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An index theorem in differential K–theory

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Let $\pi\colon X \to B$ be a proper submersion with a Riemannian structure. Given a differential K–theory class on X, we define its analytic and topological indices as differential K–theory classes on B. We prove that the two indices are the same.

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To our teacher Isadore Singer on the occasion of his 85th birthday

1 Introduction

Let $\pi\colon X \to B$ be a proper submersion of relative dimension n. The Atiyah–Singer families index theorem [7] equates the analytic and topological index maps, defined on the topological K–theory of the relative tangent bundle. Suppose that the relative tangent bundle has a spin^c–structure. This orients the map π in K–theory, and the index maps may be expressed as pushforwards $K^0(X;\mathbb{Z}) \to K^{-n}(B;\mathbb{Z})$. The topological index map π_*^{top}, which preceded the index theorem, is due to Atiyah and Hirzebruch [2]. The analytic index map π_*^{an} is defined in terms of Dirac-type operators as follows. Let $E \to X$ be a complex vector bundle representing a class $[E] \in K^0(X;\mathbb{Z})$. Choose a Riemannian structure on the relative tangent bundle, a spin^c–lift of the resulting Levi-Civita connection and a connection on E. This geometric data determines a family of fiberwise Dirac-type operators, parametrized by B. The analytic index $\pi_*^{\mathrm{an}}[E] \in K^{-n}(B;\mathbb{Z})$ is the homotopy class of that family of Fredholm operators; it is independent of the geometric data. The families index theorem asserts $\pi_*^{\mathrm{an}} = \pi_*^{\mathrm{top}}$. In the special case when B is a point, one recovers the original integer-valued Atiyah–Singer index theorem [6].

The work of Atiyah and Singer has led to many other index theorems. In this paper we prove a geometric extension of the Atiyah–Singer theorem, in which K–theory is replaced by *differential* K–theory. Roughly speaking, differential K–theory combines topological K–theory with differential forms. We define analytic and topological pushforwards in differential K–theory. The analytic pushforward $\mathrm{ind}^{\mathrm{an}}$ is constructed using the Bismut superconnection and local index theory techniques. The topological

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pushforward $\mathrm{ind}^{\mathrm{top}}$ is constructed as a refinement of the Atiyah–Hirzebruch pushforward in topological K–theory. Our main result is the following theorem.

Theorem *Let $\pi\colon X \to B$ be a proper submersion of relative dimension n equipped with a Riemannian structure and a differential* spin^c*–structure. Then $\mathrm{ind}^{\mathrm{an}} = \mathrm{ind}^{\mathrm{top}}$ as homomorphisms $\check{K}^0(X) \to \check{K}^{-n}(B)$ on differential K–theory.*

This theorem provides a topological formula (with differential forms) for geometric invariants of Dirac-type operators. We illustrate this for the determinant line bundle (Section 8) and the reduced eta-invariant (Section 9).

In the remainder of the introduction we describe the theorem and its proof in more detail. We also give some historical background.

1.1 Differential K–theory

Let u be a formal variable of degree 2 and put $\mathcal{R} = \mathbb{R}[u, u^{-1}]$. For any manifold X, its differential K–theory $\check{K}^\bullet(X)$ fits into the commutative square

$$\begin{array}{ccc} \check{K}^\bullet(X) & \longrightarrow & \Omega^\bullet(X;\mathcal{R})_K \\ \downarrow & & \downarrow \\ K^\bullet(X;\mathbb{Z}) & \longrightarrow & H^\bullet(X;\mathcal{R}). \end{array} \tag{1.2}$$

The bottom map is the Chern character $\mathrm{ch}\colon K^\bullet(X;\mathbb{Z}) \to H(X;\mathcal{R})^\bullet$; the formal variable u encodes Bott periodicity. Also, $\Omega(X;\mathcal{R})^\bullet_K$ denotes the closed differential forms whose cohomology class lies in the image of the Chern character. The right vertical map is defined by the de Rham theorem. One can define differential K–theory by positing that (1.2) be a *homotopy* pullback square (see Hopkins and Singer [28]), which is the precise sense in which differential K–theory combines topological K–theory with differential forms.

We use a geometric model for differential K–theory, defined by generators and relations. A generator $\mathcal{E}$ of $\check{K}^0(X)$ is a quadruple (E, h^E, ∇^E, ϕ), where $E \to X$ is a complex vector bundle, h^E is a Hermitian metric, ∇^E is a compatible connection, and $\phi \in \Omega(X;\mathcal{R})^{-1}/\,\mathrm{Image}(d)$. The relations come from short exact sequences of Hermitian vector bundles. There is a similar description of $\check{K}^{-1}(X)$, in which (E, h^E, ∇^E, ϕ) is additionally equipped with a unitary automorphism $U^E \in \mathrm{Aut}(E)$. One can then use periodicity to define $\check{K}^r(X)$ for any integer r.

1.3 Pushforwards for geometric submersions

Let $\pi\colon X \to B$ be a proper submersion. A Riemannian structure on π consists of an inner product on the vertical tangent bundle and a horizontal distribution on X. A differential spin^c–structure on π is a topological spin^c–structure together with a unitary connection on the characteristic line bundle associated to the spin^c–structure. This geometric data determines a local index form $\mathrm{Todd}(X/B) \in \Omega(X;\mathcal{R})^0$, the spin^c–version of the $\widehat{A}$–form.

Suppose first that the fibers of π have even dimension n. Then there is a diagram

$$\text{(1.4)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K^{-1}(X;\mathbb{R}/\mathbb{Z}) & \xrightarrow{\;j\;} & \check{K}^0(X) & \xrightarrow{\;\omega\;} & \Omega(X;\mathcal{R})^0_K & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{ind}^{\mathrm{an}}}\downarrow\downarrow{\scriptstyle \mathrm{ind}^{\mathrm{top}}} & & {\scriptstyle ?}\downarrow\downarrow{\scriptstyle ?} & & \downarrow{\scriptstyle \int_{X/B}\mathrm{Todd}(X/B)\wedge -} & & \\ 0 & \longrightarrow & K^{-n-1}(B;\mathbb{R}/\mathbb{Z}) & \xrightarrow{\;j\;} & \check{K}^{-n}(B) & \xrightarrow{\;\omega\;} & \Omega(B;\mathcal{R})^{-n}_K & \longrightarrow & 0 \end{array}$$

in which the rows are exact sequences closely related to (1.2), easily derived from the definition of differential K–theory. The left vertical arrows are the topological index $\mathrm{ind}^{\mathrm{top}}$, defined by a construction in generalized cohomology theory, and the analytic index $\mathrm{ind}^{\mathrm{an}}$, defined by the second author [33]. The main theorem of [33] is the equality of these arrows. Our analytic and topological indices are defined to fill in the ?'s in the middle vertical arrows subject to the condition that the resulting two diagrams (with analytic and topological indices) commute. Note that the right vertical arrow depends on the geometric structures; hence the same is true for the middle vertical arrows.

The analytic index is based on Quillen's notion of a superconnection [42] as generalized to the infinite-dimensional setting by Bismut [10]. To define it in our finite-dimensional model of differential K–theory we use the Bismut–Cheeger eta form [11], which mediates between the Chern character of the Bismut superconnection and the Chern character of the finite-dimensional index bundle. The resulting Definition 3.12 is then a simple extension of the $\mathbb{R}/\mathbb{Z}$ analytic index in [33].

As in topological K–theory, to define the topological index we factor π as the composition of a fiberwise embedding $X \to S^N \times B$ and the projection $S^N \times B \to B$, broadly basing our construction on Hopkins and Singer [28], Klonoff [32] and Ortiz [40]. However, our differential K–theory pushforward for an embedding, given in Definition 4.14, is new and of independent interest. For our proof of the main theorem we want the image to be defined in terms of currents instead of differential forms, so the embedding pushforward lands in the "currential" K–theory of $S^N \times B$. The definition uses the Bismut–Zhang current [13], which essentially mediates between the Chern character of a certain superconnection and a cohomologous current supported on the image of the embedding. A Künneth decomposition of the currential K–theory of $S^N \times B$ is used to give an explicit formula for the projection pushforward. The topological index is the composition of the embedding and projection pushforwards, with a modification to account for a discrepancy in the horizontal distributions. The topological index does not involve any spectral analysis, but does involve differential forms, so may be more appropriately termed the "differential topological index".

For proper submersions with odd fiber dimension, we introduce suspension and desuspension maps between even and odd differential K–theory groups. We use them to define topological and analytic pushforward maps for the odd case in terms of those for the even case.

The preceding constructions apply when B is compact. To define the index maps for noncompact B, we take a limit over an exhaustion of B by compact submanifolds. This depends on a result of independent interest which we prove in the Appendix: $\check{K}^\bullet(B)$ is isomorphic to the inverse limit of the differential K–theory groups of compact submanifolds.

1.5 Method of proof

The commutativity of the right-hand square of (1.4) for both the analytic and topological pushforwards, combined with the exactness of the bottom row, implies that for any $\mathcal{E} \in \check{K}^0(X)$ we have $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) - \mathrm{ind}^{\mathrm{top}}(\mathcal{E}) = j(\mathcal{T})$ for a unique $\mathcal{T} \in K^{-n-1}(B;\mathbb{R}/\mathbb{Z})$. We now use the basic method of proof in [33] to show that $\mathcal{T}$ vanishes. Namely, it suffices to demonstrate the vanishing of the pairing of $\mathcal{T}$ with any element of the K–homology group $K_{-n-1}(B;\mathbb{Z})$. Such pairings are given by reduced eta-invariants, assuming the family of Dirac-type operators has vector bundle kernel. After some rewriting we are reduced to proving an identity involving a reduced eta-invariant of $S^N \times B$, a reduced eta-invariant of X, and the eta form on B. The relation between the reduced eta-invariant of X and the eta form on B is an adiabatic limit result of Dai [17]. The new input is a theorem of Bismut–Zhang which relates the reduced eta-invariant of X to the reduced eta-invariant of $S^N \times B$ [13]. To handle the case when the rank of the kernel is not locally constant we follow a perturbation argument from [33], which uses a lemma of Miščenko–Fomenko [39].

1.6 Historical discussion

Karoubi's description of K–theory with coefficients [31] combines vector bundles, connections, and differential forms into a topological framework. His model of $K^{-1}(X;\mathbb{C}/\mathbb{Z})$ is essentially the same as the kernel of the map ω in (1.4). (Hermitian metrics can be added to his model to get $\mathbb{R}/\mathbb{Z}$–coefficients.) Inspired by this work, Gillet and Soulé [24] defined a group $\widehat{K}^0(X)$ in the holomorphic setting which is a counterpart of differential K–theory in the smooth setting. Faltings [18] and Gillet–Rössler–Soulé [23] proved an arithmetic Riemann–Roch theorem about these groups. Using Karoubi's description of K–theory with coefficients, the second author proved an index theorem in $\mathbb{R}/\mathbb{Z}$–valued K–theory [33]. Based on the Gillet–Soulé work, he also considered what could now be called differential flat K–theory $\widehat{K}^0_R(X)$ and differential L–theory $\widehat{L}^0_\epsilon(X)$ [34].

Differential K–theory has an antecedent in the differential character groups of Cheeger–Simons [16], which are isomorphic to integral differential cohomology groups. Independent of the developments in the last paragraph, and with physical motivation, the first author sketched a notion of differential K–theory $\check{K}^0(X)$ [20; 22]. In retrospect, differential K–theory can also be seen as a case of Karoubi's multiplicative K–theory [30] for a particular choice of subcomplexes. Generalized differential cohomology was developed by Hopkins and Singer [28]. In particular, they defined differential orientations and pushforwards in their setting, and constructed a pushforward in differential K–theory for such a map [28, Example 4.85] (but with a more elaborate notion of "differential spin^c–structure"). Klonoff [32] constructed an isomorphism between the differential K–theory group $\check{K}^0(X)$ defined by Hopkins–Singer and the one given by the finite-dimensional model used in this paper. (We remark that the argument in [32] relies on a universal connection which is not proved to exist; Ortiz [40, Section 3.3] gives a modification of the argument which bypasses this difficulty.) A topological index map for proper submersions is also developed in [32; 40] and it fits into a commutative diagram (1.4). It has many of the same ingredients as the topological index defined here and most likely agrees with it, but we have not checked the details. One notable difference is our use of currents, a key element in our proof of the main theorem. In [32], Klonoff proves a version of Proposition 8.3 and Corollary 9.39.

There are many other recent works on differential K–theory, among which we only mention two. Bunke and Schick [15] use a different definition of $\check{K}^\bullet(X)$ in which the generators are fiber bundles over X with a Riemannian structure and a differential spin^c–structure. They prove a rational Riemann–Roch-type theorem with value in rational differential cohomology. This theorem is the result of applying the differential Chern character to our index theorem; see Section 8.13. In a different direction, Simons and Sullivan [43] prove that the differential form ϕ in the definition of $\check{K}^0(X)$ can be removed, provided that one modifies the relations accordingly. In this way, differential K–theory really becomes a K–theory of vector bundles with connection. However, in our approach to the analytic and topological indices it is natural to include the form ϕ.

The differential index theorem, or rather its consequence for determinant line bundles, is used in Type I string theory to prove the anomaly cancellation known as the "Green–Schwarz mechanism" [20], at least on the level of isomorphism classes. Indeed, this application was one motivation to consider a differential K–theory index theorem, for the first author. The differential K–theory formula for the determinant line bundle reduces in special low dimensional cases to a formula in a simpler differential cohomology theory. There is a two-dimensional example relevant to worldsheet string theory [19, Section 5] and an example in four-dimensional gauge theory [21, Section 2].

1.7 Outline

We begin in Section 2 with some basic material about characteristic forms, differential K–theory and reduced eta-invariants. We also establish our notation. In Section 3 we define the analytic index for differential K–theory, in the case of vector bundle kernel. In Section 4 we construct the pushforward for differential K–theory under an embedding which is provided with a Riemannian structure and a differential spin^c–structure on its normal bundle. It lands in the currential K–theory of the image manifold. In Section 5 we construct the topological index. In Section 6 we prove our main theorem in the case of vector bundle kernel. The general case is covered in Section 7. The relationship of our index theorem to determinant line bundles, differential Riemann–Roch theorems and indices in Deligne cohomology is the subject of Section 8. In Section 9 we describe how to extend the results of the preceding sections, which were for even differential K–theory and even relative dimension, to the odd case by means of suspensions and desuspensions. In the Appendix we prove that the differential K–theory of a noncompact manifold may be computed as a limit of the differential K–theory of compact submanifolds.

More detailed explanations appear at the beginnings of the individual sections.

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2 Background material

In this section we review some standard material and clarify notation.

In Section 2.1 we describe the Chern character and the relative Chern–Simons form. One slightly nonstandard point is that we include a formal variable u of degree 2 so that the Chern character preserves integer cohomological degree as opposed to only a mod 2 degree.

In Section 2.15 we define differential K–theory in even degrees using a model which involves vector bundles, connections and differential forms. We will also need a slight extension of differential K–theory in which differential forms are replaced by de Rham currents. This "currential" K–theory is introduced in Section 2.28.

In Section 2.31 we show that on a compact odd-dimensional spin^c–manifold, the Atiyah–Patodi–Singer reduced η–invariant gives an invariant of currential K–theory. Finally, in Section 2.43 we recall Quillen's definition of superconnections and their associated Chern character forms.

2.1 Characteristic classes

Define the $\mathbb{Z}$–graded real algebra

$$\mathcal{R}^\bullet = \mathbb{R}[u, u^{-1}], \qquad \deg u = 2. \tag{2.2}$$

It is isomorphic to $K^\bullet(\mathrm{pt}; \mathbb{R})$.

Let X be a smooth manifold. Let $\Omega(X; \mathcal{R})^\bullet$ denote the $\mathbb{Z}$–graded algebra of differential forms with coefficients in $\mathcal{R}$; we use the total grading. Let $H(X; \mathcal{R})^\bullet$ denote the corresponding cohomology groups. Let $R_u\colon \Omega(X; \mathcal{R})^\bullet \otimes \mathbb{C} \to \Omega(X; \mathcal{R})^\bullet \otimes \mathbb{C}$ be the map which multiplies u by $2\pi i$.

We take $K^0(X; \mathbb{Z})$ to be the homotopy-invariant K–theory of X, ie $K^0(X; \mathbb{Z}) = [X, \mathbb{Z} \times BGL(\infty, \mathbb{C})]$. Note that one can carry out all of the usual K–theory constructions without any further assumption on the manifold X, such as compactness or finite topological type. For example, given a complex vector bundle over X, there is always another complex vector bundle on X so that the direct sum is a trivial bundle [38, Problem 5-E].

We can describe $K^0(X; \mathbb{Z})$ as an abelian group generated by complex vector bundles E over X equipped with Hermitian metrics h^E. The relations are that $E_2 = E_1 + E_3$ whenever there is a short exact sequence of Hermitian vector bundles, meaning that there is a short exact sequence

$$0 \longrightarrow E_1 \stackrel{i}{\longrightarrow} E_2 \stackrel{j}{\longrightarrow} E_3 \longrightarrow 0 \tag{2.3}$$

so that i and j^* are isometries. Note that in such a case, we get an orthogonal splitting $E_2 = E_1 \oplus E_3$.

Let ∇^E be a compatible connection on E. The corresponding Chern character form is

$$\omega(\nabla^E) = R_u \operatorname{tr}(e^{-u^{-1}(\nabla^E)^2}) \in \Omega(X;\mathcal{R})^0. \tag{2.4}$$

It is a closed form whose de Rham cohomology class $\operatorname{ch}(E) \in H(X;\mathcal{R})^0$ is independent of ∇^E. The map ch: $K^0(X;\mathbb{Z}) \longrightarrow H(X;\mathcal{R})^0$ becomes an isomorphism after tensoring the left-hand side with $\mathbb{R}$. We also put

$$c_1(\nabla^E) = -\frac{1}{2\pi i}\, u^{-1} \operatorname{tr}((\nabla^E)^2) \in \Omega(X;\mathcal{R})^0. \tag{2.5}$$

We can represent $K^0(X;\mathbb{Z})$ using $\mathbb{Z}/2\mathbb{Z}$–graded vector bundles. A generator of $K^0(X;\mathbb{Z})$ is then a $\mathbb{Z}/2\mathbb{Z}$–graded complex vector bundle $E = E_+ \oplus E_-$ on X, equipped with a Hermitian metric $h^E = h^{E_+} \oplus h^{E_-}$. Choosing unitary connections $\nabla^{E_\pm}$ and letting str denote the supertrace, we put

$$\omega(\nabla^E) = R_u \operatorname{str}(e^{-u^{-1}(\nabla^E)^2}) \in \Omega(X;\mathcal{R})^0. \tag{2.6}$$

More generally, if r is even then by Bott periodicity, we can represent a generator of $K^r(X;\mathbb{Z})$ by a complex vector bundle E on X, equipped with a Hermitian metric h^E. Again, we choose a compatible connection ∇^E. In order to define the Chern character form, we put

$$\omega(\nabla^E) = u^{r/2} R_u \operatorname{tr}(e^{-u^{-1}(\nabla^E)^2}) \in \Omega(X;\mathcal{R})^r, \tag{2.7}$$

and similarly for $\mathbb{Z}/2\mathbb{Z}$–graded generators of $K^r(X;\mathbb{Z})$.

2.8 Remark Note the factor of $u^{r/2}$. It would perhaps be natural to insert the formal variable $u^{r/2}$ in front of E but we will refrain from doing so. In any given case, it should be clear from the context what the degree is.

If ∇_1^E and ∇_2^E are two connections on a vector bundle E then there is an explicit relative Chern–Simons form $\operatorname{CS}(\nabla_1^E, \nabla_2^E) \in \Omega(X;\mathcal{R})^{-1}/\operatorname{Image}(d)$. It satisfies

$$d\,\operatorname{CS}(\nabla_1^E, \nabla_2^E) = \omega(\nabla_1^E) - \omega(\nabla_2^E). \tag{2.9}$$

More generally, if

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \tag{2.10}$$

is a short exact sequence of vector bundles with connections $\{\nabla^{E_i}\}_{i=1}^3$ then there is an explicit relative Chern–Simons form $\operatorname{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \in \Omega(X;\mathcal{R})^{-1}/\operatorname{Image}(d)$. It satisfies

$$d\,\operatorname{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = \omega(\nabla^{E_2}) - \omega(\nabla^{E_1}) - \omega(\nabla^{E_3}). \tag{2.11}$$

To construct $\mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$, put $W = [0,1] \times X$ and let $p\colon W \to X$ be the projection map. Put $F = p^* E_2$. Let ∇^F be a unitary connection on F which equals $p^* \nabla^{E_2}$ near $\{1\} \times X$ and which equals $p^*(\nabla^{E_1} \oplus \nabla^{E_3})$ near $\{0\} \times X$. Then

$$\text{(2.12)} \qquad \mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = \int_0^1 \omega(\nabla^F) \in \Omega(X;\mathcal{R})^{-1} / \operatorname{Image}(d).$$

If W is a real vector bundle on X with connection ∇^W then we put

$$\text{(2.13)} \qquad \widehat{A}(\nabla^W) = R_u \sqrt{\det\left(\frac{u^{-1}\Omega^W/2}{\sinh u^{-1}\Omega^W/2}\right)} \quad \in \Omega(X;\mathcal{R})^0,$$

where Ω^W is the curvature of ∇^W.

Suppose that W is an oriented $\mathbb{R}^n$–vector bundle on X with a Euclidean metric h^W and a compatible connection ∇^W. Let $\mathcal{B} \to X$ denote the principal $\mathrm{SO}(n)$–bundle on X to which W is associated. We say that W has a spin^c–structure if the principal $\mathrm{SO}(n)$–bundle $\mathcal{B} \to X$ lifts to a principal $\mathrm{Spin}^c(n)$–bundle $\mathcal{F} \to X$. Let $\mathcal{S}^W \to X$ be the complex spinor bundle on X that is associated to $\mathcal{F}$. It is $\mathbb{Z}/2\mathbb{Z}$–graded if n is even and ungraded if n is odd. Let $L^W \to X$ denote the characteristic line bundle on X that is associated to $\mathcal{F} \to X$ by the homomorphism $\mathrm{Spin}^c(n) \to U(1)$. (Recall that $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}/2\mathbb{Z}} U(1)$; the indicated homomorphism is trivial on the $\mathrm{Spin}(n)$ factor and is the square on the $U(1)$ factor.) Choose a unitary connection ∇^{L^W} on L^W. Then ∇^W and ∇^{L^W} combine to give a connection on $\mathcal{F}$ and hence an associated connection $\widehat{\nabla}^W$ on $\mathcal{S}^W$. We write

$$\text{(2.14)} \qquad \mathrm{Todd}(\widehat{\nabla}^W) = \widehat{A}(\nabla^W) \wedge e^{c_1(\nabla^{L^W})/2} \in \Omega(X;\mathcal{R})^0$$

The motivation for our notation comes from the case when W is the underlying real vector bundle of a complex vector bundle W'. If W' has a unitary structure then W inherits a spin^c–structure. If $\nabla^{W'}$ is a unitary connection on W' then $\mathcal{S}^W \cong \Lambda^{0,*}(W)$ inherits a connection $\widehat{\nabla}^W$ and $\mathrm{Todd}(\widehat{\nabla}^W)$ equals the Todd form of $\nabla^{W'}$ [26, Chapter 1.7].

2.15 Differential K–theory

2.16 Definition The differential K–theory group $\check{K}^0(X)$ is the abelian group coming from the following generators and relations. The generators are quadruples $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ where:

- E is a complex vector bundle on X.
- h^E is a Hermitian metric on E.

- ∇^E is an h^E–compatible connection on E.
- $\phi \in \Omega(X;\mathcal{R})^{-1}/\operatorname{Image}(d)$.

The relations are $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ whenever there is a short exact sequence (2.3) of Hermitian vector bundles and $\phi_2 = \phi_1 + \phi_3 - \mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$.

Hereafter, when we speak of a generator of $\check{K}^0(X)$, we will mean a quadruple $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ as above.

There is a homomorphism $\omega\colon \check{K}^0(X) \longrightarrow \Omega(X;\mathcal{R})^0$, given on generators by $\omega(\mathcal{E}) = \omega(\nabla^E) + d\phi$.

There is an evident extension of the definition of $\check{K}^0$ to manifolds-with-boundary.

We can also represent $\check{K}^0(X)$ using $\mathbb{Z}/2\mathbb{Z}$–graded vector bundles. A generator of $\check{K}^0(X)$ is then a quadruple consisting of a $\mathbb{Z}/2\mathbb{Z}$–graded complex vector bundle E on X, a Hermitian metric h^E on E, a compatible connection ∇^E on E and an element $\phi \in \Omega(X,\mathcal{R})^{-1}/\operatorname{Image}(d)$.

One can define $\check{K}^\bullet(X)$ by a general construction [28]; it is a 2–periodic generalized differential cohomology theory.

2.17 Remark The abelian group defined in Definition 2.16 is isomorphic to that defined by Hopkins and Singer [28]; see Klonoff [32] and Ortiz [40] for a proof.

We use the following model in arbitrary even degrees. For any even r, a generator of $\check{K}^r(X)$ is a quadruple $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ where $\phi \in \Omega(X,\mathcal{R})^{r-1}/\operatorname{Image}(d)$ has total degree $r-1$. For such a quadruple, we put

$$\omega(\mathcal{E}) = u^{r/2} R_u \operatorname{tr}(e^{-u^{-1}(\nabla^E)^2}) + d\phi \in \Omega(X;\mathcal{R})^r, \tag{2.18}$$

and similarly for $\mathbb{Z}/2\mathbb{Z}$–graded generators of $\check{K}^r(X)$.

2.19 Remark As in Remark 2.8, it would perhaps be natural to insert the formal variable $u^{r/2}$ in front of (E, h^E, ∇^E) but we will refrain from doing so.

Let $\Omega(X;\mathcal{R})^\bullet_K$ denote the union of affine subspaces of closed forms whose de Rham cohomology class lies in the image of ch: $K^\bullet(X;\mathbb{Z}) \longrightarrow H(X;\mathcal{R})^\bullet$. There are exact sequences

$$0 \longrightarrow K^{\bullet-1}(X;\mathbb{R}/\mathbb{Z}) \xrightarrow{i} \check{K}^\bullet(X) \xrightarrow{\omega} \Omega(X;\mathcal{R})^\bullet_K \longrightarrow 0, \tag{2.20}$$

$$0 \longrightarrow \frac{\Omega(X;\mathcal{R})^{\bullet-1}}{\Omega(X;\mathcal{R})^{\bullet-1}_K} \xrightarrow{j} \check{K}^\bullet(X) \xrightarrow{c} K^\bullet(X;\mathbb{Z}) \longrightarrow 0. \tag{2.21}$$

Also, $\check{K}^\bullet(X)$ is an algebra, with the product on $\check{K}^0(X)$ given by

(2.22) $$\begin{aligned}\bigl(E_1, h^{E_1}, \nabla^{E_1}, \phi_1\bigr) \cdot \bigl(E_2, h^{E_2}, \nabla^{E_2}, \phi_2\bigr) &= \bigl(E_1 \otimes E_2, h^{E_1} \otimes h^{E_2}, \nabla^{E_1} \otimes I + I \otimes \nabla^{E_2}, \\ &\qquad \phi_1 \wedge \omega(\nabla^{E_2}) + \omega(\nabla^{E_1}) \wedge \phi_2 + \phi_1 \wedge d\phi_2\bigr).\end{aligned}$$

Then with respect to the exact sequences (2.20) and (2.21),

(2.23) $$\begin{aligned} i(x)\check{b} &= i\bigl(xc(\check{b})\bigr), \\ j(\alpha)\check{b} &= j\bigl(\alpha \wedge \omega(\check{b})\bigr). \end{aligned}$$

We now describe how a differential K–theory class changes under a deformation of its Hermitian metric, its unitary connection and its differential form.

2.24 Lemma *For $i \in \{0,1\}$, let $A_i \colon X \to [0,1] \times X$ be the embedding $A_i(x) = (i,x)$. Given $\mathcal{E}' = (E', h^{E'}, \nabla^{E'}, \phi') \in \check{K}^0([0,1] \times X)$, put $\mathcal{E}_i = A_i^* \mathcal{E}' \in \check{K}^0(X)$. Then $\mathcal{E}_1 = \mathcal{E}_0 + j(\int_0^1 \omega(\mathcal{E}'))$.*

Proof We can write E' as the pullback of a vector bundle on X, under the projection map $[0,1] \times X \to X$. Thereby, E_0 and E_1 get identified with a single vector bundle E. After performing an automorphism of E', we can also assume that $h^{E'}$ is the pullback of a Hermitian metric h^E on E. Since

(2.25) $$\omega(\mathcal{E}') = \omega(\nabla^{E'}) + d\phi' = \omega(\nabla^{E'}) + dt \wedge \partial_t \phi' + d_X \phi',$$

we have

(2.26) $$\int_0^1 \omega(\mathcal{E}') = \mathrm{CS}(\nabla_1^E, \nabla_0^E) + \phi_1 - \phi_0 - d_X \int_0^1 \phi',$$

from which the lemma follows. □

2.27 Remark There is an evident extension of Lemma 2.24 to the case when $\mathcal{E}' = (E', h^{E'}, \nabla^{E'}, \phi') \in \check{K}^r([0,1] \times X)$ for r even.

2.28 Currential K–theory

Let ${}_\delta\Omega^p(X)$ denote the p–currents on X, meaning ${}_\delta\Omega^p(X) = (\Omega_c^{\dim(X)-p}(X; o))^*$, where o is the flat orientation $\mathbb{R}$–bundle on X. We think of an element of ${}_\delta\Omega^p(X)$ as a p–form on X whose components, in a local coordinate system, are distributional. Consider the cocomplex ${}_\delta\Omega(X;\mathcal{R})^\bullet = {}_\delta\Omega(X) \otimes \mathcal{R}$ equipped with the differential d of degree 1. In the definition of $\check{K}^0(X)$, suppose that we take $\phi \in {}_\delta\Omega(X)^{-1} / \operatorname{Image}(d)$.

Let ${}_\delta\check{K}^\bullet(X)$ denote the ensuing "currential" K–theory groups. With an obvious meaning for ${}_\delta\Omega(X;\mathcal{R})^n_K$, there are exact sequences

$$0 \longrightarrow K^{\bullet-1}(X;\mathbb{R}/\mathbb{Z}) \xrightarrow{i} {}_\delta\check{K}^\bullet(X) \xrightarrow{\omega} {}_\delta\Omega(X;\mathcal{R})^\bullet_K \longrightarrow 0, \tag{2.29}$$

$$0 \longrightarrow \frac{{}_\delta\Omega(X;\mathcal{R})^{\bullet-1}}{{}_\delta\Omega(X;\mathcal{R})^{\bullet-1}_K} \xrightarrow{j} {}_\delta\check{K}^\bullet(X) \xrightarrow{c} K^\bullet(X;\mathbb{Z}) \longrightarrow 0. \tag{2.30}$$

However, ${}_\delta\check{K}^\bullet(X)$ is not an algebra, since we can't multiply currents.

2.31 Reduced eta-invariants

Suppose that X is a closed odd-dimensional spin^c manifold. Let L^X denote the characteristic line bundle of the spin^c structure. We assume that X is equipped with a Riemannian metric g^{TX} and a unitary connection ∇^{L^X} on L^X. Let $\mathcal{S}^X$ denote the spinor bundle on X. Given a generator $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ for ${}_\delta\check{K}^0(X)$, let $D^{X,E}$ be the Dirac-type operator acting on smooth sections of $\mathcal{S}^X \otimes E$. Let $\overline{\eta}(D^{X,E})$ denote its reduced eta-invariant, ie

$$\overline{\eta}(D^{X,E}) = \frac{\eta(D^{X,E}) + \dim(\mathrm{Ker}(D^{X,E}))}{2} \pmod{\mathbb{Z}}. \tag{2.32}$$

2.33 Definition Define $\overline{\eta}(X,\mathcal{E}) \in u^{-(\dim(X)+1)/2} \cdot (\mathbb{R}/\mathbb{Z})$ for a given generator $\mathcal{E}$ for ${}_\delta\check{K}^0(X)$ by

$$\overline{\eta}(X,\mathcal{E}) = u^{-(\dim(X)+1)/2}\overline{\eta}(D^{X,E}) + \int_X \mathrm{Todd}(\widehat{\nabla}^{TX}) \wedge \phi \pmod{u^{-(\dim(X)+1)/2} \cdot \mathbb{Z}}. \tag{2.34}$$

Note that $\int_X \mathrm{Todd}(\widehat{\nabla}^{TX}) \wedge \phi$ is a real multiple of $u^{-(\dim(X)+1)/2}$ for dimensional reasons. Note also that $\overline{\eta}(X,\mathcal{E})$ generally depends on the geometric structure of X.

We now prove some basic properties of $\overline{\eta}(X,\mathcal{E})$.

2.35 Proposition (1) *Let W be an even-dimensional compact* spin^c*–manifold-with-boundary. Suppose that W is equipped with a Riemannian metric g^{TW} and a unitary connection ∇^{L^W}, which are products near ∂W. Let $\mathcal{F}$ be a generator for $\check{K}^0(W)$ which is a product near ∂W and let $\mathcal{E}$ be its pullback to ∂W. Then*

$$\overline{\eta}(\partial W,\mathcal{E}) = \int_W \mathrm{Todd}(\widehat{\nabla}^{TW}) \wedge \omega(\mathcal{F}) \pmod{u^{-(\dim(\partial W)+1)/2} \cdot \mathbb{Z}}. \tag{2.36}$$

(2) *The assignment* $\mathcal{E} \longrightarrow \overline{\eta}(X, \mathcal{E})$ *factors through a homomorphism* $\overline{\eta}\colon {}_{\delta}\check{K}^0(X) \to u^{-(\dim(X)+1)/2} \cdot (\mathbb{R}/\mathbb{Z})$.

(3) *If* $a \in K^{-1}(X; \mathbb{R}/\mathbb{Z})$, *then*

$$\overline{\eta}\,(X, i\,(a)) = u^{-(\dim(X)+1)/2} \langle [X], a\rangle, \tag{2.37}$$

where $[X] \in K_{-1}(X;\mathbb{Z})$ *is the (periodicity-shifted) fundamental class in* K*–homology and* $\langle [X], a\rangle \in \mathbb{R}/\mathbb{Z}$ *is the result of the pairing between* $K_{-1}(X;\mathbb{Z})$ *and* $K^{-1}(X;\mathbb{R}/\mathbb{Z})$.

Proof For part (1), write $\mathcal{F} = (F, h^F, \nabla^R, \Phi)$. By the Atiyah–Patodi–Singer index theorem [3],

$$u^{\dim(W)/2} \int_W \mathrm{Todd}(\widehat{\nabla}^{TW}) \wedge \omega(\nabla^F) - \overline{\eta}(D^{X,E}) \in \mathbb{Z}. \tag{2.38}$$

(Note that $\int_W \mathrm{Todd}(\widehat{\nabla}^{TW}) \wedge \omega(\nabla^E)$ is a real multiple of $u^{-\dim(W)/2}$ for dimensional reasons.) As

$$\begin{aligned}\int_W \mathrm{Todd}(\widehat{\nabla}^{TW}) \wedge \omega(\mathcal{F}) &= \int_W \mathrm{Todd}(\widehat{\nabla}^{TW}) \wedge (\omega(\nabla^F) + d\Phi)\\ &= \int_W \mathrm{Todd}(\widehat{\nabla}^{TW}) \wedge \omega(\nabla^F) + \int_{\partial W} \mathrm{Todd}(\widehat{\nabla}^{T\partial W}) \wedge \Phi,\end{aligned} \tag{2.39}$$

part (1) follows.

To prove part (2), suppose first that we have a relation $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ for $\check{K}^0(X)$. Put $W = [0,1] \times X$, with a product metric. If $p\colon W \to X$ is the projection map, put $F = p^*E_2$ and $h^F = p^*h^{E_2}$. Let ∇^F be a unitary connection on F which equals $p^*\nabla^{E_2}$ near $\{1\} \times X$ and which equals $p^*(\nabla^{E_1} \oplus \nabla^{E_3})$ near $\{0\} \times X$. Choose $\Phi \in \Omega(W; \mathcal{R})^{-1}/\mathrm{Im}(d)$ which equals $p^*\phi_2$ near $\{1\} \times X$ and which equals $p^*(\phi_1 + \phi_3)$ near $\{0\} \times X$. Using part (1),

$$\begin{aligned}\overline{\eta}(X, \mathcal{E}_2) - \overline{\eta}(X, \mathcal{E}_1) - \overline{\eta}(X, \mathcal{E}_3)\\ &= \int_W \mathrm{Todd}(\widehat{\nabla}^{TW}) \wedge (\omega(\nabla^F) + d\Phi)\\ &= \int_X \int_0^1 \mathrm{Todd}(\widehat{\nabla}^{TX}) \wedge (\omega(\nabla^F) + d\Phi)\\ &= \int_X \mathrm{Todd}(\widehat{\nabla}^{TX}) \wedge \big(\mathrm{CS}(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) + \phi_2 - \phi_1 - \phi_3\big)\\ &= 0\end{aligned} \tag{2.40}$$

in $u^{-(\dim(X)+1)/2} \cdot (\mathbb{R}/\mathbb{Z})$. This shows that $\overline{\eta}$ extends to a map

$$\check{K}^0(X) \to u^{-(\dim(X)+1)/2} \cdot (\mathbb{R}/\mathbb{Z}).$$

The argument easily extends if we use currents instead of forms, thereby proving part (2) of the proposition.

Part (3) follows from [33, Proposition 3]. □

2.41 Remark To prove part (2) of Proposition 2.35, we could have used the variational formula for $\overline{\eta}$ [4], which is more elementary than the Atiyah–Patodi–Singer index theorem.

More generally, if (E, h^E, ∇^E, ϕ) is a generator for ${}_\delta\check{K}^r(X)$ then we define $\overline{\eta}(X, \mathcal{E}) \in u^{(r-\dim(X)-1)/2} \cdot (\mathbb{R}/\mathbb{Z})$ by

$$\overline{\eta}(X, \mathcal{E}) = u^{(r-\dim(X)-1)/2}\overline{\eta}(D^{X,E}) + \int_X \mathrm{Todd}(\widehat{\nabla}^{TX}) \wedge \phi \pmod{u^{(r-\dim(X)-1)/2} \cdot \mathbb{Z}}. \tag{2.42}$$

2.43 Superconnections

Define the auxiliary ring

$$\mathcal{R}' = \mathbb{R}[u^{1/2}, u^{-1/2}], \tag{2.44}$$

where $u^{1/2}$ is a formal variable of degree 1 and $u^{-1/2}$ its inverse. Then $\mathcal{R} \subset \mathcal{R}'$. If E is a $(\mathbb{Z}/2\mathbb{Z})$–graded vector bundle on X then the $\Omega(X; \mathcal{R}')$–module $\Omega(X; E \otimes \mathcal{R}')$ of differential forms with values in $E \otimes \mathcal{R}'$ is $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$–graded: by form degree, degree in $\mathcal{R}'$, and degree in E. We use a quotient $(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$–grading: the integer degree is the sum of the form degree and the degree in $\mathcal{R}'$, while the mod 2 degree is the degree in E plus the mod two form degree.

2.45 Definition A *superconnection* A on E is a graded $\Omega(X; \mathcal{R}')$–derivation of $\Omega(X; E \otimes \mathcal{R}')$ of degree $(1,1)$.

Note that we can uniquely write

$$A = u^{1/2}\omega_0 + \nabla + u^{-1/2}\omega_2 + u^{-1}\omega_3 + \cdots, \tag{2.46}$$

where ∇ is an ordinary connection on E (which preserves degree) and ω_j is an $\mathrm{End}(E)$–valued j–form on X which is an even endomorphism if j is odd and an odd endomorphism if j is even. The powers of u are related to the standard scaling of a superconnection. The Chern character of A is defined by

$$\omega(A) = R_u \operatorname{Str} e^{u^{-1} A^2} \quad \in \Omega(M; \mathcal{R})^0. \tag{2.47}$$

Notice that the curvature A^2 has degree $(2,0)$ so $u^{-1} A^2$ of degree $(0,0)$ can be exponentiated. Also, there are no fractional powers of u in the result since the supertrace of an odd endomorphism of E vanishes.

3 Analytic index

In this section we define the analytic pushforward of a differential K–theory class under a proper submersion. This is an extension of the analytic pushforward in $\mathbb{R}/\mathbb{Z}$–valued K–theory that was defined in [33, Section 4]. The geometric assumptions are that we have a proper submersion $\pi\colon X \to B$ of relative dimension n, with n even, which is equipped with a Riemannian structure on the fibers and a differential spin^c–structure (in a sense that will be made precise below).

Given a differential K–theory class $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ on X, there is an ensuing family D^V of vertical Dirac-type operators. In this section we assume that $\mathrm{Ker}(D^V)$ forms a vector bundle on B. (This assumption will be lifted in Section 7). In Definition 3.12 we define the analytic pushforward $\check{\pi}_*(\mathcal{E}) \in \check{K}^{-n}(B)$ of $\mathcal{E}$, using the Bismut–Cheeger eta form.

For later purposes, we will want to extend the definition of the analytic pushforward to certain currential K–theory classes. To do so, we have to make a compatibility assumption between the singularities of the current ϕ and the fibration π. This is phrased in terms of the wave front set of the current ϕ, which is a subset of T^*X that microlocally measures the singularity locus of ϕ. For a fiber bundle $\pi\colon X \to B$ we define an analog ${}_{\mathrm{WF}}\check{K}^0(X)$ of $\check{K}^0(X)$ using currents ϕ whose wave front set has zero intersection with the conormal bundle of the fibers. Roughly speaking, this means that the singularity locus of ϕ meets the fibers of π transversely, so we can integrate ϕ fiberwise to get a smooth form on B. We then define the analytic pushforward on ${}_{\mathrm{WF}}\check{K}^0(X)$.

3.1 Construction of the analytic index

Let $\pi\colon X \to B$ be a proper submersion of relative dimension n, with n even. Recall that this is the same as saying that $\pi\colon X \to B$ is a smooth fiber bundle with compact fibers of even dimension n. Let $T^V X = \mathrm{Ker}(d\pi)$ denote the relative tangent bundle on X.

We define a *Riemannian structure on* π to be a pair consisting of a vertical metric g^{T^VX} and a horizontal distribution T^HX on X. This terminology is justified by the existence of a certain connection on T^VX which restricts to the Levi-Civita connection on each fiber of π [10, Definition 1.6]. We recall the definition. Let g^{TB} be a Riemannian metric on B. Using g^{TB} and the Riemannian structure on π, we obtain a Riemannian metric g^{TX} on X. Let ∇^{TX} be its Levi-Civita connection. Let $P\colon TX \to T^VX$ be orthogonal projection.

3.2 Definition The connection ∇^{T^VX} on T^VX is $\nabla^{T^VX} = P \circ \nabla^{TX} \circ P$. It is independent of the choice of g^{TB}.

Suppose the map π is spin^c–oriented in the sense that T^VX has a spin^c–structure, with characteristic hermitian line bundle $L^VX \to X$. A *differential* spin^c*–structure* on π is in addition a unitary connection on L^VX. Let $\mathcal{S}^VX$ denote the associated spinor bundle on X. The connections on T^VX and L^VX induce a connection $\widehat{\nabla}^{T^VX}$ on $\mathcal{S}^VX$.

Define $\pi_*\colon \Omega(X;\mathcal{R})^\bullet \to \Omega(B;\mathcal{R})^{\bullet-n}$ by

$$\pi_*(\phi) = \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^VX}) \wedge \phi. \tag{3.3}$$

Note that our π_* differs from the de Rham pushforward by the factor of $\mathrm{Todd}(\widehat{\nabla}^{T^VX})$. It will simplify later formulas if we use our slightly unconventional definition.

We recall that there is a notion of the wave front set of a current on X; it is the union of the wave front sets of its local distributional coefficients [29, Chapters 8.1 and 8.2]. The wave front set is a subset of T^*X. Let $N_V^*X = \pi^*T^*B \subset T^*X$ be the conormal bundle of the fibers. Let ${}_{\mathrm{WF}}\Omega(X;\mathcal{R})^\bullet$ denote the subspace of ${}_{\delta}\Omega(X;\mathcal{R})^\bullet$ consisting of elements whose wave front set intersects N_V^*X only at the zero section of N_V^*X. By [29, Theorem 8.2.12], Equation (3.3) defines a map

$$\pi_*\colon {}_{\mathrm{WF}}\Omega(X;\mathcal{R})^\bullet \to \Omega(B;\mathcal{R})^{\bullet-n}. \tag{3.4}$$

Let ${}_{\mathrm{WF}}\check{K}^0(X)$ be the abelian group whose generators are quadruples $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ with $\phi \in {}_{\mathrm{WF}}\Omega(X;\mathcal{R})^{-1}/\,\mathrm{Image}(d)$, and with relations as before. Then there are exact sequences

$$0 \longrightarrow K^{\bullet-1}(X;\mathbb{R}/\mathbb{Z}) \xrightarrow{i} {}_{\mathrm{WF}}\check{K}^\bullet(X) \xrightarrow{\omega} {}_{\mathrm{WF}}\Omega(X;\mathcal{R})^\bullet_K \longrightarrow 0, \tag{3.5}$$

$$0 \longrightarrow \frac{{}_{\mathrm{WF}}\Omega(X;\mathcal{R})^{\bullet-1}}{{}_{\mathrm{WF}}\Omega(X;\mathcal{R})^{\bullet-1}_K} \xrightarrow{j} {}_{\mathrm{WF}}\check{K}^\bullet(X) \xrightarrow{c} K^\bullet(X;\mathbb{Z}) \longrightarrow 0. \tag{3.6}$$

Here we use the fact that if $\alpha \in {}_{\mathrm{WF}}\Omega(X;\mathcal{R})^\bullet$ and $\alpha \in \mathrm{Image}(d\colon {}_\delta\Omega(X;\mathcal{R})^{\bullet-1} \to {}_\delta\Omega(X;\mathcal{R})^\bullet)$ then $\alpha \in \mathrm{Image}(d\colon {}_{\mathrm{WF}}\Omega(X;\mathcal{R})^{\bullet-1} \to {}_{\mathrm{WF}}\Omega(X;\mathcal{R})^\bullet)$.

Given a Riemannian structure on π and a generator $\mathcal{E}$ for $\check{K}^0(X)$, we want to define a pushforward of $\mathcal{E}$ that lives in $\check{K}^{-n}(B)$. Write $\mathcal{E} = (E, h^E, \nabla^E, \phi)$. Let $\mathcal{H}$ denote the (possibly infinite dimensional) vector bundle on B whose fiber $\mathcal{H}_b$ at $b \in B$ is the space of smooth sections of $(\mathcal{S}^V X \otimes E)\big|_{X_b}$. The bundle $\mathcal{H}$ is $\mathbb{Z}/2\mathbb{Z}$–graded. For $s > 0$, the *Bismut superconnection* A_s is

$$A_s = s u^{1/2} D^V + \nabla^{\mathcal{H}} - s^{-1} u^{-1/2} \frac{c(T)}{4}. \tag{3.7}$$

Here D^V is the Dirac-type operator acting on $\mathcal{H}_b$, $\nabla^{\mathcal{H}}$ is a certain unitary connection on $\mathcal{H}$ constructed from $\hat{\nabla}^{T^V X}$, ∇^E and the mean curvature of the fibers, and $c(T)$ is the Clifford multiplication by the curvature 2–form T of the fiber bundle. For more information, see Berline–Getzler–Vergne [9, Proposition 10.15]. We use powers of s in (3.7) in order to simplify calculations, as compared to the powers of $s^{1/2}$ used by some other authors, but there is no essential difference.

Now assume that $\mathrm{Ker}(D^V)$ forms a smooth vector bundle on B, necessarily $\mathbb{Z}/2\mathbb{Z}$–graded. There are an induced L^2–metric $h^{\mathrm{Ker}(D^V)}$ and a compatible projected connection $\nabla^{\mathrm{Ker}(D^V)}$. Note that $[\mathrm{Ker}(D^V)]$ lies in $K^{-n}(B)$. Then

$$\lim_{s \to 0} u^{-n/2} R_u \,\mathrm{STr}(e^{-u^{-1} A_s^2}) = \pi_*(\omega(\nabla^E)), \tag{3.8}$$

while

$$\lim_{s \to \infty} u^{-n/2} R_u \,\mathrm{STr}(e^{-u^{-1} A_s^2}) = \omega(\nabla^{\mathrm{Ker}(D^V)}); \tag{3.9}$$

see [9, Chapter 10]. Note that the preceding two equations lie in forms of total degree $-n$.

The Bismut–Cheeger eta-form [11] is

$$\tilde{\eta} = u^{-n/2} R_u \int_0^\infty \mathrm{STr}\left(u^{-1} \frac{dA_s}{ds} e^{-u^{-1} A_s^2} \right) ds \in \Omega(B;\mathcal{R})^{-n-1} / \mathrm{Image}(d). \tag{3.10}$$

It satisfies

$$d\tilde{\eta} = \pi_*(\omega(\nabla^E)) - \omega(\nabla^{\mathrm{Ker}(D^V)}). \tag{3.11}$$

3.12 Definition Given a generator $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ for $\check{K}^0(X)$, and assuming $\mathrm{Ker}(D^V)$ is a vector bundle, we define the *analytic index* $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) \in \check{K}^{-n}(B)$ by

$$\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) = \left(\mathrm{Ker}(D^V), h^{\mathrm{Ker}(D^V)}, \nabla^{\mathrm{Ker}(D^V)}, \pi_*(\phi) + \tilde{\eta}\right). \tag{3.13}$$

It follows from Theorem 6.2 below that the assignment $\mathcal{E} \to \mathrm{ind}^{\mathrm{an}}(\mathcal{E})$ factors through a map from $\check{K}^0(X)$ to $\check{K}^{-n}(B)$.

Given a generator $\mathcal{E}$ of ${}_{\mathrm{WF}}\check{K}^0(X)$, we define $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) \in \check{K}^{-n}(B)$ by the same formula (3.13).

3.14 Lemma *If $\mathcal{E}$ is a generator for ${}_{\mathrm{WF}}\check{K}^0(X)$ then $\omega(\mathrm{ind}^{\mathrm{an}}(\mathcal{E})) = \pi_*(\omega(\mathcal{E}))$ in $\Omega(B;\mathcal{R})^{-n}$.*

Proof From (3.11),

$$\omega(\mathrm{ind}^{\mathrm{an}}(\mathcal{E})) = \omega(\nabla^{\mathrm{Ker}(D^V)}) + d(\pi_*(\phi)+\widetilde{\eta}) = \pi_*(\omega(\nabla^E)+d\phi) = \pi_*(\omega(\mathcal{E})), \tag{3.15}$$

which proves the lemma. □

4 Pushforward under an embedding

In this section we define a pushforward on differential K–theory under a proper embedding $\iota\colon X \to Y$ of manifolds. The definition uses the data of a generator $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ of $\check{K}^0(X)$ and a Riemannian structure on the normal bundle ν of the embedding.

To motivate our definition, let us recall how to push forward ordinary K–theory under ι [1]. Suppose that the normal bundle $p\colon \nu \to X$ has even dimension r and is endowed with a spin^c–structure. Let $\mathcal{S}^\nu \to X$ denote the corresponding $\mathbb{Z}/2\mathbb{Z}$–graded spinor bundle on X. Clifford multiplication by an element in ν gives an isomorphism, on the complement of the zero-section in ν, between $p^*\mathcal{S}^\nu_+$ and $p^*\mathcal{S}^\nu_-$. The K–theory Thom class U_K is the corresponding relative class in $K^r(D(\nu), S(\nu);\mathbb{Z})$, where $D(\nu)$ denotes the closed disk bundle of ν and $S(\nu) = \partial D(\nu)$ is the sphere bundle.

Given a vector bundle E on X, the Thom homomorphism

$$K^0(X;\mathbb{Z}) \to K^r(D(\nu), S(\nu);\mathbb{Z})$$

sends $[E]$ to $p^*[E]\cdot U_K$. Transplanting this to a closed tubular neighborhood T of X in Y, we obtain a relative K–theory class in $K(T,\partial T;\mathbb{Z})$. Then excision in K–theory defines an element $\iota_*[E] \in K^r(Y;\mathbb{Z})$, which is the K–theory pushforward of $[E]$. Applying the Chern character, one finds that $\mathrm{ch}(\iota_*[E])$ is the extension to Y of the cohomology class $(p^*\,\mathrm{ch}([E]) \cup U_H)/\,\mathrm{Todd}(\nu) \in H^\bullet(D(\nu), S(\nu);\mathbb{Q})$, where $U_H \in H^r(D(\nu), S(\nu);\mathbb{Z})$ is the Thom class in cohomology.

In order to push forward classes in differential K–theory, we will need to carry along differential form information in the K–theory pushforward. There are differential form descriptions of the Thom class in cohomology, but they are not very convenient for our purposes. Instead we pass to currents and simply write the Thom homomorphism in real cohomology as the map which sends a differential form ω on X to the current $\omega \wedge \delta_X$ on Y. Following this line of reasoning, the pushforward under ι of a differential K–theory class on X is a currential K–theory class on Y. An important ingredient in its definition is a certain current γ defined by Bismut–Zhang [13].

4.1 Construction of the embedding pushforward

Let $\iota\colon X \hookrightarrow Y$ be a proper embedding of manifolds. Let r be the codimension of X in Y. We assume that r is even. Let $\delta_X \in {}_\delta\Omega(Y)^r$ denote the current of integration on X.

Let $\nu = \iota^* TY/TX$ be the normal bundle to X. We define a *Riemannian structure on ι* to be a metric g^ν on ν and a compatible connection ∇^ν on ν. Suppose the map ι carries a *differential* spin^c*–structure*, in the sense that ν has a spin^c–structure with characteristic hermitian line bundle $L^\nu \to X$ and that the line bundle is endowed with a unitary connection ∇^{L^ν}. Let $\mathcal{S}^\nu \to X$ be the spinor bundle of ν. Then $\mathcal{S}^\nu$ inherits a connection $\widehat{\nabla}^\nu$. Let $c(\xi)$ denote Clifford multiplication by $\xi \in \nu$ on $\mathcal{S}^\nu$. Let $p\colon \nu \to X$ be the vector bundle projection. Then there is a self-adjoint odd endomorphism $c \in \mathrm{End}(p^*\mathcal{S}^\nu)$ which acts on $(p^*\mathcal{S}^\nu)_\xi \cong \mathcal{S}^\nu$ as Clifford multiplication by $\xi \in \nu$.

There is a pushforward map $\iota_*\colon \Omega(X;\mathcal{R})^\bullet \to {}_\delta\Omega(Y;\mathcal{R})^{\bullet+r}$ given by

$$\iota_*(\phi) = \frac{\phi}{\mathrm{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X. \tag{4.2}$$

Note that our ι_* differs from the de Rham pushforward by the factor of $\mathrm{Todd}(\widehat{\nabla}^\nu)$. It will simplify later formulas if we use our slightly unconventional definition.

Given a Riemannian structure on ι, we want to define a map $\check{\iota}_*\colon \check{K}^0(X) \longrightarrow {}_\delta\check{K}^r(Y)$. To do so, we use a construction of Bismut and Zhang [13]. Let F be a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle on Y equipped with a Hermitian metric h^F. We assume that we are given an odd self-adjoint endomorphism V of F which is invertible on $Y-X$, and that $\mathrm{Ker}(V)$ has locally constant rank along X. Then $\mathrm{Ker}(V)$ restricts to a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle on X. It inherits a Hermitian metric $h^{\mathrm{Ker}(V)}$ from F. Let $P^{\mathrm{Ker}(V)}$ denote orthogonal projection from $F|_X$ to $\mathrm{Ker}(V)$. If F has an h^F–compatible connection ∇^F then $\mathrm{Ker}(V)$ inherits an $h^{\mathrm{Ker}(V)}$–compatible connection given by $\nabla^{\mathrm{Ker}(V)} = P^{\mathrm{Ker}(V)} \nabla^F P^{\mathrm{Ker}(V)}$. Given a connection ∇^F on F, a point $x \in X$ and a vector $\xi \in \nu_x$, lift ξ to an element $\widehat{\xi} \in T_xY$ and put

$$\partial_\xi V = P^{\mathrm{Ker}(V)} \big(\nabla^F_{\widehat{\xi}} V\big) P^{\mathrm{Ker}(V)}. \tag{4.3}$$

Then $\partial_\xi V$ is an odd self-adjoint endomorphism of $\mathrm{Ker}(V)$ which is independent of the choices of ∇^F and $\widehat{\xi}$. There is a well-defined odd self-adjoint endomorphism $\partial V \in \mathrm{End}(p^* \mathrm{Ker}(V))$ so that ∂V acts on $(p^* \mathrm{Ker}(V))_\xi \cong \mathrm{Ker}(V)$ by $\partial_\xi V$.

4.4 Lemma [13, Remark 1.1] *Given a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle E on X, equipped with a Hermitian metric h^E and a compatible connection ∇^E, there are F, h^F, ∇^F, V on Y so that*

$$\text{(4.5)}\quad \left(\mathcal{S}^\nu \otimes E, h^{\mathcal{S}^\nu} \otimes h^E, \widehat{\nabla}^\nu \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^F, c \otimes Id\right) \cong \left(\mathrm{Ker}(V), h^{\mathrm{Ker}(V)}, \nabla^{\mathrm{Ker}(V)}, \partial V\right).$$

Proof Let $D(\nu)$ denote the closed unit disk bundle of ν. Put $S(\nu) = \partial D(\nu)$. Then there is a diffeomorphism $\sigma\colon T \to D(\nu)$ between $D(\nu)$ and a closed tubular neighborhood T of X in Y. The $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle $W = \sigma^* p^*(\mathcal{S}^\nu \otimes E)$ on T is equipped with an isomorphism $J\colon W_+|_{\partial T} \to W_-|_{\partial T}$ on ∂T given by $\sigma^* c$.

By the excision isomorphism in K–theory, $K^0(Y, \overline{Y - T}) \cong K^0(T, \partial T)$. This means that after stabilization, W can be extended to a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle F on Y which is equipped with an isomorphism between $F_+|_{\overline{Y-T}}$ and $F_-|_{\overline{Y-T}}$. More explicitly, let R be a vector bundle on T so that $W_- \oplus R$ is isomorphic to $\mathbb{R}^N \times T$, for some N. Then $W_- \oplus R$ extends to a trivial $\mathbb{R}^N$–vector bundle F_- on Y. Let F_+ be the result of gluing the vector bundle $W_+ \oplus R$ (on T) with $\mathbb{R}^N \times \overline{Y - T}$ (on $\overline{Y - T}$), using the clutching isomorphism

$$\text{(4.6)}\qquad (W_+ \oplus R)\big|_{\partial T} \xrightarrow{J \oplus \mathrm{Id}} (W_- \oplus R)\big|_{\partial T} \longrightarrow \mathbb{R}^N \times \partial T$$

along ∂T.

Let h^R be a Hermitian inner product on R and let ∇^R be a compatible connection. Choose $h^{F_\pm}$ and $\nabla^{F_\pm}$ to agree with $h^{W_\pm \oplus R}$ and $\nabla^{W_\pm \oplus R}$ on T. Let $V_1 \in \mathrm{End}(F_+, F_-)$ be the result of gluing $\sigma^* c|_{W_+} \oplus \mathrm{Id}_R$ (on T) with the identity map $\mathbb{R}^N \to \mathbb{R}^N$ (on $\overline{Y - T}$). Put $V = V_1 \oplus V_1^* \in \mathrm{End}(F)$. Then (F, h^F, ∇^F, V) satisfies the claims of the lemma. □

Hereafter we assume that (F, h^F, ∇^F, V) satisfies Lemma 4.4. Note that $[F]$ lies in $K^r(Y)$. For $s > 0$, define a superconnection C_s on F by

$$\text{(4.7)}\qquad C_s = s u^{1/2} V + \nabla^F.$$

Then

$$\text{(4.8)}\qquad \lim_{s \to 0} u^{r/2} R_u \,\mathrm{str}(e^{-u^{-1} C_s^2}) = \omega(\nabla^F).$$

Also, from [13, Theorem 1.2],

$$\lim_{s\to\infty} u^{r/2} R_u \operatorname{str}(e^{-u^{-1}C_s^2}) = \frac{\omega(\nabla^E)}{\operatorname{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X \tag{4.9}$$

as currents.

4.10 Definition [13, Definition 1.3] Define $\gamma \in {}_\delta\Omega(Y;\mathcal{R})^{r-1}/\operatorname{Image}(d)$ by

$$\begin{aligned}\gamma &= u^{r/2} R_u \int_0^\infty \operatorname{str}\left(u^{-1}\frac{dC_s}{ds}e^{-u^{-1}C_s^2}\right) ds \\ &= u^{r/2} R_u \int_0^\infty \operatorname{str}\left(u^{-1/2} V e^{-u^{-1}C_s^2}\right) ds.\end{aligned} \tag{4.11}$$

The integral on the right-hand side of (4.11) is well-defined, as a current on Y, by [13, Theorem 1.2]. By [13, Remark 1.5], γ is a locally integrable differential form on Y whose wave front set is contained in ν^*.

4.12 Proposition [13, Theorem 1.4] *We have*

$$d\gamma = \omega(\nabla^F) - \frac{\omega(\nabla^E)}{\operatorname{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X. \tag{4.13}$$

4.14 Definition Given a generator $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ for $\check{K}^0(X)$, define $\breve{\iota}_*(\mathcal{E}) \in {}_\delta\check{K}^r(Y)$ to be the element represented by the quadruple

$$\breve{\iota}_*(\mathcal{E}) = \left(F, h^F, \nabla^F, \iota_*(\phi) - \gamma\right). \tag{4.15}$$

4.16 Lemma $\omega(\breve{\iota}_*(\mathcal{E})) = \iota_*(\omega(\mathcal{E}))$.

Proof We have

$$\omega(\breve{\iota}_*(\mathcal{E})) = \omega(\nabla^F) + d(\iota_*(\phi) - \gamma) = \frac{\omega(\nabla^E) + d\phi}{\operatorname{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X = \iota_*(\omega(\mathcal{E})), \tag{4.17}$$

which proves the lemma. □

4.18 Proposition (1) *The pushforward $\breve{\iota}_*(\mathcal{E}) \in {}_\delta\check{K}^r(Y)$ is independent of the choices of F, h^F, ∇^F and V, subject to (4.5).*

(2) *The assignment $\mathcal{E} \to \breve{\iota}_*(\mathcal{E})$ factors through a map $\breve{\iota}_*\colon \check{K}^0(X) \to {}_\delta\check{K}^r(Y)$.*

Proof Let G be a complex vector bundle on Y with Hermitian metric h^G and compatible connection ∇^G. If we put $F'_\pm = F_\pm \oplus G$, $h^{F'_\pm} = h^{F_\pm \oplus G}$, $\nabla^{F'_\pm} = \nabla^{F_\pm \oplus G}$ and

$$V' = V \oplus \begin{pmatrix} 0 & I_G \\ I_G & 0 \end{pmatrix}$$

then it is easy to check that γ does not change. Clearly $(F', h^{F'}, \nabla^{F'}, \iota_*(\phi) - \gamma)$ equals $(F, h^F, \nabla^F, \iota_*(\phi) - \gamma)$ in ${}_\delta\check{K}^r(Y)$.

To prove part (1), suppose that for $i \in \{0,1\}$, $(F_i, h^{F_i}, \nabla^{F_i}, V_i)$ are two different choices of data as in the statement of the proposition. Since $F_{i,+} - F_{i,-}$ represent the same class in $K^r(Y)$ for $i \in \{0,1\}$, the preceding paragraph implies that we can stabilize to put ourselves into the situation that $F_{i,+} = F_+$ and $F_{i,-} = F_-$ for some fixed $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle $F_\pm$ on Y.

For $t \in [0,1]$, let $(h^F(t), \nabla^F(t), V(t))$ be a smooth 1–parameter family of data interpolating between (h^F_0, ∇^F_0, V_0) and (h^F_1, ∇^F_1, V_1). Consider the product embedding $\iota' \colon [0,1] \times X \to [0,1] \times Y$. Let $\mathcal{E}'$ be the pullback of $\mathcal{E}$ to $[0,1] \times X$. Let F' be the pullback of F to $[0,1] \times Y$ and let $(h^{F'}, \nabla^{F'}, V')$ be the ensuing data on F' coming from the 1–parameter family. Construct $\gamma' \in {}_\delta\Omega([0,1] \times Y; \mathcal{R})^{r-1}/\operatorname{Image}(d)$ from (4.11). Put

$$\iota'_*(\mathcal{E}') = \big(F', h^{F'}, \nabla^{F'}, \iota'_*(\phi') - \gamma'\big). \tag{4.19}$$

By Remark 2.27 (or more precisely its extension to currential K–theory),

$$\big(F, h^F_1, \nabla^F_1, \iota_*(\phi) - \gamma_1\big) - \big(F, h^F_0, \nabla^F_0, \iota_*(\phi) - \gamma_0\big) = j\left(\int_0^1 \omega(\iota'_*(\mathcal{E}'))\right) \tag{4.20}$$

in ${}_\delta\check{K}^r(Y)$. However, using Lemma 4.16, $\omega(\iota'_*(\mathcal{E}')) = \iota'_*(\omega(\mathcal{E}'))$ is the pullback of $\iota_*(\omega(\mathcal{E}))$ from Y to $[0,1] \times Y$. In particular, $\int_0^1 \omega(\iota'_*(\mathcal{E}')) = 0$. This proves part (1) of the proposition.

To prove part (2), suppose that we have a relation $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ in $\check{K}^0(X)$ coming from a short exact sequence (2.3). Let $p \colon [0,1] \times X \to X$ be the projection map. Put $E' = p^* E_2$. Let $\nabla^{E'}$ be a unitary connection on E which is $p^*\nabla^{E_2}$ near $\{1\} \times X$ and which is $p^*(\nabla^{E_1} \oplus \nabla^{E_3})$ near $\{0\} \times X$. Choose $\phi' \in \Omega([0,1] \times X; \mathcal{R})^{-1}/\operatorname{Image}(d)$ which equals $p^*\phi_2$ near $\{1\} \times X$ and which equals $p^*(\phi_1 + \phi_3)$ near $\{0\} \times X$. Consider the product embedding $\iota' \colon [0,1] \times X \to [0,1] \times Y$ and construct $\iota'_*(\mathcal{E}') \in {}_\delta\check{K}^0([0,1] \times Y)$ as in Definition 4.14. For $i \in \{0,1\}$, let $A_i \colon Y \to [0,1] \times Y$ be the embedding $A_i(y) = (i, y)$. From Lemmas 2.24 and 4.16,

$$A_1^* \iota'_*(\mathcal{E}') - A_0^* \iota'_*(\mathcal{E}') = j\left(\int_0^1 \iota'_*(\omega(\mathcal{E}'))\right) = j\left(\iota_* \int_0^1 \omega(\mathcal{E}')\right) = 0, \tag{4.21}$$

since the relation $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ and Lemma 2.24 imply that $\int_0^1 \omega(\mathcal{E}')$ vanishes. Hence $\iota_*(\mathcal{E}_2) = \iota_*(\mathcal{E}_1) + \iota_*(\mathcal{E}_3)$. This proves part (2) of the proposition. □

5 Topological index

In this section we define the topological index in differential K–theory. We first consider two fiber bundles $X_1 \to B$ and $X_2 \to B$, each equipped with a Riemannian structure and a differential spin^c structure in the sense of the previous section. We now assume that we have a fiberwise isometric embedding $\iota\colon X_1 \to X_2$. The preceding section constructed a pushforward $\breve{\iota}_*\colon \breve{K}^0(X_1) \to {}_\delta\breve{K}^r(X_2)$. To define the topological index we will eventually want to compose $\breve{\iota}_*$ with the pushforward under the fibration $X_2 \to B$. However, there is a new issue because the horizontal distributions on the two fiber bundles X_1 and X_2 need not be compatible. Hence we define a correction form $\widetilde{C}$ and, in Definition 5.8, a modified embedding pushforward $\breve{\iota}_*^{\mathrm{mod}}\colon \breve{K}^0(X_1) \to {}_{\mathrm{WF}}\breve{K}^r(X_2)$.

To define the topological index, we specialize to the case when X_2 is $S^N \times B$ for some even N, equipped with a Riemannian structure coming from a fixed Riemannian metric on S^N and the product horizontal distribution. In this case we show that the pushforward of ${}_{\mathrm{WF}}\breve{K}^r(S^N \times B)$ from $S^N \times B$ to B, as defined in Definition 3.12, can be written as an explicit map $\breve{\pi}_*^{\mathrm{prod}}\colon {}_{\mathrm{WF}}\breve{K}^r(S^N \times B) \to \breve{K}^{r-N}(B)$ in terms of a Künneth-type formula for ${}_{\mathrm{WF}}\breve{K}^r(S^N \times B)$. This shows that $\breve{\pi}_*^{\mathrm{prod}}$ can be computed without any spectral analysis and, in particular, can be defined without the assumption about vector bundle kernel. Relabeling X_1 as X, we then define the topological index $\mathrm{ind}^{\mathrm{top}}\colon \breve{K}^0(X) \to \breve{K}^{-n}(B)$ by $\mathrm{ind}^{\mathrm{top}} = \breve{\pi}_*^{\mathrm{prod}} \circ \breve{\iota}_*^{\mathrm{mod}}$.

5.1 Construction of the topological index

Let $\pi_1\colon X_1 \to B$ and $\pi_2\colon X_2 \to B$ be fiber bundles over B, with compact fibers $X_{1,b}$ and $X_{2,b}$ of even dimension n_1 and n_2, respectively. Let $\iota\colon X_1 \to X_2$ be a fiberwise embedding of even codimension r, ie, ι is an embedding, $\pi_2 \circ \iota = \pi_1$, and $n_2 = n_1 + r$. Let ν be the normal bundle of X_1 in X_2. There is a short exact sequence

(5.2) $$0 \longrightarrow T^V X_1 \longrightarrow \iota^* T^V X_2 \longrightarrow \nu \longrightarrow 0,$$

of vector bundles on X_1. Suppose that π_1 and π_2 have Riemannian structures.

5.3 Definition The map ι is *compatible with the Riemannian structures on* π_1 *and* π_2 if for each $b \in B$, $\iota_b\colon X_{1,b} \to X_{2,b}$ is an isometric embedding.

The intersection of T^HX_2 with TX_1 defines a horizontal distribution $(T^HX_2)|_{X_1}$ on X_1. We do *not* assume that it coincides with T^HX_1. It follows that the orthogonal projection of $\iota^*\nabla^{T^VX_2}$ to T^VX_1 is *not* necessarily equal to $\nabla^{T^VX_1}$.

In the rest of this section we assume that ι is compatible with the Riemannian structures on π_1 and π_2. Then ν inherits a Riemannian structure from (5.2), which is split by identifying ν as the orthogonal complement to T^VX_1 in $\iota^*T^VX_2$. Namely, the metric g^ν is the quotient inner product from $g^{T^VX_2}$ and the connection ∇^ν is compressed from $\iota^*\nabla^{T^VX_2}$.

We also assume a certain compatibility of the differential spin^c–structures on π_1, π_2 and ι. To describe this compatibility, recall the discussion of spin^c–structures from Section 2.1. Over X_1 we have principal bundles $\mathcal{F}_1$, $\mathcal{F}_\nu$ and $\iota^*\mathcal{F}_2$, with structure groups $\mathrm{Spin}^c(n_1)$, $\mathrm{Spin}^c(r)$ and $\mathrm{Spin}^c(n_2)$, respectively. They project to the oriented orthonormal frame bundles $\mathcal{B}_1$, $\mathcal{B}_\nu$ and $\iota^*\mathcal{B}_2$. The embedding ι gives a reduction $\mathcal{B}_1 \times \mathcal{B}_\nu \hookrightarrow \iota^*\mathcal{B}_2$ of $\iota^*\mathcal{B}_2$ which is compatible with the inclusion $\mathrm{SO}_{n_1} \times \mathrm{SO}_r \hookrightarrow \mathrm{SO}_{n_2}$. Then we postulate that we have a lift

$$\mathcal{F}_1 \times \mathcal{F}_\nu \to \iota^*\mathcal{F}_2 \tag{5.4}$$

of $\mathcal{B}_1 \times \mathcal{B}_\nu \hookrightarrow \iota^*\mathcal{B}_2$ which is compatible with the homomorphism $\mathrm{Spin}^c(n_1) \times \mathrm{Spin}^c(r) \to \mathrm{Spin}^c(n_2)$. (The kernel of this homomorphism is a $U(1)$–factor embedded antidiagonally.) Finally, we suppose that the three spin^c–connections are compatible in the sense that the $U(1)$–connection on $\iota^*\operatorname{Ker}(\mathcal{F}_2 \to \mathcal{B}_2)$ pulls back under (5.4) to the tensor product of the $U(1)$–connections on $\operatorname{Ker}(\mathcal{F}_1 \to \mathcal{B}_1)$ and $\operatorname{Ker}(\mathcal{F}_\nu \to \mathcal{B}_2)$. Said in terms of the characteristic line bundles, there is an isomorphism $L^{T^VX_1} \otimes L^\nu \to \iota^*L^{T^VX_2}$ which is compatible with the metrics and connections.

We now prove a lemma which shows that the elements of the image of $\check{\iota}_*$ have good wave front support.

5.5 Lemma *$\nu^* \subset T^*X_2\big|_{X_1}$ intersects $N_V^*X_2 = \pi_2^*T^*B \subset T^*X_2$ only in the zero section.*

Proof Suppose that $x_1 \in X_1$ and $\xi \in \nu^*_{x_1} \cap (N_V^*X_2)_{x_1}$. Then ξ annihilates both $T_{x_1}X_1$ and $(\iota^*T^VX_2)_{x_1}$. Since $T_{x_1}X_1 \cap (\iota^*T^VX_2)_{x_1} = T^V_{x_1}X_1$, it follows easily that $T_{x_1}X_1 + (\iota^*T^VX_2)_{x_1} = (\iota^*TX_2)_{x_1}$. Thus ξ vanishes. $\square$

Hence for a generator $\mathcal{E}$ of $\check{K}^0(X_1)$, the element $\check{\iota}_*(\mathcal{E}) \in {}_\delta\check{K}^r(X_2)$ is the image of a unique element in ${}_{\mathrm{WF}}\check{K}^r(X_2)$, which we will also call $\check{\iota}_*(\mathcal{E})$.

We now define a certain correction term to take into account the possible noncompatibility between the horizontal distributions on X_1 and X_2. That is, using (5.2), we construct an explicit form $\widetilde{C} \in \Omega(X_1;\mathcal{R})^{-1}/\operatorname{Image}(d)$ so that

$$d\widetilde{C} = \iota^* \operatorname{Todd}(\widehat{\nabla}^{T^V X_2}) - \operatorname{Todd}(\widehat{\nabla}^{T^V X_1}) \wedge \operatorname{Todd}(\widehat{\nabla}^{\nu}). \tag{5.6}$$

Namely, put $W = [0,1] \times X_1$ and let $p\colon W \to X_1$ be the projection map. Put $F = p^*\iota^* T^V X_2$. Consider a spin^c–connection $\widehat{\nabla}^F$ on F which is $\iota^*\widehat{\nabla}^{T^V X_2}$ near $\{1\} \times X_1$ and which is $\widehat{\nabla}^{T^V X_1} \oplus \widehat{\nabla}^{\nu}$ near $\{0\} \times X_1$. Then

$$\widetilde{C} = \int_0^1 \operatorname{Todd}(\widehat{\nabla}^F) \in \Omega(X_1;\mathcal{R})^{-1}/\operatorname{Image}(d).$$

5.7 Lemma *Suppose that $(T^H X_2)\big|_{X_1} = T^H X_1$. Then:*

(1) *The orthogonal projection of $\iota^*\nabla^{T^V X_2}$ to $T^V X_1$ equals $\nabla^{T^V X_1}$.*

(2) $\widetilde{C} = 0$.

Proof Suppose that

$$(T^H X_2)|_{X_1} = T^H X_1.$$

Choose a Riemannian metric g^{TB} on B and construct g^{TX_2}, ∇^{TX_2}, $\nabla^{T^V X_2}$, g^{TX_1}, ∇^{TX_1} and $\nabla^{T^V X_1}$ as in Definition 3.2. Let $P_{12}\colon \iota^* TX_2 \to TX_1$ be orthogonal projection. By naturality,

$$\nabla^{TX_1} = P_{12} \circ \iota^* \nabla^{TX_2} \circ P_{12} \quad \text{and} \quad \nabla^{T^V X_1} = P_{12} \circ \iota^* \nabla^{T^V X_2} \circ P_{12}.$$

This proves part (1) of the lemma.

As $\iota^*\widehat{\nabla}^{T^V X_2} = \widehat{\nabla}^{T^V X_1} \oplus \widehat{\nabla}^{\nu}$, it follows that $\widetilde{C} = 0$. This proves part (2) of the lemma. □

5.8 Definition Define the modified pushforward $\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E}) \in {}_{\mathrm{WF}}\breve{K}^r(X_2)$ by

$$\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E}) = \breve{\iota}_*(\mathcal{E}) - j\left(\frac{\widetilde{C}}{\iota^* \operatorname{Todd}(\widehat{\nabla}^{T^V X_2}) \wedge \operatorname{Todd}(\widehat{\nabla}^{\nu})} \wedge \omega(\mathcal{E}) \wedge \delta_{X_1}\right). \tag{5.9}$$

5.10 Lemma *The following equation holds:*

$$\omega(\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E})) = \frac{\operatorname{Todd}(\widehat{\nabla}^{T^V X_1})}{\iota^* \operatorname{Todd}(\widehat{\nabla}^{T^V X_2})} \wedge \omega(\mathcal{E}) \wedge \delta_{X_1}. \tag{5.11}$$

Proof We have

$$\begin{aligned}(5.12)\quad \omega(\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E})) &= \omega(\breve{\iota}_*(\mathcal{E})) - \frac{d\widetilde{C}}{\iota^*\operatorname{Todd}(\widehat{\nabla}^{T^VX_2})\wedge\operatorname{Todd}(\widehat{\nabla}^{\nu})}\wedge\omega(\mathcal{E})\wedge\delta_{X_1}\\ &= \frac{\omega(\mathcal{E})}{\operatorname{Todd}(\widehat{\nabla}^{\nu})}\wedge\delta_{X_1}\\ &\quad - \frac{\iota^*\operatorname{Todd}(\widehat{\nabla}^{T^VX_2}) - \operatorname{Todd}(\widehat{\nabla}^{T^VX_1})\wedge\operatorname{Todd}(\widehat{\nabla}^{\nu})}{\iota^*\operatorname{Todd}(\widehat{\nabla}^{T^VX_2})\wedge\operatorname{Todd}(\widehat{\nabla}^{\nu})}\wedge\omega(\mathcal{E})\wedge\delta_{X_1}\\ &= \frac{\operatorname{Todd}(\widehat{\nabla}^{T^VX_1})}{\iota^*\operatorname{Todd}(\widehat{\nabla}^{T^VX_2})}\wedge\omega(\mathcal{E})\wedge\delta_{X_1}.\end{aligned}$$

This proves the lemma. □

In the next lemma we consider the submersion pushforward in the case of a product bundle, under the assumption that the differential K–theory class on the total space has an almost-product form.

5.13 Lemma *Let Z be a compact Riemannian* spin^c *–manifold of even dimension n with a unitary connection ∇^{L_Z} on the characteristic line bundle L^Z. Let $\pi^Z\colon Z\to \mathrm{pt}$ be the map to a point. Let B be any manifold. Let $\pi^{\mathrm{prod}}\colon Z\times B\to B$ be projection on the second factor. Let $T^H_{\mathrm{prod}}(Z\times B)$ be the product horizontal distribution on the fiber bundle $Z\times B\to B$. Let $p\colon Z\times B\to Z$ be projection on the first factor. Suppose that $\mathcal{E}^Z=(E^Z,h^{E^Z},\nabla^{E^Z},\phi^Z)$ and $\mathcal{E}^B=(E^B,h^{E^B},\nabla^{E^B},\phi^B)$ are generators for $\breve{K}^n(Z)$ and $\breve{K}^{r-n}(B)$, respectively, for some even integer r. Let $\pi^Z_*(E^Z)\in K^0(\mathrm{pt})$ denote the K–theory pushforward of $[E^Z]\in K^n(Z)$ under the map $\pi^Z\colon Z\to\mathrm{pt}$. (We can identify $\pi^Z_*(E^Z)$ with $\int_Z\operatorname{Todd}(\widehat{\nabla}^{TZ})\wedge\omega(\nabla^{E^Z})=\operatorname{Index}(D^{Z,E^Z})\in\mathbb{Z}$.) Given $\phi\in{}_{\mathrm{WF}}\Omega(Z\times B)^{r-1}/\operatorname{Image}(d)$, put $\mathcal{E}=(p^*\mathcal{E}^Z)\cdot((\pi^{\mathrm{prod}})^*\mathcal{E}^B)+j(\phi)$. Then*

$$(5.14)\qquad \breve{\pi}^{\mathrm{prod}}_*\mathcal{E}=\pi^Z_*(E^Z)\cdot\mathcal{E}^B+j(\pi^{\mathrm{prod}}_*(\phi))$$

in $\breve{K}^{r-n}(B)$.

Proof Using (2.22), we can write

$$\begin{aligned}(5.15)\quad \mathcal{E}=\Big(&p^*E^Z\otimes(\pi^{\mathrm{prod}})^*E^B,\ p^*h^{E^Z}\otimes(\pi^{\mathrm{prod}})^*h^{E^B},\\ &p^*\nabla^{E^Z}\otimes I+I\otimes(\pi^{\mathrm{prod}})^*\nabla^{E^B},\\ &p^*\phi^Z\wedge(\pi^{\mathrm{prod}})^*\omega(\nabla^{E^B})+p^*\omega(\nabla^{E^Z})\wedge(\pi^{\mathrm{prod}})^*\phi^B\\ &\qquad\qquad + p^*\phi^Z\wedge(\pi^{\mathrm{prod}})^*d\phi^B+\phi\Big).\end{aligned}$$

Also, in this product situation, we have

$$\nabla^{T^V(Z\times B)} = p^*\nabla^{TZ}, \qquad \operatorname{Ker}(D^V) = \operatorname{Ker}(D^{Z,E^Z})\otimes E^B,$$
$$h^{\operatorname{Ker}(D^V)} = h^{\operatorname{Ker}(D^{Z,E^Z})}\otimes h^{E^B}, \qquad \nabla^{\operatorname{Ker}(D^V)} = I_{\operatorname{Ker}(D^{Z,E^Z})}\otimes \nabla^{E^B}.$$

Regarding the eta form, as $[D^V,\nabla^{\mathcal{H}}] = (\nabla^{\mathcal{H}})^2 = T = 0$, we have

$$\begin{aligned}\widetilde{\eta} &= u^{(r-n)/2} R_u \int_0^\infty \operatorname{STr}\left(u^{-1}\frac{dA_s}{ds}e^{-u^{-1}A_s^2}\right) ds \\ &= u^{(r-n)/2} R_u \int_0^\infty \operatorname{STr}\left(u^{-1/2}D^V e^{-s^2(D^V)^2}\right) ds = 0\end{aligned} \tag{5.16}$$

in $\Omega(X;\mathcal{R})^{r-n-1}/\operatorname{Image}(d)$, for parity reasons. Then

$$\begin{aligned}\breve{\pi}_*^{\text{prod}}\mathcal{E} &= \Big(\operatorname{Ker}(D^{Z,E^Z})\otimes E^B, h^{\operatorname{Ker}(D^{Z,E^Z})}\otimes h^{E^B}, I_{\operatorname{Ker}(D^{Z,E^Z})}\otimes\nabla^{E^B}, \\ &\qquad \pi_*^{\text{prod}}\big(p^*\phi^Z\wedge(\pi^{\text{prod}})^*\omega(\nabla^{E^B}) + p^*\omega(\nabla^{E^Z})\wedge(\pi^{\text{prod}})^*\phi^B \\ &\qquad\qquad + p^*\phi^Z\wedge(\pi^{\text{prod}})^*d\phi^B + \phi\big)\Big) \\ &= \Big(\operatorname{Ker}(D^{Z,E^Z})\otimes E^B, h^{\operatorname{Ker}(D^{Z,E^Z})}\otimes h^{E^B}, I_{\operatorname{Ker}(D^{Z,E^Z})}\otimes\nabla^{E^B}, \\ &\qquad \pi_*^Z(\omega(\nabla^{E^Z}))\cdot\phi^B + \pi_*^{\text{prod}}(\phi)\Big) \\ &= \pi_*^Z(E^Z)\cdot\mathcal{E}^B + j(\pi_*^{\text{prod}}(\phi)).\end{aligned} \tag{5.17}$$

This proves the lemma. ☐

The next lemma is a technical result, which will be used later, about the functoriality of reduced eta invariants with respect to product structures.

5.18 Lemma *Under the hypotheses of Lemma 5.13, suppose in addition that B is an odd-dimensional closed spin^c–manifold, equipped with a Riemannian metric g^{TB} and a unitary connection ∇^{L^B}. Then*

$$\overline{\eta}(B,\breve{\pi}_*^{\text{prod}}\mathcal{E}) = \overline{\eta}(Z\times B,\mathcal{E})$$

in $u^{(r-n-\dim(B)-1)/2}\cdot(\mathbb{R}/\mathbb{Z})$.

Proof Using (5.17), we have

$$\begin{aligned}\overline{\eta}(B,\breve{\pi}_*^{\text{prod}}\mathcal{E}) &= u^{(r-n-\dim(B)-1)/2}\cdot\operatorname{Index}(D^{Z,E^Z})\cdot\overline{\eta}(D^{B,E^B}) \\ &\qquad + \int_B \operatorname{Todd}(\widehat{\nabla}^{TB})\wedge\big((\pi_*^Z(\omega(\nabla^{E^Z}))\cdot\phi^B + \pi_*^{\text{prod}}(\phi)\big).\end{aligned} \tag{5.19}$$

By separation of variables, it is easy to show that

$$\text{Index}(D^{Z,E^Z}) \cdot \overline{\eta}(D^{B,E^B}) = \overline{\eta}(D^{Z\times B, p^*E^Z \otimes \pi^* E^B}). \tag{5.20}$$

Next,

$$\begin{aligned} \int_B \text{Todd}(\widehat{\nabla}^{TB}) \wedge (\pi_*^Z(\omega(\nabla^{E^Z})) \cdot \phi^B \\ &= \left(\int_Z \text{Todd}(\widehat{\nabla}^{TZ}) \wedge \omega(\nabla^{E^Z}) \right) \cdot \int_B \text{Todd}(\widehat{\nabla}^{TB}) \wedge \phi^B \\ &= \int_{Z\times B} \text{Todd}(\widehat{\nabla}^{T(Z\times B)}) \wedge p^*\omega(\nabla^{E^Z}) \wedge (\pi^{\text{prod}})^* \phi^B. \end{aligned} \tag{5.21}$$

Also,

$$\int_B \text{Todd}(\widehat{\nabla}^{TB}) \wedge \pi_*^{\text{prod}}(\phi) = \int_{Z\times B} \text{Todd}(\widehat{\nabla}^{T(Z\times B)}) \wedge \phi. \tag{5.22}$$

Hence

$$\begin{aligned} \overline{\eta}(B, \breve{\pi}_*^{\text{prod}} \mathcal{E}) = u^{(r-n-\dim(B)-1)/2} \cdot \overline{\eta}(D^{Z\times B, p^*E^Z \otimes \pi^* E^B}) \\ + \int_{Z\times B} \text{Todd}(\widehat{\nabla}^{T(Z\times B)}) \wedge \big(p^*\omega(\nabla^{E^Z}) \wedge (\pi^{\text{prod}})^* \phi^B + \phi \big). \end{aligned} \tag{5.23}$$

On the other hand, from (5.15),

$$\begin{aligned} \overline{\eta}(Z \times B, \mathcal{E}) &= u^{(r-n-\dim(B)-1)/2} \cdot \overline{\eta}(D^{Z\times B, p^*E^Z \otimes \pi^* E^B}) \\ &\quad + \int_{Z\times B} \text{Todd}(\widehat{\nabla}^{T(Z\times B)}) \\ &\quad \wedge \Big(p^*\phi^Z \wedge (\pi^{\text{prod}})^* \omega(\nabla^{E^B}) + p^*\omega(\nabla^{E^Z}) \wedge (\pi^{\text{prod}})^* \phi^B \\ &\qquad + p^*\phi^Z \wedge (\pi^{\text{prod}})^* d\phi^B + \phi \Big) \\ &= u^{(r-n-\dim(B)-1)/2} \cdot \overline{\eta}(D^{Z\times B, p^*E^Z \otimes \pi^* E^B}) \\ &\quad + \int_{Z\times B} \text{Todd}(\widehat{\nabla}^{T(Z\times B)}) \wedge \big(p^*\omega(\nabla^{E^Z}) \wedge (\pi^{\text{prod}})^* \phi^B + \phi \big). \end{aligned} \tag{5.24}$$

This proves the lemma. □

We now work towards the construction of the topological index, beginning with a result about embedding in spheres.

5.25 Lemma *Suppose that $\pi\colon X \to B$ is a fiber bundle with X compact and even-dimensional fibers of dimension n. Suppose that π has a Riemannian structure. Given N even, let $\pi^{\text{prod}}\colon S^N \times B \to B$ be the product bundle. Then for large N, there are an embedding $\iota\colon X \to S^N \times B$ and a Riemannian metric on S^N (independent of $b \in B$) so that ι is compatible with the Riemannian structures on π and π^{prod}. (In applying Definition 5.3, we take $X_1 = X$ and $X_2 = S^N \times B$.)*

Proof Let g^{TB} be any Riemannian metric on B. Using the Riemannian structure on π, there is a corresponding Riemannian metric g^{TX} on X. Let $e\colon X \to S^N$ be any isometric embedding of X into an even-dimensional sphere with some Riemannian metric. Put $\iota(x) = (e(x), \pi(x)) \in S^N \times B$. □

Next, we establish a Künneth-type formula for the differential K–theory of $S^N \times B$. We endow S^N with an arbitrary Riemannian metric and an arbitrary unitary connection $\nabla^{L^{S^N}}$ on its characteristic line bundle L^{S^N}.

5.26 Lemma *Given $\mathcal{E} \in {}_{\mathrm{WF}}\check{K}^r(S^N \times B)$ with r and N even, consider the fibering $\pi^{\mathrm{prod}}\colon S^N \times B \to B$. Let $p\colon S^N \times B \to S^N$ be projection onto the first factor. Then there are generators $\{\mathcal{E}_i^{S^N}\}_{i=1}^2$ for $\check{K}^N(S^N)$, generators $\{\mathcal{E}_i^B\}_{i=1}^2$ for $\check{K}^{r-N}(B)$, and $\phi \in {}_{\mathrm{WF}}\Omega(S^N \times B)^{r-1}/\,\mathrm{Image}(d)$ so that*

$$\mathcal{E} = \sum_{i=1}^{2} (p^* \mathcal{E}_i^{S^N}) \cdot ((\pi^{\mathrm{prod}})^* \mathcal{E}_i^B) + j(\phi). \tag{5.27}$$

Proof By the Künneth formula in K–theory, we can write

$$c(\mathcal{E}) = \sum_{i=1}^{2} p^* e_i^{S^N} \cdot (\pi^{\mathrm{prod}})^* e_i^B \tag{5.28}$$

for additive generators $\{e_i^{S^N}\}_{i=1}^2$ of $K^N(S^N)$ and classes $\{e_i^B\}_{i=1}^2$ in $K^{r-N}(B)$. Lift the e_i's to differential K–theory classes $\mathcal{E}_i$. Then the exact sequence (3.6) implies the existence of ϕ. □

5.29 Remark It is possible to replace the compact manifold S^N in Lemma 5.26 with the noncompact affine space $\mathbb{A}^N$, provided that we use currential K–theory with compact supports. In that case the summation in (5.27) would only have a single term, and we could remove the assumption that X is compact in Lemma 5.25. We chose to avoid introducing compact supports, at the expense of having a slightly more complicated lemma.

Using Lemma 5.13, we obtain an explicit formula for the pushforward, under the product submersion $S^N \times B \to B$, of a differential K–theory class of the type considered in Lemma 5.26. We now show that the result is independent of the particular Künneth-type representation chosen.

5.30 Lemma *Given a generator $\mathcal{E}$ for ${}_{\mathrm{WF}}\check{K}^r(S^N \times B)$ with r and N even, write $\mathcal{E}$ as in (5.27). Apply the map $\check{\pi}_*^{\mathrm{prod}}$ in Lemma 5.13 to $\mathcal{E}$ in the case $Z = S^N$, to get an element of $\check{K}^{r-N}(B)$. Then the result factors through a map $\check{\pi}_*^{\mathrm{prod}}\colon {}_{\mathrm{WF}}\check{K}^r(S^N \times B) \to \check{K}^{r-N}(B)$, which is independent of the particular decomposition (5.27) chosen.*

Proof We refer to the notation in the proof of Lemma 5.26. Let $\{1, x\}$ be an additive basis of $K^0(S^N)$, where 1 is the trivial bundle of rank 1 and x has rank 0. Choose $e_1^{S^N} = u^{N/2}1$ and $e_2^{S^N} = u^{N/2}x$, where u denotes the Bott element in K–theory. Without loss of generality, we can assume that x is chosen so that $\pi_*^{S^N}(u^{N/2}x) = 1 \in \mathbb{Z}$. Given a differential K–theory class $\mathcal{E}$ as in (5.27), Lemma 5.13 implies that

$$\text{(5.31)} \qquad \breve{\pi}_*^{\text{prod}} \mathcal{E} = \mathcal{E}_2^B + j(\pi_*^{\text{prod}}(\phi)).$$

Now a different decomposition, as in (5.27), of the same differential K–theory class $\mathcal{E}$, can only arise by the changes

$$\text{(5.32)} \qquad \begin{aligned} \mathcal{E}_i^{S^N} &\longrightarrow \mathcal{E}_i^{S^N} + j(\alpha_i^{S^N}), \\ \mathcal{E}_i^{B} &\longrightarrow \mathcal{E}_i^{B} + j(\alpha_i^{B}), \\ \phi &\longrightarrow \phi - \sum_{i=1}^{2} \left(p^*\alpha_i^{S^N} \wedge (\pi^{\text{prod}})^* \omega(\mathcal{E}_i^B) \; + \; p^*\omega(\mathcal{E}_i^{S^N}) \wedge (\pi^{\text{prod}})^* \alpha_i^B \right) \end{aligned}$$

for some $\alpha_i^{S^N} \in \Omega(S^N; \mathcal{R})^{N-1} / \operatorname{Image}(d)$ and $\alpha_i^B \in \Omega(B; \mathcal{R})^{r-N-1} / \operatorname{Image}(d)$ The ensuing change in the right-hand side of (5.31) is

$$\text{(5.33)} \quad j(\alpha_2^B) - \sum_{i=1}^{2} j\left(\pi_*^{\text{prod}} \left(p^*\alpha_i^{S^N} \wedge (\pi^{\text{prod}})^* \omega(\mathcal{E}_i^B) + p^*\omega(\mathcal{E}_i^{S^N}) \wedge (\pi^{\text{prod}})^* \alpha_i^B \right) \right).$$

As $\pi_*^{S^N}(\alpha_i^{S^N}) = \pi_*^{S^N}(\omega(\mathcal{E}_1^{S^N})) = 0$ and $\pi_*^{S^N}(\omega(\mathcal{E}_2^{S^N})) = 1$, the expression in (5.33) vanishes. The lemma follows. □

The point of Lemma 5.30 is that it gives us a well-defined map $\breve{\pi}_*^{\text{prod}} \colon {}_{\text{WF}}\breve{K}^r(S^N \times B) \to \breve{K}^{r-N}(B)$ which agrees with the pushforward defined in Section 3 when applied to elements of ${}_{\text{WF}}\breve{K}^r(S^N \times B)$ that are written in the form (5.27), and which can be computed explicitly, but does not need any spectral analysis. In particular, $\breve{\pi}_*^{\text{prod}}$ is defined without any condition about vector bundle kernel. (Note that if E is a general Hermitian vector bundle on $S^N \times B$ and ∇^E is a general compatible connection on E then there is no reason that $\operatorname{Ker}(D^V)$ should form a vector bundle on B.)

We now define the topological index for compact base spaces B; the extension to proper submersions with noncompact B is described at the end of Section Section 7.

5.34 Definition Let $\pi \colon X \to B$ be a fiber bundle with X compact. Put $n = \dim(X) - \dim(B)$, which we assume to be even. Suppose that π has a Riemannian structure. Construct N and ι from Lemma 5.25. Given a generator $\mathcal{E}$ for $\breve{K}^0(X)$, construct

$\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E}) \in {}_{\mathrm{WF}}\breve{K}^{N-n}(S^N \times B)$ from Definition 5.8. Write $\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E})$ as in Equation (5.27). Using Lemma 5.30, define the *topological index* $\mathrm{ind}^{\mathrm{top}}(\mathcal{E}) \in \breve{K}^{-n}(B)$ by

$$\mathrm{ind}^{\mathrm{top}}(\mathcal{E}) = \breve{\pi}_*^{\mathrm{prod}}(\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E})). \tag{5.35}$$

5.36 Lemma $\omega(\mathrm{ind}^{\mathrm{top}}(\mathcal{E})) = \pi_*(\omega(\mathcal{E}))$.

Proof From Lemma 3.14 and Lemma 5.10,

$$\begin{aligned} \omega(\mathrm{ind}^{\mathrm{top}}(\mathcal{E})) &= \omega(\breve{\pi}_*^{\mathrm{prod}}(\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E}))) \\ &= \pi_*^{\mathrm{prod}}(\omega(\breve{\iota}_*^{\mathrm{mod}}(\mathcal{E}))) \\ &= \pi_*^{\mathrm{prod}}\left(\frac{\mathrm{Todd}(\widehat{\nabla}^{T^V X})}{\iota^* \,\mathrm{Todd}(\widehat{\nabla}^{T^V(S^N \times B)})} \wedge \omega(\mathcal{E}) \wedge \delta_X \right) \\ &= \pi_*(\omega(\mathcal{E})). \end{aligned} \tag{5.37}$$

This proves the lemma. □

5.38 Proposition *The following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{\Omega(X;\mathcal{R})^{-1}}{\Omega(X;\mathcal{R})_K^{-1}} & \xrightarrow{j} & \breve{K}^0(X) & \xrightarrow{c} & K^0(X;\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \mathrm{ind}^{\mathrm{top}}} & & \downarrow{\scriptstyle \mathrm{ind}^{\mathrm{top}}} & & \\ 0 & \longrightarrow & \dfrac{\Omega(B;\mathcal{R})^{-n-1}}{\Omega(B;\mathcal{R})_K^{-n-1}} & \xrightarrow{j} & \breve{K}^{-n}(B) & \xrightarrow{c} & K^{-n}(B;\mathbb{Z}) & \longrightarrow & 0. \end{array} \tag{5.39}$$

Proof The right-hand square commutes from our construction of $\mathrm{ind}^{\mathrm{top}}\colon \breve{K}^0(X) \to \breve{K}^{-n}(B)$; see the discussion at the beginning of Section 4 of the K–theory pushforward under an embedding. To see that the left-hand square commutes, suppose that $\phi \in \Omega(X;\mathcal{R})^{-1}/\Omega(X;\mathcal{R})_K^{-1}$. Then

$$\begin{aligned} \mathrm{ind}^{\mathrm{top}}(j(\phi)) &= \breve{\pi}_*^{\mathrm{prod}}(\breve{\iota}_*^{\mathrm{mod}}(j(\phi))) \\ &= \breve{\pi}_*^{\mathrm{prod}}\Bigg(j\Bigg(\frac{\phi}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \delta_X \\ &\qquad - \frac{\widetilde{C}}{\iota^* \,\mathrm{Todd}(\widehat{\nabla}^{T^V(S^N \times B)}) \wedge \mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge d\phi \wedge \delta_X \Bigg)\Bigg) \\ &= j\left(\int_X \left(\frac{\iota^* \,\mathrm{Todd}(\widehat{\nabla}^{T^V(S^N \times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi - \frac{\widetilde{C}}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge d\phi \right)\right) \end{aligned} \tag{5.40}$$

$$= j\left(\int_X \frac{\iota^* \operatorname{Todd}(\widehat{\nabla}^{T^V(S^N \times B)}) - d\widetilde{C}}{\operatorname{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi \right)$$
$$= j\left(\int_X \operatorname{Todd}\left(T^V X\right) \wedge \phi \right) = j(\pi_*(\phi)).$$

This proves the lemma. □

From what has been said so far, the map $\operatorname{ind}^{\mathrm{top}}\colon \check{K}^0(X) \to \check{K}^{-n}(B)$ depends on the Riemannian structure on π and, possibly, on the embedding ι. We prove in Corollary 7.36 that it is in fact independent of ι.

6 Index theorem: vector bundle kernel

In this section we prove our index theorem for families of Dirac operators, under the assumption of vector bundle kernel and compact base space.

In terms of the diagram

(6.1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^{-1}(X;\mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & \check{K}^0(X) & \xrightarrow{\omega} & \Omega(X;\mathcal{R})^0_K & \longrightarrow & 0 \\
 & & \downarrow & & \operatorname{ind}^{\mathrm{an}} \downarrow\downarrow \operatorname{ind}^{\mathrm{top}} & & \downarrow \pi_* & & \\
0 & \longrightarrow & K^{-n-1}(B;\mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & \check{K}^{-n}(B) & \xrightarrow{\omega} & \Omega(B;\mathcal{R})^{-n}_K & \longrightarrow & 0,
\end{array}$$

we know that if $\mathcal{E} \in \check{K}^0(X)$ then $\omega(\operatorname{ind}^{\mathrm{an}}(\mathcal{E}) - \operatorname{ind}^{\mathrm{top}}(\mathcal{E})) = 0$. Hence $\operatorname{ind}^{\mathrm{an}}(\mathcal{E}) - \operatorname{ind}^{\mathrm{top}}(\mathcal{E})$ is the image under j of a unique element in $K^{-n-1}(B;\mathbb{R}/\mathbb{Z})$. We now apply the method of proof of [33, Section 4] to prove that the difference vanishes, by computing its pairings with elements of $K_{-n-1}(B)$. From Proposition 2.35(2), such pairings are given by reduced eta-invariants. As in [33, Section 4], the pairing with an element of $K_{-n-1}(B)$ becomes a computation of reduced η–invariants on X after taking adiabatic limits. A new ingredient is the use of the main theorem of [13] in order to relate the reduced eta-invariants of a manifold and an embedded submanifold.

6.2 Theorem *Let $\pi\colon X \to B$ be a fiber bundle with compact fibers of even dimension. Suppose that π is equipped with a Riemannian structure and a differential* spin^c *structure. Assume that X is compact and that $\operatorname{Ker}(D^V) \to B$ is a vector bundle. Then for all $\mathcal{E} \in \check{K}^0(X)$ we have $\operatorname{ind}^{\mathrm{an}}(\mathcal{E}) = \operatorname{ind}^{\mathrm{top}}(\mathcal{E})$.*

Proof The short exact sequence (2.29), along with Lemma 3.14 and Lemma 5.36, implies that $\operatorname{ind}^{\mathrm{an}}(\mathcal{E}) - \operatorname{ind}^{\mathrm{top}}(\mathcal{E})$ lifts uniquely to an element $\mathcal{T}$ of $K^{-n-1}(B;\mathbb{R}/\mathbb{Z})$.

We want to show that this element vanishes. To do so, we use the method of proof of [33, Section 4]. From the universal coefficient theorem and the divisibility of $\mathbb{R}/\mathbb{Z}$, it suffices to show that for all $\alpha \in K_{-n-1}(B;\mathbb{Z})$, the pairing $\langle \alpha, \mathcal{T}\rangle$ vanishes in $\mathbb{R}/\mathbb{Z}$. From [27], $K_{-n-1}(B;\mathbb{Z})$ is generated by elements of the form $\alpha = f_*[M]$ where M is a closed odd-dimensional spin^c–manifold, $[M] \in K_{-n-1}(M;\mathbb{Z})$ is the fundamental class of M (shifted from $K_{\dim(M)}(M;\mathbb{Z})$ to $K_{-n-1}(M;\mathbb{Z})$ using Bott periodicity) and $f\colon M \to B$ is a smooth map. (The argument in [33, Section 4] used instead the Baum–Douglas description of K–homology [8], which essentially involves an additional vector bundle on M.) As $\langle \alpha, \mathcal{T}\rangle = \langle [M], f^*\mathcal{T}\rangle$, we can effectively pull everything back to M and so reduce to considering the case when B is an arbitrary closed odd-dimensional spin^c–manifold.

Now suppose that $\mathcal{E} = (E, h^E, \nabla^E, \phi)$. Recall from Definition 4.14 the construction of $\check{\iota}_*(\mathcal{E}) = (F, h^F, \nabla^F, \iota_*(\phi) + \gamma)$.

In the rest of this proof, all equalities will be taken modulo the integers, so will be written as congruences. We equip B with a Riemannian metric g^{TB}, and the characteristic line bundle L^B with a unitary connection ∇^{L^B}. We equip the fiber bundle $S^N \times B \to B$ with the product horizontal connection $T^H_{\text{prod}}(S^N \times B)$. Then $S^N \times B$ has the product Riemannian metric, from which the submanifold X acquires a Riemannian metric.

By Proposition 2.35 and Lemma 5.18,

$$u^{-(\dim(X)+1)/2}\langle [B], \mathcal{T}\rangle \equiv C_1 - C_2 \tag{6.3}$$

in $u^{-(\dim(X)+1)/2}\cdot(\mathbb{R}/\mathbb{Z})$, where

$$\begin{aligned} C_1 &\equiv \overline{\eta}(B, \operatorname{ind}^{\text{an}}(\mathcal{E})) \\ &\equiv u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\operatorname{Ker}(D^V)_+}) - u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\operatorname{Ker}(D^V)_-}) \\ &\qquad + \int_B \operatorname{Todd}(\widehat{\nabla}^{TB}) \wedge (\pi_*(\phi) + \widetilde{\eta}) \\ &\equiv u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\operatorname{Ker}(D^V)_+}) - u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\operatorname{Ker}(D^V)_-}) \\ &\qquad + \int_B \operatorname{Todd}(\widehat{\nabla}^{TB}) \wedge \widetilde{\eta} + \int_X \pi^* \operatorname{Todd}(\widehat{\nabla}^{TB}) \wedge \operatorname{Todd}(\widehat{\nabla}^{T^VX}) \wedge \phi \end{aligned} \tag{6.4}$$

and, using Lemma 5.18,

$$\begin{aligned} C_2 &\equiv \overline{\eta}(B, \operatorname{ind}^{\text{top}}(\mathcal{E})) \\ &\equiv \overline{\eta}(S^N \times B, \check{\iota}_*^{\text{mod}}(\mathcal{E})) \end{aligned} \tag{6.5}$$

$$\begin{aligned}
&\equiv u^{-(\dim(X)+1)/2}\overline{\eta}(D^{S^N\times B,F_+}) - u^{-(\dim(X)+1)/2}\overline{\eta}(D^{S^N\times B,F_-})\\
&\quad + \int_{S^N\times B} \mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)})\\
&\qquad \wedge \left(\iota_*\phi - \gamma - \frac{\widetilde{C}}{\iota^*\,\mathrm{Todd}(\widehat{\nabla}^{T^V(S^N\times B)}) \wedge \mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \omega(\mathcal{E}) \wedge \delta_X\right)\\
&\equiv u^{-(\dim(X)+1)/2}\overline{\eta}(D^{S^N\times B,F_+}) - u^{-(\dim(X)+1)/2}\overline{\eta}(D^{S^N\times B,F_-})\\
&\quad + \int_X \frac{\iota^*\,\mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi - \int_{S^N\times B} \mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)}) \wedge \gamma\\
&\quad - \int_X \frac{\pi^*\,\mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge (\omega(\nabla^E) + d\phi).
\end{aligned}$$

From [13, Theorem 2.2],

$$\textbf{(6.6)}\quad \overline{\eta}(D^{S^N\times B,F_+}) - \overline{\eta}(D^{S^N\times B,F_-}) \equiv \overline{\eta}(D^{X,E}) + u^{(\dim(X)+1)/2}\int_{S^N\times B} \mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)}) \wedge \gamma.$$

Thus

$$\textbf{(6.7)}\quad \begin{aligned} C_2 &\equiv u^{-(\dim(X)+1)/2}\overline{\eta}(D^{X,E}) - \int_X \frac{\pi^*\,\mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \omega(\nabla^E)\\
&\quad + \int_X \frac{\iota^*\,\mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi - \int_X \frac{\pi^*\,\mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge d\phi.\end{aligned}$$

Now

$$\textbf{(6.8)}\quad \begin{aligned}
&\int_X \frac{\iota^*\,\mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi - \int_X \frac{\pi^*\,\mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge d\phi\\
&\equiv \int_X \frac{\iota^*\,\mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi - \int_X \frac{\pi^*\,\mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge d\widetilde{C} \wedge \phi\\
&\equiv \int_X \frac{\iota^*\,\mathrm{Todd}(\widehat{\nabla}^{T(S^N\times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi\\
&\quad - \int_X \frac{\pi^*\,\mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})}\\
&\qquad \wedge \left(\iota^*\,\mathrm{Todd}(\widehat{\nabla}^{T^V(S^N\times B)}) - \mathrm{Todd}(\widehat{\nabla}^{T^VX}) \wedge \mathrm{Todd}(\widehat{\nabla}^{\nu})\right) \wedge \phi\\
&\equiv \int_X \pi^*\,\mathrm{Todd}(\widehat{\nabla}^{TB}) \wedge \mathrm{Todd}(\widehat{\nabla}^{T^VX}) \wedge \phi.
\end{aligned}$$

Then

$$\textbf{(6.9)}\quad C_1 - C_2 \equiv u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\mathrm{Ker}(D^V)_+}) - u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\mathrm{Ker}(D^V)_-}) + \int_B \mathrm{Todd}(\widehat{\nabla}^{TB}) \wedge \widetilde{\eta} - u^{-(\dim(X)+1)/2}\overline{\eta}(D^{X,E}) + \int_X \frac{\pi^* \mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \omega(\nabla^E).$$

The next lemma, stated in terms of bordisms, shows that $C_1 - C_2$ is unchanged by certain perturbations.

6.10 Lemma *Suppose that $B = \partial B'$ for some even-dimensional compact* spin^c*–manifold B'. Suppose that the structures,*

$$g^{TB},\ \nabla^{L^B},\ \pi\colon X \to B,\ T^H X,\ \iota\colon X \to S^N \times B,\ E \to X \text{ and } \nabla^E$$

extend to structures

$$g^{TB'},\ \nabla^{L^{B'}},\ \pi'\colon X' \to B',\ T^H X',\ \iota'\colon X' \to S^N \times B',\ E' \to X' \text{ and } \nabla^{E'}$$

over B', which are product-like near $B = \partial B'$. Suppose that $\mathrm{Ker}(D^V)'$ *forms a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle on B'. Then $C_1 - C_2 \equiv 0$.*

Proof From Proposition 2.35,

$$\textbf{(6.11)}\quad u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\mathrm{Ker}(D^V)_+}) - u^{-(\dim(X)+1)/2}\overline{\eta}(D^{B,\mathrm{Ker}(D^V)_-}) \equiv \int_{B'} \mathrm{Todd}(\widehat{\nabla}^{TB'}) \wedge \omega(\nabla^{\mathrm{Ker}(D^V)'}),$$

$$\textbf{(6.12)}\quad u^{-(\dim(X)+1)/2}\overline{\eta}(D^{X,E}) \equiv \int_{X'} \mathrm{Todd}(\widehat{\nabla}^{TX'}) \wedge \omega(\nabla^{E'}).$$

Also, we have that

$$\textbf{(6.13)}\quad \int_B \mathrm{Todd}(\widehat{\nabla}^{TB}) \wedge \widetilde{\eta} \equiv \int_{B'} \mathrm{Todd}(\widehat{\nabla}^{TB'}) \wedge d\widetilde{\eta}' \equiv \int_{X'} (\pi')^* \mathrm{Todd}(\widehat{\nabla}^{TB'}) \wedge \mathrm{Todd}(\widehat{\nabla}^{T^V X'}) \wedge \omega(\nabla^{E'}) - \int_{B'} \mathrm{Todd}(\widehat{\nabla}^{TB'}) \wedge \omega(\nabla^{\mathrm{Ker}(D^V)'}),$$

(6.14)
$$\begin{aligned}\int_X \frac{\pi^* \operatorname{Todd}(\widehat{\nabla}^{TB})}{\operatorname{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \omega(\nabla^E) &\equiv \int_{X'} \frac{(\pi')^* \operatorname{Todd}(\widehat{\nabla}^{TB'})}{\operatorname{Todd}(\widehat{\nabla}^{\nu'})} \wedge d\widetilde{C}' \wedge \omega(\nabla^{E'}) \\ &\equiv \int_{X'} \frac{(\pi')^* \operatorname{Todd}(\widehat{\nabla}^{TB'})}{\operatorname{Todd}(\widehat{\nabla}^{\nu'})} \\ &\qquad \wedge \Big((\iota')^* \operatorname{Todd}(\widehat{\nabla}^{T^V(S^N \times B')}) - \operatorname{Todd}(\widehat{\nabla}^{T^V X'}) \wedge \operatorname{Todd}(\widehat{\nabla}^{\nu'}) \Big) \wedge \omega(\nabla^{E'}) \\ &\equiv \int_{X'} \operatorname{Todd}(\widehat{\nabla}^{TX'}) \wedge \omega(\nabla^{E'}) \\ &\qquad - \int_{X'} (\pi')^* \operatorname{Todd}(\widehat{\nabla}^{TB'}) \wedge \operatorname{Todd}(\widehat{\nabla}^{T^V X'}) \wedge \omega(\nabla^{E'}).\end{aligned}$$

The lemma follows from combining Equations (6.11)–(6.14). □

Continuing with the proof of Theorem 6.2, we apply Lemma 6.10 with $B' = [0,1] \times B$, so $\partial B' = B_1 - B_0$. If $p\colon [0,1] \times B \to B$ is the projection map then we take all of the structures on B' to be pullbacks under p of the corresponding structures on B, except for the horizontal distribution $T^H X'$. Note that the property of having vector bundle kernel is independent of the choice of horizontal distribution. We choose $T^H X'$ to equal $p^* T^H X$ near $\{1\} \times B$, and to equal $p^*(T^H_{\mathrm{prod}}(S^N \times B))|_X$ near $\{0\} \times B$. Then Lemma 6.10 implies the computation of $C_1 - C_2$ for B_1 equals that for B_0. Thus without loss of generality, we can assume that $T^H X = (T^H_{\mathrm{prod}}(S^N \times B))|_X$. In this case, $\widetilde{C}$ vanishes from Lemma 5.7.

Next, we apply Lemma 6.10 with $B' = [0,1] \times B$ and with all of the structures on B' pulling back from B, except for the Riemannian metrics. Given $\epsilon > 0$, let $\rho\colon [0,1] \to \mathbb{R}^+$ be a smooth function which is ϵ near $\{0\}$ and which is 1 near $\{1\}$. Multiply the fiberwise metrics for the Riemannian structures $\pi'\colon [0,1] \times X \to [0,1] \times B$ and $\pi'_{\mathrm{prod}}\colon [0,1] \times S^N \times B \to [0,1] \times B$ by a factor $\rho(t)$, for $t \in [0,1]$. By doing so, we do not alter the property of having vector bundle kernel. Then Lemma 6.10 implies the computation of $C_1 - C_2$ for B_1 equals that for B_0. That is, $C_1 - C_2$ is unchanged after scaling the metrics by ϵ.

Hence it suffices to compute $C_1 - C_2$ in the limit when $\epsilon \to 0$. In this case, it is known [17, Theorem 0.1; 33, Section 4] that

(6.15)
$$\begin{aligned}&\lim_{\epsilon \to 0} \overline{\eta}(D^{X,E}) \\ &\quad \equiv \overline{\eta}(D^{B,\mathrm{Ker}(D^V)_+}) - \overline{\eta}(D^{B,\mathrm{Ker}(D^V)_-}) + u^{(\dim(X)+1)/2} \int_B \operatorname{Todd}(\widehat{\nabla}^{TB}) \wedge \widetilde{\eta}\end{aligned}$$

in $\mathbb{R}/\mathbb{Z}$. Thus $C_1 - C_2 \equiv 0$. The theorem follows. □

6.16 Corollary (1) *The assignment $\mathcal{E} \to \mathrm{ind}^{\mathrm{an}}(\mathcal{E})$ factors through a map $\check{K}^0(X) \to \check{K}^{-n}(B)$.*

(2) *The map $\mathrm{ind}^{\mathrm{top}}\colon \check{K}^0(X) \to \check{K}^{-n}(B)$ is independent of the choice of embedding ι.*

Proof Part (1) follows from Theorem 6.2 and the fact that the assignment $\mathcal{E} \to \mathrm{ind}^{\mathrm{top}}(\mathcal{E})$ factors through a map $\check{K}^0(X) \to \check{K}^{-n}(B)$. Part (2) follows from Theorem 6.2 and the fact that $\mathrm{ind}^{\mathrm{an}}$ is independent of the choice of embedding ι. □

7 Index theorem: general case

In this section we complete the proof of the differential K–theory index theorem.

In general, the kernels of a family of Dirac operators need not form a vector bundle. In such a case, the basic idea is to perform a finite-rank perturbation of the operators, in order to effectively reduce to the case of vector bundle kernel. One way to do this, used in [7] is to enlarge the domain of $(D^V)_+$ by the sections of a trivial bundle over B, in order to make a finite rank change so that $(D^V)_+$ becomes surjective; this implies vector bundle kernel. We instead follow the method of [33, Section 5], which uses a lemma of Miščenko–Fomenko (Lemma 7.13) to find a finite rank subbundle of the infinite rank bundle $\mathcal{H}$ which captures the index. Adding on this finite rank subbundle, with the opposite grading, allows one to alter the operator to make it invertible.

An additional technical issue arises in trying to construct the eta form. We want to make the D^V–term in the integrand invertible for large s, but we want to keep the small–s asymptotics of the unperturbed Bismut superconnection. As in [33, Section 5], we use the trick of "time-varying η–forms", which originated in [37].

In Section 7.1 we recall some facts about "time-varying η–invariants" and "time-varying η–forms". In Section 7.12 we review the Miščenko–Fomenko result and construct the analytic pushforward in the general case (Definition 7.27). After these preliminaries, in Section 7.31 we prove the general index theorem along the lines of the argument in the previous section. Finally, in Section 7.40 we use the limit theorem in the appendix to extend the theorem to proper fiber bundles with arbitrary base.

7.1 Eta invariants and eta forms

We first review some material from [33] about eta invariants and eta forms, which is an adaptation of [11] to the time-varying case.

Let B be a closed odd-dimensional manifold. Let $\mathcal{D}$ be a smooth 1–parameter family of first-order self-adjoint elliptic pseudodifferential operators $D(s)$ on B, such that:

- There are a $\delta > 0$ and a first-order self-adjoint elliptic pseudodifferential operator D_0 on X such that for $s \in (0, \delta)$, we have $D(s) = sD_0$.
- There are a $\Delta > 0$ and a first-order self-adjoint elliptic pseudodifferential operator D_∞ on X such that for $s \in (\Delta, \infty)$, we have $D(s) = sD_\infty$.

For $z \in \mathbb{C}$ with $\mathrm{Re}(z) \gg 0$, put

$$\eta(\mathcal{D})(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty s^z \operatorname{Tr}\left(\frac{dD(s)}{ds} e^{-D(s)^2}\right) ds. \tag{7.2}$$

7.3 Lemma [33, Lemma 2] *$\eta(\mathcal{D})(z)$ extends to a meromorphic function on $\mathbb{C}$ which is holomorphic near $z = 0$.*

Define the eta-invariant of $\mathcal{D}$ by

$$\eta(\mathcal{D}) = \eta(\mathcal{D})(0) \tag{7.4}$$

and define the reduced eta-invariant of $\mathcal{D}$ by

$$\overline{\eta}(\mathcal{D}) = \frac{\eta(\mathcal{D}) + \dim(\mathrm{Ker}(D_\infty))}{2} \pmod{\mathbb{Z}}. \tag{7.5}$$

7.6 Lemma [33, Lemma 3] *$\eta(\mathcal{D})$ only depends on D_0 and D_∞, and $\overline{\eta}(\mathcal{D})$ only depends on D_0.*

Now suppose that B additionally is a Riemannian spin^c–manifold, equipped with a spin^c–connection $\widehat{\nabla}^{TB}$ on the spinor bundle $\mathcal{S}^B$. Let E be a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle over B. We think of $[E]$ as defining an element of $K^{-n}(B)$, for some even n. If A is a superconnection on E and $s \in \mathbb{R}^+$, let A_s denote the result of multiplying each factor of u in A by s^2.

Let $\mathcal{A} = \{A(s)\}_{s \geq 0}$ be a smooth 1–parameter family of superconnections on E such that:

- There are a $\delta > 0$ and a superconnection A_0 on E such that for $s \in (0, \delta)$, we have $A(s) = (A_0)_s$.
- There are a $\Delta > 0$ and a superconnection A_∞ on E such that for $s \in (\Delta, \infty)$, we have $A(s) = (A_\infty)_s$.

Suppose that A_∞ is invertible. Define $\widetilde{\eta}(\mathcal{A})(z) \in \Omega(B; \mathcal{R})^{-n-1} / \mathrm{Image}(d)$ for $z \in \mathbb{C}$, $\mathrm{Re}(z) \gg 0$, by

$$\widetilde{\eta}(\mathcal{A})(z) = u^{-n/2} R_u \int_0^\infty z^s \operatorname{str}\left(u^{-1} \frac{dA(s)}{ds} e^{-u^{-1} A(s)^2}\right) ds. \tag{7.7}$$

7.8 Lemma [33, Lemma 4] *$\widetilde{\eta}(\mathcal{A})(z)$ extends to a meromorphic vector-valued function on $\mathbb{C}$ with simple poles. Its residue at zero vanishes in $\Omega(B;\mathcal{R})^{-n-1}/\operatorname{Image}(d)$.*

Define the eta form of $\mathcal{A}$ by

$$\widetilde{\eta}(\mathcal{A}) = \widetilde{\eta}(\mathcal{A})(0). \tag{7.9}$$

As in Lemma 7.6, $\widetilde{\eta}(\mathcal{A})$ only depends on A_0 and A_∞.

Given a superconnection A on E, let $\overline{A}$ denote the associated first-order differential operator [9, Section 3.3]. It is the essentially self-adjoint operator on $C^\infty(E \otimes \mathcal{S}^B)$ obtained by replacing the Grassmann variables in A by Clifford variables and replacing u by 1. Now given a family $\mathcal{A}$ of superconnections as above and a parameter $\epsilon > 0$, define a family of operators $\mathcal{D}^{(\epsilon)}$ by

$$D^{(\epsilon)}(s) = \overline{A(s)_{\epsilon^{-1}}}. \tag{7.10}$$

(In the fiber bundle situation, this corresponds to multiplying the fiber lengths by a factor of ϵ. The paper [11] instead expands the base, but the two approaches are equivalent.) Let $\eta(\mathcal{D}^{(\epsilon)})$ be the corresponding eta invariant. Then a generalization of [11, (A.1.7)] says that

$$\lim_{\epsilon \to 0} \overline{\eta}(\mathcal{D}^{(\epsilon)}) = u^{(\dim(B)+n+1)/2} \int_B \operatorname{Todd}(\widehat{\nabla}^{TB}) \wedge \widetilde{\eta}(\mathcal{A}) \pmod{\mathbb{Z}}. \tag{7.11}$$

7.12 Analytic pushforward

We continue with the setup of Section 3, namely a family of Dirac-type operators, except that we no longer assume that $\operatorname{Ker}(D^V)$ forms a smooth vector bundle on B. In order to deal with this more general situation, we will use a perturbation argument, following the approach of [33, Section 5]. For this, we need to assume that B is compact.

We first recall a technical lemma of Miščenko–Fomenko, along with its proof.

7.13 Lemma [39] *Suppose that B is compact. Then there are finite-dimensional vector subbundles $L_\pm \subset \mathcal{H}_\pm$ and complementary closed subbundles $K_\pm \subset \mathcal{H}_\pm$, ie*

$$\mathcal{H}_\pm = K_\pm \oplus L_\pm, \tag{7.14}$$

so that $D^V_+ \in \operatorname{Hom}(\mathcal{H}_+, \mathcal{H}_-)$ is block diagonal as a map

$$D^V_+ \colon K_+ \oplus L_+ \to K_- \oplus L_- \tag{7.15}$$

and D^V_+ restricts to an isomorphism between K_+ and K_-. (Note that $K_\pm$ may not be orthogonal to $L_\pm$.)

Proof This is proved in [39, Lemma 2.2]. For completeness, we sketch the argument. One first finds finite-dimensional vector subbundles $L'_\pm \subset \mathcal{H}_\pm$ so that the projected map

$$(L'_+)^\perp \xrightarrow{D^V_+} \mathcal{H}_- \to (L'_-)^\perp$$

is an isomorphism. With respect to the orthogonal decomposition $\mathcal{H}_\pm = (L'_\pm)^\perp \oplus L'_\pm$, write

$$D^V_+ = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{7.16}$$

where $A\colon (L'_+)^\perp \to (L'_-)^\perp$ is an isomorphism. Set

$$\begin{aligned} K_+ &= (L'_+)^\perp, \\ L_+ &= \mathrm{Image}((-A^{-1}B + I)\colon L'_+ \to \mathcal{H}_+), \\ K_- &= \mathrm{Image}((I + CA^{-1})\colon (L'_-)^\perp \to \mathcal{H}_-), \\ L_- &= L'_-. \end{aligned} \tag{7.17}$$

This proves the lemma. □

Let $i_-\colon L_- \to \mathcal{H}_-$ be the inclusion map and let $p_+\colon \mathcal{H}_+ \to L_+$ be the projection map coming from (7.14). Put $\widetilde{\mathcal{H}}_\pm = \mathcal{H}_\pm \oplus L_\mp$. Given $\alpha \in \mathbb{C}$, define $\widetilde{D}^V_+(\alpha) \in \mathrm{Hom}(\widetilde{\mathcal{H}}_+, \widetilde{\mathcal{H}}_-)$ by the matrix

$$\widetilde{D}^V_+(\alpha) = \begin{pmatrix} D^V_+ & \alpha i_- \\ \alpha p_+ & 0 \end{pmatrix}. \tag{7.18}$$

That is,

$$\widetilde{D}^V_+(\alpha)(h_+ \oplus l_-) = (D^V_+ h_+ + \alpha i_- l_- \oplus \alpha p_+ h_+). \tag{7.19}$$

7.20 Lemma *If $\alpha \neq 0$ then $\widetilde{D}^V_+(\alpha)$ is invertible.*

Proof Suppose that $\widetilde{D}^V_+(\alpha)(h_+ \oplus l_-) = 0$. As $p_+ h_+ = 0$, we know that $h_+ \in K_+$. Then $D^V_+ h_+ \in K_-$. As $D^V_+ h_+ + \alpha i_- l_- = 0$, we conclude that $D^V_+ h_+ = 0$ and $l_- = 0$. As D^V_+ is injective on K_+, it follows that $h_+ = 0$. Hence $\widetilde{D}^V_+(\alpha)$ is injective.

Now suppose that $h'_- \oplus l'_+ \in \widetilde{\mathcal{H}}_-$. With respect to (7.14), write $h'_- = k'_- + l'_-$. Put

$$\begin{aligned} h_+ &= \left(D^V_+\big|_{K_+}\right)^{-1} k'_- + \alpha^{-1} l'_+, \\ l_- &= \alpha^{-1} l'_- - \alpha^{-2} D^V_+ l'_+. \end{aligned} \tag{7.21}$$

One can check that $\widetilde{D}^V_+(h_+ \oplus l_-) = h'_- \oplus l'_+$. Thus $\widetilde{D}^V_+(\alpha)$ is surjective. □

Define $\widetilde{D}^V(\alpha) \in \mathrm{End}(\widetilde{\mathcal{H}})$ by $\widetilde{D}^V(\alpha) = \widetilde{D}^V_+(\alpha) \oplus (\widetilde{D}^V_+(\alpha))^*$, an essentially self-adjoint operator on each fiber $\widetilde{\mathcal{H}}_b$. As $\widetilde{D}^V(\alpha)$ is a finite-rank perturbation of $D^V \oplus I_L$, and $(I + (D^V)^2)^{-1}$ is compact on each fiber $\mathcal{H}_b$, it follows that $(I + (\widetilde{D}^V(\alpha))^2)^{-1}$ is compact on each fiber $\widetilde{\mathcal{H}}_b$. Lemma 7.20 now implies that if $\alpha \neq 0$ then $(\widetilde{D}^V(\alpha))^2$ has strictly positive spectrum.

Give L the projected Hermitian inner product h^L and projected compatible connection ∇^L from $\mathcal{H}$. Put $\nabla^{\widetilde{\mathcal{H}}_\pm} = \nabla^{\mathcal{H}_\pm} \oplus \nabla^{L_\mp}$. Let $\alpha\colon [0,\infty) \to [0,1]$ be a smooth function for which $\alpha(s) = 0$ if s is near 0, and $\alpha(s) = 1$ if $s \geq 1$. We view $[L]$ as defining an element of $K^{-n}(B)$.

For $s > 0$, define a superconnection $\widetilde{A}_s$ by

$$\widetilde{A}_s = s u^{1/2} \widetilde{D}^V(\alpha(s)) + \nabla^{\widetilde{\mathcal{H}}} - s^{-1} u^{-1/2} \frac{c(T)}{4}. \tag{7.22}$$

Then

$$\lim_{s\to 0} u^{-n/2} R_u \, \mathrm{STr}(e^{-u^{-1}\widetilde{A}_s^2}) = \pi_*(\omega(\nabla^E)) - \omega(\nabla^L), \tag{7.23}$$

while

$$\lim_{s\to \infty} u^{-n/2} R_u \, \mathrm{STr}(e^{-u^{-1}\widetilde{A}_s^2}) = 0. \tag{7.24}$$

Note that unlike in Section 7.1, we do not have to use zeta-function regularization because when $s \to 0$, the $\mathcal{H}$ and L factors in $\widetilde{\mathcal{H}}$ decouple and so we are reduced to the short-time asymptotics of the Bismut superconnection (3.8).

Put

$$\widetilde{\eta} = u^{-n/2} R_u \int_0^\infty \mathrm{STr}\left(u^{-1} \frac{d\widetilde{A}_s}{ds} e^{-u^{-1}\widetilde{A}_s^2} \right) ds \in \Omega(B;\mathcal{R})^{-n-1}/\,\mathrm{Image}(d). \tag{7.25}$$

It is independent of the particular choice of the function α. Also,

$$d\widetilde{\eta} = \pi_*(\omega(\nabla^E)) - \omega(\nabla^L). \tag{7.26}$$

7.27 Definition Given a generator $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ for $\check{K}^0(X)$, we define the *analytic index*

$$\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) = (L, h^L, \nabla^L, \pi_*(\phi) + \widetilde{\eta}) \tag{7.28}$$

as an element of $\check{K}^{-n}(B)$, where L is chosen as in Lemma 7.13.

Given a generator $\mathcal{E}$ of ${}_{\mathrm{WF}}\check{K}^0(X)$, we define $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) \in \check{K}^{-n}(B)$ by the same formula (7.28). We prove in Corollary 7.36 that this definition is independent of the choice of L.

7.29 Lemma *If $\mathcal{E}$ is a generator for ${}_{\mathrm{WF}}\check{K}^0(X)$ then $\omega(\mathrm{ind}^{\mathrm{an}}(\mathcal{E})) = \pi_*(\omega(\mathcal{E}))$ in $\Omega(B;\mathcal{R})^{-n}$.*

Proof We have

$$\omega(\mathrm{ind}^{\mathrm{an}}(\mathcal{E})) = \omega(\nabla^L) + d(\pi_*(\phi) + \widetilde{\eta}) = \pi_*(\omega(\nabla^E) + d\phi) = \pi_*(\omega(\mathcal{E})), \tag{7.30}$$

which proves the lemma. □

7.31 General index theorem

Continuing with the assumptions of the previous subsection, suppose that B is a closed odd-dimensional Riemannian manifold with a spin^c–structure. Let $\widehat{\nabla}^{TB}$ be a spin^c–connection on $\mathcal{S}^B$. Combining with the Riemannian structure on π and the differential spin^c–structure on π, we obtain a Riemannian metric g^{TX} on X and a spin^c–connection $\widehat{\nabla}^{TX}$ on $\mathcal{S}^X$.

As in Section 7.1, given a parameter $\epsilon > 0$, we define a family of pseudodifferential operators $\mathcal{D}^{(\epsilon)}$ (living on X) by

$$D^{(\epsilon)}(s) = \overline{(\widetilde{A}_s)_{\epsilon^{-1}}}. \tag{7.32}$$

Then the family $\mathcal{D}^{(\epsilon)}$ satisfies the formalism of Section 7.1.

To identify the operators D_0 and D_∞ corresponding to the family $\mathcal{D}^{(\epsilon)}$, let X_ϵ denote the Riemannian structure on X coming from multiplying g^{TX} in the vertical direction by ϵ. If s is near zero then $\alpha(s)$ vanishes and the superconnection $\widetilde{A}_s$ of (7.22) just becomes the direct sum of the Bismut superconnection on $\mathcal{H}$ and the connection on ΠL, the latter being L with the opposite grading. Therefore, $D_0 = D^{X_\epsilon,E} \oplus D^{B,\Pi L}$ is the sum of ordinary Dirac-type operators on X_ϵ and B. On the other hand, if $s > \Delta$ then $\alpha(s) = 1$ and $\widetilde{D}^V(\alpha(s))$ is L^2–invertible. From (7.22), D_∞ is the Dirac operator on B coupled to the superconnection $\epsilon^{-1}\widetilde{D}^V(1) + \nabla^{\mathcal{H}} - \epsilon(c(T)/4)$. If ϵ is small then the term $\epsilon^{-1}\widetilde{D}^V(1)$ dominates when computing the spectrum of D_∞, so D_∞ is an invertible first-order self-adjoint elliptic pseudodifferential operator on the disjoint union $X \sqcup B$.

Let $\overline{\eta}(\mathcal{D}^{(\epsilon)})$ be the reduced eta invariant of the rescaled family $\mathcal{D}^{(\epsilon)}$. As in Lemma 7.6, $\overline{\eta}(\mathcal{D}^{(\epsilon)})$ only depends on D_0. It follows that

$$\overline{\eta}(\mathcal{D}^{(\epsilon)}) = \overline{\eta}(D^{X_\epsilon,E}) + \overline{\eta}(D^{B,\Pi L}) = \overline{\eta}(D^{X_\epsilon,E}) - \overline{\eta}(D^{B,L}). \tag{7.33}$$

A generalization of [11, Theorem 4.35] says that

$$\lim_{\epsilon\to 0} \overline{\eta}(\mathcal{D}^{(\epsilon)}) = u^{(\dim(X)+1)/2} \int_B \mathrm{Todd}(\widehat{\nabla}^{TB}) \wedge \widetilde{\eta}(\widetilde{\mathcal{A}}) \pmod{\mathbb{Z}}. \tag{7.34}$$

We can now go through the proof of Theorem 6.2, using (7.26) and (7.33) and (7.34) in place of (3.11) and (6.15), respectively, to derive the following result.

7.35 Theorem *Suppose that $\pi\colon X \to B$ is a smooth fiber bundle with compact fibers of even dimension. Suppose that π is equipped with a Riemannian structure and a differential* spin^c*–structure. Assume that X is compact. Then for all $\mathcal{E} \in \check{K}^0(X)$, we have $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) = \mathrm{ind}^{\mathrm{top}}(\mathcal{E})$.*

7.36 Corollary (1) *The homomorphism $\mathrm{ind}^{\mathrm{top}}\colon \check{K}^0(X) \to \check{K}^{-n}(B)$ is independent of the choice of embedding ι.*

(2) *The assignment $\mathcal{E} \to \mathrm{ind}^{\mathrm{an}}(\mathcal{E})$ factors through a homomorphism $\mathrm{ind}^{\mathrm{an}}\colon \check{K}^0(X) \to \check{K}^{-n}(B)$.*

(3) *The map $\mathrm{ind}^{\mathrm{an}}\colon \check{K}^0(X) \to \check{K}^{-n}(B)$ is independent of the choice of finite-dimensional vector subbundle $L_{\pm}$.*

(4) *If D^V has vector bundle kernel then the analytic index defined in Definition 3.12 equals the analytic index defined in Definition 7.27.*

Aside from its intrinsic interest, the next proposition will be used in Section 8.

7.37 Proposition *The following diagrams commute:*

(7.38)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\Omega(X;\mathcal{R})^{-1}}{\Omega(X;\mathcal{R})^{-1}_K} & \xrightarrow{j} & \check{K}^0(X) & \xrightarrow{c} & K^0(X;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \mathrm{ind}^{\mathrm{an}}} & & \Big\downarrow{\scriptstyle \mathrm{ind}^{\mathrm{an}}} & & \\
0 & \longrightarrow & \dfrac{\Omega(B;\mathcal{R})^{-n-1}}{\Omega(B;\mathcal{R})^{-n-1}_K} & \xrightarrow{j} & \check{K}^{-n}(B) & \xrightarrow{c} & K^{-n}(B;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

(7.39)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^{-1}(X;\mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & \check{K}^0(X) & \xrightarrow{\omega} & \Omega(X;\mathcal{R})^0_K & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{ind}^{\mathrm{an}}} & & \Big\downarrow{\scriptstyle \mathrm{ind}^{\mathrm{an}}} & & \Big\downarrow{\scriptstyle \pi_*} & & \\
0 & \longrightarrow & K^{-n-1}(B;\mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & \check{K}^{-n}(B) & \xrightarrow{\omega} & \Omega(B;\mathcal{R})^{-n}_K & \longrightarrow & 0.
\end{array}$$

Proof The commuting of (7.38) follows immediately from the definition of $\mathrm{ind}^{\mathrm{an}}$. The left-hand square of (7.39) commutes from the definition of $\mathrm{ind}^{\mathrm{an}}\colon K^{-1}(X;\mathbb{R}/\mathbb{Z}) \to K^{-n-1}(B;\mathbb{R}/\mathbb{Z})$ in [33]. The right-hand square of (7.39) commutes from Lemma 7.29. □

7.40 Noncompact base

We use the limit theorem in the appendix to define the index maps for proper submersions and extend Theorem 7.35.

Suppose that $\pi\colon X \to B$ is a proper submersion of relative dimension n, with n even. Suppose that π is equipped with a Riemannian structure and a differential spin^c–structure. Let $B_1 \subset B_2 \subset \cdots$ be an exhaustion of B by compact codimension-zero submanifolds-with-boundary. From Theorem A.2, there is an isomorphism

$$\check{K}^{-n}(B) \cong \varprojlim_i \check{K}^{-n}(B_i). \tag{7.41}$$

Put $X_i = \pi^{-1}(B_i)$. Given $\mathcal{E} \in \check{K}^0(X)$, we can define $\mathrm{ind}^{\mathrm{top}}(\mathcal{E}|_{X_i}) \in \check{K}^{-n}(B_i)$ as in Section 5, after making a choice of embedding $\iota\colon X_i \to S^N \times B_i$. Clearly if $B_j \subset B_i$ then $\mathrm{ind}^{\mathrm{top}}(\mathcal{E}|_{X_j})$, as defined using the restriction of ι to X_j, is the restriction of $\mathrm{ind}^{\mathrm{top}}(\mathcal{E}|_{X_i})$ to B_j. Using the fact from Corollary 7.36 that $\mathrm{ind}^{\mathrm{top}}(\mathcal{E}|_{X_j})$ is independent of the choice of embedding, it follows that we have defined a topological index $\mathrm{ind}^{\mathrm{top}}(\mathcal{E})$ in $\check{K}^{-n}(B) \cong \varprojlim_i \check{K}^{-n}(B_i)$.

Similarly, we can define $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}|_{X_i}) \in \check{K}^{-n}(B_i)$ as in the earlier part of this section, after making a choice of the finite-dimensional vector subbundle $L_\pm$ over B_i. Clearly if $B_j \subset B_i$ then $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}|_{X_j})$, as defined using the restriction of $L_\pm$ to B_j, is the restriction of $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}|_{X_i})$ to B_j. Using the fact from Corollary 7.36 that $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}|_{X_j})$ is independent of the choice of vector subbundle, it follows that we have defined an analytic index $\mathrm{ind}^{\mathrm{an}}(\mathcal{E})$ in $\check{K}^{-n}(B) \cong \varprojlim_i \check{K}^{-n}(B_i)$.

7.42 Theorem *Suppose that $\pi\colon X \to B$ is a proper submersion with even relative dimension. Suppose that π is equipped with a Riemannian structure and a differential* spin^c*–structure. Then for all $\mathcal{E} \in \check{K}^0(X)$, we have $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) = \mathrm{ind}^{\mathrm{top}}(\mathcal{E})$.*

Proof By Theorem 7.35, we know that for each i, $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}|_{X_i}) = \mathrm{ind}^{\mathrm{top}}(\mathcal{E}|_{X_i})$ in $\check{K}^{-n}(B_i)$. Along with (7.41), the theorem follows. □

8 Relationships to earlier work

In this section we illustrate how our main index theorem relates to other work in the geometric index theory of Dirac operators. We first treat the determinant line bundle, using the holonomy theorem of [12] to show that the determinant of the analytic pushforward is the determinant bundle. (For a different approach to this question see Klonoff [32, Chapter 9]). As a consequence, our main theorem gives a "topological" construction of the determinant line bundle, equipped with its connection, up to isomorphism.

Next, in Section 8.9 we remark that when specialized to $\mathbb{R}/\mathbb{Z}$–valued K–theory, our theorem implies that the topological pushforward constructed in Section 4 coincides with the topological pushforward constructed from generalized cohomology theory.

The Chern character map, from topological K–theory to rational cohomology, has a differential refinement, going from differential K–theory to rational differential cohomology. In Section 8.13 we apply this refined Chern character map to our main theorem and recover the Riemann–Roch formula of [15].

Finally, under certain assumptions, there are geometric invariants of families of Dirac operators which live in higher-degree integral differential cohomology [35]. In Section 8.28 we point out that our index theorem computes them in terms of the topological pushforward.

8.1 Determinant line bundle

There is a map Det from $\check{K}^0(X)$ to isomorphism classes of line bundles on X, equipped with a Hermitian metric and a compatible connection. (The latter group may be identified with the integral differential cohomology group $\check{H}^2(X)$.) Given a generator $\mathcal{E} = (E, h^E, \nabla^E, \phi)$ for $\check{K}^0(X)$, its image $\mathrm{Det}(\mathcal{E})$ is represented by the line bundle $\Lambda^{\max}(E)$, equipped with the Hermitian metric $h^{\Lambda^{\max}(E)}$ and the connection

(8.2) $$\nabla^{\mathrm{Det}(\mathcal{E})} = \nabla^{\Lambda^{\max}(E)} - 2\pi i \phi_{(1)},$$

where $\phi_{(1)} \in \Omega^1(X)/\mathrm{Image}(d)$ is u times the component of $\phi \in \Omega(X;\mathcal{R})^{-1}/\mathrm{Image}(d)$ in $u^{-1}\Omega^1(X)/\mathrm{Image}(d)$. Note that changing a particular representative $\widehat{\phi} \in \Omega^1(X)$ for ϕ by an exact form df amounts to acting on the connection $\nabla^{\Lambda^{\max}(E)} - 2\pi i \widehat{\phi}$ by a gauge transformation $g = e^{2\pi i f}$.

Suppose that $\pi\colon X \to B$ is a compact fiber bundle with fibers of even dimension n, endowed with a Riemannian structure and a differential spin^c structure. If $\mathcal{E}$ is a generator for $\check{K}^0(X)$ of the form $(E, h^E, \nabla^E, 0)$ then there is a corresponding determinant line bundle $\mathrm{Det}_{\mathrm{an}}$ on B, which is equipped with a Hermitian metric h^{an} (due to Quillen [41]) and a compatible connection ∇^{an} (due to Bismut–Freed [12]); see Berline–Getzler–Vergne [9, Chapter 9.7]. The construction is analytic; for example, the construction of h^{an} uses ζ–functions built from the spectrum of the fiberwise Dirac-type operators D^V.

8.3 Proposition *After using periodicity to shift* $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) \in \check{K}^{-n}(B)$ *into* $\check{K}^0(B)$, *we have that* $\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))$ *is the inverse of* $\mathrm{Det}_{\mathrm{an}}$.

Proof Suppose first that $\mathrm{Ker}(D^V)$ is a $\mathbb{Z}/2\mathbb{Z}$–graded vector bundle on B. Using Definition 3.12 for $\mathrm{ind}^{\mathrm{an}}(\mathcal{E})$, it follows that

(8.4) $$\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E})) = \Lambda^{\max}(\mathrm{Ker}(D^V)_+) \otimes \left(\Lambda^{\max}(\mathrm{Ker}(D^V)_-)\right)^{-1}$$

is the inverse of the determinant line bundle. The connection $\nabla^{\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))}$ is

(8.5) $$\nabla^{\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))} = \nabla^{\Lambda^{\max}(\mathrm{Ker}(D^V)_+)\otimes\left(\Lambda^{\max}(\mathrm{Ker}(D^V))\right)^{-1}} - 2\pi i \widetilde{\eta}_{(1)}.$$

From (3.10),

(8.6) $$\widetilde{\eta}_{(1)} = -\frac{1}{2\pi i}\int_0^\infty \mathrm{STr}\left(D^V[\nabla^{\mathcal{H}}, D^V] e^{-s^2(D^V)^2}\right) s\, ds.$$

In this case of vector bundle kernel, h^{an} differs from the L^2–metric h^{L^2} by a factor involving the Ray-Singer analytic torsion. Let $T \in \mathrm{End}(\mathrm{Det}_{\mathrm{an}})$ be multiplication by $\sqrt{h^{L^2}/h^{\mathrm{an}}}$, so that T is an isometric isomorphism from $(\mathrm{Det}_{\mathrm{an}}, h^{L^2})$ to $(\mathrm{Det}_{\mathrm{an}}, h^{\mathrm{an}})$. Then

(8.7) $$T^*(\mathrm{Det}_{\mathrm{an}}, h^{\mathrm{an}}, \nabla^{\mathrm{an}}) = (\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))^{-1}, h^{\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))^{-1}}, \nabla^{\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))^{-1}});$$

see [9, Proof of Proposition 9.45].

If one does not assume vector bundle kernel then a direct proof of the proposition is trickier for the following reason. The usual construction of the determinant line bundle proceeds by making spectral cuts over suitable open subsets of B, constructing a line bundle with Hermitian metric and compatible connection over each open set, and then showing that these local constructions are compatible on overlaps. As $\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))$ is only defined up to isomorphism, it cannot be directly recovered from its restrictions to the elements of an open cover of B. For this reason, a direct comparison of $\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))$ and $\mathrm{Det}_{\mathrm{an}}$ is somewhat involved. Instead, we will just compare their holonomies.

Without loss of generality, we can assume that B is connected. Let $\star \in B$ be a basepoint. Let PB denote the smooth maps $c\colon [0,1] \to B$ with $c(0) = \star$. Let ΩB denote the elements of PB with $c(1) = \star$. A unitary connection on a line bundle over B gives rise to a homomorphism $H\colon \Omega B \to U(1)$, the holonomy map. Given H, we can construct a line bundle $\mathcal{L} \to B$ as $\mathcal{L} = (PB \times \mathbb{C})/\sim$, where $(c_1, z_1) \sim (c_2, z_2)$ if $c_1(1) = c_2(1)$ and $z_2 = H(c_2^{-1} \cdot c_1) z_1$. There is an evident notion of parallel transport on $\mathcal{L} \to B$ and a corresponding unitary connection. There is also a unique Hermitian inner product on $\mathcal{L} \to B$, up to overall scaling, with which the unitary connection is compatible.

Hence, given two Hermitian line bundles on B with compatible connections, if their holonomies are the same then they are isomorphic.

To compare the holonomies of the connections on $\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))$ and $\mathrm{Det}_{\mathrm{an}}$ around a closed curve, we pull the fiber bundle structure back to the curve and thereby reduce to the case when $B = S^1$. From Definition 7.27, the holonomy around S^1 of $\nabla^{\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))}$ is $e^{2\pi i \int_{S^1} \widetilde{\eta}_{(1)}}$ times the holonomy around S^1 of $\nabla^{\Lambda^{\max} L_+ \otimes (\Lambda^{\max} L_-)^{-1}}$. From (7.33) and (7.34),

$$e^{2\pi i \int_{S^1} \widetilde{\eta}_{(1)}} = e^{2\pi i (\lim_{\epsilon \to 0} \overline{\eta}(D^{X^\epsilon}, E) - \overline{\eta}(D^{S^1}, L))}. \tag{8.8}$$

By a standard computation, the holonomy around S^1 of $\nabla^{\Lambda^{\max} L_+ \otimes (\Lambda^{\max} L_-)^{-1}}$ equals $e^{2\pi i \overline{\eta}(D^{S^1}, L)}$. Thus the holonomy around S^1 of $\nabla^{\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E}))}$ is $e^{2\pi i \lim_{\epsilon \to 0} \overline{\eta}(D^{X^\epsilon}, E)}$. On the other hand, the holonomy around S^1 of $\nabla^{\mathrm{Det}_{\mathrm{an}}}$ is $e^{-2\pi i \lim_{\epsilon \to 0} \overline{\eta}(D^{X^\epsilon}, E)}$ [12, Theorem 3.16]. This proves the proposition. □

As a consequence of Theorem 7.42 and Proposition 8.3, the determinant line bundle with its Hermitian metric and compatible connection can be constructed up to isomorphism without using any spectral analysis. This was also derived in [32, Chapter 9], though with a different model of differential K–theory.

8.9 $\mathbb{R}/\mathbb{Z}$–index theory

Under the assumptions of Theorem 7.42, there is a topological index

$$\mathrm{ind}^{\mathrm{top}} \colon K^{-1}(X; \mathbb{R}/\mathbb{Z}) \to K^{-1}(B; \mathbb{R}/\mathbb{Z})$$

which can be constructed from a general procedure in generalized cohomology theory.

8.10 Proposition *The following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{-1}(X;\mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & \check{K}^0(X) & \xrightarrow{\omega} & \Omega(X;\mathcal{R})^0_K & \longrightarrow & 0 \\ & & \Big\downarrow \mathrm{ind}^{\mathrm{top}} & & \Big\downarrow \mathrm{ind}^{\mathrm{top}} & & \Big\downarrow \pi_* & & \\ 0 & \longrightarrow & K^{-n-1}(B;\mathbb{R}/\mathbb{Z}) & \xrightarrow{j} & \check{K}^{-n}(B) & \xrightarrow{\omega} & \Omega(B;\mathcal{R})^{-n}_K & \longrightarrow & 0. \end{array} \tag{8.11}$$

Proof The right-hand square commutes by Lemma 5.36. The left-hand square commutes from the fact that the diagram (7.39) commutes, along with the facts that the analytic and topological indices agree in differential K–theory (Theorem 7.42), and in $\mathbb{R}/\mathbb{Z}$–valued K–theory [33]. □

8.12 Remark In a similar vein, one might think that Theorem 7.42, along with the commuting of the right-hand squares in (7.38) and (8.11), gives a new and purely analytic proof of the Atiyah–Singer families index theorem [7]. However, such is not the case. The proof of Theorem 7.42 uses Proposition 2.35(2), whose proof uses [4, Theorem (5.3)], whose proof in turn uses the Atiyah–Singer families index theorem [4, Section 8].

8.13 Rational index theorem

For a subring $\Lambda \subset \mathbb{R}$ we define $\mathcal{R}_\Lambda = \Lambda[u,u^{-1}]$, analogous to (2.2). There is a notion of differential cohomology $\check{H}(X;\mathcal{R}_\Lambda)^\bullet$, the generalized differential cohomology theory attached to ordinary cohomology with coefficients in the graded ring $\mathcal{R}_\Lambda$. It fits into exact sequences

$$\textbf{(8.14)} \qquad 0 \longrightarrow H^{\bullet-1}(X;(\mathbb{R}/\Lambda)[u,u^{-1}]) \xrightarrow{\ i\ } \check{H}^\bullet(X;\mathcal{R}_\Lambda) \xrightarrow{\ \omega\ } \Omega(X;\mathcal{R})^\bullet_\Lambda \longrightarrow 0,$$

$$\textbf{(8.15)} \qquad 0 \longrightarrow \frac{\Omega(X;\mathcal{R})^{\bullet-1}}{\Omega(X;\mathcal{R})^{\bullet-1}_\Lambda} \xrightarrow{\ j\ } \check{H}(X;\mathcal{R}_\Lambda)^\bullet \xrightarrow{\ c\ } H^\bullet(X;\mathcal{R}_\Lambda) \longrightarrow 0,$$

where $\Omega(X;\mathcal{R})^\bullet_\Lambda$ denotes the closed $\mathcal{R}$–valued forms on X with periods in $\mathcal{R}_\Lambda = \Lambda[u,u^{-1}]$.

The differential cohomology theory $\check{H}(X;\mathcal{R}_\Lambda)^\bullet$ is essentially the same as the Cheeger–Simons theory of differential characters [16]. Namely, let $C_k(X)$ and $Z_k(X)$ denote the groups of smooth singular k–chains and k–cycles in X, respectively. Then an element of $\check{H}^k(X;\mathcal{R}_\Lambda)^l$ corresponds to a homomorphism $F\colon Z_{k-1}(X) \to u^{(l-k)/2}\cdot(\mathbb{R}/\Lambda)$ with the property that there is some $\alpha \in \Omega(X;\mathcal{R})^l$ so that for all $c \in C_k(X)$, we have $F(\partial c) = \int_c \alpha \bmod u^{(l-k)/2}\cdot\Lambda$.

Given a proper submersion $\pi\colon X \to B$ of relative dimension n which is oriented in ordinary cohomology, there is an "integration over the fiber" map $\int_{X/B}\colon \check{H}(X;\mathcal{R}_\Lambda)^\bullet \to \check{H}(B;\mathcal{R}_\Lambda)^{\bullet-n}$ [28, Section 3.4]. In short, if F is a differential character on X and $z \in Z_*(B)$ then the evaluation of $\int_{X/B}$ on z is $F(\pi^{-1}(z)) \in (\mathbb{R}/\Lambda)[u,u^{-1}]$; see also Bunke–Kreck–Schick [14].

There is a Chern character $\check{\mathrm{ch}}\colon \check{K}^\bullet(X) \to \check{H}(X;\mathcal{R}_\mathbb{Q})^\bullet$. When acting on generators of $\check{K}^0(X)$ of the form $\mathcal{E} = (E,h^E,\nabla^E,0)$, the Chern character $\check{\mathrm{ch}}(\mathcal{E})$ was defined in [16, Section 2] as a differential character. It then suffices to additionally define $\check{\mathrm{ch}}$ on Image(j), where j is the map in (2.21). For this, we may note that there is a natural map of the domain of j in (2.21) to the domain of j in (8.15), since $\Omega(X;\mathcal{R})^{\bullet-1}_K \subset \Omega(X;\mathcal{R})^{\bullet-1}_\mathbb{Q}$. Or, in the language of differential characters, we define the evaluation of $\check{\mathrm{ch}}(j(\phi))$ on a cycle $z \in Z_*(X)$ to be $\int_z \phi \mod \mathcal{R}_{\mathbb{Q}}$. The Chern character on differential K–theory was also considered in [15, Section 6].

In the proof of the next proposition we will make use of the Chern character

$$\mathrm{ch}_*\colon K_\bullet(X;\mathbb{Z}) \to H(X;\mathcal{R}_{\mathbb{Q}})_\bullet \tag{8.16}$$

on the K–homology of a space X. Recall from the proof of Theorem 6.2 that every K–homology class can be written as $u^k f_*[M]$ for some integer k and some continuous map $f\colon M \to X$ of a spin^c manifold M into X, where $[M] \in K_q(M;\mathbb{Z})$ is the fundamental class. Let $\mathrm{Todd}(M) \in H(M;\mathcal{R}_{\mathbb{Q}})^0$ be the Todd class and let $\mathrm{Todd}(M)^\vee \in H(M;\mathcal{R}_{\mathbb{Q}})_q$ be its Poincaré dual. Then

$$\mathrm{ch}_*\big(u^k f_*[M]\big) = u^k f_*\big(\mathrm{Todd}(M)^\vee\big) \in H(X;\mathcal{R}_{\mathbb{Q}})_{q+2k}. \tag{8.17}$$

The Chern character maps on cohomology and homology are compatible with the natural pairings, meaning that for $a \in K_\ell(X;\mathbb{Z})$ and $\alpha \in K^\ell(X;\mathbb{Z})$, we have

$$\langle a, \alpha\rangle = \langle \mathrm{ch}_*(a), \mathrm{ch}(\alpha)\rangle. \tag{8.18}$$

The Chern character on homology is an isomorphism after tensoring $K_\bullet(X;\mathbb{Z})$ with $\mathcal{R}_{\mathbb{Q}}$.

Recall from [16, Section 2] that there are characteristic classes in $\check{H}(X;\mathcal{R}_{\mathbb{Q}})^\bullet$. In particular, if W is a real vector bundle with a spin^c–structure and $\hat{\nabla}^W$ is a spin^c–connection on the associated spinor bundle then there is a refined Todd class $\widecheck{\mathrm{Todd}}(\hat{\nabla}^W) \in \check{H}(X;\mathcal{R}_{\mathbb{Q}})^0$.

8.19 Proposition *Let $\pi\colon X \to B$ be a proper submersion of relative dimension n, with n even. Suppose that π has a Riemannian structure and a differential* spin^c*–structure. Then for all $\mathcal{E} \in \check{K}^0(X)$,*

$$\check{\mathrm{ch}}(\mathrm{ind}^{\mathrm{top}}(\mathcal{E})) = \int_{X/B} \widecheck{\mathrm{Todd}}(\hat{\nabla}^{T^V X}) \cup \check{\mathrm{ch}}(\mathcal{E}) \in \check{H}(B;\mathcal{R}_{\mathbb{Q}})^{-n}. \tag{8.20}$$

Proof Put

$$\Delta = \check{\mathrm{ch}}(\mathrm{ind}^{\mathrm{top}}(\mathcal{E})) - \int_{X/B} \widecheck{\mathrm{Todd}}(\hat{\nabla}^{T^V X}) \cup \check{\mathrm{ch}}(\mathcal{E}). \tag{8.21}$$

From Lemma 5.36, we have $\omega(\Delta) = 0$. Then (8.14) implies that $\Delta = i(\mathcal{U})$ for some unique $\mathcal{U} \in H(B;(\mathbb{R}/\mathbb{Q})[u,u^{-1}])^{-n-1}$. Using the universal coefficient theorem and the fact that $\mathbb{R}/\mathbb{Q}$ is divisible, to show that $\mathcal{U}$ vanishes it suffices to prove the vanishing of its pairings with $H_*(B;\mathcal{R}_{\mathbb{Q}})$. We use the fact that the Chern character (8.16) is surjective after tensoring $K_*(X;\mathbb{Z})$ with $\mathcal{R}_{\mathbb{Q}}$ to argue, as in the proof of Theorem 6.2, that we can reduce to the case when B is a closed odd-dimensional spin^c–manifold and we are evaluating on the Chern character of its fundamental K–homology class.

We equip B with a Riemannian metric and a unitary connection ∇^{L^B} on the characteristic line bundle L^B. In the rest of this proof, we work modulo $\mathcal{R}_{\mathbb{Q}} = \mathbb{Q}[u, u^{-1}]$. From (8.18) and [16, Section 9],

$$\begin{aligned}(8.22)\quad & \langle \mathrm{ch}_*[B], \mathcal{U}\rangle \\ & \equiv \overline{\eta}(B; \mathrm{ind}^{\mathrm{top}}(\mathcal{E})) - \int_B \int_{X/B} \pi^* \widetilde{\mathrm{Todd}}(\hat{\nabla}^{TB}) \cup \widetilde{\mathrm{Todd}}(\hat{\nabla}^{T^V X}) \cup \check{\mathrm{ch}}(\mathcal{E}).\end{aligned}$$

From (6.5)–(6.8),

$$\begin{aligned}(8.23)\quad & \overline{\eta}(B, \mathrm{ind}^{\mathrm{top}}(\mathcal{E})) \\ & \equiv u^{-(\dim(X)+1)/2} \overline{\eta}(D^{X,E}) - \int_X \frac{\pi^* \mathrm{Todd}(\hat{\nabla}^{TB})}{\mathrm{Todd}(\hat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \omega(\nabla^E) \\ & \quad + \int_X \pi^* \mathrm{Todd}(\hat{\nabla}^{TB}) \wedge \mathrm{Todd}(\hat{\nabla}^{T^V X}) \wedge \phi \\ & \equiv \int_X \widetilde{\mathrm{Todd}}(\nabla^{TX}) \cup \check{\mathrm{ch}}(\nabla^E) - \int_X \frac{\pi^* \mathrm{Todd}(\hat{\nabla}^{TB})}{\mathrm{Todd}(\hat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \omega(\nabla^E) \\ & \quad + \int_X \pi^* \mathrm{Todd}(\hat{\nabla}^{TB}) \wedge \mathrm{Todd}(\hat{\nabla}^{T^V X}) \wedge \phi.\end{aligned}$$

Here X has the induced Riemannian metric from its embedding in $S^N \times B$. Thus

$$\begin{aligned}(8.24)\quad & \langle \mathrm{ch}_*[B], \mathcal{U}\rangle \\ & \equiv \int_X \widetilde{\mathrm{Todd}}(\nabla^{TX}) \cup \check{\mathrm{ch}}(\nabla^E) - \int_X \frac{\pi^* \mathrm{Todd}(\hat{\nabla}^{TB})}{\mathrm{Todd}(\hat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \omega(\nabla^E) \\ & \quad + \int_X \pi^* \mathrm{Todd}(\hat{\nabla}^{TB}) \wedge \mathrm{Todd}(\hat{\nabla}^{T^V X}) \wedge \phi \\ & \quad - \int_X \widetilde{\mathrm{Todd}}(\hat{\nabla}^{T^V X}) \cup \check{\mathrm{ch}}(\mathcal{E}) \\ & \equiv \int_X \widetilde{\mathrm{Todd}}(\nabla^{TX}) \cup \check{\mathrm{ch}}(\nabla^E) - \int_X \frac{\pi^* \mathrm{Todd}(\hat{\nabla}^{TB})}{\mathrm{Todd}(\hat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \omega(\nabla^E) \\ & \quad - \int_X \pi^* \widetilde{\mathrm{Todd}}(\hat{\nabla}^{TB}) \cup \widetilde{\mathrm{Todd}}(\hat{\nabla}^{T^V X}) \cup \check{\mathrm{ch}}(\nabla^E).\end{aligned}$$

As in the proof of Theorem 6.2, we can deform to the case $T^H X = (T^H(S^N \times B))|_X$ without changing the right-hand side of (8.24). In this case,

$$\text{(8.25)} \qquad \widetilde{\text{Todd}}(\hat{\nabla}^{TX}) = \pi^* \widetilde{\text{Todd}}(\hat{\nabla}^{TB}) \cup (\hat{\nabla}^{T^V X})$$

and Lemma 5.7 says that $\widetilde{C} = 0$. The proposition follows. □

8.26 Corollary *Let $\pi\colon X \to B$ be a proper submersion of relative dimension n, with n even. Suppose that π has a Riemannian structure and a differential* spin^c*–structure. Then for all $\mathcal{E} \in \check{K}^0(X)$,*

$$\text{(8.27)} \qquad \check{\text{ch}}(\text{ind}^{\text{an}}(\mathcal{E})) = \int_{X/B} \widetilde{\text{Todd}}(\hat{\nabla}^{T^V X}) \cup \check{\text{ch}}(\mathcal{E}) \in \check{H}(B; \mathcal{R}_{\mathbb{Q}})^{-n}.$$

Corollary 8.26 was proven by different means in [15, Section 6.4].

8.28 Index in Deligne cohomology

In general, the image of $\check{\text{ch}}$ lies in the rational differential cohomology group $\check{H}(X; \mathcal{R}_{\mathbb{Q}})^\bullet$ but not in the integral differential cohomology group $\check{H}(X; \mathbb{Z}[u, u^{-1}])^\bullet$. However, in some special cases one gets integral differential cohomology classes. Recall that there is a filtration of the usual K–theory $K^\bullet(X; \mathbb{Z}) = K^\bullet_{(0)}(X; \mathbb{Z}) \supset K^\bullet_{(1)}(X; \mathbb{Z}) \supset \cdots$, where $K^\bullet_{(i)}(X; \mathbb{Z})$ consists of the elements x of $K^\bullet(X; \mathbb{Z})$ with the property that for any finite simplicial complex Y of dimension less than i and any continuous map $f\colon Y \to X$, the pullback $f^* x$ vanishes in $K^\bullet(Y; \mathbb{Z})$ [2, Section 1].

Let $H^{(i)}(X; \Lambda[u, u^{-1}])^\bullet$ be the subgroup of $H(X; \Lambda[u, u^{-1}])^\bullet$ consisting of terms of X–degree equal to i, and similarly for $\check{H}^{(i)}(X; \Lambda[u, u^{-1}])^\bullet$. Given $[E] \in K^\bullet_{(i)}(X; \mathbb{Z})$, one can refine the component of $\text{ch}([E]) \in H(X; \mathbb{Q}[u, u^{-1}])^\bullet$ in $H^{(i)}(X; \mathbb{Q}[u, u^{-1}])^\bullet$ to an integer class $\text{ch}^{(i)}([E]) \in H^{(i)}(X; \mathbb{Z}[u, u^{-1}])^\bullet$. Similarly, if $[\mathcal{E}] \in \check{K}^\bullet(X)$ and $c([\mathcal{E}]) \in K^\bullet_{(i)}(X; \mathbb{Z})$ then one can refine the component of $\check{\text{ch}}([\mathcal{E}]) \in \check{H}(X; \mathbb{Q}[u, u^{-1}])^\bullet$ in $\check{H}^{(i)}(X; \mathcal{R}_{\mathbb{Q}})^\bullet$ to an element $\check{\text{ch}}^{(i)}([\mathcal{E}]) \in \check{H}^{(i)}(X; \mathbb{Z}[u, u^{-1}])^\bullet$. In general there is more than one such refinement, but if X is $(i-2)$–connected then there is a canonical choice.

Under the hypotheses of Theorem 7.42, suppose in addition that B is $(2k-2)$–connected and the Atiyah–Singer index $\text{ind}^{\text{top}}([E]) = \text{ind}^{\text{an}}([E]) \in K^{-n}(B; \mathbb{Z})$ lies in the subset $K^{-n}_{(2k)}(B; \mathbb{Z}) \subset K^{-n}(B; \mathbb{Z})$. Then an explicit cocycle in a certain integral Deligne cohomology group is constructed in [35, Section 4]. More precisely, the cocycle is a $2k$–cocycle for the Čech cohomology of the complex of sheaves

$$\text{(8.29)} \qquad \mathbb{Z} \longrightarrow \Omega^0 \longrightarrow \cdots \longrightarrow \Omega^{2k-1}$$

on B. From the viewpoint of the present paper, the cocycle constructed in [35, Section 4] represents $\check{\mathrm{ch}}^{(2k)}(\mathrm{ind}^{\mathrm{an}}(\mathcal{E})) \in \check{H}^{(2k)}(B; \mathbb{Z}[u, u^{-1}])^{-n}$. Then Theorem 7.42 implies that the same integral differential cohomology class can be computed as $\check{\mathrm{ch}}^{(2k)}(\mathrm{ind}^{\mathrm{top}}(\mathcal{E}))$.

9 Odd index theorem

In this section we extend the index-theoretic results of the previous sections to the case of odd differential K–theory classes. Because some of the arguments in the section are similar to what was already done in the even case, we state some results without proof.

In Section 9.1 we give a model for odd differential K–theory whose generators consist of a Hermitian vector bundle, a compatible connection, a unitary automorphism and an even differential form. We then construct suspension and desuspension maps between the even and odd differential K–theory classes. These are used to prove that the odd groups defined here are isomorphic to those defined from the general theory in [28].

In Section 9.25 we define the analytic and topological indices in the case of an odd differential K–theory class on the total space of a fiber bundle with even-dimensional fibers. The definition uses suspension and desuspension to reduce to the even case. It is likely that the topological index map agrees with that constructed using a more topological model [32]. We do not attempt to relate the odd analytic pushforward with the odd Bismut superconnection.

In Section 9.28 we define the analytic and topological indices in the case of an even differential K–theory class on the total space of a fiber bundle with odd-dimensional fibers. The definition uses the trick, taken from [12, Proof of Theorem 2.10], of multiplying both the base and fiber by a circle and then tensoring with the Poincaré line bundle on the ensuing torus.

In Section 9.34 we look at the result of applying a determinant map to the analytic and topological index, to obtain a map from B to S^1. We show that this map is given by the reduced eta-invariants of the fibers X_b. In particular if Z is a closed Riemannian spin^c–manifold of odd dimension n, and $\pi\colon Z \to \mathrm{pt}$ is the map to a point then for any $\mathcal{E} \in \check{K}^0(Z)$, the topological and analytic indices $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}}(\mathcal{E}), \mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}(\mathcal{E}) \in \check{K}^{-n}(\mathrm{pt})$ equal the reduced eta-invariant $\overline{\eta}(Z, \mathcal{E})$. This is a version of the main theorem in [32].

In Section 9.41 we indicate the relationship between the odd differential K–theory index and the index gerbe of [35].

9.1 Odd differential K–theory

Let X be a smooth manifold. We can describe $K^{-1}(X; \mathbb{Z})$ as an abelian group in terms of generators and relations. The generators are complex vector bundles G over X

equipped with Hermitian metrics h^G and unitary automorphisms U^G. The relations are that:

(1) $(G_2, h^{G_2}, U^{G_2}) = (G_1, h^{G_1}, U^{G_1}) + (G_3, h^{G_3}, U^{G_3})$ whenever there is a short exact sequence of Hermitian vector bundles

(9.2) $$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

so that the diagram

(9.3) $$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle U^{G_1}} & & \downarrow{\scriptstyle U^{G_2}} & & \downarrow{\scriptstyle U^{G_3}} & & \\ 0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 0 \end{array}$$

commutes.

(2) $(G, h^G, U_1^G \circ U_2^G) = (G, h^G, U_1^G) + (G, h^G, U_2^G)$.

Given a generator (G, h^G, U^G) for $K^{-1}(X;\mathbb{Z})$, let ∇^G be a unitary connection on G. Put

(9.4) $$A(t) = (1-t)\nabla^G + tU^G \cdot \nabla^G \cdot (U^G)^{-1},$$

(9.5) $$\omega(\nabla^G, U^G) = \int_0^1 R_u \operatorname{tr}(e^{-u^{-1}(dt\,\partial_t + A(t))^2}) \in \Omega(X;\mathcal{R})^{-1}.$$

Then $\omega(\nabla^G, U^G)$ is a closed form whose de Rham cohomology class $\mathrm{ch}(G, U^G) \in H(X;\mathcal{R})^{-1}$ is independent of ∇^G. The assignment $(G, U^G) \to \mathrm{ch}(G, U^G)$ factors through a map $\mathrm{ch}\colon K^{-1}(X;\mathbb{Z}) \longrightarrow H(X;\mathcal{R})^{-1}$ which becomes an isomorphism after tensoring the left-hand side with $\mathbb{R}$.

We can represent $K^{-1}(X;\mathbb{Z})$ using $\mathbb{Z}/2\mathbb{Z}$–graded vector bundles. A generator of $K^{-1}(X;\mathbb{Z})$ is then a $\mathbb{Z}/2\mathbb{Z}$–graded complex vector bundle $G = G_+ \oplus G_-$ on X, equipped with a Hermitian metric $h^G = h^{G_+} \oplus h^{G_-}$ and a unitary automorphism $U_\pm^G \in \mathrm{Aut}(G_\pm)$. Choosing compatible connections $\nabla^{G_\pm}$, and putting

(9.6) $$A_\pm(t) = (1-t)\nabla_\pm^G + tU_\pm^G \cdot \nabla_\pm^G \cdot (U^G)_\pm^{-1},$$

we put

(9.7) $$\omega(\nabla^G, U^G) = \int_0^1 R_u \operatorname{str}(e^{-u^{-1}(dt\,\partial_t + tA(t))^2}) \in \Omega(X;\mathcal{R})^{-1}.$$

If ∇_1^G and ∇_2^G are two metric-compatible connections on a Hermitian vector bundle G with a unitary automorphism U^G then there is an explicit form

$$\mathrm{CS}(\nabla_1^G, \nabla_2^G, U^G) \in \Omega(X; \mathcal{R})^{-2} / \operatorname{Image}(d)$$

so that

$$d\,\mathrm{CS}(\nabla_1^G, \nabla_2^G, U^G) = \omega(\nabla_1^G, U^G) - \omega(\nabla_2^G, U^G). \tag{9.8}$$

More generally, if we have a short exact sequence (9.2) of Hermitian vector bundles, unitary automorphisms $U^{G_i} \in \operatorname{Aut}(G_i)$, a commutative diagram (9.3) and metric-compatible connections $\{\nabla^{G_i}\}_{i=1}^3$ then there is an explicit form

$$\mathrm{CS}(\nabla^{G_1}, \nabla^{G_2}, \nabla^{G_3}, U^{G_1}, U^{G_2}, U^{G_3}) \in \Omega(X; \mathcal{R})^{-2} / \operatorname{Image}(d)$$

so that

$$\begin{aligned} d\,\mathrm{CS}(\nabla^{G_1}, \nabla^{G_2}, \nabla^{G_3}, U^{G_1}, U^{G_2}, U^{G_3}) \\ = \omega(\nabla^{G_2}, U^{G_2}) - \omega(\nabla^{G_1}, U^{G_1}) - \omega(\nabla^{G_3}, U^{G_3}). \end{aligned} \tag{9.9}$$

To construct $\mathrm{CS}(\nabla^{G_1}, \nabla^{G_2}, \nabla^{G_3}, U^{G_1}, U^{G_2}, U^{G_3})$, put $W = [0,1] \times X$ and let $p \colon W \to X$ be the projection map. Put $F = p^* G_2$, $h^F = p^* h^{G_2}$ and $U^F = p^* U^{G_2}$. Let ∇^F be a unitary connection on F which equals $p^* \nabla^{G_2}$ near $\{1\} \times X$ and which equals $p^*(\nabla^{G_1} \oplus \nabla^{G_3})$ near $\{0\} \times X$. Then

$$\mathrm{CS}(\nabla^{G_1}, \nabla^{G_2}, \nabla^{G_3}) = \int_0^1 \omega(\nabla^F, U^F) \in \Omega(X; \mathcal{R})^{-2} / \operatorname{Image}(d). \tag{9.10}$$

Also, if G is a Hermitian vector bundle with a unitary connection ∇^G and two unitary automorphisms U_1^G, U_2^G then there is an explicit form $\mathrm{CS}(\nabla^G, U_1^G, U_2^G) \in \Omega(X; \mathcal{R})^{-2} / \operatorname{Image}(d)$ so that

$$d\,\mathrm{CS}(\nabla^G, U_1^G, U_2^G) = \omega(\nabla^G, U_1^G \circ U_2^G) - \omega(\nabla^G, U_1^G) - \omega(\nabla^G, U_2^G). \tag{9.11}$$

To construct $\mathrm{CS}(\nabla^G, U_1^G, U_2^G)$, let $\Delta \subset \mathbb{A}^2$ be the simplex

$$\Delta = \{(t_1, t_2) \in \mathbb{A}^2 \colon t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1\}, \tag{9.12}$$

with the orientation induced from the canonical orientation of the affine plane $\mathbb{A}^2$. Put

$$A(t_1, t_2) = \nabla^G + t_1 U_1^G \cdot \nabla^G \cdot (U_1^G)^{-1} + t_2 (U_1^G \circ U_2^G) \cdot \nabla^G \cdot (U_1^G \circ U_2^G)^{-1}. \tag{9.13}$$

Then

$$\mathrm{CS}(\nabla^G, U_1^G, U_2^G) = \int_\Delta R_u \operatorname{tr}\left(e^{-u^{-1}(dt_1\, \partial_{t_1} + dt_2\, \partial_{t_2} + A(t_1, t_2))^2}\right). \tag{9.14}$$

9.15 Definition The differential K–theory group $\check{K}^{-1}(X)$ is the abelian group defined by the following generators and relations. The generators are quintuples $\mathcal{G} = (G, h^G, \nabla^G, U^G, \phi)$ where:

- G is a complex vector bundle on X.
- h^G is a Hermitian metric on G.
- ∇^G is an h^G–compatible connection on G.
- U^G is a unitary automorphism of G.
- $\phi \in \Omega(X;\mathcal{R})^{-2}/\operatorname{Image}(d)$.

The relations are: $\mathcal{G}_2 = \mathcal{G}_1 + \mathcal{G}_3$ whenever there is a short exact sequence (9.2) of Hermitian vector bundles, along with a commuting diagram (9.3) and $\phi_2 = \phi_1 + \phi_3 - \mathrm{CS}(\nabla^{G_1}, \nabla^{G_2}, \nabla^{G_3}, U^{G_1}, U^{G_2}, U^{G_3})$, and

$$(G, h^G, \nabla^G, U_1^G \circ U_2^G, -\mathrm{CS}(\nabla^G, U_1^G, U_2^G)) \\ = (G, h^G, \nabla^G, U_1^G, 0) + (G, h^G, \nabla^G, U_2^G, 0).$$

By a generator of $\check{K}^{-1}(X)$ we mean a quadruple $\mathcal{G} = (G, h^G, \nabla^G, U^G, \phi)$ as above. There is a homomorphism $\omega\colon \check{K}^{-1}(X) \to \Omega(X;\mathcal{R})^{-1}$ given on generators by $\omega(\mathcal{G}) = \omega(\nabla^G, U^G) + d\phi$.

There is a similar model of $\check{K}^r(X)$ for any odd r. A generator is a quintuple $\mathcal{G} = (G, h^G, \nabla^G, U^G, \phi)$ as above with only a change in degree: $\phi \in \Omega(X;\mathcal{R})^{r-1}/\operatorname{Image}(d)$. Then $\omega(\mathcal{G}) = u^{(r+1)/2}\omega(\nabla^G, U^G) + d\phi \in \Omega(X;\mathcal{R})^r$. Also, the exact sequences (2.20) and (2.21) hold in odd degrees.

There is a suspension map $S\colon \check{K}^{-1}(X) \to \check{K}^0(S^1 \times X)$ given on generators as follows. Let $\mathcal{G} = (G, h^G, \nabla^G, U^G, \phi)$ be a generator for $\check{K}^{-1}(X)$. Let $p\colon [0,1] \times X \to X$ be the projection map. Put $F = p^*G$, $h^F = p^*h^G$ and

$$\nabla^F = dt\,\partial_t + (1-t)\nabla^G + tU^G \cdot \nabla^G \cdot (U^G)^{-1}. \tag{9.16}$$

Let (E, h^E, ∇^E) be the Hermitian vector bundle with connection on $S^1 \times X$ obtained by gluing $F|_{\{0\}\times X}$ with $F|_{\{1\}\times X}$ using the automorphism U^G. Put

$$\Phi = dt \wedge \phi \in \Omega(S^1 \times X;\mathcal{R})^{-1}/\operatorname{Image}(d). \tag{9.17}$$

Then $S(\mathcal{G}) = (E, h^E, \nabla^E, \Phi) \in \check{K}^0(S^1 \times X)$. Equivalently, S is multiplication by a certain element of $\check{K}^1(S^1)$.

There is also a suspension map $S\colon \check{K}^0(X) \to \check{K}^1(S^1 \times X)$ given on generators as follows. Let $\mathcal{E} = (E, h^E, \nabla^E, \Phi)$ be a generator for $\check{K}^0(X)$. Let $p_1\colon S^1 \times X \to S^1$

and $p_2\colon S^1\times X\to X$ be the projection maps. Let $\mathcal{L}$ be the trivial complex line bundle over S^1, equipped with the product Hermitian metric $h^{\mathcal{L}}$, the product connection $\nabla^{\mathcal{L}}$ and the automorphism $U^{\mathcal{L}}$ which multiplies the fiber $\mathcal{L}_{e^{2\pi it}}$ over $e^{2\pi it}\in S^1$ by $e^{2\pi it}$. Put $G=p_1^*\mathcal{L}\otimes p_2^*E$, $h^G=p_1^*h^{\mathcal{L}}\otimes p_2^*h^E$, $\nabla^G=p_1^*\nabla^{\mathcal{L}}\otimes I+I\otimes p_2^*\nabla^E$, $U^G=p_1^*U^{\mathcal{L}}$ and

(9.18) $$\phi=p_1^*dt\wedge p_2^*\Phi\in\Omega(S^1\times X;\mathcal{R})^0/\operatorname{Image}(d).$$

Then $S(\mathcal{E})=(G,h^G,\nabla^G,U^G,\phi)\in\check{K}^0(S^1\times X)$. Again, S is multiplication by an element of $\check{K}^1(S^1)$.

The double suspension $S^2\colon\check{K}^0(X)\to\check{K}^2(T^2\times X)$ can be described explicitly as follows. Let $p_1\colon T^2\times X\to T^2$ and $p_2\colon T^2\times X\to X$ be the projection maps. Consider the trivial complex line bundle $\mathcal{M}$ on $[0,1]\times[0,1]$ with product Hermitian metric and connection $dt_1\,\partial_{t_1}+dt_2\,\partial_{t_2}-2\pi i\,t_1dt_2$. Define the hermitian line bundle $P\to T^2$ by making identifications of $\mathcal{M}$ along the boundary of $[0,1]\times[0,1]$; it is the "Poincaré" line bundle on the torus. Let h^P and ∇^P be the ensuing Hermitian metric and compatible connection on P. Given a generator $\mathcal{E}=(E,h^E,\nabla^E,\phi)$ for $\check{K}^0(X)$, its double suspension is

(9.19) $$S^2(\mathcal{E})=\left(p_1^*P\otimes p_2^*E,\,p_1^*h^P\otimes p_2^*h^E,\,p_1^*\nabla^P\otimes I+I\otimes p_2^*\nabla^E,\,dt_1\wedge dt_2\wedge\phi\right)\in\check{K}^2(T^2\times X).$$

Finally, there is a desuspension map $D\colon\check{K}^0(S^1\times X)\to\check{K}^{-1}(X)$ given on generators as follows. Let $\mathcal{E}=(E,h^E,\nabla^E,\Phi)$ be a generator for $\check{K}^0(S^1\times X)$. Picking a basepoint $\star\in S^1$, define $A\colon X\to S^1\times X$ by $A(x)=(\star,x)$. Put $G=A^*E$, $h^G=A^*h^E$ and $\nabla^G=A^*\nabla^E$. Let $U^G\in\operatorname{Aut}(G)$ be the map given by parallel transport around the circle fibers on $S^1\times X$, starting from $\{\star\}\times X$. Put

(9.20) $$\phi=\int_{S^1}\Phi\in\Omega(X;\mathcal{R})^{-2}/\operatorname{Image}(d).$$

Then

(9.21) $$D(\mathcal{E})=(G,h^G,\nabla^G,U^G,\phi)\in\check{K}^{-1}(X).$$

It is independent of the choice of basepoint $\star\in S^1$. Also, $D\circ S$ is the identity on $\check{K}^{-1}(X)$.

There are analogous suspension and desuspension maps in other degrees.

We now show how to use the suspension and desuspension maps to relate Definition 9.15 to the notion of odd differential K–theory groups from [28].

9.22 Proposition *The odd differential K–groups defined here are isomorphic to those defined by the theory of generalized differential cohomology.*

Proof Temporarily denote the geometrically defined groups in Definition 2.16 and Definition 9.15 as $\check{L}^\bullet(X)$. Then, looking at degree -1 for definiteness, the composition

$$\check{L}^{-1}(X) \underset{D}{\overset{S}{\rightleftarrows}} \check{L}^0(S^1 \times X) \cong \check{K}^0(S^1 \times X) \xrightarrow{\int_{S^1}} \check{K}^{-1}(X) \tag{9.23}$$

defines a homomorphism we claim is an isomorphism. (See Remark 2.17 for the isomorphism in the middle of (9.23).) This follows from the 5–lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{\Omega(X;\mathcal{R})^{-2}}{\Omega(X;\mathcal{R})^{-2}_K} & \xrightarrow{j} & \check{L}^{-1}(X) & \xrightarrow{c} & K^{-1}(X;\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \dfrac{\Omega(X;\mathcal{R})^{-2}}{\Omega(X;\mathcal{R})^{-2}_K} & \xrightarrow{j} & \check{K}^{-1}(X) & \xrightarrow{c} & K^{-1}(X;\mathbb{Z}) & \longrightarrow & 0 \end{array} \tag{9.24}$$

in which the rows are exact and the outer vertical arrows are the identity. □

Note that under the isomorphism $\check{L}^0(S^1 \times X) \cong \check{K}^0(S^1 \times X)$, the desuspension map corresponds to integration over the circle.

9.25 Index theorem: even-dimensional fibers, odd classes

Suppose that $\pi\colon X \to B$ is a proper submersion of relative dimension n, with n even. Suppose that π is equipped with a Riemannian structure and a differential spin^c–structure. The product submersion $\pi'\colon S^1 \times X \to S^1 \times B$ inherits a product Riemannian structure and differential spin^c–structure, so we have the analytic and topological indices $\mathrm{ind}^{\mathrm{an}}, \mathrm{ind}^{\mathrm{top}}\colon \check{K}^0(S^1 \times X) \to \check{K}^{-n}(S^1 \times B)$.

Define the analytic index

$$\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}\colon \check{K}^{-1}(X) \to \check{K}^{-n-1}(B) \tag{9.26}$$

by $\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}} = D \circ \mathrm{ind}^{\mathrm{an}} \circ S$. Define the topological index $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}}\colon \check{K}^{-1}(X) \to \check{K}^{-n-1}(B)$ by

$$\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}} = D \circ \mathrm{ind}^{\mathrm{top}} \circ S. \tag{9.27}$$

As an immediate consequence of Theorem 7.42, $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}} = \mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}$.

9.28 Index theorem: odd-dimensional fibers

Suppose that $\pi\colon X \to B$ is a proper submersion of relative dimension n, with n odd. Suppose that π is equipped with a Riemannian structure and a differential spin^c–structure.

Following [12, Proof of Theorem 2.10], we construct a new submersion $\pi'\colon T^2 \times X \to S^1 \times B$ by multiplying the base and fibers by S^1. Given $a > 0$, we endow π' with a product Riemannian structure and differential spin^c–structure, so that the circle fibers have length a. Then we have analytic and topological indices $\mathrm{ind}^{\mathrm{an}}, \mathrm{ind}^{\mathrm{top}}\colon \check{K}^2(T^2 \times X) \to \check{K}^{-n+1}(S^1 \times B)$.

Define the analytic index $\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}\colon \check{K}^0(X) \to \check{K}^{-n}(B)$ by

$$\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}} = D \circ \mathrm{ind}^{\mathrm{an}} \circ S^2. \tag{9.29}$$

Define the topological index $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}}\colon \check{K}^0(X) \to \check{K}^{-n}(B)$ by

$$\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}} = D \circ \mathrm{ind}^{\mathrm{top}} \circ S^2. \tag{9.30}$$

As an immediate consequence of Theorem 7.42, $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}} = \mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}$.

9.31 Lemma *The indices $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}}$ and $\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}$ are independent of the choice of fiber circle length a.*

Proof Given $a_0, a_1 > 0$ and $\mathcal{E} \in \check{K}^0(X)$, let $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd},0}(\mathcal{E})$ and $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd},1}(\mathcal{E})$ denote the indices in $\check{K}^{-n}(B)$ as computed using circle fibers of length a_0 and a_1, respectively. Consider the fiber bundle $\pi''\colon [0,1] \times X \to [0,1] \times B$, equipped with the product Riemannian structure and differential spin^c–structure. Let $a\colon [0,1] \to \mathbb{R}^+$ be a smooth function so that $a(t)$ is a_0 near $t = 0$, and $a(t)$ is a_1 near $t = 1$. Given $t \in [0,1]$, let the circle fiber length over $\{t\} \times B$ in the fiber bundle $[0,1] \times T^2 \times X \to [0,1] \times B$ be $a(t)$. Let $\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd},[0,1]\times B}(\mathcal{E}) \in \check{K}^{-n}([0,1] \times B)$ be the topological index of $\mathcal{E}$ as computed using the fiber bundle π''. Using Lemma 2.24 and Lemma 5.36,

$$\begin{aligned} \mathrm{ind}^{\mathrm{top}}_{\mathrm{odd},1} - \mathrm{ind}^{\mathrm{top}}_{\mathrm{odd},0} &= j\left(\int_{[0,1]} \omega(\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd},[0,1]\times B})\right) \\ &= j\left(\int_{[0,1]} \mathrm{Todd}(\hat{\nabla}^{T^V([0,1]\times T^2 \times X)}) \wedge \omega(\mathcal{E})\right). \end{aligned} \tag{9.32}$$

One can check that $\hat{\nabla}^{T^V([0,1]\times T^2\times X)}$ pulls back from a connection $\hat{\nabla}^{T^V(T^2\times X)}$ on the vertical tangent bundle of $\pi'\colon T^2 \times X \to S^1 \times B$. Thus

$$\int_{[0,1]} \mathrm{Todd}(\hat{\nabla}^{T^V([0,1]\times T^2\times X)}) \wedge \omega(\mathcal{E}) = 0. \tag{9.33}$$

This proves the lemma. □

9.34 Degree-one component of the index theorem

There is a homomorphism Det from $\check{K}^{-1}(X)$ to the space of smooth maps $[X, S^1]$. Namely, given a generator $\mathcal{E} = (E, h^E, \nabla^E, U^E, \phi)$ of $\check{K}^{-1}(X)$, its image $\mathrm{Det}(\mathcal{E}) \in [X, S^1]$ sends $x \in X$ to $e^{2\pi i \phi_{(0)}(x)} \det(U^E(x))$, where $\phi_{(0)} \in \Omega^0(X)$ denotes u times the component of $\phi \in \Omega(X; \mathcal{R})^{-2} / \mathrm{Image}(d)$ in $u^{-1}\Omega^0(X)/\mathrm{Image}(d) = u^{-1}\Omega^0(X)$.

9.35 Proposition *Let $\pi\colon X \to B$ be a proper submersion of relative dimension n, with n odd. Suppose that π is equipped with a Riemannian structure and a differential* spin^c*–structure. Given $\mathcal{E} \in \check{K}^0(X)$, after using periodicity to shift* $\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) \in \check{K}^{-n}(B)$ *into $\check{K}^{-1}(B)$, the map* $\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}(\mathcal{E}))$ *sends $b \in B$ to $u^{(n+1)/2}\overline{\eta}(X_b, \mathcal{E}|_{X_b}) \in \mathbb{R}/\mathbb{Z}$, where $X_b = \pi^{-1}(b)$.*

Proof From (9.29), we want to apply $\mathrm{Det} \circ D$ to $(\mathrm{ind}^{\mathrm{an}} \circ S^2)(\mathcal{E})$ and evaluate the result at b. Using (9.19), $S^2(\mathcal{E})$ is a certain element of $\check{K}^2(T^2 \times X)$ and then $(\mathrm{ind}^{\mathrm{an}} \circ S^2)(\mathcal{E})$ is its analytic index in $\check{K}^{-n+1}(S^1 \times B)$. In order to apply $\mathrm{Det} \circ D$ to this, and then compute the result at b, it suffices to just use the restriction of $(\mathrm{ind}^{\mathrm{an}} \circ S^2)(\mathcal{E})$ to $S^1 \times \{b\}$. After doing so, the proof of Proposition 8.3 implies the result of applying $\mathrm{Det} \circ D$ is

$$\lim_{\epsilon \to 0} \overline{\eta}\big(D^{(T^2 \times X_b)^\epsilon, p_1^* P \otimes p_2^* E|_{X_b}}\big) + u^{(n+1)/2} \int_{S^1} \int_{S^1 \times X_b} \mathrm{Todd}(\widehat{\nabla}^{T^V(T^2 \times X_b)}) \wedge p_2^* \phi. \tag{9.36}$$

For any $\epsilon > 0$, separation of variables gives

$$\begin{aligned} \overline{\eta}\big(D^{(T^2 \times X_b)^\epsilon, p_1^* P \otimes p_2^* E|_{X_b}}\big) &= \mathrm{Index}(D^{T^2, P}) \cdot \overline{\eta}(D^{X_b, E|_{X_b}}) \\ &= \overline{\eta}(D^{X_b, E|_{X_b}}). \end{aligned} \tag{9.37}$$

Then the evaluation of $\mathrm{Det}(\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}(\mathcal{E}))$ at b is

$$\overline{\eta}(D^{X_b, E|_{X_b}}) + u^{(n+1)/2} \int_{X_b} \mathrm{Todd}(\widehat{\nabla}^{TX_b}) \wedge \phi = u^{\frac{n+1}{2}} \overline{\eta}(X_b, \mathcal{E}|_{X_b}). \tag{9.38}$$

This proves the proposition. □

9.39 Corollary [32] *Suppose that Z is a closed Riemannian* spin^c*–manifold of odd dimension n with a* spin^c*–connection $\widehat{\nabla}^{TZ}$. Let $\pi\colon Z \to \mathrm{pt}$ be the mapping to a point. Given $\mathcal{E} = (E, h^E, \nabla^E, \phi) \in \check{K}^0(Z)$, we have*

$$\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}(\mathcal{E}) = \mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}}(\mathcal{E}) = \overline{\eta}(Z, \mathcal{E}) \tag{9.40}$$

in $\check{K}^{-n}(\mathrm{pt}) = u^{-(n+1)/2} \cdot (\mathbb{R}/\mathbb{Z})$.

9.41 Index gerbe

Let $\pi\colon X \to B$ be a proper submersion of relative dimension n, with n odd. Suppose that π is equipped with a Riemannian structure and a differential spin^c–structure. If E is a Hermitian vector bundle on X with compatible connection ∇^E then one can construct an index gerbe [35], which is an abelian gerbe-with-connection. Isomorphism classes of such gerbes-with-connection are in bijection with the differential cohomology group $\check{H}^3(X;\mathbb{Z})$. On the other hand, taking $\mathcal{E} = (E, h^E, \nabla^E, 0) \in \check{K}^0(X)$, the component of $\check{\mathrm{ch}}(\mathrm{ind}^{\mathrm{an}}_{\mathrm{odd}}(\mathcal{E})) = \check{\mathrm{ch}}(\mathrm{ind}^{\mathrm{top}}_{\mathrm{odd}}(\mathcal{E})) \in \check{H}(B;\mathbb{Q}[u,u^{-1}])^\bullet$ in $\check{H}^3(B;\mathbb{Q})$ comes from an element of the group $\check{H}^3(B;\mathbb{Z})$. Presumably this is the class of the index gerbe. It should be possible to prove this by comparing the holonomies around surfaces in B, along the lines of the proof of Proposition 8.3.

Appendix: Limits in differential K–theory

Let X be a topological space and let $X_1 \subset X_2 \subset \cdots$ be an increasing sequence of compact subspaces whose union is X. Milnor [38] proved that for every cohomology theory h and every integer q, there is an exact sequence

(A.1) $$0 \longrightarrow \varprojlim_i{}^1\, h^q(X_i) \longrightarrow h^q(X) \longrightarrow \varprojlim_i h^q(X_i) \longrightarrow 0,$$

in which the quotient is the (inverse) limit and the kernel is its first derived functor; see Hatcher [25, Section 3.F], May [36, Section 19.4] for modern expositions and Atiyah–Segal [5, Section 4] for the specific case of K–theory. In this appendix we prove that the *differential* cohomology of the union of compact manifolds is isomorphic to the inverse limit of the differential cohomologies: there is no $\varprojlim^1$ term. So as to not introduce new notation, we present the argument for differential K–theory, which is the case of interest for this paper. We remark that for integral differential cohomology the theorem is immediate if we use the isomorphism with Cheeger–Simons differential characters, as a differential character is determined by its restriction to compact submanifolds. The universal coefficient theorem for K–theory [45] plays an analogous role in the following proof.

A.2 Theorem *Let X be a smooth manifold and $X_1 \subset X_2 \subset \cdots$ an increasing sequence of compact codimension-zero submanifolds-with-boundary whose union is X. Then for any integer q, the restriction maps induce an isomorphism*

(A.3) $$\check{K}^q(X) \longrightarrow \varprojlim_i \check{K}^q(X_i).$$

The proof is based on the exact sequence (2.20) and the following lemmas.

A.4 Lemma *The restriction maps induce an isomorphism*

$$\Omega(X;\mathcal{R})^q_K \longrightarrow \varprojlim_i \Omega(X_i;\mathcal{R})^q_K. \tag{A.5}$$

Proof We first note that the universal coefficient theorem for K–theory implies that $K^q(X)/\text{torsion} \cong \operatorname{Hom}\big(K_q(X),\mathbb{Z}\big)$. Representing K–homology classes by spin^c–manifolds, as in the proof of Theorem 6.2, we see that the cohomology class of a closed differential form $\omega \in \Omega(X;\mathcal{R})^q$ is in the image of the Chern character if and only if for every closed Riemannian spin^c–manifold W and every smooth map $f\colon W \to X$, we have

$$\int_W \operatorname{Todd}(W) \wedge f^*\omega \in u^{(q-\dim(W))/2}\cdot\mathbb{Z}. \tag{A.6}$$

For each such choice of W and f, we know that $f(W) \subset X_i$ for some i.

Returning to the map (A.5), a differential form is determined pointwise, so the map is injective. For the same reason, an element $\{\omega_i\}$ in the inverse limit on the right-hand side glues to a global differential form ω, and $d\omega = 0$ since d is local. Since $\omega|_{X_i} \in \Omega(X_i;\mathcal{R})^q_K$ for each i, the previous paragraph implies that $\omega \in \Omega(X;\mathcal{R})^q_K$. This proves the lemma. □

A.7 Lemma *The restriction maps induce an isomorphism*

$$K^{q-1}(X;\mathbb{R}/\mathbb{Z}) \longrightarrow \varprojlim_i K^{q-1}(X_i;\mathbb{R}/\mathbb{Z}). \tag{A.8}$$

Proof The Ext term in the universal coefficient theorem for K–theory vanishes since $\mathbb{R}/\mathbb{Z}$ is divisible. Applying the universal coefficient theorem twice and the fact that homology commutes with colimits, we obtain

$$\begin{aligned} K^{q-1}(X;\mathbb{R}/\mathbb{Z}) &\cong \operatorname{Hom}\big(K_{q-1}(X),\mathbb{R}/\mathbb{Z}\big) \\ &\cong \operatorname{Hom}\big(\varinjlim_i K_{q-1}(X_i),\mathbb{R}/\mathbb{Z}\big) \\ &\cong \varprojlim_i \operatorname{Hom}\big(K_{q-1}(X_i),\mathbb{R}/\mathbb{Z}\big) \\ &\cong \varprojlim_i K^{q-1}(X_i;\mathbb{R}/\mathbb{Z}). \end{aligned} \tag{A.9}$$

□

Proof of Theorem A.2 Milnor's exact sequence (A.1) and Lemma A.7 imply that

$$\varprojlim_i{}^1 K^{q-1}(X_i; \mathbb{R}/\mathbb{Z}) = 0. \tag{A.10}$$

Thus the limit of the short exact sequence (2.20) is a short exact sequence [44, Section 3.5]. The restriction maps then fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{q-1}(X;\mathbb{R}/\mathbb{Z}) & \longrightarrow & \check{K}^q(X) & \longrightarrow & \Omega(X;\mathcal{R})^q_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_i K^{q-1}(X_i;\mathbb{R}/\mathbb{Z}) & \longrightarrow & \varprojlim_i \check{K}^q(X_i) & \longrightarrow & \varprojlim_i \Omega(X_i;\mathcal{R})^q_K & \longrightarrow & 0 \end{array} \tag{A.11}$$

in which the rows are exact and, by the lemmas, the outer vertical arrows are isomorphisms. The theorem now follows from the 5–lemma. □

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