Functiones et Approximatio
59.1 (2018), 99–115
doi: 10.7169/facm/1710

FILTRATIONS OF INFINITESIMAL GENERATORS

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Dedicated to the memory of
Paweł Domański

Abstract: We study the problem of characterizing membership of normalized holomorphic functions of the disc to the class of infinitesimal generators and their sectorial analytical extension. We provide new formulas and applications to dynamics of the corresponding semigroups using filtrations of the class of infinitesimal generators.

Keywords: semigroups of holomorphic functions, infinitesimal generators, filtrations

1. Preliminaries and motivations

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, denote by $\mathrm{Hol}(\mathbb{D},\mathbb{C})$ the set of holomorphic functions on $\mathbb{D}$, and by $\mathrm{Hol}(\mathbb{D})$ the set of holomorphic self-maps of $\mathbb{D}$.

Definition 1.1. *A family* $\{\phi_t\}_{t\geqslant 0}\subset \mathrm{Hol}(\mathbb{D})$ *is called a one-parameter continuous semigroup (or just semigroup) if*

(a) $\phi_{t+s}=\phi_t\circ\phi_s,\ t,s\geqslant 0$*; and*
(b) $\lim_{t\to 0^+}\phi_t=\mathrm{id}_{\mathbb{D}}$*, where* $\mathrm{id}_{\mathbb{D}}$ *is the identity map on* $\mathbb{D}$ *and the limit is taken with respect to the topology of uniform convergence on compact sets in* $\mathbb{D}$*.*

A remarkable result of Berkson and Porta [4] asserts that each one-parameter semigroup of holomorphic self-maps of $\mathbb{D}$ is locally uniformly differentiable with respect to the parameter $t\geqslant 0$, and, moreover, if

$$f=\lim_{t\to 0^+}\frac{1}{t}\left(\mathrm{id}_{\mathbb{D}}-\phi_t\right),$$

The second and third authors were partially supported by the Ministerio de Economía y Competitividad and the European Union (FEDER), project MTM2015-63699-P, and by La Consejería de Economía y Conocimiento de la Junta de Andalucía.

2010 Mathematics Subject Classification: primary: 37C10, 30C35; secondary: 30D05, 37C25 then ϕ_t is the solution of the Cauchy problem:

$$\frac{\partial \phi_t(z)}{\partial t} + f(\phi_t(z)) = 0 \qquad \text{and} \qquad \phi_0(z) = z \in \mathbb{D}. \tag{1.1}$$

The function f is called the *infinitesimal generator* of the semigroup $\{\phi_t\}_{t\geqslant 0} \subset \mathrm{Hol}(\mathbb{D})$. The class of all holomorphic generators is denoted by $\mathcal{G}$.

From the continuous version of the Denjoy-Wolff Theorem (see, for example, [17] and [15]) a semigroup $\{\phi_t\}_{t\geqslant 0}$ has at most one interior common fixed point $\tau \in \mathbb{D}$, and this point is an attractive point of $\{\phi_t\}_{t\geqslant 0}$ in the sense that

$$\tau = \lim_{t\to\infty} \phi_t(z), \qquad z \in \mathbb{D},$$

if and only if $\{\phi_t\}_{t\geqslant 0}$ contains neither an elliptic automorphism of $\mathbb{D}$ nor the identity mapping. This point τ is called the *Denjoy-Wolff point* for the semigroup $\{\phi_t\}_{t\geqslant 0}$. It follows from the uniqueness of solutions to the Cauchy problem that this point τ must be the (unique) null point of f in $\mathbb{D}$.

Up to Möbius transformations T_τ $\left(T_\tau(z) = \frac{\tau - z}{1 - z\overline{\tau}},\ \tau \in \mathbb{D}\right)$ of the unit disc for a semigroup having an interior fixed point, one can always require the condition $f(0) = 0$, or, what is the same, $\phi_t(0) = 0$ for all $t \geqslant 0$.

A necessary and sufficient condition (see [4]) for a holomorphic function f such that $f(0) = 0$ to be an infinitesimal generator is

$$\mathrm{Re}\,\frac{f(z)}{z} \geqslant 0, \qquad z \in \mathbb{D}\setminus\{0\}. \tag{1.2}$$

In this paper, we reduce to consider the class $\mathcal{A}$ of functions which are holomorphic on the open unit disc $\mathbb{D}$ and normalized by $f(0) = f'(0) - 1 = 0$, and denote by $\mathcal{G}_0$ the subset of $\mathcal{A}$ consisting of infinitesimal generators, that is,

$$\mathcal{A} := \{f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C}) :\ f(0) = f'(0) - 1 = 0\} \qquad \text{and} \qquad \mathcal{G}_0 := \mathcal{A} \cap \mathcal{G}. \tag{1.3}$$

The class $\mathcal{G}_0$ is very important for the study of non-autonomous problems, such as Loewner theory (see, *e.g.*, [8, 6]). For this class $\mathcal{G}_0$, we consider the following problems:

1. Criteria for membership to the class $\mathcal{G}_0$.
2. Analytic extension of a semigroup in its parameter into a domain in the complex plane.
3. The study of the asymptotic behavior of semigroups.

The first problem has been investigated (also for the entire class $\mathcal{G}$ and in several variables) by many authors, *e.g.*, [2, 3, 4, 5, 9, 16] and [17] and, in essentially all cases, the answer is given by suitable inequalities in terms of intrinsic objects (hyperbolic metric, hyperbolic distance, Green function) or extrinsic ones (Euclidean inequalities). For our purpose we recall the so-called *Abate's formula* [1]: a function $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is an infinitesimal generator if and only if for all $z \in \mathbb{D}$,

$$\mathrm{Re}\left[2f(z)\overline{z} + (1 - |z|^2)f'(z)\right] \geqslant 0. \tag{1.4}$$

However, often in practice given $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$, it is hard to verify (1.2) or (1.4) (or the other equivalent conditions). For instance, it is not trivial to check using (1.2) or (1.4) whether the function $f(z) = -z - 2\log(1-z)$ is an infinitesimal generator. However, we will show in Theorem 1.3 that a sufficient condition for $g \in \mathcal{A}$ to be an infinitesimal generator is that $\operatorname{Re} g'(z) \geqslant 0$ for all $z \in \mathbb{D}$. Hence, since $\operatorname{Re} f'(z) = \operatorname{Re} \frac{1+z}{1-z} > 0, z \in \mathbb{D}$, it follows at once that, in fact, f is an infinitesimal generator.

The condition $\operatorname{Re} f'(z) \geqslant 0$ for $f \in \mathcal{A}$ implies by the Noshiro-Warschawski Theorem [8, 11] that f is univalent. Since not all infinitesimal generators are univalent, the condition $\operatorname{Re} f'(z) \geqslant 0$ is far from being a necessary condition for membership in $\mathcal{G}_0$. Therefore, such a condition defines a natural subclass of $\mathcal{G}_0$. How can we move in some natural way from such a subclass to the full class $\mathcal{G}_0$?

In order to properly answer the previous question, we introduce the notion of filtration:

Definition 1.2. *A filtration of $\mathcal{G}_0$ is a family $\mathfrak{F} = \{\mathfrak{F}_s\}_{s\in[a,b]}$, $\mathfrak{F}_s \subseteq \mathcal{G}_0$, where $a, b \in [-\infty, +\infty]$, $a < b$, such that $\mathfrak{F}_s \subseteq \mathfrak{F}_t$ whenever $a \leqslant s \leqslant t \leqslant b$. Moreover, we say that*

- *the filtration $\{\mathfrak{F}_s\}_{s\in[a,b]}$ is strict if $\mathfrak{F}_s \subsetneq \mathfrak{F}_t$ for $s < t$; and*
- *the filtration $\{\mathfrak{F}_s\}_{s\in[a,b]}$ is exhaustive if $\mathfrak{F}_b = \mathcal{G}_0$.*

In this paper, we define and study some natural filtrations of $\mathcal{G}_0$ which are related to the questions (1) and (2) above and also detect some dynamical properties of the associated semigroups. For instance, a natural filtration, due to (1.2), is the one given by $\{f \in \mathcal{A} : \operatorname{Re} \frac{f(z)}{z} \geqslant -a, \forall z \in \mathbb{D} \setminus \{0\}\}_{a\in(-\infty,0]}$. Membership to one of these classes reflects on the ratio of convergence of the associated semigroup to 0 (see Proposition 2.7).

The main result of this paper, which clearly is a consequence of Theorem 4.7, is the following:

Theorem 1.3. *Let $f \in \mathcal{A}$ satisfy the condition*

$$\operatorname{Re}\left[\alpha|z|^2\frac{f(z)}{z} + (1-\alpha)(1-|z|^2)f'(z)\right] \geqslant 0, \qquad z \in \mathbb{D}\setminus\{0\}, \tag{1.5}$$

for some $\alpha \in [0,1]$. Then f is a holomorphic generator. Moreover,

(i) *If inequality* (1.5) *holds for some $\alpha \in [0, 2/3]$, then it also holds for all $\beta \in (\alpha, 1]$.*
(ii) *For every $\beta \in (0, 2/3]$ there exists f_β which satisfies* (1.5) *for $\alpha = \beta$ but does not satisfy* (1.5) *for $\alpha \in [0, \beta)$.*
(iii) *If $f \in \mathcal{A}$ is an infinitesimal generator, then inequality* (1.5) *holds for all $\alpha \in [2/3, 1]$.*

In other words, the filtration of $\mathcal{G}_0$ defined by (1.5) for $\alpha \in [0, 2/3]$ is strict and exhaustive.

Another result on the same line we prove is the following (see Theorems 4.12 and 4.15):

Theorem 1.4. *Let $f \in \mathcal{A}$ satisfy the condition*

$$\operatorname{Re}\left[\alpha \frac{f(z)}{z} + (1-\alpha) f'(z)\right] \geqslant 0, \qquad z \in \mathbb{D} \setminus \{0\}, \tag{1.6}$$

for some $\alpha \in [0,1]$. Then f is a holomorphic generator. Moreover,

(i) *If inequality* (1.6) *holds for some $\alpha \in [0,1)$, then it also holds for all $\beta \in (\alpha, 1]$.*
(ii) *For every $\beta \in (0,1]$ there exists f_β which satisfies* (1.6) *for $\alpha = \beta$ but does not satisfy* (1.6) *for $\alpha \in [0, \beta)$.*
(iii) *If $\{\phi_t\}_{t \geqslant 0} \subset \operatorname{Hol}(\mathbb{D})$ is the semigroup generated by $f \in \mathcal{A}$ satisfying* (1.6) *with some $\alpha \in [0,1)$, then the following estimate of convergence holds:*

$$|\phi_t(z)| \leqslant e^{-\varkappa(\alpha)t}|z|, \qquad \text{where} \quad \varkappa(\alpha) = \int_0^1 \frac{1-t^{1-\alpha}}{1+t^{1-\alpha}}\, dt > 0. \tag{1.7}$$

In particular, the filtration of $\mathcal{G}_0$ defined by (1.6) for $\alpha \in [0,1]$ is strict and exhaustive.

Note that both inequalities (1.5) and (1.6) are the same for $\alpha = 0$ and $\alpha = 1$. For $\alpha = 0$ we have the class of univalent functions satisfying the Noshiro-Warschawski condition, while for $\alpha = 1$ we get the whole class $\mathcal{G}_0 = \mathcal{A} \cap \mathcal{G}$. However, the classes of functions characterized by inequalities (1.5) and (1.6) are different for $\alpha \in (0,1)$.

In order to make a filtration of $\mathcal{G}_0$ useful, for instance to get estimates like (1.7), we introduce the following notion:

Definition 1.5. *Let $\mathcal{F} \subset \mathcal{G}_0$. We say that a function $f_* \in \mathcal{G}_0$ is totally extremal for $\mathcal{F}$ if*

(i) *$f_* \in \mathcal{F}$;*
(ii) *for every $\lambda \in \mathbb{C}$, $r \in [0,1]$ and $f \in \mathcal{F}$,*

$$\min_{|z|=r} \operatorname{Re}\left(\lambda \frac{f(z)}{z}\right) \geqslant \min_{|z|=r} \operatorname{Re}\left(\lambda \frac{f_*(z)}{z}\right).$$

Condition (ii) can be explained as follows. Given $f_* \in \mathcal{F}$, write $f_*(z) = zp_*(z)$. Then f_* is totally extremal for $\mathcal{F}$ if and only if for every $r \in [0,1]$ and $f \in \mathcal{F}$ with $f(z) = zp(z)$, the image $p(\mathbb{D}_r)$ of the disc of radius r lies in the convex hull of $p_*(\mathbb{D}_r)$.

If an infinitesimal generator is totally extremal in $\mathcal{F}$, the semigroup it generates has some extremal dynamical behavior among those semigroup generated by infinitesimal generators of the class $\mathcal{F}$, from which the reason of the name.

Definition 1.6. *We say that a filtration $\mathfrak{F} = \{\mathfrak{F}_s\}_{s \in [a,b]}$ admits a net $\{f_s\}_{s \in [a,b]}$ of totally extremal functions if for every $s \in [a,b]$, the function f_s is totally extremal for the class $\mathfrak{F}_s$.*

In the paper, we will consider also the problem of finding nets of totally extremal functions for the filtrations we consider.

2. Exponentially squeezing filtration

The issue of the asymptotic behavior of semigroups can be found in many sources (see, for instance, the books [9, 17] and references therein). In the context of this paper we need the following notion.

Definition 2.1. *We denote by* $\mathfrak{C} = \{\mathfrak{C}_\alpha\}_{\alpha\in[0,1]}$ *the family of sets defined by*

$$\mathfrak{C}_\alpha := \left\{ f \in \mathcal{A} : \ \operatorname{Re} f(z)\overline{z} \geqslant (1-\alpha)|z|^2 \quad \forall z \in \mathbb{D} \right\}.$$

Clearly, $\mathfrak{C}$ is a filtration, which we call the *exponentially squeezing filtration*. By definition, $\mathfrak{C}_1 = \mathcal{G}_0$, while the class $\mathfrak{C}_0$ is trivial since it reduces to the identity function.

Theorem 2.2. *The filtration* $\mathfrak{C}$ *is strict and exhaustive and admits a net* $\{f_\alpha\}_{\alpha\in(0,1]}$ *of totally extremal functions, where*

$$f_\alpha(z) = z\,\frac{1+(1-2\alpha)z}{1+z}.$$

Proof. The filtration $\mathfrak{C}$ is exhaustive since $\mathfrak{C}_1 = \mathcal{G}_0$.

On the other hand, a function $f(z) = zp(z)$ belongs to the class $\mathfrak{C}_\alpha$, $\alpha \in (0,1]$, if and only if $p(0) = 1$ and $\operatorname{Re} p(z) \geqslant 1-\alpha$. Since the function p_α defined by

$$p_\alpha(z) = \frac{f_\alpha(z)}{z} = \frac{1+(1-2\alpha)z}{1+z}$$

maps the open unit disc $\mathbb{D}$ conformally onto the half-plane $\{w \in \mathbb{C} : \ \operatorname{Re} w > 1-\alpha\}$, we conclude that $f_\alpha \notin \mathfrak{C}_\beta$ whenever $\alpha < \beta$, so the filtration $\mathfrak{C}$ is strict. In addition, this subordination implies that $f(z) = zp(z) \in \mathfrak{C}_\alpha$ if and only if $p \prec p_\alpha$, that is, f_α is totally extremal for $\mathfrak{C}_\alpha$. ■

Next elementary lemma will be useful to bound the argument of Carathéodory functions associated with certain infinitesimal generators.

Lemma 2.3. *Let* $0 < R < C < +\infty$. *Then*

$$\max_{|w-C|=R} |\arg w| = \arctan\left(\frac{R}{\sqrt{C^2-R^2}}\right).$$

Since the image of $|z| = r$ by the function p_α (defined in the proof of Theorem 2.2) is the circle centered at $\frac{1-r^2(1-2\alpha)}{1-r^2}$ and radius $\frac{2\alpha r}{1-r^2}$, Lemma 2.3 and Theorem 2.2 imply:

Corollary 2.4.

(i) *If* $f \in \mathcal{G}_0$, *then* $f \in \mathfrak{C}_\alpha$ *if and only if for any* $r \in [0,1)$,

$$\min_{|z|=r} \operatorname{Re}\frac{f(z)}{z} \geqslant \operatorname{Re}\frac{f_\alpha(r)}{r} = \frac{1+(1-2\alpha)r}{1+r}.$$

(ii) *If* $f \in \mathfrak{C}_\alpha$, *then for any* $r \in [0,1)$,

$$\max_{|z|=r} \left| \arg \frac{f(z)}{z} \right| \leqslant \max_{|z|=r} \left| \arg \frac{f_\alpha(z)}{z} \right| = \arctan \left(\frac{2\alpha r}{\sqrt{(1-r^2)(1-(1-2\alpha)^2 r^2)}} \right).$$

Remark 2.5. It is well known (see, *e.g.*, [8, p. 73]) that if f is a normalized convex function then, for every $z \in \mathbb{D}$, $z \neq 0$,

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}.$$

Hence, every normalized convex function is an infinitesimal generator and belongs to the class $\mathfrak{C}_{\frac{1}{2}}$.

The study of the asymptotic behavior of the semigroups is mostly connected with establishing the local or global rates of convergence or the growth estimates of the semigroup with respect to the parameter. A well known result of Gurganus [13] states that for all $z \in \mathbb{D}$, the asymptotic behavior of a semigroup $\{\phi_t\}$, generated by $f \in \mathcal{G}_0$ can be described as follows:

$$|z| \exp\left(-s\frac{1+|z|}{1-|z|}\right) \leqslant |\phi_s(z)| \leqslant |z| \exp\left(-s\frac{1-|z|}{1+|z|}\right), \qquad s \in [0,\infty).$$

However, these estimates are not uniform on the whole disc. Also, it might happen that for specific subclasses of $\mathcal{G}_0$ the rate of convergence can be improved. Now we show that this is the case for the elements of the exponentially squeezing filtration. To do this we need the following notion.

Definition 2.6. *A continuous semigroup* $\{\phi_t\}_{t\geqslant 0} \subset \operatorname{Hol}(\mathbb{D})$ *is said to be exponentially squeezing with squeezing ratio* $a \geqslant 0$ *if* $|\phi_t(z)| \leqslant e^{-at}|z|$ *for all* $z \in \mathbb{D}$.

Proposition 2.7. *Given* $f \in \mathcal{G}_0$, *the following are equivalent:*

(1) $\{\phi_t\}_{t\geqslant 0}$ *is exponentially squeezing with squeezing ratio* $a \in [0,1]$,
(2) $f \in \mathfrak{C}_{1-a}$.

Proof. Given $z \in \mathbb{D}$, let $g(t) := |\phi_t(z)|^2 - e^{-2at}|z|^2$. Note that $g(0) = 0$ and differentiating in t we obtain

$$g'(t) = -2 \operatorname{Re} f(\phi_t(z))\overline{\phi_t(z)} + 2a|\phi_t(z)|^2 - 2ag(t).$$

If (1) holds, then $g(t) \leqslant 0$. Therefore, from

$$g'(0) = \lim_{t\to 0^+} \frac{g(t)-g(0)}{t} = \lim_{t\to 0^+} \frac{g(t)}{t} \leqslant 0,$$

we get (2).

Now, assume (2) holds. Setting $h(t) = g'(t) + 2ag(t)$ and solving the differential equation $g'(t) + 2ag(t) - h(t) = 0$ with the initial value $g(0) = 0$, we obtain $g(t) = e^{-2at}\int_0^t e^{2as}h(s)ds \leqslant 0$, since $h(t) \leqslant 0$ for all $t \geqslant 0$. Hence (1) holds. ∎

In turn, Proposition 2.7 and Remark 2.5 imply the following fact.

Corollary 2.8. *Let $\{\phi_t\}_{t\geqslant 0}$ be a semigroup generated by a convex function. Then this semigroup is exponentially squeezing with squeezing ratio $\frac{1}{2}$.*

Example 2.9. Consider the Cauchy problem:

$$\frac{\partial u(t,z)}{\partial t} + \log(1+u(t,z)) = 0, \qquad u(0,z) = z \in \mathbb{D}.$$

Since $f(z) = \log(1+z)$ is a convex function, its solution satisfies the estimate

$$|u(t,z)| \leqslant e^{-t/2}|z|.$$

3. Sectorial filtration

We next consider the following filtration, which we call the *sectorial filtration*.

Definition 3.1. *We denote by $\mathfrak{S} = \{\mathfrak{S}_\alpha\}_{\alpha\in[0,1]}$ the family of sets defined by*

$$\mathfrak{S}_\alpha := \left\{ f \in \mathcal{A} : \left|\arg \frac{f(z)}{z}\right| \leqslant \frac{\pi\alpha}{2} \quad \forall z \in \mathbb{D}\setminus\{0\} \right\}.$$

This definition immediately implies that $\mathfrak{S}$ is a filtration with $\mathfrak{S}_1 = \mathcal{G}_0$ and $\mathfrak{S}_0 = \{\mathrm{id}_{\mathbb{D}}\}$.

Theorem 3.2. *The filtration $\mathfrak{S}$ is strict and exhaustive and admits a net $\{f_\alpha\}_{\alpha\in(0,1]}$ of totally extremal functions, where*

$$f_\alpha(z) = z\left(\frac{1-z}{1+z}\right)^\alpha.$$

Proof. The filtration $\mathfrak{S}$ is exhaustive since $\mathfrak{S}_1 = \mathcal{G}_0$. A function $f(z) = zp(z)$ belongs to the class $\mathfrak{S}_\alpha$, $\alpha \in (0,1]$ if and only if $p(0) = 1$ and $|\arg p(z)| \leqslant \frac{\pi\alpha}{2}$. Since the function p_α defined by $p_\alpha(z) = \frac{f_\alpha(z)}{z} = \left(\frac{1-z}{1+z}\right)^\alpha$, maps the open unit disc $\mathbb{D}$ conformally onto the sector $\{w \in \mathbb{C} : |\arg w| \leqslant \frac{\pi\alpha}{2}\}$, we conclude that $f_\alpha \notin \mathfrak{S}_\beta$ whenever $\alpha < \beta$, so the filtration $\mathfrak{S}$ is strict. In addition, this subordination implies that $f(z) = zp(z) \in \mathfrak{S}_\alpha$ if and only if $p \prec p_\alpha$, that is, f_α is totally extremal for $\mathfrak{S}_\alpha$. ■

As an immediate consequence of this result, and using again Lemma 2.3, we have the following.

Corollary 3.3.

(i) *If $f \in \mathcal{G}_0$, then $f \in \mathfrak{S}_\alpha$ if and only if for any $r \in [0,1)$,*

$$\max_{|z|=r}\left|\arg\frac{f(z)}{z}\right| \leqslant \max_{|z|=r}\left|\arg\frac{f_\alpha(z)}{z}\right| = \alpha \arctan\left(\frac{2r}{1-r^2}\right).$$

(ii) *If $f \in \mathfrak{S}_\alpha$, then for any $r \in [0,1)$,*

$$\min_{|z|=r} \operatorname{Re}\frac{f(z)}{z} \geqslant \operatorname{Re}\frac{f_\alpha(r)}{r} = \left(\frac{1-r}{1+r}\right)^\alpha.$$

4. Linear filtrations

In this section, we deal with filtrations defined by expressions that depend linearly on a function f and its derivative f'. Such filtrations can be determined as follows. Let $\mathcal{K}$ be the set consisting of all functions $k : [0,1) \times [0,1) \to [0,+\infty)$ such that for every $r \in [0,1)$ the function $[0,1) \ni \alpha \mapsto k(\alpha, r)$ is non-decreasing. We set $k(1,r) := \lim_{\alpha \to 1^-} k(\alpha, r)$, possibly infinite.

Definition 4.1. *Given $k \in \mathcal{K}$, we denote by $\mathfrak{F}[k] = \{\mathfrak{F}_\alpha[k]\}_{\alpha\in[0,1]}$ the family of sets defined by*

$$\mathfrak{F}_\alpha[k] := \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left[k(\alpha, |z|)\frac{f(z)}{z} + f'(z) \right] \geqslant 0 \quad \forall z \in \mathbb{D} \setminus \{0\} \right\}. \tag{4.1}$$

Theorem 4.2. *For any $k \in \mathcal{K}$, the family $\mathfrak{F}[k]$ is a filtration of $\mathcal{G}_0$. If, in addition, for some $\gamma_0 \in [0,1]$, we have*

$$k(\gamma_0, r) \geqslant \frac{1+r^2}{1-r^2}, \tag{4.2}$$

then $\mathfrak{F}_{\gamma_0}[k] = \mathcal{G}_0$. Hence, the filtration $\{\mathfrak{F}_\alpha[k]\}_{\alpha\in[0,\gamma]}$ is exhaustive for every $\gamma \in [\gamma_0, 1]$.

Proof. Let $f \in \mathfrak{F}_\alpha[k]$ for some $\alpha \in [0,1]$. Let $p(z) := \frac{f(z)}{z}$. To see that $\mathfrak{F}_\alpha[k] \subseteq \mathcal{G}_0$, it is enough to show that $\mathrm{Re}\, p(z) \geqslant 0$ for all $z \in \mathbb{D}$. Equation (4.1) can be rewritten as

$$(1 + k(\alpha, |z|))\, \mathrm{Re}\, p(z) + \mathrm{Re}\, zp'(z) \geqslant 0, \qquad z \in \mathbb{D}. \tag{4.3}$$

By [17, Lemma 3.5.3], $\mathrm{Re}\, p(z) \geqslant 0$ for all $z \in \mathbb{D}$, and, hence, $\mathfrak{F}_\alpha[k] \subset \mathcal{G}_0$ for all $\alpha \in [0,1]$.

Fix $0 \leqslant \alpha < \beta \leqslant 1$ and let $f(z) = zp(z) \in \mathfrak{F}_\alpha[k]$. By (4.3), for all $z \in \mathbb{D}$,

$$\mathrm{Re}\, zp'(z) \geqslant -1 - k(\alpha, |z|)\, \mathrm{Re}\, p(z).$$

Hence, since $k(\cdot, |z|)$ is non-negative and non-decreasing, for all $z \in \mathbb{D}$, $z \neq 0$, we have

$$\begin{aligned} \mathrm{Re}\left[k(\beta, |z|)\frac{f(z)}{z} + f'(z) \right] &= \mathrm{Re}\left[(1 + k(\beta, |z|))p(z) + zp'(z) \right] \\ &\geqslant \mathrm{Re}\left[(1 + k(\beta, |z|))p(z) - (1 + k(\alpha, |z|))p(z) \right] \\ &= \left[k(\beta, |z|) - k(\alpha, |z|) \right] \mathrm{Re}\, p(z) \geqslant 0. \end{aligned}$$

Therefore, $\mathfrak{F}_\alpha[k] \subseteq \mathfrak{F}_\beta[k]$, that is, $\mathfrak{F}[k]$ is a filtration of $\mathcal{G}_0$.

Assume (4.2) holds. We show that any $f \in \mathcal{G}_0$ belongs to $\mathfrak{F}_\gamma[k]$. Indeed, given $f \in \mathcal{G}_0$, let $p(z) := \frac{f(z)}{z}$. By (1.2), $\mathrm{Re}\, p(z) \geqslant 0$ for all $z \in \mathbb{D}$. It is well-known that

$|zp'(z)| \leqslant \dfrac{2\operatorname{Re} p(z)}{1-|z|^2}$ (see, for example, [11] or [12]). Therefore, according to our assumption

$$\begin{aligned}\operatorname{Re}\left[k(\gamma,|z|)\frac{f(z)}{z}+f'(z)\right] &= \operatorname{Re}\left[(k(\gamma,|z|)+1)\,p(z)+zp'(z)\right]\\ &\geqslant \left[(k(\gamma,|z|)+1)-\frac{2}{1-|z|^2}\right]\operatorname{Re} p(z)\geqslant 0.\end{aligned}$$

The proof is complete. ■

Setting $k(\alpha, r) := \alpha$, we immediately get the following.

Corollary 4.3. *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $f'(0) = 1$. If $\operatorname{Re} f'(z) \geqslant 0$ for all $z \in \mathbb{D}$, then $f \in \mathcal{G}_0$.*

Question 4.4. *What conditions on $k(\alpha, |z|)$ imply that the filtration $\mathfrak{F}[k]$ is strict?*

Now we see how linear filtrations can be used to get some information on the boundary behavior of a semigroup.

Let $f \in \mathcal{G}_0$. We say that the boundary point $\zeta = 1 \in \partial\mathbb{D}$ is a *boundary singularity of order* $\lambda \in (0, 1]$ for f if

$$\lim_{r\to 1^-}\frac{f(r)}{(1-r)^{\lambda}} = \omega \neq 0, \infty \tag{4.4}$$

and

$$\lim_{r\to 1^-}\frac{f'(r)}{(1-r)^{\lambda-1}} = -\lambda\omega. \tag{4.5}$$

By [7, Theorem 6.4], for every $\lambda \in (0, 1]$ there exists $f_\lambda \in \mathcal{G}_0$ having a boundary singularity of order λ at 1. A simple example of a function that satisfies (4.4) and (4.5) is $f(z) = z(1-z)^{\lambda}$, $\lambda \in (0, 1]$. Since $\operatorname{Re} \frac{f(z)}{z} \geqslant 0$, it follows immediately that

$$\operatorname{Re}\omega \geqslant 0. \tag{4.6}$$

Proposition 4.5. *Let $k \in \mathcal{K}$. Assume moreover that for every $\alpha \in [0, 1)$ the limit*

$$\ell(\alpha) := \lim_{r\to 1^-} k(\alpha, r)(1-r)$$

exists and $\ell(\alpha) \neq \infty$. If $f \in \mathfrak{F}_\alpha[k]$ has a boundary singularity of order $\lambda \in (0, 1]$ at 1 then

$$\ell(\alpha) \geqslant \lambda.$$

Proof. Setting $z = r \in (0, 1)$ in (4.1) and dividing by $(1-r)^{\lambda-1}$, we obtain

$$\operatorname{Re}\left[k(\alpha,r)(1-r)\frac{f(r)}{r(1-r)^{\lambda}}+\frac{f'(r)}{(1-r)^{\lambda-1}}\right]\geqslant 0.$$

Taking limit as $r \to 1^-$, we obtain

$$\operatorname{Re}[\ell(\alpha)\omega - \lambda\omega] \geqslant 0.$$

Hence, $(\ell(\alpha) - \lambda)\operatorname{Re}\omega \geqslant 0$. By (4.6), we have the result. ■

4.1. Hyperbolic filtration

A particularly interesting filtration of type $\mathfrak{F}[k]$ is the one obtained by taking $k_1 : [0,1) \times [0,1) \to \mathbb{R}^+$ defined by

$$k_1(\alpha, r) := \frac{\alpha r^2}{(1-\alpha)(1-r^2)}.$$

In this case, condition (4.1) can be rewritten as

$$\operatorname{Re}\left[\alpha|z|^2 \frac{f(z)}{z} + (1-\alpha)(1-|z|^2)f'(z)\right] \geqslant 0, \qquad z \in \mathbb{D} \setminus \{0\}. \tag{4.7}$$

Definition 4.6. *We say that $f \in \mathcal{A}$ belongs to $\mathfrak{H}_\alpha$ if f satisfies* (4.7)*. We denote by $\mathfrak{H} = \{\mathfrak{H}_\alpha\}_{\alpha \in [0,1]}$ and call it the hyperbolic filtration.*

The reason for the name comes from the fact that the inequalities defining the filtration can be obtained by suitably differentiating a convex combination of inequalities containing the infinitesimal hyperbolic metric and the hyperbolic distance:

$$\omega(\phi_t(z), \phi_t(w)) \leqslant \omega(z,w) \qquad \text{or} \qquad \kappa(\phi_t(z); f(\phi_t(z))) \leqslant \kappa(z; f(z)),$$

which hold for every semigroup $(\phi_t(z))$ with infinitesimal generator f, where ω denotes the hyperbolic distance and κ the infinitesimal hyperbolic metric on $\mathbb{D}$.

Note also that for $\alpha \in [0,1)$, the sufficient condition for exhaustiveness of Theorem 4.2 does not hold, while obviously $\mathfrak{H}_1 = \mathcal{G}_0$. Hence, $\{\mathfrak{H}_\alpha\}_{\alpha \in [0,\gamma]}$ is an exhaustive filtration of $\mathcal{G}_0$ with some $\gamma \leqslant 1$. In fact, for $\alpha = \frac{2}{3}$, equation (4.7) reduces to the Abate's formula (1.4). Therefore,

$$\mathfrak{H}_\alpha = \mathcal{G}_0 \qquad \text{for all } \ \alpha \in \left[\frac{2}{3}, 1\right].$$

Theorem 4.7. *The filtration $\{\mathfrak{H}_\alpha\}_{\alpha \in [0, \frac{2}{3}]}$ is strict and exhaustive.*

Proof. As we already observed, $\{\mathfrak{H}_\alpha\}_{\alpha \in [0,\gamma]}$ is an exhaustive filtration for some $\gamma \leqslant \frac{2}{3}$. Hence, it suffices to show that $\{\mathfrak{H}_\alpha\}_{\alpha \in [0, \frac{2}{3}]}$ is strict.

To end this, we have to find for every given $\beta \in [0, \frac{2}{3}]$, some function $f_\beta \in \mathfrak{H}_\beta$ such that $f_\beta \notin \mathfrak{H}_\alpha$ for $\alpha < \beta$. To this end, set

$$f_\eta(z) := z(1 + \eta z^2).$$

A straightforward computation shows that

$$\begin{aligned}\operatorname{Re}&\left[\alpha|z|^2 \frac{f_\eta(z)}{z} + (1-\alpha)(1-|z|^2)f_\eta'(z)\right] \\ &= (1-\alpha) + (2\alpha - 1)|z|^2 + \eta[\alpha|z|^2 + 3(1-\alpha)(1-|z|^2)] \operatorname{Re} z^2 \\ &\geqslant (1-\alpha) + (2\alpha - 1)|z|^2 - \eta[\alpha|z|^2 + 3(1-\alpha)(1-|z|^2)]|z|^2.\end{aligned}$$

Hence, noticing that we have equality in the last inequality for $z = ir$, $r \in [0,1)$, we see that $f_\eta \in \mathfrak{H}_\alpha$ if and only if

$$\eta \leqslant \frac{(1-\alpha)+(2\alpha-1)r^2}{\alpha r^4 + 3(1-\alpha)(1-r^2)r^2} =: \psi(\alpha, r), \qquad r \in (0,1). \tag{4.8}$$

Now, for $\alpha \in [0, \frac{2}{3}]$, let $r(\alpha) \in [0,1]$ be such that

$$\psi(\alpha, r(\alpha)) = \min_{r\in(0,1]} \psi(\alpha, r).$$

Then, by (4.8), it follows that

$$f_\eta \in \mathfrak{H}_\alpha \iff \eta \leqslant \psi(\alpha, r(\alpha)). \tag{4.9}$$

Now fix $r \in (0,1]$ and write

$$\psi(\alpha, r) = \frac{A\alpha + B}{C\alpha + D},$$

where $A = 2r^2 - 1$, $B = 1 - r^2$, $C = r^4 - 3(1-r^2)r^2$, and $D = 3(1-r^2)r^2$. Differentiating in α, we obtain

$$\frac{\partial \psi(\alpha, r)}{\partial \alpha} = \frac{AD - CB}{(C\alpha + D)^2}.$$

Since

$$AD - CB = 2r^4(1-r)^2,$$

the function $[0, \frac{2}{3}] \ni \alpha \mapsto \psi(\alpha, r)$ is strictly increasing for $r \in (0,1)$.

It is easy to check that for $\alpha = \frac{2}{3}$, the minimum is attained at $r(\frac{2}{3}) = 1$ and $\psi(\frac{2}{3}, 1) = 1$. While, if $\alpha = 0$, the minimum is attained at $r = 1$ and is $\psi(0,1) = \frac{1}{3}$.

We claim that $r(\alpha) \in (0,1)$ and $\psi(\alpha, r(\alpha)) < 1$ if $\alpha \in (0, \frac{2}{3})$.

To see this, note that $\psi(\alpha, 1) = 1$ for every $\alpha \in (0, \frac{2}{3}]$ and $\psi(\alpha, r) \to +\infty$ for $r \to 0^+$. Hence, it is enough to show that there exists $r \in (0,1)$ such that $\psi(\alpha, r) < 1$. Indeed, this is equivalent to say that there exists $r \in (0,1)$ such that the function

$$\begin{aligned}\Phi(r) &:= (1-\alpha) + (2\alpha - 1)r^2 - [\alpha r^4 + 3(1-\alpha)(1-r^2)r^2] \\ &= (1-\alpha) + (5\alpha - 4)r^2 + (3 - 4\alpha)r^4 < 0.\end{aligned}$$

Since $\Phi(1) = 0$ and $\Phi'(1) = 2(2 - 3\alpha) > 0$ as $\alpha \in [0, \frac{2}{3})$, it follows that for $r < 1$ and r sufficiently close to 1, $\Phi(r) < 0$; hence, $\psi(\alpha, r) < 1$, and the claim is proved.

Now set $\eta(\alpha) := \psi(\alpha, r(\alpha))$. Then by (4.9), $f_{\eta(\alpha)} \in \mathfrak{H}_\alpha$ for all $\alpha \in [0, \frac{2}{3}]$. Moreover, let $\beta, \alpha \in [0, \frac{2}{3}]$ be such that $\alpha < \beta$. Then, taking into account that $\alpha \mapsto \psi(\alpha, r)$ is strictly increasing for every fixed $r \in (0,1)$, that $r(\beta) \in (0,1)$ for all $\beta \in (0, \frac{2}{3})$, and that $\psi(\alpha, r(\alpha)) < 1 = \psi(1, r(1))$, we have

$$\eta(\beta) = \psi(\beta, r(\beta)) > \psi(\alpha, r(\beta)) \geqslant \psi(\alpha, r(\alpha)).$$

Hence, $f_{\eta(\beta)}$ does not satisfy (4.9) and so $f_{\eta(\beta)} \notin \mathfrak{H}_\alpha$. ∎

Obviously, Theorem 4.7 implies Theorem 1.3. At the same time, the following question is still open.

Question 4.8. *Does the filtration $\mathfrak{H}$ admit a net of totally extremal functions?*

4.2. Analytic filtration

Now we concentrate on the filtration of type $\mathfrak{F}[k]$ obtained by choosing $k = k_0 : [0,1) \times [0,1) \to \mathbb{R}^+$, where

$$k_0(\alpha, r) := \frac{\alpha}{1-\alpha}\,.$$

In this case, condition (4.1) can be rewritten as

$$\operatorname{Re}\left[\alpha\frac{f(z)}{z} + (1-\alpha)f'(z)\right] \geqslant 0, \qquad z \in \mathbb{D}\setminus\{0\}. \tag{4.10}$$

Definition 4.9. *We denote by $\mathfrak{A}_\alpha$ the set of all $f \in \mathcal{A}$ which satisfy* (4.10)*. We let $\mathfrak{A} = \{\mathfrak{A}_\alpha\}_{\alpha\in[0,1]}$ and call it the* analytic filtration*.*

Lemma 4.10. *A function $f \in \mathcal{A}$ belongs to $\mathfrak{A}_\alpha$ if and only if*

$$f(z) = z\int_0^1 q\left(t^{1-\alpha}z\right)dt \tag{4.11}$$

for some function q in the Carathéodory class.

Proof. If $f \in \mathfrak{A}_\alpha$, then the function

$$q(z) := \alpha\frac{f(z)}{z} + (1-\alpha)f'(z), \qquad z \in \mathbb{D}, \tag{4.12}$$

belongs to Carathéodory class. Expanding f and q in this equality in power series around zero, one can see that (4.11) holds. On the other hand, given q of the Carathéodory class, the function f defined by (4.11) satisfies (4.12) and then it belongs to $f \in \mathfrak{A}_\alpha$. ■

Recall that the family $\mathfrak{A}$ forms a filtration by Theorem 4.2, while the sufficient condition for exhaustiveness of Theorem 4.2 is satisfied only if $\alpha = 1$. It is not clear *a priori* whether $\mathfrak{A}_\alpha = \mathcal{G}_0$ for some $\alpha < 1$.

To study this filtration in more detail, we denote

$$p_\alpha(z) := \int_0^1 \frac{1-t^{1-\alpha}z}{1+t^{1-\alpha}z}\,dt \qquad \text{and} \qquad f_\alpha(z) = zp_\alpha(z). \tag{4.13}$$

Note that

$$\inf_{z\in\mathbb{D}} \operatorname{Re} p_\alpha(z) = \int_0^1 \frac{1-t^{1-\alpha}}{1+t^{1-\alpha}}\,dt =: \varkappa(\alpha), \tag{4.14}$$

where function $\varkappa$ is decreasing and maps $[0,1]$ onto $[0, 2\ln 2 - 1]$.

The proof of the first statement of the following assertion was suggested by P. Gumenyuk.

Proposition 4.11.

(i) *For every* $\alpha \in [0,1]$, *the function* p_α *is univalent.*

(ii) *For every* $\alpha \in [0,1]$, $f_\alpha \in \mathfrak{A}_\alpha$. *Moreover, if* $f \in \mathcal{G}_0$ *then* $f \in \mathfrak{A}_\alpha$ *if and only if for all* $z \in \mathbb{D}$, $z \neq 0$,

$$\operatorname{Re}\left[\alpha \frac{f(z)}{z} + (1-\alpha) f'(z)\right] \geqslant \operatorname{Re}\left[\alpha \frac{f_\alpha(|z|)}{|z|} + (1-\alpha) f'_\alpha(|z|)\right]. \qquad (4.15)$$

Proof. To prove (i), consider the function $P(\theta) = \operatorname{Re}\left[\frac{1-t^{1-\alpha}re^{i\theta}}{1+t^{1-\alpha}re^{i\theta}}\right]$, where $r \in (0,1)$. Differentiating we get

$$P'(\theta) = 2\operatorname{Im}\left[\frac{t^{1-\alpha}re^{i\theta}}{(1+t^{1-\alpha}re^{i\theta})^2}\right] = 2\operatorname{Im}\left[\frac{z}{(1+z)^2}\right]\Bigg|_{z=t^{1-\alpha}re^{i\theta}}.$$

Thus, the function $P(\theta)$ is increasing on $[0,\pi]$ and decreasing on $[\pi, 2\pi]$. Therefore, for every fixed $r \in (0,1)$, the function $\operatorname{Re} p_\alpha(re^{i\theta})$ is also increasing on $[0,\pi]$ and decreasing on $[\pi, 2\pi]$. This implies that for every fixed $w \in \mathbb{C}$, the variation of $\arg(p_\alpha(z) - w)$ on each circle $|z| = r$ is not greater than 2π. Hence, p_α is univalent.

Now we have to show that $f_\alpha \in \mathfrak{A}_\alpha$. Indeed, since the function q_1 defined by $q_1(z) = \frac{1-z}{1+z}$ belongs to the Carathéodory class, this statement follows from Lemma 4.10. Furthermore, since the function f_α is a solution to the differential equation $\alpha\frac{u(z)}{z} + (1-\alpha)zu'(z) = q_1(z)$, inequality (4.15) follows by Harnack's inequality. ■

Theorem 4.12. *The filtration* $\mathfrak{A}$ *is strict and exhaustive and admits the univalent net of totally extremal functions* $\{f_\alpha\}$ *defined in* (4.13).

Proof. We already know by Proposition 4.11 that $f_\alpha \in \mathfrak{A}_\alpha$.

If $f \in \mathfrak{A}_\alpha$, it can be represented by (4.11). By the subordination $q \prec q_1$, where $q_1(z) = \frac{1-z}{1+z}$, we conclude that for every $\lambda \in \mathbb{C}$,

$$\min_{|z|=r} \operatorname{Re}\left(\lambda \frac{f(z)}{z}\right) \geqslant \min_{|z|=r} \operatorname{Re}\left(\lambda p_\alpha(z)\right), \qquad (4.16)$$

so the family of functions $\{f_\alpha\}$ forms a net of totally extremal functions for the filtration $\mathfrak{A}$. The functions of this net are univalent by Lemma 4.11.

To complete the proof, it is enough to show that $f_\beta \notin \mathfrak{A}_\alpha$ whenever $0 \leqslant \alpha < \beta \leqslant 1$. Indeed, we have already seen $\inf_{z\in\mathbb{D}} \operatorname{Re}\frac{f_\beta(z)}{z} = \varkappa(\beta) < \varkappa(\alpha)$, while $\operatorname{Re}\frac{f(z)}{z} \geqslant \varkappa(\alpha)$ for any $f \in \mathfrak{A}_\alpha$. ■

Now we discuss other consequences of Theorem 4.12.

Corollary 4.13. *Let* $0 \leqslant \alpha \leqslant \beta \leqslant 1$ *and* $f \in \mathfrak{A}_\alpha$. *Then, for all* $z \in \mathbb{D} \setminus \{0\}$,

$$\operatorname{Re}\left[\beta \frac{f(z)}{z} + (1-\beta) f'(z)\right] \geqslant \operatorname{Re}\left[\beta \frac{f_\alpha(|z|)}{|z|} + (1-\beta) f'_\alpha(|z|)\right].$$

Proof. Take $p(z) = \frac{f(z)}{z}$, $z \in \mathbb{D}$, $z \neq 0$. Inequality (4.16) and the very definition of p_α imply that $\operatorname{Re} p(z) \geqslant \operatorname{Re} p_\alpha(|z|)$ for all $z \in \mathbb{D}$. This fact and inequality (4.15) show

$$\begin{aligned}
&\operatorname{Re}\left[\beta\frac{f(z)}{z} + (1-\beta)f'(z)\right] \\
&\qquad = \operatorname{Re}\left[p(z) + (1-\beta)zp'(z)\right] \\
&\qquad = \frac{1-\beta}{1-\alpha}\operatorname{Re}\left[\left(\frac{1-\alpha}{1-\beta} - 1\right)p(z) + p(z) + (1-\alpha)zp'(z)\right] \\
&\qquad \geqslant \frac{1-\beta}{1-\alpha}\operatorname{Re}\left[\left(\frac{1-\alpha}{1-\beta} - 1\right)p_\alpha(|z|) + p_\alpha(|z|) + (1-\alpha)|z|p'_\alpha(|z|)\right] \\
&\qquad = \operatorname{Re}\left[\beta\frac{f_\alpha(|z|)}{|z|} + (1-\beta)f'_\alpha(|z|)\right]. \qquad \blacksquare
\end{aligned}$$

Theorem 4.14. *Let $\alpha \in [0,1]$ and $f \in \mathfrak{A}_\alpha$. Then*

$$\sup_{z\in\mathbb{D}}\left|\arg\frac{f(z)}{z}\right| \leqslant \sup_{z\in\mathbb{D}}\left|\arg\frac{f_\alpha(z)}{z}\right| \leqslant \frac{\pi b}{2},$$

where $b = b(\alpha)$ is the unique solution in $[0,1]$ to the equation

$$\frac{\pi b}{2} + \arctan\left((1-\alpha)b\right) = \frac{\pi}{2}.$$

Proof. The first inequality immediately follows by Theorem 4.12. Consider the function p_α defined in (4.13). A straightforward calculation shows that

$$p_\alpha(z) + (1-\alpha)zp'_\alpha(z) = \frac{1-z}{1+z}.$$

Hence, by [14, Theorem 3.1c] applied with $\lambda = 1-\alpha$, we have $p_\alpha(z) \prec \left(\frac{1-z}{1+z}\right)^b$, and the assertion follows. $\blacksquare$

Using numerical computations we get $b(0) \approx 0.6383$ and $b(0.5) \approx 0.7669$.

Membership to the analytic filtration allows to control the rate of convergence to the origin. Indeed, Proposition 2.7 and the inequality (4.16) in the proof of Theorem 4.12 directly imply:

Theorem 4.15. *Let $f \in \mathfrak{A}_\alpha$. Then the semigroup $\{\phi_t(z)\}_{t\geqslant 0}$ generated by f is exponentially squeezing with squeezing ratio*

$$a = \int_0^1 \frac{1-t^{1-\alpha}}{1+t^{1-\alpha}}\,dt =: \varkappa(\alpha) > 0.$$

Thus, Theorem 1.4 follows from Theorems 4.12 and 4.15.

5. Analytic extension of semigroups

In this section, for a given semigroup of holomorphic self-mappings on the unit disc $\mathbb{D}$, we consider the problem of its analytic extension in a complex parameter.

Given $\theta \in \left(0, \frac{\pi}{2}\right]$, we denote

$$\Lambda(\theta) = \{\zeta \in \mathbb{C} : |\arg \zeta| < \theta\} . \tag{5.1}$$

Definition 5.1. *A family $\{F_\zeta\}_{\zeta\in\Lambda}$ of holomorphic self-mappings of $\mathbb{D}$ indexed by a parameter ζ in a sector $\Lambda := \Lambda(\theta) \cup \{0\}$ of the complex plane is said to be a one-parameter analytic semigroup if*

(i) *$\zeta \mapsto F_\zeta$ is analytic in Λ;*
(ii) *$\lim_{\Lambda\ni\zeta\to 0} F_\zeta = F_0 = I$;*
(iii) *$F_{\zeta_1+\zeta_2} = F_{\zeta_1} \circ F_{\zeta_2}$ whenever $\zeta_1, \zeta_2 \in \Lambda$.*

Recall that the family of sets $\mathfrak{S} = \{\mathfrak{S}_\alpha\}_{\alpha\in[0,1]}$ (see Definition 3.1) forms a filtration of the class $\mathcal{G}_0$. Our approach to analytic extension of semigroups of holomorphic self-mappings is based on the following result [10].

Theorem 5.2. *Let $\{F_t\}_{t\geqslant 0}$ be a semigroup of holomorphic self-mappings of $\mathbb{D}$ generated by $f \in \mathcal{G}_0$, and let $\alpha \in [0, 1)$. Then this semigroup extends analytically to the sector $\Lambda\left(\frac{\pi(1-\alpha)}{2}\right)$ in $\mathbb{C}$ if and only if $f \in \mathfrak{S}_\alpha$.*

As an immediate consequence of this theorem and Theorem 4.14, we have the following.

Corollary 5.3. *Let $\alpha \in [0, 1)$. Then $\mathfrak{A}_\alpha \subset \mathfrak{S}_b$, where $b = b(\alpha)$ is the unique solution in $[0, 1)$ to the equation*

$$\arctan\left((1-\alpha)b\right) = \frac{\pi(1-b)}{2} .$$

Consequently, the semigroup generated by $f \in \mathfrak{A}_\alpha$ extends analytically to the sector $\Lambda\left(\frac{\pi(1-b(\alpha))}{2}\right)$.

Now we consider the restriction of the semigroup generated by $f \in \mathcal{G}_0$ to the disc $\mathbb{D}_r$ of radius $r \in (0, 1)$. It is easy to see that this restriction is the semigroup generated by function $f_r \in \mathcal{G}_0$ defined by $f_r(z) = \frac{1}{r} f(rz)$. Obviously, the restricted semigroup can be analytically extended to a sector wider than the sector of analyticity for the original semigroup. More precisely, using Corollaries 2.4 and 3.3 and Theorem 4.14, one concludes the following.

Corollary 5.4. *Let $f \in \mathcal{G}_0$ and $r \in (0, 1)$. The restriction of the semigroup generated by f to the disc $\mathbb{D}_r$ can be analytically extended to the sector $\Lambda\left(\frac{\pi}{2} - \theta\right)$, where θ is defined as follows:*

(a) *If $f \in \mathfrak{C}_\alpha$, then* $\theta = \arctan\left(\dfrac{2\alpha r}{\sqrt{(1-r^2)(1-(1-2\alpha)^2 r^2)}}\right)$;

(b) *If $f \in \mathfrak{S}_\alpha$, then* $\theta = \alpha \arctan\left(\frac{2r}{1-r^2}\right)$;

(c) *If $f_\alpha \in \mathfrak{A}_\alpha$, then* $\theta = \max\limits_{|z|=r} \left|\arg\left(\displaystyle\int_0^1 \frac{1-t^{1-\alpha}z}{1+t^{1-\alpha}z}\, dt\right)\right|$.

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Received: 22 October 2017; **revised:** 1 April 2018