# A cartesian presentation of weak $n$–categories

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We propose a notion of weak $(n+k,n)$–category, which we call $(n+k,n)$–$\Theta$–spaces. The $(n+k,n)$–$\Theta$–spaces are precisely the fibrant objects of a certain model category structure on the category of presheaves of simplicial sets on Joyal's category $\Theta_n$. This notion is a generalization of that of complete Segal spaces (which are precisely the $(\infty,1)$–$\Theta$–spaces). Our main result is that the above model category is cartesian.

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## 1 Introduction

In this note, we propose a definition of weak $n$–category, and more generally, weak $(n+k,n)$–category for all $0 \leq n < \infty$ and $-2 \leq k \leq \infty$, called $(n+k,n)$–$\Theta$–*spaces*. The collection of $(n+k,n)$–$\Theta$–spaces forms a category $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$, and there is a notion for a morphism in this category to be an equivalence. The category $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$ together with the given class of equivalences has the following desirable property: it is cartesian closed, in a way compatible with the equivalences. More precisely, we have the following.

(1) The category $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$ is cartesian closed; in other words, it has products $Y \times Z$ and function objects $Z^Y$ for any pair of objects $Y, Z$ in $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$, so that $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}(X \times Y, Z) \approx \Theta_n\mathrm{Sp}_k^{\mathrm{fib}}(X, Z^Y)$.

(2) If $f\colon X \to Y$ is an equivalence in $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$, then so are $f \times Z\colon X \times Z \to Y \times Z$ and $Z^f\colon Z^Y \to Z^X$.

The category $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$ will be defined as the full subcategory of fibrant objects in a Quillen model category $\Theta_n\mathrm{Sp}_k$. The underlying category of $\Theta_n\mathrm{Sp}_k$ is the category $s\mathrm{PSh}(\Theta_n)$ of simplicial presheaves on a certain category $\Theta_n$. We equip this category with a model category structure, obtained as the Bousfield localization of the injective model structure on presheaves with respect to a certain set of morphisms $\mathscr{T}_{n,k}$. We will prove that $\Theta_n\mathrm{Sp}_k$ is a *cartesian model category*, ie, the model category structure is nicely compatible with the internal function objects of $s\mathrm{PSh}(\Theta_n)$. Then $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$ is the full subcategory of *fibrant* objects in $\Theta_n\mathrm{Sp}_k$; equivalences in $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$ are just levelwise weak equivalences of presheaves.

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For $n = 0$, the category $\Theta_0$ is the terminal category, so that $s\mathrm{PSh}(\Theta_0)$ is the category of simplicial sets $\mathrm{Sp}$. An $(\infty, 0)$–$\Theta$–space is precisely a Kan complex, and a $(k, 0)$–$\Theta$–space is precisely a $k$–truncated Kan complex, ie, a Kan complex with homotopy groups vanishing above dimension $k$.

For $n = 1$, the category $\Theta_1$ is the category $\Delta$ of finite ordinals, so that $s\mathrm{PSh}(\Theta_1)$ is the category of simplicial spaces. An $(\infty, 1)$–$\Theta$–space is precisely a *complete Segal space*, in the sense of our paper [16].

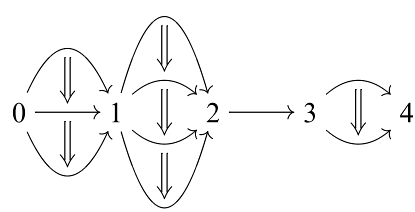
The category $\Theta_n$ which we use was introduced by Joyal [12], as part of an attempt to define a notion of weak $n$–category generalizing the notion of quasicategory. Sketches of these ideas can be found in Leinster [14] and Cheng [7]. The category $\Theta_n$ has also been studied by Berger [3; 4], with particular application to the theory of iterated loop spaces.

## 1.1 The categories $\Theta_n$

We will give an informal description of Joyal's categories $\Theta_n$ here, suitable for our purposes; our description is essentially the same as that given in [4, Section 3]. It is most useful for us to regard $\Theta_n$ as a full subcategory of **St**–$n$–**Cat**, the category of strict $n$ categories. Thus, $\Theta_0$ is the full subcategory of **St**–0–**Cat** = Set consisting of the terminal object. The category $\Theta_1$ is the full subcategory of **St**–1–**Cat** consisting of objects $[n]$ for $n \geq 0$, where $[n]$ represents the free strict 1–category on the diagram

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow (n-1) \longrightarrow n.$$

Thus, $\Theta_1 = \Delta$, the usual simplicial indexing category. The category $\Theta_2$ is the full subcategory of **St**–2–**Cat** consisting of objects which are denoted $[m]([n_1], \ldots, [n_m])$ for $m, n_1, \ldots, n_m \geq 0$. This represents the strict 2–category $C$ which is "freely generated" by objects $\{0, 1, \ldots, m\}$ and morphism categories $C(i-1, i) = [n_i]$. For instance, the object $[4]([2], [3], [0], [1])$ in $\Theta_2$ corresponds to the "free 2–category" on the following picture:



In general, the objects of $\Theta_n$ are of the form $[m](\theta_1,\dots,\theta_m)$, where $m\geq 0$ and the $\theta_i$ are objects of $\Theta_{n-1}$; this object corresponds to the strict $n$–category $C$ "freely generated" by objects $\{0,\dots,m\}$ and a strict $(n-1)$–category of morphisms $C(i-1,i)=\theta_i$. The morphisms of $\Theta_n$ are functors between strict $n$–categories.

We make special note of objects $O_0,\dots,O_n$ in $\Theta_n$. These are defined recursively by $O_0=[0]$ and $O_k=[1](O_{k-1})$ for $k=1,\dots,n$. Thus, the object $O_k$ in $\Theta_n$ corresponds to the "freestanding $k$–cell" in **St**–$n$–**Cat**.

## 1.2 Informal description of $\Theta$–spaces

We will start by describing $\Theta_n\mathrm{Sp}^{\mathrm{fib}}_{\infty}$, the category of $(\infty,n)$–$\Theta$–spaces. Let Sp denote the category of simplicial sets. We will regard objects of Sp as "spaces"; the following definitions are perfectly sensible if objects of Sp are taken to be actual topological spaces (compactly generated).

An object of $\Theta_n\mathrm{Sp}^{\mathrm{fib}}_{\infty}$ is a simplicial presheaf on $\Theta_n$ (ie, a functor $X\colon\Theta_n^{\mathrm{op}}\to\mathrm{Sp}$), satisfying three conditions:

(i) an *injective fibrancy* condition,

(ii) a *Segal* condition, and

(iii) a *completeness* condition.

A morphism $f\colon X\to Y$ of $\Theta_n\mathrm{Sp}^{\mathrm{fib}}_{\infty}$ is a morphism of simplicial presheaves; the morphism $f$ is said to be an *equivalence* (or *weak equivalence*) if it is a "levelwise" weak equivalence of simplicial presheaves, ie, if $f(\theta)\colon X\theta\to Y\theta$ is a weak equivalence of simplicial sets for all $\theta\in\mathrm{ob}\,\Theta$.

The injective fibrancy condition (i) says that $X$ has a right lifting property with respect to maps in $s\mathrm{PSh}(\Theta_n)$ which are both monomorphisms and levelwise weak equivalences; that is, $X$ is fibrant in the *injective* model structure on $s\mathrm{PSh}(\Theta_n)$.

The Segal condition (ii) says that for all objects $\theta$ of $\Theta_n$, the space $X(\theta)$ is weakly equivalent to an inverse limit of a certain diagram of spaces $X(O_k)$; taken together with the injective fibrancy condition, this inverse limit is in fact the homotopy inverse limit of the given diagram. For $n=1$, the Segal condition amounts to requiring that the "Segal map"

$$X([m])\to X([1])\underset{X([0])}{\times}\cdots\underset{X([0])}{\times}X([1])\approx X(O_1)\underset{X(O_0)}{\times}\cdots\underset{X(O_0)}{\times}X(O_1)$$

be a weak equivalence for all $m \geq 2$. As an example of how the Segal condition works for $n = 2$, the space $X([4]([2],[3],[0],[1]))$ is required to be weakly equivalent to

$$\Big(X(O_2) \underset{X(O_1)}{\times} X(O_2)\Big) \underset{X(O_0)}{\times} \Big(X(O_2) \underset{X(O_1)}{\times} X(O_2) \underset{X(O_1)}{\times} X(O_2)\Big) \underset{X(O_0)}{\times} X(O_1) \underset{X(O_0)}{\times} X(O_2).$$

The completeness condition (iii) says that the space $X(O_k)$ should behave like the "moduli space" of $k$–cells in a $(\infty, n)$–category. That is, if the points of $X(O_k)$ correspond to individual $k$–cells, such points should be connected by a path in $X(O_k)$ if they represent "equivalent" $k$–cells, there should be a homotopy between paths for every "equivalence between equivalences", and so on. It turns out that the way to enforce this is to require that, for $k = 1, \dots, n$, the map $X(i_k)\colon X(O_{k-1}) \to X(O_k)$ which encodes "send a $(k-1)$–cell to its identity $k$–morphism" should induce a weak equivalence of spaces

$$X(O_{k-1}) \to X(O_k)^{\text{equiv}}.$$

Here $X(O_k)^{\text{equiv}}$ is the union of those path components of $X(O_k)$ which consist of $k$–morphisms which are "$k$–equivalences". Thus, the completeness condition asserts that the moduli space of $(k-1)$–cells is weakly equivalent to the moduli space of $k$–equivalences.

The category $\Theta_n\mathrm{Sp}_k$ of $(n+k, n)$–$\Theta$–spaces is obtained by imposing an additional

(iv) *$k$–truncation* condition.

To state this, we need the moduli space $X(\partial O_m)$ of "parallel pairs of $(m-1)$–morphisms" in $X$. This space is defined inductively as an inverse limit of the spaces $X(O_m)$, so that

$$X(\partial O_m) \overset{\text{def}}{=} \lim\big(X(O_{m-1}) \to X(\partial O_{m-1}) \leftarrow X(O_{m-1})\big),$$

with $X(\partial O_0) = 1$. Then the $k$–truncation condition asserts that the fibers of $X(O_n) \to X(\partial O_n)$ are $k$–types, ie, have vanishing homotopy groups in all dimensions greater than $k$.

The above definition is examined in detail in Section 11.

## 1.3 Presentations and enriched model categories

Our construction of a cartesian model category structure is a special case of a general procedure, which associates to certain kinds of model categories $M$ a new model category $M$–$\Theta\mathrm{Sp}$; we may regard this as being analogous to the procedure which associates to a category $V$ with finite products the category $V$–Cat of categories enriched over $V$.

Specifically, suppose we are given a pair $(C, \mathscr{S})$ consisting of a small category $C$ and a set $\mathscr{S}$ of morphisms in $s\mathrm{PSh}(C)$; this data is called a *presentation*, following the treatment of Dugger [8]. (Here, $s\mathrm{PSh}(C)$ denotes the category of presheaves of simplicial sets on $C$.) Let $M = s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathscr{S}}$ denote the model category structure on $s\mathrm{PSh}(C)$ obtained by Bousfield localization of the injective model structure with respect to $\mathscr{S}$. We define a new presentation $(\Theta C, \mathscr{S}_\Theta)$, and thus obtain a model category $M\text{–}\Theta\mathrm{Sp} \overset{\mathrm{def}}{=} s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}_{\mathscr{S}_\Theta}$. The category $\Theta C$ is a "wreath product" of $\Delta$ with $C$, as defined by [4] (see Section 3), while the set $\mathscr{S}_\Theta$ consists of some maps built from elements of $\mathscr{S}$, together with certain "Segal" and "completeness" maps (this set is described in Section 8).

Our main result is the following theorem.

**1.4 Theorem** *Let $M = s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathscr{S}}$ for some presentation $(C, \mathscr{S})$. If $M$ is a cartesian model category, then $M\text{–}\Theta\mathrm{Sp}$ is also a cartesian model category.*

This theorem is a straightforward generalization of the main theorem of [16], which proves the theorem for the special case $(C, \mathscr{S}) = (1, \varnothing)$ (in which case $M = \mathrm{Sp}$, and thus $M\text{–}\Theta\mathrm{Sp}$ is the category of simplicial spaces with the complete Segal space model structure.)

The model categories for $(n+k, n)$–$\Theta$–spaces are obtained iteratively, so that

$$\Theta_{n+1}\mathrm{Sp}_k \overset{\mathrm{def}}{=} (\Theta_n\mathrm{Sp}_k)\text{–}\Theta\mathrm{Sp},$$

starting with $\Theta_0\mathrm{Sp}_k = \mathrm{Sp}_k$, where $\mathrm{Sp}_k$ is the Bousfield localization of $\mathrm{Sp}$ whose fibrant objects are Kan complexes which are $k$–types. Applying the theorem inductively shows that $\Theta_n\mathrm{Sp}_k$ are cartesian model categories. The category $\Theta_n\mathrm{Sp}_k^{\mathrm{fib}}$ is defined to be the full subcategory of fibrant objects in the model category $\Theta_n\mathrm{Sp}_k$.

## 1.5 The relationship between $M$–$\Theta$Sp and $M$–Cat

If $M$ is a cartesian model category, then we may certainly consider $M$–Cat, the category of small categories enriched over $M$. Given an object $X$ of $M$–Cat, let $hX$ denote the ordinary category whose objects are the same as $X$, and whose morphism sets are given by $hX(a,b) \overset{\mathrm{def}}{=} hM(1, X(a,b))$, where $hM$ denotes the homotopy category of $M$. Let us say that a morphism $f\colon X \to Y$ of objects of $M$–Cat is a *weak equivalence* if

(1) for each pair of objects $a, b$ of $X$, the induced map $X(a,b) \to Y(fa, fb)$ is a weak equivalence in $M$, and

(2) the induced functor $hX \to hY$ is an equivalence of 1–categories.

We can make the following conjecture.

**1.6 Conjecture** *Let $M = s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathscr{S}}$ for some presentation $(C, \mathscr{S})$, and suppose that $M$ is a cartesian model category. Then there is a model category structure on $M$–Cat with the above weak equivalences, and a Quillen equivalence*

$$M\text{–Cat} \approx M\text{–}\Theta\mathrm{Sp}.$$

For the case of $M = \mathrm{Sp}$, the conjecture follows from theorems of Bergner [5; 6].

## 1.7 Why is this a good notion of weak $n$–category?

We propose that $(n+k, n)$–$\Theta$–spaces are a model for weak $(n+k, n)$–categories. Some points in its favor are the following.

(1) Our notion of $(\infty, 1)$–$\Theta$–spaces is precisely what we called a *complete Segal space* in [16]. This is recognized as a suitable model for $(\infty, 1)$–categories, due to work of Bergner [6] and Joyal and Tierney [13].

(2) As noted above in Section 1.5, the definition of $(n+k, n)$–$\Theta$–spaces is a special case of a more general construction, which conjecturally models "homotopy theories enriched over a cartesian model category". In particular, a consequence of our conjecture would be a Quillen equivalence

$$(\Theta_n\mathrm{Sp}_k)\text{–Cat} \approx (\Theta_n\mathrm{Sp}_k)\text{–}\Theta\mathrm{Sp} = \Theta_{n+1}\mathrm{Sp}_k.$$

That is, $(n+1+k, n+1)$–$\Theta$–spaces are (conjecturally) "the same" as categories enriched over $(n+k, n)$–$\Theta$–spaces.

(3) Our notion satisfies the "homotopy hypothesis". There is an evident notion of groupoid object in $\Theta_n\mathrm{Sp}_k$, and we show (Section 11.25) show that the full subcategory of such groupoid objects models $\mathrm{Sp}_{n+k}$, the homotopy theory of $(n+k)$–types.

(4) More generally, it is understood that $n$–tuply monoidal $k$–groupoids should correspond to "$k$–types of $E_n$–spaces", where $E_n$ is a version of the little $n$–cubes operad; furthermore, $n$–tuply *groupal* $k$–groupoids should correspond to "$k$–types of $n$–fold loop spaces" (see, for instance, Baez and Dolan [1, Section 3]). In terms of our models, $n$–tuply groupal $k$–groupoids are objects $X$ of $\Theta_n\mathrm{Sp}_k$ for which (i) $X(O_j) \approx 1$ for $j < n$, and (ii) $X(O_n)^{\mathrm{equiv}} \approx X(O_n)$, and one would conjecture that

the full subcategory of such objects in $\Theta_n\mathrm{Sp}_k$ should model $k$–types of $n$–fold loop spaces. That this is in fact the case is apparent from the results of Berger [4].

As noted above, the theory of $\Theta$–spaces is consciously a generalization of the theory of complete Segal spaces, which is one of a family of models for $(\infty, 1)$–categories based on simplicial objects. A reasonable approach to producing a generalization of these ideas is to use multisimplicial objects; proposals for this include Tamsamani's theory of weak $n$–categories [17], the Segal $n$–categories of Hirschowitz and Simpson [10], and more recent work by Barwick [2] and Lurie [15] on multisimplicial generalizations of complete Segal spaces. Although all these constructions appear to give good models for $(\infty, n)$–categories, it is not clear to me that any of them result in a Quillen model category which satisfies all of the following: (i) it models the homotopy theory of $(\infty, n)$–categories with the correct notion of equivalence, (ii) it is a cartesian model category, and (iii) it is a simplicial model category. It does appear that the Hirschowitz–Simpson model satisfies (i) and (ii), but it does not satisfy (iii). The multisimplicial complete Segal space model of Barwick and Lurie does satisfy (i) and (iii), but does not appear to satisfy (ii) (when $n > 1$).

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# 2 Cartesian model categories and cartesian presentations

## 2.1 Cartesian closed categories

A category $V$ is said to be *cartesian closed* if it has finite products, and if for all $X, Y \in \mathrm{ob}\, V$ there is an *internal function object* $Y^X$, which comes equipped with a natural isomorphism

$$V(T, Y^X) \approx V(T \times X, Y).$$

Examples of cartesian categories include the category of sets, and the category of presheaves of sets on a small category $C$.

We will write $\varnothing$ for some chosen initial object in a cartesian closed category $V$.

## 2.2 Cartesian model categories

We will say that a Quillen model category $M$ is *cartesian* if it is cartesian closed as a category, if the terminal object is cofibrant, and if the following equivalent properties hold.

(1) If $f\colon A \to A'$ and $g\colon B \to B'$ are cofibrations in $M$, then the induced map $h\colon A \times B' \amalg_{A\times B} A' \times B \to A' \times B'$ is a cofibration; if in addition either $f$ or $g$ is a weak equivalence then so is $h$.

(2) If $f\colon A \to A'$ is a cofibration and $p\colon X \to X'$ is a fibration in $M$, then the induced map $q\colon (X')^{A'} \to (X')^{A} \times_{X^A} X^{A'}$ is a fibration; if in addition either $f$ or $p$ is a weak equivalence then so is $q$.

(This notion is a bit stronger than Hovey's notion of symmetric monoidal model category as applied to cartesian closed categories [11, 4.2]; he does not require the unit object to be cofibrant, but rather imposes a weaker condition.)

## 2.3 Spaces

Let Sp denote the category of simplicial sets, equipped with the standard Quillen model structure. We call this the model category of *spaces*. It is standard that Sp is a cartesian model category.

We will often use topological flavored language when discussing objects of Sp, even though such are objects are not topological spaces but simplicial sets. Thus, a "point" in a "space" is really a 0–simplex of a simplicial set, a "path" is a 1–simplex, and so on.

## 2.4 Simplicial presheaves

Let $C$ be a small category, and let $s\mathrm{PSh}(C)$ denote the category of *simplicial presheaves* on $C$, ie, the category of contravariant functors $C^{\mathrm{op}} \to \mathrm{Sp}$.

A simplicial presheaf $X$ is said to be *discrete* if each $X(c)$ is a discrete simplicial set; the full subcategory of discrete objects in $s\mathrm{PSh}(C)$ is equivalent to the category of presheaves of *sets* on $C$, and we will implicitly identity the two.

Let $F_C\colon C \to s\mathrm{PSh}(C)$ denote the *Yoneda functor*; thus $F_C$ sends an object $c \in \mathrm{ob}\, C$ to the presheaf $F_C c = C(-, c)$. Observe that the presheaf $F_C c$ is discrete. When the context is understood we may write $F$ for $F_C$.

Let $\Gamma_C\colon s\mathrm{PSh}(C) \to \mathrm{Sp}$ denote the *global sections functor*, which sends a functor $X\colon C^{\mathrm{op}} \to \mathrm{Sp}$ to its limit. The functor $\Gamma$ is right adjoint to the functor $\mathrm{Sp} \to s\mathrm{PSh}(C)$ which sends a simplicial set $K$ to the constant presheaf with value $K$ at each object of $C$. Note that if $C$ has a terminal object [0], then $\Gamma X \approx X([0])$. When the context is understood we may write $\Gamma X$ for $\Gamma_C X$.

For $X, Y$ in $s\mathrm{PSh}(C)$, we write $\mathrm{Map}_C(X, Y) \overset{\mathrm{def}}{=} \Gamma(Y^X)$; this is called the *mapping space*. Thus, $s\mathrm{PSh}(C)$ is enriched over Sp. Note that if $c \in \mathrm{ob}\, C$, then we have

$$X(c) \approx \Gamma(X^{F(c)}) \approx \mathrm{Map}_C(F(c), X).$$

When the context is understood we may write $\mathrm{Map}(X, Y)$ for $\mathrm{Map}_C(X, Y)$.

## 2.5 Model categories for simplicial presheaves

Say that a map $f\colon X \to Y \in s\mathrm{PSh}(C)$ is a *levelwise weak equivalence* if each map $f(c)\colon X(c) \to Y(c)$ is a weak equivalence in Sp for all $c \in \mathrm{ob}\, C$. There are two standard model category structures we can put on $s\mathrm{PSh}(C)$ with these weak equivalences, called the projective and injective structures; they are Quillen equivalent to each other.

The *projective* structure is characterized by requiring that $f\colon X \to Y \in s\mathrm{PSh}(C)$ be a fibration if and only if $f(c)\colon X(c) \to Y(c)$ is one in Sp for all $c \in \mathrm{ob}\, C$. We write $s\mathrm{PSh}(C)^{\mathrm{proj}}$ for the category of presheaves of simplicial sets on $C$ equipped with the projective model structure.

The *injective* structure is characterized by requiring that $f\colon X \to Y \in s\mathrm{PSh}(C)$ is a cofibration if and only if $f(c)\colon X(c) \to Y(c)$ is one in Sp for all $c \in \mathrm{ob}\, C$. We write $s\mathrm{PSh}(C)^{\mathrm{inj}}$ for the category of presheaves of simplicial sets on $C$ equipped with the injective model structure.

The identity functor provides a Quillen equivalence $s\mathrm{PSh}(C)^{\mathrm{proj}} \rightleftarrows s\mathrm{PSh}(C)^{\mathrm{inj}}$.

Both the projective and injective model category structures are cofibrantly generated and proper, and have functorial factorizations. They are also both simplicial model categories.

Given object $X, Y$ in $s\mathrm{PSh}(C)$, we write $h\mathrm{Map}_C(X, Y)$ for the *derived mapping space* of maps from $X$ to $Y$. This is a homotopy type in Sp, defined so that for any cofibrant approximation $X^c \to X$ and fibrant approximation $Y \to Y^f$, the derived mapping space $h\mathrm{Map}_C(X, Y)$ is weakly equivalent to the space of maps $\mathrm{Map}_C(X^c, Y^f)$. Note that in the above, we may pick either the injective or projective model category structures in order to make our replacements.

## 2.6 The injective model structure

The injective model structure has a few additional properties which are of importance to us. In particular,

(1) every object of $s\mathrm{PSh}(C)^{\mathrm{inj}}$ is cofibrant, and

(2) every *discrete* object of $s\mathrm{PSh}(C)^{\mathrm{inj}}$ is fibrant.

Furthermore, we have the following.

**2.7 Proposition** *The model category $s\mathrm{PSh}(C)^{\mathrm{inj}}$ is a cartesian model category.*

**Proof** This is immediate from characterization (1) of cartesian model categories. □

## 2.8 Presentations

A *presentation* is a pair $(C, \mathcal{S})$ consisting of a small category $C$ and a set $\mathcal{S} = \{s\colon S \to S'\}$ of morphisms in $s\mathrm{PSh}(C)$. We say that an object $X$ of $s\mathrm{PSh}(C)$ is *$\mathcal{S}$–local* if for all morphisms $s\colon S \to S'$ in $\mathcal{S}$, the induced map

$$h\mathrm{Map}(s, X)\colon h\mathrm{Map}(S', X) \to h\mathrm{Map}(S, X)$$

is a weak equivalence of spaces. We say that a morphism $f\colon A \to B$ in $s\mathrm{PSh}(C)$ is an *$\mathcal{S}$–equivalence* if the induced map

$$h\mathrm{Map}(f, X)\colon h\mathrm{Map}(B, X) \to h\mathrm{Map}(A, X)$$

is a weak equivalence of spaces for all $\mathcal{S}$–local objects $X$. The collection of $\mathcal{S}$–equivalences is denoted $\overline{\mathcal{S}}$; we have that $\mathcal{S} \subset \overline{\mathcal{S}}$.

Let $(C, \mathcal{S})$ be a presentation, let $X$ be an object of $s\mathrm{PSh}(C)$, and let $X \to X^f$ denote a fibrant replacement of $X$ in the *injective* model structure. Since every object is cofibrant in the injective model structure, we have that $X$ is $\mathcal{S}$–local if and only if $\mathrm{Map}(S', X^f) \to \mathrm{Map}(S, X^f)$ is a weak equivalence for all $s \in \mathcal{S}$.

## 2.9 Cartesian presentations

Let $(C, \mathcal{S})$ be a presentation. Given an object in $X$ of $s\mathrm{PSh}(C)$, we say it is *$\mathcal{S}$–cartesian local* if for all $s\colon S \to S'$ in $\mathcal{S}$, the induced map

$$Y^s\colon Y^{S'} \to Y^S$$

is a levelwise weak equivalence, where $X \to Y$ is some choice of fibrant replacement in $s\mathrm{PSh}(C)^{\mathrm{inj}}$.

**2.10 Proposition** *Let $X$ be an object of $s\mathrm{PSh}(C)$, and choose some fibrant replacement $X \to Y$ in $s\mathrm{PSh}(C)^{\mathrm{inj}}$. Then $X$ is $\mathcal{S}$–cartesian local if and only if for all $c \in \mathrm{ob}\, C$, the function object $Y^{F(c)}$ is $\mathcal{S}$–local.*

**Proof** Immediate from the isomorphism

$$Y^S(c) \approx \mathrm{Map}(F(c), Y^S) \approx \mathrm{Map}(S, Y^{F(c)}). \qquad \square$$

Observe that every $\mathcal{S}$–cartesian local object is necessarily $\mathcal{S}$–local, since $\mathrm{Map}(S, Y) \approx \mathrm{Map}(1, Y^S)$; however, the converse need not hold. We say that a presentation $(C, \mathcal{S})$ is a *cartesian presentation* if every $\mathcal{S}$–local object is $\mathcal{S}$–cartesian local.

**2.11 Proposition** *Let $(C, \mathcal{S})$ be a presentation. The following are equivalent.*

(1) *$(C, \mathcal{S})$ is a cartesian presentation.*

(2) *For all $\mathcal{S}$–fibrant $X$ in $s\mathrm{PSh}(C)$ and all $c \in \mathrm{ob}\, C$, the object $X^{F(c)}$ is $\mathcal{S}$–local.*

(3) *For all $s\colon S \to S' \in \mathcal{S}$ and all $c \in \mathrm{ob}\, C$, the map $s \times \mathrm{id}\colon S \times Fc \to S' \times Fc$ is in $\overline{\mathcal{S}}$.*

**Proof** Immediate from Proposition 2.10. □

**2.12 Proposition** *If $(C, \mathcal{S})$ is a cartesian presentation, then $f, g \in \overline{\mathcal{S}}$ imply $f \times g \in \overline{\mathcal{S}}$.*

## 2.13 Localization

Given a presentation $(C, \mathcal{S})$, we write $s\mathrm{PSh}(C)^{\mathrm{proj}}_{\mathcal{S}}$ and $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathcal{S}}$ for the model category structures on $s\mathrm{PSh}(C)$ obtained by Bousfield localization of the original projective and injective model structures on $s\mathrm{PSh}(C)$. These model categories are again Quillen equivalent to each other. We will set out the details in the case of the injective model structure.

**2.14 Proposition** *Given a presentation $(C, \mathcal{S})$ there exists a cofibrantly generated, left proper, simplicial model category structure on $s\mathrm{PSh}(C)$ which is characterized by the following properties.*

(1) *The cofibrations are exactly the monomorphisms.*

(2) *The fibrant objects are precisely the injective fibrant objects which are $\mathcal{S}$–local. (We call these the $\mathcal{S}$–*fibrant *objects.)*

(3) *The weak equivalences are precisely the $\mathcal{S}$–equivalences.*

*Furthermore, we have the following:*

(4) *A levelwise weak equivalence $g\colon X \to Y$ is an $\mathcal{S}$–equivalence, and the converse holds if both $X$ and $Y$ are $\mathcal{S}$–local.*

(5) *An $\mathcal{S}$–fibration $g\colon X \to Y$ is an injective fibration, and the converse holds if both $X$ and $Y$ are $\mathcal{S}$–fibrant.*

**Proof** This is an example of [9, Theorem 4.1.1], since $s\mathrm{PSh}(C)^{\mathrm{inj}}$ is a left proper cellular model category. □

We will write $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathcal{S}}$ for the above model structure, which is called the *$\mathcal{S}$–local injective model structure*.

Observe that if $(C, \mathcal{S})$ and $(C, \mathcal{S}')$ are two presentations on $C$ such that the $\mathcal{S}$–local objects are precisely the same as the $\mathcal{S}'$–local objects, then $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathcal{S}} = s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathcal{S}'}$.

## 2.15 Quillen pairs between localizations

**2.16 Proposition** *Suppose that $(C,\mathcal{S})$ and $(D,\mathcal{T})$ are presentations, and that we have a Quillen pair $G_\#\colon s\mathrm{PSh}(C)^{\mathrm{inj}} \rightleftarrows s\mathrm{PSh}(D)^{\mathrm{inj}}$ (with $G_\#$ the left adjoint). Then*

$$G_\#\colon s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathcal{S}} \rightleftarrows s\mathrm{PSh}(D)^{\mathrm{inj}}_{\mathcal{T}} \;:G^*$$

*is a Quillen pair if and only if either of the two following equivalent statements hold.*

(1) *For all $s\in\mathcal{S}$, $G_\# s\in\overline{\mathcal{T}}$.*

(2) *For all $\mathcal{T}$–fibrant objects $Y$ in $s\mathrm{PSh}(D)$, $G^*Y$ is $\mathcal{S}$–fibrant.*

**Proof** This is straightforward from the definitions. □

## 2.17 $\mathcal{S}$–equivalences and homotopy colimits

The following proposition says that the class of $\mathcal{S}$–equivalences is closed under homotopy colimits. We refer to Hirschhorn [9] for background on homotopy colimits.

**2.18 Proposition** *Let $D$ be a small category, and let $(C,\mathcal{S})$ be a presentation. Suppose that $\alpha\colon G\to H$ is a natural transformation of functors $D\to s\mathrm{PSh}(C)^{\mathrm{inj}}$, and consider the induced map*

$$\mathrm{hocolim}_D\,\alpha\colon \mathrm{hocolim}_D\,G\to\mathrm{hocolim}_D\,H$$

*on homotopy colimits, where these homotopy colimits are computed in the injective model structure on $s\mathrm{PSh}(C)$. If $\alpha(d)\in\overline{S}$ for all $d\in\mathrm{ob}\,D$, then $\mathrm{hocolim}_D\,\alpha\in\overline{S}$.*

**Proof** In general, the map $h\mathrm{Map}_C(\mathrm{hocolim}_D\,H,X)\to h\mathrm{Map}(\mathrm{hocolim}_D\,G,X)$ is weakly equivalent to the map $\mathrm{holim}_D\,h\mathrm{Map}(H,X)\to\mathrm{holim}_D\,h\mathrm{Map}(G,X)$; the result follows by considering the case when $X$ is $\mathcal{S}$–local. □

We later use (in Section 6), we record the following fact.

**2.19 Proposition** *Let $C$ be a small category, and let $X$ be an object of $s\mathrm{PSh}(C)^{\mathrm{inj}}$. Suppose $\mathcal{P}$ is a finite set of subobjects $K\subseteq X$ in $s\mathrm{PSh}(C)$. If $\mathrm{colim}_{K\in\mathcal{P}}\,K\to X$ is an isomorphism (regarding $\mathcal{P}$ as a finite poset), then*

$$\mathrm{hocolim}_{K\in\mathcal{P}}\,K\to X$$

*is a levelwise weak equivalence, where homotopy colimit is computed using the injective model structure.*

**Proof** Since $(\operatorname{hocolim}_{K\in\mathcal{P}} K)(c) \approx \operatorname{hocolim}_{K\in\mathcal{P}}(K(c))$, we can reduce to the case when $C = 1$; that is, we may assume $X$ is an object of Sp.

Suppose $X$ is a set and $\mathcal{P}$ is a collection of subsets of $X$ such that $\operatorname{colim}_{K\in\mathcal{P}} K \approx X$. It is straightforward to show that for all $K \in \mathcal{P}$, the map $\operatorname{colim}_{\mathcal{P}_{<L}} K \to L$ is a monomorphism, where $\mathcal{P}_{<L} \subseteq \mathcal{P}$ denotes the poset of proper subobjects of $L$. The same therefore holds true for a collection of subobjects of a simplicial sets satisfying the same properties. Thus the functor $\mathcal{P} \to \mathrm{Sp}$ determined by the collection of subobjects of $X$, is cofibrant in the projective model structure on $s\mathrm{PSh}(\mathcal{P}^{\mathrm{op}})$, and so the colimit of this functor is the homotopy colimit. □

Finally, we record the following fact, which we use in Section 5. For a category $C$ and an object $A$ in $C$, we write $A\backslash C$ for the slice category of objects under $A$ in $C$.

**2.20 Proposition** *Let $C$ be a small category, and let $(D,\mathcal{S})$ be a presentation. Suppose that $\alpha\colon G \to H$ is a natural transformation of functors $s\mathrm{PSh}(C) \to s\mathrm{PSh}(D)$. Suppose the following hold.*

(1) *The functors $X \mapsto (G(\varnothing) \to G(X))\colon s\mathrm{PSh}(C)^{\mathrm{inj}} \to G(\varnothing)\backslash s\mathrm{PSh}(D)^{\mathrm{inj}}$ and $X \mapsto (H(\varnothing) \to H(X))\colon s\mathrm{PSh}(C)^{\mathrm{inj}} \to H(\varnothing)\backslash s\mathrm{PSh}(D)^{\mathrm{inj}}$ are left Quillen functors.*

(2) *The map $\alpha(\varnothing)\colon G(\varnothing) \to H(\varnothing)$ is a monomorphism, and is in $\overline{\mathcal{S}}$.*

(3) *The maps $\alpha(Fc)\colon G(Fc) \to H(Fc)$ are in $\overline{\mathcal{S}}$ for all $c \in \operatorname{ob} C$.*

*Then $\alpha(X) \in \overline{\mathcal{S}}$ for all objects $X$ of $s\mathrm{PSh}(C)$.*

**Proof** In the special case in which $\alpha(\varnothing)\colon G(\varnothing) \to H(\varnothing)$ is an isomorphism, note that since (i) every object of $s\mathrm{PSh}(C)$ is levelwise weakly equivalent to a homotopy colimit of some diagram of free objects, and (ii) left Quillen functors preserve homotopy colimits, the result follows using Proposition 2.18.

For the general case, factor $\alpha$ into $G \xrightarrow{\beta} K \xrightarrow{\gamma} H$ where $K(X) = G(X) \cup_{G(\varnothing)} H(\varnothing)$. The map $\beta(X)$ is a pushout of the $\mathcal{S}$–local equivalence $\alpha(\varnothing)$ along the map $G(\varnothing) \to G(X)$ which is an injective cofibration by (i); thus $\beta(X) \in \overline{\mathcal{S}}$. The special case described above applies to show that $\gamma(X) \in \overline{\mathcal{S}}$. Thus, the composite $\alpha(X) \in \overline{\mathcal{S}}$, as desired. □

## 2.21 Cartesian presentations give cartesian model categories

**2.22 Proposition** *The model category $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathcal{S}}$ is cartesian if and only if $(C,\mathcal{S})$ is a cartesian presentation. In particular, $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathcal{S}}$ is a cartesian model category if for all $\mathcal{S}$–fibrant $Y$ and all $c \in \operatorname{ob} C$, the object $Y^{F(c)}$ is $\mathcal{S}$–fibrant.*

**Proof** It is clear that the terminal object is cofibrant in $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathscr{S}}$, so it suffices to show that $(C,\mathscr{S})$ is a cartesian presentation if and only if condition (1) of Section 2.2 holds. Let $f\colon A\to A'$ and $g\colon B\to B'$ be cofibrations in $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathscr{S}}$. It is clear that the map

$$h\colon A\times B'\amalg_{A\times B} A'\times B\to A'\times B'$$

is a cofibration in any case, so it suffices to show that "$W$ is $\mathscr{S}$–fibrant implies $W$ is $\mathscr{S}$–cartesian fibrant" is equivalent to "$g\in\overline{\mathscr{S}}$ implies $h\in\overline{\mathscr{S}}$". Since $s\mathrm{PSh}(C)^{\mathrm{inj}}$ is a cartesian model category, for a cofibration $g$ as above and all injective fibrant $W$ we have that $W^g$ is a levelwise weak equivalence if and only if $\mathrm{Map}(h,W)$ is a weak equivalence. The result follows by considering the case of $\mathscr{S}$–fibrant $W$. □

### 2.23 $k$–Types

Observe that $\mathrm{Sp}\approx s\mathrm{PSh}(1)^{\mathrm{inj}}$. For any integer $k\geq -2$, let

$$\mathrm{Sp}_k \overset{\text{def}}{=} s\mathrm{PSh}(1)^{\mathrm{inj}}_{\{\partial\Delta^{k+2}\to\Delta^{k+2}\}}.$$

This is called the model category of *$k$–types*. The fibrant objects are precisely the fibrant simplicial sets whose homotopy groups vanish in dimensions greater than $k$. This is a cartesian model category.

## 3 The $\Theta$ construction

The $\Theta$ construction was introduced by Berger in [4], where, with good cause, he calls it the "categorical wreath product over $\Delta$"; what we are calling $\Theta C$, he calls $\Delta\wr C$.

### 3.1 The category $\Delta$

We write $\Delta$ for the standard category of finite ordinals whose objects are $[m]=\{0,1,\ldots,m\}$ for $m\geq 0$ and morphisms are weakly monotone maps. We will use the following transparent notation to describe particular maps in $\Delta$; we write

$$\delta^{k_0k_1\cdots k_m}\colon [m]\to[n]$$

for the function defined by $i\mapsto k_i$.

We call a morphism $\delta\colon[m]\to[n]\in\Delta$ an *injection* or *surjection* if it is so as a map of sets. We say that $\delta$ is *sequential* if

$$\delta(i-1)+1\geq\delta(i)\qquad\text{for all } i=1,\ldots,m.$$

Observe that every surjection is sequential.

## 3.2 The category $\Theta C$

Let $C$ be a category. We define a new category $\Theta C$ as follows. The objects of $\Theta C$ are tuples of the form $([m], c_1, \dots, c_m)$, where $[m]$ is an object of $\Delta$ and $c_1, \dots, c_m$ are objects of $C$. It will be convenient to write $[m](c_1, \dots, c_m)$ for this object, and to write $[0]$ for the unique object with $m = 0$.

Morphisms $[m](c_1, \dots, c_m) \to [n](d_1, \dots, d_m)$ are tuples $(\delta, \{f_{ij}\})$ consisting of

(i) a morphism $\delta\colon [m] \to [n]$ of $\Delta$, and

(ii) for each pair $i, j$ of integers such that $1 \le i \le m$, $1 \le j \le n$, and $\delta(i-1) < j \le \delta(i)$, a morphism $f_{ij}\colon c_i \to d_j$ of $C$.

In other words,

$$(\Theta C)([m](c_1, \dots, c_m), [n](d_1, \dots, d_n)) \approx \coprod_{\delta\colon [m]\to[n]} \prod_{i=1}^{m} \prod_{j=\delta(i-1)+1}^{\delta(i)} C(c_i, d_j).$$

The composite

$$[m](c_1, \dots, c_m) \xrightarrow{(\delta, \{f_{ij}\})} [n](d_1, \dots, d_n) \xrightarrow{(\epsilon, \{g_{jk}\})} [p](e_1, \dots, e_p)$$

is the pair $(\epsilon\delta, \{h_{ik}\})$, where $h_{ik} = g_{jk} f_{ij}$ for the unique value of $j$ for which $f_{ij}$ and $g_{jk}$ are both defined.

Pictorially, it is convenient to represent an object of $\Theta C$ as a sequence of arrows labelled by objects of $C$. For instance, $[3](c_1, c_2, c_3)$ would be drawn

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} 2 \xrightarrow{c_3} 3.$$

An example of a morphism $[3](c_1, c_2, c_3) \to [4](d_1, d_2, d_3, d_4)$ is the picture

$$\begin{tikzcd}
 & 0 \arrow[r, "c_1"] \arrow[dl, dotted] \arrow[d, phantom, "f_{11}\;\; f_{12}"] & 1 \arrow[r, "c_2"] \arrow[d, dotted] & 2 \arrow[r, "c_3"] \arrow[dl, dotted] \arrow[d, phantom, "f_{33}"] & 3 \arrow[dl, dotted] \\
0 \arrow[r, "d_1"'] & 1 \arrow[r, "d_2"'] & 2 \arrow[r, "d_3"'] & 3 \arrow[r, "d_4"'] & 4
\end{tikzcd}$$

where the dotted arrows describe the map $\delta^{0223}\colon [3] \to [4]$, and the squiggly arrows represent morphisms $f_{11}\colon c_1 \to d_1$, $f_{12}\colon c_1 \to d_2$, $f_{33}\colon c_3 \to d_3$ in $C$.

Observe that (as suggested by our notation) there are functors

$$[m]\colon C^{\times m} \to \Theta C$$

for $m \geq 0$, which are defined in the evident way on objects, and which to a morphism $(g_i\colon c_i \to d_i)_{i=1,\dots,m}$ assign the morphism $(\mathrm{id}, \{f_{ij}\})$ where $f_{ii} = g_i$.

If $C$ is a small category, then so is $\Theta C$, and it is apparent that $\Theta$ describes a 2–functor $\mathrm{Cat} \to \mathrm{Cat}$.

## 3.3 A notation for morphisms in $\Theta C$

We use the following notation for certain maps in $\Theta C$. Suppose

$$(\delta, \{f_{ij}\})\colon [m](c_1, \dots, c_m) \to [n](d_1, \dots, d_n)$$

is a morphism in $\Theta C$ such that for each $i = 1, \dots, m$, the sequence of maps

$$(f_{ij}\colon c_i \to d_j)_{j=\delta(i-1)+1,\dots,\delta(i)}$$

identifies $c_i$ as the product of the $d_j$'s in $C$. Then we simply write $\delta$ for this morphism. Note that even if $C$ is a category which does not have all products, this notation is always sensible if $\delta \in \Delta$ is injective and sequential.

**3.4 Remark** If $C$ is a category with finite products, morphisms in $\Theta C$ amount to pairs $(\delta, \{f_i\}_{i=1,\dots,m})$, where

$$f_i\colon c_i \to d_{\delta(i-1)+1} \times \cdots \times d_{\delta(i)}.$$

In this case, our special notation is to write $\delta$ for $(\delta, \{\mathrm{id}\}_{i=1,\dots,m})$.

There is a variant of the $\Theta$ construction which works when $C$ is a monoidal category. If $C$ is a monoidal category, we can define a category $\Theta^{\mathrm{mon}} C$ with the same objects as $\Theta C$, but with morphisms $[m](c_1, \dots, c_m) \to [n](d_1, \dots, d_n)$ corresponding to tuples $(\delta, \{f_i\}_{i=1,\dots,m})$ where

$$f_i\colon c_i \to d_{\delta(i-1)+1} \otimes \cdots \otimes d_{\delta(i)}.$$

It seems likely that this variant notion should be useful for producing presentations of categories enriched over general monoidal model categories.

## 3.5 The categories $\Theta_n$

For $n \geq 0$ we define categories $\Theta_n$ by setting $\Theta_0 = 1$ (the terminal category), and defining $\Theta_n \stackrel{\text{def}}{=} \Theta\Theta_{n-1}$. One sees immediately that $\Theta_1$ is isomorphic to $\Delta$.

**3.6 Remark** The category $\Theta_n$ can be identified as a category of finite planar trees of level $\leq n$ [12]. The opposite category $\Theta_n^{\mathrm{op}}$ is isomorphic to the category of "combinatorial $n$–disks" in the sense of Joyal [12]; see Cheng [7, Chapter 7] and Berger [4].

## 3.7 $\Theta$ and enriched categories

If $V$ is a cartesian closed category and $\varnothing$ is an initial object of $V$, it is straightforward to show that

(1) for every object $v \in \operatorname{ob} V$, the product $\varnothing \times v$ is an initial object of $V$, and

(2) for an object $v \in \operatorname{ob} V$, the set $\hom_V(v, \varnothing)$ is nonempty if and only if $v$ is initial.

Suppose that $V$ is a cartesian closed category with a chosen initial object $\varnothing$. The *tautological functor*

$$\tau\colon \Theta V \to V\text{–Cat}$$

is defined as follows. For an object $[m](v_1, \ldots, v_m)$, we let $C = \tau([m](v_1, \ldots, v_m))$ be the $V$–category with object set $C_0 = \{0, 1, \ldots, m\}$, and with morphism objects

$$C(p,q) = \begin{cases} \varnothing & \text{if } p > q, \\ 1 & \text{if } p = q, \\ v_{p+1} \times \cdots \times v_q & \text{if } p < q. \end{cases}$$

The unique maps $1 \to C(p,p)$ define "identity maps", and composition $C(p,q) \times C(q,r) \to C(p,r)$ is defined in the evident way. It is clear how to define $\tau$ on morphisms.

**3.8 Remark** The functor $\tau$ is not fully faithful. For instance, there is a $V$–functor $f\colon \tau([1](\varnothing)) \to \tau([1](\varnothing))$ which on objects sends $0 \in [1]$ to $1 \in [1]$ and vice versa; this map $f$ is not in the image of $\tau$.

For a full subcategory $W$ of $V$, we will write $\tau\colon \Theta W \to V\text{–Cat}$ for the evident composite $\Theta W \to \Theta V \xrightarrow{\tau} V\text{–Cat}$.

**3.9 Proposition** [4, Proposition 3.5] *If $W$ is a full subcategory of $V$ which does not contain any initial objects of $V$, then $\tau\colon \Theta W \to V$–Cat is fully faithful.*

**Proof** The fact that only an initial object can map to an initial object in $V$ implies that for $c_i, d_j \in \operatorname{ob} W$, a functor $F\colon \tau([m](c_1, \ldots, c_m)) \to \tau([n](d_1, \ldots, d_n))$ is necessarily given on objects by a weakly monotone function $\delta\colon \{0, \ldots, m\} \to \{0, \ldots, n\}$. Given $\delta$, the functor $F$ determines and is determined by morphisms $f_{ij}\colon c_i \to d_j$ for $i = 1, \ldots, m$, $j = \delta(i-1)+1, \ldots, \delta(i)$. $\square$

**3.10 Corollary** *For each $n \geq 0$, the functor $\tau_n\colon \Theta_n \to$ **St**–$n$–**Cat** defined inductively as the composite*

$$\Theta_n \xrightarrow{\Theta\tau_{n-1}} \Theta(\mathbf{St}\text{–}(n\text{-}1)\text{–}\mathbf{Cat}) \xrightarrow{\tau} \mathbf{St}\text{–}n\text{–}\mathbf{Cat}$$

*is fully faithful.*

Thus, we can identify $\Theta_n$ with a full subcategory of **St**–$n$–**Cat**.

# 4 Presheaves of spaces over $\Theta C$

In the next few sections we will be especially concerned with the category $s\mathrm{PSh}(\Theta C)$ of simplicial presheaves on $\Theta C$. In this section we describe two essential constructions. First, we describe an adjoint pair of functors $(T_\#, T^*)$ between simplicial presheaves on $\Theta C$ and simplicial presheaves on $\Delta = \Theta 1$. Next, we describe a functor $V$, called the "intertwining functor", which relates $\Theta(s\mathrm{PSh}(C))$ and $s\mathrm{PSh}(\Theta C)$.

## 4.1 The functors $T^*$ and $T_\#$

Let $T\colon \Delta \to s\mathrm{PSh}(\Theta C)$ be the functor defined by

$$(T[n])([m](c_1,\ldots,c_m)) \overset{\text{def}}{=} \Delta([m],[n]),$$

Observe that if $C$ has a terminal object $t$, then $T[n] \approx F_{\Theta C}[n](t,\ldots,t)$.

Let $T^*\colon s\mathrm{PSh}(\Theta C) \to s\mathrm{PSh}(\Delta)$ denote the functor defined by

$$(T^*X)[m] \overset{\text{def}}{=} \mathrm{Map}_{s\mathrm{PSh}(\Theta C)}(T[m], X).$$

The functor $T^*$ preserves limits, and has a left adjoint $T_\#\colon s\mathrm{PSh}(\Delta) \to s\mathrm{PSh}(\Theta C)$.

**4.2 Proposition** *On objects $X$ in $s\mathrm{PSh}(\Delta)$, the object $T_\# X$ is given by*

$$(T_\# X)[m](c_1,\ldots,c_m) \approx X[m].$$

**Proof** A straightforward calculation. □

**4.3 Corollary** *The functor $T_\#\colon s\mathrm{PSh}(\Delta)^{\mathrm{inj}} \to s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}$ preserves small limits, cofibrations and weak equivalences; in particular, it is the left adjoint of a Quillen pair.*

We will regard $T^*X$ as the "underlying simplicial space" of the object $X$ in $s\mathrm{PSh}(\Theta C)$.

## 4.4 The intertwining functor $V$

The *intertwining functor*

$$V\colon \Theta(s\mathrm{PSh}(C)) \to s\mathrm{PSh}(\Theta C)$$

is a functor which extends the Yoneda functor $F_{\Theta C}\colon \Theta C \to s\mathrm{PSh}(\Theta C)$; it will play a crucial role in what follows.

Recall from Remark 3.4 that since $s\mathrm{PSh}(C)$ has finite products, a morphism

$$[m](A_1, \dots, A_m) \to [n](B_1, \dots, B_n)$$

in $\Theta(s\mathrm{PSh}(C))$ amounts to a pair $(\delta, \{f_j\}_{j=1,\dots,m})$ where $\delta\colon [m] \to [n]$ in $\Delta$, and

$$f_j\colon A_j \to \prod_{j=\delta(k-1)+1}^{\delta(k)} B_k \qquad \text{in } s\mathrm{PSh}(C).$$

On objects $[m](A_1, \dots, A_m)$ in $\Theta(s\mathrm{PSh}(C))$ the functor $V$ is defined by

$$\big(V[m](A_1, \dots, A_m)\big)([q](c_1, \dots, c_q)) = \coprod_{\delta\in\Delta([q],[m])} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} A_j(c_i).$$

To a morphism $(\sigma, \{f_j\})\colon [m](A_1, \dots, A_m) \to [n](B_1, \dots, B_n)$ we associate the map of presheaves defined by

$$\coprod_{\delta\in\Delta([q],[m])} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} A_j(c_i) \to \coprod_{\delta'\in\Delta([q],[n])} \prod_{i=1}^{q} \prod_{k=\delta'(i-1)+1}^{\delta'(i)} B_k(c_i)$$

which sends the summand associated to $\delta$ to the summand associated to $\delta' = \sigma\delta$ by a map which is a product of maps of the form $f_j(c_i)$.

Observe that

$$\begin{aligned}(V[m](Fd_1, \dots, Fd_m))([q](c_1, \dots, c_q)) &\approx \coprod_{\delta\colon [q]\to[m]} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} C(c_i, d_j) \\ &\approx (\Theta C)([q](c_1, \dots, c_q), [m](d_1, \dots, d_m)) \\ &\approx F_{\Theta C}[m](d_1, \dots, d_m)([q](c_1, \dots, c_q)).\end{aligned}$$

Thus we obtain a natural isomorphism $\nu\colon F_{\Theta C} \to V(\Theta F_C)$ of functors $\Theta C \to s\mathrm{PSh}(\Theta C)$.

In this paper, we are proposing the category $s\mathrm{PSh}(\Theta C)$ as a model for $s\mathrm{PSh}(C)$–enriched categories. In this light, the object $V[m](A_1, \dots, A_m)$ of $s\mathrm{PSh}(\Theta C)$ may be thought of as a model of the $s\mathrm{PSh}(C)$–enriched category freely generated by the $s\mathrm{PSh}(C)$–enriched graph

$$(0) \xrightarrow{A_1} (1) \xrightarrow{A_2} \cdots \xrightarrow{A_{m-1}} (m-1) \xrightarrow{A_m} (m).$$

The following proposition describes how the intertwining functor interacts with colimits. Recall that for an object $X$ of a category $C$, $A\backslash X$ denotes the slice category of objects under $X$ in $C$.

**4.5 Proposition** *The intertwining functor $V\colon \Theta(s\mathrm{PSh}(C)) \to s\mathrm{PSh}(\Theta C)$ has the following properties. Fix $m, n \geq 0$ and objects $A_1, \dots, A_m$, $B_1, \dots, B_n$ of $s\mathrm{PSh}(C)$.*

(1) *The map $(V\delta^{0,\dots,m}, V\delta^{m+1+1,\dots,m+1+n})$ which sends*

$$V[m](A_1, \dots, A_m) \amalg V[n](B_1, \dots, B_n) \to V[m+1+n](A_1, \dots, A_m, \varnothing, B_1, \dots, B_n)$$

*is an isomorphism.*

(2) *The functor*

$$X \mapsto V[m+1+n](A_1, \dots, A_m, X, B_1, \dots, B_n)\colon$$
$$s\mathrm{PSh}(C) \to V[m+1+n](A_1, \dots, A_m, \varnothing, B_1, \dots, B_n)\backslash s\mathrm{PSh}(\Theta C)$$

*is a left adjoint.*

**Proof** For $p = 0, \dots, q+1$, let $G(p) \subseteq \Delta([q], [m+1+n])$ be defined by

$$G(p) = \begin{cases} \{\, \delta \mid \delta(0) \geq m+1 \,\} & \text{if } p = 0, \\ \{\, \delta \mid \delta(p-1) \leq m,\ \delta(p) \geq m+1 \,\} & \text{if } 1 \leq p \leq q, \\ \{\, \delta \mid \delta(q) \leq m \,\} & \text{if } p = q+1. \end{cases}$$

Thus the $G(p)$ determine a partition of the set $\Delta([q], [m+1+n])$. The coproduct which defines $V[m](A_1, \dots, A_m, X, B_1, \dots, B_n)(\theta)$ for $\theta = [q](c_1, \dots, c_q)$ decomposes into factors according to this partition of $\Delta([q], [m+1+n])$. Under this decomposition, the factor corresponding to $p = 0$ is

$$\coprod_{\delta \in G(0)} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} B_{j-(m+1)}(c_i) \approx V[n](B_1, \dots, B_n)(\theta),$$

the factor corresponding to $p = q+1$ is

$$\coprod_{\delta \in G(q+1)} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} A_j(c_i) \approx V[m](A_1, \dots, A_m)(\theta),$$

while the factor corresponding to $p$ where $1 \le p \le q$ is

$$\coprod_{\delta \in G(p)} \left( \prod_{i=1}^{p} \prod_{j=\delta(i-1)+1}^{\min(\delta(i),m)} A_j(c_i) \right) \times X(c_p) \times \left( \prod_{i=p}^{q} \prod_{j=\max(\delta(i-1),m)+2}^{\delta(i)} B_{j-(m+1)}(c_i) \right).$$

From this claim (1) is immediate, as is the observation that the functor described in (2) preserves colimits, and so has a right adjoint. □

**4.6 Proposition** *For all $m, n \ge 0$ and objects $A_1, \ldots, A_m$, $B_1, \ldots, B_n$ in $s\mathrm{PSh}(C)$, the functor*

$$X \mapsto V[m+1+n](A_1, \ldots, A_m, X, B_1, \ldots, B_n)\colon$$
$$s\mathrm{PSh}(C) \to V[m+1+n](A_1, \ldots, A_m, \varnothing, B_1, \ldots, B_n)\backslash s\mathrm{PSh}(\Theta C)$$

*preserves cofibrations and weak equivalences, and thus is the left adjoint of a Quillen pair.*

**Proof** A straightforward calculation, using the decomposition given in the proof of Proposition 4.5. □

**4.7 Remark** It can be shown that $V$ is the left Kan extension of $F_{\Theta C}$ along $\Theta F_C$.

## 4.8 A product decomposition

We will need to make use of the following description of the product $V[1](A) \times V[1](B)$ in $s\mathrm{PSh}(\Theta C)$.

**4.9 Proposition** *The map*

$$\mathrm{colim}\big(V[2](A,B) \xleftarrow{V\delta^{02}} V[1](A \times B) \xrightarrow{V\delta^{02}} V[2](B,A)\big) \to V[1](A) \times V[1](B),$$

*induced by*

$$(V\delta^{011}, V\delta^{001})\colon V[2](A,B) \to V[1](A) \times V[1](B)$$

*and* $$(V\delta^{001}, V\delta^{011})\colon V[2](B,A) \to V[1](A) \times V[1](B),$$

*is an isomorphism.*

**Proof** This is a straightforward calculation. It may be helpful to contemplate the diagram

$$\begin{tikzcd} \bullet \arrow[r, "A"] \arrow[d, "B"'] \arrow[dr, "A\times B"] & \bullet \arrow[d, "B"] \\ \bullet \arrow[r, "A"'] & \bullet \end{tikzcd}$$

to grok the argument. □

## 4.10 Subobjects of $V[m](c_1,\dots,c_m)$

Observe that $V[m](1,\dots,1)\approx T_\# F[m]$; we write $\pi\colon V[m](A_1,\dots,A_m)\to T_\# F[m]$ for the map induced by projection to the terminal object in $s\mathrm{PSh}(C)$. Given a subobject $f\colon K\subseteq F[m]$ in $s\mathrm{PSh}(\Delta)$, we define $V_K(A_1,\dots,A_m)$ to be the inverse limit of the diagram

$$V[m](A_1,\dots,A_m)\xrightarrow{\pi} T_\# F[m]\xleftarrow{T_\# f} T_\# K.$$

Explicitly,

$$V_K(A_1,\dots,A_m)([q](c_1,\dots,c_q))$$

is the subobject of

$$V[m](A_1,\dots,A_m)([q](c_1,\dots,c_q))$$

coming from summands associated to $\delta\colon [q]\to[m]$ such that $F\delta\colon F[q]\to F[m]$ factors through $K\subseteq F[m]$ in $s\mathrm{PSh}(\Delta)$. Note that $V_{F[m]}(A_1,\dots,A_m)\approx V[m](A_1,\dots,A_m)$.

These subobjects will be used in Section 6.

## 4.11 Mapping objects

Given an object $X$ in $s\mathrm{PSh}(\Theta C)$, an ordered sequence $x_0,\dots,x_m$ of points in $X[0]$, and a sequence $c_1,\dots,c_m\in\mathrm{ob}\,C$, we define $M_X(x_0,\dots,x_m)(c_1,\dots,c_m)$ to be the pullback of the diagram

$$\{(x_0,\dots,x_m)\}\to X[0]^{\times m+1}\xleftarrow{(X\delta^0,\dots,X\delta^m)} X[m](c_1,\dots,c_m).$$

Allowing the objects $c_1,\dots,c_m$ to vary gives us a presheaf $M_X(x_0,\dots,x_m)$ in $s\mathrm{PSh}(C^{\times m})$. We will be especially interested in $M_X(x_0,x_1)$, an object of $s\mathrm{PSh}(C)$, which we will refer to as a *mapping object* for $X$.

We can use the intertwining functor $V$ to get a fancier version of the mapping objects, as follows. Again, given $X$ in $s\mathrm{PSh}(\Theta C)$ and $x_0,\dots,x_m\in X[0]$, and also given objects $A_1,\dots,A_m$ in $s\mathrm{PSh}(C)$, we define $\widetilde{M}_X(x_0,\dots,x_q)(A_1,\dots,A_q)$ to be the pullback of the diagram

$$\begin{aligned}&\{(x_0,\dots,x_m)\}\to X[0]^{\times m+1}\\ &\qquad\approx \mathrm{Map}(V[m](\varnothing,\dots,\varnothing),X)\leftarrow \mathrm{Map}(V[m](A_1,\dots,A_m),X)\end{aligned}$$

where the right-hand map is induced by the maps $X\delta^i_*$. Allowing the objects $A_1,\dots,A_m$ to vary gives us a functor $\widetilde{M}_X(x_0,\dots,x_m)\colon s\mathrm{PSh}(C)^{\times m}\to s\mathrm{PSh}(\Theta C)$. Observe that

$$M_X(x_0,\dots,x_m)(c_1,\dots,c_m)\approx \widetilde{M}_X(x_0,\dots,x_m)(Fc_1,\dots,Fc_m),$$

and that

$$\widetilde{M}_X(x_0,\dots,x_{m+1+n})(A_1,\dots,A_m,\varnothing,B_1,\dots,B_n) \\ \approx \widetilde{M}_X(x_0,\dots,x_m)(A_1,\dots,A_m)\times\widetilde{M}_X(x_{m+1+1},\dots,x_{m+1+n})(B_1,\dots,B_n).$$

**4.12 Lemma** *There is a natural isomorphism*

$$\widetilde{M}_X(x_0,x_1)(A)\approx \operatorname{Map}_C(A,M_X(x_0,x_1)).$$

**Proof** This follows using the natural isomorphisms

$$\widetilde{M}_X(x_0,x_1)(Fc)\approx M_X(x_0,x_1)(c)\approx \operatorname{Map}_C(Fc,M_X(x_0,x_1))$$

and the fact that $V[1]\colon s\mathrm{PSh}(C)\to V[1](\varnothing)\backslash s\mathrm{PSh}(\Theta C)$ preserves colimits Proposition 4.5, which implies that $A\mapsto \widetilde{M}_X(x_0,x_1)(A)$ takes colimits to limits. □

# 5 Segal objects

In this section, we examine the properties of a certain class of objects in $s\mathrm{PSh}(\Theta C)$, called *Segal objects*. In the case that $C=1$, these are the *Segal spaces* of [16]. We work with a fixed small category $C$.

## 5.1 Segal maps and Segal objects

Let $\mathrm{Se}_C$ denote the set of morphisms in $s\mathrm{PSh}(\Theta C)$ of the form

$$\mathrm{se}^{(c_1,\dots,c_m)}\overset{\text{def}}{=}(F\delta^{01},\dots,F\delta^{m-1,m})\colon G[m](c_1,\dots,c_m)\to F[m](c_1,\dots,c_m)$$

where

$$G[m](c_1,\dots,c_m)\overset{\text{def}}{=}\operatorname{colim}\big(F[1](c_1)\xleftarrow{F\delta^1}F[0]\xrightarrow{F\delta^0}\cdots\xleftarrow{F\delta^1}F[0]\xrightarrow{F\delta^0}F[1](c_m)\big)$$

for $m\geq 2$ and $c_1,\dots,c_m\in\operatorname{ob}C$. It is straightforward to check that an injective fibrant $X$ in $s\mathrm{PSh}(\Theta C)$ is $\mathrm{Se}_C$–fibrant if and only if each of the induced maps

$$X[m](c_1,\dots,c_m)\to\lim\big(X[1](c_1)\xrightarrow{X\delta^1}X[0]\xleftarrow{X\delta^0}\cdots\xrightarrow{X\delta^1}X[0]\xleftarrow{X\delta^0}X[1](c_m)\big)$$

is a weak equivalence. Equivalently, an injective fibrant $X$ is $\mathrm{Se}_C$–fibrant if and only if the evident maps

$$M_X(x_0,\dots,x_m)(c_1,\dots,c_m)\to M_X(x_0,x_1)(c_1)\times\cdots\times M_X(x_{m-1},x_m)(c_m)$$

are weak equivalences.

A *Segal object* is a $\mathrm{Se}_C$–fibrant object in $s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}$, that is, a fibrant object in $s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}_{\mathrm{Se}_C}$.

## 5.2 More maps of Segal type

For an object $[m](A_1,\dots,A_m)$ of $\Theta(s\mathrm{PSh}(C))$, we obtain a map in $s\mathrm{PSh}(\Theta C)$ of the form

$$\mathrm{se}^{[m](A_1,\dots,A_m)}\colon V_{G[m]}(A_1,\dots,A_m)\to V[m](A_1,\dots,A_m).$$

induced by $\mathrm{se}^{(1,\dots,1)}\colon G[m]\to F[m]$ in $s\mathrm{PSh}(\Delta)$, where $V_{G[m]}$ is as defined in Section 4.10. Observe that

$$V_{G[m]}(A_1,\dots,A_m)\approx \operatorname{colim}\Bigl(V[1](A_1)\xleftarrow{\delta^1}V[0]\xrightarrow{\delta^0}\cdots\xleftarrow{\delta^1}V[0]\xrightarrow{\delta^0}V[1](A_m)\Bigr).$$

Note also that if $\delta\colon[m]\to[n]$ in $\Delta$ is injective and sequential, then $F\delta\colon F[m]\to F[n]$ carries $G[m]$ into $G[n]$, and thus we obtain an induced map

$$\delta_*\colon V_{G[m]}(A_{\delta(1)},\dots,A_{\delta(m)})\to V_{G[n]}(A_1,\dots,A_m).$$

It is straightforward to check that $V_{G[m]}\colon s\mathrm{PSh}(C)^{\times m}\to s\mathrm{PSh}(\Theta C)$ satisfies formal properties similar to $V[m]\colon s\mathrm{PSh}(C)^{\times m}\to s\mathrm{PSh}(\Theta C)$. Namely,

(1) for all $A_1,\dots,A_m,B_1,\dots,B_n$ objects of $s\mathrm{PSh}(C)$, The map

$$(\delta_*^{0,\dots,m},\delta_*^{m+1+1,\dots,m+1+n})\colon V_{G[m]}(A_1,\dots,A_m)\amalg V_{G[n]}(B_1,\dots,B_n)\\ \to V_{G[m+1+n]}(A_1,\dots,A_m,\varnothing,B_1,\dots,B_n)$$

is an isomorphism, and

(2) for all $m,n\geq 0$ and objects $A_1,\dots,A_m,B_1,\dots,B_n$ in $s\mathrm{PSh}(C)$, the functor

$$X\mapsto V_{G[m+1+n]}(A_1,\dots,A_m,X,B_1,\dots,B_n)\colon\\ s\mathrm{PSh}(C)\to V_{G[m+1+n]}(A_1,\dots,A_m,\varnothing,B_1,\dots,B_n)\backslash s\mathrm{PSh}(\Theta C)$$

is a left Quillen functor.

We record the following fact.

**5.3 Proposition** *For all objects $[m](A_1,\dots,A_m)$ of $\Theta(s\mathrm{PSh}(C))$, we have*

$$\mathrm{se}^{[m](A_1,\dots,A_m)}\in\overline{\mathrm{Se}}_C,$$

*where $\overline{\mathrm{Se}}_C$ is the class of $\mathrm{Se}_C$–local equivalences.*

**Proof** We prove this by induction on $m\geq 0$. Let $\mathcal{D}$ denote the class of objects $[m](A_1,\dots,A_m)$ in $\Theta(s\mathrm{PSh}(C))$ such that $\mathrm{se}^{[m](A_1,\dots,A_m)}\in\overline{\mathrm{Se}}_C$.

We observe the following.

(1) All objects of the form $[0]$ and $[1](A)$ are in $\mathcal{D}$, since $\mathrm{se}^{[0]}$ and $\mathrm{se}^{[1](A)}$ are isomorphisms.

(2) All objects of the form $[m](Fc_1,\dots,Fc_m)$ for all $c_1,\dots,c_m \in \mathrm{ob}\, C$ are in $\mathcal{D}$, since $\mathrm{se}^{[m](Fc_1,\dots,Fc_m)} = \mathrm{se}^{(c_1,\dots,c_m)}$.

For $m \geq 1$ and $0 \leq j \leq m$, let $\mathcal{E}_{m,j}$ denote the class of objects of the form

$$[m](A_1,\dots,A_j,Fc_{j+1},\dots,Fc_m) \quad \text{or} \quad [m](A_1,\dots,A_j,\varnothing,Fc_{j+2},\dots,Fc_m),$$

where $A_1,\dots,A_j$ in $s\mathrm{PSh}(C)$ and $c_{j+1},\dots,c_m \in \mathrm{ob}\, C$. We need to prove that $\mathcal{E}_{m,m} \subseteq \mathcal{D}$ for all $m$. Observation (1) says that this is so for $m = 0$ and $m = 1$, while observation (2) says that $\mathcal{E}_{m,0} \subseteq \mathcal{D}$ for all $m$. The proof will be completed once we show that for all $m \geq 2$ and $1 \leq j \leq m$, $\mathcal{E}_{m,j-1} \subseteq \mathcal{D}$ implies $\mathcal{E}_{m,j} \subseteq \mathcal{D}$.

Consider the transformation $\alpha\colon G \to H$ of functors $s\mathrm{PSh}(C) \to s\mathrm{PSh}(\Theta C)$ defined by the evident inclusion

$$\alpha\colon V_{G[m]}(A_1,\dots,A_{j-1},-,Fc_{j+1},\dots,Fc_m) \\ \to V[m](A_1,\dots,A_{j-1},-,Fc_{j+1},\dots,Fc_m).$$

The functors $G$ and $H$ produce left Quillen functors $s\mathrm{PSh}(C)^{\mathrm{inj}} \to G(\varnothing)\backslash s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}$ and $s\mathrm{PSh}(C)^{\mathrm{inj}} \to H(\varnothing)\backslash s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}$, and it is clear from the explicit description of $V$ that $\alpha(\varnothing)$ is a monomorphism. Since $\mathcal{E}_{m,j-1} \subseteq \mathcal{D}$, we have that $\alpha(\varnothing) \in \overline{\mathrm{Se}}_C$ and $\alpha(Fc_j) \in \overline{\mathrm{Se}}_C$ for all objects $c_j$ of $C$. Thus Proposition 2.20 shows that $\alpha(A_j) \in \overline{\mathrm{Se}}_C$ for all objects $A_j$ of $s\mathrm{PSh}(C)$, and thus $[m](A_1,\dots,A_j,Fc_{j+1},\cdots,Fc_m) \in \mathcal{D}$; that is, $\mathcal{E}_{m,j} \subseteq \mathcal{D}$, as desired. □

**5.4 Corollary** *Let $X$ be a $\mathrm{Se}_C$–fibrant object of $s\mathrm{PSh}(\Theta C)$, let $x_0,\dots,x_m \in X[0]$, and let $A_1,\dots,A_m$ be objects of $s\mathrm{PSh}(C)$. Then the map*

$$\mathrm{Map}_{\Theta C}(V[m](A_1,\dots,A_m),X) \\ \to \mathrm{Map}_{\Theta C}(V[1](A_1),X) \times_{X[0]} \cdots \times_{X[0]} \mathrm{Map}_{\Theta C}(V[1](A_m),X)$$

*induced by $V\delta^{i-1,i}$ for $1 \leq i \leq m$ is a weak equivalence, and the map*

$$\widetilde{M}_X(x_0,\dots,x_m)(A_1,\dots,A_m) \to \widetilde{M}_X(x_0,x_1)(A_1) \times \cdots \times \widetilde{M}_X(x_{m-1},x_m)(A_m)$$

*is a weak equivalence.*

# 6 $(\Theta C, \mathrm{Se}_C)$ is a cartesian presentation

We now prove the following result.

**6.1 Proposition** *For any small category $C$, the pair $(\Theta C, \mathrm{Se}_C)$ is a cartesian presentation, and thus $s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathrm{Se}_C}$ is a cartesian model category.*

Our proof is an adaptation of the proof we gave in [16, Section 10] for the case $C = 1$; it follows after Proposition 6.6 below.

## 6.2 Covers

Let $[m]$ be an object of $\Delta$, and let $K \subseteq F[m]$ be a subobject in $s\mathrm{PSh}(\Delta)$. We say that $K$ is a *cover* of $F[m]$ if

(i) for all sequential $\delta\colon [1] \to [m]$, the map $F\delta\colon F[1] \to F[m]$ factors through $K$, and

(ii) there exists a (necessarily unique) dotted arrow making the diagram commute in every diagram of the form:

$$\begin{tikzcd} F[1] \arrow[r] \arrow[d, "F\delta^{0n}"'] & K \arrow[d] \\ F[n] \arrow[r] \arrow[ur, dotted] & F[m] \end{tikzcd}$$

It is immediate that

(0) the identity map id: $F[m] \to F[m]$ is a cover,

(1) the subobject $G[m] \subseteq F[m]$ generated by the images of the maps $F\delta^{i-1,i}\colon F[1] \to F[m]$ is a cover (called the *minimal cover*),

(2) if $\delta\colon [p] \to [m]$ is sequential, and $K \subseteq F[m]$ is a cover, then the pullback $\delta^{-1}K \subseteq F[p]$ of $K$ along $F\delta$ is a cover of $F[p]$, and

(3) if $\delta\colon [p] \to [m]$ and $\delta'\colon [p] \to [n]$ are sequential, and $M \subseteq F[m]$ and $N \subseteq F[n]$ are covers, then the pullback $(\delta, \delta')^{-1}(M \times N)$ of $M \times N$ along $(F\delta, F\delta')\colon F[p] \to F[m] \times F[n]$ is a cover of $F[p]$.

## 6.3 Covers produce $\mathrm{Se}_C$–equivalences

Recall that given a subobject $K \subseteq F[m]$ in $s\mathrm{PSh}(\Delta)$, and a sequence $A_1, \dots, A_m$ of $s\mathrm{PSh}(C)$, we have defined (in Section 4.10) a subobject $V_K(A_1, \dots, A_m)$ of $V[m](A_1, \dots, A_m)$ in $s\mathrm{PSh}(\Theta C)$.

**6.4 Proposition** *If $K \subseteq F[m]$ is a cover in $s\mathrm{PSh}(\Delta)$, then $V_K(A_1, \dots, A_m) \to V[m](A_1, \dots, A_m)$ is in $\overline{\mathrm{Se}}_C$ for all $A_1, \dots, A_m$ objects of $s\mathrm{PSh}(C)$.*

**Proof** Since $V_{G[m]}(A_1, \dots, A_m) \to V[m](A_1, \dots, A_m)$ is in $\overline{\mathrm{Se}}_C$ by Proposition 5.3, it suffices to show that $V_{G[m]}(A_1, \dots, A_m) \to V_K(A_1, \dots, A_m)$ is in $\overline{\mathrm{Se}}_C$ for covers $K \subset F[m]$ which are proper inclusions. We will prove this using induction on $m$.

Given a subobject $K \subseteq F[m]$ in $s\mathrm{PSh}(\Delta)$, let $\mathcal{P}_K$ denote the category whose objects are injective sequential maps $\delta\colon [p] \to [m]$ such that $F\delta$ factors through $K$, and whose morphisms $([p] \to [m]) \to ([p'] \to [m])$ are arrows $[p] \to [p']$ in $\Delta$ making the evident triangle commute. The category $\mathcal{P}_K$ is a poset. For each $\delta\colon [p] \to [m] \in \mathcal{P}_K$ we have a natural square:

$$\begin{array}{ccc} V_{\delta^{-1}K}(A_1, \dots, A_m) & \longrightarrow & V_{G[m]}(A_1, \dots, A_m) \\ \downarrow & & \downarrow \\ V_{F[p]}(A_1, \dots, A_m) & \longrightarrow & V_K(A_1, \dots, A_m) \end{array}$$

Observe that since $\delta$ is a monomorphism, the map $F[p] \to F[m]$ is a monomorphism; we have abused notation and written $F[p]$ for this subobject.

We have that the maps

$$\mathrm{hocolim}_{\mathcal{P}_K} V_{\delta^{-1}K}(A_1, \dots, A_m) \to V_{G[m]}(A_1, \dots, A_m)$$

and

$$\mathrm{hocolim}_{\mathcal{P}_K} V_{F[p]}(A_1, \dots, A_m) \to V_K(A_1, \dots, A_m)$$

are levelwise weak equivalences in $s\mathrm{PSh}(\Theta C)$ by Proposition 2.19, since the corresponding maps from colimits over $\mathcal{P}_K$ are isomorphisms. Since the inclusion $K \subset F[m]$ is proper, $p < m$ for all objects of $\mathcal{P}_K$, and so each $V_{\delta^{-1}K}(A_1, \dots, A_m) \to V_{F[p]}(A_1, \dots, A_m) \in \overline{\mathrm{Se}}_C$ by the induction hypothesis, the result follows using Proposition 2.18. $\square$

## 6.5 Proof that $(\Theta C, \mathrm{Se}_C)$ is cartesian

**6.6 Proposition** *If $M \subseteq F[m]$ and $N \subseteq F[n]$ are covers in $s\mathrm{PSh}(\Delta)$, then*

$$V_M(A_1, \dots, A_m) \times V_N(B_1, \dots, B_n) \to V[m](A_1, \dots, A_m) \times V[n](B_1, \dots, B_m)$$

*is an $\mathrm{Se}_C$–equivalence.*

**Proof** Let $\mathcal{Q}_{m,n}$ denote the category whose objects are pairs of maps $(\delta\colon [p]\to[m], \delta'\colon [p]\to[n])$ in $\Delta$ such that $\delta$ and $\delta'$ are surjective (and thus sequential), and $(F\delta, F\delta')\colon F[p]\to F[m]\times F[n]$ is a monomorphism. The category $\mathcal{Q}_{m,n}$ is a poset, and we have $\operatorname{colim}_{\mathcal{Q}_{m,n}} F[p]\to F[m]\times F[n]$ is an isomorphism in $s\mathrm{PSh}(\Delta)$. For each object $(\delta,\delta')$ of $\mathcal{Q}_{m,n}$ there is a natural square

$$\begin{tikzcd} V_{(\delta,\delta')^{-1}M\times N}(C_1,\dots,C_p) \arrow[r] \arrow[d] & V_M(A_1,\dots,A_m)\times V_N(B_1,\dots,B_n) \arrow[d] \\ V[p](C_1,\dots,C_p) \arrow[r] & V[m](A_1,\dots,A_m)\times V[n](B_1,\dots,B_n) \end{tikzcd}$$

where $C_i = A_{\delta(i)}$ or $B_{\delta'(i)}$ according to whether $\delta(i)>\delta(i-1)$ or $\delta'(i)>\delta'(i-1)$. We have that $\operatorname{hocolim}_{\mathcal{Q}_{m,n}} V_{(\delta,\delta')^{-1}M\times N}\to V_M\times V_N$ and $\operatorname{hocolim}_{\mathcal{Q}_{m,n}} V_{F[p]}\to V_{F[m]}\times V_{F[n]}$ are levelwise weak equivalences in $s\mathrm{PSh}(\Theta C)$ by Proposition 2.19. By Proposition 6.4, each of the maps $V_{(\delta,\delta')^{-1}M\times N}\to V_{F[p]}$ is in $\overline{\mathrm{Se}}_C$, and thus the result follows using Proposition 2.18. □

Now we can give the proof of the proposition stated at the beginning of the section.

**Proof of Proposition 6.1** To prove that $(\Theta C, \mathrm{Se}_C)$ is cartesian, it suffices to show that $\mathrm{se}^{(c_1,\dots,c_m)}\times F\theta\colon G[m](c_1,\dots,c_m)\times F\theta\to F[m](c_1,\dots,c_m)\times F\theta$ is in $\overline{\mathrm{Se}}_C$ for all $m\geq 2$, $c_1,\dots,c_m\in \operatorname{ob} C$ and $\theta\in\operatorname{ob}\Theta C$. This is a special case of Proposition 6.6. □

## 6.7 Presentations of the form $(\Theta C, \mathrm{Se}_C\cup\mathcal{U})$

Let $\mathcal{U}$ be a set of morphisms in $s\mathrm{PSh}(\Theta C)$.

**6.8 Proposition** *The presentation $(\Theta C, \mathrm{Se}_C\cup\mathcal{U})$ is cartesian if and only if for all $X$ in $s\mathrm{PSh}(\Theta C)$ which are $(\mathrm{Se}_C\cup\mathcal{U})$–fibrant, and for all $c\in\operatorname{ob} C$, the function object $X^{F[1](c)}$ is $\mathcal{U}$–local.*

**Proof** By Proposition 2.11 and Proposition 6.1, it is enough to show that if $X$ is $(\mathrm{Se}_C\cup\mathcal{U})$–fibrant, then $X^{F\theta}$ is $\mathcal{U}$–local for all $\theta\in\operatorname{ob}\Theta C$. Since $X$ is $\mathrm{Se}_C$–local, every $X^{F\theta}$ is weakly equivalent to a homotopy fiber product of the form

$$X^{F[1](c_1)}\times_X\cdots\times_X X^{F[1](c_m)},$$

and thus the result follows. □

# 7 Complete Segal objects

In this section, we examine the properties of a certain class of Segal objects in $s\mathrm{PSh}(\Theta C)$, called *complete Segal objects*. In the case that $C = 1$, these are precisely the *complete Segal spaces* of [16]. We show below that complete Segal objects are the fibrant objects of a cartesian model category, generalizing a result of [16, Section 12].

## 7.1 The underlying Segal space of a Segal object

Recall the Quillen pair $T_\#\colon s\mathrm{PSh}(\Delta) \rightleftarrows s\mathrm{PSh}(\Theta C) : T^*$ of Section 4.1. Given an object $X$ of $s\mathrm{PSh}(\Theta C)$, we will call $T^*X$ in $s\mathrm{PSh}(\Delta)$ its *underlying simplicial space*; according to the following proposition, it is reasonable to call $T^*X$ the *underlying Segal space* of $X$ if $X$ is itself a Segal object.

**7.2 Proposition** *If $X$ is an $\mathrm{Se}_C$–fibrant object in $s\mathrm{PSh}(\Theta C)$, then $T^*X$ is an $\mathrm{Se}_1$–fibrant object in $s\mathrm{PSh}(\Theta C) = s\mathrm{PSh}(\Delta)$. That is, $T^*X$ is a Segal space in the sense of [16].*

**Proof** The map

$$\mathrm{se}^{[m](1,\dots,1)}\colon \operatorname{colim}\big(V[1](1) \leftarrow V[0] \rightarrow \cdots \leftarrow V[0] \rightarrow V[1](1)\big) \rightarrow V[m](1,\dots,1)$$

is isomorphic to $T_\#\mathrm{se}^{(1,\dots,1)}\colon T_\#G[m] \rightarrow T_\#F[m]$. □

## 7.3 The homotopy category of a Segal object

Recall that if $X$ in $s\mathrm{PSh}(\Delta)$ is a Segal space, then we define its homotopy category $hX$ as follows. The objects of $hX$ are points of $X[0]$, and morphisms are given by

$$hX(x_0,x_1) \stackrel{\text{def}}{=} \pi_0 M_X(x_0,x_1).$$

It is shown in [16] that $hX$ is indeed a category; composition is defined and its properties are verified using the isomorphisms $\pi_0 M_X(x_0,\dots,x_m) \approx hX(x_0,x_1) \times \cdots \times hX(x_{m-1},x_m)$ which hold for a Segal space.

For an $\mathrm{Se}_C$–fibrant object $X$ in $s\mathrm{PSh}(\Theta C)$, we define its *homotopy category* $hX$ to be the homotopy category of $T^*X$. Explicitly, objects of $hX$ are points in $X[0]$, and morphisms are

$$hX(x_0,x_1) \stackrel{\text{def}}{=} \pi_0 \Gamma_C M_X(x_0,x_1).$$

## 7.4 The enriched homotopy category of a Segal object

The homotopy category $hX$ described above can be refined to a homotopy category enriched over the homotopy category $hs\mathrm{PSh}(C)$ of presheaves of spaces on $C$. This *$hs\mathrm{PSh}(C)$–enriched homotopy category* is denoted $\underline{hX}$, and is defined as follows.

We take $\operatorname{ob}\underline{hX} = \operatorname{ob} hX$. Given objects $x_0, x_1$ of $hX$, recall that the *function object* of maps $x_0 \to x_1$ is the object $M_X(x_0, x_1)$ of $s\mathrm{PSh}(C)$. For objects $x_0, x_1$ in $hX$, we set $\underline{hX}(x_0, x_1) \stackrel{\text{def}}{=} M_X(x_0, x_1)$ viewed as an object in the homotopy category $hs\mathrm{PSh}(C)$. To make this a category, let $\Delta_m \colon C \to C^{\times m}$ denote the "diagonal" functor, and let $\Delta_m^* \colon s\mathrm{PSh}(C^{\times m}) \to s\mathrm{PSh}(C)$ denote the functor which sends $F \mapsto F\Delta_m$. The functor $\Delta_m^*$ preserves weak equivalences and products. Observe that since $X$ is a Segal object, there are evident weak equivalences

$$\Delta_m^* M_X(x_0, \dots, x_m) \to M_X(x_0, x_1) \times \cdots \times M_X(x_{m-1}, x_m)$$

in $s\mathrm{PSh}(C)$. Thus we obtain "identity" and "composition" maps

$$1 \approx \Delta_0^* M_X(x_0) \to M_X(x_0, x_0),$$

$$M_X(x_0, x_1) \times M_X(x_1, x_2) \xleftarrow{\sim} \Delta_2^* M_X(x_0, x_1, x_2) \to M_X(x_0, x_2)$$

in $hs\mathrm{PSh}(C)$, and it is straightforward to check that these make $\underline{hX}$ into an $hs\mathrm{PSh}(C)$–enriched category. Furthermore, we see that $hs\mathrm{PSh}(C)(1, \underline{hX}(x_0, x_1)) \approx hX(x_0, x_1)$.

## 7.5 Equivalences in a Segal object

Recall that if $X$ in $s\mathrm{PSh}(\Delta)$ is a Segal space, then we say that a point in $X[1]$ is an *equivalence* if it projects to an isomorphism in the homotopy category $hX$. We write $M_X^{\mathrm{equiv}}(x_0, x_1)$ for the subspace of $M_X(x_0, x_1)$ consisting of path components which project to isomorphisms in $hX$, and we let $X^{\mathrm{equiv}}$ denote the subspace of $X[1]$ consisting of path components which contain points from $M_X^{\mathrm{equiv}}(x_0, x_1)$ for some $x_0, x_1 \in \operatorname{ob} hX$. Thus $M_X^{\mathrm{equiv}}(x_0, x_1)$ is the fiber of $(X\delta^0, X\delta^1) \colon X^{\mathrm{equiv}} \to X[0] \times X[0]$ over $(x_0, x_1)$; observe that the map $X\delta^{00} \colon X[0] \to X[1]$ factors through $X^{\mathrm{equiv}} \subseteq X[1]$.

These definitions transfer to Segal objects. Thus, if $X$ is a Segal object in $s\mathrm{PSh}(\Theta C)$, we say that a point in $\Gamma_C M_X(x_1, x_2)$ is an *equivalence* if it projects to an isomorphism in the homotopy category $hX$. We define $M_X^{\mathrm{equiv}}(x_0, x_1)$ to be the subspace of $\Gamma_C M_X(x_0, x_1)$ consisting of path components which project to isomorphisms in $hX$. The space of equivalences $X^{\mathrm{equiv}}$ is defined to be the subspace of $\Gamma_C X[1]$ consisting of path components which contain points from $M_X^{\mathrm{equiv}}(x_0, x_1)$ for some $x_0, x_1 \in X[0]$; thus $M_X^{\mathrm{equiv}}(x_0, x_1)$ is the fiber of $X^{\mathrm{equiv}} \to X[0] \times X[0]$ over $(x_0, x_1)$. Observe that the map $X\delta^{00} \colon X[0] \to \Gamma_C X[1]$ factors through $X^{\mathrm{equiv}} \subseteq \Gamma_C X[1]$.

## 7.6 The set $\mathrm{Cpt}_C$

Let $E$ be the object in $s\mathrm{PSh}(\Delta)$ which is the "discrete nerve of the free-standing isomorphism", as in [16, Section 6]. Let $p\colon E \to F[0]$ be the evident projection map, and let $i\colon F[1] \to E$ be the inclusion of one of the nonidentity arrows. We recall the following result.

**7.7 Proposition** [16, Theorem 6.2] *If $X$ in $s\mathrm{PSh}(\Delta)$ is a Segal space, then the map*

$$\mathrm{Map}(i, X)\colon \mathrm{Map}(E, X) \to \mathrm{Map}(F[1], X) \approx X[1]$$

*factors through $X^{\mathrm{equiv}} \subseteq X[1]$ and induces a weak equivalence $\mathrm{Map}(E, X) \to X^{\mathrm{equiv}}$.*

We define $\mathrm{Cpt}_C$ to be the set consisting of the single map

$$T_\# p\colon T_\# E \to T_\# F[0].$$

We say that $X$ in $s\mathrm{PSh}(C)$ is a *complete Segal object* if it is $(\mathrm{Se}_C \cup \mathrm{Cpt}_C)$–fibrant. As a consequence of Proposition 7.7, we have the following.

**7.8 Proposition** *Let $X$ be a Segal object of $s\mathrm{PSh}(\Theta C)$. The map*

$$\mathrm{Map}(T_\# E, X) \to \mathrm{Map}(T_\# F[1], X) \approx T^* X[1]$$

*factors through $X^{\mathrm{equiv}} \subseteq T^* X[1]$, and induces a weak equivalence $\mathrm{Map}(T_\# E, X) \to X^{\mathrm{equiv}}$ of spaces. Thus, a Segal object $X$ is a complete Segal object if and only if $X[0] \to X^{\mathrm{equiv}}$ is a weak equivalence of spaces.*

**7.9 Remark** In Section 10 we give another formulation of the completeness condition, in which the simplicial space $E$ is replaced by a smaller one $Z$, so that variants of Proposition 7.7 and Proposition 7.8 hold with $E$ replaced by $Z$. Either formulation works just as well for our purposes.

## 7.10 Fully faithful maps

If $X$ and $Y$ are Segal objects in $s\mathrm{PSh}(\Theta C)$, we say that a map $f\colon X \to Y$ is *fully faithful* if for all $c \in \mathrm{ob}\, C$ the square

$$\begin{tikzcd} X[1](c) \arrow[r] \arrow[d] & Y[1](c) \arrow[d] \\ X[0] \times X[0] \arrow[r] & Y[0] \times Y[0] \end{tikzcd}$$

is a homotopy pullback square.

**7.11 Proposition** *Let $f\colon X \to Y$ be a map between Segal objects in $s\mathrm{PSh}(\Theta C)$. The following are equivalent.*

(1) *$f$ is fully faithful.*

(2) *For all $c \in \mathrm{ob}\, C$ and all $x_0, x_1$ points of $X[0]$, the map $M_X(x_0, x_1)(c) \to M_X(fx_0, fx_1)(c)$ induced by $f$ is a weak equivalence of spaces.*

(3) *The induced map $\underline{hX} \to \underline{hY}$ of enriched homotopy categories is fully faithful, ie, $\underline{hX}(x_0, x_1) \to \underline{hY}(x_0, x_1)$ is an isomorphism in $hs\mathrm{PSh}(C)$ for all points $x_0, x_1$ of $X[0]$.*

**7.12 Proposition** *Suppose $X$ is a Segal object in $s\mathrm{PSh}(C)$. Then the map $X \approx X^{T_\# F[0]} \to X^{T_\# F[1]}$ induced by $T_\# F\delta^{00}$ is fully faithful.*

**Proof** Observe that $T_\# F[1] \approx V[1](1)$, and that the statement will be proved if we can show that for all $c \in \mathrm{ob}\, C$, the square obtained by applying $\mathrm{Map}_{\Theta C}(-, X)$ to the square

$$\begin{tikzcd}
V[1](Fc) & V[1](Fc) \times V[1](1) \arrow[l, "\mathrm{proj}"'] \\
V[1](\varnothing) \arrow[u, "V[1](\mathrm{incl})"] & V[1](\varnothing) \times V[1](1) \arrow[l, "\mathrm{proj}"] \arrow[u, "V[1](\mathrm{incl})\times \mathrm{id}"']
\end{tikzcd}$$

is a homotopy pullback of spaces. Using the product decomposition Proposition 4.9, we obtain a diagram

$$\begin{tikzcd}
V[2](Fc, 1) & V[1](Fc \times 1) \arrow[l, "V\delta^{02}"'] \arrow[r, "V\delta^{02}"] & V[2](1, Fc) \\
V[2](\varnothing, 1) \arrow[u] \arrow[d, "V\delta^{011}"'] & V[1](\varnothing \times 1) \arrow[l, "V\delta^{02}"'] \arrow[r, "V\delta^{02}"] \arrow[u] \arrow[d] & V[2](1, \varnothing) \arrow[u] \arrow[d, "V\delta^{112}"] \\
V[1](\varnothing) & V[1](\varnothing) \arrow[l] \arrow[r] & V[1](\varnothing)
\end{tikzcd} \tag{7.13}$$

in which taking colimits of rows provides the diagram

$$V[1](Fc) \times V[1](1) \xleftarrow{V[1](\mathrm{incl})\times \mathrm{id}} V[1](\varnothing) \times V[1](1) \xrightarrow{\mathrm{proj}} V[1](\varnothing).$$

The horizontal morphisms of (7.13) are monomorphisms, therefore the colimits of the rows are in fact homotopy colimits in $s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}$. Thus, it suffices to show that $\mathrm{Map}_{\Theta C}(V[1](Fc), X)$ maps by a weak equivalence to the homotopy limit of $\mathrm{Map}_{\Theta C}(-, X)$ applied to the above diagram. We claim that in fact that the evident projection maps induce weak equivalences from $\mathrm{Map}_{\Theta C}(V[1](Fc), X)$ to the inverse limits of each of the columns of $\mathrm{Map}_{\Theta C}(–, X)$ applied to the diagram.

This is clear for the middle column: the map $V[1](\varnothing\times 1)\to V[1](\varnothing)$ is an isomorphism, so the colimit of the middle column is isomorphic to $V[1](Fc)$. We will show the proof for the left-hand column, leaving the right-hand column for the reader. Consider the diagram:

$$\begin{tikzcd}
V[1](Fc) \arrow[r, "V\delta^{01}"] & V[2](Fc,1) \arrow[r, "V\delta^{011}"] & V[1](Fc) \\
& V[2](\varnothing,1) \arrow[u] \arrow[r] & V[1](\varnothing) \arrow[u] \\
V[0] \arrow[uu, "V\delta^{1}"] \arrow[r, "V\delta^{1}"'] & V[1](1) \arrow[u, "V\delta^{12}"] \arrow[r] & V[0] \arrow[u, "V\delta^{1}"']
\end{tikzcd}$$

We want to show that $\mathrm{Map}_{\Theta C}(-, X)$ carries the upper-right square to a homotopy pullback. The lower-right square is a pushout square (use the isomorphism $V[2](\varnothing,1)\approx V[0]\amalg V[1](1)$), as is the outer square; thus they are homotopy pushouts (of spaces) since the vertical maps are monomorphisms. Applying $\mathrm{Map}_{\Theta C}(-, X)$ to the diagram takes these two squares to homotopy pullbacks of spaces; this operation also takes the left-hand rectangle to a homotopy pullback of spaces, since $X$ is a Segal object. Thus we can conclude that $\mathrm{Map}_{\Theta C}(-, X)$ carries the upper-right square to a homotopy pullback of spaces, as desired. □

Say that a map $f\colon X\to Y$ of spaces is a *homotopy monomorphism* if it is injective on $\pi_0$, and induces a weak equivalence between each path component of $X$ and the corresponding path component of $Y$. Say a map $f\colon X\to Y$ of objects of $s\mathrm{PSh}(C)$ is a homotopy monomorphism if each $f(c)\colon X(c)\to Y(c)$ is a homotopy monomorphism of spaces.

**7.14 Lemma** *If $X$ is a Segal object in $s\mathrm{PSh}(\Theta C)$, then $X^{T_\# i}\colon X^{T_\# E}\to X^{T_\# F[1]}$ is a homotopy monomorphism in $s\mathrm{PSh}(\Theta C)$.*

**Proof** For $\theta\in\operatorname{ob}\Theta C$, the map $X^{T_\# E}(\theta)\to X^{T_\# F[1]}(\theta)$ is isomorphic to

$$\mathrm{Map}(T_\# E, X^{F\theta})\to \mathrm{Map}(T_\# F[1], X^{F\theta}),$$

which since $X^{F\theta}$ is a Segal object, is weakly equivalent to the map $(X^{F\theta})^{\mathrm{equiv}}\to T^*(X^{F\theta})[1]$, which is a homotopy monomorphism. □

**7.15 Proposition** *If $X$ is a Segal object in $s\mathrm{PSh}(\Theta C)$, then the map $X^{T_\# q}\colon X \approx X^{T_\# F[0]} \to X^{T_\# E}$ is fully faithful.*

**Proof** It is straightforward to check that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps of Segal objects in $s\mathrm{PSh}(C)$ such that $gf$ is fully faithful and $g$ is a homotopy monomorphism, then $f$ is fully faithful. Apply this observation to $X \to X^{T_\# E} \to X^{T_\# F[1]}$, using Proposition 7.12 and Lemma 7.14. □

## 7.16 Essentially surjective maps

If $X$ and $Y$ are Segal objects in $s\mathrm{PSh}(\Theta C)$, we say that a map $f\colon X \to Y$ is *essentially surjective* if the induced functor $hf\colon hX \to hY$ on homotopy categories (Section 7.3) is essentially surjective, ie, if every object of $hY$ is isomorphic to an object in the image of $hf$.

**7.17 Proposition** *Suppose $X$ is a Segal object in $s\mathrm{PSh}(\Theta C)$. Then the map*

$$X^{T_\# q}\colon X \approx X^{T_\# F[0]} \to X^{T_\# E}$$

*is essentially surjective.*

**Proof** Observe that since $T_\#$ preserves products Corollary 4.3, the map

$$T^*(X^{T_\# q})\colon T^*(X^{T_\# F[0]}) \to T^*(X^{T_\# F[1]})$$

is isomorphic to the map

$$(T^*X)^q\colon T^*X \approx (T^*X)^{F[0]} \to (T^*X)^E.$$

Thus we are reduced to the case when $C = 1$, and $X$ is a Segal space, in which case the result follows from [16, Lemma 13.9] □

**7.18 Lemma** *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps of Segal objects in $s\mathrm{PSh}(\Theta C)$ such that* (i) *$gf$ is fully faithful and* (ii) *$f$ is fully faithful and essentially surjective, then $g$ is fully faithful.*

**Proof** We need to show for all $y_0, y_1$ points of $Y[0]$ that $\underline{hY}(y_0, y_1) \to \underline{hZ}(gy_0, gy_1)$ is an isomorphism in $hs\mathrm{PSh}(C)$. Since $f$ is essentially surjective, we may choose points $x_0, x_1$ in $X[0]$ so that $fx_i \approx y_i$, $i = 0, 1$, as objects of $hY$. □

**7.19 Proposition** *If $X$ is a Segal object in $s\mathrm{PSh}(C)$, then the map $X^{T_\# i}\colon X^{T_\# E} \to X^{T_\# F[1]}$ is fully faithful.*

**Proof** Apply Lemma 7.18 to $X \to X^{T_\# E} \to X^{T_\# F[1]}$, using Proposition 7.12, Proposition 7.15, and Proposition 7.17. □

### 7.20 $(\Theta C, \mathrm{Se}_C \cup \mathrm{Cpt}_C)$ is a cartesian presentation

**7.21 Proposition** *$(\Theta C, \mathrm{Se}_C \cup \mathrm{Cpt}_C)$ is a cartesian presentation.*

**Proof** By Proposition 6.8, it suffices to show that if $X$ is a complete Segal object, then $X^{F[1](c)}$ is $\mathrm{Cpt}_C$–local for all $c \in \mathrm{ob}\, C$. That is, we must show that $X^{F[1](c)} \approx \mathrm{Map}_{\Theta C}(T_\# F[0], X^{F[1](c)}) \to \mathrm{Map}_{\Theta C}(T_\# E, X^{F[1](c)})$ is a weak equivalence of spaces, or equivalently that

$$(X^{T_\# F[0]})[1](c) \to (X^{T_\# E})[1](c)$$

is a weak equivalence of spaces. This is immediate from the fact that $X^{T_\# F[0]} \to X^{T_\# E}$ is fully faithful Proposition 7.15 and the fact that $X[0] \approx X^{T_\# F[0]}[0] \to X^{T_\# E}[0] \approx X^{\mathrm{equiv}}$ is a weak equivalence, since $X$ is a complete Segal object. □

## 8 The presentation $(\Theta C, \mathscr{S}_\Theta)$

In this section, we consider what happens when we start with a presentation $(C, \mathscr{S})$. In this case, we define a new presentation $(\Theta C, \mathscr{S}_\Theta)$ which depends on $(C, \mathscr{S})$, by

$$\mathscr{S}_\Theta \overset{\mathrm{def}}{=} \mathrm{Se}_C \cup \mathrm{Cpt}_C \cup V[1](\mathscr{S}),$$

where $V[1](\mathscr{S}) \overset{\mathrm{def}}{=} \{\, V[1](f) \mid f \in \mathscr{S} \,\}$, and where $\mathrm{Se}_C$, and $\mathrm{Cpt}_C$ are as defined in Section 5.1 and Section 7.6.

Say that two model categories $M_1$ and $M_2$ are *equivalent* if there is an equivalence $E\colon M_1 \to M_2$ of categories which preserves and reflects cofibrations, fibrations, and weak equivalences (this is much stronger than Quillen equivalence). If $M$ is a model category equivalent to one of the form $s\mathrm{PSh}(C)_{\mathscr{S}}$ for some presentation $(C, \mathscr{S})$, then we write

$$M\text{–}\Theta\mathrm{Sp} \overset{\mathrm{def}}{=} s\mathrm{PSh}(\Theta C)_{\mathscr{S}_\Theta}.$$

We call $M$–$\Theta$Sp the model category of *$\Theta$–spaces over $M$*.

In the rest of this section, we prove the following result, which is the precise form of Theorem 1.4.

**8.1 Theorem** *If $(C, \mathscr{S})$ is a cartesian presentation, then $(\Theta C, \mathscr{S}_\Theta)$ is a cartesian presentation, so that $s\mathrm{PSh}(\Theta C)_{\mathscr{S}_\Theta}$ is a cartesian model category.*

## 8.2 $V[1](\mathcal{S})$–fibrant objects

The $V[1](\mathcal{S})$–fibrant objects are precisely the injective fibrant objects whose mapping spaces are $\mathcal{S}$–fibrant. Explicitly, we have the following.

**8.3 Proposition** *An injectively fibrant object $X$ in $s\mathrm{PSh}(\Theta C)$ is $V[1]\mathcal{S}$–fibrant if and only if for each $x_1, x_2 \in X[0]$, the object $M_X(x_1, x_2)$ is an $\mathcal{S}$–fibrant object of $s\mathrm{PSh}(C)$.*

## 8.4 Proof of Theorem 8.1

It is clear that Theorem 8.1 follows from Proposition 6.8, Proposition 7.21, and the following.

**8.5 Proposition** *If $(C, \mathcal{S})$ is a cartesian presentation, then $(\Theta C, \mathrm{Se}_C \cup V[1](\mathcal{S}))$ is a cartesian presentation.*

In the remainder of the section we prove this proposition Proposition 8.5.

In light of Proposition 8.3 and Proposition 6.8, it is enough to show that if $(C, \mathcal{S})$ is a cartesian presentation and $X$ is $\mathrm{Se}_C \cup V[1](\mathcal{S})$–fibrant, and if $Y = X^{F[1](d)}$ for some $d \in \mathrm{ob}\, C$, then $M_Y(g_0, g_1)$ is an $\mathcal{S}$–fibrant object of $s\mathrm{PSh}(C)$, for all points $g_0, g_1$ in $X^{F[1](d)}[0]$.

Let $c$ and $d$ be objects in $C$, and consider the following diagram in $s\mathrm{PSh}(\Theta C)$.

$$\begin{tikzcd}
V[2](Fc, Fd) & V[1](Fc \times Fd) \arrow[l, "V\delta^{02}"'] \arrow[r, "V\delta^{02}"] & V[2](Fd, Fc) \\
V[2](\varnothing, Fd) \arrow[u] & V[1](\varnothing) \arrow[l, "V\delta^{02}"] \arrow[r, "V\delta^{02}"'] \arrow[u] & V[2](Fd, \varnothing) \arrow[u]
\end{tikzcd} \tag{8.6}$$

By Proposition 4.9, taking colimits along the rows gives the map

$$f\colon V[1](Fd) \amalg V[1](Fd) \approx V[1](\varnothing) \times V[1](Fd) \to V[1](Fc) \times V[1](Fd)$$

induced by $\varnothing \to Fc$. (Recall that $V[1](\varnothing) \approx 1 \amalg 1$.)

Now $\mathrm{Map}_{\Theta C}(f, X)$ is isomorphic to the map

$$(Y\delta^0, Y\delta^1)\colon Y[1](c) \to Y[0] \times Y[0]$$

whose fiber over $(g_0, g_1)$ is $M_Y(g_0, g_1)$.

Applying $\mathrm{Map}_{s\mathrm{PSh}(\Theta C)}(-, X)$ to the diagram (8.6) gives:

**(8.7)**
$$\begin{tikzcd} X[2](c,d) \arrow[r] \arrow[d] & \mathrm{Map}(V[1](Fc \times Fd), X) \arrow[d] & X[2](d,c) \arrow[l] \arrow[d] \\ X[0] \times X[1](d) \arrow[r] & X[0] \times X[0] & X[1](d) \times X[0] \arrow[l] \end{tikzcd}$$

The space $M_Y(g_0, g_1)$ is the pullback of the diagram obtained by taking fibers of each of the vertical maps of (8.7), over points $(x_{00}, g_1)$, $(x_{00}, x_{11})$, and $(g_0, x_{11})$ respectively, where $x_{ij} = (X\delta^j)(g_i)$. The vertical maps of (8.7) are fibrations of spaces, and thus the pullback of fibers is a homotopy pullback. Thus, it suffices to show that the fiber of each of the vertical maps, viewed as a functor of $c$, is an $\mathscr{S}$–fibrant object of $s\mathrm{PSh}(C)$.

We claim that these fibers, as presheaves on $C$, are weakly equivalent to the presheaves $M_X(x_{00}, x_{10})$, $(M_X(x_{00}, x_{11}))^{Fd}$, and $M_X(x_{01}, x_{11})$ respectively. The objects $M_X(x_{00}, x_{10})$, $M_X(x_{00}, x_{11})$, and $M_X(x_{01}, x_{11})$ are $\mathscr{S}$–fibrant by the hypothesis that $X$ is $V[1](\mathscr{S})$–fibrant, Since $(C, \mathscr{S})$ is a cartesian presentation, it follows that $(M_X(x_{00}, x_{11}))^{Fd}$ is $\mathscr{S}$–fibrant. Thus, we complete the proof of the proposition by proving this claim.

The left-hand vertical arrow of (8.7) factors

$$X[2](c,d) \xrightarrow{(X\delta^{01}, X\delta^{12})} X[1](c) \times_{X[0]} X[1](d) \xrightarrow{((X\delta^0)\pi_1, \pi_2)} X[0] \times X[1](d).$$

The first map is a weak equivalence since $X$ is $\mathrm{Se}_C$–local, so it suffices to examine the fibers of the second map over $(x_0, g_1)$. It is straightforward to check that this fiber is isomorphic to $M_X(x_{00}, x_{10})(c)$.

The right-hand vertical arrow of (8.7) is analysed similarly, so that its fibers are weakly equivalent to $M_X(x_{01}, x_{11})(c)$.

For the middle vertical arrow of (8.7), Lemma 4.12 allows us to identify the fiber over $(x_{00}, x_{11})$ with

$$\mathrm{Map}_{s\mathrm{PSh}(C)}(Fc \times Fd, M_X(x_{00}, x_{11})) \approx (M_X(x_{00}, x_{11}))^{Fd}(c).$$

# 9 Groupoid objects

Let $\mathrm{Gpd}_C$ be the set consisting of the morphism

$$T_\# q\colon T_\# F[1] \to T_\# E.$$

We say that a Segal object is a *Segal groupoid* if it is $\mathrm{Gpd}_C$–local; likewise, a complete Segal object is called a *complete Segal groupoid* if it is $\mathrm{Gpd}_C$–local.

**9.1 Lemma** *If $X$ is a Segal object in $s\mathrm{PSh}(\Theta C)$, then $X$ is $\mathrm{Gpd}_C$–local if and only if $X^{T_\# i}\colon X^{T_\# E} \to X^{T_\# F[1]}$ is a levelwise weak equivalence in $s\mathrm{PSh}(\Theta C)$.*

**Proof** The "if" part is immediate. To prove the "only if" part, note that for any Segal object $Y$, the map $Y^{T_\# E} \to Y^{T_\# F[1]}$ is fully faithful by Proposition 7.19. If $X$ is $\mathrm{Se}_C \cup \mathrm{Gpd}_C$–fibrant, then $X^{T_\# i}[0]\colon X^{T_\# E}[0] \to X^{T_\# F[1]}[0]$ is a weak equivalence of spaces, and thus $X^{T_\# i}$ must be a levelwise weak equivalence in $s\mathrm{PSh}(\Theta C)$. □

**9.2 Proposition** *The presentations*

$$(\Theta C, \mathrm{Se}_C \cup \mathrm{Gpd}_C) \quad \textit{and} \quad (\Theta C, \mathrm{Se}_C \cup \mathrm{Cpt}_C \cup \mathrm{Gpd}_C)$$

*are cartesian presentations. If $(C, \mathscr{S})$ is a cartesian presentation, then*

$$(\Theta C, \mathrm{Se}_C \cup \mathrm{Gpd}_C \cup V[1]\mathscr{S}) \quad \textit{and} \quad (\Theta C, \mathrm{Se}_C \cup \mathrm{Cpt}_C \cup \mathrm{Gpd}_C \cup V[1]\mathscr{S})$$

*are cartesian presentations.*

**Proof** We only need to show that $(\Theta C, \mathrm{Se}_C \cup \mathrm{Gpd}_C)$ is a cartesian presentation; the other results follow using Proposition 7.21 and Theorem 8.1.

To show that $(\Theta C, \mathrm{Se}_C \cup \mathrm{Gpd}_C)$ is a cartesian presentation, we need to show Proposition 6.8 that if $X$ is $\mathrm{Se}_C \cup \mathrm{Gpd}_C$–fibrant, then $Y = X^{F[1](c)}$ is $\mathrm{Gpd}_C$–local for all $c \in \mathrm{ob}\, C$. The map $\mathrm{Map}_{\Theta C}(T_\# i, Y)\colon \mathrm{Map}_{\Theta C}(T_\# E, Y) \to \mathrm{Map}_{\Theta C}(T_\# F[1], Y)$ is isomorphic to $(X^{T_\# i})[1](c)\colon (X^{T_\# E})[1](c) \to (X^{T_\# F[1]})[1](c)$, which is a weak equivalence by Lemma 9.1. □

Given a presentation $(C, \mathscr{S})$ with $M = s\mathrm{PSh}(C)^{\mathrm{inj}}_{\mathscr{S}}$, let

$$\Theta_{\mathrm{Gpd}}(C, \mathscr{S}) = (\Theta C, \mathscr{S}_\Theta \cup \mathrm{Gpd}_C)$$

and

$$M\text{–}\Theta\mathrm{Gpd} \stackrel{\mathrm{def}}{=} s\mathrm{PSh}(\Theta C)^{\mathrm{inj}}_{\mathscr{S}_\Theta \cup \mathrm{Gpd}_C}.$$

## 10 Alternate characterization of complete Segal objects

In the section we consider a characterization of the "completeness" property in the definition of a complete Segal space, which is a bit more elementary than the one given in [16]. The results of this section are not needed elsewhere in this paper.

Let $E$ in $s\mathrm{PSh}(\Delta)$ be the "discrete nerve" of the groupoid with two uniquely isomorphic objects $x$ and $y$, let $p\colon E \to F[0]$ denote the projection, and let $i\colon F[1] \to E$ denote the map which picks out the morphism from $x$ to $y$. In [16, Proposition 6.2] it is shown that if $X$ is a Segal space, then $\mathrm{Map}(i, X)\colon \mathrm{Map}(E, X) \to \mathrm{Map}(F[1], X) \approx X[1]$ factors through a weak equivalence $\mathrm{Map}(E, X) \to X^{\mathrm{equiv}}$. From this, we see that a Segal space $X$ is complete if and only if $\mathrm{Map}(p, X)$ is a weak equivalence [16, Proposition 6.4].

The proof of [16, Proposition 6.2] is long and technical. Also, the result is not entirely satisfying, because $E$ is an "infinite dimensional" object, in the sense that as a simplicial space it is constructed from infinitely many cells, which appear in all dimensions (see [16, Section 11]). It is possible to replace $E$ with the finite subobject $E^{(k)}$ for $k \geq 3$ (see [16, Proposition 11.1]), but this is also not very satisfying.

Here we prove a variant of [16, Proposition 6.2] where $E$ is replaced by an object $Z$, which is a finite cell object. The idea is based on the following observation: in a category enriched over spaces, the homotopy equivalences $g\colon X \to Y$ are precisely those morphisms for which there exist morphisms $f, h\colon Y \to X$ and homotopies $\alpha\colon gf \sim 1_Y$ and $\beta\colon hg \sim 1_X$, and that for a given homotopy equivalence $g$ the "moduli space" of such data $(f, h, \alpha, \beta)$ is weakly contractible.

Define an object $Z$ in $s\mathrm{PSh}(\Delta)$ to be the colimit of the diagram

$$F[3] \xleftarrow{(\delta^{02},\delta^{13})} F[1] \amalg F[1] \xrightarrow{\delta^{00} \amalg \delta^{00}} F[0] \amalg F[0].$$

Let $p\colon Z \to F[0]$ be the evident projection map, and let $i\colon F[1] \to Z$ be the composite of $\delta^{12}\colon F[1] \to F[3]$ with the quotient map $F[3] \to Z$.

**10.1 Proposition** *Let $X$ be a Segal space (ie, an $\mathrm{Se}_1$–fibrant object of $s\mathrm{PSh}(\Delta)$). The map $\mathrm{Map}(Z, X) \to \mathrm{Map}(F[1], X)$ factors through $X^{\mathrm{equiv}} \subseteq X[1]$, and induces a weak equivalence $\mathrm{Map}(Z, X) \to X^{\mathrm{equiv}}$ of spaces.*

*Thus, a Segal space $X$ is a complete Segal space if and only if the square*

$$\begin{tikzcd}
X[0] \arrow[r, "X\delta^{0000}"] \arrow[d, "{(X\delta^{0}, X\delta^{0})}"'] & X[3] \arrow[d, "{(X\delta^{02}, X\delta^{13})}"] \\
X[0] \times X[0] \arrow[r, "X\delta^{00} \times X\delta^{00}"'] & X[1] \times X[1]
\end{tikzcd}$$

*is a homotopy pullback.*

**Proof** Consider the following commutative diagram:

$$\begin{tikzcd}
P \arrow[r, "d"] \arrow[d, tail, "b"'] & T \arrow[r, "e"] \arrow[d, tail, "c"'] & X^{\mathrm{equiv}} \arrow[d, tail, "j"] \\
Q \arrow[r, "a"] \arrow[d] & X[3] \arrow[r, "X\delta^{12}"] \arrow[d, "(X\delta^{02}, X\delta^{13})"'] & X[1] \arrow[d, "(X\delta^{1}, X\delta^{0})"] \\
X[0]\times X[0] \arrow[r, "X\delta^{00}\times X\delta^{00}"'] & X[1]\times X[1] \arrow[r, "X\delta^{1}\times X\delta^{0}"'] & X[0]\times X[0]
\end{tikzcd}$$

Here, the objects $Q$, $T$, and $P$ are defined to be the pullbacks of the lower left, upper right, and upper left squares respectively; each of these squares is a homotopy pullback of spaces, since $(X\delta^{02}, X\delta^{13})$ and $j$ are fibrations. (The lower right square is in general *not* a pullback or a homotopy pullback.) The maps $b$, $c$, and $j$ are homotopy monomorphisms. Observe that $Q \approx \mathrm{Map}(Z, X)$, and so we want to prove that $(X\delta^{12})a$ factors through a weak equivalence $k\colon Q \to X^{\mathrm{equiv}}$.

The result will follow by showing (i) that the horizontal map $(X\delta^{12})a\colon Q \to X_1$ factors through the inclusion $j\colon X^{\mathrm{equiv}} \to X_1$ by a map $k\colon Q \to X^{\mathrm{equiv}}$ (and thus $b\colon P \to Q$ is a weak equivalence), and (ii) that the right hand rectangle is a homotopy pullback, ie, that $T \approx \mathrm{holim}(X_h \to X_0 \times X_0 \leftarrow X_1 \times X_1)$. Condition (ii) implies that $ed\colon P \to X^{\mathrm{equiv}}$ is a weak equivalence, since it is a homotopy pullback of the identity map of $X_0 \times X_0$. Since $fb = ed$, it follows that $k$ is a weak equivalence, as desired.

The proof of (i) is straightforward. If $H$ is a point in $Q$, let $g \overset{\mathrm{def}}{=} ((X\delta^{12})a)(H)$ in $X[1]$. Then by construction the class $[g]$ in the homotopy category $hX$ admits both a left and a right inverse, and thus $g$ is a point of $X^{\mathrm{equiv}}$. (See the discussion in [16, Section 5.5].)

To prove (ii), let

$$T' = \lim(X^{\mathrm{equiv}} \xrightarrow{(X\delta^{1}, X\delta^{0})j} X[0]\times X[0] \xleftarrow{X\delta^{1}\times X\delta^{0}} X[1]\times X[1]).$$

Since $X\delta^1$ and $X\delta^0$ are fibrations, this is a homotopy pullback. We need to show that $t\colon T \to T'$ is a weak equivalence. Let $\pi'\colon X[1]\times X[1] \to X[0]\times X[0]\times X[0]\times X[0]$, be the map defined by

$$\pi'(u, v) \overset{\mathrm{def}}{=} ((X\delta^0)u, (X\delta^0)v, (X\delta^1)u, (X\delta^1)v).$$

Let $\pi\colon T' \to (X[0])^4$ be the composite of $\pi'$ with the tautological map $T' \to X[1]\times X[1]$. Note that both $\pi$ and $\pi t$ are fibrations of spaces.

Let $\underline{x} = (x_0, x_1, x_2, x_3)$ be a tuple of points in $X[0]$. The fiber of $\pi$ over $\underline{x}$ is the space

$$T'_{\underline{x}} \overset{\text{def}}{=} M_X(x_0, x_2) \times M_x^{\text{equiv}}(x_1, x_2) \times M_X(x_1, x_3).$$

The fiber of $\pi t$ over $\underline{x}$ is the limit

$$T_{\underline{x}} \overset{\text{def}}{=} \lim\bigl(M_X(x_0, x_1, x_2, x_3) \to M_X(x_1, x_2) \leftarrowtail M_X^{\text{equiv}}(x_1, x_2)\bigr).$$

To prove the proposition, we need to show that for all $\underline{x}$, the map $t_{\underline{x}}\colon T_{\underline{x}} \to T'_{\underline{x}}$ induced by $t$ is a weak equivalence.

Given a point $f$ in $M_X(x_0, x_1)$, we write $M_X(x_0, x_1)_f$ for the path component of $M_X(x_0, x_1)$ containing $f$. Given sequence of points $f_i$ in $M_X(x_{i-1}, x_i)$, we write $M_X(x_0, \dots, x_n)_{f_1, \dots, f_n}$ for the path component of $M_X(x_0, \dots, x_n)$ which projects to $M_X(x_0, x_1)_{f_1} \times \cdots \times M_X(x_{n-1}, x_n)_{f_n}$ under the Segal map. We claim that if $f \in M_X(x_0, x_1)$, $g \in M_X^{\text{equiv}}(x_1, x_2)$, and $h \in M_X(x_2, x_3)$, then the maps

$$\zeta\colon M_X(x_0, x_1, x_2, x_3)_{f,g,h} \to M_X(x_0, x_1, x_2)_{f,g} \times M_X(x_1, x_3)_{h \circ g}$$

and $\quad \eta\colon M_X(x_0, x_1, x_2)_{f,g} \times M_X(x_1, x_3)_{h\circ g}$

$$\to M_X(x_0, x_2)_{g \circ f} \times M_X(x_1, x_2)_g \times M_X(x_1, x_3)_{h \circ g}$$

are weak equivalences. This is a straightforward calculation, using the ideas of [16, Proposition 11.6]. The map $t_{\underline{x}}$ is the disjoint union of maps $\eta\zeta$ over the appropriate path components, and thus the proposition is proved. □

# 11 $(n+k, n)$–$\Theta$–spaces

In this section, we do three things. First, we make precise the "informal description" of $(n+k, n)$–$\Theta$–spaces given in Section 1.2. Next, we identify the "discrete" $(\infty, n)$–$\Theta$–spaces Proposition 11.24. Finally, we show that "groupoids" in $(n+k, n)$–$\Theta$–spaces are essentially the same as $(n+k)$–truncated spaces Proposition 11.27, thus proving the "homotopy hypothesis" for these models.

## 11.1 Functor associated to a presheaf on $\Theta_n$

For an object $X$ of $s\mathrm{PSh}(\Theta_n)$, let $\bar{X}\colon s\mathrm{PSh}(\Theta_n)^{\mathrm{op}} \to \mathrm{Sp}$ denote the functor defined by

$$\bar{X}(K) \overset{\text{def}}{=} \mathrm{Map}_{\Theta_n}(K, X).$$

The construction $X \mapsto \bar{X}$ is the Yoneda embedding of $s\mathrm{PSh}(\Theta_n)$ into the category of Sp–enriched functors $s\mathrm{PSh}(\Theta_n)^{\mathrm{op}} \to \mathrm{Sp}$. The object $X$ is recovered from the functor $\bar{X}$ by the formula $X(\theta) \approx \bar{X}(F\theta)$.

## 11.2 The discrete nerve

Given a strict $n$–category $C$, we define the *discrete nerve* of $C$ to be the presheaf of sets $\operatorname{dnerve} C$ on $\Theta_n$ defined by

$$(\operatorname{dnerve} C)(\theta) = \mathbf{St}\text{–}n\text{–}\mathbf{Cat}(\tau_n \theta, C).$$

Since we can regard presheaves of sets as a full subcategory of discrete presheaves of simplicial sets, we will regard dnerve as a functor dnerve: $\mathbf{St}\text{–}n\text{–}\mathbf{Cat} \to s\mathrm{PSh}(\Theta_n)$. This functor is fully faithful. Finally, note that there is a natural isomorphism $F \approx \operatorname{dnerve} \tau$, where $\tau_n \colon \Theta_n \to \mathbf{St}\text{–}n\text{–}\mathbf{Cat}$ is the inclusion functor of Corollary 3.10, and $F \colon \Theta_n \to s\mathrm{PSh}(\Theta_n)$ is the Yoneda embedding of $\Theta_n$.

## 11.3 The suspension and inclusion functors

For all $n \geq 1$ there is a *suspension* functor

$$\sigma \colon \Theta_{n-1} \to \Theta_n$$

defined on objects by $\sigma(\theta) \overset{\text{def}}{=} [1](\theta)$. Composing suspension functors gives functors $\sigma^k \colon \Theta_{n-k} \to \Theta_n$ for $0 \leq k \leq n$.

For all $n \geq 1$ there is an *inclusion* functor

$$\tau \colon \Theta_{n-1} \to \Theta_n$$

which is the restriction of the standard inclusion $\mathbf{St}\text{–}n-1\text{–}\mathbf{Cat} \to \mathbf{St}\text{–}n\text{–}\mathbf{Cat}$ to $\Theta_{n-1}$. Composing inclusion functors gives functors $\tau^k \colon \Theta_{n-k} \to \Theta_n$ for $0 \leq k \leq n$.

## 11.4 The category $\Theta_n\mathbf{Sp}_k$

For $0 \leq n < \infty$, let $\mathcal{T}_{n,\infty}$ be the set of morphisms in $s\mathrm{PSh}(\Theta_n)$ defined by

$$\mathcal{T}_{0,\infty} = \varnothing,$$
$$\mathcal{T}_{n,\infty} = \mathrm{Se}_{\Theta_{n-1}} \cup \mathrm{Cpt}_{\Theta_{n-1}} \cup V[1](\mathcal{T}_{n-1,\infty}) \text{ for } n > 0.$$

If also given $-2 \leq k < \infty$, let $\mathcal{T}_{n,k}$ be the set of morphisms in $s\mathrm{PSh}(\Theta_n)$ defined by

$$\mathcal{T}_{0,k} = \{\partial\Delta^{k+2} \to \Delta^{k+2}\},$$
$$\mathcal{T}_{n,k} = \mathrm{Se}_{\Theta_{n-1}} \cup \mathrm{Cpt}_{\Theta_{n-1}} \cup V[1](\mathcal{T}_{n-1,k}) \text{ for } n > 0.$$

In the notation of Section 8, $\mathcal{T}_{n,k} = (\mathcal{T}_{n-1,k})_\Theta$ for $n > 0$.

**11.5 Proposition** *For all $0 \leq n < \infty$ and $-2 \leq k \leq \infty$, the presentation $(\Theta_n, \mathcal{T}_{n,k})$ is cartesian.*

**Proof** Immediate from Theorem 8.1. □

Let $\Theta_n\mathrm{Sp}_k \overset{\mathrm{def}}{=} s\mathrm{PSh}(\Theta_n)^{\mathrm{inj}}_{\mathscr{T}_{n,k}}$; we call this the $(n+k,n)$–$\Theta$–*space model category*. We show that the fibrant objects of $\Theta_n\mathrm{Sp}_k$ are precisely the $(n+k,n)$–$\Theta$–spaces described in Section 1.2.

## 11.6 The structure of the sets $\mathscr{T}_{n,k}$

For $n \geq 0$ and $-2 \leq k < \infty$, we have

$$\mathscr{T}_{n,\infty} = \mathscr{T}_n^{\mathrm{Se}} \cup \mathscr{T}_n^{\mathrm{Cpt}}$$

and

$$\mathscr{T}_{n,k} = \mathscr{T}_{n,\infty} \cup \{(V[1])^n(\partial\Delta^{k+2} \to \Delta^{k+2})\},$$

where

$$\mathscr{T}_n^{\mathrm{Se}} = \{\, (V[1])^k(\mathrm{se}^{\theta_1,\dots,\theta_r}) \mid 0 \leq k < n,\ r \geq 2,\ \theta_1,\dots,\theta_r \in \mathrm{ob}\,\Theta_{n-k} \,\}$$

and

$$\mathscr{T}_n^{\mathrm{Cpt}} = \{\, (V[1])^k(T_\# p) \mid 0 \leq k < n \,\}.$$

**11.7 Proposition** *For $\theta_1,\dots,\theta_r \in \mathrm{ob}\,\Theta_{n-k}$, the map $(V[1])^k(\mathrm{se}^{\theta_1,\dots,\theta_r})$ is isomorphic to the map*

$$\mathrm{colim}\big(F\sigma^k[1](\theta_1) \leftarrow F\sigma^k[0] \to \cdots \leftarrow F\sigma^k[0] \to F\sigma^k[1](\theta_r)\big) \to F\sigma^k[r](\theta_1,\dots,\theta_r),$$

*induced by applying $F\sigma^k$ to the maps $\delta^{i-1,i}\colon [1](\theta_i) \to [r](\theta_1,\dots,\theta_r)$.*

**Proof** Immediate using Lemma 11.10 and Proposition 4.5. □

## 11.8 The objects $O_k$ and $\partial O_k$

Fix $n \geq 0$. We write $O_k$ for the discrete nerve of the free-standing $k$–cell in **St**–$n$–**Cat**. It follows that $O_k \approx F\sigma^k[0] \approx F[1]([1](\cdots[1]([0])))$, where $\sigma^k\colon \Theta_{n-k} \to \Theta_n$. Note that our usage of $O_k$ here is slightly different than that described in the introduction, where $O_k$ was used to mean the object of $\Theta_n$, rather than the object of $s\mathrm{PSh}(\Theta_n)$.

If $k > 0$, then the free-standing $k$–cell in **St**–$n$–**Cat** is a $k$–morphism between two parallel $(k-1)$–cells. Let $s_k, t_k\colon O_{k-1} \to O_k$ denote the map between discrete nerves induced by the inclusion of the two parallel $(k-1)$–cells. Equivalently, $s_k$ and $t_k$ are the maps obtained by applying $\sigma^{k-1}$ to the maps $\delta^0, \delta^1\colon [0] \to [1]$ of $\Theta_{n-k}$.

Let $\partial O_k$ denote the maximal proper subobject of $O_k$; that is, $\partial O_k \subset O_k$ is the largest sub–$\Theta_n$–presheaf of $O_k$ which does not contain the "tautological section" $\iota \in O_k(\sigma^k[0])$. Let $e_k\colon \partial O_k \to O_k$ denote the inclusion.

**11.9 Proposition** *For $1 \le k \le n$, the map*

$$\operatorname{colim}\bigl(O_{k-1} \xleftarrow{e_{k-1}} \partial O_{k-2} \xrightarrow{e_{k-1}} O_{k-1}\bigr) \to \partial O_k$$

*defined by $s_k, t_k\colon O_{k-1} \to O_k$ is an isomorphism in $s\mathrm{PSh}(\Theta_n)$.*

By abuse of notation, we write $s_k, t_k\colon O_{k-1} \to \partial O_k$ for the inclusion of the two copies of $O_{k-1}$.

It is clear that $\partial O_k$ is isomorphic to the discrete nerve of the "free-standing pair of parallel $(k-1)$–cells". Observe that $\partial O_0 = \varnothing$.

**11.10 Lemma** *For $\theta \in \operatorname{ob}\Theta_{n-1}$, the object $V[1](F\theta) \approx F([1](\theta)) \approx F\sigma(\theta)$ as objects of $s\mathrm{PSh}(\Theta_n)$.*

**Proof** A straightforward calculation. □

**11.11 Proposition** *For $1 \le k < n$, the functor $V[1]\colon s\mathrm{PSh}(\Theta_{n-1}) \to s\mathrm{PSh}(\Theta_n)$ carries the diagram*

$$s_k, t_k\colon O_{k-1} \rightrightarrows O_k \leftarrow \partial O_k \ :e_k$$

*up to isomorphism to the diagram*

$$s_{k+1}, t_{k+1}\colon O_k \rightrightarrows O_{k+1} \leftarrow \partial O_{k+1} \ :e_{k+1}.$$

*Furthermore, $V[1](\varnothing) = V[1](\partial O_0) \approx \partial O_1$.*

**Proof** Again, a straightforward calculation using Lemma 11.10 and Proposition 4.5. □

## 11.12 Mapping objects between pairs of parallel $(k-1)$–cells

Let $X$ be a $\mathcal{T}_n^{\mathrm{Se}}$–fibrant object in $s\mathrm{PSh}(\Theta_n)$. We call the space $\overline{X}(O_k) = \mathrm{Map}_{\Theta_n}(O_k, X)$ the *moduli space of $k$–cells* of $X$. We call the space $\overline{X}(\partial O_k) = \mathrm{Map}_{\Theta_n}(\partial O_k, X)$ the *moduli space of pairs of parallel $(k-1)$–cells*. (These are the spaces denoted $X(O_k)$ and $X(\partial O_k)$ in the introduction.)

Observe that the maps $s_k, t_k\colon O_{k-1} \to \partial O_k$ determine an isomorphism

$$\overline{X}(\partial O_k) \xrightarrow{\sim} \overline{X}(O_{k-1}) \times_{\overline{X}(\partial O_{k-1})} \overline{X}(O_{k-1}).$$

In particular, $\overline{X}(\partial O_k) \to \overline{X}(O_{k-1}) \times \overline{X}(O_{k-1})$ is a monomorphism, so that a point of $\overline{X}(\partial O_k)$ can be named by a suitable pair of points in $\overline{X}(O_{k-1})$.

Suppose $1 \le k \le n$, and suppose given $(f_0, f_1) \in \overline{X}(\partial O_k)$. We write $\underline{\mathrm{map}}_X(f_0, f_1)$ for the object of $s\mathrm{PSh}(\Theta_{n-k})$ defined by

$$\underline{\mathrm{map}}_X(f_0, f_1)(\theta) \overset{\mathrm{def}}{=} \lim\bigl(\overline{X}(V[1]^k(F\theta)) \to \overline{X}(V[1]^k(\varnothing)) \approx \overline{X}(\partial O_k) \leftarrow \{(f_0, f_1)\}\bigr).$$

Observe that these objects can be obtained by iterating the mapping object construction of Section 4.11. In particular, if $(x_0, x_1) \in X[0] \times X[0] \approx \overline{X}(\partial O_1)$, then $\underline{\mathrm{map}}_X(x_0, x_1) \approx M_X(x_0, x_1)$ as objects of $s\mathrm{PSh}(\Theta_{n-1})$.

**11.13 Lemma** *If $X$ is a $\mathcal{T}_n^{\mathrm{Se}}$–fibrant object of $s\mathrm{PSh}(\Theta_n)$, then $\underline{\mathrm{map}}_X(f_0, f_1)$ is a $\mathcal{T}_{n-k}^{\mathrm{Se}}$–fibrant object of $s\mathrm{PSh}(\Theta_{n-k})$.*

**Proof** Immediate from the fact that $\mathcal{T}_n^{\mathrm{Se}} \supseteq V[1]^k(\mathcal{T}_{n-k}^{\mathrm{Se}})$. □

## 11.14 The moduli space $X(O_k)^{\mathrm{equiv}}$ of $k$–equivalences

Let $X$ be a $\mathcal{T}_n^{\mathrm{Se}}$–fibrant object of $s\mathrm{PSh}(\Theta_n)$, and suppose $1 \le k \le n$. Given a $k$–cell in $X$, ie, a point $g$ in $\overline{X}(O_k)$, let

$$b_0 = (\overline{X}s_k)(g) \quad \text{and} \quad b_1 = (\overline{X}t_k)(g)$$

be the "source" and "target" $(k-1)$–cells of $g$, and let

$$a_0 = (\overline{X}s_{k-1})b_0 = (\overline{X}s_{k-1})b_1 \quad \text{and} \quad a_1 = (\overline{X}t_{k-1})b_0 = (\overline{X}t_{k-1})b_1$$

be the "source" and "target" $(k-2)$–cells of $b_0$ and $b_1$. Let $Y = \underline{\mathrm{map}}_X(a_0, a_1)$ as an object of $s\mathrm{PSh}(\Theta_{n-k+1})$; the presheaf $Y$ is $\mathcal{T}_{n-k+1}^{\mathrm{Se}}$–fibrant by Lemma 11.13. Then $b_0$ and $b_1$ are "objects" of $Y$, (that is, points in $Y[0] = \overline{Y}(O_0)$), and $g$ is a "1–cell" of $Y$, (that is, a point of $\overline{Y}(O_1)$). Recall (Section 7.3) that $g$ thus represents an element $[g]$ of the homotopy category $hY$ of $Y$.

Say that a $k$–cell $g$ of $X$ is a *$k$–equivalence* if it represents an isomorphism in the homotopy category $hY$ of $Y = \underline{\mathrm{map}}_X(a_0, a_1)$. Let $\overline{X}(O_k)^{\mathrm{equiv}} \subseteq \overline{X}(O_k)$ denote the union of path components of $\overline{X}(O_k)$ which contain $k$–equivalences.

## 11.15 Characterization of $\mathcal{T}_n^{\mathrm{Se}} \cup \mathcal{T}_n^{\mathrm{Cpt}}$–fibrant objects

Recall the map $i\colon F[1] \to E$ of Section 7.6.

**11.16 Proposition** *For all $1 \le k \le n$, the map*

$$\overline{X}(V[1]^{k-1}(T_\# i))\colon \overline{X}(V[1]^{k-1}(T_\# E)) \to \overline{X}(V[1]^{k-1}(T_\# F[1])) \approx \overline{X}(O_k)$$

*factors through the subspace $\overline{X}(O_k)^{\mathrm{equiv}} \subseteq \overline{X}(O_k)$ and induces a weak equivalence $\overline{X}(V[1]^{k-1}(T_\# E)) \to \overline{X}(O_k)^{\mathrm{equiv}}$.*

**Proof** Let $(a_0, a_1)$ be a point in $\bar{X}(\partial O_{k-1})$ and let $Y = \underline{\mathrm{map}}_X(a_0, a_1)$. Since $Y$ is $\mathcal{T}^{\mathrm{Se}}_{n-k}$–fibrant, it is in particular $\mathrm{Se}_{\Theta_{n-k-1}}$–fibrant, and thus the map

$$\mathrm{Map}_{\Theta_{n-k}}(T_\# i, Y)\colon \mathrm{Map}_{\Theta_{n-k}}(T_\# E, Y) \to \mathrm{Map}_{\Theta_{n-k}}(T_\# F[1], Y) \approx T^* Y[1]$$

factors through $Y^{\mathrm{equiv}} \subseteq T^* Y[1]$, and induces a weak equivalence $\mathrm{Map}(T_\# E, Y) \to Y^{\mathrm{equiv}}$ of spaces Proposition 7.8.

Now consider the diagram:

$$\begin{tikzcd}
\bar{X}(V[1]^{k-1}(T_\# E)) \arrow[rr, "\bar{X}(V[1]^{k-1}(T_\# i))"] \arrow[dr] & & \bar{X}(V[1]^{k-1}(T_\# F[1])) \arrow[dl] \\
& \bar{X}(V[1]^{k-1}(\varnothing)) &
\end{tikzcd}$$

Over $(a_0, a_1) \in \bar{X}(\partial O_{k-1}) \approx \bar{X}(V[1]^{k-1}(\varnothing))$, the map induced by $\bar{X}(V[1]^{k-1}(T_\# i))$ on fibers is isomorphic to the map $\mathrm{Map}_{\Theta_{n-k}}(T_\# i, Y)$, and the result follows. □

**11.17 Corollary** *Let $X$ be a $\mathcal{T}^{\mathrm{Se}}_n$–fibrant object of $s\mathrm{PSh}(\Theta_n)$. Then $X$ is $\mathcal{T}^{\mathrm{Cpt}}_n$–fibrant if and only if the maps $\bar{X}(i_k)\colon \bar{X}(O_{k-1}) \to \bar{X}(O_k)^{\mathrm{equiv}}$ are weak equivalences for $1 \le k \le n$.*

**Proof** Immediate from the structure of $\mathcal{T}^{\mathrm{Cpt}}_n$ Section 11.6. □

Thus, the $\mathcal{T}^{\mathrm{Se}}_n \cup \mathcal{T}^{\mathrm{Cpt}}_n$–fibrant objects of $s\mathrm{PSh}(\Theta_n)$ are precisely the $(\infty, n)$–$\Theta$–spaces.

We record the following.

**11.18 Proposition** *If $X$ is a $(\infty, n)$–$\Theta$–space, and $(f_0, f_1)$ in $\bar{X}(\partial O_k)$ is a pair of parallel $(k-1)$–cells of $X$, then $\underline{\mathrm{map}}_X(f_0, f_1)$ is an $(\infty, n-k)$–$\Theta$–space.*

## 11.19 Characterization of $k$–truncated objects

Let $X$ be an $(\infty, n)$–$\Theta$–space (ie, a $\mathcal{T}_{n,\infty} = \mathcal{T}^{\mathrm{Se}}_n \cup \mathcal{T}^{\mathrm{Cpt}}_n$–fibrant object in $s\mathrm{PSh}(\Theta_n)$). Let $(f_0, f_1)$ be a point in $\bar{X}(\partial O_n)$. Then $\underline{\mathrm{map}}_X(f_0, f_1)$ is an object of $s\mathrm{PSh}(\Theta_0) \approx \mathrm{Sp}$. Furthermore, if $K$ is a space, then the fiber of $\bar{X}(V[1]^n(K)) \to \bar{X}(V[1]^n(\varnothing)) \approx \bar{X}(\partial O_n)$ over $(f_0, f_1)$ is naturally isomorphic to $\mathrm{Map}(K, \underline{\mathrm{map}}_X(f_0, f_1))$.

**11.20 Proposition** *A $\mathcal{T}_{n,\infty}$–fibrant object $X$ of $s\mathrm{PSh}(\Theta_n)$ is $\mathcal{T}_{n,k}$–fibrant if and only if for all $(f_0, f_1)$ in $\bar{X}(\partial O_n)$, the space $\underline{\mathrm{map}}_X(f_0, f_1)$ is $k$–truncated.*

**Proof** On fibers over $(f_0, f_1)$, the map

$$\bar{X}(V[1]^n(\Delta^{k+2})) \to \bar{X}(V[1]^n(\partial\Delta^{k+2}))$$

induces the map

$$\mathrm{Map}(\Delta^{k+2}, \underline{\mathrm{map}}_X(f_0, f_1)) \to \mathrm{Map}(\partial\Delta^{k+2}, \underline{\mathrm{map}}_X(f_0, f_1))$$

of spaces. □

## 11.21 Rigid $n$–categories

The following proposition characterizes the discrete $\mathcal{T}_n^{\mathrm{Se}}$–fibrant objects of $s\mathrm{PSh}(\Theta_n)$.

**11.22 Proposition** *The functor* dnerve *induces an equivalence between* **St**–$n$–**Cat** *and the full subcategory of discrete $\mathcal{T}_n^{\mathrm{Se}}$–fibrant objects of $s\mathrm{PSh}(\Theta_n)$.*

**Proof** A discrete presheaf $X$ is $\mathcal{S}$–fibrant if and only if $\mathrm{Map}(s, X)\colon \mathrm{Map}(S', X) \to \mathrm{Map}(S, X)$ is an isomorphism for all $s\colon S \to S'$ in $\mathcal{S}$. It is clear that if $\mathcal{S} = \mathcal{T}_n^{\mathrm{Se}}$, then this condition amounts to requiring that $X$ be in the essential image of dnerve. □

Let $C$ be a strict $n$–category. We define the following notions for cells in $C$, by downwards induction.

(1) Let $g\colon x \to y$ be a $k$–morphism in $C$. If $1 \le k < n$, we say $g$ is a *$k$–equivalence* if there exist $k$–cells $f, h\colon y \to x$ in $C$ such that $gf \sim 1_y$ and $hg \sim 1_x$. If $k = n$, we say $g$ is a $k$–equivalence if it is a $k$–isomorphism.

(2) Let $f, g\colon x \to y$ be two parallel $k$–cells in $C$. If $0 \le k < n$, we say that $f$ and $g$ are *equivalent*, and write $f \sim g$, if there exists a $(k+1)$–equivalence $h\colon f \to g$. If $k = n$, we say that $f$ and $g$ are equivalent if there are equal.

**11.23 Proposition** *Let $C$ be a strict $n$–category. The following are equivalent.*

(1) *For all $1 \le k \le n$, every $k$–equivalence is an identity $k$–morphism.*

(2) *For all $1 \le k \le n$, every $k$–isomorphism is an identity $k$–morphism.*

**Proof** It is clear that (1) implies (2). To show that (2) implies (1), use downward induction on $k$. □

By a *rigid $n$–category*, we mean a strict $n$–category $C$ satisfying either of the equivalent conditions of Proposition 11.23.

**11.24 Proposition** *Let $C$ be a strict $n$–category. The discrete nerve* $\operatorname{dnerve} C$ *is an $(\infty,n)$–$\Theta$–space (ie, is $\mathcal{T}_n^{\mathrm{Se}} \cup \mathcal{T}_n^{\mathrm{Cpt}}$–fibrant) if and only if $C$ is a rigid $n$–category.*

**Proof** Let $C$ be a strict $n$–category. By Proposition 11.23 $\operatorname{dnerve} C$ is $\mathcal{T}_n^{\mathrm{Se}}$–fibrant. It will also be $\mathcal{T}_n^{\mathrm{Cpt}}$–fibrant if and only if $\overline{X}(O_k)^{\mathrm{equiv}} \to \overline{X}(O_k)$ is an isomorphism for all $1 \leq k \leq n$, and the result follows from the observation that $\overline{X}(O_k)^{\mathrm{equiv}}$ corresponds precisely to the set of $k$–isomorphisms in $C$. □

## 11.25 Groupoids and the homotopy hypothesis

For $n \geq 0$, let

$$\mathcal{T}_0^{\mathrm{Gpd}} \overset{\mathrm{def}}{=} \varnothing,$$

$$\mathcal{T}_n^{\mathrm{Gpd}} \overset{\mathrm{def}}{=} \mathrm{Gpd}_{\Theta_{n-1}} \cup V[1](\mathcal{T}_{n-1}^{\mathrm{Gpd}}),$$

using the definition of $\mathrm{Gpd}_C$ of Section 9. Explicitly, we have

$$\mathcal{T}_n^{\mathrm{Gpd}} \approx \{\, V[1]^k(T_\# q) \mid 0 \leq k < n \,\},$$

where $T_\# q\colon T_\# F[1] \to T_\# E$ is as in Section 9.

Let
$$\Theta_n\mathrm{Gpd}_k \overset{\mathrm{def}}{=} s\mathrm{PSh}(\Theta_n)^{\mathrm{inj}}_{\mathcal{T}_{n,k} \cup \mathcal{T}_n^{\mathrm{Gpd}}}.$$

We call this the $(n+k,n)$–$\Theta$–*groupoid model category*. The fibrant objects of $\Theta_n\mathrm{Gpd}_k$ are called $(n+k,n)$–$\Theta$–*groupoids*; they form a full subcategory of the category of $(n+k,n)$–$\Theta$–spaces.

**11.26 Proposition** *Let $n \geq 0$. Let $X$ be a $(\infty,n)$–$\Theta$–space. The following are equivalent.*

(1) *The object $X$ is $\mathcal{T}_n^{\mathrm{Gpd}}$–fibrant.*

(2) *For all $0 \leq k < n$, the maps $\overline{X}(i_k)\colon \overline{X}(O_k) \to \overline{X}(O_{k+1})$ are weak equivalences of spaces.*

(3) *For all $\theta \in \operatorname{ob}\Theta_n$, the map $Xp\colon X[0] \to X\theta$ induced by $p\colon \theta \to [0]$ is a weak equivalence of spaces.*

**Proof** It is clear from the definition of $\mathcal{T}_n^{\mathrm{Gpd}}$ that (1) and (2) are equivalent. It is immediate that (3) implies (2); it remains to show that (1) implies (3), which we will show by induction on $n$. Note that there is nothing to prove if $n = 0$. Since $\mathcal{T}_n^{\mathrm{Gpd}} \supseteq \mathrm{Gpd}_{\Theta_{n-1}}$, we see that the object $T^*X$ is a $(\infty,1)$–$\Theta$–groupoid, which is to say, a groupoid-like complete Segal space, and thus all maps $(T^*X)\delta^0\colon (T^*X)[0] \to (T^*X)[m]$ are weak equivalences of spaces [16, Corollary 6.6]. Therefore, $X[0] \to X\theta$ is a weak equivalence for all $\theta = [m]([0], \dots, [0])$, $m \geq 0$. Now consider the diagram

$$\begin{tikzcd} X[0] \arrow[r, "\sim"] \arrow[dr, "a"'] & X[m]([0],\dots,[0]) \arrow[r, "\sim"] \arrow[d] & X[1]([0]) \times_{X[0]} \cdots \times_{X[0]} X[m]([0]) \arrow[d, "b"] \\ & X[m](\theta_1,\dots,\theta_m) \arrow[r, "\sim"'] & X[1](\theta_1) \times_{X[0]} \cdots \times_{X[0]} X[1](\theta_m) \end{tikzcd}$$

where $\theta_1, \dots, \theta_m \in \operatorname{ob}\Theta_{n-1}$. To show that $a$ is a weak equivalence, it suffices to show that $b$ is, so it suffices to show that $X[1](p)\colon X[1]([0]) \to X[1](\theta)$ is a weak equivalence of spaces for all $\theta \in \Theta_{n-1}$. Consider the diagram:

$$\begin{tikzcd} \overline{X}(V[1](F[0])) \arrow[rr, "\overline{X}(V[1](Fp))"] \arrow[dr] & & \overline{X}(V[1](F\theta)) \arrow[dl] \\ & \overline{X}(V[1](\varnothing)) & \end{tikzcd}$$

The map $\overline{X}(V[1](Fp))$ is isomorphic to $X[1](p)\colon X[1]([0]) \to X[1](\theta)$. Let $(x_0, x_1)$ be a point in $\overline{X}(\partial O_1) \approx \overline{X}(V[1](\varnothing))$; the map induced on fibers over $(x_0, x_1)$ by $\overline{X}(V[1](Fp))$ is isomorphic to

$$\underline{\mathrm{map}}_X(x_0, x_1)(p)\colon \underline{\mathrm{map}}_X(x_0, x_1)([0]) \to \underline{\mathrm{map}}_X(x_0, x_1)(\theta).$$

It is clear that $\underline{\mathrm{map}}_X(x_0, x_1)$ is a $(\infty, n-1)$–$\Theta$–groupoid, and thus $\underline{\mathrm{map}}_X(x_0, x_1)(p)$ is a weak equivalence of spaces by the inductive hypothesis. □

Let $c_\#\colon \mathrm{Sp} \rightleftarrows s\mathrm{PSh}(\Theta_n)$ denote the adjoint pair where the left adjoint $c_\#$ sends a space $X$ to the constant presheaf with value $X$.

**11.27 Proposition** (1) *The adjoint pair*

$$c_\#\colon \mathrm{Sp} \rightleftarrows s\mathrm{PSh}(\Theta_n)^{\mathrm{inj}}_{\mathcal{T}_{n,\infty} \cup \mathcal{T}_n^{\mathrm{Gpd}}} :c^*$$

*is a Quillen equivalence.*

(2) *For all $-2 \leq k < \infty$ the adjoint pair*

$$c_\#\colon \mathrm{Sp}_{n+k} \rightleftarrows s\mathrm{PSh}(\Theta_n)^{\mathrm{inj}}_{\mathcal{T}_{n,k} \cup \mathcal{T}_n^{\mathrm{Gpd}}} :c^*$$

*is a Quillen equivalence.*

**Proof** We first consider (1). Observe that $c_\#$ preserves cofibrations, and that caries all spaces to $\mathcal{T}_{n,\infty} \cup \mathcal{T}_n^{\mathrm{Gpd}}$–fibrant objects by Proposition 11.27, and thus $c_\#$ preserves weak equivalences. Therefore the pair is a Quillen pair, and it is a straightforward consequence of Proposition 11.27 that the natural map $X \to c^* c_\# X$ is always weak equivalence, the natural map $c_\# c^* Y \to Y$ is a weak equivalence for all $\mathcal{T}_{n,\infty} \cup \mathcal{T}_n^{\mathrm{Gpd}}$–local objects $Y$, and thus the pair is a Quillen equivalence.

The proof that we get a Quillen equivalence in (2) proceeds in the same way, once we observe that $c_\#$ carries $n+k$–truncated spaces to $\mathcal{T}_{n,k} \cup \mathcal{T}_n^{\mathrm{Gpd}}$–local objects. □

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