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# ON THE DIOPHANTINE EQUATION $y^p = f(x_1, x_2, ..., x_r)$

Raghavendran Srikanth, Sivanarayanapandian Subburam

**Abstract:** In this paper, we study the Diophantine equation

$$y^p = f(x_1, x_2, ..., x_r),$$

where $f(x_1, x_2, ..., x_r)$ is a real polynomial in variables $x_1, x_2, ..., x_r$ in $R$, a group of real numbers under the usual addition $+$, having the least element property.

**Keywords:** Diophantine equation, monic polynomial.

## 1. Introduction

In 1999, Poulakis [1] produced an algorithm to solve the Diophantine equation $y^2 = x^4+a_1x^3+a_2x^2+a_3x+a_4$. In 2000, Szalay [7] gave an upper bound for the solutions of $y^2 = f(x)$, where $f(x) = a_0x^n+a_1x^{n-1}+\cdots+a_n$ is an integer polynomial and $n$ is even. In 2002, Szalay [8] generalized the work by the equation $y^p = f(x)$, where $\deg f(x)$ is multiple of $p$. In 2008, Sankaranarayanan and Saradha [2] provided an upper bound for the integral solutions of $f(x) = g(y)$, where $f(x)$ and $g(y)$ are integer polynomials in variables $x$ and $y$ with $\gcd(\deg(f(x)), \deg(g(y))) > 1$. In 2012, Srikanth and Subburam [3] improved the method of Szalay [8]. In 2013, Szalay [9] delt with the general equation $z^2 = f(x, y)$. In 2014-15, Subburam and Thangadurai [5] and [6] gave upper bounds for the solutions of the equation $ax^3 + by + c = xyz$. In 2015, Subburam [4] studied the integral solutions of $(y - q_1)(y - q_2)\cdots(y - q_n) = f(x)$.

Our aim in this paper is to prove the following theorems. Here, we use the notations: $\mathbb{R}$ is the set of all real numbers, $R \subset \mathbb{R}$ having the least element property a group under the usual addition $+$, $\delta$ the least positive element of $R$, for any element $x \in \mathbb{R}$, $|x|_R$ the largest element of $R$ with $|x|_R \leqslant x$, $p$ a prime, $f(x_1, ..., x_t)$ a polynomial such that

$$f(x_1, ..., x_t) = B(x_1, ..., x_t)^p + C(x_1, ..., x_t)$$

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for some polynomials $B(x_1,...,x_t)$ and $C(x_1,...,x_t)$ in variables $x_1$, $x_2,...$, $x_t$ with coefficients in $\mathbb{R}$, $S$ the set of all positive integers $s \leqslant t$ such that $f(x_1,...,x_t)$ is a monic polynomial in variable $x_s$ of degree $\deg_{x_s}(f(x_1,...,x_t))$. For any elements $\psi, x_1, x_2, ..., x_t \in R$, $K_\psi = \{x\psi^{-1} : x \in R\}$,

$$P_i(x_1,...,x_t) = -(|\psi B(x_1,...,x_t)|_R - \alpha_i)^p + (\psi B(x_1,...,x_t))^p + \psi^p C(x_1,...,x_t)$$

and

$$Q_i(x_1,...,x_t) = (|\psi B(x_1,...,x_t)|_R + \alpha_i)^p - (\psi B(x_1,...,x_t))^p - \psi^p C(x_1,...,x_t)$$

where $\psi^{-1}$ is the inverse of $\psi$ in $\mathbb{R} \setminus \{0\}$ under usual multiplication, $\alpha_i = i\delta$ and $i = 0, 1, ...$

In 2013, Szalay [7] proved that if $(x, y, z)$ is an integral solution of the equation $z^2 = f(x, y)$, where $f(x, y)$ is an integer polynomial, then $P_1(x, y) > 0$ and $Q_1(x, y) > 0$ implies that $C(x, y) = 0$. This result is generalized in the following theorem:

**Theorem 1.** *Let $r$ be a positive integer. If $P_i(x_1,...,x_t) = 0$ and $Q_i(x_1,...,x_t) = 0$ have no solution in $R^t$ for the integers $i$ with $1 \leqslant i \leqslant r-1$ and if $(x_1,...,x_t,y) \in R^t \times K_\psi$ is a solution of the equation*

$$y^p = f(x_1,...,x_t),$$

*then each of*

$$(1) \qquad P_r(x_1,...,x_t) > 0 \quad \text{and} \quad Q_r(x_1,...,x_t) > 0$$

$$(2) \qquad P_r(x_1,...,x_t) < 0 \quad \text{and} \quad Q_r(x_1,...,x_t) < 0$$

*implies that*

$$\psi^p f(x_1,...,x_t) - |\psi B(x_1,...,x_t)|_R^p = 0.$$

**Theorem 2.** *Let $r$ be a positive integer. If $s \in S$, $\deg_{x_s} C(x_1,...,x_t) < \deg_{x_s} B(x_1,...,x_t)^{p-1}$, $\psi B(R,...,R) \subset R$, and if*

$$\mathbb{P}_i(x_1,...,x_t) = -(\psi B(x_1,...,x_t) - \alpha_i)^p + (\psi B(x_1,...,x_t))^p + \psi^p C(x_1,...,x_t) = 0$$

*and*

$$\mathbb{Q}_i(x_1,...,x_t) = (\psi B(x_1,...,x_t) + \alpha_i)^p - (\psi B(x_1,...,x_t))^p - \psi^p C(x_1,...,x_t) = 0$$

*have no solution in $R^t$ for all integers $i$ with $0 \leqslant i \leqslant r-1$, then all solutions $(x_1,...,x_t,y) \in R^t \times K_\psi$ of the equation*

$$y^p = f(x_1,...,x_t)$$

*satisfy*

$$\min \vartheta(s) \leqslant x_s \leqslant \max \vartheta(s),$$

*where*

$$\vartheta(s) = \{x_s \in \mathbb{R} : \mathbb{P}_r \text{ or } \mathbb{Q}_r \text{ or } C = 0 \text{ for some } x_1,...,x_{s-1},x_{s+1},... \in \mathbb{R}\}.$$

We have the following corollary, which generalize the works of Szalay [8] and Srikanth-Subburam [3], from Theorem 1.

**Corollary 1.** *Let $r$ be a positive integer. If $\psi B(R) \subset R$, $\deg(C(x)) < \deg(B(x)^{p-1})$ and if $\mathbb{P}_i(x) = 0$ and $\mathbb{Q}_i(x) = 0$ have no solutions in $R^t$ for the integer $i$ with $1 \leqslant i \leqslant r-1$, then all solutions $(x, y) \in R \times K_\psi$ of the equation*

$$y^p = f(x)$$

*satisfy*

$$\min \vartheta \leqslant x \leqslant \max \vartheta,$$

*where $\vartheta = \{\alpha \in \mathbb{R} : C(\alpha) = 0 \;\; or \;\; \mathbb{P}_r(\alpha) = 0 \;\; or \;\; \mathbb{Q}_r(\alpha) = 0\}$.*

## 2. Proof of Theorem 1

Let $(x_1, ..., x_t, y) \in R^t \times K_\psi$ be a solution of the equation

$$y^p = f(x_1, ..., x_t).$$

Assume that $P_r(x_1, ..., x_t) > 0$ and $Q_r(x_1, ..., x_t) > 0$. Then

$$(|\psi B(x_1, ..., x_t)|_R - \alpha_r)^p < (\psi B(x_1, ..., x_t))^p + \psi^p C(x_1, ..., x_t)$$

and

$$(\psi B(x_1, ..., x_t))^p + \psi^p C(x_1, ..., x_t) < (|\psi B(x_1, ..., x_t)|_R + \alpha_r)^p.$$

This implies that

$$(|\psi B(x_1, ..., x_t)|_R - \alpha_r)^p < (\psi y)^p < (|\psi B(x_1, ..., x_t)|_R + \alpha_r)^p.$$

Therefore, we have

$$(|\psi B(x_1, ..., x_t)|_R - \alpha_r) < \pm\psi y < (|\psi B(x_1, ..., x_t)|_R + \alpha_r).$$

Since $y \in K_\psi$, $\pm\psi y \in R$. Also, it is clear that $|\psi B(x_1, ..., x_t)|_R - \alpha_r$ and $|\psi B(x_1, ..., x_t)|_R + \alpha_r$ are in $R$, since $|\psi B(x_1, ..., x_t)|_R$ and $\alpha_r$ are in $R$. Therefore

$$\pm\psi y = |\psi B(x_1, ..., x_t)|_R - \alpha_i$$

or

$$\pm\psi y = |\psi B(x_1, ..., x_t)|_R + \alpha_i$$

or

$$\pm\psi y = |\psi B(x_1, ..., x_t)|_R$$

for some $i = 1, 2, ..., r-1$. From this, we can write that

$$P_i(x_1, ..., x_t) = \psi^p f(x_1, ..., x_t) - (|\psi B(x_1, ..., x_t)|_R - \alpha_i)^p = 0$$

or

$$Q_i(x_1,...,x_t) = -\psi^p f(x_1,...,x_t) + (|\psi B(x_1,...,x_t)|_R + \alpha_i)^p = 0$$

or

$$\psi^p f(x_1,...,x_t) - |\psi B(x_1,...,x_t)|_R^p = 0.$$

Since $P_i(X_1,...,X_t) = 0$ and $Q_i(X_1,...,X_t) = 0$ have no solution in $R$ for the positive integers $i$ with $1 \leqslant i \leqslant r-1$, we have

$$\psi^p f(x_1,...,x_t) - |\psi B(x_1,...,x_t)|_R^p = 0.$$

Assume that $P_r(x_1,...,x_t) < 0$ and $Q_r(x_1,...,x_t) < 0$. Then

$$(|\psi B(x_1,...,x_t)|_R - \alpha_r)^p > (\psi B(x_1,...,x_t))^p + \psi^p C(x_1,...,x_t)$$

and

$$(\psi B(x_1,...,x_t))^p + \psi^p C(x_1,...,x_t) > (|\psi B(x_1,...,x_t)|_R + \alpha_r)^p.$$

This implies that

$$(|\psi B(x_1,...,x_t)|_R + \alpha_r)^p < (\psi y)^p < (|\psi B(x_1,...,x_t)|_R - \alpha_r)^p.$$

If $p$ is odd, then

$$(|\psi B(x_1,...,x_t)|_R + \alpha_r)^p < (|\psi B(x_1,...,x_t)|_R - \alpha_r)^p.$$

implies that

$$(|\psi B(x_1,...,x_t)|_R + \alpha_r) < (|\psi B(x_1,...,x_t)|_R - \alpha_r),$$

which is a contradiction. If $p$ is even, then

$$(|\psi B(x_1,...,x_t)|_R + \alpha_r)^2 < (|\psi B(x_1,...,x_t)|_R - \alpha_r)^2.$$

implies that $|\psi B(x_1,...,x_t)|_R < 0$. Therefore

$$(-|\psi B(x_1,...,x_t)|_R - \alpha_r)^2 < (\delta\psi y)^2 < (-|\psi B(x_1,...,x_t)|_R + \alpha_r)^2,$$

where $\delta = \pm 1$ with $\delta\psi y > 0$. Since $-|\psi B(x_1,...,x_t)|_R + \alpha_r > 0$, we get

$$-|\psi B(x_1,...,x_t)|_R - \alpha_r < \delta\psi y < -|\psi B(x_1,...,x_t)|_R + \alpha_r.$$

That is, we have (1). Therefore we get

$$\psi^p f(x_1,...,x_t) - |\psi B(x_1,...,x_t)|_R^p = 0.$$

This proves the result.

## 3. Proof of Theorem 2

Let $s \in S$. Suppose that there is a solution $(x_1, ..., x_t, y) \in R^t \times K_\psi$ of the equation

$$y^p = f(x_1, ..., x_t)$$

such that

$$\min \vartheta(s) > x_s \text{ and } x_s > \max \vartheta(s),$$

where

$$\vartheta(s) = \{x_s \in \mathbb{R} : \mathbb{P}_r \text{ or } \mathbb{Q}_r \text{ or } C = 0 \text{ for some } x_1, ..., x_{s-1}, x_{s+1}, ... \in \mathbb{R}\}.$$

Then one of the following four cases is true:

(1) $\mathbb{P}_r(x_1, ..., x_t) > 0$ and $\mathbb{Q}_r(x_1, ..., x_t) > 0$
(2) $\mathbb{P}_r(x_1, ..., x_t) > 0$ and $\mathbb{Q}_r(x_1, ..., x_t) < 0$
(3) $\mathbb{P}_r(x_1, ..., x_t) < 0$ and $\mathbb{Q}_r(x_1, ..., x_t) > 0$
(4) $\mathbb{P}_r(x_1, ..., x_t) < 0$ and $\mathbb{Q}_r(x_1, ..., x_t) < 0$.

Since we have $\deg_{X_s} C(X_1, ..., X_t) < \deg_{X_s} B(X_1, ..., X_t)^{p-1}$, we conclude that $\deg_{X_s}(\mathbb{P}_r(X_1, ..., X_t)) = \deg_{X_s}(\mathbb{Q}_r(X_1, ..., X_t))$ and the leading coefficients of the polynomials $\mathbb{P}_r(X_1, ..., X_t)$ and $\mathbb{Q}_r(X_1, ..., X_t)$ in variable $X_s$ are the same. Therefore the cases (2) and (3) are impossible. So we have (1) and (4). Since $\psi B(R, ..., R) \subset R$ and, $\mathbb{P}_i(X_1, ..., X_t) = 0$ and $\mathbb{Q}_i(X_1, ..., X_t) = 0$ have no solutions in $R$ for all $i$ with $1 \leqslant i \leqslant r-1$, by Theorem 1, we get that $C(x_1, ..., x_t) = 0$. This is a contradiction.This proves the theorem.

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**Addresses:** R. Srikanth: Department of Mathematics, SASTRA University, Thanjavur, Tamil Nadu, India;
S. Subburam: Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA University, Kumbakonam, Tamil Nadu, India.

**E-mail:** srikanth@maths.sastra.edu, ssrammaths@yahoo.com

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