The quantum content of the gluing equations

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The gluing equations of a cusped hyperbolic 3–manifold M are a system of polynomial equations in the shapes of an ideal triangulation $\mathcal{T}$ of M that describe the complete hyperbolic structure of M and its deformations. Given a Neumann–Zagier datum (comprising the shapes together with the gluing equations in a particular canonical form) we define a formal power series with coefficients in the invariant trace field of M that should (a) agree with the asymptotic expansion of the Kashaev invariant to all orders, and (b) contain the nonabelian Reidemeister–Ray–Singer torsion of M as its first subleading "1–loop" term. As a case study, we prove topological invariance of the 1–loop part of the constructed series and extend it into a formal power series of rational functions on the PSL(2, $\mathbb{C}$) character variety of M. We provide a computer implementation of the first three terms of the series using the standard `SnapPy` toolbox and check numerically the agreement of our torsion with the Reidemeister–Ray–Singer for all 59924 hyperbolic knots with at most 14 crossings. Finally, we explain how the definition of our series follows from the quantization of 3–dimensional hyperbolic geometry, using principles of topological quantum field theory. Our results have a straightforward extension to any 3–manifold M with torus boundary components (not necessarily hyperbolic) that admits a regular ideal triangulation with respect to some PSL(2, $\mathbb{C}$) representation.

1 Introduction

1.1 The Kashaev invariant and perturbative Chern–Simons theory

The *Kashaev invariant* $\langle K \rangle_N \in \mathbb{C}$ of a knot K in 3–space (for $N = 2, 3, \dots$) is a powerful sequence of complex numbers determined by the *Jones polynomial* of the knot (see [46]) and its cablings; see Turaev [70] and Witten [72]. The *Volume Conjecture* of Kashaev and Murakami–Murakami [49; 50; 55] relates the Kashaev invariant of a hyperbolic knot K with the hyperbolic volume $\mathrm{Vol}(M)$ of its complement $M = S^3 \setminus K$ (see Thurston [66]):

$$(1\text{-}1) \qquad \lim_{N \to \infty} \frac{1}{N} \log |\langle K \rangle_N| = \frac{\mathrm{Vol}(M)}{2\pi}.$$

A generalization of the Volume Conjecture (see Gukov [39]) predicts a full asymptotic expansion of the Kashaev invariant to all orders in $1/N$:

$$(1\text{-}2) \qquad\qquad \langle K \rangle_N \overset{N \to \infty}{\sim} \mathcal{Z}_M(2\pi i/N)$$

for a suitable formal power series

$$(1\text{-}3) \quad \mathcal{Z}_M(\hbar) = \exp\left(\frac{1}{\hbar} S_{M,0} - \frac{3}{2} \log \hbar + S_{M,1} + \sum_{n \geq 2} \hbar^{n-1} S_{M,n} \right), \quad \hbar = \frac{2\pi i}{N}.$$

The formal power series $\mathcal{Z}_M(\hbar)$ in (1-3) is conjectured to coincide with the perturbative partition function of Chern–Simons theory with complex gauge group $\mathrm{SL}(2, \mathbb{C})$ along the discrete faithful representation ρ_0 of the hyperbolic manifold M. Combining such an interpretation with further conjectures of the first author, Gukov, Lenells and Zagier [17] and the second author and Lê [34; 36] one predicts the following.

- $S_{M,0} = i\,(\mathrm{Vol}_M + i\,\mathrm{CS}_M) \in \mathbb{C}/(4\pi^2 \mathbb{Z})$ is the complexified volume of M (cf Thurston [67] and Neumann [56]).

- $S_{M,1}$ is related (see Witten [73], Bar-Natan and Witten [3] and Gukov and Murakami [40]) to the nonabelian Ray–Singer torsion (see De Loera, Rambau and Santos [63]), which ought to equal (cf Müller [54]) the combinatorial nonabelian Reidemeister torsion. More precisely, by Dubois and the second author [19, Conjecture 1.8] we should have

$$(1\text{-}4) \qquad\qquad \tau_M^{\mathrm{R}} = 4\pi^3 \exp(-2S_{M,1}) \in E_M^*,$$

  where τ_M^{R} is the nonabelian Reidemeister–Ray–Singer torsion of M with respect to the meridian (see Porti [62] and Dubois [18]), and E_M is the invariant trace field of M.

- For $n \geq 2$, the *n–loop invariants* $S_{M,n}$ are conjectured to lie in the invariant trace field E_M [17; 34].

The generalization (1-2) of the Volume Conjecture has been numerically verified for a few knots using either state integral formulas for Chern–Simons theory when available [17] or a numerical computation of the Kashaev invariant and its numerical asymptotics, lifted to algebraic numbers; see the second author and Zagier [37; 31; 32].

Our goal is to provide an exact, combinatorial definition of the formal power series $\mathcal{Z}_M(\hbar)$ via formal Gaussian integration using the shape parameters and the Neumann–Zagier matrices of a regular ideal triangulation of M. Our definitions

- express the putative torsion $\exp(-2S_{M,1})$ and the *n–loop invariants* $S_{M,n}$ manifestly in terms of the shape parameters z_i and the gluing matrices of a regular ideal triangulation $\mathcal{T}$ of M;

- manifestly deduce that the putative torsion and the n–loop invariants for $n \geq 2$ are elements of the invariant trace field;

- explain the difference of $\mathcal{Z}_M(\hbar)$ for pairs of geometrically similar knots studied by Zagier and the second author;

- provide an effective way to compute the n–loop invariants using standard commands of the SnapPy toolbox [13], as demonstrated for $n = 1, 2, 3$ for hyperbolic knots with at most 14 crossings;

- allow efficient tests of the asymptotics of the Volume Conjecture (1-2), the "1–loop Conjecture" (1-4) and other conjectures in Quantum Topology.

We note that we only define $\exp(-2S_{M,1})$ up to a sign, and $S_{M,2}$ modulo $\mathbb{Z}/24$. All higher n–loop invariants are defined unambiguously.

Although we give a purely combinatorial definition of $\mathcal{Z}_M(\hbar)$ without any knowledge of state integrals or Chern–Simons theory with complex gauge group, in Section 5 we explain how our definition follows from the state integral model of the first author [15] and its perturbative expansion.

1.2 The Neumann–Zagier datum

All manifolds and all ideal triangulations in this paper will be oriented. The volume of a hyperbolic manifold M, appearing in the Volume Conjecture and contributing to $S_{M,0}$, is already known to have a simple expression in terms of shape parameters of a regular ideal triangulation, ie, one that recovers the complete hyperbolic structure of M. (For extended discussion on regular triangulations, see Section 4.) If $\mathcal{T} = \{\Delta_i\}_{i=1}^N$ is a regular ideal triangulation of M with shape parameters $z_i \in \mathbb{C} \setminus \{0, 1\}$ for $i = 1, \ldots, N$, then (cf Dupont and Sah [23] and Neumann and Zagier [58])

$$(1\text{-}5) \qquad \mathrm{Vol}(M) = \sum_{i=1}^N D(z_i),$$

where $D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$ is the Bloch–Wigner dilogarithm function. This formula can also be interpreted as calculating the image of the class $[M] := \sum_i [z_i] \in \mathcal{B}$ of M in the Bloch group $\mathcal{B}$ under the natural map $D: \mathcal{B} \to \mathbb{R}$. An analogous formula, using the class of M in the "extended" Bloch group $\hat{\mathcal{B}}$, gives the full complexified volume $S_{M,0}$; see Neumann [56; 57], Goette and Zickert [38; 75].

It is natural to ask whether the class of M in $\mathcal{B}$ determines not only $S_{M,0}$ but the higher $S_{M,n}$ as well. This question was posed to the authors several years ago by D Zagier. Subsequent computations [37; 32] indicated that a positive answer was not possible.

For example, there is a family of pairs of pretzel knots $((-2, 3, 3+2p), (-2, 3, 3-2p))$ for $p = 2, 3, \ldots$, as well as the figure-eight knot and its sister, which all have the same class in the Bloch group (and classes differing by 6–torsion in the extended Bloch group), but different invariants $S_{M,n}$ for $n \geq 1$.

The extra information necessary to determine the $S_{M,n}$ can be described as follows. Recall that if $\mathcal{T}$ is a regular ideal triangulation of M with N tetrahedra, its shapes $z = (z_1, \ldots, z_N)$ satisfy a system of polynomial equations, one equation for every edge, and one imposing parabolic holonomy around the meridian of the cusp [66; 58]. Let us set

$$(1\text{-}6) \qquad\qquad z_i' = (1 - z_i)^{-1}, \quad z_i'' = 1 - z_i^{-1}.$$

The equations can then be written in the form

$$(1\text{-}7) \qquad z^A z''^B := \prod_{j=1}^{N} z_j^{A_{ij}} (1 - z_j^{-1})^{B_{ij}} = \pm 1, \quad i = 1, \ldots, N,$$

where A and B are $N \times N$ square matrices with integer entries, which we call the *Neumann–Zagier matrices* following [58].

Definition 1.1 If $\mathcal{T}$ is a regular ideal triangulation of M, its *Neumann–Zagier datum* (resp. *enhanced Neumann–Zagier datum*) is given by the triple

$$\beta_{\mathcal{T}} = (z, A, B), \quad \text{resp.} \quad \hat{\beta}_{\mathcal{T}} = (z, A, B, f),$$

where z is a solution to the gluing equations and f is a combinatorial flattening of $\mathcal{T}$, a collection of integers that we define in Section 2.4.

As we will discuss in detail in Section 2, implicit in the above definition is the dependence of $\beta_{\mathcal{T}}$ and $\hat{\beta}_{\mathcal{T}}$ on the following choices:

(1) a pair of opposite edges for every oriented ideal tetrahedron (a so-called choice of *quad type*).

(2) An edge of $\mathcal{T}$.

(3) A meridian loop in the boundary of M in general position with respect to $\mathcal{T}$.

(4) A combinatorial flattening.

1.3 The 1–loop invariant

Definition 1.2 Given a one-cusped hyperbolic manifold M with regular ideal triangulation $\mathcal{T}$ and enhanced Neumann–Zagier datum $\hat{\beta}_{\mathcal{T}}$ we define

$$(1\text{-}8) \qquad \tau_{\mathcal{T}} := \pm \frac{1}{2} \det(A \Delta_{z''} + B \Delta_z^{-1}) z^{f''} z''^{-f} \in E_M/\{\pm 1\},$$

where $\Delta_z := \operatorname{diag}(z_1, \ldots, z_N)$ and $\Delta_{z''} := \operatorname{diag}(z_1'', \ldots, z_N'')$ are diagonal matrices, and $z^{f''} z''^{-f} := \prod_i z_i^{f_i''} z_i''^{-f_i}$.

Note that $\tau_{\mathcal{T}}$ takes value in the invariant trace field of M and is only defined up to a sign. In Section 3 we will show the following.

Theorem 1.3 $\tau_{\mathcal{T}}$ *is independent of the quad type of* $\mathcal{T}$, *the chosen edge of* $\mathcal{T}$, *the choice of a meridian loop, and the choice of a combinatorial flattening.*

We now consider the dependence of $\tau_{\mathcal{T}}$ on the choice of a regular ideal triangulation of M. It is well known that the set $\mathcal{X}$ of ideal triangulations of a cusped hyperbolic manifold is nonempty (see Casler [10]) and connected by 2–3 moves; see for example Matveev [52; 53] and Piergallini [59]. That is, a sequence of 2–3 moves can be used to take any one ideal triangulation to any other. The subset $\mathcal{X}_{\rho_0}$ of $\mathcal{X}$ of regular triangulations is also nonempty; see Section 4. Topologically, these are the triangulations without any univalent edges; see Champanerkar [11], Boyd, Dunfield and Rodriguez-Villegas [7], Dunfield and the second author [22] and Tillmann [69]. We will prove the following in Section 3.

Theorem 1.4 $\tau_{\mathcal{T}}$ *is constant on every component of* $\mathcal{X}_{\rho_0}$ *connected by 2–3 moves.*

1.4 Expectations

We may pose some questions and conjectures about the 1–loop invariant $\tau_{\mathcal{T}}$ and the structure of the set $\mathcal{X}_{\rho_0}$. Let us begin with two questions whose answers are unfortunately unknown.

Question 1.5 Is $\mathcal{X}_{\rho_0}$ connected by 2–3 moves?

Question 1.6 Is $\tau_{\mathcal{T}}$ constant on the set $\mathcal{X}_{\rho_0}$?

Clearly, a positive answer to the first question implies a positive answer to the second.

Despite the unknown answer to the above questions, with additional effort we can still define a distinguished component of $\mathcal{X}_{\rho_0}$, and thus obtain a topological invariant of M. Namely, let $\mathcal{X}_M^{\mathrm{EP}} \subset \mathcal{X}_{\rho_0}$ denote the subset that consists of regular refinements of the canonical (Epstein–Penner) ideal cell decomposition of M [25]. $\mathcal{X}_M^{\mathrm{EP}}$ is canonically associated to a cusped hyperbolic manifold M. A detailed description of $\mathcal{X}_M^{\mathrm{EP}}$ is given by the second author, Hodgson, Rubinstein and Segerman in [35, Section 6]. $\mathcal{X}_M^{\mathrm{EP}}$ generically consists of a single element. In Section 4.2 we will show the following.

Proposition 1.7 $\mathcal{X}_M^{\mathrm{EP}}$ *lies in a connected component of* $\mathcal{X}_{\rho_0}$. *Consequently, the value of* $\tau_{\mathcal{T}}$ *on* $\mathcal{X}_M^{\mathrm{EP}}$ *is a topological invariant* τ_M *of* M.

Admittedly, it would be more natural to show that $\tau_{\mathcal{T}}$ is constant on all of $\mathcal{X}_{\rho_0}$. Proposition 1.7 appears to be an artificial way to construct a much needed topological invariant of cusped hyperbolic 3–manifolds.

Our next conjecture compares our torsion τ_M with the nonabelian Reidemeister torsion τ_M^{R} of M with respect to the meridian defined in [62; 18].

Conjecture 1.8 *For all hyperbolic knot complements we have* $\tau_M^{\mathrm{R}} = \pm \tau_M$.

Numerical evidence for the above conjecture is presented in Appendix D using Dunfield's computation of τ_M^{R} via `SnapPy` [21]. Observe that both sides of the equation in Conjecture 1.8 are algebraic numbers (defined up to a sign) that are elements of the invariant trace field of M. Moreover, if M has a regular ideal triangulation with N ideal tetrahedra and its fundamental group is generated with r elements, then τ_M and τ_M^{R} are essentially given by the determinant of square matrices of size N and $3r - 3$, respectively. It is still unclear to us how to relate these two matrices or their determinants.

By definition, $\tau_M^{\mathrm{R}} \in E_M^*$. Thus, a mild but important corollary of Conjecture 1.8 is that τ_M is nonzero. This is a crucial ingredient, necessary for the definition of the higher loop invariants $S_{M,n}$ using perturbation theory.

1.5 The higher-loop invariants

In this section we define the higher loop invariants $S_{\mathcal{T},n}$ for $n \geq 2$. They are analyzed in detail in Section 5, using a state integral (5-2). The result, however, may be summarized as follows. Let us introduce a formal power series

$$(1\text{-}9)\quad \psi_\hbar(x;z) = \exp\left(\sum_{n,k,2n+k-2>0} \frac{\hbar^{n+k/2-1}(-x)^k B_n}{n!k!} \mathrm{Li}_{2-n-k}(z^{-1}) \right)$$
$$\in \mathbb{Q}(z)[\![x, \hbar^{1/2}]\!],$$

where B_n is the nth *Bernoulli number* (with $B_1 = +1/2$), and $L_m(z)$ is the mth polylogarithm. Note that $\text{Li}_m(z) \in (1-z)^{-m-1}\mathbb{Z}[z]$ is a rational function for all nonpositive integers m. This formal series comes from the asymptotic expansion of the quantum dilogarithm function after removal of its two leading asymptotic terms; see Barnes [4], Faddeev and Kashaev [28; 27]. The quantum dilogarithm is the Chern–Simons partition function of a single tetrahedron and its asymptotics are studied in detail in Section 5.

We fix an enhanced Neumann–Zagier datum $\hat{\beta}_{\mathcal{T}} = (z, A, B, f)$ of an oriented 1–cusped manifold M and a regular ideal triangulation $\mathcal{T}$ with N tetrahedra. Let $v = Af + Bf''$. We assume that

$$\det(B) \neq 0, \quad \tau_M \neq 0.$$

The condition $\det(B) \neq 0$ is always satisfied with a suitable labeling of shapes; see Lemma A.3. In that case, Lemma A.2 implies that

$$(1\text{-}10) \qquad \mathcal{H} = -B^{-1}A + \Delta_{z'},$$

is a symmetric matrix, where $\Delta_{z'} = \text{diag}(z'_1, \ldots, z'_N)$. We define

$$(1\text{-}11) \quad f_{\mathcal{T},\hbar}(x; z) = \exp\left(-\frac{\hbar^{1/2}}{2} x^T B^{-1} v + \frac{\hbar}{8} f^T B^{-1} A f\right) \prod_{i=1}^{N} \psi_\hbar(x_i, z_i)$$
$$\in \mathbb{Q}(z)[\![x, \hbar^{1/2}]\!],$$

where $x = (x_1, \ldots, x_N)^T$ and $z = (z_1, \ldots, z_N)$. Assuming that $\mathcal{H}$ is invertible, a formal power series $f_\hbar(x) \in \mathbb{Q}(z)[\![x, \hbar^{1/2}]\!]$ has a *formal Gaussian integration*, given by (cf Bessis, Itzykson and Zuber [6]),

$$(1\text{-}12) \qquad \langle f_\hbar(x) \rangle = \frac{\int dx\, e^{-1/2 x^T \mathcal{H} x} f_\hbar(x)}{\int dx\, e^{-1/2 x^T \mathcal{H} x}}.$$

This integration is defined by expanding $f_\hbar(x)$ as a series in x, and then formally integrating each monomial, using the quadratic form $\mathcal{H}^{-1}$ to contract x–indices pairwise.

Definition 1.9 With the above conventions, we define

$$(1\text{-}13) \qquad \exp\left(\sum_{n=2}^{\infty} S_{\mathcal{T},n}(z)\, \hbar^{n-1}\right) := \langle f_{\mathcal{T},\hbar}(x; z) \rangle.$$

Remark 1.10 Notice that the result involves only integral powers of $\hbar$ and each term is a rational function in the complex numbers z. Moreover, $S_{\mathcal{T},n} \in \tau_{\mathcal{T}}^{-3n+3} \mathbb{Q}[z, z', z'']$

for all $n \geq 2$. This follows from the fact that the connected Feynman diagrams that contribute to $S_{\mathcal{T},n}$ have at most $3n - 3$ edges and each edge (contracted by $\mathcal{H}^{-1}$) contributes a factor of $\det(\mathcal{H})^{-1}$. Thus, we can also write

$$(1\text{-}14) \qquad \exp\left(\sum_{n=2}^{\infty} S_{\mathcal{T},n} \hbar^{n-1} \right) = 1 + \sum_{n=1}^{\infty} \frac{\widetilde{S}_{\mathcal{T},n}}{\tau_{\mathcal{T}}^{3n}} \hbar^n,$$

where $\widetilde{S}_{\mathcal{T},n} \in \mathbb{Q}[z, z', z'']$. Experimentally, it appears that $\widetilde{S}_{\mathcal{T},n}$ have lower complexity than $S_{\mathcal{T},n}$; see Appendix D.

1.6 Feynman diagrams

A convenient way to organize the above definition is via Feynman diagrams, using Wick's Theorem to express each term $S_{\mathcal{T},n}$ as a finite sum of connected diagrams with at most n loops, where the number of loops of a connected graph is its first Betti number. This is well known and explained in detail, eg by Hori, Katz, Klemm, Pandharipande, Thomas, Vafa, Vakil and Zaslow in [45, Chapter 9], by Polyak in [60] and in [6].

The Feynman rules for computing the $S_{\mathcal{T},n}$, described in Section 5, turn out to be the following.[1] One draws all connected graphs D with vertices of all valencies, such that

$$(1\text{-}15) \qquad L(D) := (\#1\text{–vertices} + \#2\text{–vertices} + \#\text{loops}) \leq n.$$

In each diagram, the edges represent an $N \times N$ propagator

$$(1\text{-}16) \qquad\qquad \text{propagator}: \quad \Pi = \hbar \mathcal{H}^{-1},$$

while each k–vertex comes with an N–vector of factors $\Gamma_i^{(k)}$,

$$(1\text{-}17) \quad \Gamma_i^{(k)} = (-1)^k \sum_{p=\alpha_k}^{\alpha_k + n - L(D)} \frac{\hbar^{p-1} B_p}{p!} \mathrm{Li}_{2-p-k}(z_i^{-1}) + \begin{cases} -\frac{1}{2}(\boldsymbol{B}^{-1} v)_i & k = 1, \\ 0 & k \geq 2, \end{cases}$$

where $\alpha_k = 1$ (resp. 0) if $k = 1, 2$ (resp. $k \geq 3$). The diagram D is then evaluated by contracting the *vertex factors* $\Gamma_i^{(k)}$ with propagators, multiplying by a standard *symmetry factor*, and taking the $\hbar^{n-1}$ part of the answer. In the end, $S_{M,n}$ is the sum of evaluated diagrams, plus an additional *vacuum* contribution

$$(1\text{-}18) \qquad \Gamma^{(0)} = \frac{B_n}{n!} \sum_{i=1}^{N} \mathrm{Li}_{2-n}(z_i^{-1}) + \begin{cases} \frac{1}{8} f \cdot \boldsymbol{B}^{-1} A f & n = 2, \\ 0 & n \geq 3. \end{cases}$$

To illustrate the above algorithm, we give the explicit formulas for S_2 and S_3 below.

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[1] To derive these from (1-12), one should first rescale $x \to \hbar^{-1/2} x$.

1.7 The 2–loop invariant

The six diagrams that contribute to $S_{M,2}$ are shown in Figure 1, together with their symmetry factors.
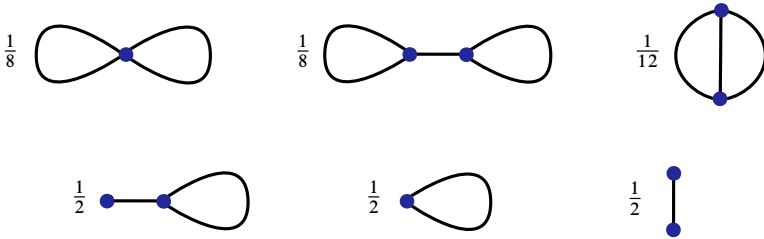

Figure 1: Diagrams contributing to $S_{M,2}$ with symmetry factors. The top row of diagrams have exactly two loops, while the bottom row have fewer loops and additional 1–vertices and 2–vertices.

Their evaluation gives the following formula for $S_{\mathcal{T},2}$:

$$(1\text{-}19) \quad S_{\mathcal{T},2} = \mathrm{coeff}\Big[\frac{1}{8}\Gamma_i^{(4)}(\Pi_{ii})^2 + \frac{1}{8}\Pi_{ii}\Gamma_i^{(3)}\Pi_{ij}\Gamma_j^{(3)}\Pi_{jj}$$
$$+ \frac{1}{12}\Gamma_i^{(3)}(\Pi_{ij})^3\Gamma_j^{(3)} + \frac{1}{2}\Gamma_i^{(1)}\Pi_{ij}\Gamma_j^{(3)}\Pi_{jj}$$
$$+ \frac{1}{2}\Gamma_i^{(2)}\Pi_{ii} + \frac{1}{2}\Gamma_i^{(1)}\Pi_{ij}\Gamma_j^{(1)}, \hbar\Big] + \Gamma^{(0)},$$

where all the indices i and j are implicitly summed from 1 to N and $\mathrm{coeff}[f(\hbar),\hbar]$ denotes the coefficient of $\hbar$ of a power series $f(\hbar)$. Concretely, the 2–loop contribution from the vacuum energy is $\Gamma^{(0)} = \frac{1}{8}f^T B^{-1}A f - \frac{1}{12}\sum_i z_i'$, and the four vertices that appear only contribute at leading order,

$$(1\text{-}20) \quad \Gamma_i^{(1)} = \frac{z_i' - (B^{-1}\nu)_i}{2}, \qquad \Gamma_i^{(2)} = \frac{z_i z_i'^2}{2},$$
$$\Gamma_i^{(3)} = -\frac{z_i z_i'^2}{\hbar}, \qquad \Gamma_i^{(4)} = -\frac{z_i(1+z_i)z_i'^3}{\hbar}.$$

We expect $S_{\mathcal{T},2}$ to be well-defined modulo $\mathbb{Z}/24$, and this is exactly what happens in hundreds of examples that we computed.

1.8 The 3–loop invariant

For the next invariant $S_{\mathcal{T},3}$, all the diagrams of Figure 1 contribute, collecting the coefficient of $\hbar^2$ of their evaluation. In addition, there are 34 new diagrams that satisfy

the inequality (1-15); they are shown in Figures 2 and 3. Calculations indicate that the 3–loop invariant $S_{\mathcal{T},3}$ is well-defined, independent of the regular triangulation $\mathcal{T}$. The invariants $S_{\mathcal{T},0}, \tau_{\mathcal{T}}, S_{\mathcal{T},2}, S_{\mathcal{T},3}$ have been programmed in `Mathematica` and take as input a Neumann–Zagier datum available from `SnapPy` [13].

For the 4–loop invariant, there are 291 new diagrams. A `python` implementation will be provided in the future. For large n, one expects about $n!^2 C^n$ diagrams to contribute to S_n.

Remark 1.11 Note that the n–loop invariant for $n \geq 3$ is independent of the combinatorial flattening and in fact depends only on $(\boldsymbol{B}^{-1}\boldsymbol{A}, \boldsymbol{B}^{-1}v, z)$.
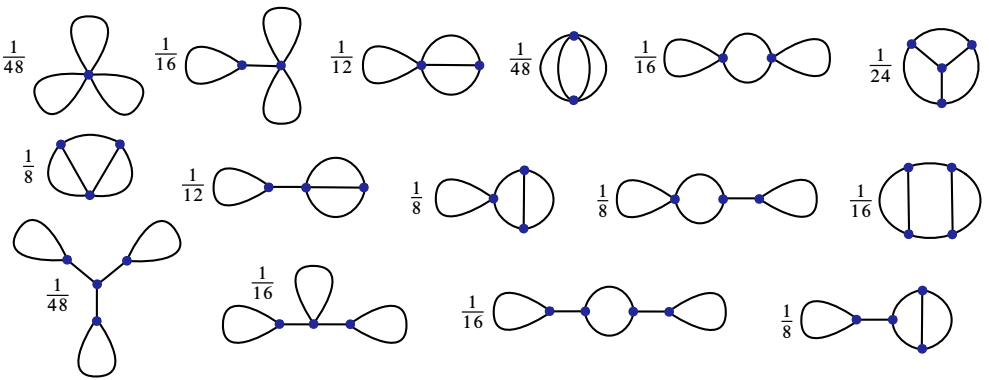

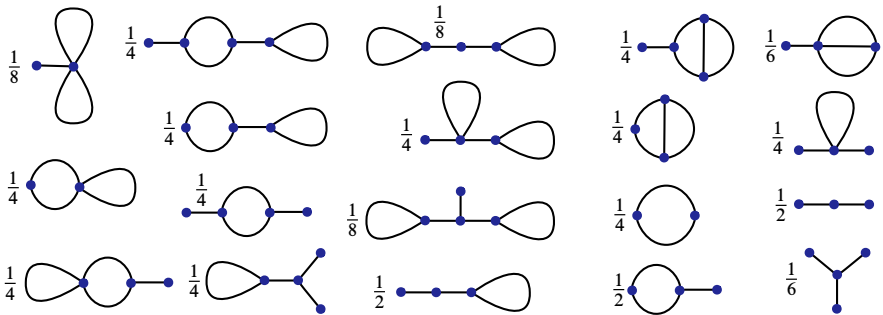Figure 2: Diagrams with three loops contributing to S_3

Figure 3: Diagrams with 1–vertices and 2–vertices contributing to S_3

1.9 The Feynman diagrams are stable graphs

During a Master's Class in Aarhus in February 2013, the second author observed that the Feynman graphs of our paper can be identified with the stable graphs which appear

in the *Topological Recursion* of Eynard–Orantin and the graphs that appear in the intersection theory of the moduli space of curves; see [26] and Aganagic, Bouchard and Klemm [1, Figure 1]. We thank Bertrand Eynard and Nicolas Orantin for delivering the Master's Class in Aarhus and Jorgen Andersen for organizing it.

Definition 1.12 A *stable graph* G is an abstract connected graph (with no cyclic order of the edges around a vertex) with the property that every vertex v of G is attached a genus g_v and a degree (ie, valency) n_v such that $2g_v - 2 + n_v > 0$. The total degree of a stable graph is given by $\sum_v (2g_v - 2 + n_v) = 2g - 2 + n$ where n is the number of external legs and g is defined to be the genus of G.

Let $\mathcal{G}(g, n)$ denote the (finite) set of stable graphs of genus g with n external legs. Then

$$S_g = \sum_{G \in \mathcal{G}(g,0)} \frac{1}{|\mathrm{Aut}(G)|} \langle G \rangle.$$

To explain where the genus comes in our Feynman graphs, observe that $\Gamma_i^{(k)}$ from Equation (1-17) for each vertex v has $k = n_v$, and $2g_v = p$ that contributes to $\hbar^{2g_v - 1} B_{2g_v} / (2g_v)! \mathrm{Li}_{2 - 2g_v - n_v}$. In other words, $2g_v$ extracts the monomial h^{p-1} from $\Gamma^{(k)}$. Notice that since $B_{\mathrm{odd}} = 0$ (for odd greater than 1) then we must have $p = \mathrm{even} = 2g_v$.

1.10 Generalizations

There are several natural extensions of the results presented above. First, one could attempt to prove the independence of the all-loop invariants $Z_{\mathcal{T}}(\hbar)$, including the entire series of $S_{\mathcal{T}, n}$s, under 2–3 moves and different choices of Neumann–Zagier datum. This was done nonrigorously in [15], but a full mathematical argument in the spirit of Theorems 1.3 and 1.4 is still missing. We hope to address this in future work.

In a different direction, one can extend the formulas for $\tau_{\mathcal{T}}$ and $S_{\mathcal{T}, n}$ to

- manifolds with multiple cusps,
- representations other than the discrete faithful,
- representations with nonparabolic meridian holonomy,
- nonhyperbolic manifolds.

The only truly necessary condition is that a 3–manifold M have a topological ideal triangulation $\mathcal{T}$ that — upon solving gluing equations and using a developing map — reproduces some desired representation $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$. We call such an ideal triangulation ρ–regular, and in Section 4 we will briefly discuss most of the above generalizations. In particular, we will demonstrate in Sections 4.6 and 5.5 how to extend $\tau_{\mathcal{T}}, S_{\mathcal{T},n}$ to rational functions on the character variety of a (topologically) cusped manifold. The generalization to multiple cusps is also quite straightforward, but left out mainly for simplicity of exposition.

2   Mechanics of triangulations

We begin by reviewing the gluing rules for ideal hyperbolic tetrahedra and the equations that determine their shape parameters. We essentially follow the classic [66; 58], but find it helpful to work with additive logarithmic (rather than multiplicative) forms of the gluing equations. Recall that all manifolds and all ideal triangulations are *oriented*.

2.1   Ideal tetrahedra

Combinatorially, an *oriented ideal tetrahedron* Δ is a topological ideal tetrahedron with three complex *shape parameters* (z, z', z'') assigned to pairs of opposite edges (Figure 4). The shapes always appear in the same cyclic order (determined by the orientation) around every vertex, and they satisfy

$$(2\text{-}1\mathrm{a}) \qquad\qquad zz'z'' = -1,$$

$$(2\text{-}1\mathrm{b}) \qquad\qquad z'' + z^{-1} - 1 = 0.$$

In other words, $z' = 1/(1-z)$ and $z'' = 1 - z^{-1}$. We call the tetrahedron *nondegenerate* if none of the shapes take values in $\{0, 1, \infty\}$, ie, $z, z', z'' \in \mathbb{C}^*\backslash\{1\}$. It is sufficient to impose this on a single one of the shapes.

Borrowing common terminology from the theory of normal surfaces, cf Burton [8], Kang and Rubenstein [47; 48] and Tillman [68], we define the *quadrilateral type* (in short, *quad type*) of Δ to be the distinguished pair of opposite edges labelled by z.
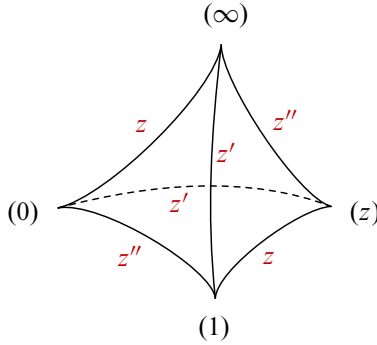
Figure 4: An ideal tetrahedron

Clearly, there is a threefold choice of quad type for any oriented ideal tetrahedron. Different choices correspond to a cyclic permutation of the vector $(z, z,', z'')$, which leaves relations (2-1a) invariant.

Geometrically, the shape parameters determine a $\mathrm{PSL}(2, \mathbb{C})$ structure on Δ. Equivalently, they determine a hyperbolic structure, possibly of negative volume. We can then describe the ideal hyperbolic tetrahedron Δ as the convex hull of four ideal points in hyperbolic three-space $\mathbb{H}^3$, whose cross ratio is z (or z', or z''). Each shape z fixes the complexified dihedral angle on the edge it labels, via

$$(2\text{-}2) \qquad z = \exp(\text{torsion} + i \text{ angle}),$$

and similarly for z', z''.

2.2 The gluing matrices

We now discuss an important combinatorial invariant of ideal triangulations, namely the gluing and Neumann–Zagier matrices, their symplectic properties, and the notion of a combinatorial flattening. Although these notions are motivated by hyperbolic geometry (namely the gluing of ideal tetrahedra around their faces and edges to describe a complete hyperbolic structure on a cusped manifold), we stress that these notions make sense for arbitrary 3–manifolds with torus boundary, and for triangulations whose gluing equations may not have solutions in $\mathbb{C} \setminus \{0, 1\}$.

Let M be an oriented one-cusped manifold with an ideal triangulation $\mathcal{T} = \{\Delta_i\}_{i=1}^N$ and a choice of quad type.

The choice of quad, combined with the orientation of $\mathcal{T}$ and M allow us to attach variables (Z_i, Z_i', Z_i'') to each tetrahedron Δ_i. An Euler characteristic argument shows

that the triangulation has N edges E_I, $I = 1, \ldots, N$. For each edge E_I we introduce a gluing equation of the form

$$(2\text{-}3) \qquad E_I: \quad \sum_{i=1}^{N} (G_{Ii} Z_i + G'_{Ii} Z'_i + G''_{Ii} Z''_i) = 2\pi i, \quad I = 1, \ldots, N,$$

where $G_{Ii} \in \{0, 1, 2\}$ (resp., G'_{Ii}, G''_{Ii}) is the number of times an edge of tetrahedron Δ_i with parameter Z_i (resp., Z'_i, Z''_i) is incident to the edge E_I in the triangulation. In addition, we impose the equations

$$(2\text{-}4) \qquad\qquad\qquad Z_i + Z'_i + Z''_i = i\pi,$$

for $i = 1, \ldots, N$. Equations (2-3) are not all independent. For a one-cusped manifold, every edge begins and ends at the cusp, which implies $\sum_{I=1}^{N} G_{Ii} = \sum_{I=1}^{N} G'_{Ii} = \sum_{I=1}^{N} G''_{Ii} = 2$, and therefore that the sum of the left-hand sides of Equations (2-3) equals $2\pi i N$. This is the only linear dependence in case of one cusp. In general, there is one relation per cusp of M, as follows from [56, Theorem 4.1].

An oriented peripheral simple closed curve μ (such as a meridian) on the boundary of M, in general position with the triangulation of the boundary torus that comes from $\mathcal{T}$, also gives rise to a gluing equation. We assume that the curve is simple (has no self intersections), and set the signed sum of edge parameters on the dihedral angles subtended by the curve to zero. A parameter is counted with a plus sign (resp. minus sign) if an angle is subtended in a counterclockwise (resp. clockwise) direction as viewed from the boundary. These rules are the same as described in [56], and demonstrated in Section 2.6 below.

Let us choose such a peripheral curve μ. We choose a meridian if M is a knot complement. The gluing equation associated to μ then takes the form

$$(2\text{-}5) \qquad \mu: \quad \sum_{i=1}^{N} (G_{N+1,i} Z_i + G'_{N+1,i} Z'_i + G''_{N+1,i} Z''_i) = 0,$$

with $G_{N+1,i}, G'_{N+1,i}, G''_{N+1,i} \in \mathbb{Z}$.

2.3 The Neumann–Zagier matrices

The matrices G, G' and G'' have both symmetry and redundancy. We have already observed that any one of the edge constraints (2-3) can be removed. Let us then ignore the edge $I = N$. We can also use (2-4) to eliminate one of the three shapes for each tetrahedron. We choose this canonically to be Z'_i, though which pair of edges is

labelled Z_i' depends on the choice of quad type for the tetrahedron. Then the first $N-1$ edge equations and the meridian (μ) equation are equivalent to

$$(2\text{-}6) \qquad \sum_{j=1}^{N}(A_{ij}Z_j + B_{ij}Z_j'') = i\pi v_i, \quad i = 1, \ldots, N,$$

where

$$(2\text{-}7) \qquad A_{ij} = \begin{cases} G_{ij} - G_{ij}' & I \neq N, \\ G_{N+1,j} - G_{N+1,j}' & i = N, \end{cases}$$

$$B_{ij} = \begin{cases} G_{ij}'' - G_{ij}' & I \neq N, \\ G_{N+1,j}'' - G_{N+1,j}' & i = N, \end{cases}$$

$$(2\text{-}8) \qquad v_i := \begin{cases} 2 - \sum_{j=1}^{N} G_{ij}' & i \neq N, \\ -\sum_{j=1}^{N} G_{N+1,j}' & i = N. \end{cases}$$

We will generally assume Z, Z'' and v to be column vectors, and we will write $AZ + BZ'' = i\pi v$. The matrices (G, G', G'') as well as (A, B, v) can easily be obtained from SnapPy [13], as is illustrated in Appendix D.

The Neumann–Zagier matrices A and B have a remarkable property: they are the top two blocks of a $2N \times 2N$ symplectic matrix [58]. It follows that

$$(2\text{-}9) \qquad AB^T = BA^T,$$

and that the $N \times 2N$ block (AB) has full rank. This symplectic property is crucial for defining the state integral of [15], for defining our formal power series invariant $\mathcal{Z}_M(\hbar)$, and for the combinatorial proofs of topological invariance of the 1–loop invariant. A detailed discussion of the symplectic properties of the Neumann–Zagier matrices A, B is given in Appendix A.

2.4 Combinatorial flattenings

We now have all ingredients to define what is a combinatorial flattening.

Definition 2.1 Given an ideal triangulation $\mathcal{T}$ of M, a *combinatorial flattening* is a collection of $3N$ integers $(f_i, f_i', f_i'') \in \mathbb{Z}^3$ for $i = 1, \ldots, N$ that satisfy

$$(2\text{-}10a) \qquad f_i + f_i' + f_i'' = 1, \quad i = 1, \ldots, N,$$

$$(2\text{-}10b) \qquad \sum_{i=1}^{N}(G_{Ii}f_i + G_{Ii}'f_i' + G_{Ii}''f_i'') = \begin{cases} 2 & I = 1, \ldots, N, \\ 0 & I = N+1. \end{cases}$$

Note that if we eliminate f' using Equation (2-10a), a flattening is a pair of vectors $(f, f'') \in \mathbb{Z}^{2N}$ that satisfies

$$(2\text{-}11) \qquad\qquad A f + B f'' = \nu.$$

Evidently, Equation (2-11) is a system of linear Diophantine equations. Neumann proved in [56, Lemma 6.1] that every ideal triangulation $\mathcal{T}$ has a flattening.

Remark 2.2 Combinatorial flattenings should not be confused with the (geometric) flattenings of [57, Definition 3.1]. The latter flattenings are coherent choices of logarithms for the shape parameters z, z', z'' of a complex solution to the gluing equations. On the other hand, our combinatorial flattenings are independent of a solution to the gluing equations. In the rest of the paper, the term *flattening* will mean a *combinatorial flattening* in the sense of Definition 2.1.

2.5 The shape solutions to the gluing equations

If we exponentiate the equations (2-4), and set $(z_i, z_i', z_i'') = (e^{Z_i}, e^{Z_i'}, e^{Z_i''})$, we obtain that (z_i, z_i', z_i'') satisfy Equation (2-1a). If we combine the exponentiated equations (2-6) with the nonlinear relation (2-1b) for each tetrahedron, we obtain

$$(2\text{-}12) \qquad\qquad z^A z''^B = z^A (1 - z^{-1})^B = (-1)^\nu,$$

where $z^A := \prod_j z_j^{A_{ij}}$. These N equations in N variables are just the gluing equations of Thurston [66] and Neumann and Zagier [58], and fully capture the constraints imposed by the gluing. For hyperbolic M, a triangulation $\mathcal{T}$ is regular precisely when one of the solutions to (2-12) corresponds to the complete hyperbolic structure.

2.6 Example: 4_1

As an example, we describe the enhanced Neumann–Zagier datum of the figure-eight knot complement M. It has a well known regular ideal triangulation $\mathcal{T}$ consisting of $N = 2$ tetrahedra, to which we assign logarithmic shape parameters (Z, Z', Z'') and (W, W', W'').

A map of the boundary of the cusp neighborhood is shown in Figure 5. We have chosen one of 3^2 possible cyclic labelings by Z s and W s (ie one of 3^2 possible quad types). Each of the edges intersects the cusp twice, so it is easy to read off from Figure 5 that the edge constraints (2-3) are

$$E_1 : \quad 2Z + Z'' + 2W + W'' = 2\pi i,$$
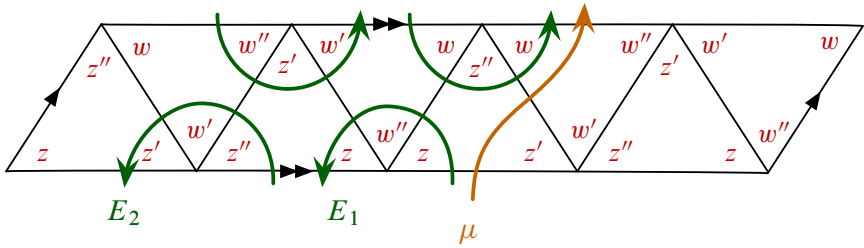$$E_2 : \quad 2Z' + Z'' + 2W' + W'' = 2\pi i.$$

Figure 5: The boundary of the cusp neighborhood for the figure-eight knot

The sum of the left-hand sides is automatically $4\pi i$, so we can choose to ignore the second constraint. If we choose the meridian path μ as in Figure 5, the meridian constraint (2-5) is

$$\mu: \quad -Z' + W = 0.$$

Putting together the first edge constraint and the meridian into matrices, we have

$$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} Z' \\ W' \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z'' \\ W'' \end{pmatrix} = i\pi \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Using $Z + Z' + Z'' = W + W' + W'' = i\pi$ to eliminate Z' and W', we get

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Z'' \\ W'' \end{pmatrix} = i\pi \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

From this last expression, we can read off

(2-13) $$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The two gluing equations (2-12) are then

(2-14) $$z^2 w^2 z'' w'' = 1, \quad z w z'' = -1,$$

with $z'' = 1 - z^{-1}$ and $w'' = 1 - w^{-1}$. The solution for the complete hyperbolic structure is $z = w = e^{i\pi/3}$.

Finally, a flattening $(f_z, f_z', f_z''; f_w, f_w', f_w'') \in \mathbb{Z}^6$, ie an integer solution to the equations $Af + Bf'' = v$ and $f + f' + f'' = 1$, is given by

(2-15) $$(f_z, f_z', f_z''; f_w, f_w', f_w'') = (0, 1, 0; 1, 0, 0).$$

It is easy to see that every flattening has the form $(a, b, 1-a-b; b, a, 1-a-b)$ for integers a, b.

3 Topological invariance of our torsion

Given a one-cusped hyperbolic manifold M with regular triangulation $\mathcal{T} = \{\Delta_i\}_{i=1}^N$ and Neumann–Zagier datum $\hat{\beta}_\mathcal{T} = (z, A, B, f)$, we have proposed the nonabelian torsion is given by

$$(3\text{-}1) \qquad \tau_\mathcal{T} := \pm\frac{1}{2}\det(A\Delta_{z''} + B\Delta_z^{-1})z^{f''}z''^{-f} \in E_M/\{\pm 1\},$$

where $\Delta_z = \mathrm{diag}(z_1, \ldots, z_N)$, and similarly for $\Delta_{z''}$. Since $(z, z', z'') \in E_M$ we must have $\tau_\mathcal{T} \in E_M$ as well.

After a brief example of how the formula (3-1) works, we will proceed to prove Theorems 1.3 and 1.4 on the topological invariance of $\tau_\mathcal{T}$. We saw in Section 2 that the Neumann–Zagier datum depends not only on a triangulation $\mathcal{T}$, but also on a choice of

(1) quad type for $\mathcal{T}$,

(2) one edge of $\mathcal{T}$ whose gluing equation is redundant,

(3) normal meridian path,

(4) flattening f.

We will begin by showing $\tau_\mathcal{T}$ is independent of these four choices, and then show it is invariant under 2–3 moves, so long as the 2–3 moves connect two regular triangulations.

The four choices here are independent, and can be studied in any order. However, in order to prove independence of flattening, it is convenient to use a quad type for which the matrix B is nondegenerate. Such a quad type can always be found (Lemma A.3), but is not automatic. Therefore, we will first show invariance under change of quad type, and then proceed to the other choices. It is interesting to note that of all the arguments that follow (including the 2–3 move), independence of flattening is the only one that requires the use of the full gluing equations $z^A z''^B = (-1)^\nu$.

3.1 Example: 4_1 continued

To illustrate the Equation (1-8), consider the figure-eight knot complement again. From Section 2.6, we already have one possible choice for the Neumann–Zagier

matrices (2-13) and a generic flattening (2-15). We use them to obtain

$$
\begin{aligned}
\pm\tau_{\mathbf{4_1}} &= \frac{1}{2}\det\left[\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}z'' & 0\\ 0 & w''\end{pmatrix} + \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}z^{-1} & 0\\ 0 & w^{-1}\end{pmatrix}\right]w''^{-1}\\[4pt]
&= \frac{1}{2w''}\det\begin{pmatrix}2z'' + z^{-1} & 2w'' + w^{-1}\\ z'' + z^{-1} & w''\end{pmatrix}\\[4pt]
&= \frac{1}{2w''}\det\begin{pmatrix}z'' + 1 & w'' + 1\\ 1 & w''\end{pmatrix}\\[4pt]
&= \frac{1}{2}(z'' - w''^{-1}) = \frac{1}{2}\sqrt{-3},
\end{aligned}
$$

(3-2)

where at intermediate steps we used $z'' + z^{-1} - 1 = w'' + w^{-1} - 1 = 0$, and at the end we substituted the discrete faithful solution $z = z'' = w = w'' = e^{i\pi/3}$.

The invariant $\tau_{\mathbf{4_1}}$ belongs to the invariant trace field $E_{\mathbf{4_1}} = \mathbb{Q}(\sqrt{-3})$, and agrees with the torsion of the figure-eight knot complement [21].

3.2 Independence of a choice of quad type

Now, let us fix a manifold M, a triangulation $\mathcal{T}$ with N tetrahedra, and an enhanced Neumann–Zagier datum $\hat{\beta}_{\mathcal{T}} = (z, A, B, f)$.

To prove independence of quad type, it is sufficient to check that $\tau_{\mathcal{T}}$ is invariant under a cyclic permutation of the first triple of shape parameters (z_1, z_1', z_1''), while holding fixed the choice of meridian loop and redundant edge. Let us write $z = (z_1, \ldots, z_N)$, $v = (v_1, \ldots, v_N)^T$, $f = (f_1, \ldots, f_N)^T$ and

(3-3) $$A = (a_1, a_2, \ldots, a_N), \quad B = (b_1, b_2, \ldots, b_N),$$

in column notation. After the permutation, a new Neumann–Zagier datum is given by $(\tilde{z}, \tilde{A}, \tilde{B}, \tilde{f})$ where

(3-4) $$\tilde{z} = (z_1', z_2, \ldots, z_N), \quad \tilde{z}' = (z_1'', z_2', \ldots, z_N'), \quad \tilde{z}'' = (z_1, z_2'', \ldots, z_N''),$$

$$\tilde{A} = (-b_1, a_2, \ldots, a_N), \quad \tilde{B} = (a_1 - b_1, b_2, \ldots, b_N),$$

(3-5)

$$\tilde{v} = (n_1 - b_1, n_2, \ldots, n_N)^T.$$

The new shapes satisfy $\tilde{z}^{\tilde{A}}\tilde{z}''^{\tilde{B}} = (-1)^{\tilde{v}}$. We also naturally obtain a new flattening $(\tilde{f}, \tilde{f}', \tilde{f}'')$ by permuting

(3-6) $$\tilde{f} = (f_1', f_2, \ldots, f_N)^T, \quad \tilde{f}' = (f_1'', f_2', \ldots, f_N')^T,$$

$$\tilde{f}'' = (f_1, f_2'', \ldots, f_N'')^T;$$

this is an integer solution to $\tilde{A}\tilde{f} + \tilde{B}\tilde{f}'' = \tilde{v}$ and $\tilde{f} + \tilde{f}' + \tilde{f}'' = 1$.

The torsion $\tau_{\mathcal{T}}$ (1-8) consists of two parts, a determinant and a monomial correction. By making use of the relations $z_1 + z_1'^{-1} - 1 = 0$ and $z_1 z_1' z_1'' = -1$, we find the determinant with the permuted Neumann–Zagier datum to be

$$
\begin{aligned}
(3\text{-}7) \quad \det(&\widetilde{A}\Delta_{\widetilde{z}''} + \widetilde{B}\Delta_{\widetilde{z}}^{-1}) \\
&= \det(-b_1 z_1 + (a_1 - b_1)z_1'^{-1}, a_2 z_2'' + b_2 z_2^{-1}, \dots, a_N z_N'' + b_N z_N^{-1}) \\
&= \det(a_1 z_1'^{-1} - b_1, a_2 z_2'' + b_2 z_2^{-1}, \dots, a_N z_N'' + b_N z_N^{-1}) \\
&= -z_1 \det(a_1 z_1'' + b_1 z_1^{-1}, a_2 z_2'' + b_2 z_2^{-1}, \dots, a_N z_N'' + b_N z_N^{-1}) \\
&= -z_1 \det(A\Delta_{z''} + B\Delta_z^{-1}).
\end{aligned}
$$

By simply using $z_1 z_1' z_1'' = -1$ and $f_1 + f_1' + f_1'' = 1$, we also see that the monomial correction transforms as

$$
\begin{aligned}
(3\text{-}8) \qquad \widetilde{z}^{\widetilde{f}''} \widetilde{z}''^{-\widetilde{f}} &= z^{f''} z''^{-f} \frac{z_1'^{f_1} z_1^{-f_1'}}{z_1^{f_1''} z_1''^{-f_1}} \\
&= z^{f''} z''^{-f} (-1)^{f_1} \frac{(z_1 z_1'')^{-f_1} z_1^{f_1 + f_1'' - 1}}{z_1^{f_1''} z_1''^{-f_1}} \\
&= z^{f''} z''^{-f} (-1)^{f_1} z_1^{-1}.
\end{aligned}
$$

The extra factors $z_1^{\pm 1}$ in the two parts of the torsion precisely cancel each other, leading in the end to

$$
(3\text{-}9) \quad \det(\widetilde{A}\Delta_{\widetilde{z}''} + \widetilde{B}\Delta_{\widetilde{z}}^{-1})\widetilde{z}^{\widetilde{f}''} \widetilde{z}''^{-\widetilde{f}} = (-1)^{f_1 + 1} \det(A\Delta_{z''} + B\Delta_z^{-1})z^{f''} z''^{-f}.
$$

This is just as desired, showing that the torsion is invariant up to a sign.

3.3 Independence of a choice of edge

We fix $M, \mathcal{T}, \hat{\beta}_{\mathcal{T}} = (z, A, B, f)$. In order to choose matrices A, B, we must ignore the redundant gluing equation corresponding to an edge of $\mathcal{T}$. This was discussed in Section 2.3. Suppose, then, that we choose a different edge to ignore. For example, if we choose the $(N-1)^{\text{st}}$ rather than the Nth (and keep the same quad type and meridian path), then we obtain new Neumann–Zagier matrices $\widetilde{A}, \widetilde{B}$, which are related to the original ones as

$$
(3\text{-}10) \qquad \widetilde{A} = P_{(N-1,N)}A, \quad \widetilde{B} = P_{(N-1,N)}B,
$$

where

$$(3\text{-}11) \qquad P_{(N-1,N)} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ -1 & -1 & \cdots & -1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Similarly, eliminating the Ith rather than the Nth edge constraint is implemented by multiplying with a matrix $P_{(I,N)}$ whose I^{th} row is filled with -1s. Any such matrix satisfies $\det P_{(I,N)} = -1$.

In the formula for $\tau_{\mathcal{T}}$, only the determinant part is affected by a change of edge. Then

$$(3\text{-}12) \qquad \det(\tilde{A}\Delta_{z''} + \tilde{B}\Delta_z^{-1}) = \det(P_{(I,N)}(A\Delta_{z''} + B\Delta_z^{-1}))$$
$$= -\det(A\Delta_{z''} + B\Delta_z^{-1}),$$

leading to invariance of $\tau_{\mathcal{T}}$, up to the usual sign.

3.4 Independence of a choice of meridian path

Recall that an ideal triangulation on M induces a triangulation of its boundary torus ∂M. Consider two simple closed meridian loops in ∂M in general (normal) position with respect to the triangulation of ∂M. Recall that these paths are drawn on the triangulated 2–dimensional torus ∂M where faces of tetrahedra correspond to edges in the 2–dimensional triangulation, and edges of tetrahedra to vertices. In particular, for a one-cusped manifold M, every edge of the triangulation intersects a pair of vertices on the boundary ∂M.

We can deform one of our meridian paths into the other by using repeated applications of the fundamental move shown in Figure 6, locally pushing a section of the path across a vertex of ∂M. Thus, it suffices to assume that the two paths only differ by one such move. Suppose that we cross the Ith edge (by Section 3.3 we may assume that $I \neq N$), which has a combinatorial gluing constraint

$$(3\text{-}13) \qquad X_I := \sum_{i=1}^{N}(G_{Ii}Z_i + G'_{Ii}Z'_i + G''_{Ii}Z''_i) = 2\pi i,$$

and that the two tetrahedra where the paths enter and exit the vicinity of the edge have parameters (Z, Z', Z'') and (W, W', W''), as in the figure. We do not exclude the possibility that (Z, Z', Z'') and (W, W', W'') both coincide with the same triple
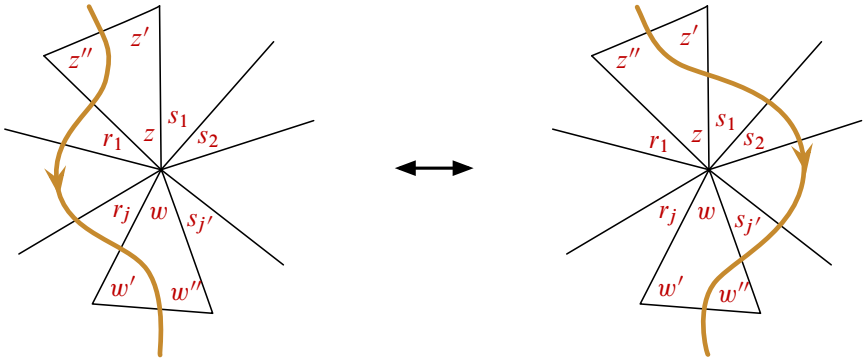
Figure 6: The fundamental move for changing a meridian path. Here, we deform through an edge E_I with gluing constraint $X_I = Z + W + R_1 + \cdots + R_j + S_1 + \cdots + S_{j'} = 2\pi i$.

(Z_i, Z_i', Z_i''), in some cyclic permutation. Then the difference in the logarithmic meridian equations (2-5) for the two paths will be

$$(3\text{-}14) \qquad \delta_\mu = \pm(X_I - (Z + Z' + Z'') - (W + W' + W'')).$$

Note that two logarithmic meridian constraints that differ by (3-14) are compatible and equivalent, since upon using the additional equations $X_I = 2\pi i$ and $Z + Z' + Z'' = W + W' + W'' = i\pi$, we find that $\delta_\mu = 0$. A discretized version of this observation demonstrates that the same flattening satisfies both discretized meridian constraints.[2]

If we compute matrices A, B using one meridian path and $\widetilde{A}, \widetilde{B}$ using the other — keeping quad type, flattening, and edge the same — the change (3-14) implies

$$(3\text{-}15) \qquad \widetilde{A} = P_I^{(\mu)\pm 1} A, \quad \widetilde{B} = P_I^{(\mu)\pm 1} B,$$

where $P_I^{(\mu)}$ is the $SL(N, \mathbb{Z})$ matrix

$$(3\text{-}16) \qquad P_I^{(\mu)} = I + E_{NI},$$

ie the identity plus an extra entry "1" in the Nth (meridian) row and I^{th} column. Since $\det P_I^{(\mu)} = 1$, this immediately shows that $\det(\widetilde{A}\Delta_{z''} + \widetilde{B}\Delta_z^{-1}) = \det(A\Delta_{z''} + B\Delta_z^{-1})$, and so the a change in the meridian path cannot affect $\tau_{\mathcal{T}}$.

---

[2]Note that this would not be the case if we allowed self-intersections of the meridian loops.

3.5 Independence of a choice of flattening

Now suppose that we choose two flattenings (f, f', f'') and $(\widetilde{f}, \widetilde{f}', \widetilde{f}'')$, both satisfying

$$(3\text{-}17) \qquad \boldsymbol{A}f + \boldsymbol{B}f'' = \nu, \quad f + f' + f'' = 1,$$

$$(3\text{-}18) \qquad \boldsymbol{A}\widetilde{f} + \boldsymbol{B}\widetilde{f}'' = \nu, \quad \widetilde{f} + \widetilde{f}' + \widetilde{f}'' = 1.$$

We may assume that we have a quad type with $\boldsymbol{B}$ nondegenerate. Indeed, by the result of Section 3.2, flattening invariance in one quad type implies flattening invariance in any quad type. Moreover, by Lemma A.3 of Appendix A, a quad type with nondegenerate $\boldsymbol{B}$ always exists. We also note that when $\boldsymbol{B}$ is invertible the matrix $\boldsymbol{B}^{-1}\boldsymbol{A}$ is symmetric (Lemma A.2).

The determinant in $\tau_{\mathcal{T}}$ is insensitive to the change of flattening. The monomial, on the other hand, can be manipulated as follows. Let us choose logarithms (Z, Z', Z'') of the shape parameters such that $\boldsymbol{A}Z + \boldsymbol{B}Z'' = i\pi\nu$. Then, assuming that $\boldsymbol{B}$ is nondegenerate, we compute

$$\frac{z^{\widetilde{f}''} z''^{-\widetilde{f}}}{z^{f''} z''^{-f}} = \exp[Z \cdot (f'' - \widetilde{f}'') - Z'' \cdot (f - \widetilde{f})]$$

$$= \exp[-Z \cdot \boldsymbol{B}^{-1}\boldsymbol{A}(f - \widetilde{f}) - (i\pi\boldsymbol{B}^{-1}\nu - \boldsymbol{B}^{-1}\boldsymbol{A}Z) \cdot (f - \widetilde{f})]$$

$$= \exp[-i\pi\boldsymbol{B}^{-1}\nu \cdot (f - \widetilde{f})]$$

$$= \exp[-i\pi f'' \cdot (f - \widetilde{f}) - i\pi\boldsymbol{B}^{-1}\boldsymbol{A}f \cdot (f - \widetilde{f})]$$

$$= \exp[-i\pi f'' \cdot (f - \widetilde{f}) + i\pi f \cdot (f'' - \widetilde{f}'')]$$

$$= \exp[i\pi(f'' \cdot \widetilde{f} - f \cdot \widetilde{f}'')] = \pm 1.$$

Therefore, the monomial can change at most by a sign, and $\tau_{\mathcal{T}}$ is invariant as desired. This completes the proof of Theorem 1.3. □

3.6 Invariance under 2–3 moves

We finally come to the proof of Theorem 1.4, ie the invariance of $\tau_{\mathcal{T}}$ under 2–3 moves. We set up the problem as in Figure 7. Namely, we suppose that M has two different regular triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, with N and $N+1$ tetrahedra, respectively, which are related by a local 2–3 move. Let us denote the respective (triples of) shape parameters as

$$(3\text{-}19) \qquad Z := (X_1, X_2, Z_3, \ldots, Z_N), \quad \widetilde{Z} := (W_1, W_2, W_3, Z_3, \ldots, Z_N).$$

We fix a quad type, labeling the five tetrahedra involved in the 2–3 move as in the figure. We will also assume that when calculating Neumann–Zagier matrices A and B, we choose to ignore an edge that is *not* the central one of the 2–3 *bipyramid*.
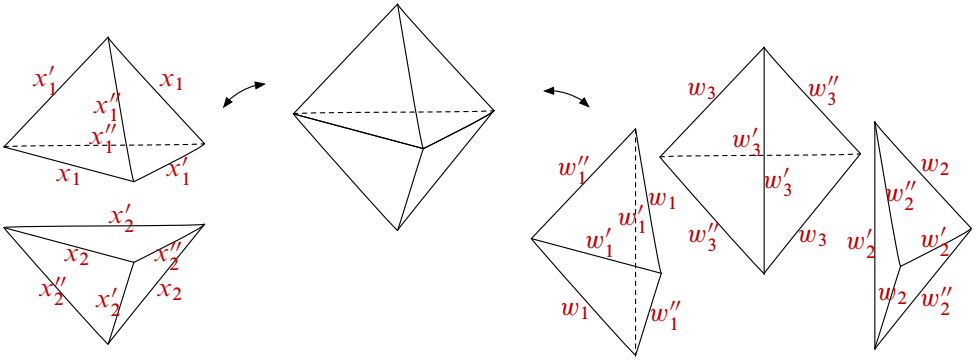

Figure 7: The geometry of the 2–3 move: a bipyramid split into two tetrahedra for triangulation $\bigcup_{i=1}^{N} \Delta_i$, and three for triangulation $\bigcup_{i=1}^{N+1} \widetilde{\Delta}_i$.

There are nine linear relations among the shapes of the tetrahedra involved in the move; three come from adding dihedral angles on the equatorial edges of the bipyramid

$$(3\text{-}20) \qquad W_1' = X_1 + X_2, \quad W_2' = X_1' + X_2'', \quad W_3' = X_1'' + X_2',$$

and six from the longitudinal edges

$$(3\text{-}21) \qquad \begin{aligned} X_1 &= W_2 + W_3'', \quad X_1' = W_3 + W_1'', \quad X_1'' = W_1 + W_2'', \\ X_2 &= W_2'' + W_3, \quad X_2' = W_1'' + W_2, \quad X_2'' = W_3'' + W_1. \end{aligned}$$

Moreover, due to the central edge of the bipyramid, there is an extra gluing constraint in $\widetilde{\mathcal{T}}$:

$$(3\text{-}22) \qquad W_1' + W_2' + W_3' = 2\pi i.$$

After exponentiating the relations (3-20)–(3-22), and also using $z_i z_i' z_i'' = -1$ and $z_i'' + z_i^{-1} - 1 = 0$ for every tetrahedron Δ_i and $\widetilde{\Delta}_i$, we find a birational map between the shape parameters in the two triangulations. Explicitly,

$$(3\text{-}23) \qquad \left\{ w_1' = x_1 x_2, \; w_2' = \frac{1 - x_2^{-1}}{1 - x_1}, \; w_3' = \frac{1 - x_1^{-1}}{1 - x_2} \right\} \quad \text{or}$$

$$\left\{ x_1 = \frac{1 - w_2'^{-1}}{1 - w_3'}, \; x_2 = \frac{1 - w_3'^{-1}}{1 - w_2'} \right\}.$$

Note that the birational map is well-defined and one-to-one as long as no shape parameters $(x_1, x_2, w_1, w_2, w_3)$ equal 0, 1, or ∞. This condition is satisfied *so long as* triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ are both regular. (A necessary condition is that no univalent edges are created on one side or the other of the 2–3 move; this is also sufficient when considering the discrete faithful representation of M.)

We must also choose a flattening in the two triangulations. Let us suppose for $\bigcup_{i=1}^{N+1} \widetilde{\Delta}_i$ we have a flattening with (triples of) integer parameters $\widetilde{f} = (d_1, d_2, d_3, f_3, \ldots, f_N)$. This automatically determines a flattening $f = (e_1, e_2, f_3, \ldots, f_N)$ for the $\bigcup_{i=1}^{N} \Delta_i$ triangulation, by simply setting

$$
(3\text{-}24) \qquad
\begin{aligned}
e_1 &= d_2 + d_3'', & e_1' &= d_3 + d_1'', & e_1'' &= d_1 + d_2'', \\
e_2 &= d_2'' + d_3, & e_2' &= d_1'' + d_2, & e_2'' &= d_3'' + d_1.
\end{aligned}
$$

This is a discretized version of the six longitudinal relations (3-21). One can check that expected relations such as $e_1 + e_1' + e_1'' = 1$ are satisfied by virtue of the discretized edge constraint $d_1' + d_2' + d_3' = 2$ (cf (3-22)).

We have all the data needed to calculate $\tau_{\mathcal{T}}$. Let us start with determinants. In the triangulation $\bigcup_{i=1}^{N} \Delta_i$, we write the matrices A and B schematically in columns as

$$
(3\text{-}25) \qquad A = (a_1, a_2, a_i), \quad B = (b_1, b_2, b_i),
$$

with a_i meaning $(a_3, a_4, \ldots, a_N)$ and similarly for b_i. This leads to a determinant

$$
(3\text{-}26) \qquad \det\left(A \Delta_{z''} + B \Delta_z^{-1}\right) = \det(a_1 x_1'' + \tfrac{b_1}{x_1}, a_2 x_2'' + \tfrac{b_2}{x_2}, a_i z_i'' + \tfrac{b_i}{z_i}).
$$

Alternatively, in the triangulation $\bigcup_{i=1}^{N+1} \widetilde{\Delta}_i$, the matrices $\widetilde{A}$ and $\widetilde{B}$ have one extra row and one extra column. The extra gluing condition (3-22) causes the extra row in both $\widetilde{A}$ and $\widetilde{B}$ to contain three -1s. Altogether, the matrices take the form

$$
(3\text{-}27) \qquad \widetilde{A} = \begin{pmatrix} -1 & -1 & -1 & 0 \\ b_1 + b_2 & a_1 & a_2 & a_i \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 0 & a_2 + b_1 & a_1 + b_2 & b_i \end{pmatrix},
$$

so that

$$
\begin{aligned}
\widetilde{A} \Delta_{\widetilde{z}''} + \widetilde{B} \Delta_{\widetilde{z}}^{-1} &= \begin{pmatrix} -w_1'' - \tfrac{1}{w_1} & -w_2'' - \tfrac{1}{w_2} & -w_3'' - \tfrac{1}{w_3} & 0 \\ (b_1 + b_2) w_1'' & a_1 w_2'' + \tfrac{a_2 + b_1}{w_2} & a_2 w_3'' + \tfrac{a_1 + b_2}{w_3} & a_i z_i'' + \tfrac{b_i}{z_i} \end{pmatrix} \\
&= \begin{pmatrix} -1 & -1 & -1 & 0 \\ (b_1 + b_2) w_1'' & a_1 w_2'' + \tfrac{a_2 + b_1}{w_2} & a_2 w_3'' + \tfrac{a_1 + b_2}{w_3} & a_i z_i'' + \tfrac{b_i}{z_i} \end{pmatrix}.
\end{aligned}
$$

It is then straightforward to check, using the map (3-23), that

$$
\text{(3-28)} \quad (\widetilde{A}\Delta_{\widetilde{z}''} + \widetilde{B}\,\Delta_{\widetilde{z}}^{-1})
\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{w_3' x_2''}{x_1''} & 0 \\ 0 & \frac{w_2' x_1''}{x_2''} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}
$$

$$
= \begin{pmatrix} -1 & 0 \\ * & A\Delta_{z''} + B\Delta_z^{-1} \end{pmatrix}.
$$

The determinant of the last matrix on the left hand side is $1 - w_2' w_3' = 1 - w_1'^{-1} = w_1$. Therefore,

$$
\text{(3-29)} \qquad \det(\widetilde{A}\Delta_{\widetilde{z}''} + \widetilde{B}\,\Delta_{\widetilde{z}}^{-1}) = -w_1^{-1}\det(A\Delta_{z''} + B\Delta_z^{-1}).
$$

We should also consider the monomial correction. However, with flattenings related as in (3-24), and with shapes related by the exponentiated version of (3-21), it is easy to check that

$$
\text{(3-30)} \qquad \widetilde{z}^{\widetilde{f}''}\widetilde{z}''^{-\widetilde{f}} = w_1(-1)^{d_2'' - e_1''}z^{f''}z''^{-f}.
$$

We have thus arrived at the desired result; by combining (3-29) and (3-30), we find

$$
\text{(3-31)} \qquad \det(\widetilde{A}\Delta_{\widetilde{z}''} + \widetilde{B}\,\Delta_{\widetilde{z}}^{-1})\widetilde{z}^{\widetilde{f}''}\widetilde{z}''^{-\widetilde{f}} = \pm\det(A\Delta_{z''} + B\Delta_z^{-1})z^{f''}z''^{-f},
$$

so we have that $\tau_{\mathcal{T}}$ is invariant under the 2–3 move. This completes the proof of Theorem 1.4. $\qquad\qquad\square$

4 Torsion on the character variety

Having given a putative formula for the nonabelian torsion of a cusped hyperbolic manifold M at the discrete faithful representation ρ_0, it is natural to ask whether the formula generalizes to other settings. In this section, we extend the torsion formula to general representations $\rho\colon \pi_1(M) \longrightarrow (\mathrm{P})\mathrm{SL}(2,\mathbb{C})$ for manifolds M with torus boundary, essentially by letting the shapes z be functions of ρ. We also find that some special results hold when M is hyperbolic and the representations lie on the geometric component X_M^{geom} of the $\mathrm{SL}(2,\mathbb{C})$ character variety.

We will begin with a short review of what it means for a combinatorial ideal triangulation to be *regular* with respect to a general representation ρ. We will also finally prove Proposition 1.7. Recall that Proposition 1.7 identified a canonical connected subset

$\mathcal{X}_M^{\mathrm{EP}}$ of the set of regular triangulations $\mathcal{X}_{\rho_0}$ of a hyperbolic 3–manifold M. This result allowed us to construct the topological invariant τ_M.

We then proceed to define an enhanced Neumann–Zagier datum $\widehat{\beta}_{\mathcal{T}} = (z, A, B, f)$ suitable for a general representation ρ, and propose a generalization of the torsion formula:

$$(4\text{-}1) \qquad \tau_{\mathcal{T}}(\rho) := \pm \frac{1}{2} \det(A\,\Delta_{z''} + B\,\Delta_z^{-1}) z^{f''} z''^{-f}.$$

This formula looks identical to (1-8). However, the shape parameters here are promoted to functions $z \to z(\rho)$ of the representation ρ, which satisfy a well known deformed version of the gluing equations. Moreover, the flattening f is slightly more restricted than it was previously. We will prove the following in Section 4.5.

Theorem 4.1 *The formula for $\tau_{\mathcal{T}}(\rho)$ is independent of the choice of enhanced Neumann–Zagier datum, and is invariant under 2–3 moves connecting ρ–regular triangulations.*

When M is hyperbolic, it turns out that ρ_0–regular triangulations are ρ–regular for all but finitely many representations $\rho \in X_M^{\mathrm{geom}}$. Then we can create a topological invariant τ_M that is a function on X_M^{geom} just as in Proposition 1.7, by evaluating $\tau_{\mathcal{T}}(\rho)$ on any triangulation in the canonical subset $\mathcal{X}_M^{\mathrm{EP}} \subseteq \mathcal{X}_{\rho_0}$.

In general, there is a rational map from the character variety X_M to the zero-locus Y_M of the A–polynomial $A_M(\ell, m) = 0$ (see Cooper, Culler, Gillet, Long and Shalen [12]), for any M with torus boundary. Therefore, the shapes z and the torsion $\tau_{\mathcal{T}}$ are algebraic functions on components of the A–polynomial curve Y_M. When M is hyperbolic and $\rho \in X_M^{\mathrm{geom}}$, somewhat more is true: the shapes are *rational* function on the geometric component Y_M^{geom} (Proposition B.1). Then

$$(4\text{-}2) \qquad \tau_M \in C(Y_M^{\mathrm{geom}}) = \mathbb{Q}(m)[\ell]/(A_M^{\mathrm{geom}}(\ell, m)),$$

where $A_M^{\mathrm{geom}}(\ell, m)$ is the geometric factor of the A–polynomial. We will give a simple example of the function τ_M for the figure-eight knot in Section 4.6.

4.1 A review of ρ–regular ideal triangulations

In this section we discuss the ρ–regular ideal triangulations that are needed to generalize our torsion invariant. Let M denote a 3–manifold with nonempty boundary and let $\rho \colon \pi_1(M) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ be a $\mathrm{PSL}(2, \mathbb{C})$ representation of its fundamental group. Let $\mathcal{X}$ denote the set of combinatorial ideal triangulations $\mathcal{T}$ of M. Matveev and Piergallini independently showed that every two elements of $\mathcal{X}$ with at least two ideal

tetrahedra are connected by a sequence of 2–3 moves (and their inverses) [52; 59]. For a detailed exposition, see [53] and Benedetti and Petronio [5].

Given an ideal triangulation $\mathcal{T}$, let $V_{\mathcal{T}}$ denote the affine variety of nondegenerate solutions (ie, solutions in $\mathbb{C} \setminus \{0, 1\}$) of the gluing equations of $\mathcal{T}$ corresponding to its edges. There is a developing map

$$(4\text{-}3) \qquad\qquad\qquad V_{\mathcal{T}} \longrightarrow X_M,$$

where $X_M := \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2, \mathbb{C}))/\mathrm{PSL}(2, \mathbb{C})$ denotes the affine variety of all $\mathrm{PSL}(2, \mathbb{C})$ representations of $\pi_1(M)$.

Definition 4.2 Fix a $\mathrm{PSL}(2, \mathbb{C})$–representation of M. We say that $\mathcal{T} \in \mathcal{X}$ is ρ–*regular* if ρ is in the image of the developing map (4-3).

Let $\mathcal{X}_\rho \subset \mathcal{X}$ denote the set of all ρ–regular ideal triangulations of M. When M is hyperbolic, let ρ_0 denote its discrete faithful representation ρ_0 and let $X_M^{\mathrm{geom}} \subset X_M$ denote the geometric component of its character variety [66; 58]. We then have the following result.

Lemma 4.3   (a) $\mathcal{T} \in \mathcal{X}_{\rho_0}$ *if and only if* $\mathcal{T}$ *has no homotopically peripheral (ie, univalent) edges.*

   (b)   *If* $\mathcal{T} \in \mathcal{X}_{\rho_0}$, *then* $\mathcal{T} \in \mathcal{X}_\rho$ *for all but finitely many* $\rho \in X_M^{\mathrm{geom}}$.

Proof   Part (a) has been observed several times; see [11], [7, Section 10.3], [69, Theorem 2.3] and also [22, Remark 3.4]. For part (b), fix $\mathcal{T} \in \mathcal{X}_{\rho_0}$. Observe that $\mathcal{T}$ is ρ–regular if the image of every edge[3] of $\mathcal{T}$ under ρ does not commute with the image under ρ of the peripheral subgroup of M. This is an algebraic condition on ρ, and moreover, when $\rho \in X_M^{\mathrm{geom}}$ is analytically nearby ρ_0, the condition is satisfied. It follows that the set of points of X_M^{geom} that satisfy the above condition is Zariski open. On the other hand, X_M^{geom} is an affine curve [66; 58]. It follows that $\mathcal{T}$ is ρ–regular for all but finitely many $\rho \in X_M^{\mathrm{geom}}$.   □

4.2   The Epstein–Penner cell decomposition and its triangulations

Now we consider the canonical ideal cell decomposition of a hyperbolic manifold M with cusps [25], and finally prove Proposition 1.7. It is easy to see that every convex ideal polyhedron can be triangulated into ideal tetrahedra with nondegenerate shapes; see for instance Hodgson, Rubinstein and Segerman [44]. One wishes to know that

---

[3]Note that every edge can be completed to a closed loop by adding a path on the boundary T^2. The choice of completion does not matter for studying commutation with the peripheral subgroup.

every two such triangulations are related by a sequence of 2–3 moves. This is a combinatorial problem of convex geometry which we summarize below. For a detailed discussion, the reader may consult the book by De Loera, Rambau and Santos [14] and references therein.

Fix a convex polytope P in $\mathbb{R}^d$. One can consider the set of triangulations of P. When $d = 2$, P is a polygon and it is known that every two triangulations are related by a sequence of flips. For general d, flips are replaced by *geometric bistellar moves*. When $d \geq 5$, it is known that the graph of triangulations (with edges given by geometric bistellar flips) is not connected, and has isolated vertices. For $d = 3$, it is not known whether the graph is connected.

The situation is much better when one considers *regular triangulations* of P. In that case, the corresponding graph of regular triangulations is connected, an in fact it is the edge set of the *secondary polytope* of P. When $d = 3$ and P is convex and in general position, then the only geometric bistellar move is the 2–3 move where the added edge that appears in the move is an edge that connects two vertices of P. When $d = 3$ and P is not in general position, the same conclusion holds as long as one allows for tetrahedra that are flat, ie, lie on a 2–dimensional plane.

Returning to the Epstein–Penner ideal cell decomposition, let $\mathcal{X}_M^{\text{EP}}$ denote the set of regular (in the sense of polytopes and in the sense of ρ_0) ideal triangulations of the ideal cell decomposition. The above discussion together with the fact that no edge of the ideal cell decomposition is univalent, implies that $\mathcal{X}^{\text{EP}}$ is a connected subset of $\mathcal{X}_{\rho_0}$. This concludes the proof of Proposition 1.7.

A detailed discussion on the canonical set $\mathcal{X}_M^{\text{EP}}$ of ideal triangulations of a cusped hyperbolic 3–manifold M is given in [35, Section 6].

4.3 Neumann–Zagier datum and the geometric component

Let M be a manifold with torus boundary and $\mathcal{T}$ a (combinatorial) ideal triangulation. The Neumann–Zagier datum $\beta_{\mathcal{T}} = (z, A, B)$ may be generalized for representations $\rho \in X_M$ besides the discrete faithful.

To begin, choose a representation $\rho \colon \pi_1(M) \longrightarrow \text{PSL}(2, \mathbb{C})$, and, if desired, a lift to $\text{SL}(2, \mathbb{C})$. Let (μ, λ) be meridian and longitude cycles[4] on ∂M, and let $(m^{\pm 1}, \ell^{\pm 1})$ be the eigenvalues of $\rho(\mu)$ and $\rho(\lambda)$, respectively. For example, for the lift of the

---

[4]Recall again that these cycles are only canonically defined for knot complements. In general there is some freedom in choosing them, but the torsion depends in a predictable way on the choice, cf Yamaguchi [74].

discrete faithful representation to $SL(2, \mathbb{C})$, we have $(m, \ell) = (1, -1)$; see Calegari [9]. These eigenvalues define a map

$$(4\text{-}4) \qquad\qquad X_M \longrightarrow (\mathbb{C}^*)^2/\mathbb{Z}_2,$$

whose image is a curve Y_M, the zero-locus of the A–polynomial $A_M(\ell, m) = 0$ [12]. We will denote the representation ρ as ρ_m to emphasize its meridian eigenvalue.

Now, given a triangulation $\mathcal{T}$, and with A, B, and ν defined as in Section 2.3, the gluing equations (2-12) can easily be deformed to account for $m \neq 1$. Namely, we find [58]

$$(4\text{-}5) \qquad\qquad \prod_{j=1}^{N} z_j{}^{A_{ij}} z_j''{}^{B_{ij}} = (-1)^{\nu_i} m^{2\delta_{iN}}.$$

The developing map (4-3) maps every solution of these equations to a representation $\pi_1(M) \longrightarrow PSL(2, \mathbb{C})$ with meridian eigenvalue $\pm m$. The triangulation $\mathcal{T}$ is ρ_m–regular if and only if ρ_m is in the image of this map. We can similarly express the longitude eigenvalue as a product of shape parameters

$$(4\text{-}6) \qquad\qquad \prod_{j=1}^{N} z_j{}^{2C_j} z_j''{}^{2D_j} = (-1)^{2\nu_\lambda} \ell^2,$$

for some $2C_j, 2D_j, 2\nu_\lambda \in \mathbb{Z}$. Then, if $\mathcal{T}$ is a ρ_m–regular triangulation, the irreducible component of Y_M containing ρ_m is explicitly obtained by eliminating all shapes z_j from (4-5)–(4-6).

In general, the shapes z_j are algebraic functions on components of the variety Y_M. However, if M is hyperbolic and $\mathcal{T}$ is regular for all but finitely many representations on the geometric component Y_M^{geom}, then the shapes z_j become rational functions, $z_j \in C(Y_M^{\text{geom}})$. We provide a proof of this fact in Appendix B. The field of functions $C(Y_M^{\text{geom}})$ may be identified with $\mathbb{Q}(m)[\ell]/(A^{\text{geom}}(\ell, m))$, and the functions $z_j(\ell, m)$ can easily be obtained from equations (4-5)–(4-6).

4.4 Flattening compatible with a longitude

In this section we define a restricted combinatorial flattening that is compatible with a longitude.

Recall what is a combinatorial flattening of an ideal triangulation $\mathcal{T}$ from Definition 2.1. Given a simple peripheral curve λ on the boundary of M that represents a longitude — in particular, having intersection number one with the chosen meridian μ — we can

construct the sum of combinatorial edge parameters along λ, just as in Section 2.2. It takes the form

$$(4\text{-}7) \qquad \lambda : \quad \sum_{i=1}^{N}(\boldsymbol{G}_{N+2,i}Z_i + \boldsymbol{G}'_{N+2,i}Z'_i + \boldsymbol{G}''_{N+2,i}Z''_i),$$

for integer vectors $\boldsymbol{G}_{N+1}$, $\boldsymbol{G}'_{N+2}$, $\boldsymbol{G}''_{N+2}$. Just as we obtained $\boldsymbol{A}$, $\boldsymbol{B}$, and ν from the edge and meridian equations (with or without deformation), we may also now define

$$C_i = \frac{1}{2}(\boldsymbol{G}_{N+2,i}-\boldsymbol{G}'_{N+2,i}), \quad D_i = \frac{1}{2}(\boldsymbol{G}''_{N+2,i} - \boldsymbol{G}'_{N+2,i}),$$

$$(4\text{-}8)$$

$$\nu_\lambda = -\frac{1}{2}\sum_{i=1}^{N}\boldsymbol{G}'_{N+2,i}.$$

Definition 4.4 A combinatorial flattening (f, f', f'') is compatible with a longitude if in addition to Equations (2-10a)–(2-10b), it also satisfies

$$(4\text{-}9) \qquad \boldsymbol{G}_{N+2,i} f + \boldsymbol{G}'_{N+2,i} f' + \boldsymbol{G}''_{N+2,i} f'' = 0.$$

Equivalently, a combinatorial flattening compatible with the longitude is a vector $(f, f'') \in \mathbb{Z}^{2N}$ that satisfies

$$(4\text{-}10) \qquad \boldsymbol{A} f + \boldsymbol{B} f'' = \nu, \quad C \cdot f + D \cdot f'' = \nu_\lambda.$$

A combinatorial flattening compatible with a longitude always exists [56, Lemma 6.1].

In the context of functions on the character variety, it is natural to deform the meridian gluing equation, and simultaneously to introduce a longitude gluing equation, in the form

$$(4\text{-}11\text{a}) \qquad \mu : \quad \sum_{i=1}^{N}(\boldsymbol{G}_{N+1,i}Z_i + \boldsymbol{G}'_{N+1,i}Z'_i + \boldsymbol{G}''_{N+1,i}Z''_i) = 2u,$$

$$(4\text{-}11\text{b}) \qquad \lambda : \quad \sum_{i=1}^{N}(\boldsymbol{G}_{N+2,i}Z_i + \boldsymbol{G}'_{N+2,i}Z'_i + \boldsymbol{G}''_{N+2,i}Z''_i) = 2v,$$

for some complex parameters u and v. Upon exponentiation, these equations reduce to the expected (4-5)–(4-6) if

$$(4\text{-}12) \qquad m^2 = e^{2u}, \quad \ell^2 = e^{2v}.$$

(It is easy to show, following [58], that C and D as defined by (4-10) are the correct exponents for the exponentiated longitude Equation (4-6).) If M is a knot complement

and we want to lift from $\mathrm{PSL}(2, \mathbb{C})$ to $\mathrm{SL}(2, \mathbb{C})$ representations, we should take $m = e^u$ and $\ell = -e^v$ and divide (4-11a) by two before exponentiating. This provides the correct way to take a square root of the exponentiated gluing equation, cf [9].

The remarkable symplectic property of A and B may be extended to C and D, even though in general C, D are vectors of half-integers rather than integers. Namely, there exists a completion of $(A\,B)$ to a full symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ such that the *bottom rows* of C and D are the vectors C and D [58]. In particular, this means that

$$(4\text{-}13) \qquad A_N \cdot D - B_N \cdot C = 1,$$

where A_N, B_N are the bottom (meridian) rows of A, B.

4.5 Invariance of the generalized torsion

We finally have all the required ingredients for the generalized torsion formula. Let M be a three-manifold with torus boundary, and $\rho_m \colon \pi_1(M) \to (\mathrm{P})\mathrm{SL}(2, \mathbb{C})$ a representation with meridian eigenvalue m. Let $\mathcal{T}$ be a ρ_m–regular triangulation of M, which exists by Lemma 4.3 at least for a dense set of representations on the geometric component of the character variety. Choose an enhanced Neumann–Zagier datum (z, A, B, f), with $z = z(\rho_m)$ satisfying the deformed gluing equations (4-5) and f satisfying (4-10). Then, as in (4-1), we define

$$\tau_{\mathcal{T}}(\rho_m) := \pm \frac{1}{2} \det(A \Delta_{z''} + B \Delta_z^{-1}) z^{f''} z''^{-f}.$$

We can now prove Theorem 4.1.

Repeating verbatim the arguments of Section 3, it is easy to see that $\tau_{\mathcal{T}}$ is independent of a choice of quad type, a choice of an edge of $\mathcal{T}$ and a choice of a meridian loop. The crucial observation is that the equations $A Z + B Z'' = i\pi v$ (including the meridian equation) are never used in the respective proofs. Therefore, deforming the meridian equation by $u \neq 0$ does not affect anything. For the same reason, it is not hard to see that the formula is invariant under ρ_m–regular 2–3 moves, by repeating the argument of Section 3.6.

The only nontrivial verification required is that $\tau_{\mathcal{T}}$ is independent of the choice of flattening. This does use the gluing equations in a crucial way. We check it now for $m \neq 1$.

Choose logarithms (Z, Z', Z'') of the shape parameters and a logarithm u of m such that $Z + Z' + Z'' = i\pi$ and

$$(4\text{-}14) \qquad A Z + B Z'' = 2u + i\pi v,$$

where $\boldsymbol{u}$ denote the N–dimensional vector $(0, 0, \ldots, 0, u)^T$. By independence of quad type and Lemma A.3, we may assume we are using a quad type with nondegenerate $\boldsymbol{B}$. Now, suppose that (f, f', f'') and $(\tilde{f}, \tilde{f}', \tilde{f}'')$ are two different generalized flattenings. Then

$$
\begin{aligned}
(Z \cdot f'' &- Z'' \cdot f) - (Z \cdot \tilde{f}'' - Z'' \cdot \tilde{f}) \\
&= Z \cdot (f'' - \tilde{f}'') + \boldsymbol{B}^{-1}(AZ - i\pi v - 2\boldsymbol{u}) \cdot (f - \tilde{f}) \\
&= Z \cdot (f'' - \tilde{f}'') + Z \cdot \boldsymbol{B}^{-1} A(f - \tilde{f}) - i\pi \boldsymbol{B}^{-1} v \cdot (f - \tilde{f}) - 2\boldsymbol{B}^{-1} \boldsymbol{u} \cdot (f - \tilde{f}) \\
&= -i\pi \boldsymbol{B}^{-1} v \cdot (f - \tilde{f}) - 2\boldsymbol{B}^{-1} \boldsymbol{u} \cdot (f - \tilde{f}) \\
&= i\pi (f'' \cdot \tilde{f} - f \cdot \tilde{f}'') - 2\boldsymbol{B}^{-1} \boldsymbol{u} \cdot (f - \tilde{f}),
\end{aligned}
$$

by manipulations similar to those of Section 3.5. The new term $2\boldsymbol{B}^{-1} \boldsymbol{u} \cdot (f - \tilde{f})$ is now dealt with by completing the Neumann–Zagier matrices $(\boldsymbol{A}\,\boldsymbol{B})$ to a full symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2N, \mathbb{Q})$, whose bottom row agrees with (C, D). The symplectic condition implies that $\boldsymbol{A}\boldsymbol{D}^T - \boldsymbol{B}\boldsymbol{C}^T = I$, or $\boldsymbol{B}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{D}^T - \boldsymbol{C}^T$. Then

$$
\begin{aligned}
\boldsymbol{B}^{-1}\boldsymbol{u} \cdot (f - \tilde{f}) &= \boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{D}^T\boldsymbol{u} \cdot (f - \tilde{f}) - \boldsymbol{C}^T\boldsymbol{u} \cdot (f - \tilde{f}) \\
&= \boldsymbol{u} \cdot \boldsymbol{D}\boldsymbol{B}^{-1}\boldsymbol{A}(f - \tilde{f}) - \boldsymbol{u} \cdot \boldsymbol{C}(f - \tilde{f}) \\
&= -\boldsymbol{u} \cdot (\boldsymbol{C}(f - \tilde{f}) + \boldsymbol{D}(f'' - \tilde{f}'')) \\
&= -u(C \cdot (f - \tilde{f}) + D \cdot (f'' - \tilde{f}'')).
\end{aligned}
$$

In this last equation, only the bottom row of $\boldsymbol{C}$ and $\boldsymbol{D}$ appears, due to the contraction with $\boldsymbol{u} = (0, 0, \ldots, 0, u)$. But this bottom row is precisely what enters the generalized flattening equations (4-10); since both flattenings satisfy these equations, we must have $\boldsymbol{B}^{-1}\boldsymbol{u} \cdot (f - \tilde{f}) = 0$. Therefore, upon exponentiating, we find

$$
(4\text{-}15) \qquad z^{f''} z''^{-f} = (-1)^{f'' \cdot \tilde{f} - f \cdot \tilde{f}''} z^{\tilde{f}''} z''^{-\tilde{f}} = \pm z^{\tilde{f}''} z''^{-\tilde{f}},
$$

which demonstrates that $\tau_{\mathcal{T}}$ is independent of the choice of flattening. Theorem 4.1 follows. □

4.6 Example: 4_1 continued

We briefly demonstrate the generalized torsion formula, using representations on the geometric component of the character variety $X_{\boldsymbol{4_1}}$ for the figure-eight knot complement.

We may consider the same triangulation as in Section 2.6. The edge and meridian equations (2-14) are deformed to

$$
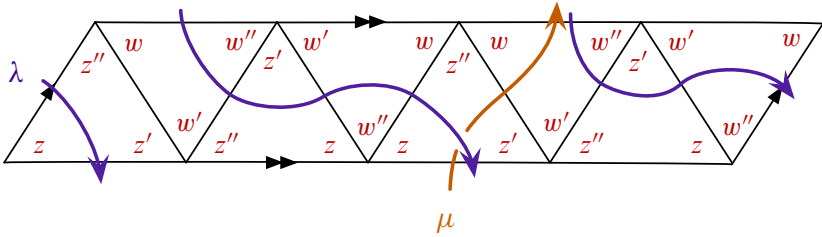(4\text{-}16) \qquad z^2 w^2 z'' w'' = 1, \quad z w z'' = -m^2,
$$

Figure 8: Longitude path for the figure-eight knot complement

with $z'' = 1 - z^{-1}$, $w'' = 1 - w^{-1}$ as usual. In addition, there is a longitude equation that may be read off from the longitude path in Figure 8. In logarithmic (combinatorial) form, we have $-2Z + 2Z' = 2v$, or

$$(4\text{-}17) \qquad\qquad -4Z - 2Z'' = 2v - 2\pi i,$$

from which we identify

$$(4\text{-}18) \qquad\qquad C = (-2, 0), \quad D = (-1, 0), \quad v_\lambda = -1.$$

Dividing (4-17) by two and exponentiating, we find

$$(4\text{-}19) \qquad\qquad z^{-2} z''^{-1} = \ell,$$

with $\ell = -e^v$. This is the appropriate square root of (4-6) for lifting the geometric representations to $SL(2, \mathbb{C})$. We can easily check it: by eliminating shape parameters from (4-16) and (4-19), we recover the geometric $SL(2, \mathbb{C})$ A–polynomial for the figure-eight knot,

$$(4\text{-}20) \qquad A_{\mathbf{4_1}}^{\text{geom}}(\ell, m) = m^4 - (1 - m^2 - 2m^4 - m^6 + m^8)\ell + m^4 \ell^2.$$

We may also use equations (4-16)–(4-19) to express the shape parameters as functions of ℓ and m. We find

$$(4\text{-}21) \qquad\qquad z = -\frac{m^2 - m^{-2}}{1 + m^2 \ell}, \quad w = \frac{m^2 + \ell}{m^2 - m^{-2}}.$$

These are functions on the curve $Y_{\mathbf{4_1}}^{\text{geom}} = \{A_{\mathbf{4_1}}^{\text{geom}}(\ell, m) = 0\}$.

The flattening (2-15) does *not* satisfy the new longitude constraint $C \cdot f + D \cdot f'' = v_\lambda$, so we must find one that does. The choice

$$(4\text{-}22) \qquad\qquad (f_z, f_z', f_z''; f_w, f_w', f_w'') = (0, 0, 1; 0, 0, 1)$$

will work. Repeating the calculation of Section 3.1 with the same A and B but the new generalized flattening, we now obtain

$$
\begin{aligned}
\tau_{\mathbf{4_1}}(\rho_m) &= \pm\frac{1}{2}\det\begin{pmatrix} z'' + 1 & w'' + 1 \\ 1 & w'' \end{pmatrix} zw \\
&= \pm\frac{1}{2}(z''w'' - 1)zw \\
&= \pm\frac{1 - m^2 - 2m^4 - m^6 + m^8 - 2m^4\ell}{2m^4(m^2 - m^{-2})}.
\end{aligned}
$$

(4-23)

This is in full agreement with the torsion found by [40; 17]. Note that for fixed m there are *two* choices of representation ρ_m on the geometric component of the character variety; they correspond to the two solutions of $A_{\mathbf{4_1}}^{\mathrm{geom}}(\ell, m) = 0$ in ℓ.

Remark 4.5 It is interesting to observe that the numerator of Equation (4-23) is exactly $\partial A_{\mathbf{4_1}}^{\mathrm{geom}}/\partial\ell$. That the numerator of the geometric torsion typically carries a factor of $\partial A_{\mathbf{4_1}}^{\mathrm{geom}}/\partial\ell$ might be gleaned from the structure of "$\widehat{A}$–polynomials" in [17] and Gukov and Sułkowski [41], and will also be explored elsewhere.

5 The state integral and higher loops

Our explicit formulas for the torsion $\tau_{\mathcal{T}}$, as well as higher invariants $S_{\mathcal{T},n}$, have been obtained from a state integral model for analytically continued $\mathrm{SL}(2, \mathbb{C})$ Chern–Simons theory. In this section, we will review the state integral, and analyze its asymptotics in order to rederive the full asymptotic expansion

$$
(5\text{-}1) \qquad \mathcal{Z}_{\mathcal{T}}(\hbar) = \hbar^{-3/2}\exp\left[\frac{1}{\hbar}S_{\mathcal{T},0} + S_{\mathcal{T},1} + \hbar S_{\mathcal{T},2} + \hbar^2 S_{\mathcal{T},3} + \cdots\right],
$$

and to unify the formulas of previous sections. We should point out that this section is *not* analytically rigorous, but serves as a motivation for our definition of the all-loop invariants, and provides a glimpse into the calculus of (complex, finite-dimensional) state integrals.

The basic idea of a state integral is to cut a manifold M into canonical pieces (ideal tetrahedra); to assign a simple partition function to each piece (a quantum dilogarithm); and then to multiply these simple partition functions together and integrate out over boundary conditions in order to obtain the partition function of the glued manifold M. A state integral provides a *finite-dimensional reduction* of the full Feynman path integral on M.

Currently, there are two flavors of $SL(2, \mathbb{C})$ state integrals in the literature. The first, introduced by Hikami in [42; 43], studied in [17], and made mathematically rigorous by Andersen and Kashaev in [2], is based on a 3–dimensional lift of the 2–dimensional quantum Teichmüller theory in Kashaev's formalism [51]. It uses variables associated to faces of tetrahedra. The second, developed in [15], explicitly uses shape parameters — associated to edges of tetrahedra — and constitutes a 3d lift of Teichmüller theory in the Fock–Chekhov formalism [30]. The two types of state integrals should be equivalent, though this has only been demonstrated in isolated examples so far; see Spiridonov and Vartanov [65].

It is the second state integral that we employ in this paper, due to its explicit dependence on shape parameters. Indeed, suppose that M is an oriented one-cusped hyperbolic manifold with a ρ_0–regular triangulation $\mathcal{T}$ and enhanced Neumann–Zagier datum $\widehat{\beta}_{\mathcal{T}} = (z, A, B, f)$, with $A f + B f'' = \nu$. We must also assume that B is nondegenerate, which (Lemma A.3) is always possible. Then we will[5] show in Appendix C that the state integral of [15] takes the form

$$(5\text{-}2) \qquad \mathcal{Z}_{\mathcal{T}}(\hbar) = \sqrt{\frac{8\pi^3}{\hbar^3 \det B}} \int \frac{d^N Z}{(2\pi\hbar)^{N/2}} e^x \prod_{i=1}^{N} \psi_\hbar(Z_i),$$

where $x = \frac{1}{\hbar}[\frac{1}{2}(i\pi + \frac{\hbar}{2})^2 f \cdot B^{-1}\nu - (i\pi + \frac{\hbar}{2})Z \cdot B^{-1}\nu + \frac{1}{2}Z \cdot B^{-1}A Z]$ and $\psi_\hbar(Z)$ is a noncompact quantum dilogarithm [4; 27], the Chern–Simons partition function of a single tetrahedron. The integration variables Z_i are, literally, the logarithmic shape parameters of $\mathcal{T}$.

The integration contour of (5-2) is unspecified. A complete, nonperturbative definition of $\mathcal{Z}_{\mathcal{T}}(\hbar)$ requires a choice of contour, and the choice leading to invariance under 2–3 moves (etc.) may be quite subtle. However, a formal asymptotic expansion of the state integral as in (5-1) *does not require* a choice of contour. It simply requires a choice of critical point for the integrand. Then the asymptotic series may be developed via *formal Gaussian integration* in an infinitesimal neighborhood of the critical point.

We will show in Section 5.1 that all the leading order critical points of (5-2) are logarithmic solutions to the gluing equations

$$(5\text{-}3) \qquad \text{critical points} \quad \longleftrightarrow \quad z^A (1 - z^{-1})^B = (-1)^\nu,$$

with $z = \exp(Z)$. In particular, the critical points are isolated. Then, choosing the discrete faithful solution to (5-3), we formally expand the state integral to find that

---

[5]Here we multiply (C-23) (at $u = 0$) by an extra, canonical normalization factor $(2\pi/\hbar)^{3/2}$, in order to precisely match the asymptotics of the Kashaev invariant at the discrete faithful representation.

- $S_{\mathcal{T},0}$, the evaluation of leading order part of the integrand at the critical point, is the complex volume of M;

- $\exp(-2S_{\mathcal{T},1})$ is expressed as the determinant of a Hessian matrix

$$\mathcal{H} = -\boldsymbol{B}^{-1}\boldsymbol{A} + \Delta_{1-z}^{-1},$$

  with a suitable monomial correction, and reproduces the torsion (1-8);

- the higher $S_{\mathcal{T},n}$ are obtained via a finite-dimensional Feynman calculus, and explicitly appear as rational functions of shape parameters.

It follows from the formalism of [15], reviewed in Appendix C, that the state integral (5-2) is only well-defined up to multiplicative prefactors of the form

$$(5\text{-}4) \qquad \exp\left(\frac{\pi^2}{6\hbar}a + \frac{i\pi}{4}b + \frac{\hbar}{24}c\right), \quad a, b, c \in \mathbb{Z}.$$

This means that we only obtain $(S_{\mathcal{T},0}, \tau_{\mathcal{T}} = 4\pi^3 e^{-2S_{\mathcal{T},1}}, S_{\mathcal{T},2})$ modulo $(\frac{\pi^2}{6}\mathbb{Z}, i, \frac{1}{24}\mathbb{Z})$, respectively; however, all the higher invariants $S_{\mathcal{T},n\geq 3}$ should be unambiguous. Moreover, in Section 3 we saw that the ambiguity in $\tau_{\mathcal{T}}$ could be lifted[6] to a sign ± 1. Although the construction of the asymptotic series (5-1) appears to depend on $\mathcal{T}$, we certainly expect the following.

Conjecture 5.1 *The invariants $\{S_{\mathcal{T},n}\}_{n=0}^{\infty}$ are independent of the choice of regular triangulation and Neumann–Zagier datum (including the choice of quad type with $\det \boldsymbol{B} \neq 0$, etc.), up to the ambiguity (5-4), and thus constitute topological invariants of M.*

We now proceed to analyze the critical points and asymptotics of (5-2) in greater detail. In Section 5.5, we will also generalize the state integral to arbitrary representations, with nonunit meridian eigenvalue $m = e^u \neq 1$, and give an example of $S_{\mathcal{T},2}(m)$, $S_{\mathcal{T},3}(m)$ as functions on the character variety Y_M^{geom} for the figure-eight knot.

5.1 Critical points

We begin by showing that the critical points of (5-2) are indeed solutions to the gluing equations. For this purpose, we need to know the quantum dilogarithm $\psi_\hbar(Z)$. The latter is given by [17, Equation 3.22]

$$(5\text{-}5) \qquad \psi_\hbar(Z) = \prod_{r=1}^{\infty} \frac{1 - q^r e^{-Z}}{1 - (^L q)^{-r+1} e^{-L Z}},$$

---

[6]It may also be possible to lift the ambiguities in $S_{\mathcal{T},0}$ and $S_{\mathcal{T},2}$ by using ordered triangulations, as in [57; 75].

for $|q| < 1$, where

$$(5\text{-}6) \qquad q := \exp \hbar, \quad {}^L q := \exp \frac{-4\pi^2}{\hbar}, \quad {}^L Z := \frac{2\pi i}{\hbar} Z.$$

The quantum dilogarithm $\psi_\hbar(Z)$ coincides with the restriction to $|q| < 1$ of Faddeev's quantum dilogarithm [27], as follows from [17, Equation 3.23]. $\psi_\hbar(Z)$ is the Chern–Simons wavefunction of a single tetrahedron [15]. The quantum dilogarithm has an asymptotic expansion as $\hbar \to 0$, given by (cf [17, Equation 3.26])

$$(5\text{-}7) \quad \psi_\hbar(Z) \overset{\hbar \to 0}{\sim} \exp \sum_{n=0}^{\infty} \frac{B_n \hbar^{n-1}}{n!} \widetilde{\mathrm{Li}}_{2-n}(e^{-Z})$$

$$= \exp \left[\frac{1}{\hbar} \widetilde{\mathrm{Li}}_2(e^{-Z}) + \frac{1}{2} \widetilde{\mathrm{Li}}_1(e^{-Z}) - \frac{\hbar}{12} z' + \frac{\hbar^3}{720} z(1+z) z'^3 + \cdots \right],$$

where B_n is the nth Bernoulli number, with $B_1 = 1/2$.

The coefficients of strictly positive powers of $\hbar$ (ie $n \geq 2$) in the expansion are rational functions of $z = e^Z$, but the two leading asymptotics — the logarithm and dilogarithm — are multivalued and have branch cuts. In contrast, the function $\psi_\hbar(Z)$ itself is a meromorphic function on $\mathbb{C}$ for any fixed $\hbar \neq 0$. Branch cuts in its asymptotics arise when families of poles collide in the $\hbar \to 0$ limit. In the case of purely imaginary $\hbar$ with $\operatorname{Im} \hbar > 0$ (a natural choice in the analytic continuation of SU(2) Chern–Simons theory), a careful analysis of this pole-collision process leads to branch cuts for $\widetilde{\mathrm{Li}}_2$ and $\widetilde{\mathrm{Li}}_1$ that are different from the standard ones (Figure 9). We indicate the modified analytic structure of these two functions (really functions of Z rather than e^{-Z}) with an extra tilde.

Now, the critical points of the integrand, at leading order[7] in the $\hbar$ expansion, are solutions to

$$0 = \frac{\partial}{\partial Z_i} \left(-\frac{\pi^2}{2} f \cdot \boldsymbol{B}^{-1} v - i\pi Z \cdot \boldsymbol{B}^{-1} v + \frac{1}{2} Z \cdot \boldsymbol{B}^{-1} A Z + \sum_i \widetilde{\mathrm{Li}}_2(e^{-Z_i}) \right)$$

$$= -i\pi (\boldsymbol{B}^{-1} v)_i + (\boldsymbol{B}^{-1} A Z)_i - \widetilde{\mathrm{Li}}_1(e^{-Z_i}),$$

in other words,

$$(5\text{-}8) \qquad A Z + \boldsymbol{B}(-\widetilde{\mathrm{Li}}_1(e^{-Z})) = i\pi v.$$

Since $\exp[-\widetilde{\mathrm{Li}}_1(e^{-Z_i})] = 1 - z_i^{-1}$, we see that every solution to (5-8) is a particular logarithmic lift of a solution to the actual gluing equations $z^A (1 - z^{-1})^B = (-1)^v$.

---

[7]We treat all subleading terms as perturbations. The exact location of the critical point will acquire perturbative corrections, described in Section 5.4.
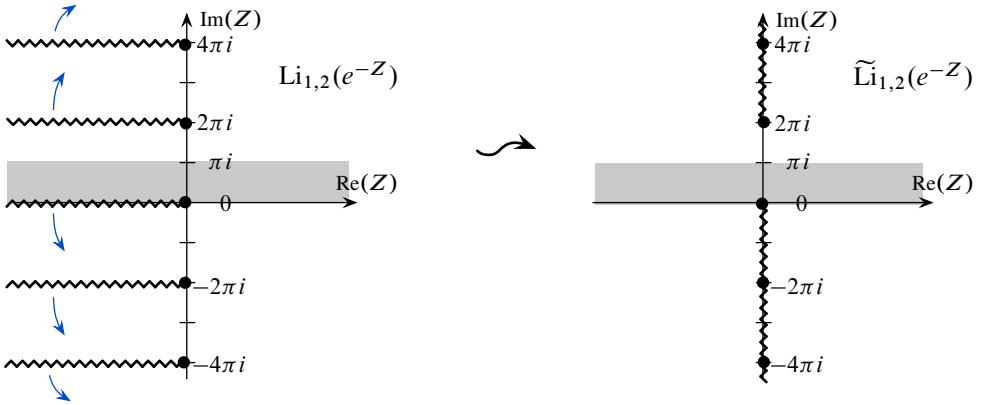
Figure 9: Rotating the standard branch cuts of $\mathrm{Li}_2(e^{-Z})$ and $\mathrm{Li}_1(e^{-Z})$ to produce $\widetilde{\mathrm{Li}}_2(e^{-Z})$ and $\widetilde{\mathrm{Li}}_1(e^{-Z})$, as functions of Z. The shaded region indicates where the standard logarithms of shape parameters for the discrete faithful representation lie.

It is a lift that precisely satisfies the logarithmic constraints (2-6) of Section 2, with $Z''_i = -\widetilde{\mathrm{Li}}_1(e^{-Z_i})$.

When $0 \leq \mathrm{Im}\, Z_i \leq \pi$, the branches of the standard logarithms and dilogarithms agree with those of the modified ones. In particular, given the discrete faithful solution to $z^A(1 - z^{-1})^B = (-1)^n$, taking standard logarithms immediately produces a solution to (5-8). Therefore, the discrete faithful representation always corresponds to a critical point of the state integral.

5.2 Volume

By substituting a solution to (5-8) back into the $\hbar^{-1}$ (leading order) part of the integrand, we obtain the following formula for the complex volume of a representation:

$$(5\text{-}9) \quad S_{\mathcal{T},0} = -\frac{\pi^2}{2} f \cdot \boldsymbol{B}^{-1} v - i\pi Z \cdot \boldsymbol{B}^{-1} v$$
$$+ \frac{1}{2} Z \cdot \boldsymbol{B}^{-1} A Z + \sum_i \widetilde{\mathrm{Li}}_2(e^{-Z_i}) \quad \left(\mathrm{mod}\ \frac{\pi^2}{6}\right).$$

Some manipulation involving the flattening can be used to recast this as

$$(5\text{-}10) \quad \boxed{S_{\mathcal{T},0} = -\frac{1}{2}(Z - i\pi f) \cdot (Z'' + i\pi f'') + \sum_i \widetilde{\mathrm{Li}}_2(e^{-Z_i})} \quad \left(\mathrm{mod}\ \frac{\pi^2}{6}\right),$$

where $Z_i'' := -\widetilde{\mathrm{Li}}_1(e^{-Z_i})$. It is straightforward to verify that this formula is independent of the choice of quad type, choice of edge of $\mathcal{T}$, choice of meridian loop, choice of flattening, and 2–3 moves defines a topological invariant, which agrees with the complex Chern–Simons invariant of M. Since the complex volume in this form has already been studied at length in the literature, we suppress the details here.

At the discrete faithful representation, we can remove the "tildes" from the logarithm and dilogarithm. If we consider the discrete faithful solution to $z^A(1-z^{-1})^B = (-1)^n$, and take standard logarithms

$$Z_i = \log z_i, \quad Z_i'' = \log(1 - z_i^{-1}) \quad (\text{with } 0 \leq \mathrm{Im}\, Z, \mathrm{Im}\, Z'' \leq \pi),$$

we find

$$(5\text{-}11) \qquad S_{\mathcal{T},0} = i\,(\mathrm{Vol}(M) - i\,\mathrm{CS}(M))$$

$$= -\frac{1}{2}(Z - i\pi f) \cdot (Z'' + i\pi f'') + \sum_i \mathrm{Li}_2(e^{-Z_i}) \quad \left(\mathrm{mod}\ \frac{\pi^2}{6}\right).$$

This is a version of the simple formula for the complex volume given in [56]. It is known that the ambiguity in the volume can be lifted from $\pi^2/6$ to $2\pi^2$ using more refined methods; see [57; 38; 75] and Dupont and Zickert [24].

5.3 Torsion revisited

Next, we can derive our torsion formula (1-8). The torsion comes from the $\hbar^0$ part in the asymptotic expansion of the state integral, which has several contributions.

From formal Gaussian integration around a critical point (5-8), we get a determinant $(2\pi\hbar)^{N/2}(\det \mathcal{H})^{-1/2}$, where

$$(5\text{-}12) \quad \mathcal{H}_{ij} = -\frac{\partial^2}{\partial Z_i \partial Z_j}\left(-\frac{\pi^2}{2}f \cdot \boldsymbol{B}^{-1}v - i\pi Z \cdot \boldsymbol{B}^{-1}v\right.$$

$$\left. + \frac{1}{2}Z \cdot \boldsymbol{B}^{-1}AZ + \sum_i \widetilde{\mathrm{Li}}_2(e^{-Z_i})\right)$$

$$= \boxed{(-\boldsymbol{B}^{-1}A + \Delta_{z'})_{ij}}$$

is the Hessian matrix of the exponent (at leading order $\hbar^{-1}$). Here we define that $\Delta_{z'} := \mathrm{diag}(z_1', \ldots, z_N')$, with $z_i' = (1 - z_i)^{-1}$ as usual. Multiplying the determinant is the $\hbar^0$ piece of the integrand, evaluated at the critical point. From the $\hbar^0$ part of the

quadratic exponential, we get

$$(5\text{-}13) \quad \exp\left(\frac{i\pi}{2} f \cdot \boldsymbol{B}^{-1} v - \frac{1}{2} Z \cdot \boldsymbol{B}^{-1} v\right)$$

$$= \exp\left(\frac{1}{2} f \cdot (\boldsymbol{B}^{-1} A Z + Z'') - \frac{1}{2} Z \cdot (\boldsymbol{B}^{-1} A f + f'')\right)$$

$$= \exp\left(-\frac{1}{2} Z \cdot f'' + \frac{1}{2} Z'' \cdot f\right)$$

$$= (z^{f''} z''^{-f})^{-1/2},$$

whereas from the quantum dilogarithm at order $\hbar^0$ we find

$$(5\text{-}14) \qquad \exp\left(\frac{1}{2} \sum_i \widetilde{\mathrm{Li}}_1(e^{-Z_i})\right) = \pm \prod_i \frac{1}{\sqrt{1 - z_i^{-1}}} = \pm \det \Delta_{z''}^{-1/2}.$$

Combining the determinant $(2\pi\hbar)^{N/2} (\det \mathcal{H})^{-1/2}$, the corrections (5-13)–(5-14), and the overall prefactor $\sqrt{8\pi^3/\det \boldsymbol{B}} \, (2\pi\hbar)^{-N/2}$ in the integral (5-2) itself, we finally obtain

$$(5\text{-}15) \qquad e^{S_1} = \sqrt{\frac{8\pi^3}{\det \boldsymbol{B} \det(-\boldsymbol{B}^{-1} A + \Delta_{z'}) \det \Delta_{z''} z^{f''} z''^{-f}}}$$

$$= \sqrt{\frac{-8\pi^3}{\det(A \Delta_{z''} + \boldsymbol{B} \Delta_z^{-1}) z^{f''} z''^{-f}}},$$

up to multiplication by a power of i; or

$$(5\text{-}16) \qquad \tau_M := 4\pi^3 e^{-2S_1} = \pm \frac{1}{2} \det(A \Delta_{z''} + \boldsymbol{B} \Delta_z^{-1}) z^{f''} z''^{-f},$$

just as in (1-8). Despite the fact that the original state integral only made sense for nondegenerate $\boldsymbol{B}$, the final formula for the torsion is well-defined for any $\boldsymbol{B}$.

5.4 Feynman diagrams and higher loops

The remainder of the invariants $S_{\mathcal{T},n}$ can be obtained by continuing the saddle-point (stationary phase) expansion of the state integral to higher order. The calculation can be systematically organized into a set of Feynman rules (cf [45, Chapter 9], [6] and [60]). The resulting formulas — summarized in Section 1 — are explicit algebraic functions of the exponentiated shape parameters z_i, and belong to the invariant trace field E_M.

To proceed, we should first recenter the integration around a critical point. Thus, we replace $Z \to Z + \zeta$ and integrate over ζ, assuming Z to be a solution to (5-8). Using

[17, Equation 3.26], we expand as follows:

$$(5\text{-}17) \quad Z_{\mathcal{T}}(\hbar) = \sqrt{\frac{8\pi^3}{\hbar^3 \det \boldsymbol{B}}} \int \frac{d^N \zeta}{(2\pi\hbar)^{N/2}} \prod_{i=1}^{N} \psi_\hbar(Z_i + \zeta_i)$$

$$\times e^{\frac{1}{\hbar}[\frac{1}{2}(i\pi+\frac{\hbar}{2})^2 f \cdot \boldsymbol{B}^{-1} v - (i\pi+\frac{\hbar}{2})(Z+\zeta)\cdot \boldsymbol{B}^{-1} v + \frac{1}{2}(Z+\zeta)\cdot \boldsymbol{B}^{-1} A(Z+\zeta)]}$$

$$\sim \sqrt{\frac{8\pi^3}{\hbar^3 \det \boldsymbol{B}}} e^{\Gamma^{(0)}(Z)}$$

$$\times \int \frac{d^N \zeta}{(2\pi\hbar)^{N/2}} \exp\left[-\frac{1}{2\hbar}\zeta \cdot \mathcal{H}(Z) \cdot \zeta + \sum_{k=1}^{\infty} \sum_{i=1}^{N} \frac{\Gamma_i^{(k)}(Z)}{k!} \zeta_i^k\right].$$

In this form, the first coefficient $\Gamma^{(0)}(Z)$ can be identified with an overall *vacuum energy*, while the rest of the $\Gamma_i^{(k)}(Z)$ are *vertex factors*.

Every $\Gamma^{(k)}(Z)$ here is a series in $\hbar$, in general starting with a $1/\hbar$ term. However, $\Gamma_i^{(1)}$ must vanish at leading order $\hbar^{-1}$ precisely because Z is a solution to the leading order critical point equations; and we have also already extracted the leading $\hbar^{-1}$ piece of $\Gamma_i^{(2)}$ as the Gaussian integration measure $-(1/2\hbar)\zeta\mathcal{H}\zeta$. Typically, 1–vertices and 2–vertices are absent from a Feynman calculus. Here, however, they appear because our critical point equation and the Hessian (respectively) are only accurate at leading order, and incur $\hbar$–corrections. (Note that the 1–vertices and 2–vertices are counted separately in (5-19) below.)

The vacuum energy $\Gamma^{(0)}$ contributes to every $S_{\mathcal{T},n}$, $n \geq 0$. Its leading order $\hbar^{-1}$ term is just the complex volume (5-10), while the $\hbar^0$ piece contains the corrections (5-13)–(5-14) to the torsion. At higher order in $\hbar$, we have

$$(5\text{-}18) \quad \Gamma^{(0)}(Z) = \frac{1}{\hbar}S_0 + \hbar^0(\cdots) + \frac{\hbar}{8} f \cdot \boldsymbol{B}^{-1} A f$$

$$+ \sum_{n=2}^{\infty} \frac{\hbar^{n-1} B_n}{n!} \sum_{i=1}^{N} \text{Li}_{2-n}(z_i^{-1}) \quad \left(\text{mod } \frac{\hbar}{24}\right).$$

Each S_n, $n \geq 2$, is calculated by taking the $\hbar^{n-1}$ part of $\Gamma^{(0)}$, and adding to it an appropriate sum of Feynman diagrams. The rules for the diagrams are derived from (5-17) as follows. There are vertices of all valencies $k = 1, 2, \ldots$, with a vertex factor given by $\Gamma_i^{(k)}$. One draws all connected diagrams (graphs) with

$$(5\text{-}19) \qquad \boxed{\# \text{loops} + \# 1\text{–vertices} + \# 2\text{–vertices} \leq n.}$$

Each k–vertex is assigned a factor $\Gamma_i^{(k)}$, and each edge is assigned a propagator

$$(5\text{-}20) \qquad \text{propagator}: \quad \Pi_{ij} := \hbar \mathcal{H}_{ij}^{-1} = \hbar(-\boldsymbol{B}^{-1}\boldsymbol{A} + \Delta_{z'})_{ij}^{-1}.$$

The diagrams are then evaluated by contracting the vertex factors with propagators, and multiplying by a standard *symmetry factor*. In each diagram, one should restrict to the $\hbar^{n-1}$ term in its evaluation.

Explicitly, using the asymptotic expansion (5-7) of the quantum dilogarithm, we find that the vertices are

$$(5\text{-}21\text{a}) \qquad 1\text{–vertex}: \quad \Gamma_i^{(1)} = -\frac{1}{2}(\boldsymbol{B}^{-1}v)_i - \sum_{n=1}^{\infty} \frac{\hbar^{n-1}B_n}{n!}\mathrm{Li}_{1-n}(z_i^{-1})$$

$$= -\frac{1}{2}(\boldsymbol{B}^{-1}v)_i + \frac{z_i'}{2} + \cdots,$$

$$(5\text{-}21\text{b}) \qquad 2\text{–vertex}: \quad \Gamma_i^{(2)} = \sum_{n=1}^{\infty} \frac{\hbar^{n-1}B_n}{n!}\mathrm{Li}_{-n}(z_i^{-1})$$

$$= \frac{z_i z_i'^2}{2} - \frac{\hbar}{12}z_i(1+z_i)z_i'^3 + \cdots,$$

$$(5\text{-}21\text{c}) \qquad k\text{–vertex}: \quad \Gamma_i^{(k)} = (-1)^k \sum_{n=0}^{\infty} \frac{\hbar^{n-1}B_n}{n!}\mathrm{Li}_{2-n-k}(z_i^{-1}) \qquad (k \geq 3).$$

Note that in $\Gamma_i^{(1)}$ we could also write $\boldsymbol{B}^{-1}v = \boldsymbol{B}^{-1}\boldsymbol{A}f + f''$. When the inequality (5-19) is saturated, only the leading order ($\hbar^{-1}$ or $\hbar^0$) terms of the vertex factors (5-21) need be considered. Otherwise, subleading $\hbar$–corrections may be necessary.

Examples of 2–loop and 3–loop Feynman diagrams were given in Figures 1–3 of Section 1, along with the entire evaluated expression for $S_{\mathcal{T},2}$.

5.5 n–loop invariants on the character variety

Just as we extended the torsion formula to general representations $\rho \in X_M$ in Section 4, we may now generalize the entire state integral. The basic result for the higher invariants $S_{\mathcal{T},n}$ is that their formulas remain completely unchanged. The shapes z_i simply become functions of the representation ρ, and satisfy the deformed gluing equations (4-5)–(4-6). One must also make sure to use a generalized flattening whenever it occurs, just as in Section 4.

We note that, for a hyperbolic knot complement $M = S^3 \backslash K$, the generalized Chern–Simons state integral $\mathcal{Z}_M(u; \hbar)$ is expected to match the asymptotic expansion of the

colored Jones polynomials $J_N(K; q)$. Specifically, one should consider the limit

(5-22) $$N \to \infty, \quad \hbar \to 0, \quad q^N = e^{N\hbar} = e^{2u} \text{ fixed,}$$

where $m = e^u$ is the meridian eigenvalue for a geometric representation ρ_m in the neighborhood of the discrete faithful. This is the full *Generalized Volume Conjecture* of [39].

To see how formulas for the generalized invariants $S_{\mathcal{T},n}$, $n \geq 0$, come about, consider the state integral at general meridian eigenvalue $m = e^u$. From (C-23) of Appendix C, we find

(5-23) $$\mathcal{Z}_{\mathcal{T}}(u; \hbar) = \sqrt{\frac{8\pi}{\hbar^3 \det B}} \int \frac{d^N Z}{(2\pi\hbar)^{N/2}} \prod_{i=1}^{N} \psi_\hbar(Z_i) e^{-\frac{1}{2\hbar} Z \cdot B^{-1} A Z} e^x,$$

where

$$x = \tfrac{1}{\hbar}[2u \cdot DB^{-1}u + (2\pi i + \hbar)f \cdot B^{-1}u + \tfrac{1}{2}(i\pi + \tfrac{\hbar}{2})^2 f \cdot B^{-1}v - Z \cdot B^{-1}(2u + (i\pi + \tfrac{\hbar}{2})v)],$$

$u := (0, \ldots, 0, u)$ and D is the block appearing in any completion of the Neumann–Zagier matrices $(A\,B)$ to $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2N, \mathbb{Q})$, such that the bottom row D of D appears in the longitude gluing equation $C \cdot Z + D \cdot Z'' = v + 2\pi i v_\lambda$ (Section 4.4). Indeed, since we are contracting with u, only this bottom row of D really matters in (5-23).

The critical points of the state integral are now given by

(5-24) $$AZ + BZ'' = 2u + i\pi v,$$

with $Z'' := -\widetilde{\mathrm{Li}}_1(e^{-Z})$. As expected, this is the logarithmic form of the deformed gluing equation (4-5). Thus, all critical points correspond to representations $\rho = \rho_m \in X_M$. The multivalued nature of this equation must be carefully studied to make sure desired solutions actually exist. However, for example, representations on the geometric component X_M^{geom} always exist in a neighborhood of the discrete faithful representation, if we choose $u = \log m$ to be close to zero (and use a regular triangulation).

We then start expanding the state integral around a critical point, setting

(5-25) $$\mathcal{Z}_{\mathcal{T}}(u; \hbar) \sim \hbar^{-\frac{3}{2}} \exp\left[\frac{1}{\hbar} S_{\mathcal{T},0}(u) + S_{\mathcal{T},1}(u) + \hbar S_{\mathcal{T},2}(m) + \hbar^2 S_{\mathcal{T},3}(m) + \cdots\right].$$

The leading contribution $S_{\mathcal{T},0}(\rho)$ is given, following some standard manipulations using the generalized flattening, by

(5-26) $$S_{\mathcal{T},0}(u) = uv(u) - \frac{1}{2}(Z - i\pi f) \cdot (Z'' + i\pi f'') + \sum_{i=1}^{N} \widetilde{\mathrm{Li}}_2(e^{-Z}) \quad \left(\mathrm{mod}\, \frac{\pi^2}{6}\right).$$

Here we write $S_{\mathcal{T},0}$ as a function of the logarithmic meridian eigenvalue u, though a fixed choice of representation ρ will implicitly fix the choice of longitude eigenvalue $v = \log(-\ell)$ as well. Expression (5-26) is a holomorphic version of the complex volume of a cusped manifold with deformed cusp. Explicitly,

$$(5\text{-}27) \qquad S_{\mathcal{T},0}(u) = i\,(\mathrm{Vol}_M(u) + i\,\mathrm{CS}_M(u)) - 2v\Re(u).$$

This is the correct form of the complex volume to use in the Generalized Volume Conjecture; cf [40].

At first subleading order, we rederive the generalized torsion formula. The calculation is identical to that of Section 5.3, with the exception of the correction (5-13) coming from the $\hbar^0$ part of the exponential. This correction now becomes

$$(5\text{-}28) \qquad \exp\left[f\cdot\boldsymbol{B}^{-1}u + \frac{i\pi}{2}f\cdot\boldsymbol{B}^{-1}v - \frac{1}{2}Z\cdot\boldsymbol{B}^{-1}v\right].$$

To simplify this correction, we must use $\boldsymbol{A}f + \boldsymbol{B}f'' = v$ and a deformed gluing equation $\boldsymbol{A}Z + \boldsymbol{B}Z'' = 2u + i\pi v$. The u–dependent part of the gluing equation cancels the new u–dependent term in (5-28), ultimately leading to the same result

$$\exp\left[f\cdot\boldsymbol{B}^{-1}u + \frac{i\pi}{2}f\cdot\boldsymbol{B}^{-1}v - \frac{1}{2}Z\cdot\boldsymbol{B}^{-1}v\right] = (z^{f''}z''^{-f})^{-1/2},$$

and therefore the same torsion[8]

$$(5\text{-}29) \qquad \tau_{\mathcal{T}} = 4\pi e^{-2S_{\mathcal{T},1}} = \frac{1}{2}\det(\boldsymbol{A}\Delta_{z''} + \boldsymbol{B}\Delta_z^{-1})z^{f''}z''^{-f}.$$

Finally, we can produce a generalized version of the Feynman rules of Section 5.4. We note, however, that the u–dependent terms in (5-23) do not contribute to either the vacuum energy $\Gamma^{(0)}$ (at order $\hbar^1$ or higher), the propagator, or the vertex factors $\Gamma_i^{(k)}$. Therefore, the Feynman rules must look exactly the same. The only difference is that the critical point Equation (5-24) requires us to use shape parameters that satisfy the generalized gluing equations.

5.6 Example: 4_1 completed

We may demonstrate the power of the Feynman diagram approach by computing the first two subleading corrections $S_{\mathcal{T},2}$ and $S_{\mathcal{T},3}$ for the figure-eight knot complement.

---

[8]The normalization of the torsion here differs from the torsion at the discrete faithful by a factor of π^2. In fact, we intentionally changed the normalization of the entire state integral (5-23) by π^2. This is because we wanted the state integral to match the asymptotics of the colored Jones polynomials exactly, and the asymptotics happen to jump by π^2 when $u \neq 0$, cf [40].

We can use the same Neumann–Zagier datum described in Section 2.6, along with the generalized flattening of Section 4.6. Let us specialize to representations ρ_m on the geometric component of the character variety. Then the two shapes z, w are expressed as functions on the A–polynomial curve,

$$(5\text{-}30) \qquad z = -\frac{m^2 - m^{-2}}{1 + m^2 \ell}, \quad w = \frac{m^2 + \ell}{m^2 - m^{-2}},$$

as in (4-21).

The 2–loop invariant is explicitly given in (1-19) of Section 1. Evaluating this expression in `Mathematica`, we find

$$S_{\mathbf{4_1},2} = -\frac{w^3(z+1) + w^2((11-8z)z - 4) + w(z-1)(z(z+12) - 5) + (z-2)(z-1)^2}{12(w+z-1)^3}.$$

Upon using (5-30) to substitute rational functions for z and w, the answer may be most simply expressed as

$$(5\text{-}31) \quad \widetilde{S}_{\mathbf{4_1},2} = \frac{S_{\mathbf{4_1},2} + 1/8}{\tau_{\mathbf{4_1}}^3} = -\frac{1}{192}(m^{-6} - m^{-4} - 2m^{-2} + 15 - 2m^2 - m^4 + m^6),$$

where we have divided by a power of the torsion as suggested in (1-14) of Section 1. We have also absorbed a constant $1/8$, recalling that our formula is only well-defined modulo $\mathbb{Z}/24$.

In a similar way, we may calculate the 3–loop invariant, finding unambiguously

$$(5\text{-}32) \qquad \widetilde{S}_{\mathbf{4_1},3} = \frac{S_{\mathbf{4_1},3}}{\tau_{\mathbf{4_1}}^6} = \frac{1}{128}(m^{-6} - m^{-4} - 2m^{-2} + 5 - 2m^2 - m^4 + m^6).$$

These answers agree perfectly with the findings of [17], and the comparison there to the asymptotics of the colored Jones polynomials at general u. Moreover, at the discrete faithful representation we obtain

$$(5\text{-}33) \qquad S_{\mathbf{4_1},2} = \frac{11i}{72\sqrt{3}} = -\frac{11}{192\tau_{\mathbf{4_1}}^3}, \quad S_{\mathbf{4_1},3} = -\frac{1}{54} = \frac{1}{128\tau_{\mathbf{4_1}}^6},$$

in agreement with known asymptotics of the Kashaev invariant.

Appendix A: Symplectic properties of A and B

The $N \times N$ Neumann–Zagier matrices A and B form the top half of a symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2N, \mathbb{Q})$ [58]. In this section we discuss some elementary properties of symplectic matrices.

Lemma A.1 *The $N \times 2N$ matrix $(A B)$ is the upper half of a symplectic matrix if and only if $A B^T$ is symmetric and $(A B)$ has maximal rank N.*

Proof It is easy to see that the rows of $(A B)$ have zero symplectic product (with respect to the standard symplectic form on $\mathbb{Q}^{2N}$ if and only if $A B^T$ is symmetric. In addition they span a vector space of rank N if and only if $(A B)$ has maximal rank N. The result follows. □

Lemma A.2 *If $(A B)$ is the upper half of a symplectic matrix and B is nondegenerate, then $B^{-1} A$ is symmetric.*

Proof Lemma A.1 implies that $A B^T$ is symmetric, and so is $(B)^{-1} A B^T ((B)^{-1})^T$. □

It is not true in general that B is invertible. However, after a possible change of quad type, we can assume that B is invertible. This is the content of the next lemma.

Lemma A.3 (a) *Suppose $(A B)$ is the upper half of a symplectic $2N \times 2N$ matrix. If A has rank r, then any r linearly independent columns of A and their complementary $N - r$ columns in B form a basis for the column space of $(A B)$.*

(b) *There always exists a choice of quad type for which B is nondegenerate (for any fixed choice of redundant edge and meridian path).*

Proof For (a) let $\mathrm{rank}(A) = r \leq N$. Without loss of generality, we may suppose that the first r columns of A are linearly independent. We want to show that, together with the last $N - r$ columns of B, they form a matrix of rank N.

If we simultaneously multiply both A and B on the left by any nonsingular matrix $U \in \mathrm{GL}(N, \mathbb{R})$, both the symplectic condition and the columns are preserved. This follows from the fact that $\left(\begin{smallmatrix} U & 0 \\ 0 & U^{-1,T} \end{smallmatrix} \right) \in \mathrm{Sp}(2N, \mathbb{R})$. By allowing such a transformation, we may assume that A takes the block form

(A-1) $$A = \begin{pmatrix} I_{r \times r} & A_2 \\ 0 & 0 \end{pmatrix}$$

for some A_2. Similarly, we split B into blocks of size r and $N - r$,

(A-2) $$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

Since $(A B)$ has full (row) rank, we see that the bottom $N - r$ rows of B must be linearly independent, ie $\mathrm{rank}(B_3 B_4) = N - r$. From the symplectic condition of

Lemma A.1, we also find that $B_3 + B_4 A_2^T = 0$, so that $\text{rank}(B_3 B_4) \leq \text{rank}(B_4)$. This then implies that B_4 itself must have maximal rank $N - r$. Therefore, the last $N - r$ columns of $\boldsymbol{B}$ are linearly independent, and also independent of the first N columns of $\boldsymbol{A}$; ie the matrix $\left(\begin{smallmatrix} I_{r \times r} & B_2 \\ 0 & B_4 \end{smallmatrix}\right)$ has maximal rank as desired. This concludes the proof of part (a).

For part (b) let us denote the columns of $\boldsymbol{A}$ and $\boldsymbol{B}$ as a_i and b_i. A change of quad type corresponding to a cyclic permutation $Z_i \mapsto Z_i' \mapsto Z_i'' \mapsto Z_i$ on the ith tetrahedron permutes the ith columns of $\boldsymbol{A}$ and $\boldsymbol{B}$ as $(a_i, b_i) \mapsto (b_i - a_i, -a_i)$. Therefore, given N complementary columns of $(\boldsymbol{A}\,\boldsymbol{B})$ that have full rank, we can use such permutations to move all the columns (up to a sign) into $\boldsymbol{B}$. $\qquad\square$

Appendix B: The shape parameters are rational functions on the character variety

In this Appendix, we prove that the shape parameters of a regular ideal triangulation are rational functions on Y_M^{geom}, the geometric component of the $\text{SL}_2(\mathbb{C})$ A–polynomial curve.

Proposition B.1 *Fix a regular ideal triangulation $\mathcal{T}$ of a one-cusped hyperbolic manifold M. Then every shape parameter of $\mathcal{T}$ is a rational function on Y_M^{geom}.*

Proof The proof is a little technical, and follows from work of Dunfield [20, Corollary 3.2], partially presented in [7, Appendix]. For completeness, we give the details of the proof here. We thank N Dunfield for a careful explanation of his proof to us.

Consider the affine variety $R(M, \text{SL}(2, \mathbb{C})) = \text{Hom}(\pi_1, \text{SL}(2, \mathbb{C}))$ and its algebro-geometric quotient $X_{M, \text{PSL}(2, \mathbb{C})}$ by the conjugation action of $\text{PSL}(2, \mathbb{C})$. Following Dunfield from the Appendix to [7], let $\bar{R}(M, \text{SL}(2, \mathbb{C}))$ denote the subvariety of $R(M, \text{SL}(2, \mathbb{C})) \times P^1(\mathbb{C})$ consisting of pairs (ρ, z) where z is a fixed point of $\rho(\pi_1(\partial M))$. Let $\bar{X}_{M, \text{SL}(2, \mathbb{C})}$ denote the algebrogeometric quotient of $\bar{R}(M, \text{SL}(2, \mathbb{C}))$ under the diagonal action of $\text{SL}(2, \mathbb{C})$ by conjugation and Möbius transformations respectively. We will call elements $(\rho, z) \in \bar{R}(M, \text{SL}(2, \mathbb{C}))$ *augmented representations*. Their images in the augmented character variety $\bar{X}(M, \text{SL}(2, \mathbb{C}))$ will be called *augmented characters* and will be denoted by square brackets $[(\rho, z)]$. Likewise, replacing $\text{SL}(2, \mathbb{C})$ by $\text{PSL}(2, \mathbb{C})$, we can define the character variety $X_{M, \text{PSL}(2, C)}$ and its augmented version $\bar{X}_{M, \text{PSL}(2, \mathbb{C})}$.

The advantage of the augmented character variety $\bar{X}_{M, \text{SL}(2, \mathbb{C})}$ is that given $\gamma \in \pi_1(\partial M)$ there is a regular function e_γ that sends $[(\rho, z)]$ to the eigenvalue of $\rho(\gamma)$ corresponding

to z, using Lemma B.3 below. In contrast, in $X_{M,\mathrm{SL}(2,\mathbb{C})}$ only the trace $e_\gamma + e_\gamma^{-1}$ of $\rho(\gamma)$ is well-defined. Likewise, in $\bar{X}_{M,\mathrm{PSL}(2,C)}$ (resp. $X_{M,\mathrm{SL}(2,\mathbb{C})}$) only e_γ^2 (resp. $e_\gamma^2 + e_\gamma^{-2}$) is well-defined.

From now on, we will restrict to the geometric component of the character variety $X_{M,\mathrm{PSL}(2,\mathbb{C})}$ and we will fix a regular ideal triangulation $\mathcal{T}$. In [20, Theorem 3.1] Dunfield proves that the natural restriction map

$$X_{M,\mathrm{PSL}(2,\mathbb{C})} \longrightarrow X_{\partial M,\mathrm{PSL}(2,\mathbb{C})}$$

of affine curves is of degree 1. The variety $X_{\partial M,\mathrm{PSL}(2,\mathbb{C})}$ is an affine curve in $(\mathbb{C}^*)^2/\mathbb{Z}_2$ and let $V_{M,\mathrm{PSL}(2,\mathbb{C})} \subset (\mathbb{C}^*)^2$ denote the preimage of $X_{\partial M,\mathrm{PSL}(2,\mathbb{C})}$ of the $2:1$ map $(\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^2/\mathbb{Z}_2$. The commutative diagram

(B-1)
$$
\begin{array}{ccc}
\bar{X}_{M,\mathrm{PSL}(2,\mathbb{C})} & \longrightarrow & V_{M,\mathrm{PSL}(2,\mathbb{C})} \\
\downarrow & & \downarrow \\
X_{M,\mathrm{PSL}(2,\mathbb{C})} & \longrightarrow & X_{\partial M,\mathrm{PSL}(2,\mathbb{C})}
\end{array}
$$

has both vertical maps of degree 2, and the bottom horizontal map of degree 1. Thus, it follows that the top horizontal map is of degree 1. In [7, Section 10.3] Dunfield constructs a degree 1 developing map

$$V_{\mathcal{T}} \longrightarrow \bar{X}_{M,\mathrm{PSL}(2,\mathbb{C})},$$

which combined with the previous discussion gives a chain of birational curve isomorphisms

(B-2) $$V_{\mathcal{T}} \longrightarrow \bar{X}_{M,\mathrm{PSL}(2,\mathbb{C})} \longrightarrow V_{M,\mathrm{PSL}(2,\mathbb{C})}.$$

Since the shape parameters are rational (in fact coordinate) functions on $V_{\mathcal{T}}$, it follows that they are rational functions on $V_{M,\mathrm{PSL}(2,\mathbb{C})}$. Using the regular map $V_{M,\mathrm{SL}(2,\mathbb{C})} \longrightarrow V_{M,\mathrm{PSL}(2,\mathbb{C})}$, we obtain that the shape parameters are rational functions on $V_{M,\mathrm{SL}(2,\mathbb{C})}$. $\qquad\square$

Proposition B.1 has the following concrete corollary.

Corollary B.2 *Given a regular ideal triangulation $\mathcal{T}$ with N tetrahedra, there is a solution of the shape parameters in $\mathbb{Q}(m,\ell)/(A(m,\ell))$.*

Lemma B.3 *Suppose*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$

and $c \neq 0$. Then, λ is an eigenvalue of A if and only if $z = (\lambda - 2d)/(2c)$ is a fixed point of the corresponding Möbius transformation in $P^1(\mathbb{C})$.

Appendix C: Deriving the state integral

In this Appendix, we explain the connection between the quantization formalism of [15] and the special state integrals (5-2) and (5-23) that led to all the formulas in the present paper. We will first review classical "symplectic gluing" of tetrahedra, then extend gluing to the quantum setting and construct the state integral. There are multiple points in the construction that have yet to be made mathematically rigorous, which we will try to indicate.

C.1: Symplectic gluing

The main idea of [15] is that gluing of tetrahedra should be viewed, both classically and quantum mechanically, as a process of symplectic reduction.

Suppose we have a one-cusped manifold M with a triangulation $\mathcal{T} = \{\Delta_i\}_{i=1}^N$. Classically, each tetrahedron Δ_i comes with a phase space

$$(C\text{-}1) \qquad \mathcal{P}_{\partial\Delta_i} = \{\text{flat } SL(2,\mathbb{C}) \text{ connections on } \partial\Delta_i\}$$
$$\approx \{(Z_i, Z_i', Z_i'') \in \mathbb{C} \backslash (2\pi i \mathbb{Z}) \mid Z_i + Z_i' + Z_i'' = i\pi\},$$

with (holomorphic) symplectic structure

$$(C\text{-}2) \qquad\qquad\qquad \omega_{\partial\Delta_i} = dZ \wedge dZ'',$$

and a Lagrangian submanifold[9]

$$(C\text{-}3) \qquad \mathcal{L}_{\Delta_i} = \{\text{flat } SL(2,\mathbb{C}) \text{ connections that extend to } \Delta_i\}$$
$$= \{e^{Z''} + e^{-Z} - 1 = 0\} \subset \mathcal{P}_{\partial\Delta_i}.$$

When gluing the tetrahedra together, we first form a product

$$(C\text{-}4) \qquad \mathcal{L}_\times = \mathcal{L}_{\Delta_1} \times \cdots \times \mathcal{L}_{\Delta_N} \subset \mathcal{P}_\times = \mathcal{P}_{\partial\Delta_1} \times \cdots \times \mathcal{P}_{\partial\Delta_N}.$$

The edge constraints $X_I := \sum_{i=1}^N (G_{Ii} Z_i + G_{Ii}' Z_i' + G_{Ii}'' Z_i'') - 2\pi i$ from (2-3) are functions on the product phase space $\mathcal{P}_\times$, and can be used as (holomorphic) moment maps to generate $N - 1$ independent translation actions t_I. Recall [58] that the

---

[9]Explicitly, $\mathcal{P}_{\partial\Delta_i}$ is a space of flat connections on a 4–punctures sphere with parabolic holonomy at the four punctures; while $\mathcal{L}_{\Delta_i}$ is the subspace with trivial holonomy, hence connections that extend into the bulk of the tetrahedron. See, eg, the first author, Gaiotto and Gukov [16, Section 2].

logarithmic meridian and longitude holonomies (u, v) are also functions on $\mathcal{P}_\times$, which Poisson-commute with all the edges X_I, and so are fixed under these translations. Then the phase space of M is a symplectic quotient,

(C-5) $\qquad \mathcal{P}_{\partial M} = \{\text{flat SL}(2, \mathbb{C}) \text{ connections on } \partial M \simeq T^2\} \approx \{(u, v) \in \mathbb{C}\}$

$\qquad\qquad = \mathcal{P}_\times /\!\!/ (t_I),$

and the A–polynomial of M (more properly, components of the A–polynomial for which the triangulation is regular) is the result of pulling the Lagrangian $\mathcal{L}_\times$ through the quotient,

(C-6) $\qquad \mathcal{L}_M = \text{``}\mathcal{L}_\times /\!\!/ (t_I)\text{''} \approx \{A_M(e^v, e^u) = 0\} \subset \mathcal{P}_{\partial M}.$

This is quite easy to check using equations (4-5) and (4-6).

C.2: Quantization

Quantum mechanically, we have that each tetrahedron has a Hilbert space $\mathcal{H}_{\Delta_i}$, a wavefunction $\mathcal{Z}_{\Delta_i}(Z_i)$ and a quantum operator $\hat{\mathcal{L}}_{\partial \Delta_i}$ that annihilates the wavefunction. The symplectic-gluing procedure extends to the quantum setting, with appropriate quantum generalizations of all the above operations. Roughly, one forms a product wavefunction

(C-7) $\qquad \mathcal{Z}_\times(Z_1, \ldots, Z_N) = \mathcal{Z}_{\Delta_1} \otimes \cdots \otimes \mathcal{Z}_{\Delta_N} \in \mathcal{H}_\times = \mathcal{H}_{\partial \Delta_1} \otimes \cdots \otimes \mathcal{H}_{\partial \Delta_N},$

and restricts the product Hilbert space using $N - 1$ new polarizations coming from the edge constraints. The resulting restricted wavefunction is $\mathcal{Z}_M(u)$, and it is annihilated by a quantized version of the A–polynomial; see the second author [39; 33].

To make this more precise, let M again be an oriented one-cusped manifold, and choose a triangulation $\mathcal{T} = \{\Delta_i\}_{i=1}^N$ (regular with respect to some desired family of representations), a quad type, a redundant edge, and a meridian path, just as in Section 2.

To each tetrahedron Δ_i we associate a boundary Hilbert space $\mathcal{H}_{\partial \Delta_i}$. It is some extension[10] of $L^2(\mathbb{R})$ that includes the wavefunction

(C-8) $\qquad\qquad \mathcal{Z}_{\Delta_i}(Z_i; \hbar) := \psi_\hbar(Z_i),$

where $\psi_\hbar(Z_i)$ is Faddeev's quantum dilogarithm (5-5) [27]. We also associate to Δ_i an algebra of operators

(C-9) $\qquad\qquad \hat{\mathcal{A}}_{\partial \Delta_i} = \mathbb{C}\langle \hat{Z}_i, \hat{Z}_i', \hat{Z}_i'' \rangle / (\hat{Z}_i + \hat{Z}_i' + \hat{Z}_i'' = i\pi + \tfrac{\hbar}{2}),$

---

[10]This space has not been mathematically defined yet; constructions of (eg) [2] might prove useful for achieving this.

with commutation relations

$$(C\text{-}10) \qquad [\hat{Z}_i, \hat{Z}'_i] = [\hat{Z}'_i, \hat{Z}''_i] = [\hat{Z}''_i, \hat{Z}_i] = \hbar.$$

Then the quantization of the Lagrangian (C-3) annihilates the wavefunction,

$$(C\text{-}11) \qquad \hat{\mathcal{L}}_{\Delta_i} := e^{\hat{Z}''_i} + e^{-\hat{Z}_i} - 1, \quad \hat{\mathcal{L}}_{\Delta_i} \mathcal{Z}_{\Delta_i} = 0,$$

where the operators act in the representation

$$(C\text{-}12) \qquad \begin{aligned} \hat{Z}_i = Z_i, & \qquad \hat{Z}''_i = \hbar \partial_{Z_i}; \quad \text{or} \\ e^{\hat{Z}_i} \mathcal{Z}(Z_i) = e^{Z_i} \mathcal{Z}(Z_i), & \quad e^{\hat{Z}''_i} \mathcal{Z}(Z_i) = \mathcal{Z}(Z_i + \hbar). \end{aligned}$$

In order to glue the tetrahedra together, we start by forming the product wavefunction $\mathcal{Z}_\times(Z_1, \ldots, Z_N) = \mathcal{Z}_{\Delta_1}(Z_1) \cdots \mathcal{Z}_{\Delta_N}(Z_N)$. This is an element of a product Hilbert space (C-7). Acting on this product Hilbert space is the product $\hat{\mathcal{A}}_\times$ of algebras (C-9), which is simply generated by all the $\hat{Z}_i, \hat{Z}'_i, \hat{Z}''_i$, with canonical commutation relations (C-10) (and operators from distinct tetrahedra always commuting).

Now, following the notation of Sections 2.2 and 4.4, we can define N operators $\hat{X}_I \in \hat{\mathcal{A}}_\times$, one for each independent edge, and one for the meridian:

$$(C\text{-}13) \quad \hat{X}_I := \begin{cases} \sum_{i=1}^{N} (G_{Ii}\hat{Z}_i + G'_{Ii}\hat{Z}'_i + G''_{Ii}\hat{Z}''_i) - 2\pi i - \hbar & I = 1, \ldots, N-1, \\ G_{N+1,i}\hat{Z}_i + G'_{N+1,i}\hat{Z}'_i + G''_{N+1,i}\hat{Z}''_i & I = N. \end{cases}$$

Similarly, we may define an operator

$$(C\text{-}14) \qquad \hat{P}_N := \tfrac{1}{2}(G_{N+2,i}\hat{Z}_i + G'_{N+2,i}\hat{Z}'_i + G''_{N+2,i}\hat{Z}''_i)$$

corresponding to the longitude. Due to the symplectic structure found in [58], we know that we may complete the set $\{\hat{X}_1, \ldots, \hat{X}_N, \hat{P}_N\}$ to a full canonical basis of the algebra $\hat{\mathcal{A}}_\times$. We do this by adding $N-1$ additional operators $\hat{P}_I$, which are linear combinations of the $\hat{Z}$s, such that

$$(C\text{-}15) \qquad [\hat{P}_I, \hat{X}_j] = \delta_{Ij}\hbar, \quad [\hat{P}_I, \hat{P}_j] = [\hat{X}_I, \hat{X}_j] = 0, \quad 1 \le I, j \le N.$$

The operators $\hat{X}_I, \hat{P}_I$ have a simple interpretation in terms of a generalized Neumann–Zagier datum. Namely, if we complete $(A\,B)$ and the rows C, D (of Section 4.4) to a full symplectic matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then

$$(C\text{-}16) \qquad \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \hat{Z} \\ \hat{Z}'' \end{pmatrix} - \left(i\pi + \tfrac{\hbar}{2}\right) \begin{pmatrix} \nu \\ \nu_P \end{pmatrix}.$$

Here ν is precisely the vector of N integers which was introduced in (2-8), while $\nu_P = (*, \ldots, *, \nu_\lambda)$, with ν_λ from (4-8). The first $N - 1$ entries of ν_P depend on the precise completion of the canonical basis (or the symplectic matrix), and ultimately drop out of the gluing construction.

C.3: Quantum reduction

Classically, in order to glue we would want to set the $N - 1$ edge constraints $X_I \to 0$, and the meridian $X_N \to 2u$. In Section C.1, these functions were actually used as moment maps to perform a symplectic reduction. Now we should do the same thing quantum mechanically. In order to reduce the product wavefunction $\mathcal{Z}_\times(Z_1, \ldots, Z_N)$ of (C-7) to the final wavefunction $\mathcal{Z}_M(u)$ of the glued manifold M, we must transform the wavefunction to a representation (or "polarization") in which the operators $\hat{X}_I$ act diagonally (by multiplication). In this representation, the wavefunction depends explicitly on the X_I. The "reduction" then simply requires fixing $X_I \to (0, \ldots, 0, 2u)$. Schematically,

$$(\text{C-17}) \quad \mathcal{Z}_\times(Z_1, \ldots, Z_N) \xrightarrow{\text{transform}} \tilde{\mathcal{Z}}_\times(X_1, \ldots, X_N) \overset{\text{fix}}{\mapsto} \mathcal{Z}_M(u) = \tilde{\mathcal{Z}}_\times(0, \ldots, 0, 2u).$$

The transformation from $\mathcal{Z}_\times$ to $\tilde{\mathcal{Z}}_\times$ is accomplished — formally — with the Weil representation $\mathcal{R}$ of the affine symplectic group; see Shale [64] and Weil [71]. In particular, we need $\mathcal{R}(\alpha)$ for the affine symplectic transformation α in (C-16). In [15, Section 6], it was discussed in detail how to find $\mathcal{R}(\alpha)$ by factoring the matrix of (C-16) into generators. Then, for example, an "S–type" element of the symplectic group acts via Fourier transform

$$(\text{C-18}) \qquad \mathcal{R}\!\left(\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\right)\!: f(Z) \mapsto \tilde{f}(W) = \int \frac{d^N Z}{(2\pi i \hbar)^{N/2}} e^{\frac{1}{\hbar} Z \cdot W} f(Z),$$

whereas a "T–type" element acts as multiplication by a quadratic exponential

$$(\text{C-19}) \qquad \mathcal{R}\!\left(\begin{pmatrix} I & 0 \\ T & I \end{pmatrix}\right)\!: f(Z) \mapsto \tilde{f}(W) = e^{\frac{1}{2\hbar} W^T T W} f(W).$$

Affine shifts act either by translation or multiplication by a linear exponential.

In the present case, there is a convenient trick that allows us to find $\mathcal{R}(\alpha)$ without decomposing α into generators. We assume that the block $\boldsymbol{B}$ of the symplectic matrix is nondegenerate, since we know we can always choose a quad type with this property. For the moment, let us also suppose that the affine shifts vanish, $\nu = \nu_P = 0$. Then the Weil action is

$$(\text{C-20}) \qquad \mathcal{R}(\alpha)\colon \mathcal{Z}_\times(Z) \mapsto \tilde{\mathcal{Z}}_\times(X) = \frac{1}{\sqrt{\det \boldsymbol{B}}} \int \frac{d^N Z}{(2\pi i \hbar)^{N/2}} e^x \, \mathcal{Z}_\times(Z),$$

where $x = \frac{1}{2\hbar}(X \cdot \boldsymbol{DB}^{-1}X - 2Z \cdot \boldsymbol{B}^{-1}X + Z \cdot \boldsymbol{B}^{-1}\boldsymbol{A}Z)$. In particular, it can easily be verified that this correctly intertwines an action of operators $(\widehat{Z}_i = Z_i, \widehat{Z}''_i = \hbar\partial_{Z_i})$ on $\mathcal{Z}_\times(Z)$ with an action of operators $(\widehat{X}_I = X_I, \widehat{P}_I = \hbar\partial_{X_I})$ on $\widetilde{\mathcal{Z}}_\times(X)$. For example,

$$
\begin{aligned}
\int d^N Z e^x (\boldsymbol{A}\widehat{Z} + \boldsymbol{B}\widehat{Z}'')\mathcal{Z}_\times(Z) &= \int d^N Z e^x (\boldsymbol{A}Z + \hbar\boldsymbol{B}\partial_Z)\mathcal{Z}_\times(Z) \\
&= \int d^N Z[(\boldsymbol{A}Z - \hbar\boldsymbol{B}\partial_Z)e^x]\mathcal{Z}_\times(Z) \\
&= \int d^N Z X e^x \mathcal{Z}_\times(Z) \\
&= \widehat{X}\widetilde{\mathcal{Z}}_\times(X).
\end{aligned}
$$

Nonzero affine shifts ν and ν_P further modify the result to

$$
\begin{aligned}
\widetilde{\mathcal{Z}}_\times(X) = \frac{1}{\sqrt{\det \boldsymbol{B}}} \int \frac{d^N Z}{(2\pi i\hbar)^{N/2}} \exp\Bigg[&-\frac{1}{\hbar}X \cdot \Big(i\pi + \frac{\hbar}{2}\Big)\nu_P \\
&+ \frac{1}{2\hbar}\Big(\Big(X + \Big(i\pi + \frac{\hbar}{2}\Big)\nu\Big) \cdot \boldsymbol{DB}^{-1}\Big(X + \Big(i\pi + \frac{\hbar}{2}\Big)\nu\Big) \\
&- 2Z \cdot \boldsymbol{B}^{-1}\Big(X + \Big(i\pi + \frac{\hbar}{2}\Big)\nu\Big) + Z \cdot \boldsymbol{B}^{-1}\boldsymbol{A}Z\Big)\Bigg]\mathcal{Z}_\times(Z),
\end{aligned}
$$

and then, after setting $X \to 2\boldsymbol{u} = (0, \dots, 0, 2u)$ as in (C-17), we find

$$
\begin{aligned}
\text{(C-21)} \quad \mathcal{Z}_M(u) = \frac{1}{\sqrt{\det \boldsymbol{B}}} \int \frac{d^N Z}{(2\pi i\hbar)^{N/2}} \exp\Bigg[&-\frac{1}{\hbar}(2\pi i + \hbar)\nu_\lambda u \\
&+ \frac{1}{2\hbar}\Big(\Big(2\boldsymbol{u} + \Big(i\pi + \frac{\hbar}{2}\Big)\nu\Big) \cdot \boldsymbol{DB}^{-1}\Big(2\boldsymbol{u} + \Big(i\pi + \frac{\hbar}{2}\Big)\nu\Big) \\
&- 2Z \cdot \boldsymbol{B}^{-1}\Big(2\boldsymbol{u} + \Big(i\pi + \frac{\hbar}{2}\Big)\nu\Big) + Z \cdot \boldsymbol{B}^{-1}\boldsymbol{A}Z\Big)\Bigg]\prod_{i=1}^N \psi_\hbar(Z_i).
\end{aligned}
$$

This is the partition function of the one-cusped manifold M, modulo a multiplicative ambiguity of the form $\exp[\frac{\pi^2}{6}a + \frac{i\pi}{4}b + \frac{1}{24}c]$ for $a, b, c \in \mathbb{Z}$, which we will say more about in Section C.5. By *construction*, this partition function is annihilated by the quantum $\widehat{A}$–polynomial of M.

C.4: Introducing a flattening

In order to obtain the state integral (5-23) appearing in the paper, we can introduce a generalized flattening (as in Section 4.4) and use it to simplify (C-21). Note that the discrete-faithful state integral (5-2) follows immediately from (5-23) upon setting $\boldsymbol{u} = (0, \dots, 0, u) \to 0$.

Suppose, then, that we have integers (f, f'') that satisfy

$$\text{(C-22)} \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f \\ f'' \end{pmatrix} = \begin{pmatrix} \nu \\ \nu_P \end{pmatrix},$$

for some ν_P whose last entry is ν_λ. We will *assume* that a completed symplectic matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ can be chosen in $\mathrm{Sp}(2N, \mathbb{Z})$ rather than in $\mathrm{Sp}(2N, \mathbb{Q})$. In that case, since f and f'' are vectors with integer entries, it follows that $\nu_P \in \mathbb{Z}^N$. Then,

$$-\nu_\lambda u + \nu \cdot DB^{-1}u = -(Cf + Df'') \cdot u + (D^T A f + D^T B f'') \cdot B^{-1}u = f \cdot B^{-1}u,$$

where we used the symplectic identities $D^T B = B^T D$ and $D^T A = I + B^T C$; and

$$\begin{aligned}
\nu \cdot DB^{-1}\nu &= \nu \cdot D(f'' + B^{-1}Af) \\
&= \nu \cdot (Df'' + Cf + B^{-1,T}f) \\
&= f \cdot B^{-1}\nu + \nu \cdot \nu_P \\
&= f \cdot B^{-1}\nu \ (\text{mod } \mathbb{Z}),
\end{aligned}$$

in a similar way. These relations allow us to write the state integral (C-21) as

$$\text{(C-23)} \qquad \mathcal{Z}_M(u) = \frac{1}{\sqrt{\det B}} \int \frac{d^N Z}{(2\pi\hbar)^{N/2}} e^x \prod_{i=1}^N \psi_\hbar(Z_i),$$

where $x = \frac{1}{\hbar}[2u \cdot DB^{-1}u + (2\pi i + \hbar)f \cdot B^{-1}u + \frac{1}{2}(i\pi + \frac{\hbar}{2})^2 f \cdot B^{-1}\nu - ZB^{-1}(2u + (i\pi + \frac{\hbar}{2})\nu)]$, just as in (5-23). (We drop a factor of $\sqrt{i}$ from the measure, since it can be absorbed in the overall normalization ambiguity.)

C.5: Normalization and invariance

The normalization of Chern–Simons state integrals has always been a subtle issue. For the integral of [15], ambiguities in the normalization come from two sources: the projectivity of the Weil representation, and the incomplete invariance of the integral (even formally) under a change of "quad type" and a 2–3 move.

Let us consider the Weil representation first. We will assume that all symplectic matrices are in $\mathrm{Sp}(2N, \mathbb{Z})$, and that all shifts involve integers (like ν and ν_P) times $i\pi + \frac{\hbar}{2}$. This assumption (which, again, is only an observed property) allows us to improve on the estimates of [15, Equation (6.6)]. The Weil representation becomes a projective unitary representation of $\mathrm{ISp}(2N, \mathbb{Z}) \simeq \mathrm{Sp}(2N, \mathbb{Z}) \ltimes [(i\pi + \frac{\hbar}{2})\mathbb{Z}]^{2N}$ on $L^2(\mathbb{R}^N)$, for $\hbar$ pure imaginary. Our Hilbert space $\mathcal{H}_\Delta^{\otimes 2N}$ is very close to $L^2(\mathbb{R}^N)$, so we may hope that the Weil representation is also unitary projective there. The most severe projective

ambiguity arises from a violation of expected commutation relations between shifts and
T–type transformations such as (C-19). This leads to projective factors of the form

$$
\text{(C-24)} \qquad \exp\left[\frac{1}{2\hbar}\left(i\pi + \frac{\hbar}{2}\right)^2 a\right] = \exp\left[\left(-\frac{\pi^2}{2\hbar} + \frac{i\pi}{2} + \frac{\hbar^2}{8}\right)a\right], \quad a \in \mathbb{Z}.
$$

With the exception of factors like this, unitarity with respect to the norm

$$
\|f\|^2 = \int \frac{d^N Z}{(\pm 2\pi i\hbar)^{N/2}} |f(Z)|^2
$$

may be used to normalize Weil transformations. For example, the factor in (C-20),
$[(2\pi i\hbar)^N \det \boldsymbol{B}]^{-1/2}$, follows easily from formal manipulations on the integral trans-
formation to demonstrate unitarity.

The lack of complete invariance under a change of quad type (cyclic permutation
invariance) and a 2–3 move can also ruin the normalization of the state integral. The
change of quad type was analyzed, formally, in [15, Section 6.2.1]. A cyclic permutation
of a tetrahedron is accomplished by an affine version of the element $ST \in \text{Sp}(2N;\mathbb{Z})$,
under the Weil representation. The single-tetrahedron wavefunction transforms as

$$
\text{(C-25)} \quad \psi_\hbar(Z) \mapsto \int \frac{dZ}{\sqrt{2\pi i\hbar}} e^{\frac{1}{2\hbar}(Z^2 + 2ZZ' - (2\pi i + \hbar)Z)} \psi_\hbar(Z)
$$
$$
= e^{\frac{\pi^2}{6\hbar} \pm \frac{i\pi}{4} - \frac{\hbar}{24}} \psi_\hbar(Z').
$$

The last equality follows from the Fourier transform of the quantum dilogarithm; see
Faddeev, Kashaev and Volkov [29] and Ponsot and Teschner [61]. This shows that the
tetrahedron wavefunction is *invariant* under permutations, up to a factor

$$
\text{(C-26)} \qquad \exp\left[\left(\frac{\pi^2}{6\hbar} \pm \frac{i\pi}{4} - \frac{\hbar}{24}\right)a\right], \quad a \in \mathbb{Z}.
$$

The analysis of the 2–3 move is slightly more involved. It was done in terms of operator
algebra in [15], and then explained in terms of wavefunctions in [16, Section 6.2].
The main idea is that a 2–3 move can be done locally during the gluing procedure, by
performing a formal, "local" transformation on the state integral. The crucial property
involved is the Ramanujan-like identity for the quantum dilogarithm [29; 61], which
expresses three quantum dilogarithms as an integral of two; for example,

$$
\text{(C-27)} \quad \psi_\hbar(W_1')\psi_\hbar(W_2')\psi_\hbar(W_3')|_{W_1' + W_2' + W_3' = 2\pi i + \hbar}
$$
$$
\sim \int \frac{dZ}{\sqrt{2\pi i\hbar}} e^{\frac{1}{2\hbar}(Z^2 + 2W_2'Z - (2\pi i + \hbar)(W_1' + W_2' + Z))} \psi_\hbar(-Z)\psi_\hbar(Z - W_1')
$$

which holds up to a factor that is again of the type (C-26).

Putting together all three effects, we find that we might be able to control the overall normalization of the state integral up to a factor of the form

$$(\text{C-28}) \qquad \exp\left[\frac{\pi^2}{6\hbar}a + \frac{i\pi}{4}b + \frac{\hbar}{24}c\right], \quad a, b, c \in \mathbb{Z}.$$

Appendix D: Computer implementation and computations

An enhanced Neumann–Zagier datum is a tuple $(z, \boldsymbol{A}, \boldsymbol{B}, f)$ attached to a regular ideal triangulation of a cusped hyperbolic manifold M. The program `SnapPy` [13] in its `python` and `sage` implementation computes the gluing matrices $\boldsymbol{G}, \boldsymbol{G}', \boldsymbol{G}''$ of Sections 2.4 and 4.4; and therefore it can easily compute an enhanced Neumann–Zagier datum $\hat{\beta}_{\mathcal{T}} = (z, \boldsymbol{A}, \boldsymbol{B}, f)$. The shape parameters z are algebraic numbers computed numerically to arbitrary precision (eg, 10000 digits) or exactly as algebraic numbers.

A `Mathematica` module of the authors computes (numerically or exactly) the n–loop invariants $S_{\mathcal{T},n}$ for $n = 0, 2, 3$ as well as our torsion $\tau_{\mathcal{T}}$ given as input the Neumann–Zagier datum. As an example, consider the hyperbolic knot $\boldsymbol{9_{12}}$ with volume $8.836642343\ldots$ and the `SnapPy` ideal triangulation with 10 tetrahedra. Its invariant trace field $E_{\boldsymbol{9_{12}}}$ is $\mathbb{Q}(x)$ where $x = -0.06265158\ldots + i\,1.24990458\ldots$ is a root of

$$x^{17} - 8x^{16} + 32x^{15} - 89x^{14} + 195x^{13} - 353x^{12} + 542x^{11} - 719x^{10} + 834x^9$$
$$- 851x^8 + 764x^7 - 605x^6 + 421x^5 - 253x^4 + 130x^3 - 55x^2 + 18x - 3 = 0.$$

$E_{\boldsymbol{9_{12}}}$ is of type $[1, 8]$ with discriminant $3 \cdot 298171 \cdot 5210119 \cdot 156953399$. Our torsion is

$$\tau_{\boldsymbol{9_{12}}} = \frac{1}{2}(15 - 7x - 15x^2 + 55x^3 - 67x^4 + 81x^5 - 43x^6 - 112x^7 + 303x^8 - 488x^9$$
$$+ 606x^{10} - 595x^{11} + 464x^{12} - 289x^{13} + 143x^{14} - 49x^{15} + 8x^{16})$$
$$= -3.133657804174628986\ldots + 14.061239582208047255\ldots i.$$

The two and three-loop invariants simplify considerably when multiplied by $\tau_{\boldsymbol{9_{12}}}^3$ and $\tau_{\boldsymbol{9_{12}}}^6$ respectively and are given by

$$S_{\boldsymbol{9_{12}},2}\tau_{\boldsymbol{9_{12}}}^3 = \frac{1}{2^6 \cdot 3}(36263 - 194718x + 503316x^2 - 971739x^3 + 1582041x^4$$
$$- 2152164x^5 + 2372779x^6 - 2109742x^7 + 1426659x^8$$
$$- 484152x^9 - 374803x^{10} + 836963x^{11} - 859483x^{12}$$
$$+ 621288x^{13} - 326550x^{14} + 109607x^{15} - 16840x^{16})$$
$$= 398.62270435384630954\ldots + 948.91209325049603870\ldots i,$$

$$S_{\mathbf{9}_{12},3}\tau_{\mathbf{9}_{12}}^{6}$$

$$
\begin{aligned}
= \tfrac{1}{2^7}(&2320213 - 19092785x^1 + 72589953x^2 - 186402605x^3 + 382362100x^4 \\
&- 661985976x^5 + 982969902x^6 - 1258919324x^7 + 1402544816x^8 \\
&- 1359436057x^9 + 1134208276x^{10} - 803313515x^{11} + 473961630x^{12} \\
&- 225394732x^{13} + 80872920x^{14} - 19104127x^{15} + 2161102x^{16})
\end{aligned}
$$

$$= 71793.64335382669630\ldots + 204530.00105728258992\ldots i.$$

The norm $(N_1, N_2, N_3) = (N(\tau_{\mathbf{9}_{12}}), N(S_{\mathbf{9}_{12},2}\tau_{\mathbf{9}_{12}}^{3}), N(S_{\mathbf{9}_{12},3}\tau_{\mathbf{9}_{12}}^{6}))$ of the above algebraic numbers is given by

$$N_1 = \frac{3 \cdot 298171 \cdot 5210119 \cdot 156953399}{2^{17}},$$

$$N_2 = \frac{173137 \cdot 2497646101 \ldots \ldots 5575954409 \quad (70 \text{ digits})}{2^{102} \cdot 3^{17}},$$

$$N_3 = \frac{1601979456 \ldots \ldots 5984185143 \quad (100 \text{ digits})}{2^{119}}.$$

Recall that although $S_{2,\mathbf{9}_{12}}$ is defined modulo an integer multiple of $1/24$, $S_{3,\mathbf{9}_{12}}$ is defined without ambiguity and the numerator N_3 is a prime number of 103 digits.

For a computation of the Reidemeister torsion τ_M^{R} of the discrete faithful representation of a cusped hyperbolic manifold M, we use a theorem of Yamaguchi [74] to identify it with

$$\tau_M^{\mathrm{R}} = \frac{1}{c_M} \frac{d\tau_M^{\mathrm{R}}(t)}{dt}\bigg|_{t=1}$$

where c_M is the cusp shape of M and $\tau_M^{\mathrm{R}}(t) \in E_M[t^{\pm 1}]$ is the *torsion polynomial* of M using the adjoint representation of $\mathrm{SL}(2, \mathbb{C})$. Using the `hypertorsion` package of N Dunfield (see [21]), we can compute τ_M^{R} as follows:

```
cd Genus-Comp
sage:import snappy, hypertorsion

def torsion(manifold, precision=100):
    M = snappy.Manifold(manifold)
    p = hypertorsion.hyperbolic_adjoint_torsion(M, precision)
    q = p.derivative()
    rho = hypertorsion.polished_holonomy(M, precision)
    z = rho.cusp_shape()
    torsion = q(1)/z.conjugate()
    return [M.name(), torsion]
```

For the above example, we have

```
sage: torsion("9_12",500)
['L105002',  -3.133657804174628986\ldots
+ 14.061239582208047255\ldots*I]
```

numerically confirming Conjecture 1.8. Further computations gives a numerical confirmation of Conjecture 1.8 to 1000 digits for all 59924 hyperbolic knots with at most 14 crossings.

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