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DIRICHLET SERIES FROM THE INFINITE DIMENSIONAL POINT OF VIEW

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Dedicated to the memory of our
missed friend Paweł Domański

Abstract: A classical result of Harald Bohr linked the study of convergent and bounded Dirichlet series on the right half plane with bounded holomorphic functions on the open unit ball of the space c_0 of complex null sequences. Our aim here is to show that many questions in Dirichlet series have very natural solutions when, following Bohr's idea, we translate these to the infinite dimensional setting. Some are new proofs and other new results obtained by using that point of view.

Keywords: Dirichlet series, Bohr transform, holomorphic function, Banach space.

1. Introduction

At the beginning of the 20th century the study of Dirichlet series attracted the attention of many mathematicians. H. Bohr did a systematic study of the convergence of ordinary Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$. These series define holomorphic functions in half-planes. Among other questions he was interested in comparing the planes on which a given Dirichlet series converges absolutely and the ones in which that series converges uniformly. To attack this problem, Bohr had a very deep idea. It was to consider the sequence of primer numbers $\mathfrak{p} = (\mathfrak{p}_n)$, $1 < \mathfrak{p}_1 < \mathfrak{p}_2 < \ldots$, and by using the fundamental theorem of arithmetics each $n \in \mathbb{N}$ has a unique representation $n = \mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}\ldots\mathfrak{p}_n^{\alpha_n} = \mathfrak{p}^{\alpha}$ with $\alpha = (\alpha_1, \ldots, \alpha_N, 0, \ldots) \in \mathbb{N}_0^{(\mathbb{N})}$, being $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$. Here $\mathbb{N}_0^{(\mathbb{N})}$ is the set of the sequences of elements of $\mathbb{N}_0$ that are 0 except for a finite number of entries. Thus Bohr associated to each Dirichlet series

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a formal power series in infinitely many variables in the following way:

$$\sum_{n=1}^{\infty} a_n n^{-s} \quad \overset{a_n = a_{\mathfrak{p}^\alpha} = c_\alpha}{\leftrightsquigarrow} \quad \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha .$$

Let us see what is the correspondence that Bohr stated.

A formal power series in infinitely many variables is a series of the form

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha ,$$

where $c_\alpha \in \mathbb{C}$ and for $z = (z_n)_{n=1}^{\infty}$ and $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, $\alpha = (\alpha_1, \dots, \alpha_N, 0, \dots)$

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_N^{\alpha_N}$$

If $\mathfrak{P}$ denotes the algebra of all formal power series $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha$ and $\mathfrak{D}$ denotes the algebra of all Dirichlet series, the Bohr transform is defined:

$$\begin{array}{cccc} \mathfrak{B} : & \mathfrak{P} & \longrightarrow & \mathfrak{D} \\ & \sum c_\alpha z^\alpha & \xrightarrow{a_{\mathfrak{p}^\alpha} = c_\alpha} & \sum a_n n^{-s} \end{array}$$

This mapping is clearly bijective, linear and multiplicative; hence an algebra homomorphism. Up to here there is no topological structure in any of the sets. Bohr thought about that transform due to the fact that in 1909 Hilbert in [36] had introduced the concept of infinite dimensional holomorphic function. His insight was to translate a problem of one complex variable into another of infinite dimensional nature. In modern terms and notation what happens is the following. Consider the Banach space c_0 of all null complex sequences and denote by $H_\infty(B_{c_0})$ the Banach algebra of all functions $f : B_{c_0} \to \mathbb{C}$ that are bounded and holomorphic (meaning to be complex FrĂŠchet-differentiable) on B_{c_0} endowed with the norm $\|f\|_\infty = \sup_{z \in B_{c_0}} |f(z)|$ (or $\|f\|_{B_{c_0}}$ when we want to emphasize the setting).

A mapping $P : X \to Y$ (between Banach spaces) is an m-homogeneous polynomial if there exists an m-linear mapping $L : X \times \cdots \times X \to Y$ such that $P(x) = L(x, \dots, x)$ for every $x \in X$. The space of all continuous m-homogeneous polynomials from X into Y will be denoted by $\mathcal{P}(^m X; Y)$ and by $\mathcal{P}(^m X)$ if Y is the complex field.

Let f be a function $f : B_X \to Y$ (B_X is the open unit ball of X). A very well-known characterization of f to be complex FrĂŠchet-differentiable (holomorphic) on B_X is the following: for each $a \in B_X$ there exists a sequence $P_m^a : X \to Y$ of continuous m-homogeneous polynomials such that the series $\sum_{m=0}^{\infty} P_m^a(x - a)$ converges absolutely and uniformly to $f(x)$ on some neighbourhood of a. The series $\sum_{m=0}^{\infty} P_m^a$ is called the Taylor series of f at a. Moreover, if P_m denotes P_m^0, then $f(x) = \sum_{m=0}^{\infty} P_m(x)$ for every $x \in B_X$.

A continuous m-homogeneous polynomial $P \in \mathcal{P}(^m X, Y)$ has an extension to the bidual X^{**} of X, $\hat{P} \in \mathcal{P}(^m X^{**}, Y^{**})$, which is called the *Aron-Berner extension* of P (see [1]). In fact, $\hat{P}$ is defined in the following way. Let A be the symmetric m-linear mapping associated to P, A can be extended to an m-linear mapping $\hat{A}$ from X^{**} into Y^{**} in such a way that for each fixed j, $1 \leqslant j \leqslant m$, and for each fixed $x_1, \ldots, x_{j-1} \in X$ and $z_{j+1}, \ldots, z_m \in X^{**}$, the linear mapping $z \to \hat{A}(x_1, \ldots, x_{j-1}, z, z_{j+1}, \ldots, z_m)$, $z \in X^{**}$, is (w^*, w^*)-continuous. In other words, we define $\hat{A}(x_1, \ldots, x_{j-1}, z, z_{j+1}, \ldots, z_m)$ to be the weak-star limit of the net $(\hat{A}(x_1, \ldots, x_{j-1}, x_\alpha, z_{j+1}, \ldots, z_m))$ for a weak-star convergent net $(x_\alpha) \subset X$ to z. By this (w^*, w^*)-continuity A can be extended to an m-linear mapping $\hat{A}$ from X^{**} into Y^{**}, beginning with the last variable and working backwards to the first. Then the restriction $\hat{P}(z) = \hat{A}(z, \ldots, z)$ is called the Aron-Berner extension of P. Davie and Gamelin [22] proved that if $f : B_X \to Y$ is a bounded holomorphic function, $f(x) = \sum_{m=0}^{\infty} P_m(x)$ for $x \in B_X$, then $\hat{f}(z) = \sum_{m=0}^{\infty} \hat{P}_m(z)$ is a well-defined holomorphic function on $B_{X^{**}}$ and $\|f\|_{B_X} = \|\hat{f}\|_{B_{X^{**}}}$. In particular, for c_0 we have an isometry from $\mathcal{P}(^m c_0)$ into $\mathcal{P}(^m \ell_\infty)$ and from $H_\infty(B_{c_0})$ into $H_\infty(B_{\ell_\infty})$. On the other hand, let $\mathbb{C}_+$ denote be the right half plane $[\operatorname{Re} s > 0]$ and by $\mathcal{H}_\infty = \mathcal{H}_\infty(\mathbb{C}_+)$ we denote the Banach algebra of all Dirichlet series that are convergent and bounded on $[\operatorname{Re} s > 0]$, again endowed with the respective supremum norm denoted $\|\cdot\|_\infty$ or $\|\cdot\|_{[\operatorname{Re} s>0]}$. The following holds.

Theorem 1.1. *The Bohr transform $\mathfrak{B} : H_\infty(B_{c_0}) \longrightarrow \mathcal{H}_\infty$ is an isometric isomorphism of Banach algebras.*

This result was proved by Hedenmalm, Lindqvist and Seip in 1997 ([30]). A self-contained direct proof can be found in [25]. The connection between a bounded holomorphic function f on the open unit ball of c_0 and a formal power series $\sum_\alpha c_\alpha(f) z^\alpha$ is done in the following way. If f belongs to $H_\infty(B_{c_0})$ and α belongs to $\mathbb{N}_0^{(\mathbb{N})}$ as we can identify $\mathbb{N}_0^{(\mathbb{N})} = \cup_{N=1}^{\infty} \mathbb{N}_0^N$ and $\mathbb{D}^N$ with $\mathbb{D}^N \times \{0\} \subset B_{c_0}$, then if $\alpha = (\alpha_1, \ldots, \alpha_N, 0, \ldots)$, the Cauchy integral formula for holomorphic functions on $\mathbb{D}^N$ gives

$$c_\alpha(f) = \frac{1}{(2\pi i)^N} \int_{|\zeta_1|=r_1} \cdots \int_{|\zeta_N|=r_N} \frac{f(\zeta_1, \ldots, \zeta_N, 0, \ldots)}{\zeta_1^{\alpha_1+1} \cdots \zeta_N^{\alpha_N+1}} d\zeta_N \cdots d\zeta_1 \tag{1.1}$$

for every choice of $0 < r_j < 1$ with $j = 1, \ldots, N$.

Actually more can be said. Another way to look at this transformation $\mathfrak{B}$ is to understand it as the evaluation of the function f in $H_\infty(B_{c_0})$ in the sequence $z = (1/\mathfrak{p}^s) = (1/\mathfrak{p}_n^s)_{n=1}^{\infty}$, since the following equality holds

$$\mathfrak{B}(f)(s) = f\Big(\frac{1}{\mathfrak{p}^s}\Big), \tag{1.2}$$

for any $s \in [\operatorname{Re} s > 0]$. This result can be found in [25]. Even more, the series

$\sum c_\alpha(f)\Big(\frac{1}{\mathfrak{p}^s}\Big)^\alpha$ is always summable for $s \in [\operatorname{Re} s > 1/2]$ and then

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(f)\Big(\frac{1}{\mathfrak{p}^s}\Big)^\alpha = \sum_{n=1}^\infty a_n \frac{1}{n^s}.$$

See again [25]. The inverse of the Bohr transform is called the *Bohr lift*,

$$\begin{array}{cccc} \mathfrak{L} = \mathfrak{B}^{-1}: & \mathfrak{D} & \longrightarrow & \mathfrak{P}, \\ & \sum a_n n^{-s} & \xrightarrow{c_\alpha = a_{\mathfrak{p}^\alpha}} & \sum c_\alpha z^\alpha \end{array}$$

and then the preceding equality reads as follows

$$\mathfrak{B}^{-1}(D)\big(\tfrac{1}{\mathfrak{p}^s}\big) = \sum_n a_n \frac{1}{n^s}.$$

Bohr in 1912 defined the *abscissa of absolute convergence* $\sigma_a(D)$ of a Dirichlet series $D = \sum_{n=1}^\infty a_n n^{-s}$

$$\sigma_a(D) = \inf\big\{\sigma \in \mathbb{R} \colon D \text{ converges absolutely in } [\operatorname{Re} s \geqslant \sigma]\big\} \in [-\infty, \infty],$$

and defined the *abscissa of uniform convergence* $\sigma_u(D)$ of a Dirichlet series $D = \sum_{n=1}^\infty a_n n^{-s}$

$$\sigma_u(D) = \inf\big\{\sigma \in \mathbb{R} \colon D \text{ converges uniformly in } [\operatorname{Re} s \geqslant \sigma]\big\} \in [-\infty, \infty],$$

and also the abscissa of boundedness

$$\sigma_b(D) = \inf\big\{\sigma \in \mathbb{R} \colon D \text{ converges and it is bounded on } [\operatorname{Re} s \geqslant \sigma]\big\} \in [-\infty, \infty].$$

One of his key fundamental result was to prove the equality $\sigma_b(D) = \sigma_u(D)$. Moreover, using the Bohr transform, he was able to obtain that

$$\sigma_a(D) - \sigma_u(D) \leqslant \frac{1}{2},$$

for every Dirichlet series $D = \sum_{n=1}^\infty a_n n^{-s}$. But he was unable to find any Dirichlet series such that $\sigma_a(D) - \sigma_u(D) > 0$. Toepliz, the same year, using the Bohr lift for two homogeneous polynomials on c_0, found a Dirichlet series such that $\sigma_a(D) - \sigma_u(D) \geqslant 1/4$. Finally in 1931, in a remarkable paper, Bohenblust and Hille [15], again through the Bohr lift, found a Dirichlet series D such that

$$\sigma_a(D) - \sigma_u(D) = \frac{1}{2}.$$

The modern theory of Dirichlet series connecting its study, i.e. complex analysis and analytic number theory with harmonic and functional analysis, began with the Acta Mathematica paper [34] by H. Helson and D. Lowdenslager [34] of 1958 and more recently in 1997 with the seminal paper [30]. Since then a lot of research has been given in Dirichlet series, see [4, 6, 7, 8, 10, 11, 12, 13, 14, 16, 18, 19, 20, 23, 29, 31, 32, 33, 37, 38, 39, 41, 42, 43], the book [40] and the forthcoming [25] where the proof of all unproven claims in this paper can be found. But most of the results are based in techniques of one complex variable and of harmony analysis.

The object of this paper is to show how the point of view of infinite dimensional holomorphy introduced by Harald Bohr together with modern tools of that theory can produce new insights in Dirichlet series, by giving a different, but very natural new proofs for some known results in Dirichlet series and even new results. But also the other way around, thanks to the theory of Dirichlet series we are going to produce new results on Banach spaces of bounded holomorphic functions on the open unit ball of c_0. In other words, as the Bohr transform and the Bohr lift allow us to jump from Dirichlet series on a half plane to power series in infinitely many variables and the other way around, we plan to show that some problems that in one of the settings looks difficult to deal with from the other point of view are easier to treat and even better to understand what is going on.

2. Controlling the supremum of a Dirichlet series on a half-plane

Before using infinite dimensional techniques let us begin with the following question. Is there a general relationship between the supremum on a half plane of a bounded Dirichlet series and the supremum on its corresponding boundary line even if the Dirichlet series has no continuous extension to that boundary? The answer is yes and given in the next result, where we use an extension of the classical technique for Dirichlet series.

Given a Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ that is convergent and bounded on $[\operatorname{Re} s > 0]$, let us recall that for $t \in \mathbb{R}$

$$\overline{\lim}|D(it)| = \inf_{r>0} \sup_{s\in\mathbb{D}^+(n^{it},r)} |D(s)|,$$

where $\mathbb{D}^+(n^{it}, r) = \{s \in \mathbb{C} : |s - n^{it}| < r,\ \operatorname{Re} s > 0\}$.

Lemma 2.1. *For a Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ that is convergent and bounded on $[\operatorname{Re} s > 0]$ we have*

$$\sup_{[\operatorname{Re} s>0]} \Big|\sum_{n=1}^{\infty} a_n n^{-s}\Big| = \sup_{t\in\mathbb{R}} \overline{\lim}|D(it)|.$$

Proof. Let us write

$$A = \sup_{t\in\mathbb{R}} \overline{\lim}|D(it)| \qquad \text{and} \qquad B = \sup_{[\operatorname{Re} s>0]} \Big|\sum_{n=1}^{\infty} a_n \frac{1}{n^s}\Big|.$$

By definition, $A \leqslant B$. For the converse inequality let us fix $\varepsilon > 0$ and consider the function

$$g_\varepsilon(s) := e^{-\varepsilon\sqrt{s}} \sum_{n=1}^{\infty} a_n \frac{1}{n^s}, \qquad \operatorname{Re} s > 0,$$

where $\sqrt{s}$ denotes the principal square root of s. Then g_ε is a holomorphic function on $[\operatorname{Re} s > 0]$. Taking now $s = re^{i\alpha} \in \mathbb{C}$ with $\operatorname{Re} s > 0$, we have

$$|g_\varepsilon(s)| = e^{-\varepsilon \operatorname{Re}\sqrt{s}} \Big| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \Big| = e^{-\varepsilon\sqrt{r}\cos\frac{\alpha}{2}} \Big| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \Big| \leqslant B e^{-\varepsilon\sqrt{r}\cos\frac{\pi}{4}}$$

and this tends to 0 as $r \to \infty$. Then there exists $R > 0$ such that $|g_\varepsilon(re^{i\alpha})| \leqslant A$ for $r \geqslant R$. Taking now $\Delta = \{s \in \mathbb{C} : \ \operatorname{Re} s > 0, \ |s| < R\}$, since we have $\overline{\lim}|g_\varepsilon(s)| \leqslant A$ for every s in the boundary of Δ and again for each $t \in \mathbb{R}$

$$|\overline{\lim} g_\varepsilon(it)| = e^{-\varepsilon \operatorname{Re}\sqrt{it}}\, \overline{\lim}|D(it)| \leqslant A e^{-\varepsilon\sqrt{r}\cos\frac{\pi}{4}} \leqslant A.$$

As ∞ is not accessible from the open set Δ, by the maximum modulus principle for subharmonic functions [28, Theorem 1], we have $|g_\varepsilon(s)| \leqslant A$ for all $s \in \Delta$. This altogether gives $|g_\varepsilon(s)| \leqslant A$ for every $\operatorname{Re} s \geqslant 0$. Letting $\varepsilon \to 0$ gives the conclusion. ■

Corollary 2.2. *Let $D(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ be a Dirichlet series that is convergent and bounded on* $[\operatorname{Re} s > 0]$ *which has continuous extension to* $[\operatorname{Re} s \geqslant 0]$. *Then*

$$\sup_{[\operatorname{Re} s>0]} \Big| \sum_{n=1}^{\infty} a_n n^{-s} \Big| = \sup_{t\in\mathbb{R}} |D(n^{it})|.$$

Corollary 2.3. *Let $D(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ be a Dirichlet series such that $\sigma_b(D) < \infty$ and consider $\sigma > \sigma_b(D)$. Then*

$$\sup_{[\operatorname{Re} s>\sigma]} \Big| \sum_{n=1}^{\infty} a_n n^{-s} \Big| = \sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^{\infty} a_n n^{-\sigma+it} \Big|.$$

3. Infinite dimensional approach to Dirichlet series

Above we have given a general result on Dirichlet series that clearly implies (Corollary 2.2) that for a *Dirichlet polynomial* $\sum_{n=1}^{N} a_n n^{-s}$ the supremum of its absolute value on $[\operatorname{Re} s > 0]$ coincides with the supremum of its absolute value on the imaginary line $[\operatorname{Re} s = 0]$. Let us now give a proof of this result based in several complex variables techniques. This proof is by no means new, since it can be traced to the classical work of H. Bohr. It has as an essential ingredient the Bohr's fundamental lemma (see [25, Highlight 3.2]), which is one of the most important application of the Bohr transform and lift, stating that for every Dirichlet polynomial the Bohr lift is isometric, and in a precise way reads as follows:

For every $N \in \mathbb{N}$ and $a_1, \dots, a_N \in \mathbb{C}$

$$\sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{it} \Big| = \sup_{w\in\mathbb{T}^{\pi(N)}} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(N)} \\ 1\leqslant \mathfrak{p}^\alpha \leqslant N}} a_{\mathfrak{p}^\alpha} w^\alpha \Big|.$$

where $\pi(N)$ denotes the prime counting function, it counts the number of primes that are smaller than or equal to N.

This lemma is one of the most important tools in the theory of Dirichlet series and the Bohr transform plays a crucial role to prove it.

Proposition 3.1. *For $a_1, \dots, a_N \in \mathbb{C}$ we have*

$$\sup_{[\operatorname{Re} s>0]} \Big| \sum_{n=1}^{N} a_n n^{-s} \Big| = \sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

Proof. A classical result by Kronecker states that if $\big\{1, \theta_1, \dots, \theta_N\big\}$ are $\mathbb{Z}$-linearly independent set of real numbers, then the sequence $z_j = \big(e^{2\pi i j\theta_1}, \dots, e^{2\pi i j\theta_N}\big)$, $j \in \mathbb{N}$, is dense in $\mathbb{T}^N$. In particular, the set $\big\{(\mathfrak{p}_1^{it}, \dots, \mathfrak{p}_N^{it}) \colon t \in \mathbb{R}\big\}$ is dense in $\mathbb{T}^N$. Hence, by the continuity of a polynomial we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big| &= \sup_{t\in\mathbb{R}} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(N)} \\ 1\leqslant \mathfrak{p}^\alpha \leqslant N}} a_{\mathfrak{p}^\alpha} (\mathfrak{p}_1^{-it})^{\alpha_1} \cdots (\mathfrak{p}_{\pi(N)}^{-it})^{\alpha_{\pi(N)}} \Big| \\ &= \sup_{w\in\mathbb{T}^N} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(N)} \\ 1\leqslant \mathfrak{p}^\alpha \leqslant N}} a_{\mathfrak{p}^\alpha} w^\alpha \Big|. \end{aligned}$$

On the other hand

$$\begin{aligned} \sup_{[\operatorname{Re} s>0]} \Big| \sum_{n=1}^{N} a_n \frac{1}{n^s} \Big| &= \sup_{t\in\mathbb{R},\ \sigma>0} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(N)} \\ 1\leqslant \mathfrak{p}^\alpha \leqslant N}} a_{\mathfrak{p}^\alpha} \big(\frac{1}{\mathfrak{p}_1^{\sigma+it}}\big)^{\alpha_1} \cdots \big(\frac{1}{\mathfrak{p}_{\pi(N)}^{\sigma+it}}\big)^{\alpha_{\pi(N)}} \Big| \\ &\leqslant \sup_{w\in\mathbb{D}^N} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(N)} \\ 1\leqslant \mathfrak{p}^\alpha \leqslant N}} a_{\mathfrak{p}^\alpha} w^\alpha \Big|. \end{aligned}$$

But, by the maximum modulus theorem for several variables we have that

$$\sup_{w\in\mathbb{T}^N} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(N)} \\ 1\leqslant \mathfrak{p}^\alpha \leqslant N}} a_{\mathfrak{p}^\alpha} w^\alpha \Big| = \sup_{w\in\mathbb{D}^N} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(N)} \\ 1\leqslant \mathfrak{p}^\alpha \leqslant N}} a_{\mathfrak{p}^\alpha} w^\alpha \Big|,$$

from where the conclusion follows. ∎

Now we want to consider the following apparently silly question. Let $D(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$ be a non constant Dirichlet series such that $\sigma_b(D) < \infty$. Is the function $M(\sigma) = \sup_{[\operatorname{Re} s>\sigma]} \Big| \sum_{n=1}^{\infty} a_n n^{-s} \Big|$ strictly decreasing for $\sigma > \sigma_b(D)$?

A naive answer would be that this result is true as an immediate consequence of the maximum modulus theorem. But a maximum modulus argument cannot be applied, in general, to bounded holomorphic functions on half planes as the next example shows.

Example 3.2. Defined $g(s) = \frac{2s}{s+1}$ that is holomorphic and bounded on the closed right half plane, we have

$$|g(s_0)| < 2 = \sup_{[\operatorname{Re} s > \sigma]} |g(s)|$$

for every $\sigma \geqslant 0$ and every $s_0 \in [\operatorname{Re} s \geqslant 0]$. In other words $M(\sigma) = 2$ is a constant function. A proof of our claim is the following: the function $1 + z$ attains the maximum on the closed unit disc only at 1 and

$$\sup_{z \in A} |1 + z| = 2,$$

for $A \subset \mathbb{D}$ if and only if $1 \in \overline{A}$. Finally as $g(s) = 1 + \phi(s)$, where ϕ is the Cayley transform that maps the half plane $[\operatorname{Re} s > 0]$ onto $\mathbb{D}$. We have that ϕ is going to map $[\operatorname{Re} s > \sigma]$ (for $\sigma > 0$) into a disc strictly contained in $\mathbb{D}$ and tangent to the torus at 1. But $\phi^{-1}(1) = \infty$ and this is why g does not attain a maximum at any point in $[\operatorname{Re} s \geqslant 0]$.

The answer to the above question is yes and its proof with one complex variable techniques is based on the use of properties of subharmonic functions. Here we are going to present a different proof taking a detour through $H_\infty(B_{c_0})$.

Theorem 3.3. *Let $D(s) = \sum_{n=1}^\infty a_n \frac{1}{n^s}$ be a non constant Dirichlet series in $\mathcal{H}_\infty$. We have*

$$\sup_{[\operatorname{Re} s > \sigma]} \Big| \sum_{n=1}^\infty a_n n^{-s} \Big| > \sup_{[\operatorname{Re} s > \eta]} \Big| \sum_{n=1}^\infty a_n n^{-s} \Big|,$$

for every $0 < \sigma < \eta$.

Proof. Let define $D_\sigma(s) = D(s + \sigma) = \sum_{n=1}^\infty \frac{a_n}{n^\sigma} \frac{1}{n^s}$ and $D_\eta(s) = D(s + \eta) = \sum_{n=1}^\infty \frac{a_n}{n^\eta} \frac{1}{n^s}$. Clearly D_σ and D_η belong to $\mathcal{H}_\infty$, and

$$\sup_{[\operatorname{Re} s > \sigma]} \Big| \sum_{n=1}^\infty a_n n^{-s} \Big| = \sup_{[\operatorname{Re} s > 0]} \Big| D_\sigma(s) \Big| = \|D_\sigma\|_{[\operatorname{Re} s > 0]},$$

$$\sup_{[\operatorname{Re} s > \eta]} \Big| \sum_{n=1}^\infty a_n n^{-s} \Big| = \sup_{[\operatorname{Re} s > 0]} \Big| D_\eta(s) \Big| = \|D_\eta\|_{[\operatorname{Re} s > 0]}.$$

Moreover, if $f, f_\sigma, f_\eta : B_{c_0} \to \mathbb{C}$ are the Bohr lift of D, D_σ and D_η, i.e. the unique bounded holomorphic functions on the open unit ball of c_0 such that their monomial coefficients are $c_\alpha(f) = a_{p^\alpha}$, $c_\alpha(f_\sigma) = \frac{a_{p^\alpha}}{p^{\alpha\sigma}}$ and $c_\alpha(f_\eta) = \frac{a_{p^\alpha}}{p^{\alpha\eta}}$, we have

$$\|D_\sigma\|_{[\operatorname{Re} s > 0]} = \|f_\sigma\|_{B_{c_0}} \qquad \text{and} \qquad \|D_\eta\|_{[\operatorname{Re} s > 0]} = \|f_\eta\|_{B_{c_0}}.$$

By c_{00} we denote the dense subspace of all finite sequences in c_0. Clearly

$$f_\eta(z) = f\left(\frac{1}{p^\eta} z\right) = f_\sigma\left(\frac{p^\sigma}{p^\eta} z\right) := f_\sigma\left(\frac{z_1}{2^{\eta-\sigma}}, \frac{z_2}{3^{\eta-\sigma}}, \dots, \frac{z_k}{p_k^{\eta-\sigma}}, \dots\right),$$

for every $z = (z_1, z_2, \dots, 0, \dots) \in B_{c_{00}}$ and, by density, for every sequence $z = (z_1, z_2, \dots) \in B_{c_0}$.

Hence

$$\|f_\eta\|_{B_{c_0}} = \|f_\sigma\|_{\frac{1}{p^{\eta-\sigma}}B_{c_0}} \leqslant \|f_\sigma\|_{\frac{1}{2^{\eta-\sigma}}B_{c_0}}.$$

If we now denote by $\hat{f}_\eta$ and $\hat{f}_\sigma$ the Aron-Berner extensions of f_η and f_σ respectively, we have by the Davie and Gamelin result ([22])

$$\|\hat{f}_\eta\|_{B_{\ell_\infty}} = \|f_\eta\|_{B_{c_0}} \leqslant \|f_\sigma\|_{\frac{1}{2^{\eta-\sigma}}B_{c_0}} = \|\hat{f}_\sigma\|_{\frac{1}{2^{\eta-\sigma}}B_{\ell_\infty}}.$$

But $\hat{f}_\sigma$ is $w(\ell_\infty,\ell_1)$-continuous on $\frac{1}{2^{\eta-\sigma}}\overline{B}_{\ell_\infty}$ that is a $w(\ell_\infty,\ell_1)$-compact. Hence $|\hat{f}_\sigma|$ has a maximum in that compact set, i.e. there exists $z_0 \in \ell_\infty$ with $\|z_0\| \leqslant \frac{1}{2^{\eta-\sigma}}$ and

$$\|\hat{f}_\sigma\|_{\frac{1}{2^{\eta-\sigma}}B_{\ell_\infty}} = |\hat{f}_\sigma(z_0)|.$$

Now we have two possibilities. Either the holomorphic mapping $\varphi : 2^{\eta-\sigma}\mathbb{D} \to \mathbb{C}$ defined by $\varphi(\lambda) = f_\sigma(\lambda z_0)$ is constant or not. If φ is constant this implies that $|\hat{f}_\sigma|$ has a maximum on $\frac{1}{2^{\eta-\sigma}}B_{\ell_\infty}$ at 0 that is an interior point. Hence, by the Maximum Modulus Theorem applied to the restriction of that function to any complex line crossing zero, $\hat{f}_\sigma$ is constant on the whole B_{ℓ_∞}, thus f_σ is constant on B_{c_0}. This implies that D_σ is constant, which clearly implies that D is a constant function. This is a contradiction with our hypothesis.

If φ is not constant, by the Maximum Modulus Theorem, there exists $2^{\eta-\sigma} > |\lambda_0| > 1$ such that

$$|\hat{f}_\sigma(\lambda_0 z_0)| > |\hat{f}_\sigma(z_0)|.$$

In that case

$$\begin{aligned}\sup_{[\operatorname{Re} s>\sigma]}\Big|\sum_{n=1}^{\infty} a_n n^{-s}\Big| &= \|D_\sigma\|_{[\operatorname{Re} s>0]} = \|\hat{f}_\sigma\|_{B_{\ell_\infty}} \geqslant |\hat{f}_\sigma(\lambda_0 z_0)| \\ &> \|\hat{f}_\sigma\|_{\frac{1}{2^{\eta-\sigma}}B_{\ell_\infty}} = \|\hat{f}_\eta\|_{B_{\ell_\infty}} \\ &= \|D_\eta\|_{[\operatorname{Re} s>0]} = \sup_{[\operatorname{Re} s>\eta]}\Big|\sum_{n=1}^{\infty} a_n n^{-s}\Big|,\end{aligned}$$

and the conclusion follows. ■

Another application of the Bohr transform will allow us to give some interesting answers to the following question. Consider a Dirichlet series $D = \sum_{n=1}^\infty a_n n^{-s}$ that is convergent and bounded on $[\operatorname{Re} s > 0]$, in other words that $D \in \mathcal{H}_\infty$. Can we find non trivial subsequences (n_k) such that the series $\sum_{k=1}^\infty a_{n_k} n_k^{-s}$ converges? And if yes, when can we say that the new series is bounded?

For a fixed N consider the index set $I_N = \{\mathfrak{p}_1^{\alpha_1}\cdots\mathfrak{p}_N^{\alpha_N} : \alpha \in \mathbb{N}_0^N\}$. We are going to use the following notation. We will say that the series $\sum_{n\in I_N} a_n n^{-s}$ is convergent at s if the Dirichlet series $\sum_{n=1}^\infty b_n n^{-s}$ is convergent at s, where (b_n) is the sequence defined by $b_n = a_n$ if $n = \mathfrak{p}_1^{\alpha_1}\cdots\mathfrak{p}_N^{\alpha_N}$ and $b_n = 0$ otherwise.

Given $n = \mathfrak{p}_1^{\alpha_1}\cdots\mathfrak{p}_k^{\alpha_k}$, we write $\Omega(n) = \alpha_1 + \cdots + \alpha_k$, and if $D = \sum_{n=1}^\infty a_n n^{-s}$ is a formal Dirichlet series, we define in a similar way to above the convergence of the Dirichlet series $D_m = \sum_{\Omega(n)=m} a_n n^{-s}$.

Theorem 3.4. *Let $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series that is convergent and bounded on* $[\operatorname{Re} s > 0]$. *The following hold:*

(1) *The series $\sum_{k=1}^{\infty} a_{p_k} \mathfrak{p}_k^{-s}$ is convergent on* $[\operatorname{Re} s > 0]$.
(2) *More in general, if $N \in \mathbb{N}$ then the Dirichlet series $\sum_{n \in I_N} a_n n^{-s}$ is convergent on* $[\operatorname{Re} s > 0]$ *and* $\sup_{[\operatorname{Re} s>0]} |\sum_{n \in I_N} a_n n^{-s}| \leqslant \sup_{[\operatorname{Re} s>0]} |\sum_{n=1}^{\infty} a_n \frac{1}{n^s}|$.
(3) *Given $m \in \mathbb{N}$, the Dirichlet series $D_m = \sum_{\Omega(n)=m} a_n n^{-s}$ is convergent on* $[\operatorname{Re} s > 0]$ *and* $\sup_{[\operatorname{Re} s>0]} |D_m| \leqslant \sup_{[\operatorname{Re} s>0]} |\sum_{n=1}^{\infty} a_n \frac{1}{n^s}|$.

Proof. 3.4.(2). Fix $N \in \mathbb{N}$ and let $f = \mathfrak{B}^{-1}(D)$. We have that f belongs to $H_\infty(B_{c_0})$. Let us define $g : B_{c_0} \to \mathbb{C}$ by $g(z) = f(z_1, \ldots, z_N, 0, \ldots)$. Clearly g is holomorphic on B_{c_0} and

$$\sup_{z \in B_{c_0}} |g(z)| \leqslant \sup_{z \in B_{c_0}} |f(z)| = \sup_{[\operatorname{Re} s>0]} |D(s)|. \tag{3.3}$$

Observe that in the last equality we are using that, in our setting, the Bohr transform $\mathfrak{B}$ is an isometry. On the other hand, $c_\alpha(g) = c_\alpha(f) = a_{\mathfrak{p}^\alpha}$ for every $\alpha \in \mathbb{N}_0^N$ and $c_\alpha(g) = 0$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})} \setminus \mathbb{N}_0^N$. Thus, $\sum_{n=1}^{\infty} b_n n^{-s} = \mathfrak{B}(g)$ is a Dirichlet series such that $b_n = a_n$ if $n = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_N^{\alpha_N}$ and $b_n = 0$ otherwise. By Theorem 1.1, $\mathfrak{B}(g)$ is convergent and bounded on $[\operatorname{Re} s > 0]$ and

$$\sup_{[\operatorname{Re} s>0]} \left| \sum_{n=1}^{\infty} b_n \frac{1}{n^s} \right| = \sup_{z \in B_{c_0}} |g(z)|. \tag{3.4}$$

Combining (3.3) and (3.4) we get the conclusion.

Now we proof 3.4.(3). Since $f = \mathfrak{B}^{-1}(D)$ is holomorphic on the unit ball B_{c_0} there exists a sequence of m-homogeneous continuous polynomials $P_m : c_0 \to \mathbb{C}$ such that

$$f(z) = \sum_{m=0}^{\infty} P_m(z),$$

for every $z \in B_{c_0}$. An appropriate application of the equality (1.1) shows that $c_\alpha(P_m) = c_\alpha(f)$ if $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots)$ with $|\alpha| = \alpha_1 + \ldots + \alpha_r = m$ and $c_\alpha(P_m) = 0$ otherwise. Moreover, Cauchy inequalities imply that

$$\sup_{z \in B_{c_0}} |P_m(z)| \leqslant \sup_{z \in B_{c_0}} |f(z)| = \sup_{[\operatorname{Re} s>0]} |D(s)|. \tag{3.5}$$

As a consequence P_m belongs to $H_\infty(B_{c_0})$. Hence $\sum_{n=1}^{\infty} b_n n^{-s} = \mathfrak{B}(P_m) \in \mathcal{H}_\infty$, and

$$\sup_{[\operatorname{Re} s>0]} \left| \sum_{n=1}^{\infty} b_n \frac{1}{n^s} \right| = \sup_{z \in B_{c_0}} |P_m(z)|. \tag{3.6}$$

Again, by definition of the Bohr transform, $b_n = a_n$ if $n = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_r^{\alpha_r}$ with $\alpha_1 + \ldots + \alpha_r = m$ and $b_n = 0$ otherwise. Finally, (3.5) and (3.6) give the desired inequality. ■

The Hilbert criterion for bounded holomorphic functions on the open unit ball of c_0 (whose proof can be found in [25]) has the following corollary: *if $f : B_{c_0} \to \mathbb{C}$ is a continuous function such that $f_N : \mathbb{C}^N \to \mathbb{C}$ defined by $f_N(z_1, \dots, z_N) = f(z_1, \dots, z_N, 0 \dots)$ is holomorphic for every N and $\sup_N \|f_N\|_{D^N} < +\infty$, then $f \in H_\infty(B_{c_0})$ and $\|f\|_{B_{c_0}} = \sup_N \|f_N\|_{D^N}$.* This result, together with above theorem and Theorem 1.1, give the following statement that is itself a Hilbert criterion for bounded Dirichlet series.

Corollary 3.5. *Given a Dirichlet series $\sum a_n n^{-s}$, the following are equivalent:*

(1) $\sum a_n n^{-s} \in \mathcal{H}_\infty$.
(2) $\sup_N \left\| \sum_{n \in I_N} a_n n^{-s} \right\|_\infty < \infty$.

In this case the supremum in (2) *coincides with* $\| \sum a_n n^{-s} \|_\infty$.

By using Taylor series expansion at 0 of any holomorphic function on the open unit ball of c_0 and the Cauchy inequalities, we obtain next corollary.

Corollary 3.6. *Let $D = \sum a_n n^{-s} \in \mathcal{H}_\infty$ and $D_m = \sum_{\Omega(n)=m} a_n n^{-s}, m \in \mathbb{N}$. Then*

(1) *For every m we have $D_m \in \mathcal{H}_\infty$ and $\|D_m\|_\infty \leqslant \|D\|_\infty$.*
(2) *For every $\sigma > 0$, $D = \sum_{m=0}^\infty D_m$ uniformly on* $[\operatorname{Re} s \geqslant \sigma]$.

Actually there is a close relationship between (1) and (2) of above result. That is the content of next proposition whose proof relies again in the properties of the Taylor series expansion of a holomorphic function.

Proposition 3.7. *Let $D = \sum a_n n^{-s}$ be a formal Dirichlet series and $D_m = \sum_{\Omega(n)=m} a_n n^{-s}$, $m \in \mathbb{N}$. If there exists $M > 0$ such that for every m we have $D_m \in \mathcal{H}_\infty$ and $\|D_m\|_\infty \leqslant M$, then for every $\sigma > 0$, $D(s) = \sum a_n n^{-s}$ converges uniformly on* $[\operatorname{Re} s \geqslant \sigma]$. *Also $D(s) = \sum_{m=0}^\infty D_m(s)$ for every s so that* $\operatorname{Re} s > 0$.

Proof. Let $P_m = \mathfrak{B}^{-1}(D_m)$, we know that $P_m \in \mathcal{P}(^m c_0)$ and that $\|P_m\| = \|D_m\|_{[\operatorname{Re} s > 0]} \leqslant M$ for every m. Hence the series $f(z) = \sum_{m=0}^\infty P_m(z)$ converges absolutely and uniformly on rB_{c_0} for every $0 < r < 1$. Furthermore, f is bounded on rB_{c_0} as

$$|f(rz)| \leqslant \sum_{m=0}^\infty |P_m(rz)| = \sum_{m=0}^\infty r^m |P_m(z)| \leqslant \sum_{m=0}^\infty r^m \|P_m\| \leqslant \frac{M}{1-r},$$

for every $0 < r < 1$ and every $z \in B_{c_0}$. Now consider $0 < \delta < \sigma$ and let $\Phi_\delta : c_0 \to c_0$, defined by $\Phi_\delta(z) = (\frac{z_k}{\mathfrak{p}_k^\delta})$. Clearly, it is a linear and continuous mapping. Moreover, it is a compact mapping since $\Phi_\delta(\bar{B}_{c_0})$ is a compact subset of B_{c_0}. Thus there exists $0 < r < 1$ such that $\Phi_\delta(\bar{B}_{c_0}) \subset rB_{c_0}$. Consequently $f \circ \Phi_\delta$ belongs to $H_\infty(B_{c_0})$. Now, we consider $D_\delta = \mathfrak{B}(f \circ \Phi_\delta)$ that belongs to $\mathcal{H}_\infty$. But $D_\delta(s) = \sum_{n=1}^\infty \frac{a_n}{n^\delta} \frac{1}{n^s}$, and this series converges uniformly on $[\operatorname{Re} s \geqslant \sigma - \delta]$. Thus, D converges uniformly on $[\operatorname{Re} s \geqslant \sigma]$.

Finally, by applying (1.2) to D and each D_m, we have for every s so that $\operatorname{Re} s > 0$

$$D(s) = f\left(\frac{1}{\mathfrak{p}^s}\right) = \sum_{m=0}^{\infty} P_m\left(\frac{1}{\mathfrak{p}^s}\right) = \sum_{m=0}^{\infty} D_m(s). \qquad \blacksquare$$

The following theorem, proved by F. Bayart in [8, Lemma 18], is a kind of Montel theorem for Dirichlet series. We present now a proof based on the infinite dimensional holomorphy point of view.

Theorem 3.8. *Let (D_n) be a bounded sequence in $\mathcal{H}_\infty$. Then there exists a subsequence (D_{n_k}) that converges to some $D \in \mathcal{H}_\infty$ uniformly on each half plane $[\operatorname{Re} s > \sigma]$ with $\sigma > 0$.*

Proof. Let g_n be the Bohr lift of D_n, i.e. $g_n = \mathfrak{B}^{-1}(D_n)$. By Theorem 1.1, we have that $\|g_n\|_{B_{c_0}} = \|D_n\|_{[\operatorname{Re} s>0]}$. Thus (g_n) is a bounded sequence in $H_\infty(B_{c_0})$. By Montel's theorem there exists a subsequence (g_{n_k}) that converges to some $g \in H_\infty(B_{c_0})$ uniformly on the compact subsets of B_{c_0}. We take $D = \mathfrak{B}(g)$. We see now how Theorem 1.1 actually transfers this convergence on compact sets of B_{c_0} to convergence on half planes of Dirichlet series. Given $\sigma > 0$, the set $K_\sigma = \{x = (x_n) \in B_{c_0} : |x_n| \leqslant \frac{1}{\mathfrak{p}_n^\sigma} \text{ for all } n \in \mathbb{N}\}$ is a compact subset of B_{c_0}. Then (1.2) gives

$$\sup_{s \in [\operatorname{Re} s \geqslant \sigma]} |D_{n_k}(s) - D(s)| = \sup_{s \in [\operatorname{Re} s \geqslant \sigma]} |(g_{n_k} - g)\big(\tfrac{1}{\mathfrak{p}^s}\big)| \leqslant \sup_{x \in K_\sigma} |(g_{n_k} - g)(x)|$$

for every k. Since $\sup_{x \in K_\sigma} |(g_{n_k} - g)(x)|$ converges to 0, we obtain our claim. $\blacksquare$

Up to now we have not used that the Bohr transform is an isometric isomorphism of Banach algebras. Now we are going to give some applications of this fact. Let $\mathbb{D}$ be the open unit disc and denote by $\mathcal{A}(\mathbb{D})$ the disc algebra, consisting of all continuous functions $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ which are holomorphic on the open unit disc $\mathbb{D}$. On the other hand, recall that $\mathbb{C}_+$ denote the right half plane $[\operatorname{Re} s > 0]$ and denote by $\mathcal{A}(\mathbb{C}_+)$ the set of all Dirichlet series $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ which are convergent on $\mathbb{C}_+$ and define a uniformly continuous function on that half plane.

If X is a Banach algebra with identity, let $\mathcal{M}(X)$ be the *maximal ideal space* (also called *the spectrum*) of X, i.e. $\mathcal{M}(X)$ consists of all non-zero complex valued homomorphisms from X to $\mathbb{C}$. By [27, II.1.10], $\mathcal{M}(\mathcal{A}(\mathbb{D})) = \{\delta_z : z \in \overline{\mathbb{D}}\}$, where δ_z is the evaluation at z, $\delta_z(f) = f(z)$ for all $f \in \mathcal{A}(\mathbb{D})$. It is well-known that the set of polynomials is dense in $\mathcal{A}(\mathbb{D})$. On the other hand, by [2, Theorem 2.3], the Dirichlet polynomials are dense in $\mathcal{A}(\mathbb{C}_+)$, i.e.

$$\mathcal{A}(\mathbb{C}_+) = \overline{\operatorname{span}\left\{\frac{1}{n^s} : n \in \mathbb{N}\right\}}^{\|\cdot\|_\infty},$$

so the set of trigonometric polynomials is also dense in $\mathcal{A}(\mathbb{C}_+)$. Then a natural question appears:

Is it true that $\mathcal{M}(\mathcal{A}(\mathbb{C}_+)) = \{\delta_s : \operatorname{Re} s \geqslant 0\}$? The answer is negative and we are going to use the Bohr lift to prove that.

Theorem 3.9. *The set of evaluations* $\{\delta_s : Re\, s \geqslant 0\}$ *is a proper subset of* $\mathcal{M}(\mathcal{A}(\mathbb{C}_+))$.

Proof. By [2, Theorem 2.5], $\mathcal{A}(\mathbb{C}_+)$ is isometrically isomorphic to the algebra $\mathcal{A}_u(B_{c_0})$ of all holomorphic and uniformly continuous functions on the open unit ball B_{c_0} of the space c_0. The proof is done by showing that the Bohr lift

$$\mathfrak{L} : \mathcal{A}(\mathbb{C}_+) \longrightarrow \mathcal{A}_u(B_{c_0})$$

is an isomorphic isometry. Taking the transpose of it we have that

$$\mathfrak{L}^t : \mathcal{M}(\mathcal{A}_u(B_{c_0})) \longrightarrow \mathcal{M}(\mathcal{A}(\mathbb{C}_+)).$$

On the other hand, the Aron-Berner extension in the case of c_0, says that every bounded holomorphic function f on the open unit ball of c_0 can be extended in an unique way to $\hat{f}$ as a bounded holomorphic function on B_{ℓ_∞}, the open unit ball of the Banach space ℓ_∞ of bounded complex sequences endowed with the supremum norm. Moreover, this extension is multiplicative and preserves the norm. Additionally if $f \in \mathcal{A}_u(B_{c_0})$, then this process of extension can be done to the closed unit ball $\overline{B}_{\ell_\infty}$. Hence defined $\widetilde{\delta_a}(f) = \hat{f}(a)$, we have that $\widetilde{\delta_a} \in \mathcal{M}(\mathcal{A}_u(B_{c_0}))$ for every $a \in \overline{B}_{\ell_\infty}$. Actually $\mathcal{M}(\mathcal{A}_u(B_{c_0})) = \{\widetilde{\delta_z} : z \in \overline{B}_{\ell_\infty}\}$ (see [3]). We claim that if we take any $a = (a_n)_{n\in\mathbb{N}} \in \overline{B}_{\ell_\infty}$ satisfying that for some coordinate, say $a_N = 0$, then

$$\widetilde{\delta_z} \in \mathcal{M}(\mathcal{A}(\mathbb{C}_+)) \setminus \{\delta_s : \mathrm{Re}\, s \geqslant 0\}.$$

Indeed, we can assume $a_1 = 0$. If our claim is not true, then there exists s_0 with $\mathrm{Re}\, s_0 \geqslant 0$ such that $\mathfrak{L}^t(\widetilde{\delta_a}) = \delta_{s_0}$. Hence $\mathfrak{L}^t(\widetilde{\delta_a})(\frac{1}{2^s}) = \delta_{s_0}(\frac{1}{2^s}) = \frac{1}{2^{s_0}}$. But

$$\mathfrak{L}^t(\widetilde{\delta_a})\left(\frac{1}{2^s}\right) = \widetilde{\delta_a}\left(\mathfrak{L}\left(\frac{1}{2^s}\right)\right) = \widetilde{\delta_a}(z_1),$$

thus

$$\mathfrak{L}^t(\widetilde{\delta_a})\left(\frac{1}{2^s}\right) = \widetilde{\delta_a}(z_1) = a_1 = 0.$$

Therefore $\frac{1}{2^{s_0}} = 0$. A contradiction. ∎

In [21] the authors study spaces of multiple Dirichlet series and their properties. They used the Bohr transform and lift to get the main results.

A k-multiple Dirichlet series is a series of the form

$$\sum_{m_1,\ldots,m_k=1}^{\infty} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}},$$

where $\{a_{m_1,\ldots,m_k}\} \subset \mathbb{C}$ is the k-multiple sequence of coefficients of the series, and $s_1,\ldots,s_k \in \mathbb{C}$ are complex variables.

A k-multiple series is *regularly convergent* if it is convergent and all of its j-dimensional subseries are convergent, where a j-dimensional subseries is a series of the same multiple sequence in which we take the sum over j indexes $m_{i_1}, \dots, m_{i_j}$, where the other indexes $m_{l_1}, \dots, m_{l_{k-j}}$ remain fixed.

In an analogous way to the one dimensional case, in [21] is introduced the Banach algebra $\mathcal{A}(\mathbb{C}_+^k)$ of all k-multiple Dirichlet series which are regularly convergent on $\mathbb{C}_+^k$ and define uniformly continuous functions on $\mathbb{C}_+^k$. Since the algebra of the disc $\mathcal{A}(\mathbb{D})$ and the algebra of the bidisc $\mathcal{A}(\mathbb{D}^2)$ are not topologically isomorphic, as it was proved in [35], it is natural to guess that the algebra $\mathcal{A}(\mathbb{C}_+)$ is not topologically isomorphic to $\mathcal{A}(\mathbb{C}_+^2)$. But the answer is just the opposite.

Theorem 3.10. *The Banach algebras $\mathcal{A}(\mathbb{C}_+^k)$, $k \in \mathbb{N}$, are all isometrically isomorphic.*

The proof of this result is given in [21, Theorem 3.5] where it is shown that the natural extension of the Bohr transform and lift to several variables implies that for each k, the Banach algebra $\mathcal{A}(\mathbb{C}_+^k)$, $k \in \mathbb{N}$, is isometrically isomorphic to $\mathcal{A}_u(B_{c_0^k})$ that in turn, in a straightforward way, it is isometrically isomorphic as Banach algebra to $\mathcal{A}_u(B_{c_0})$.

This is another example of a problem that from the point of view of Dirichlet series in several variables looks difficult to attach and it is transparent in the infinite dimensional setting.

We finally introduce the concept of ℓ_1-multiplier for $\mathcal{H}_\infty$-Dirichlet series. A sequence (b_n) of complex numbers is said to be an ℓ_1-multiplier for $\mathcal{H}_\infty$ whenever

$$\sum_{n=1}^{\infty} |a_n b_n| < \infty$$

for all $\sum_n a_n n^{-s} \in \mathcal{H}_\infty$. Recall that a sequence (b_n) of complex numbers is said to be completely multiplicative whenever $b_{nm} = b_n b_m$ for all n, m.

The Bohr transform links the concept of completely multiplicative ℓ_1-multipliers with concept of sets of monomial convergence for functions in $H_\infty(B_{c_0})$

$$\mathrm{mon} H_\infty(B_{c_0}) = \Big\{ z \in \mathbb{C}^{\mathbb{N}} \ : \ \forall\, f \in H_\infty(B_{c_0}) : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \big| c_\alpha(f) z^\alpha \big| < \infty \Big\}.$$

introduced and first studied in [26]. The connection is the following.

Let (b_n) be a completely multiplicative sequence of complex numbers, and $1 \leqslant p \leqslant \infty$. Then (b_n) is an ℓ_1-multiplier for $\mathcal{H}_\infty$ if and only if $(b_{\mathfrak{p}_k}) \in \mathrm{mon} H_\infty(B_{c_0})$.

For each bounded sequence $z = (z_n)$ of complex numbers we define

$$\boldsymbol{b}(z) = \Big(\limsup_{n \to \infty} \frac{1}{\log n} \sum_{j=1}^{n} z_j^{*2} \Big)^{1/2},$$

where $z^* = (z_n^*)$ is the decreasing rearrangement of z.

We recall that $\ell_{\frac{2m}{m-1},\infty} = \{ z = (z_n)_n \colon \|z\| = \sup_n z_n^* n^{\frac{m-1}{2m}} < \infty \}$.

Theorem 3.11. *Let (b_n) be a completely multiplicative sequence of complex numbers, $1 \leqslant p < \infty$ and $m \in \mathbb{N}$.*

(1) *(b_n) is an ℓ_1-multiplier for $\mathcal{H}_\infty^m = \left\{ \sum a_n n^{-s} \in \mathcal{H}_\infty : a_n \neq 0 \Rightarrow \Omega(n) = m \right\}$ if and only if $(b_{\mathfrak{p}_j}) \in \ell_{\frac{2m}{m-1},\infty}$.*

(2) *(b_n) is an ℓ_1-multiplier for $\mathcal{H}_\infty$ provided we have that $|b_{\mathfrak{p}_j}| < 1$ for all j and $\boldsymbol{b}\big((b_{\mathfrak{p}_j})\big) < 1$.*
Conversely, if (b_n) is ℓ_1-multiplier for $\mathcal{H}_\infty$, then $|b_{\mathfrak{p}_j}| < 1$ for all j and $\boldsymbol{b}\big((b_{\mathfrak{p}_j})\big) \leqslant 1$.

This result is obtained in [9], by describing the monomial sets of convergence for $H_\infty(B_{c_0})$ and applying that study to these spaces of Dirichlet series. The key is the next theorem on monomial convergence given also in [9].

Theorem 3.12. *For each $z \in \mathbb{D}^{\mathbb{N}}$ the following two statements hold:*

(1) *If $\limsup_{n\to\infty} \dfrac{1}{\log n} \sum_{j=1}^{n} z_j^{*2} < 1$, then $z \in \operatorname{mon} H_\infty(B_{c_0})$.*

(2) *If $z \in \operatorname{mon} H_\infty(B_{c_0})$, then $\limsup_{n\to\infty} \dfrac{1}{\log n} \sum_{j=1}^{n} z_j^{*2} \leqslant 1$; moreover, here the converse implication is false.*

For m-homogeneous polynomials on c_0 the set of convergence is completely described. Defined

$$\operatorname{mon}\mathcal{P}(^m c_0) = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \forall P \in \mathcal{P}(^m c_0) : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \left| c_\alpha(P) z^\alpha \right| < \infty \right\},$$

we have

Theorem 3.13 ([9]).

$$\operatorname{mon}\mathcal{P}(^m c_0) = \ell_{\frac{2m}{m-1},\infty}.$$

4. Dirichlet series approach to infinite dimensional holomorphy

Now we want to show that the classical theory of Dirichlet series can give new information on $H_\infty(B_{c_0})$, the space of all holomorphic and bounded functions on B_{c_0}.

Theorem 4.1. *If $f : B_{c_0} \to \mathbb{C}$ is holomorphic and bounded, then its monomial coefficients $(c_\alpha(f))_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ is an element of $\ell_2(\mathbb{N}_0^{(\mathbb{N})})$ and its norm is less than or equal to $\sup_{z \in B_{c_0}} |f(z)|$.*

Proof. Let $D = \sum_{n=1}^\infty a_n \frac{1}{n^s} = \mathfrak{B}(f)$. We know that $\sum a_n n^{-s} \in \mathcal{H}_\infty$. Now, Carlson's inequality, [25, Proposition 1.21], implies that

$$\left(\sum_{n=1}^\infty |a_n|^2 \right)^{1/2} \leqslant \sup_{[\operatorname{Re} s > 0]} |D(s)| = \sup_{z \in B_{c_0}} |f(z)|.$$

But we have that $c_\alpha(f) = a_{\mathfrak{p}^\alpha}$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$. ■

This result was greatly improved working in the context of Sidon sets in [23, Corollary 2] where it is proved that for every $c < 1/\sqrt{2}$ it holds that

$$\sum_{n=1}^{\infty} |a_n| n^{-\frac{1}{2}} e^{c\sqrt{\log n \log\log n}} < \infty, \tag{4.7}$$

for every $\sum a_n n^{-s} \in \mathcal{H}_\infty$. (See also [5]).

Given $N \geqslant 2$ the Sidon number $S(N)$ is defined as the best constant $C > 0$ such that for every Dirichlet polynomial $\sum_{n=1}^{N} a_n n^{-s}$

$$\sum_{n=1}^{N} |a_n| \leqslant C \sup_{[\operatorname{Re} s>0]} \Big| \sum_{n=1}^{N} \frac{a_n}{n^s} \Big| .$$

Actually (4.7) is a corollary of [23, Theorem 3] where (improving a result by R. de la Bretèche in [17]) it is established the following:

$$S(N) = \sqrt{N} \exp\Big\{ \Big(-\frac{1}{\sqrt{2}} + o(1) \Big) \sqrt{\log N \log\log N} \Big\},$$

when $N \to \infty$.

As an immediate consequence we obtain the next result.

Theorem 4.2. *Let J be a finite subset of $\mathbb{N}_0^{(\mathbb{N})}$, $N(J) = \max\{\mathfrak{p}^\alpha : \alpha \in J\}$, and let $\sum_{\alpha\in J} c_\alpha z^\alpha$ be a polynomial of finite type on c_0. We have*

$$\sum_{\alpha\in J} |c_\alpha| \leqslant S(N(J)) \Big\| \sum_{\alpha\in J} c_\alpha z^\alpha \Big\|_{B_{c_0}},$$

when $\operatorname{card}(J) \to \infty$.

In the previous section, we have described the results obtained in [9] on the set of convergence of the monomial expansion of all bounded holomorphic function on the open unit ball of c_0. In particular, Theorem 3.13 states $\operatorname{mon}\mathcal{P}(^m c_0) = \ell_{\frac{2m}{m-1},\infty}$. Let us show how the use of a deep property of Dirichlet series allows us to give an alternative proof of the inclusion

$$\ell_{\frac{2m}{m-1},\infty} \subset \operatorname{mon}\mathcal{P}(^m c_0).$$

Proof. Let $P \in \mathcal{P}(^m c_0)$ and $D = \mathfrak{B}(P) \in \mathcal{H}_\infty$. We have that $\|D\|_{[\operatorname{Re} s>0]} = \|P\|$ and, moreover, that $D(s) = \sum_{\Omega(n)=m} a_n \frac{1}{n^s}$ where recall that $a_{\mathfrak{p}^\alpha} = c_\alpha(P)$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$. But we know from the theory of Dirichlet series, [5, Theorem 1.4], that there is a constant $K_m > 0$ so that for every bounded Dirichlet series of that type it holds the following inequality

$$\sum_{\Omega(n)=m} |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} \leqslant K_m \sup_{t\in\mathbb{R}} \Big| \sum_{\Omega(n)=m} a_n n^{it} \Big|. \tag{4.8}$$

On the other hand, for every finite family $(x_j)_j$ with $x_j \geqslant 0$ and all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = m$, a simple application of the binomial formula yields

$$(x^\alpha)^{1/m} \leqslant \sum_j \alpha_j x_j. \tag{4.9}$$

Now, if $z \in \ell_{\frac{2m}{m-1},\infty}$ we have (z^* denotes the decreasing rearrangement of z) $\sup_n z_n^* n^{\frac{m-1}{2m}} = \|z\| < \infty$. Then

$$z_n^* \leqslant \|z\| \frac{1}{n^{\frac{m-1}{2m}}} \leqslant \|z\| \Big(\frac{\log(n \log n)}{n \log n}\Big)^{\frac{m-1}{2m}} \leqslant \|z\| A \Big(\frac{\log(\mathfrak{p}_n)}{\mathfrak{p}_n}\Big)^{\frac{m-1}{2m}}$$

where in the last step A is a constant that come from applying the Prime Number Theorem. Hence

$$\begin{aligned}
\sum_{|\alpha|=m} |c_\alpha| (z^*)^\alpha &\leqslant (\|z\|A)^m \sum_{|\alpha|=m} |c_\alpha| \bigg(\Big(\frac{\log(\mathfrak{p}_n)}{\mathfrak{p}_n}\Big)^{\frac{m-1}{2m}} \bigg)^\alpha \\
&= (\|z\|A)^m \sum_{|\alpha|=m} |c_\alpha| \frac{\Big(\big[(\log \mathfrak{p}_n)^\alpha\big]^{1/m}\Big)^{\frac{m-1}{2}}}{(\mathfrak{p}^\alpha)^{\frac{m-1}{2m}}} \\
&\leqslant (\|z\|A)^m \sum_{|\alpha|=m} |c_\alpha| \frac{\Big(\sum \alpha_k \log \mathfrak{p}_k\Big)^{\frac{m-1}{2}}}{(\mathfrak{p}^\alpha)^{\frac{m-1}{2m}}} \\
&= (\|z\|A)^m \sum_{\Omega(n)=m} |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} \\
&\leqslant (\|z\|A)^m \cdot K_m \sup_{t \in \mathbb{R}} \Big| \sum_{\Omega(n)=m} a_n n^{it} \Big| \\
&= (\|z\|A)^m \cdot K_m \|P\|.
\end{aligned}$$

Above, to obtain the first inequality we apply (4.9). The second inequality is consequence of (4.8).

This shows that $z^* \in \mathrm{mon}\mathcal{P}({}^m c_0)$. Now, since $z \in \ell_{\frac{2m}{m-1},\infty}$, we have that z belongs to c_0, and there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $(z_n) = (z^*_{\sigma(n)})$. But by [24, page 550], if a sequence is in $\mathrm{mon}\mathcal{P}({}^m c_0)$, then every rearrangement by a permutation is also there. Hence we have $z \in \mathrm{mon}\mathcal{P}({}^m c_0)$ and we are done. ■

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