this section we generalize work of Badzioch [2] and the first author [7] concerning rigidification of simplicial algebras over algebraic theories. These results, which give us a convenient framework for understanding fixed-object simplicial categories, were used to establish the Quillen equivalence between the model structures for simplicial categories and Segal categories. More specifically, we described simplicial categories as product-preserving functors from an algebraic theory $\mathcal{T}_{\mathcal{O}\mathcal{C}\mathrm{at}}$ to $\mathcal{S}\mathcal{S}$ets and showed that Segal categories with a fixed object set were equivalent to such functors which preserve products only up to homotopy. Here, we want to consider categories enriched, not in $\mathcal{S}\mathcal{S}$ets, but in $\Theta_n Sp$; in the fixed-object case, we can describe these enriched categories as product-preserving functors $\mathcal{T}_{\mathcal{O}\mathcal{C}\mathrm{at}} \to \Theta_n Sp$. Therefore, we need to generalize results about such functors so that they continue to hold when we replace $\mathcal{S}\mathcal{S}$ets with $\Theta_n Sp$.

We begin with a review of algebraic theories.

Definition 5.1 [7] Given a set S, an S–*sorted algebraic theory* (or *multisorted theory*) $\mathcal{T}$ is a small category with objects $T_{\underline{\alpha}^n}$ where $\underline{\alpha}^n = \langle \alpha_1, \dots, \alpha_n \rangle$ for $\alpha_i \in S$ and $n \geq 0$ varying, and such that each $T_{\underline{\alpha}^n}$ is equipped with an isomorphism

$$T_{\underline{\alpha}^n} \cong \prod_{i=1}^{n} T_{\alpha_i}.$$

For a particular $\underline{\alpha}^n$, the entries α_i can repeat, but they are not ordered. In other words, $\underline{\alpha}^n$ is a an n–element subset with multiplicities. There exists a terminal object T_0 corresponding to the empty subset of S.

5.2 Strict and homotopy $\mathcal{T}$–algebras in $\Theta_n Sp$

Definition 5.3 Given an S–sorted theory $\mathcal{T}$, a *(strict)* $\mathcal{T}$–*algebra in* $\Theta_n Sp$ is a product-preserving functor $A \colon \mathcal{T} \to \Theta_n Sp$. In other words, the canonical map

$$A(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} A(T_{\alpha_i}),$$

induced by the projections $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ for all $1 \leq i \leq n$, is an isomorphism in $\Theta_n Sp$.

We denote the category of strict $\mathcal{T}$–algebras in $\Theta_n Sp$ by $\mathcal{A}lg^{\mathcal{T}}_{\Theta_n}$.

Definition 5.4 Given an S–sorted theory $\mathcal{T}$, a *homotopy* $\mathcal{T}$–*algebra in* $\Theta_n Sp$ is a functor $X \colon \mathcal{T} \to \Theta_n Sp$ which preserves products up to homotopy, ie, for all $\alpha \in S^n$, the canonical map

$$X(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} X(T_{\alpha_i})$$

induced by the projection maps $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ for each $1 \leq i \leq n$ is a weak equivalence in $\Theta_n Sp$.

Given an S–sorted theory $\mathcal{T}$ and $\alpha \in S$, there is an evaluation functor

$$U_\alpha\colon \mathcal{A}lg^{\mathcal{T}}_{\Theta_n} \to \Theta_n Sp$$

given by

$$U_\alpha(A) = A(T_\alpha).$$

Define a weak equivalence in the category $\mathcal{A}lg^{\mathcal{T}}_{\Theta_n}$ to be a map $f\colon A \to B$ such that $U_\alpha(f)\colon U_\alpha(A) \to U_\alpha(B)$ is a weak equivalence in $\Theta_n Sp$ for all $\alpha \in S$. Similarly, define a fibration of $\mathcal{T}$–algebras to be a map f such that $U_\alpha(f)$ is a fibration in $\mathcal{M}$ for all α. Then define a cofibration to be a map with the left lifting property with respect to the maps which are fibrations and weak equivalences.

The following theorem is a generalization of a result by Quillen [28, II.4].

Proposition 5.5 *There is a model structure on the category $\mathcal{A}lg^{\mathcal{T}}_{\Theta_n}$ with weak equivalences and fibrations given by evaluation functors U_α for all $\alpha \in S$.*

Proof The proof follows just as it does for algebras in $\mathcal{SS}$ets [7, 4.7]. □

Let $\Theta_n Sp^{\mathcal{T}}_f$ denote the category of functors $\mathcal{T} \to \Theta_n Sp$ with model structure given by levelwise weak equivalences and fibrations. Similarly, let $\Theta_n Sp^{\mathcal{T}}_c$ denote the same category with model structure given by levelwise weak equivalences and cofibrations. Since the objects of $\Theta_n Sp$ are simplicial presheaves, in particular presheaves of sets, we can regard the set of maps

$$P = \left\{ p_{\underline{\alpha}^n}\colon \coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -) \right\}$$

as defining a set of maps in $\Theta_n Sp$ given by constant diagrams. Then, we have model structures $\mathcal{L}(\Theta_n Sp)^{\mathcal{T}}_f$ and $\mathcal{L}(\Theta_n Sp)^{\mathcal{T}}_c$ given by localizing the model structures $\Theta_n Sp^{\mathcal{T}}_f$ and $\Theta_n Sp^{\mathcal{T}}_c$ with respect to this set of maps. The following proposition generalizes [7, 4.9].

Proposition 5.6 *There is a model category structure $\mathcal{L}(\Theta_n Sp^{\mathcal{T}})$ on the category $\Theta_n Sp^{\mathcal{T}}$ with weak equivalences the P–local equivalences, cofibrations as in $\mathcal{SS}\mathrm{ets}^{\mathcal{T}}_f$, and fibrations the maps which have the right lifting property with respect to the maps which are cofibrations and weak equivalences.*

Here, we use a slight modification of this theorem as follows. We define a model structure analogous to $\mathcal{L}(\Theta_n Sp^{\mathcal{T}})$ but on the category of functors $\mathcal{T} \to \Theta_n Sp$ which send T_0 to $\Delta[0]$, as in [9, 3.11]. We denote this category by $\mathcal{L}(\Theta_n Sp)^{\mathcal{T}}_*$.

Proposition 5.7 *Consider the category of functors $\mathcal{T} \to \Theta_n Sp$ such that the image of T_0 is $\Delta[0]$. There is a model category structure, on $\mathcal{L}(\Theta_n Sp)^{\mathcal{T}}_{*}$ in which the in which the fibrant objects are homotopy $\mathcal{T}$–algebras in $\Theta_n Sp$.*

The main theorem of this section is the following, and its proof follows just as in the case of $\mathcal{SS}$ets.

Theorem 5.8 *There is a Quillen equivalence of model categories*

$$L\colon \mathcal{L}(\Theta_n Sp)^{\mathcal{T}}_{*,f} \rightleftarrows \mathcal{A}lg^{\mathcal{T}}_{\Theta_n} :N.$$

5.9 Algebras over the theory of categories with fixed object set

We now look at the algebraic theory that is of use here, namely the theory $\mathcal{T}_{\mathcal{OC}\mathrm{at}}$ of categories with fixed object set $\mathcal{O}$. Consider the category $\mathcal{OC}at$ whose objects are the small categories with a fixed object set $\mathcal{O}$ and whose morphisms are the functors which are the identity on the objects. There is a theory $\mathcal{T}_{\mathcal{OC}at}$ associated to this category. Given an element $(\alpha, \beta) \in \mathcal{O} \times \mathcal{O}$, consider the directed graph with vertices the elements of $\mathcal{O}$ and with a single edge starting at α and ending at β. The objects of $\mathcal{T}_{\mathcal{OC}\mathrm{at}}$ are isomorphism classes of categories which are freely generated by coproducts of such directed graphs, where coproducts are taken in the category of categories with fixed object set $\mathcal{O}$. In other words, this theory is $(\mathcal{O} \times \mathcal{O})$–sorted.

A product-preserving functor $\mathcal{T}_{\mathcal{OC}\mathrm{at}} \to \mathcal{S}$ets is essentially a category with object set $\mathcal{O}$. In the comparison between simplicial categories and Segal categories with a fixed object set, we use simplicial algebras $\mathcal{T}_{\mathcal{OC}\mathrm{at}} \to \mathcal{SS}$ets, which correspond to simplicial categories, or categories enriched over simplicial sets, with fixed object set $\mathcal{O}$. Here, we regard strictly product-preserving functors $\mathcal{T}_{\mathcal{OC}\mathrm{at}} \to \Theta_n Sp$ as categories enriched over $\Theta_n Sp$ with object set $\mathcal{O}$.

When $\Theta_n Sp$ is additionally a cofibrantly generated model category of simplicial presheaves, then we can consider the model structure $\mathcal{A}lg^{\mathcal{T}_{\mathcal{OC}\mathrm{at}}}_{\Theta_n}$ and the related model structure for homotopy algebras, $\mathcal{L}(\Theta_n Sp)^{\mathcal{T}_{\mathcal{OC}\mathrm{at}}}$. The homotopy algebras can be regarded as a weaker version of categories enriched over $\Theta_n Sp$, yet not as weak as the Segal category objects that we considered in the previous section; our goal is to show they are all equivalent nonetheless.

The argument of proof used for the following proof is identical to the case of $\mathcal{SS}$ets in [9, Sections 4 and 5].

Theorem 5.10 *There is a Quillen equivalence of model categories*

$$\mathcal{L}(\Theta_n Sp)^{\mathcal{T}_{\mathcal{OC}\mathrm{at}}}_{\mathcal{O},f} \rightleftarrows \mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathcal{O},f}.$$

We also note the following comparison between our two model structures for Segal objects in $\Theta_n Sp$.

Proposition 5.11 *The identity functor gives a Quillen equivalence*

$$\mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathcal{O},f} \rightleftarrows \mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathcal{O},c}.$$

Proof The proof follows since weak equivalences are the same in both model structures and all the cofibrations in $\mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathcal{O},f}$ are cofibrations in $\mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathcal{O},c}$. □

6 Two model structures for Segal category objects

In this section, we remove the restriction that the sets in degree zero are fixed and establish two different model structures. Unlike in the fixed object case, we can no longer obtain the desired model structures as a localization of ones with levelwise weak equivalences. Nonetheless, one will be "injective-like," in that the cofibrations will be given levelwise, and the other will be "projective-like," in that the cofibrations will look as if they given by a localization of a projective structure. We begin by constructing sets of generating cofibrations for each.

6.1 Generating cofibrations for the injective-like model structure

Thus, we begin with the generating cofibrations for the Reedy model structure on $\Theta_n Sp_c^{\Delta^{op}}$, which are given by

$$A \times \Delta[p] \cup B \times \partial\Delta[p] \to B \times \Delta[p],$$

where $A \to B$ ranges over all generating cofibrations in $\Theta_n Sp$ and $p \geq 0$. Since the localization does not change the cofibrations, we can use the Reedy generating cofibrations as a generating set for $\Theta_n Sp$. Recall that a map $X \to Y$ is an acyclic fibration in $\mathcal{SS}\mathrm{ets}^{\Theta_n}$ if, for any object $[q](c_1,\dots,c_q)$, the map $X_{(c_1,\dots,c_q)} \to P_{(c_1,\dots,c_q)}$ is an acyclic fibration of simplicial sets, where $P_{(c_1,\dots,c_q)}$ is the pullback in the diagram

$$\begin{tikzcd}
P_{(c_1,\dots,c_q)} \arrow[r] \arrow[d] & Y_{(c_1,\dots,c_q)} \arrow[d] \\
M_{(c_1,\dots,c_q)}X \arrow[r] & M_{(c_1,\dots,c_q)}Y.
\end{tikzcd}$$

Here $M_{(c_1,\dots,c_q)}X$ denotes the matching object for X at $[q](c_1,\dots,c_q)$ and analogously for Y [17, 15.2.5].

The map $X_{(c_1,\dots,c_q)} \to P_{(c_1,\dots,c_q)}$ is an acyclic fibration of simplicial sets precisely when it has the left lifting property with respect to the generating cofibrations for the standard model structure on $\mathcal{SS}$ets, ie, with respect to the maps $\partial\Delta[m] \to \Delta[m]$ for all $m \geq 0$. Now, notice that

$$X_{(c_1,\dots,c_q)} = \mathrm{Map}(\Theta[q](c_1,\dots,c_q), X),$$
$$M_{(c_1,\dots,c_q)}X = \mathrm{Map}(\partial\Theta[q](c_1,\dots,c_q), X),$$

where $\Theta[q](c_1,\dots,c_q)$ is the analogue of $\Delta[q]$ in $\mathcal{SS}$ets, ie, the representable object for maps into $[q](c_1,\dots,c_q)$, and $\partial\Theta[q](c_1,\dots,c_q)$ is the analogue of $\partial\Delta[q]$. Thus, we get that

$$P_{(c_1,\dots,c_p)} = \mathrm{Map}(\Theta[q](c_1,\dots,c_q), Y) \times_{\mathrm{Map}(\partial\Theta[q](c_1,\dots,c_q),Y)} \mathrm{Map}(\partial\Theta[q](c_1,\dots,c_q), X).$$

Putting all this information together, and letting $\vec{c} = (c_1,\dots,c_q)$, we see that $X \to Y$ is an acyclic fibration in $\Theta_n Sp$ precisely when it has the right lifting property with respect to all maps

$$\partial\Delta[m] \times \Theta[q](\vec{c}) \cup \Delta[m] \times \partial\Theta[q](\vec{c}) \to \Delta[m] \times \Theta[q](\vec{c}).$$

Recall that $(\Theta_n Sp)^{\Delta^{\mathrm{op}}}_{\mathrm{disc}}$ denotes the category of functors $\Delta^{\mathrm{op}} \to \Theta_n Sp$ such that the image of $[0]$ is discrete, and that $(-)_r$ denotes the reduction functor which, applied to any object of $(\Theta_n Sp)^{\Delta^{\mathrm{op}}}$, forces the Θ_n–space at level 0 to be discrete. We have a preliminary set of possible generating cofibrations for this category, given by

$$((\partial\Delta[m] \times \Theta[q](\vec{c}) \cup \Delta[m] \times \partial\Theta[q](\vec{c}) \times \Delta[p] \cup (\Delta[m] \times \Theta[q](\vec{c}) \times \partial\Delta[p])_r \to ((\Delta[m] \times \Theta[q](\vec{c}) \times \Delta[p])_r.$$

As arose in [10, Section 4], some of these maps are not still monomorphisms after applying the reduction functor. It suffices to take all maps as above where $m = q = p = 0$, and where $m, q \geq 0$ and $p \geq 1$. All other maps where $p = 0$ either result in isomorphisms (which are unnecessary to include) or maps which are not isomorphisms. For example, when $p = q = 0$ and $m = 1$, we obtain $\Delta[0] \amalg \Delta[0] \to \Delta[0]$ after reduction, which is not a monomorphism. Define

$$I_c = \{((\partial\Delta[m] \times \Theta[q](\vec{c}) \cup \Delta[m] \times \partial\Theta[q](\vec{c})) \times \Delta[p] \cup (\Delta[m] \times \Theta[q](\vec{c})) \times \partial\Delta[p])_r \to ((\Delta[m] \times \Theta[q](\vec{c})) \times \Delta[p])_r \mid m, q \geq 0,\ p \geq 1\},$$

which will be a set of generating cofibrations for one of our model structures.

6.2 Generating cofibrations for the projective-like model structure

However, this reduction process does not work as well when we seek to find generating cofibrations for a model structure analogous to the projective model structure on $(\Theta_n Sp)^{\Delta^{\mathrm{op}}}$, in which the generating cofibrations are of the form

$$A \times \Delta[p] \to B \times \Delta[p],$$

where $p \geq 0$ and $A \to B$ is a generating cofibration in $\Theta_n Sp$. For some of the maps $A \to B$ (in particular when, using the description of such maps above, $m = 1$ or $q = 1$), reduction does not give the correct map.

Thus, we also need to consider another set, first to prove a technical lemma for our first model structure, and then to be a set of generating cofibrations for the second model structure. For any object A in $\Theta_n Sp$ and $p \geq 0$, define the object $A_{[p]}$ to be the pushout of the diagram

$$\begin{tikzcd} A \times (\Delta[p])_0 \arrow[r] \arrow[d] & A \times \Delta[p] \arrow[d] \\ (\Delta[p])_0 \arrow[r] & A_{[p]}. \end{tikzcd}$$

Define the set

$$I_f = \{A_{[p]} \to B_{[p]} \mid p \geq 0, A \to B \text{ a generating cofibration in } \Theta_n Sp\}.$$

6.3 Lifting properties for the sets I_c and I_f

Let X be a $\Theta_n Sp$–Segal precategory, and consider the map $X \to \operatorname{cosk}_0 X$. Denote by $X_p(v_0, \dots, v_p)$ the fiber of the map

$$X_p \to (\operatorname{cosk}_0 X)_p = X_0^{p+1}.$$

Then, for any object A or B as given above (noting that these objects are small in $\Theta_n Sp$), we get

$$\begin{aligned} \operatorname{Hom}(A_{[p]}, X) &= \operatorname{Hom}(A \times \Delta[p] \amalg_{A \times \Delta[p]_0} \Delta[p]_0, X) \\ &= \operatorname{Hom}(A, X_p) \times_{\operatorname{Hom}(A, X_0^{p+1})} X_0^{p+1} \\ &= \coprod_{v_0, \dots, v_p} \operatorname{Hom}(A, X_p(v_0, \dots, v_p)). \end{aligned}$$

(Notice that by our assumption that X is a $\Theta_n Sp$–Segal precategory, X_0 is a discrete object of $\Theta_n Sp$ and therefore our abuse of terminology that it has "elements" $v_0, \dots, v_p$ makes sense.)

We make use of the following facts about fibrations in $\Theta_n Sp$. We give the proof in Section 8.

Proposition 6.4 *Suppose that X, X', Y, and Y' are objects of $\Theta_n Sp$.*

(1) *If X and Y are both discrete constant diagrams, then any map $X \to Y$ is a fibration.*
(2) *If $X \to Y$ and $X \to Y$ be fibrations, then $X \amalg X' \to Y \amalg Y'$ is a fibration.*

The following lemma is the higher analogue of [10, 4.1].

Lemma 6.5 *Suppose that a map $f\colon X \to Y$ of Segal precategory objects has the right lifting property with respect to the maps in I_f. Then the map $X_0 \to Y_0$ is surjective, and each map*

$$X_p(v_0, \dots, v_p) \to Y_p(f v_0, \dots, f v_p)$$

is an acyclic fibration in $\Theta_n Sp$ for each $p \geq 1$ and $(v_0, \dots, v_p) \in X_0^{p+1}$.

Proof Using our description of the generating cofibrations of $\Theta_n Sp$, when $m = q = 0$, we get the map $\varnothing \to \Delta[0]$. The fact that $X \to Y$ has the right lifting property with respect to $\varnothing_{[0]} \to \Delta[0]_{[0]}$ implies that $X_0 \to Y_0$ is surjective.

To prove the remaining part of the statement, we need to show that a dotted arrow lift exists in all diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X_p(v_0, \dots, v_p) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y_p(f v_0, \dots, f v_p) \end{tikzcd}$$

for all choices of $p \geq 1$ and $A \to B$. By our hypothesis, we have the existence of dotted arrow lifts

$$\begin{tikzcd} A_{[p]} \arrow[r] \arrow[d] & X \arrow[d] \\ B_{[p]} \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

The existence of such a lift is equivalent to the surjectivity of the map $\mathrm{Hom}(B_{[p]}, X) \to P$, where P is the pullback in the diagram

$$\begin{tikzcd} \mathrm{Hom}(B_{[p]}, X) \arrow[r] & P \arrow[r] \arrow[d] & \mathrm{Hom}(A_{[p]}, X) \arrow[d] \\ & \mathrm{Hom}(B_{[p]}, Y) \arrow[r] & \mathrm{Hom}(A_{[p]}, Y). \end{tikzcd}$$

But, as we just showed above, we get

$$\operatorname{Hom}(B_{[p]}, X) = \coprod_{v_0,\dots,v_p} \operatorname{Hom}(B, X_p(v_0,\dots,v_p)),$$

and analogously for the other objects in the diagram. Looking at each component for each $(v_0,\dots,v_p)$ separately, we can check that surjectivity of this map does indeed give us the lift that we require. □

Lemma 6.6 *Suppose that $f\colon X \to Y$ is a map in $(\Theta_n Sp)^{\Delta^{op}}_{disc}$ with the right lifting property with respect to the maps in I_c. Then*

1. *the map $f_0\colon X_0 \to Y_0$ is surjective,*
2. *for every $m \geq 1$ and $(v_0,\dots,v_m) \in X_0^{n+1}$, the map*

$$X_m(v_0,\dots,v_m) \to Y_m(fv_0,\dots,fv_m)$$

is a weak equivalence in $\Theta_n Sp$.

Proof Since f has the right lifting property with respect to the maps in the set I_c, it has the right lifting property with respect to all cofibrations. In particular, f has the right lifting property with respect to the maps in the set I_f. Therefore, the result follows by Lemma 6.5. □

6.7 The injective-like model structure

In order to give a precise definition of our weak equivalences, we need to define a "localization" functor L on the category $\Theta_n Sp^{\Delta^{op}}_{disc}$ such that, for any object X, LX is a Segal space object which is also a Segal category object weakly equivalent to X in $\mathcal{L}_S \Theta_n Sp^{\Delta^{op}}$.

To begin, recall the object $G(p)$ as defined in Section 4, and consider one choice of generating acyclic cofibrations in $\mathcal{L}_S \Theta_n Sp^{\Delta^{op}}$, namely, the set

$$\{C \times \Delta[p] \cup D \times G(p) \to D \times \Delta[p]\},$$

where $p \geq 0$ and $C \to D$ is a generating acyclic cofibration in $\Theta_n Sp$. Using these maps, we can use the small object argument to construct a localization functor.

However, the maps with $p = 0$ are problematic because taking pushouts along them, as given by the small object argument, results in objects which are no longer Segal category objects. Thus, we consider maps as above, but with the restriction that $p \geq 1$. To show that the "localization" functor that results from this smaller set of maps is sufficient, in that it still gives us a Segal space object, we can use an argument just like the one given in [10, Section 5].

Now, we make the following definitions in $\Theta_n Sp^{\Delta^{op}}_{disc}$.

- Weak equivalences are the maps $f\colon X \to Y$ such that the induced map $LX \to LY$ is a Dwyer–Kan equivalence of Segal space objects. (We call such maps *Dwyer–Kan equivalences*.)
- Cofibrations are the monomorphisms.
- Fibrations are the maps with the right lifting property with respect to the maps which are both cofibrations and weak equivalences.

Lemma 6.8 *Suppose that $f\colon X \to Y$ is a map in $(\Theta_n Sp)^{\Delta^{op}}_{disc}$ with the right lifting property with respect to the maps in I_c. Then f is a Dwyer–Kan equivalence.*

Proof Suppose that $f\colon X \to Y$ has the right lifting property with respect to the maps in I_c. By Lemma 6.6, $f_0\colon X_0 \to Y_0$ is surjective and each map

$$X_m(v_0, \dots, v_m) \to Y_m(f v_0, \dots, f v_m)$$

is a weak equivalence in $\Theta_n Sp$ for $m \geq 1$ and $(v_0, \dots, v_m) \in X_0^{m+1}$. To prove that f is a Dwyer–Kan equivalence, it remains to show that, for any $x, y \in X_0$, $\operatorname{map}_{LX}(x, y) \to \operatorname{map}_{LY}(fx, fy)$ is a weak equivalence in $\Theta_n Sp$.

First, we construct a factorization of f as follows. Define ΦY to be the pullback in the diagram

$$\begin{tikzcd}
\Phi Y \arrow[r] \arrow[d] & Y \arrow[d] \\
\operatorname{cosk}_0(X_0) \arrow[r] & \operatorname{cosk}_0(Y_0).
\end{tikzcd}$$

Then $(\Phi Y)_0 = X_0$ and, for every $m \geq 1$ and $(v_0, \dots, v_m) \in X_0^{m+1}$, there is an isomorphisms of mapping objects

$$(\Phi Y)_0(v_0, \dots, v_m) \cong Y_m(f v_0, \dots, f v_m).$$

Then $X \to \Phi Y$ is a Reedy weak equivalence and hence a Dwyer–Kan equivalence. Therefore, it remains to prove that $\Phi Y \to Y$ is a Dwyer–Kan equivalence, via an inductive argument on the skeleta of Y.

For any $p \geq 0$, consider the map $\Phi(\operatorname{sk}_p Y) \to \operatorname{sk}_p Y$. If $p = 0$, then $\Phi(\operatorname{sk}_0 Y)$ and $\operatorname{sk}_0 Y$ are actually $\Theta_n Sp$–Segal objects which can be observed to be Dwyer–Kan equivalent.

Therefore, assume that the map $\Phi(\mathrm{sk}_{p-1}Y) \to \mathrm{sk}_{p-1}Y$ is a Dwyer–Kan equivalence and consider the map $\Phi(\mathrm{sk}_p Y) \to \mathrm{sk}_p Y$.

We know that $\mathrm{sk}_p Y$ is obtained from $\mathrm{sk}_{p-1}Y$ via iterations of pushouts along maps $A_{[m]} \to B_{[m]}$ for $A \to B$ a generating cofibration in $\Theta_n Sp$. Since we need a more precise formulation, we recall that generating cofibrations in $\Theta_n Sp$ are of the form

$$\partial\Delta[m] \times \Theta[q](c_1, \dots, c_q) \cup \Delta[m] \times \partial\Theta[q](c_1, \dots, c_q) \to \Delta[m] \times \Theta[q](c_1, \dots, c_q)$$

for $m, q \geq 0$ and $c_1, \dots, c_q$ objects of Θ_{n-1}. So, we have the pushout diagram

$$\begin{tikzcd}
\Delta[m] \times \Theta[q](c_1, \dots, c_q) \times \Delta[p]_0 \arrow[r] \arrow[d] & \Delta[m] \times \Theta[q](c_1, \dots, c_q) \times \Delta[p] \arrow[d] \\
\Delta[p]_0 \arrow[r] & (\Delta[m] \times \Theta[q](c_1, \dots, c_q))_{[p]}.
\end{tikzcd}$$

Similarly, we obtain $(\partial\Delta[m] \times \Theta[q](c_1, \dots, c_q) \cup \Delta[m] \times \partial\Theta[q](c_1, \dots, c_q))_{[p]}$.

For simplicity, assume that we require only one pushout to obtain $\mathrm{sk}_p Y$ from $\mathrm{sk}_{p-1}Y$; here we further simplify by considering the case where $m = q = 0$, although the argument can be extended more generally. For this case, we have the pushout diagram

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & \mathrm{sk}_{p-1}Y \arrow[d] \\
\Delta[p] \arrow[r] & \mathrm{sk}_p Y.
\end{tikzcd}$$

Since we know by our inductive hypothesis that $\Phi(\mathrm{sk}_{p-1}Y) \to \mathrm{sk}_{p-1}Y$ is a Dwyer–Kan equivalence, it suffices to establish that $\Phi\Delta[p] \to \Delta[p]$ is a Dwyer–Kan equivalence. In the setting where these are levelwise discrete simplicial spaces, this fact was established in [10, Section 9]. The argument given there continues to hold in the present case, making use of the fact that the model structure for $\Theta_n Sp$–Segal spaces is cartesian (Theorem 3.14). $\square$

Theorem 6.9 *There is a cofibrantly generated model category structure* $\mathcal{L}\Theta_n Sp^{\Delta^{op}}_{\mathrm{disc},c}$ *on the category of* $\Theta_n Sp$*–Segal precategories with the above weak equivalences, fibrations, and cofibrations.*

For the proof, we use the following result.

Theorem 6.10 (Beke [5, 4.1]) *Let* $\mathcal{M}$ *be a locally presentable category,* W *a full accessible subcategory of* $\mathrm{Mor}(\mathcal{M})$ *satisfying the two-out-of-three property, and* I *a set of morphisms in* $\mathcal{M}$*. Suppose that*

(1) $\mathrm{inj}(I) \subseteq W$,

(2) $W \cap \mathrm{cof}(I)$ *is closed under pushouts and transfinite composition.*

Proof of Theorem 6.9 It suffices to check the conditions of the above theorem. It is not too hard to show that condition (1) is satisfied with W the class of weak equivalences as defined. However, to prove the remaining two statements we need the set

$$I_c = \{(A \times \Delta[p] \cup B \times \partial\Delta[p])_r \to (B \times \Delta[p])_r\},$$

where $A \to B$ are the generating cofibrations in $\Theta_n Sp$.

Condition (2) was established in Lemma 6.8.

For condition (3), first notice that elements of $\mathrm{cof}(I_c)$ are monomorphisms. Now suppose that $X \to Y$ is a weak equivalence which is in $\mathrm{cof}(I_c)$, and suppose

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & W \end{tikzcd}$$

is a pushout diagram. Then notice that in the diagram

$$\begin{tikzcd} \mathrm{map}_{LX}(x,y) \arrow[r] \arrow[d] & \mathrm{map}_{LZ}(x,y) \arrow[d] \\ \mathrm{map}_{LY}(x,y) \arrow[r] & \mathrm{map}_{LW}(x,y) \end{tikzcd}$$

again has the left hand vertical map a cofibration and weak equivalence in $\Theta_n Sp$, and is again a pushout diagram. Furthermore, using the definition of homotopy category in a $\Theta_n Sp$–Segal category, it can be shown that the analogous diagram of homotopy categories is again a pushout diagram. Therefore, weak equivalences which are in $\mathrm{cof}(I_c)$ are preserved by pushouts. A similar argument using mapping objects and homotopy categories establishes that such maps are preserved by transfinite compositions. □

6.11 The projective-like model structure

We now define another model structure with the same weak equivalences, but for which the cofibrations are given by transfinite compositions of pushouts along the maps of the generating set I_f, and the fibrations are then determined.

Theorem 6.12 *There is a model structure $\mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathrm{disc},f}$ on the category of Segal precategory objects with weak equivalences the Dwyer–Kan equivalences and the cofibrations given by iterated pushouts along the maps of the set I_f.*

Proof As before, we show that the conditions of Theorem 6.10 are satisfied. Condition (1) continues to hold from the previous model structure. A similar proof can be used to establish condition (2), using Lemma 6.5 and a proof analogous to the one for Lemma 6.8. Condition (3) works as in the other model structure. □

7 Quillen equivalences between Segal category objects and enriched categories

We now establish Quillen equivalences between the models given in the previous sections.

Proposition 7.1 *The identity functor induces a Quillen equivalence*

$$\mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathrm{disc},c} \rightleftarrows \mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathrm{disc},f}.$$

Proof The identity map from $\mathcal{L}\Theta_n Sp^{\Delta^{op}}_{f,\mathrm{disc}}$ to $\mathcal{L}\Theta_n Sp^{\Delta^{op}}_{c,\mathrm{disc}}$ preserves cofibrations and acyclic cofibrations, so we get a Quillen pair. The fact that it is a Quillen equivalence follows then from the fact that weak equivalences are the same in both categories. □

Proposition 7.2 *There is a Quillen pair*

$$F\colon \mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{f,\mathrm{disc}} \rightleftarrows \Theta_n Sp\text{–}Cat\colon R.$$

The idea here is that the local objects of $\mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{f,\mathrm{disc}}$ are "weak," in that the condition for local is given by a weak equivalence of mapping spaces. However, the objects in the image of R are "strict," in that they satisfy the same condition but with an isomorphism of mapping spaces. To formalize this relationship, needed to prove this proposition, we give the following definition.

Definition 7.3 Let $\mathcal{D}$ be a small category, $\mathcal{C}$ a simplicial category, and $\mathcal{C}^{\mathcal{D}}$ the category of functors $\mathcal{D} \to \mathcal{C}$. Let S be a set of morphisms in $\mathcal{S}\mathrm{Sets}^{\mathcal{D}}$. An object Y of $\mathcal{C}^{\mathcal{D}}$ is *strictly S–local* if for every morphism $f\colon A \to B$ in S, the induced map on function complexes

$$f^*\colon \mathrm{Map}(B,Y) \to \mathrm{Map}(A,Y)$$

is an isomorphism of simplicial sets. A map $g\colon C \to D$ in $\mathcal{C}^{\mathcal{D}}$ is a *strict S–local equivalence* if for every strictly S–local object Y in $\mathcal{C}^{\mathcal{D}}$, the induced map

$$g^*\colon \mathrm{Map}(D,Y) \to \mathrm{Map}(C,Y)$$

is an isomorphism of simplicial sets.

Here, we consider functors $\Delta^{op} \to \Theta_n Sp$ which are discrete at level zero. Notice that a category enriched in $\Theta_n Sp{-}Cat$ can be regarded as a strictly local object in this category when we localize with respect to the map φ described in Section 4. Recall that a Segal category object is a (nonstrictly) local object when regarded as a Segal space object $\Delta^{op} \to \Theta_n Sp$. Thus, the enriched nerve functor can be regarded as an inclusion map

$$R\colon \Theta_n Sp{-}Cat \to \Theta_n Sp^{\Delta^{op}}.$$

Although we are working in the subcategory of functors which are discrete at level zero, we can still use the following lemma to obtain a left adjoint functor F to our inclusion map R, since the construction will always produce a diagram with discrete set at level zero when applied to such a diagram.

Lemma 7.4 *For any small category $\mathcal{D}$ and any model category $\mathcal{M}$, consider the category of all diagrams $X\colon \mathcal{D} \to \mathcal{M}$ and the category of strictly local diagrams with respect to the set of maps $S = \{f\colon A \to B\}$. The forgetful functor from the category of strictly local diagrams to the category of all diagrams has a left adjoint.*

Proof This lemma was proved in [7, 5.6] in the case where $\mathcal{M} = \mathcal{SS}ets$, but the proof continues to hold if we use a more general simplicial category. □

We define $F\colon \mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathrm{disc},f} \to \Theta_n Sp{-}Cat$ to be this left adjoint to the inclusion map of strictly local diagrams.

Proof of Proposition 7.2 To prove this proposition, we modify the approach given in the proof of the analogous result when $n = 1$ [10, 8.3]. We first show that F preserves cofibrations. Since F is a left adjoint functor, we know that it preserves colimits, so it suffices to show that F takes the maps in the set I_f to cofibrations in $\Theta_n Sp{-}Cat$.

Let $*$ denote the terminal object in $(\Theta_n Sp)^{\Delta^{op}}$. Since cofibrations are inclusions in $\Theta_n Sp$, the map $\varnothing \to *$ is a cofibration, and $\varnothing_{[0]} \to *_{[0]}$ is already local; in fact it corresponds to the generating cofibration $\varnothing \to \{x\}$ in $\Theta_n Sp{-}Cat$.

For any generating cofibration $A \to B$, localizing the map $A_{[1]} \to B_{[1]}$ results in the generating cofibration $UA \to UB$ of $\Theta_n Sp{-}Cat$. Localizing any other map of I_f results in a map in $\Theta_n Sp{-}Cat$ which is a colimit of maps of this form, and therefore F preserves cofibrations.

To show that F preserve acyclic cofibrations, we use the Quillen equivalence in the fixed-object set situation; the argument given in [10, 8.3] still holds in this more general setting. □

To prove that this Quillen pair is a Quillen equivalence, we use the following theorem, which is the analogue of [10, 8.5].

Lemma 7.5 *For every cofibrant object X in $\mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{\mathrm{disc},f}$, the map $X \to FX$ is a Dwyer–Kan equivalence.*

Proof Consider an object in $\mathcal{L}\Theta_n Sp^{\Delta^{op}}_{\mathrm{disc},f}$ of the form $\coprod_i B_{[p_i]}$, where B is the target of a generating cofibration of $\Theta_n Sp$, and let Y be a fibrant object of $\mathcal{L}\Theta_n Sp^{\Delta^{op}}_{\mathrm{disc},f}$. Then notice that $(\Delta[p] \times B)_k = \Delta[p]_k \times B$ since B is regarded as a constant simplicial diagram. Then

$$\begin{aligned}
\mathrm{Map}(\Delta[m], \Delta[p] \times B) &\cong \mathrm{Map}(\Delta[m], \Delta[p]) \times \mathrm{Map}(\Delta[m], B) \\
&\cong \mathrm{Map}(G(m), \Delta[p]) \times \mathrm{Map}(G(m), B) \\
&\cong \mathrm{Map}(G(m), \Delta[p], B),
\end{aligned}$$

so $\Delta[p] \times B$ is strictly local. By its construction, it follows that $\coprod_i B_{[p_i]}$ is also strictly local. In particular, the map

$$\coprod_i B[p_i] \to F\Big(\coprod_i B[p_i]\Big)$$

is a Dwyer–Kan equivalence.

Now, suppose that X is any cofibrant object. Then it can be written as a colimit of objects of the above form, and we can assume that it can be written as

$$X \simeq \mathrm{colim}_{\Delta^{op}} X_j,$$

where $X_j = \coprod_I B[p_i]$. Then, using arguments about mapping spaces and strictly local objects as in [10, 8.5], we can show that

$$\mathrm{Map}(X, Y) \simeq \mathrm{Map}(FX, Y)$$

for any strictly local fibrant object Y, completing the proof. □

Theorem 7.6 *The Quillen pair*

$$F\colon \mathcal{L}(\Theta_n Sp)^{\Delta^{op}}_{f,\mathrm{disc}} \rightleftarrows \Theta_n Sp\text{–}\mathcal{C}at\colon R.$$

is a Quillen equivalence.

Proof To prove this result, we can use Lemma 7.5 to prove that F reflects weak equivalences between cofibrant objects. Then, we show that for any fibrant $\Theta_n Sp$–category, the map $F((RY)^c) \to Y$ is a Dwyer–Kan equivalence, where $(RY)^c$ denotes a cofibrant replacement of RY. The proof follows just as in the $n = 1$ case [10, 8.6]. □

8 Fibrations in $\Theta_n Sp$

In this section we give the proof of Proposition 6.4, establishing properties of fibrations in $\Theta_n Sp$.

We begin with the case where $n = 1$, so that $\Theta_n Sp$ is just $\mathcal{CSS}$, the model structure for complete Segal spaces.

Proposition 8.1 *The statement of Proposition 6.4 holds when $n = 1$.*

Proof Recall that the generating acyclic cofibrations in $\mathcal{CSS}$ are of the form

$$V[m,k] \times \Delta[p]^t \cup \Delta[m] \times G(p)^t \to \Delta[m] \times \Delta[p]^t$$

or

$$V[m,k] \times E^t \cup \Delta[m] \times \Delta[0]^t \to \Delta[m] \times E^t,$$

where $m \geq 1$, $0 \leq k \leq m$, $p \geq 0$, and E denotes the nerve of the category with two objects and a single isomorphism between them.

Suppose that X and Y are discrete constant simplicial spaces. To show that any map $X \to Y$ is a fibration, it suffices to prove that it has the right lifting property with respect to these two kinds of generating acyclic cofibrations, which is equivalent to the existence of dotted arrow lifts in the diagrams of simplicial sets

$$\begin{tikzcd}
V[m,k] \arrow[r] \arrow[d] & X_n \arrow[d] & \\
\Delta[m] \arrow[r] \arrow[ur, dashed] & P \arrow[r] \arrow[d] & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[d] \\
& Y_n \arrow[r] & Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1,
\end{tikzcd}
\qquad
\begin{tikzcd}
V[m,k] \arrow[r] \arrow[d] & \mathrm{Map}(E^t, X) \arrow[d] & \\
\Delta[m] \arrow[r] \arrow[ur, dashed] & Q \arrow[r] \arrow[d] & X_0 \arrow[d] \\
& \mathrm{Map}(E^t, Y) \arrow[r] & Y_0,
\end{tikzcd}$$

where P and Q denote the pullbacks of their respective lower square diagrams. In the first diagram, since X and Y are constant, $X_0 = X_1 = X_n$ and $Y_0 = Y_1 = Y_n$ for all $n \geq 2$, so $P = X_n$ and the right hand vertical map in the upper square is an isomorphism. Therefore, the necessary lift exists. Similarly, in the second diagram, we can again use the fact that X and Y are discrete to show that $\mathrm{Map}(E^t, X) = X_0$ and $\mathrm{Map}(E^t, Y) = Y_0$, from which it follows that $Q = X_0$ and the right hand vertical map in the upper diagram is an isomorphism, implying the existence of the desired lift. Therefore, we have established that (1) holds in $\mathcal{CSS}$.

For (2), suppose that $X \to Y$ and $X' \to Y'$ have the right lifting property with respect to the two kinds of generating acyclic cofibrations. For the first kind, we need to find a dotted arrow lift in any diagram of the form

$$\begin{tikzcd} V[m,k] \arrow[r] \arrow[d] & (X \amalg X')_n \arrow[d] & \\ \Delta[m] \arrow[r] \arrow[ur, dashed] & P \arrow[r] \arrow[d] & (X \amalg X')_1 \times_{(X \amalg X')_0} \cdots \times_{(X \amalg X')_0} (X \amalg X')_1 \arrow[d] \\ & (Y \amalg Y')_n \arrow[r] & (Y \amalg Y')_1 \times_{(Y \amalg Y')_0} \cdots \times_{(Y \amalg Y')_0} (Y \amalg Y')_1 . \end{tikzcd}$$

However, since all maps in sight are given by coproducts of maps, we can rewrite the right hand vertical map in the lower diagram as

$$(X_1 \times_{X_0} \cdots \times_{X_0} X_1) \amalg (X'_1 \times_{X'_0} \cdots \times_{X'_0} X'_1) \to (Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1) \amalg (Y'_1 \times_{Y'_0} \cdots \times_{Y'_0} Y'_1).$$

Since $\Delta[m]$ is connected, finding a lift reduces to finding a lift on one of the components, which holds since we have assumed that each component map $X \to X'$ or $X' \to Y'$ is a fibration. A similar argument can be used to establish the right lifting property with respect to the second type of acyclic cofibration. □

The proof of Proposition 6.4 can then be established via the following inductive result.

Proposition 8.2 *If conditions (1) and (2) from Proposition 6.4 hold for $\Theta_{n-1}Sp$, $n \geq 2$, then they hold for $\Theta_n Sp$.*

Proof The generating acyclic cofibrations of $\Theta_n Sp$ are of three kinds:

$$V[m,k] \times \Theta_p(c_1, \dots, c_p) \cup \Delta[m] \times G(p)(c_1, \dots, c_p) \to \Delta[m] \times \Theta_p(c_1, \dots, c_p)$$

for $m \geq 1$, $0 \leq k \leq m$, $p \geq 0$, and $c_1, \dots, c_p$ objects of Θ_{n-1},

$$V[m,k] \times T_\# \Delta[0] \cup \Delta[m] \times T_\# E \to \Delta[m] \times T_\# \Delta[0],$$

for m, k as before, and

$$V[m,k] \times V[1](B) \cup \Delta[m] \times V[1](A) \to \Delta[m] \times V[1](B),$$

where $A \to B$ is a map in $\mathcal{T}_{n-1}$, the set of generating cofibrations for $\Theta_{n-1}Sp$.

Let us first consider the case where $X \to Y$ is a map between discrete constant objects. Showing that this map has the right lifting property with respect to the first two kinds of generating acyclic cofibrations is analogous to the proof of Proposition 8.1. For the third kind, we need to show the existence of a dotted arrow lift in any diagram of the form

$$\begin{tikzcd}
V[m,k] \arrow[r] \arrow[d] & \mathrm{Map}(V[1](B),X) \arrow[d] & \\
\Delta[m] \arrow[r] \arrow[ur, dashed] & P \arrow[r] \arrow[d] & \mathrm{Map}(V[1](A),X) \arrow[d] \\
& \mathrm{Map}(V[1](B),Y) \arrow[r] & \mathrm{Map}(V[1](A),Y),
\end{tikzcd}$$

where P denotes the pullback of the lower square.

Now, recall from [31] that we can define the mapping object $M_X(x_0,x_1)(c_1)$ to be the object of $\Theta_{n-1}Sp$ defined as the pullback in the diagram

$$\begin{tikzcd}
M_X(x_0,x_1)(c_1) \arrow[r] \arrow[d] & X[1](c_1) \arrow[d] \\
(x_0,x_1) \arrow[r] & X[0]\times X[0].
\end{tikzcd}$$

Furthermore, we get

$$\mathrm{Map}(V[1](B),X) = \coprod_{x_0,x_1} \mathrm{Map}(B, M_X(x_0,x_1)),$$

and analogously for other objects in the above diagram. Since we have reduced the problem to the world of $\Theta_{n-1}Sp$, our inductive hypothesis shows that the necessary lift exists. Hence, condition (1) holds.

The same kind of argument, and again using the ideas of the proof of Proposition 8.1, we can verify that condition (2) holds as well. □

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Department of Mathematics, University of California Riverside
900 University Avenue, Riverside, CA 92521, USA

Department of Mathematics, University of Illinois at Urbana-Champaign
273 Altgeld Hall, MC-382, 1409 West Green Street, Urbana, IL 61801, USA

bergnerj@member.ams.org, rezk@math.uiuc.edu

http://www.math.ucr.edu/~jbergner/, http://www.math.uiuc.edu/~rezk/

Proposed: Paul Goerss
Seconded: Peter Teichner, Ralph Cohen

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