Functiones et Approximatio
57.2 (2017), 151–184
doi: 10.7169/facm/1622

A HIGHER RANK SELBERG SIEVE WITH AN ADDITIVE TWIST AND APPLICATIONS

M. Ram Murty, Akshaa Vatwani

To the memory of Atle Selberg,
with respect and admiration

Abstract: We develop an axiomatic formulation of the higher rank version of the classical Selberg sieve with an "additive twist" and provide asymptotic formulas for the same. As an application of this higher rank sieve, we obtain improvements of results of Heath-Brown and Ho-Tsang on almost prime k-tuples.

Keywords: the Selberg sieve, almost prime k-tuples.

1. Introduction

Atle Selberg [16] was the first to suggest the existence of a "higher rank" version of his celebrated sieve method (see page 351 of [16] and page 245 of [15]). This has been applied by Maynard [9] and the Polymath project [14] to give, among other results, a simplified proof of the Yitang Zhang's ground-breaking result [18] regarding bounded gaps between primes. In [17], the second author gives an axiomatic formulation of the higher rank sieve as well as applications to various problems.

In this paper, we develop the theory of the Selberg sieve with an "additive twist", the general theory of which seems to be new even in the rank one case, though the ideas are nascent in some of Selberg's work. Thus, our main theorems, which can be seen as a natural extension of this progression of ideas, are contained in Theorems 3.3 and 4.3 below.

Though the theory can be employed to a number of problems, in this paper we restrict our attention to the well-known prime k-tuples conjecture. The remarkable advantage of the general formulation leads us naturally to realize the culmination of Selberg's idea hinted in the two papers cited earlier, namely [15] and [16].

Research of the first author partially supported by an NSERC Discovery grant and a Simons fellowship. Research of the second author partially supported by an Ontario Graduate Scholarship.

2010 Mathematics Subject Classification: primary: 11N05; secondary: 11N35, 11N36

As an application of our general theory, we prove Theorem 5.13, which we state below. Let τ denote the divisor function and

$$\mathscr{H} = \{h_1, \dots, h_k\}$$

be an admissible set, by which we mean that h_i's are distinct non-negative integers and for every prime p, there is a residue class $b_p \pmod p$ such that $b_p \notin \mathscr{H} \pmod p$.

Theorem. *There exists ρ_k such that there are $\gg x(\log\log x)^{-1}(\log x)^{-k}$ integers $n \leqslant x$ satisfying: the product $\prod_{i=1}^{k}(n+h_i)$ is square-free and*

$$\sum_{i=1}^{k} \tau(n+h_i) \leqslant \lfloor \rho_k \rfloor .$$

For large k, we have $\rho_k \sim \frac{3}{4}k^2$.

This improves upon Theorem 1 of Heath-Brown [5], which gives $\rho_k \sim \frac{3}{2}k^2$. Ho and Tsang [6] obtained the above result with $\rho_k \sim k^2$. Similar computations have also been performed by Maynard [10]. It is worth noting that the above mentioned results do not use the higher rank sieve and rely on the combinatorial version of the classical sieve. Note that for $k = 2$, the best result is due to Chen [1].

As a consequence, we obtain bounds on the number of distinct prime divisors of the product $\prod_{i=1}^{k}(n+h_i)$ as well as for each $n+h_i$ individually, which are stated as Corollaries 5.14 and 5.15.

Our derivation of the above results uses the k-rank Selberg sieve along with the divisor function analogue of the Bombieri-Vinogradov theorem. More precisely, let $(a, q) = 1$, and set

$$E(x,q,a) = \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} \tau(n) - \frac{1}{\phi(q)} \sum_{\substack{n \leqslant x \\ (n,q)=1}} \tau(n), \tag{1.1}$$

where ϕ is Euler's function. Then for any $A > 0$ and any $\theta < 2/3$,

$$\sum_{q \leqslant x^\theta} \max_{(a,q)=1} |E(x,q,a)| \ll \frac{x}{(\log x)^A}. \tag{1.2}$$

In fact, the sharper result that

$$|E(x,q,a)| \ll q^{-1/4} x^{1/2+\epsilon}$$

for any $\epsilon > 0$ was known to Selberg (see p.237 of [15]) as well as Hooley [7] and Linnik (unpublished). It is conjectured that (1.2) is valid for any $\theta < 1$. Fouvry and Iwaniec [2] have investigated this conjecture in a variety of directions. In particular, they showed that if a is fixed, then for any $\epsilon, A > 0$, we have

$$\sum_{\substack{x^{\frac{2}{3}+\epsilon} < q < x^{1-\epsilon} \\ (a,q)=1}} |E(x,q,a)| \ll \frac{x}{(\log x)^A},$$

where the implied constant depends on ϵ, A and a. For other variants of the Bombieri-Vinogradov theorem, we refer the reader to [11].

2. Notation

We repeat much of the notation and terminology set up in [17] for convenience of the reader. A k-tuple of integers $\underline{d} := (d_1, \dots, d_k)$ is said to be square-free if the product of its components is square-free. For a real number R, the inequality $\underline{d} \leqslant R$ means that $\prod_i d_i \leqslant R$. The notion of divisibility among tuples is defined component-wise, that is,

$$\underline{d} \mid \underline{n} \iff d_i \mid n_i \text{ for all } 1 \leqslant i \leqslant k.$$

It follows that the notion of congruence among tuples, modulo a tuple, is also defined component-wise. On the other hand, we say a scalar q divides the tuple $\underline{d}$ if q divides the product $\prod_i d_i$. However when we explicitly write the congruence relation $\underline{d} \equiv \underline{e} \pmod{q}$, we mean that it holds for each component. When we say that a tuple $\underline{d}$ divides a scalar q, we mean that $\prod_i d_i$ divides q. For a square-free tuple, this is equivalent to each component dividing q.

We do not invoke any special notation for vector functions, that is, functions acting on k-tuples. It will be evident from its argument whether a function is a vector or scalar function. Most of the functions that we deal with are multiplicative. A vector function is said to be multiplicative if all its component functions are multiplicative. In this context, we define the function $f(\underline{d})$ to mean the product of its component (multiplicative) functions acting on the corresponding components of the tuple, that is,

$$f(\underline{d}) = \prod_{i=1}^{k} f_i(d_i).$$

The identity function acting on a tuple $\underline{d}$ is denoted by $\underline{d}$ itself. In this case, $\underline{d}$ would represent the product $\prod_{i=1}^{k} d_i$. It will be clear from the context whether we mean the above product or the vector tuple itself. Similarly, when we write a tuple raised to some power, we interpret it as the appropriate function acting on the tuple. For example, $\underline{d}^2 = \prod_{i=1}^{k} d_i^2$. Similarly, we define for k-tuples $\underline{d}$ and $\underline{\alpha}$, $\underline{d}^{(\underline{\alpha})} = \prod_{i=1}^{k} d_i^{\alpha_i}$.

Some more vector functions that will be used by us are the Euler phi function, as well as the lcm and gcd functions. For example,

$$[\underline{d}, \underline{e}] := \prod_{i=1}^{k} [d_i, e_i].$$

We also use the notation $[\underline{d}, \underline{e}] | \underline{n}$ to mean $[d_i, e_i] | n_i$ for $1 \leqslant i \leqslant k$. When written as the argument of a vector function, $[\underline{d}, \underline{e}]$ will denote the tuple whose components are $[d_i, e_i]$. The meaning of the use will be clear from the context.

Similarly, a vector function $\nu(\underline{d})$ is called additive if all its components ν_i are additive, in which case, we define

$$\nu(\underline{d}) = \sum_{i=1}^{k} \nu_i(d_i).$$

We use the convention $n \sim N$ to denote $N \leqslant n < 2N$. Alternatively, $f(x) \sim g(x)$ may also denote that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. The meaning will be clear from the context. Moreover, if we have an expression of the form $f(x) = (1 + o(1))cg(x)$, where c is a constant independent of x, it is understood that the case $c = 0$ implies that $f(x) = o(g(x))$.

Furthermore, we let $\tau(n)$ denote the number of divisors of the integer n and $\omega(n)$ denote the number of distinct prime factors of n. The greatest integer less than or equal to x is denoted as $\lfloor x \rfloor$. Throughout this paper, δ denotes a positive quantity which can be made as small as needed.

We employ the following multi-index notation to denote mixed derivatives of a function on k-tuples, $\mathcal{F}(\underline{t})$.

$$\mathcal{F}^{(\underline{\alpha})}(\underline{t}) := \frac{\partial^{\alpha} \mathcal{F}(t_1, \ldots, t_k)}{(\partial t_1)^{\alpha_1} \ldots (\partial t_k)^{\alpha_k}}, \tag{2.1}$$

for any k-tuple $\underline{\alpha}$ with $\alpha := \sum_{j=1}^{k} \alpha_j$.

3. The higher rank Selberg sieve revisited

In this section, we summarize the salient features of the higher rank Selberg sieve discussed in [17]. Our exposition is concise for the sake of brevity and the reader is encouraged to peruse Section 3.2 of the above mentioned paper. Given a set $\mathcal{S}$ of k-tuples, $\mathcal{S} = \{\underline{n} = (n_1, \ldots, n_k)\}$, in [17], we undertook a systematic study of sums of the form

$$\sum_{\underline{n} \in \mathcal{S}} w_{\underline{n}} \Big(\sum_{\underline{d} | \underline{n}} \lambda_{\underline{d}} \Big)^2, \tag{3.1}$$

where $w_{\underline{n}}$ is a 'weight' attached to the tuples $\underline{n}$ and $\lambda_{\underline{d}}$'s are parameters to be chosen. Throughout this section, the condition $\underline{n} \in \mathcal{S}$ is understood to hold without being explicitly stated. We impose the following hypotheses on this sum:

H1. If a prime p divides a tuple $\underline{n}$ such that p divides n_i and n_j, with $i \neq j$, then p must lie in some fixed finite set of primes $\mathcal{P}_0$.

This hypothesis allows us to perform the 'W trick'. We fix some $W = \prod_{p<D_0} p$, with D_0 depending on $\mathcal{S}$, such that $p \in \mathcal{P}_0$ implies that $p|W$. We then fix some tuple of residue classes $\underline{b} \pmod W$, with $(b_i, W) = 1$ for all i and restrict $\underline{n}$ to be congruent to $\underline{b}$ in the above sum.

H2. With $W, \underline{b}$ as in H1, the function $w_{\underline{n}}$ satisfies

$$\sum_{\substack{\underline{d} | \underline{n} \\ \underline{n} \equiv \underline{b} \ (\mathrm{mod}\ W)}} w_{\underline{n}} = \frac{X}{f(\underline{d})} + r_{\underline{d}},$$

for some multiplicative function f and some quantity X depending on the set $\mathcal{S}$.

H3. The components of f satisfy

$$f_j(p) = \frac{p}{\alpha_j} + O(p^t), \qquad \text{with } t < 1,$$

for some fixed $\alpha_j \in \mathbb{N}$, α_j independent of X, k.

We denote the tuple $(\alpha_1, \ldots, \alpha_k)$ as $\underline{\alpha}$ and the sum of the components $\sum_{j=1}^{k} \alpha_j$ as α.

H4. There exists $\theta > 0$ and $Y \ll X$ such that

$$\sum_{[\underline{d},\underline{e}] < Y^\theta} |r_{[\underline{d},\underline{e}]}| \ll \frac{Y}{(\log Y)^A},$$

for any $A > 0$, as $Y \to \infty$.

Henceforth, we assume D_0 (and hence W) $\to \infty$ as $X \to \infty$. The big oh and little oh notation used henceforth is understood to be with respect to $X \to \infty$, unless stated otherwise. Moreover, the implied constants may depend on those parameters which are independent of X (such as the function f, parameters A, α_j, β_j etc) but not those quantities which do depend on X (such as D_0, W, R, Y) With all this in place, we state the main results of the higher rank sieve obtained in [17].

Lemma 3.1. *Set R to be some fixed power of X and let $D_0 = o(\log \log R)$. Let f be a multiplicative function satisfying* H3 and $\mathcal{G}, \mathcal{H} : [0, \infty)^k \to \mathbb{R}$ be smooth functions with compact support. We denote

$$\mathcal{G}\left(\frac{\log \underline{d}}{\log R}\right) := \mathcal{G}\left(\frac{\log d_1}{\log R}, \cdots, \frac{\log d_k}{\log R}\right)$$

and similarly for $\mathcal{H}$. Let the dash over the sum mean that we sum over k-tuples $\underline{d}$ and $\underline{e}$ with $[\underline{d}, \underline{e}]$ square-free and co-prime to W. Then,

$$\sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])} \mathcal{G}\left(\frac{\log \underline{d}}{\log R}\right) \mathcal{H}\left(\frac{\log \underline{e}}{\log R}\right) = (1 + o(1)) C(\mathcal{G}, \mathcal{H})^{(\underline{\alpha})} \frac{c(W)}{(\log R)^\alpha},$$

where

$$\begin{aligned} C(\mathcal{G}, \mathcal{H})^{(\underline{\alpha})} &= \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^{k} \frac{t_j^{\alpha_j - 1}}{(\alpha_j - 1)!}\right) \mathcal{G}(\underline{t})^{(\underline{\alpha})} \mathcal{H}(\underline{t})^{(\underline{\alpha})} d\underline{t}, \\ c(W) &:= \prod_{p|W} \frac{p^\alpha}{\phi(p)^\alpha}. \end{aligned}$$

Here $\mathcal{G}(\underline{t})^{(\underline{\alpha})}$ and $\mathcal{H}(\underline{t})^{(\underline{\alpha})}$ are as in the notation of (2.1).

Let $\eta_{\mathcal{G}}$, $\eta_{\mathcal{H}}$ be shifted Fourier transforms of $\mathcal{G}$ and $\mathcal{H}$ respectively. More precisely, let

$$\eta_{\mathcal{G}}(\underline{u}) = \int_{\mathbb{R}^k} (\mathcal{G}(\underline{t}) \exp(\underline{t})) \exp(i\underline{u} \cdot \underline{t}) d\underline{t},$$

where $\exp(\underline{t}) = \prod_{j=1}^n e^{t_j}$ and the dot denotes dot product of tuples. We have a similar expression for $\eta_{\mathcal{H}}(\underline{u})$.

Lemma 3.2. *Let $\underline{a}$ denote the tuple $(a_1, \dots, a_k)$ and let $a = \sum_j a_j$. We follow the same notation for $\underline{b}$ and $\underline{c}$ and the notation of (2.1) for the relevant mixed derivatives. Then the integral*

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v}) \prod_{j=1}^k \frac{(1+iu_j)^{a_j}(1+iv_j)^{b_j}}{(1+iu_j+1+iv_j)^{c_j}} d\underline{u}d\underline{v}$$

is given by

$$C(\mathcal{G},\mathcal{H})^{(\underline{a},\underline{b},\underline{c})} := (-1)^{a+b} \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k \frac{t_j^{c_j-1}}{(c_j-1)!} \right) \mathcal{G}(\underline{t})^{(\underline{a})} \mathcal{H}(\underline{t})^{(\underline{b})} d\underline{t}.$$

We now choose $\lambda_{\underline{d}}$ in terms of a fixed symmetric smooth function $\mathcal{F} : [0,\infty)^k \to \mathbb{R}$, supported on the simplex $\Delta_k(1) := \{(t_1, \dots, t_k) \in [0,\infty)^k : t_1 + \ldots + t_k \leqslant 1\}$, as follows

$$\lambda_{\underline{d}} = \mu(\underline{d}) \mathcal{F}\left(\frac{\log d_1}{\log R}\right) := \mu(d_1)\ldots\mu(d_k)\mathcal{F}\left(\frac{\log d_1}{\log R}, \dots, \frac{\log d_k}{\log R}\right). \tag{3.2}$$

Theorem 3.3. *Let $\lambda_{\underline{d}}$'s be as chosen above. Suppose hypotheses H1 to H3 hold and H4 holds with $Y = X$. Set $R = X^{\theta/2-\delta}$ for small $\delta > 0$ and let $D_0 = o(\log\log R)$. Then,*

$$\sum_{\underline{n} \equiv \underline{b} \ (\mathrm{mod}\ W)} w_{\underline{n}} \Bigg(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}} \Bigg)^2 = (1+o(1)) C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})} c(W) \frac{X}{(\log R)^\alpha},$$

with

$$C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})} = \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^k \frac{t_j^{\alpha_j-1}}{(\alpha_j-1)!} \right) \left(\mathcal{F}^{(\underline{\alpha})}(\underline{t})\right)^2 d\underline{t}, \qquad c(W) := \frac{W^\alpha}{\phi(W)^\alpha}.$$

4. The higher rank sieve with an additive function

In [17], one could have considered a more general setting for the sieve, in which the weights $w_{\underline{n}}$ satisfy:

H2'.
$$\sum_{\substack{\underline{d}|\underline{n} \\ \underline{n}\equiv\underline{b} \pmod W}} w_{\underline{n}} = \frac{X}{f(\underline{d})} + \frac{X^*}{f_*(\underline{d})}\nu(\underline{d}) + r_{\underline{d}},$$

where f and f_* are multiplicative functions and ν is an additive function. In other words, each component $\nu_j(d_j)$ of $\nu(\underline{d})$ is an additive function.

H2 used earlier can be thought of as H2' with $X^* = 0$. However, there do arise situations when the expressions we need to analyze are of this more general form, as we shall see. One could consider even more general situations as hinted by Selberg [16], but we do not do so here. The reader may consult p.37 of [3] for a cursory discussion of this kind of setting.

This expression motivates the analysis of sums of the following form.

Lemma 4.1. *Let f, g, h be multiplicative vector functions and ν be an additive vector function. Let $\underline{d}$ and $\underline{e}$ denote k-tuples as usual. We define by S^* the twisted sum*

$$\sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}\nu([\underline{d},\underline{e}]),$$

where the dash over the initial sum means that we are summing over tuples $\underline{d}$ and $\underline{e}$ such that $[\underline{d},\underline{e}]$ is square-free. Assuming that all the sums involved are absolutely convergent, we have

$$S^* = S - \sum_q S(q)\left(\sum_{j=1}^{k} \frac{\nu_j(q)g_j(q)}{f_j(q)} + \frac{\nu_j(q)h_j(q)}{f_j(q)} - \frac{\nu_j(q)g_j(q)h_j(q)}{f_j(q)}\right),$$

where the summation in the second term runs over all primes q, and

$$S = \sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}, \qquad S(q) = \sum_{\substack{\underline{d},\underline{e} \\ q\nmid[\underline{d},\underline{e}]}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}.$$

Proof. Fix a prime q. We denote by $S^*(q)$ the sum

$$\sum_{\substack{\underline{d},\underline{e} \\ q\nmid[\underline{d},\underline{e}]}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}\nu([\underline{d},\underline{e}]) \tag{4.1}$$

Then as $[d_i, e_i]$, $[d_j, e_j]$ are co-prime for all $i \neq j$, q can divide only one of the $[d_j, e_j]$'s if it divides the tuple $[\underline{d},\underline{e}]$. Hence,

$$S^* = S^*(q) + \sum_{j=1}^{k} \sum_{\substack{\underline{d},\underline{e} \\ q|[d_j,e_j]}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}\nu([\underline{d},\underline{e}]).$$

Again as before, for each j, the condition $q \mid [d_j, e_j]$ leads to three cases:

(a) $q \mid d_j$, $q \nmid e_j$
(b) $q \nmid d_j$, $q \mid e_j$
(c) $q \mid d_j$, $q \mid e_j$.

For each case, the dash over the sum indicates that q cannot divide $[d_i, e_i]$ for any $i \neq j$. The sum for case (a) is given by,

$$\sum_{j=1}^{k} \sideset{}{'}\sum_{\substack{\underline{d},\underline{e} \\ q|d_j, q\nmid e_j}} \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}\nu([\underline{d},\underline{e}])$$

We write $d_j = qd'_j$, so that $[d_j, e_j] = q[d'_j, e_j]$ and hence $\nu([d_j, e_j]) = \nu_j(q) + \nu_j([d'_j, e_j])$. Then with obvious notation, the sum for case (a) is equal to

$$-\sum_{j=1}^{k} \frac{g_j(q)\nu_j(q)}{f_j(q)} \sum_{\substack{\underline{d}',\underline{e} \\ q\nmid[\underline{d}',\underline{e}]}} \frac{\mu(\underline{d}')\mu(\underline{e})g(\underline{d}')h(\underline{e})}{f([\underline{d}',\underline{e}])}$$
$$-\sum_{j=1}^{k} \frac{g_j(q)}{f_j(q)} \sum_{\substack{\underline{d}',\underline{e} \\ q\nmid[\underline{d}',\underline{e}]}} \frac{\mu(\underline{d}')\mu(\underline{e})g(\underline{d}')h(\underline{e})}{f([\underline{d}',\underline{e}])}\nu_j([\underline{d}',\underline{e}]),$$

that is,

$$-\sum_{j=1}^{k} \frac{g_j(q)\nu_j(q)}{f_j(q)} S(q) - \sum_{j=1}^{k} \frac{g_j(q)}{f_j(q)} S^*(q).$$

Due to the additive function, we have obtained an extra term compared to the expression for case (a) in Lemma 3.2 of [17]. Cases (b) and (c) yield exactly the same expressions, with g_j replaced by h_j and $-g_jh_j$ respectively.

Thus

$$S^* = \left(1 - \sum_{j=1}^{k}\left(\frac{g_j(q)}{f_j(q)} + \frac{h_j(q)}{f_j(q)} - \frac{g_j(q)h_j(q)}{f_j(q)}\right)\right) S^*(q) \tag{4.2}$$
$$-\sum_{j=1}^{k}\left(\frac{g_j(q)\nu_j(q)}{f_j(q)} + \frac{h_j(q)\nu_j(q)}{f_j(q)} - \frac{g_j(q)h_j(q)\nu_j(q)}{f_j(q)}\right) S(q).$$

By Lemma 3.2 of [17], we know that S has an Euler product. Note that the first term in the parenthesis is in fact the Euler factor in this Euler product, corresponding to the prime q. Let us denote it by P_q. Now, fixing a prime $q' \neq q$ and repeating this process for $S^*(q)$, we obtain

$$S^*(q) = \left(1 - \sum_{j=1}^{k}\left(\frac{g_j(q')}{f_j(q')} + \frac{h_j(q')}{f_j(q')} - \frac{g_j(q')h_j(q')}{f_j(q')}\right)\right) S^*(q,q') \tag{4.3}$$
$$-\sum_{j=1}^{k}\left(\frac{g_j(q')\nu_j(q')}{f_j(q')} + \frac{h_j(q')\nu_j(q')}{f_j(q')} - \frac{g_j(q')h_j(q')\nu_j(q')}{f_j(q')}\right) S(q,q'),$$

where $S^*(q,q')$ and $S(q,q')$ are the respective sums with the primes q and q' eliminated in the sense of (4.1).

Note that the Euler product for $S(q,q')$ is the Euler product for S with P_q and $P_{q'}$ – the Euler factors corresponding to q and q', removed. We plug the above expression for $S^*(q)$ into the expression for S^* preceding it, to obtain

$$\begin{aligned} S^* &= P_q P_{q'} S^*(q,q') \\ &\quad - \sum_{j=1}^{k} \left(\frac{g_j(q')\nu_j(q')}{f_j(q')} + \frac{h_j(q')\nu_j(q')}{f_j(q')} - \frac{g_j(q')h_j(q')\nu_j(q')}{f_j(q')} \right) P_q S(q,q') \\ &\quad - \sum_{j=1}^{k} \left(\frac{g_j(q)\nu_j(q)}{f_j(q)} + \frac{h_j(q)\nu_j(q)}{f_j(q)} - \frac{g_j(q)h_j(q)\nu_j(q)}{f_j(q)} \right) S(q). \end{aligned}$$

Noting that $P_q S(q,q') = S(q')$ and continuing this process over all primes gives S^* equal to

$$\prod_p P_p - \sum_q \left(\sum_{j=1}^{k} \left(\frac{g_j(q)\nu_j(q)}{f_j(q)} + \frac{h_j(q)\nu_j(q)}{f_j(q)} - \frac{g_j(q)h_j(q)\nu_j(q)}{f_j(q)} \right) \right) S(q).$$

The product $\prod_p P_p$ is simply the Euler product for S. Thus, S^* is expressible in terms of S and $S(q)$ (with q running over all primes), both of whose Euler products are known. ■

We proceed to the general sieve as in the previous section, with hypotheses H1, H3 and H4. However, the hypothesis H2 is now replaced by H2'. We also impose an additional hypothesis H5 on our additive function ν, which is akin to H3 for the multiplicative function f.

H5. For each j, we have

$$\sum_p \frac{\nu_j(p)}{p^{1+\delta}} = \frac{\beta_j}{\delta} + O(1), \qquad \sum_p \frac{|\nu_j(p)|}{p^{1+\delta}} \ll_\delta \frac{1}{\delta} \tag{4.4}$$

as $\delta \to 0$.

We will be concerned with this sum for $p > D_0$. Then,

$$\sum_{p>D_0} \frac{\nu_j(p)}{p^{1+\delta}} = \frac{\beta_j}{\delta} + O\left(\sum_{p<D_0} \frac{|\nu_j(p)|}{p^{1+\delta}} \right).$$

Moreover,

$$\sum_{p<D_0} \frac{|\nu_j(p)|}{p^{1+\delta}} \leqslant e \sum_{p<D_0} \frac{|\nu_j(p)|}{p^{1+\delta}} e^{\left(-\frac{\log p}{\log D_0}\right)} \leqslant e \sum_p \frac{|\nu_j(p)|}{p^{1+\frac{1}{\log D_0}+\delta}}.$$

From (4.4), as $D_0 \to \infty$, we have

$$\sum_p \frac{|\nu_j(p)|}{p^{1+\frac{1}{\log D_0}}} = O(\log D_0).$$

Hence,

$$\sum_{p>D_0} \frac{\nu_j(p)}{p^{1+\delta}} = \frac{\beta_j}{\delta} + O(\log D_0).$$

In practice one usually finds that for a fixed j, $\nu_j(p)$ is of the same sign for all primes p, so that the two conditions of (4.4) can be reconciled into a single condition.

The following result is the analogue of Lemma 3.1 in the additive function case. Let $\alpha = \sum_{j=1}^k \alpha_j$, $W = \prod_{p<D_0} p$ and $D_0 \to \infty$ as $X \to \infty$, as before.

Lemma 4.2. *Set R to be some fixed power of X and let $D_0 = o(\log\log R)$. Let f be a multiplicative function and ν be an additive function satisfying H3 and H5 respectively. Let $\mathcal{G}$, $\mathcal{H}$ be smooth functions with compact support. Let all notation be as in Lemma 3.2 and Theorem 3.3. Then, the sum*

$$\sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])} \nu([\underline{d},\underline{e}]) \mathcal{G}\left(\frac{\log \underline{d}}{\log R}\right) \mathcal{H}\left(\frac{\log \underline{e}}{\log R}\right)$$

is given by (as $R \to \infty$)

$$(1+o(1)) \frac{c(W)}{(\log R)^{\alpha-1}} \sum_{j=1}^k \beta_j \alpha_j C_j^*(\mathcal{G},\mathcal{H})^{(\underline{\alpha})} + O\left(\frac{\log D_0}{(\log R)^{\alpha}}\right),$$

where,

$$C_j^*(\mathcal{G},\mathcal{H})^{(\underline{\alpha})} = C(\mathcal{G},\mathcal{H})^{(\underline{\alpha},\underline{\alpha},\underline{\alpha}+e_j)} - C(\mathcal{G},\mathcal{H})^{(\underline{\alpha}-e_j,\underline{\alpha},\underline{\alpha})} - C(\mathcal{G},\mathcal{H})^{(\underline{\alpha},\underline{\alpha}-e_j,\underline{\alpha})}$$

and the tuple $\underline{\alpha} \pm e_j$ is $(\alpha_1, \ldots, \alpha_j \pm 1, \ldots, \alpha_k)$.

Proof. Following the proof of Lemma 3.4 of [17], we extend the functions $\mathcal{G}$ and $\mathcal{H}$ to smooth compactly supported functions on $\mathbb{R}^k$. Then by Fourier inversion, we have

$$\begin{aligned} \mathcal{G}(\underline{t}) &= \int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u}) \exp\left(-(\underline{1}+i\underline{u}) \cdot \underline{t}\right) d\underline{u}, \\ \mathcal{H}(\underline{t}) &= \int_{\mathbb{R}^k} \eta_{\mathcal{H}}(\underline{v}) \exp\left(-(\underline{1}+i\underline{v}) \cdot \underline{t}\right) d\underline{v}, \end{aligned} \tag{4.5}$$

where $\eta_{\mathcal{G}}$ and $\eta_{\mathcal{H}}$ are shifted Fourier transforms of $\mathcal{G}$ and $\mathcal{H}$ respectively and the dot denotes dot product of tuples. $\eta_{\mathcal{G}}$ and $\eta_{\mathcal{H}}$ are Fourier transforms of smooth functions with compact support and are hence rapidly decaying smooth functions, satisfying the bounds

$$|\, \eta_{\mathcal{G}}(\underline{t}) \,| \ll (1+|\underline{t}|)^{-A_1}, \qquad |\, \eta_{\mathcal{H}}(\underline{t}) \,| \ll (1+|\underline{t}|)^{-A_2}, \tag{4.6}$$

for any $A_1, A_2 > 0$.

The required sum can be written as

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})Z^*(\underline{u},\underline{v})d\underline{u}d\underline{v}, \tag{4.7}$$

where

$$Z^*(\underline{u},\underline{v}) = {\sum_{\underline{d},\underline{e}}}' \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])}\nu([\underline{d},\underline{e}])\frac{1}{\underline{d}^{(1+i\underline{u})/\log R}}\frac{1}{\underline{e}^{(1+i\underline{v})/\log R}}.$$

Consider the Euler product for $Z(\underline{u},\underline{v})$ given by equation (3.7) of [17]:

$$Z(\underline{u},\underline{v}) = (1+o(1))\prod_{p>D_0}\left(1-\sum_{j=1}^{k}\frac{\alpha_j}{p}\left(\frac{1}{p^{\frac{1+iu_j}{\log R}}}+\frac{1}{p^{\frac{1+iv_j}{\log R}}}-\frac{1}{p^{\frac{1+iu_j}{\log R}+\frac{1+iv_j}{\log R}}}\right)\right). \tag{4.8}$$

Let $Z(\underline{u},\underline{v})_q$ be $Z(\underline{u},\underline{v})$ with the Euler factor corresponding to some prime $q\nmid W$ removed. Applying Lemma 4.1 gives the following expression for $Z^*(\underline{u},\underline{v})$ in terms of $Z(\underline{u},\underline{v})$ and $Z(\underline{u},\underline{v})_q$:

$$-\sum_{q\nmid W}Z(\underline{u},\underline{v})_q\left(\sum_{j=1}^{k}\frac{\nu_j(q)}{f_j(q)}\left(\frac{1}{q^{\frac{1+iu_j}{\log R}}}+\frac{1}{q^{\frac{1+iv_j}{\log R}}}-\frac{1}{q^{\frac{1+iu_j+1+iv_j}{\log R}}}\right)\right)+Z(\underline{u},\underline{v})$$

Noting from equation (3.7) of [17], the Euler factor for the prime q in the Euler product for $Z(\underline{u},\underline{v})$, we have,

$$Z(\underline{u},\underline{v}) = (1+O(q^{-1}))Z(\underline{u},\underline{v})_q.$$

As $1/(1+O(q^{-1})) = (1+O(q^{-1}))$, one obtains

$$Z(\underline{u},\underline{v})_q = (1+O(q^{-1}))Z(\underline{u},\underline{v}).$$

Thus, $Z^*(\underline{u},\underline{v})$ equals,

$$\begin{aligned}&-Z(\underline{u},\underline{v})\sum_{j=1}^{k}\sum_{q\nmid W}\frac{\nu_j(q)}{f_j(q)}\left(\frac{1}{q^{\frac{1+iu_j}{\log R}}}+\frac{1}{q^{\frac{1+iv_j}{\log R}}}-\frac{1}{q^{\frac{1+iu_j+1+iv_j}{\log R}}}\right)\\&+|Z(\underline{u},\underline{v})|\sum_{j=1}^{k}O\left(\sum_{q>D_0}\frac{|\nu_j(q)|}{q|f_j(q)|}\right)+Z(\underline{u},\underline{v}).\end{aligned}$$

The hypothesis H3 on f gives

$$\frac{|\nu_j(q)|}{|f_j(q)|} = O\left(\frac{|\nu_j(q)|}{q}\right), \qquad \sum_{q\nmid W}\frac{\nu_j(q)}{f_j(q)} = \sum_{q>D_0}\frac{\alpha_j\nu_j(q)}{q}+O\left(\sum_{q>D_0}\frac{|\nu_j(q)|}{q^{2-t}}\right),$$

with $t<1$. As done in the comments following H5, we can show that

$$\sum_{q<D_0} \frac{|\nu_j(q)|}{q} \ll \sum_{q} \frac{|\nu_j(q)|}{q^{1+\frac{1}{\log D_0}}} \ll \log D_0. \tag{4.9}$$

Keeping this in mind, we apply partial summation on

$$\sum_{q>D_0} \frac{|\nu_j(q)|}{q} \frac{1}{q^{1-t}},$$

to get

$$\sum_{q>D_0} \frac{|\nu_j(q)|}{q^{2-t}} \ll \frac{\log D_0}{D_0^{1-t}}. \tag{4.10}$$

Since $0 \leqslant t < 1$, the above term is $o(1)$ as $D_0 \to \infty$.

We conclude that as $D_0 \to \infty$, $Z^*(\underline{u}, \underline{v})$ is given by

$$\begin{aligned} &-(1+o(1))Z(\underline{u},\underline{v}) \sum_{j=1}^{k} \left(\sum_{q>D_0} \frac{\nu_j(q)\alpha_j}{q} \left(\frac{1}{q^{\frac{1+iu_j}{\log R}}} + \frac{1}{q^{\frac{1+iv_j}{\log R}}} - \frac{1}{q^{\frac{1+iu_j+1+iv_j}{\log R}}} \right) \right) \\ &+ Z(\underline{u},\underline{v}) + |Z(\underline{u},\underline{v})| O\left(\frac{\log D_0}{D_0} \right). \end{aligned}$$

By partial summation and the bound (4.9), we have

$$\sum_{q>D_0} \frac{|\nu_j(q)|}{q} \frac{1}{q^{\frac{1+u_j}{\log R}}} \ll (\log D_0) D_0^{-\frac{1+u_j}{\log R}} \ll \log D_0,$$

as $R \to \infty$. Similar bounds can be obtained for each summand in the first term above. This shows that $|Z^*(\underline{u},\underline{v})| \ll (\log D_0)|Z(\underline{u},\underline{v})|$. Recalling the bound $|Z(\underline{u},\underline{v})| \ll (\log R)^{O(1)}$ given by (3.9) of [17], we see that

$$|Z^*(\underline{u},\underline{v})| \ll (\log R)^{O(1)}. \tag{4.11}$$

Using the same argument as in Lemma 3.4 of [17] , one can show that the contribution from the region $|\underline{u}|$ or $|\underline{v}| \geqslant (\log R)^{\epsilon}$ to the integral (4.7) is $O((\log R)^{-A})$ for any $A>0$. In the region $|\underline{u}|, |\underline{v}| \leqslant (\log R)^{\epsilon}$, we apply H5 with δ being terms of the type $(1+iu_j)/\log R$, to obtain that

$$\begin{aligned} Z^*(\underline{u},\underline{v}) &= (1+o(1))(\log R) Z(\underline{u},\underline{v}) \\ &\quad \times \sum_{j=1}^{k} \beta_j \alpha_j \left(\frac{1}{1+iu_j+1+iv_j} - \frac{1}{1+iu_j} - \frac{1}{1+iv_j} \right) \\ &\quad + O(\log D_0)|Z(\underline{u},\underline{v})|. \end{aligned}$$

We then use the known result (cf. (3.10), [17])

$$Z(\underline{u},\underline{v}) = (1+o(1))c(W)\frac{1}{(\log R)^{\alpha}}\prod_{j=1}^{k}\frac{(1+iu_j)^{\alpha_j}(1+iv_j)^{\alpha_j}}{(1+iu_j+1+iv_j)^{\alpha_j}}, \tag{4.12}$$

to simplify the above expression, finally obtaining

$$\begin{aligned} Z^*(\underline{u},\underline{v}) = {} & (1+o(1))c(W)\frac{1}{(\log R)^{\alpha-1}}\left(L_2(\underline{u},\underline{v}) - L_1(\underline{u},\underline{v}) - L_1(\underline{v},\underline{u})\right) \\ & + O(\log D_0)|Z(\underline{u},\underline{v})|, \end{aligned} \tag{4.13}$$

with

$$\begin{aligned} L_1(\underline{u},\underline{v}) &= \sum_{j=1}^{k} L_1(\underline{u},\underline{v})^{(j)} \\ &= \sum_{j=1}^{k}\beta_j\alpha_j\frac{(1+iu_j)^{\alpha_j-1}(1+iv_j)^{\alpha_j}}{(1+iu_j+1+iv_j)^{\alpha_j}}\prod_{l\neq j}\frac{(1+iu_l)^{\alpha_l}(1+iv_l)^{\alpha_l}}{(1+iu_l+1+iv_l)^{\alpha_l}}, \\ L_2(\underline{u},\underline{v}) &= \sum_{j=1}^{k} L_2(\underline{u},\underline{v})^{(j)} \\ &= \sum_{j=1}^{k}\beta_j\alpha_j\frac{(1+iu_j)^{\alpha_j}(1+iv_j)^{\alpha_j}}{(1+iu_j+1+iv_j)^{\alpha_j+1}}\prod_{l\neq j}\frac{(1+iu_l)^{\alpha_l}(1+iv_l)^{\alpha_l}}{(1+iu_l+1+iv_l)^{\alpha_l}}. \end{aligned} \tag{4.14}$$

Plugging all this into (4.7), we are led to evaluate new integrals of the form

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})L_1(\underline{u},\underline{v})d\underline{u}d\underline{v}, \qquad \int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})L_2(\underline{u},\underline{v})d\underline{u}d\underline{v}.$$

Note that we have extended the integrals to be over all of $\mathbb{R}^k$. This can be done since the $O((\log R)^{-A})$ contribution from the complementary region gets absorbed into the $o(1)$ term of (4.13).

Now, using Lemma 3.2, we have

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})L_1(\underline{u},\underline{v})^{(j)}d\underline{u}d\underline{v} = \beta_j\alpha_j C(\mathcal{G},\mathcal{H})^{(\underline{\alpha}-e_j,\underline{\alpha},\underline{\alpha})},$$

where $\underline{\alpha}$ is as before and $\underline{\alpha}-e_j = (\alpha_1,\ldots,\alpha_j-1,\ldots,\alpha_k)$. Similarly,

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})L_1(\underline{v},\underline{u})^{(j)}d\underline{u}d\underline{v} = \beta_j\alpha_j C(\mathcal{G},\mathcal{H})^{(\underline{\alpha},\underline{\alpha}-e_j,\underline{\alpha})},$$

and

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})L_2(\underline{u},\underline{v})^{(j)}d\underline{u}d\underline{v} = \beta_j\alpha_j C(\mathcal{G},\mathcal{H})^{(\underline{\alpha},\underline{\alpha},\underline{\alpha}+e_j)},$$

where $\underline{\alpha}+e_j = (\alpha_1,\ldots,\alpha_j+1,\ldots,\alpha_k)$.

Putting together all these evaluated integrals gives for the main term of (4.13), the term involving $C_j^*(\mathcal{G},\mathcal{H})^{(\underline{\alpha})}$ in the lemma. We have not considered yet the error term of (4.13). In the region $|\underline{u}|, |\underline{v}| < (\log R)^\epsilon$, the expression (4.12) gives

$$|Z(\underline{u},\underline{v})| = (1+o(1))c(W)\frac{1}{(\log R)^\alpha}\left|\prod_{j=1}^{k}\frac{(1+iu_j)^{\alpha_j}(1+iv_j)^{\alpha_j}}{(1+iu_j+1+iv_j)^{\alpha_j}}\right|, \qquad (4.15)$$

Plugging $|Z(\underline{u},\underline{v})|$ into (4.7) leads to

$$(1+o(1))\frac{c(W)}{(\log R)^\alpha}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})\left|\prod_{j=1}^{k}\frac{(1+iu_j)^{\alpha_j}(1+iv_j)^{\alpha_j}}{(1+iu_j+1+iv_j)^{\alpha_j}}\right|d\underline{u}d\underline{v}.$$

As this integral is absolutely convergent due to rapid decay of the integrand, in this region we obtain

$$|Z(\underline{u},\underline{v})|O(\log D_0) \ll \frac{c(W)\log D_0}{(\log R)^\alpha} = o(1)$$

In the region $|\underline{u}|$ or $|\underline{v}| \geqslant (\log R)^\epsilon$, the bound (4.11) as well as rapid decay bounds (4.6) mean that one can pull out any negative power of $\log R$ out of the integrand, and hence the error obtained in the former region dominates.

We find that the required sum is

$$(1+o(1))\frac{c(W)}{(\log R)^{\alpha-1}}\sum_{j=1}^{k}\beta_j\alpha_j C_j^*(\mathcal{G},\mathcal{H})^{(\underline{\alpha})},$$

as required. ■

Theorem 4.3. *Let $\lambda_{\underline{d}}$'s be chosen as in* (3.2)*. We assume hypotheses H1, H2', H4 and H5. We also assume that both functions, f and f_* arising from H2' satisfy H3 with α_j's and α_j^*'s respectively. Choose $R = X^{\theta/2-\delta}$ and assume that $D_0 = o(\log\log R)$. Then,*

$$\sum_{\underline{n}\equiv\underline{b} \pmod W} w_{\underline{n}}\left(\sum_{\underline{d}|\underline{n}}\lambda_{\underline{d}}\right)^2 = (1+o(1))\frac{c(W)X}{(\log R)^\alpha}C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}$$
$$+(1+o(1))\frac{c^*(W)X^*}{(\log R)^{\alpha^*-1}}\sum_{j=1}^{k}\beta_j\alpha_j^* C_j^*(\mathcal{F},\mathcal{F})^{(\underline{\alpha}^*)}$$

where $C_j^*(\mathcal{F},\mathcal{F})^{(\underline{\alpha}^*)}$ denotes the quantity

$$C(\mathcal{F},\mathcal{F})^{(\underline{\alpha}^*,\underline{\alpha}^*,\underline{\alpha}^*+e_j)} - C(\mathcal{F},\mathcal{F})^{(\underline{\alpha}^*-e_j,\underline{\alpha}^*,\underline{\alpha}^*)} - C(\mathcal{F},\mathcal{F})^{(\underline{\alpha}^*,\underline{\alpha}^*-e_j,\underline{\alpha}^*)},$$

and

$$c(W) = \frac{W^\alpha}{\phi(W)^\alpha}, \qquad c^*(W) = \frac{W^{\alpha^*}}{\phi(W)^{\alpha^*}}.$$

All notation is as in Lemmas 3.1, 3.2, 4.2 *and* $\alpha^* := \sum_j \alpha_j^*$.

Proof. We proceed exactly as in the proof of Theorem 3.6 of [17], by expanding the square, interchanging the order of summation, applying the W-trick and finally using H2'. This gives us the following expression for the above sum,

$$X \sum_{\underline{d},\underline{e}<R}{}' \frac{\lambda_{\underline{d}}\lambda_{\underline{e}}}{f([\underline{d},\underline{e}])} + X^* \sum_{\underline{d},\underline{e}<R}{}' \frac{\lambda_{\underline{d}}\lambda_{\underline{e}}}{f_*([\underline{d},\underline{e}])}\nu([\underline{d},\underline{e}]) + O\Bigg(\sum_{\underline{d},\underline{e}<R}{}' |\lambda_{\underline{d}}||\lambda_{\underline{e}}||r_{[\underline{d},\underline{e}]}|\Bigg).$$

Now, we have two main terms. Using the given choice of $\lambda_{\underline{d}}$'s, the first term can be analyzed as in Theorem 3.6 of [17] to obtain

$$(1+o(1))C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}c(W)\frac{X}{(\log R)^{\alpha}}.$$

The second term yields

$$X^* \sum_{\underline{d},\underline{e}<R}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f_*([\underline{d},\underline{e}])}\nu([\underline{d},\underline{e}])\mathcal{F}\left(\frac{\log \underline{d}}{\log R}\right)\mathcal{F}\left(\frac{\log \underline{e}}{\log R}\right).$$

By Lemma 4.2, this is given by

$$(1+o(1))\left(\sum_{j=1}^{k} \beta_j \alpha_j^* C_j^*(\mathcal{F},\mathcal{F})^{(\underline{\alpha}^*)}\right) c^*(W)\frac{X^*}{(\log R)^{\alpha^*-1}}.$$

To complete the proof, we note that the choice of R along with H4 ensures that the error term is negligible. ■

Note that a general version of the above theorem holds, without the condition

$$(\log D_0/\log R) = o(1).$$

In that case, in addition to the two terms already obtained in the theorem, we would get another term, namely

$$O(\log D_0)c(W)\frac{X^*}{(\log R)^{\alpha}}.$$

5. Application to almost prime k-tuples

We apply the higher rank sieve to the well-known prime k-tuples problem, by making use of the divisor function τ to sieve out primes.

A set $\mathscr{H}$ of distinct non-negative integers is said to be admissible if for every prime p, there is a residue class $b_p \pmod p$ such that $b_p \notin \mathscr{H} \pmod p$. We will work with a fixed admissible set of size k, $\mathscr{H} = \{h_1, \ldots, h_k\}$. We use the '$W$ trick' to remove the effect of small primes, that is we restrict n to be in a fixed residue class b modulo W, where $W = \prod_{p<D_0} p$ and b is chosen so that $b + h_i$ is co-prime to W for each h_i. This choice of b is possible because of admissibility of the set $\mathscr{H}$.

One can choose $D_0 = \log\log\log N$, so that $W \sim (\log\log N)^{(1+o(1))}$ by an application of the prime number theorem. We consider the expressions,

$$S_1 = \sum_{\substack{n\sim N\\ n\equiv b \ (\mathrm{mod}\ W)}} \Big(\sum_{d_j|n+h_j \forall j} \lambda_{\underline{d}} \Big)^2,$$

$$S_2 = \sum_{\substack{n\sim N\\ n\equiv b \ (\mathrm{mod}\ W)}} \Big(\sum_{j=1}^{k} \tau(n+h_j) \Big) \Big(\sum_{d_j|n+h_j \forall j} \lambda_{\underline{d}} \Big)^2.$$

For ρ positive, if we denote by $S(N,\rho)$ the quantity

$$\rho S_1 - S_2,$$

then we have the following key observation.

Proposition 5.1. *Given a positive number ρ, if*

$$\rho S_1 - S_2 > 0$$

for all large N, then there are infinitely many integers n such that

$$\sum_{j=1}^{k} \tau(n+h_j) \leqslant \lfloor \rho \rfloor,$$

where $\lfloor \rho \rfloor$ denotes the greatest integer less than or equal to ρ.

Proof. As $\lambda_{\underline{d}}$'s are non-negative, we see that if $S(N,\rho) > 0$, there must exist $n \sim N$ such that

$$\rho - \sum_{j=1}^{k} \tau(n+h_j) > 0.$$

As this happens for all large N, this inequality holds for infinitely many integers n. As each $\tau(n+h_j)$ is an integer, this completes the proof. ■

The asymptotic formula for S_1 was derived in Lemma 4.2 of [17]. We proceed to derive an asymptotic formula for S_2. Recall the definition of $E(x,q,a)$ in Section 1. Given $\theta > 0$, we say that θ is *permissible* for the divisor function if for any $A > 0$, $(a,q) = 1$, we have

$$\sum_{q<x^\theta} \max_{(a,q)=1} \max_{y\leqslant x} |E(y,q,a)| \ll \frac{x}{(\log x)^A}. \tag{5.1}$$

We define the level of distribution θ_0 of the divisor function to be the supremum of all the permissible values of θ. Henceforth we will work with a fixed permissible θ and set $R = N^{\theta/2-\delta}$, for some small $\delta > 0$. In the context of our problem, θ is assumed to be less than 1.

5.1. Asymptotic formula for S_2

We write

$$S_2 = \sum_{m=1}^{k} S_2^{(m)}, \qquad S_2^{(m)} = \sum_{\substack{n \sim N \\ n \equiv b \ (\mathrm{mod}\ W)}} \tau(n+h_m) \Bigg(\sum_{d_j | n+h_j \forall j} \lambda_{\underline{d}} \Bigg)^2,$$

and proceed to obtain an asymptotic formula for $S_2^{(m)}$. This calls upon the theory of the sieve with an additive function, as we will see.

We begin with some preliminary propositions.

Proposition 5.2. *Let $q \geqslant 1$ and $(a,q)=1$. Then,*

$$\sum_{\substack{n \leqslant x \\ n \equiv a \ (\mathrm{mod}\ q)}} \tau(n) = \frac{\phi(q)}{q^2} x \Bigg(\log x + c + 2 \sum_{p|q} \frac{\log p}{p-1} \Bigg) + E(x,q,a) + O\left(q^{\epsilon - 1}\sqrt{x}\right),$$

with $c = 2\gamma - 1$, where γ is Euler's constant.

Proof. We have,

$$\sum_{\substack{n \leqslant x \\ n \equiv a \ (\mathrm{mod}\ q)}} \tau(n) = \frac{1}{\phi(q)} \sum_{\substack{n \leqslant x \\ (n,q)=1}} \tau(n) + E(x,q,a).$$

Lemma 16 on p. 234 of [15] gives

$$\frac{1}{\phi(q)} \sum_{\substack{n \leqslant x \\ (n,q)=1}} \tau(n) = \frac{\phi(q)}{q^2} x \Bigg(\log x + c + 2 \sum_{p|q} \frac{\log p}{p-1} \Bigg) + O\Bigg(\frac{\sqrt{x}}{\phi(q)} \prod_{p|q} \left(1 + \frac{1}{\sqrt{p}}\right)^2 \Bigg)$$

Now,

$$\prod_{p|q} \left(1 + \frac{1}{\sqrt{p}}\right)^2 \ll 3^{\omega(q)} \ll q^{\epsilon},$$

where $\omega(q)$ denotes the number of distinct prime factors of q. Hence this error term is $O(q^{\epsilon-1}\sqrt{x})$. This completes the proof. ■

The function τ is not completely multiplicative, but it is not very far from being so. We state some results that will make this more precise.

Proposition 5.3. *We have,*

$$\tau(m)\tau(n) = \sum_{d|m,n} \tau\left(\frac{mn}{d^2}\right)$$

Proof. The result is trivial when $(m,n)=1$. It is enough to prove it in the case $m=p^a$, $n=p^b$ for some prime p. Assume without loss of generality that $a\leqslant b$. Then the left hand side of the result gives $\tau(p^a)\tau(p^b)=(a+1)(b+1)$. The right hand side gives

$$\begin{aligned}\sum_{j=0}^{a}\tau\left(p^{a+b-2j}\right) &= \sum_{j=0}^{a}(a+b+1-2j) = (a+b+1)(a+1)-a(a+1)\\ &= (a+1)(b+1),\end{aligned}$$

completing the proof. ∎

We now state the following two variable version of Möbius inversion (c.f. Lemma 2.1 of [13], which can be proved in the usual manner.

Lemma 5.4. *Let*

$$F(m,n)=\sum_{d|m,n}G(m/d,n/d).$$

Then,

$$G(m,n)=\sum_{d|m,n}\mu(d)F(m/d,n/d)$$

and conversely.

Applying this lemma to Proposition 5.3 gives the following expression for $\tau(mn)$ in terms of $\tau(m)$ and $\tau(n)$. It can also be proved directly following the method of Proposition 5.3.

Proposition 5.5. *We have,*

$$\tau(mn)=\sum_{d|m,n}\mu(d)\tau(m/d)\tau(n/d).$$

The following proposition allows us to obtain a more general form of Proposition 5.2, without the condition $(a,q)=1$.

Proposition 5.6. *Let $q\geqslant 1$ be square-free, $(a,q)=\delta$. Then,*

$$\sum_{\substack{n\leqslant x\\ n\equiv a \ (\mathrm{mod}\ q)}}\tau(n)=\sum_{d|\delta}\mu(d)\tau(\delta/d)\sum_{\substack{n'\leqslant x/\delta d\\ n'\equiv a_d \ (\mathrm{mod}\ q'),}}\tau(n')$$

where $q'=q/\delta$ and $a_d\equiv a_1\overline{\delta d} \pmod{q'}$. In particular, $(a_d,q')=1$.

Proof. We write $n=n_1\delta$ and $q=q'\delta$. As q is square-free, we have $(q',\delta)=1$. Letting $\overline{\delta}$ denote the inverse of δ modulo q' and $a_1\equiv a\overline{\delta} \pmod{q'}$, we have,

$$\sum_{\substack{n\leqslant x\\ n\equiv a \ (\mathrm{mod}\ q)}}\tau(n)=\sum_{\substack{n_1\leqslant x/\delta\\ n_1\equiv a_1 \ (\mathrm{mod}\ q')}}\tau(n_1\delta).$$

Note that $(q', a_1) = 1$. Using Proposition 5.5 and then interchanging summation, we obtain that the above sum equals

$$\sum_{\substack{n_1 \leqslant x/\delta \\ n_1 \equiv a_1 \ (\mathrm{mod}\ q')}} \sum_{d | \delta, n_1} \mu(d)\tau(\delta/d)\tau(n_1/d) = \sum_{d|\delta} \mu(d)\tau(\delta/d) \sum_{\substack{n_1 \leqslant x/\delta \\ n_1 \equiv a_1 \ (\mathrm{mod}\ q') \\ n_1 = dn'}} \tau(n_1/d)$$

$$= \sum_{d|\delta} \mu(d)\tau(\delta/d) \sum_{\substack{n' \leqslant x/d\delta \\ n' \equiv a_d \ (\mathrm{mod}\ q')}} \tau(n'),$$

where $a_d \equiv a_1 \overline{d} \,(\mathrm{mod}\ q')$, which exists as $(\delta, q') = 1$ implies $(d, q') = 1$. Here, $(a_d, q') = 1$ as needed. ■

Proposition 5.7. *Let $q \geqslant 1$ be square-free, $(a, q) = \delta$. Let $q' = q/\delta$. Then,*

$$\sum_{\substack{n \leqslant x \\ n \equiv a \ (\mathrm{mod}\ q)}} \tau(n)$$
$$= \left(\frac{\delta\tau(\delta)}{\phi(\delta)} \sum_{d|\delta} \frac{\mu(d)}{d\tau(d)} \right) \frac{\phi(q)}{q^2} x \left(\log x - \log\delta + \sum_{p|\delta} \frac{\log p}{2p-1} + c + 2\sum_{p|q'} \frac{\log p}{p-1} \right)$$
$$+ E'(x, q, a) + O\left(\delta^{1/2} q^{\epsilon - 1} \sqrt{x} \right).$$

Here,

$$E'(x, q, a) = \tau(\delta) \sum_{d|\delta} \frac{\mu(d)}{\tau(d)} E(x/\delta d, q', a_d),$$

with $a_d \equiv a\overline{\delta d} \,(\mathrm{mod}\ q')$.

Proof. Combining Propositions 5.2 and 5.6 for the sum

$$\sum_{\substack{n \leqslant x \\ n \equiv a \ (\mathrm{mod}\ q)}} \tau(n),$$

we obtain the following main term

$$\sum_{d|\delta} \mu(d)\tau(\delta/d) \frac{\phi(q')}{q'^2} \frac{x}{\delta d} \left(\log x - \log(\delta d) + c + 2\sum_{p|q'} \frac{\log p}{p-1} \right).$$

Here as q is square-free, so are δ and each d dividing δ. Hence, $\tau(\delta/d) = \tau(\delta)/\tau(d)$. Similarly, $\phi(q') = \phi(q)/\phi(\delta)$. With some simplification, this gives

$$\frac{\delta\tau(\delta)}{\phi(\delta)} \sum_{d|\delta} \frac{\mu(d)}{d\tau(d)} \frac{\phi(q)}{q^2} x \left(\log x - \log\delta - \log d + c + 2\sum_{p|q'} \frac{\log p}{p-1} \right)$$

We examine the term involving $\log d$. We would like to express the relevant series as a product, namely,

$$\sum_{d|\delta} \frac{\mu(d)}{d\tau(d)} \log d = \left(\sum_{d|\delta} \frac{\mu(d)}{d\tau(d)} \right) h(\delta),$$

for some suitable function h. Consider the Dirichlet series

$$f_\delta(s) := \sum_{d|\delta} \frac{\mu(d)}{d^s \tau(d)}.$$

Then

$$f'_\delta(s) = -\sum_{d|\delta} \frac{\mu(d)\log d}{d^s\tau(d)} = f_\delta(s) \sum_{p|\delta} \left(\frac{\log p}{2p^s - 1} \right),$$

by logarithmic differentiation of the Euler product of $f_\delta(s)$. Thus

$$f'_\delta(1) =: -\sum_{d|\delta} \frac{\mu(d)\log d}{d\tau(d)} = \sum_{d|\delta} \frac{\mu(d)}{d\tau(d)} \sum_{p|\delta} \left(\frac{\log p}{2p - 1} \right).$$

Plugging this into the obtained expression gives the desired main term.

The error terms are given by

$$\tau(\delta) \sum_{d|\delta} \frac{\mu(d)}{\tau(d)} E(x/\delta d, q', a_d) + O\bigg(\frac{\tau(\delta)}{\sqrt{\delta}} \sum_{d|\delta} \frac{1}{\tau(d)\sqrt{d}} (q')^{\epsilon-1}\sqrt{x} \bigg).$$

We denote the first term by $E'(x,q,a)$. As

$$\sum_{d|\delta} \frac{1}{\tau(d)\sqrt{d}} \ll \tau(\delta) \ll \delta^{\epsilon/2},$$

the second error term gives

$$O\left(\delta^{\epsilon - 1/2} (q')^{\epsilon-1} \sqrt{x} \right) = O\left(\delta^{1/2} q^{\epsilon-1} \sqrt{x} \right). \qquad \blacksquare$$

We now seek to show that the error term $E'(x,q,a)$ appearing in the proposition above satisfies an average bound of the type (5.1), as this will be needed in our estimation of S_2.

In order to do this, let us define some notation. We define

$$E^*(x,q) := \max_{(a,q)=1} \max_{y \leqslant x} |E(y,q,a)|, \qquad \widetilde{E}^*(x,q) := \max_{a \pmod q} \max_{y \leqslant x} |E'(y,q,a)|.$$

Then (5.1) can be rewritten as

$$\sum_{q < x^\theta} E^*(x,q) \ll \frac{x}{(\log x)^A}. \tag{5.2}$$

We wish to prove that

$$\sum_{q<x^\theta} \widetilde{E}^*(x,q) := \sum_{q<x^\theta} \max_{a \ (\mathrm{mod}\ q)} \max_{y\leqslant x} |E'(y,q,a)| \ll \frac{x}{(\log x)^A}. \tag{5.3}$$

We begin with the following result giving a bound for each individual $E'(y,q,a)$, $y \leqslant x$.

Proposition 5.8. *Let $q \geqslant 1$ be square-free, $\delta = (a,q)$ and $q' = q/\delta$. Then, for any $y \leqslant x$,*

$$|E'(y,q,a)| \leqslant \tau(\delta)^2 E^*(x/\delta, q').$$

Proof. By definition of $E'(y,q,a)$,

$$|E'(y,q,a)| \leqslant \tau(\delta) \sum_{d|\delta} \frac{1}{\tau(d)} |E(y/\delta d, q', a_d)|,$$

with $a_d \equiv a\overline{\delta d} \ (\mathrm{mod}\ q')$. We know that $(a_d, q') = 1$. Notice that

$$|E(y/\delta d, q', a_d)| \leqslant \max_{(a,q')=1} \max_{z\leqslant x/\delta} |E(z,q',a)|$$

The right hand side of the above inequality is simply $E^*(x/\delta, q')$, giving

$$|E'(y,q,a)| \leqslant E^*(x/\delta, q')\tau(\delta) \sum_{d|\delta} \frac{1}{\tau(d)}.$$

It is clear that $\sum_{d|\delta} \frac{1}{\tau(d)} \leqslant \sum_{d|\delta} 1 \leqslant \tau(\delta)$, thereby completing the proof. ■

Theorem 5.9. *Let $q \geqslant 1$ be square-free. For any $A > 0$, we have*

$$\sum_{q<x^\theta} \max_{y\leqslant x} \max_{(a,q)=1} |E'(y,q,a)| \ll \frac{x}{(\log x)^A},$$

for any permissible θ.

Proof. Using the result of the previous proposition with the same notation, we have

$$\widetilde{E}^*(x,q) := \max_{a \ (\mathrm{mod}\ q)} \max_{y\leqslant x} |E'(y,q,a)| \leqslant \max_{a \ (\mathrm{mod}\ q)} \tau(\delta)^2 \max_{y\leqslant x} E^*(x/\delta, q/\delta).$$

Clearly, the condition $\max_{y\leqslant x}$ is redundant. As $\delta := (a,q)$ is the only parameter in the right hand side that depends on the choice of residue class $a \ (\mathrm{mod}\ q)$, the condition $\max_{a \ (\mathrm{mod}\ q)}$ can be replaced with $\max_{\delta|q}$, to give

$$\widetilde{E}^*(x,q) \leqslant \max_{\delta|q} \tau(\delta)^2 \max_{y\leqslant x} E^*(x/\delta, q/\delta).$$

Then,

$$\sum_{q\leqslant x^\theta} \widetilde{E}^*(x,q) \leqslant \sum_{q\leqslant x^\theta} \max_{\delta|q} \tau(\delta)^2 \max_{y\leqslant x} E^*(x/\delta, q/\delta) \leqslant \sum_{q\leqslant x^\theta} \sum_{\delta|q} \tau(\delta)^2 \max_{y\leqslant x} E^*(x/\delta, q')$$
$$\leqslant \sum_{\delta\leqslant x^\theta} \tau(\delta)^2 \sum_{q'\leqslant x^\theta/\delta} E^*(x/\delta, q'),$$

after interchanging summation. Using $x^\theta/\delta \leqslant (x/\delta)^\theta$, for $\theta < 1$, and (5.2) gives

$$\sum_{q\leqslant x^\theta} \widetilde{E}^*(x,q) \leqslant \sum_{\delta\leqslant x^\theta} \tau(\delta)^2 \frac{x}{\delta} \frac{1}{(\log(x/\delta))^A},$$

for any $A > 0$. In the given range of δ, $\log(x/\delta) \gg \log x$. One can use the Tauberian theorem (cf. Ex. 4.4.17 of [12]) or even elementary estimations to obtain $\sum_{n\leqslant x} \tau(n)^2 \sim cx(\log x)^3$ for some constant c, followed by partial summation to get

$$\sum_{n\leqslant x} \frac{\tau(n)^2}{n} \ll (\log x)^4.$$

Thus,

$$\sum_{q\leqslant x^\theta} \widetilde{E}^*(x,q) \ll \sum_{\delta\leqslant x^\theta} \frac{\tau(\delta)^2}{\delta} \frac{x}{(\log x)^A} \ll \frac{x(\log x)^4}{(\log x)^A},$$

This completes the proof of the theorem. ■

We are ready to derive an asymptotic expression for $S_2^{(m)}$. The following can be compared to Theorem 2.1 of Li-Pan [8]. It must be noted that the expression for $S_2^{(m)}$ obtained by Li-Pan contains a sign error for the term $\boldsymbol{\beta}_2$. This sign is crucial; if the expression obtained by [8] were correct, one could show that it leads to infinitude of twin primes for any $\theta > 2/3$. Moreover, it would violate the analogue of the Elliott-Halberstam conjecture for the divisor function. Namely, one could then prove that the divisor function cannot have level of distribution greater than 4/5. This is contrary to expected heuristic reasoning.

Lemma 5.10. *With $\lambda_{\underline{d}}$'s as chosen in* (3.2) *in terms of $\mathcal{F}$, and $R = N^{\theta/2-\delta}$, we have as $N \to \infty$,*

$$S_2^{(m)} := \sum_{\substack{n\sim N \\ n\equiv b \ (\mathrm{mod}\ W)}} \tau(n+h_m) \Bigg(\sum_{d_j|n+h_j\forall j} \lambda_{\underline{d}} \Bigg)^2$$
$$= (1+o(1)) \frac{W^{k-1}}{\phi(W)^k} \frac{N}{(\log R)^k} \left(\frac{\log N}{\log R} \boldsymbol{\alpha}^{(m)} - \boldsymbol{\beta}_1^{(m)} - 4\boldsymbol{\beta}_2^{(m)} \right),$$

with

$$\boldsymbol{\alpha}^{(m)} := \int_{\Delta_k(1)} t_m \left(\mathcal{F}^{(\underline{1}+e_m)}(\underline{t})\right)^2 dt_1 \dots dt_k,$$
$$\boldsymbol{\beta}_1^{(m)} := \int_{\Delta_k(1)} t_m^2 \left(\mathcal{F}^{(\underline{1}+e_m)}(\underline{t})\right)^2 dt_1 \dots dt_k,$$

and

$$\boldsymbol{\beta}_2^{(m)} := \int_{\Delta_k(1)} t_m \mathcal{F}^{(\underline{1}+e_m)}(\underline{t}) \mathcal{F}^{(\underline{1})}(\underline{t}) dt_1 \dots dt_k.$$

We use the usual notation (2.1) *as before.*

Proof. We begin by establishing the setting of the sieve. The tuple $\underline{n}$ in this case is $(n+h_1, \dots, n+h_k)$, where h_i are elements of the fixed set $\mathscr{H}$. The set $\mathcal{S}$ is given by

$$\mathcal{S} = \{\underline{n} = (n+h_1, \dots, n+h_k) : n \sim N\}.$$

The choice of W and $\underline{b}$ was stated at the beginning of this section. The weights in this case are given by

$$w_{\underline{n}} = \tau(n+h_m).$$

Clearly, hypothesis H1 holds as $\mathscr{H}$ is a fixed set. We try to show H2'. For this, consider the sum

$$\sum_{\substack{\underline{d} | \underline{n} \\ \underline{n} \equiv \underline{b} \ (\mathrm{mod}\ W)}} w_{\underline{n}} = \sum_{\substack{n \sim N \\ d_j | n+h_j \forall j \\ n \equiv b \ (\mathrm{mod}\ W)}} \tau(n+h_m)$$

For the above expression, we can use the same argument as in Lemma 4.2 of [17] to conclude that $(d_i, d_j) = 1$ for all $i \neq j$, and $(d_j, W) = 1$ for all j. We rewrite the above sum as

$$\sum_{\substack{N+h_m \leqslant n' < 2N+h_m \\ n' \equiv h_m - h_j \ (\mathrm{mod}\ d_j) \forall j \\ n' \equiv b+h_m \ (\mathrm{mod}\ W)}} \tau(n'),$$

where $n' = n + h_m$. Note that if a prime p divides $h_m - h_j$, for $j \neq m$, then p divides W. Hence $(d_j, W) = 1$ implies that $h_m - h_j$ must be co-prime to d_j for $j \neq m$. It is also clear that $b + h_m$ is co-prime to W. Then, using the Chinese remainder theorem, the above sum can be written as the sum over a single residue class a modulo $q = W \prod_j d_j$, with $(a, q) = d_m$ to obtain

$$\sum_{\substack{N+h_m \leqslant n' < 2N+h_m \\ n' \equiv a \ (\mathrm{mod}\ q)}} \tau(n').$$

Letting q' denote q/d_m, we apply Proposition 5.7 to write this sum as

$$\frac{d_m\tau(d_m)}{\phi(d_m)}\sum_{t|d_m}\frac{\mu(t)}{t\tau(t)}\frac{\phi(q)}{q^2}N\Bigg(\log N-\log d_m+\sum_{p|d_m}\frac{\log p}{2p-1}+c+2\sum_{p|q'}\frac{\log p}{p-1}\Bigg)$$
$$+E'(N,q,a)+O\left(d_m^{1/2}q^{\epsilon-1}\sqrt{N}\right). \quad (5.4)$$

We first deal with the main term (the first term) of this sum. Further simplification using the definition of q along with

$$\sum_{t|d_m}\frac{\mu(t)}{t\tau(t)}=\prod_{p|d_m}\left(1-\frac{1}{2p}\right),$$

gives,

$$\frac{\phi(W)}{W^2}N\frac{d_m\tau(d_m)}{\phi(d_m)}\prod_{p|d_m}\left(1-\frac{1}{2p}\right)\prod_{j=1}^{k}\frac{\phi(d_j)}{d_j^2}\log N \quad (5.5)$$
$$+\frac{\phi(W)}{W^2}N\frac{d_m\tau(d_m)}{\phi(d_m)}\prod_{p|d_m}\left(1-\frac{1}{2p}\right)\prod_{j=1}^{k}\frac{\phi(d_j)}{d_j^2}\left(c+2\sum_{p|q'}\frac{\log p}{p-1}-\sum_{p|d_m}\frac{\log p}{2p-1}\right)$$
$$-\frac{\phi(W)}{W^2}N\log d_m\frac{d_m\tau(d_m)}{\phi(d_m)}\prod_{p|d_m}\left(1-\frac{1}{2p}\right)\prod_{j=1}^{k}\frac{\phi(d_j)}{d_j^2}.$$

As all the d_j's are co-prime to W as well as mutually co-prime to each other, we can rewrite

$$2\sum_{p|q'}\frac{\log p}{p-1}=\sum_{p|W}\frac{2\log p}{p-1}+\sum_{j\neq m}\sum_{p|d_j}\frac{2\log p}{p-1}.$$

It is clear that the latter sum, being the sum of component additive functions, is an additive function of the k-tuple $\underline{d}$.

Thus H2' is satisfied, with

$$X=\frac{\phi(W)}{W^2}N\Bigg(\log N+c+\sum_{p|W}\frac{2\log p}{p-1}\Bigg),\qquad X^*=-\frac{\phi(W)}{W^2}N,$$

$$f(\underline{d})=f_*(\underline{d})=\frac{\phi(d_m)}{d_m\tau(d_m)}\prod_{p|d_m}\left(\frac{2p}{2p-1}\right)\prod_{j=1}^{k}\frac{d_j^2}{\phi(d_j)},$$

$$\nu(\underline{d})=\log d_m+\sum_{p|d_m}\frac{\log p}{2p-1}-\sum_{j\neq m}\sum_{p|d_j}\frac{2\log p}{p-1}.$$

Moreover, the error terms in (5.4) give

$$r_{\underline{d}}=E'(N,q,a)+O\left(d_m^{1/2}q^{\epsilon-1}\sqrt{N}\right).$$

It is clear that H3 holds for f and f_* with

$$\alpha_j = \alpha_j^* = \begin{cases} 1 & \text{if } j = 1, \dots, k, j \neq m \\ 2 & \text{if } j = m. \end{cases}$$

This brings us to H5 for the additive function ν, given by

$$\nu_j(p) = -\frac{2\log p}{p-1} \quad \text{for } j \neq m, \qquad \nu_m(p) = \log p - \frac{\log p}{2p-1}.$$

Noting that $\nu_m(p) > 0$ for all primes p, we see that if one fixes a component j then the sign of ν_j is fixed, which means that the absolute value sign in the statement of H5 is irrelevant. The behaviour of $-\zeta'(s)/\zeta(s)$ as $s \to 1^+$ as well as the absolute convergence of the series $\sum_p \frac{\log p}{p^2}$ shows that H5 holds for ν with

$$\beta_j = \begin{cases} 0 & \text{if } j = 1, \dots, k, j \neq m \\ 1 & \text{if } j = m. \end{cases}$$

Recall that we are using θ to denote the level of distribution of the divisor function (see (5.1)). To verify H4 it suffices to show that for any $A > 0$,

$$\sum_{[\underline{d},\underline{e}]<N^\theta} E'(N,q,a) + O\left(\sum_{[\underline{d},\underline{e}]<N^\theta} [d_m, e_m]^{1/2} q^{\epsilon-1}\sqrt{N} \right) \ll \frac{N}{(\log N)^A}, \tag{5.6}$$

Denoting $\prod_{j\neq m}[d_j, e_j]$ as $[\underline{d},\underline{e}]'$, we have

$$\sum_{[\underline{d},\underline{e}]<N^\theta} [d_m, e_m]^{1/2} q^{\epsilon-1} \ll W^{\epsilon-1} \sum_{[d_m,e_m]<N^\theta} [d_m, e_m]^{\epsilon-1/2} \sum_{[\underline{d},\underline{e}]'<N^\theta} ([\underline{d},\underline{e}]')^{\epsilon-1}$$

Using Proposition 3.1 of [17] and partial summation along with the fact that the average order of $\tau_3(n)$ is $(\log n)^2$, we get

$$\sum_{[d_m,e_m]<N^\theta} [d_m, e_m]^{\epsilon-1/2} \ll \sum_{r<N^\theta} r^{\epsilon-1/2}\tau_3(r) \ll (N^\theta)^{\epsilon+1/2}(\log N)^2.$$

Similarly,

$$\sum_{[\underline{d},\underline{e}]'<N^\theta} ([\underline{d},\underline{e}]')^{\epsilon-1} \ll \sum_{r<N^\theta} r^{\epsilon-1}\tau_{3(k-1)}(r) \ll (N^\theta)^{\epsilon}(\log N)^{3k}.$$

As ϵ can be made arbitrarily small and $W \ll (\log\log N)^2$, we obtain for the second term of (5.6), the estimate

$$\sum_{[\underline{d},\underline{e}]<N^\theta} [d_m, e_m]^{1/2} q^{\epsilon-1}\sqrt{N} \ll N^{\epsilon'+\theta/2}\sqrt{N},$$

for any $\epsilon' > 0$. As $\theta < 1$, this term is indeed of the order of $N/(\log N)^A$ for any $A > 0$ as required. The first term of (5.6) can be written as

$$\sum_{q<WN^\theta} E'(N,q,a) \ll \sum_{q<N^{\theta+\epsilon}} E'(N,q,a)$$

for any $\epsilon > 0$, due to the choice of W. For small enough $\epsilon > 0$, $\theta + \epsilon$ is permissible (being less than the level of distribution θ_0), and one can apply Theorem 5.9, to obtain that the first term of (5.6) is $O(N/(\log N)^A)$ as well. Thus H4 holds.

As the choice of D_0 gives

$$\frac{\log D_0}{\log R} = o(1),$$

we are now in a position to apply Theorem 4.3 with $X, X^*, \underline{\alpha}, \underline{\alpha}^*, \underline{\beta}$ as given above, $\alpha = \alpha^* = k+1$ and $c(W) = c^*(W) = W^{k+1}/\phi(W)^{k+1}$. After some simplification, this gives the following asymptotic formula (as $R \to \infty$) for the required sum

$$\begin{aligned}&(1+o(1))\frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}\left(\frac{\log N + c + \sum_{p|W}\frac{2\log p}{p-1}}{\log R}C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}\right)\\&-(1+o(1))\frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}\left(2C_m^*(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}\right).\end{aligned}$$

Here $\underline{\alpha}$ is the tuple $(1,\ldots,1,2,1,\ldots,1)$, with 2 in the m-th place. As $W = \prod_{p<D_0} p$,

$$\sum_{p|W}\frac{2\log p}{p-1} \ll \sum_{p|W} 1 \ll D_0$$

By the choice of D_0 made in the beginning of this section, it is clear that $D_0 = o(\log R)$. Hence, the asymptotic formula for the required sum as $R \to \infty$, becomes

$$(1+o(1))\frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}\left(\frac{\log N}{\log R}C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})} - 2C_m^*(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}\right).$$

Using Lemma 3.2 keeping in mind the notation (2.1), we obtain that

$$C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})} = \boldsymbol{\alpha}^{(m)}(\mathcal{F}).$$

Similarly, by definition (see Theorem 4.3), $C_m^*(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}$ equals

$$C(\mathcal{F},\mathcal{F})^{(\underline{\alpha},\underline{\alpha},\underline{\alpha}+e_m)} - C(\mathcal{F},\mathcal{F})^{(\underline{\alpha}-e_m,\underline{\alpha},\underline{\alpha})} - C(\mathcal{F},\mathcal{F})^{(\underline{\alpha},\underline{\alpha}-e_m,\underline{\alpha})},$$

which equals

$$\frac{\boldsymbol{\beta}_1^{(m)}}{2} + 2\boldsymbol{\beta}_2^{(m)},$$

after applying Lemma 3.2. This completes the proof. ■

Noting that condition H4 holds for both S_1 and S_2 when θ is permissible for the divisor function, we put together the above lemma with the asymptotic formula for S_1 obtained in Lemma 4.2 of [17] to obtain

Lemma 5.11. *The quantity $S(N,\rho) := \rho S_1 - \sum_{m=1}^{k} S_2^{(m)}$ as $N \to \infty$ is given by*

$$(1+o(1))\frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}\left(\rho I(\mathcal{F}) - \frac{\boldsymbol{\alpha}^*}{(\theta/2-\delta)} + \boldsymbol{\beta}_1^* + 4\boldsymbol{\beta}_2^*\right),$$

with

$$\boldsymbol{\alpha}^* = \sum_{i=1}^{k}\boldsymbol{\alpha}^{(m)}, \qquad \boldsymbol{\beta}_1^* = \sum_{i=1}^{k}\boldsymbol{\beta}_1^{(m)}, \qquad \boldsymbol{\beta}_2^* = \sum_{i=1}^{k}\boldsymbol{\beta}_2^{(m)}.$$

5.2. Choice of the test function

Inspired by the choice of the function in [4], let us choose the function $\mathcal{F}$ to be

$$\mathcal{F}(t_1,\ldots,t_k) = (-1)^k\frac{\ell!}{(k+\ell)!}\left(1-\sum_{i=1}^{k} t_i\right)^{\ell+k} \tag{5.7}$$

when $\sum_{j=1}^{k} t_j \leqslant 1$, and zero otherwise. Then by (2.1) ,

$$\mathcal{F}^{(\underline{1})}(t_1,\ldots,t_k) = \left(1-\sum_{i=1}^{k} t_i\right)^{\ell}, \tag{5.8}$$

We wish to compute all the integrals appearing in Lemma 5.11 with this choice of function.

In order to compute the integral $I(\mathcal{F})$, we begin by considering the integral

$$I_{\ell,k} = \int_{\Delta_k(1)}\left(1-\sum_{i=1}^{k} t_i\right)^{\ell} d\underline{t}. \tag{5.9}$$

Then

$$\begin{aligned}
I_{\ell,k} &= \int_0^1 \cdots \int_0^{1-\sum_{i=2}^{k} t_i}\left(1-\sum_{i=1}^{k} t_i\right)^{\ell} dt_1 dt_2 \ldots dt_k \\
&= \frac{1}{\ell+1}\int_0^1 \cdots \int_0^{1-\sum_{i=3}^{k} t_i}\left(1-\sum_{i=2}^{k} t_i\right)^{\ell+1} dt_2 dt_3 \ldots dt_k \\
&= \frac{1}{\ell+1} I_{\ell+1,k-1}
\end{aligned}$$

This gives a recursion formula for $I_{\ell,k}$. Moreover,

$$I_{\ell+k-1,1} = \int_0^1 (1-t_k)^{\ell+k-1}\, dt_k = \frac{1}{\ell+k}.$$

Thus, $I_{\ell,k} = \ell!/(\ell+k)!$. By definition of $I(\mathcal{F})$ in Lemma 4.2 of [17], one has

$$I(\mathcal{F}) = I_{2\ell,k} = \frac{(2\ell)!}{(2\ell+k)!}. \tag{5.10}$$

We have from (5.8),

$$\mathcal{F}^{(\underline{1}+e_m)}(t_1,\dots,t_k) = -\ell\left(1-\sum_{i=1}^k t_i\right)^{\ell-1}, \tag{5.11}$$

Let us compute the integrals α^* and β_2^*. Consider first the integral

$$J_{\ell,k} = \int_{\Delta_k(1)} \sum_{i=1}^k t_i \left(1-\sum_{i=1}^k t_i\right)^{\ell} d\underline{t}.$$

Then

$$I_{\ell,k} - J_{\ell,k} = \int_{\Delta_k(1)} \left(1-\sum_{i=1}^k t_i\right)\left(1-\sum_{i=1}^k t_i\right)^{\ell} d\underline{t} = I_{\ell+1,k}$$

Using known values for the integrals $I_{\ell,k}$ and $I_{\ell+1,k}$,

$$J_{\ell,k} = k\frac{\ell!}{(\ell+k+1)!}.$$

Using definitions in Lemmas 5.10 and 5.11, we have

$$\begin{aligned}\boldsymbol{\alpha}^*(\mathcal{F}) &= \ell^2\int_{\Delta_k(1)}\left(\sum_{i=1}^k t_i\right)\left(1-\sum_{i=1}^k t_i\right)^{2\ell-2} d\underline{t},\\ \boldsymbol{\beta}_2^*(\mathcal{F}) &= -\ell\int_{\Delta_k(1)}\left(\sum_{i=1}^k t_i\right)\left(1-\sum_{i=1}^k t_i\right)^{2\ell-1} d\underline{t}.\end{aligned}$$

From (5.8) and (5.11) for our choice of function, this gives

$$\boldsymbol{\alpha}^* = \ell^2 J_{2\ell-2,k} = \ell^2 k\frac{(2\ell-2)!}{(2\ell+k-1)!}, \qquad \boldsymbol{\beta}_2^* = -\ell J_{2\ell-1,k} = -\ell\, k\frac{(2\ell-1)!}{(2\ell+k)!}. \tag{5.12}$$

We are now left to compute the integral $\boldsymbol{\beta}_1^*$. In order to do this, we first consider the integral $\boldsymbol{\beta}_1^{(m)}$ defined in Lemma 5.10. For our choice of function, using (5.11), we have

$$\boldsymbol{\beta}_1^{(m)} = \ell^2\int_{\Delta_k(1)} t_m^2\left(1-\sum_{i=1}^k t_i\right)^{2\ell-2} d\underline{t}$$

Let σ_m denote $\sum_{i\neq m} t_i$. Integration by parts with respect to t_m gives us

$$\int_0^{1-\sigma_m} t_m^2 \left(1-\sum_{i=1}^k t_i\right)^{2\ell-2} dt_m = -\frac{t_m^2}{2\ell-1}\left(1-\sum_{i=1}^k t_i\right)^{2\ell-1}\Bigg|_{t_m=1-\sigma_m}^{t_m=0} + \frac{2}{2\ell-1}\int_0^{1-\sigma_m} t_m \left(1-\sum_{i=1}^k t_i\right)^{2\ell-1} dt_m$$

The first term evaluated over the given limits is zero. Let $\underline{t}^{(m)}$ denote the k-tuple $\underline{t}$ with the mth component removed. Then,

$$\begin{aligned}\boldsymbol{\beta}_1^{(m)} &= \ell^2\int_0^1\cdots\int_0^{1-\sum_{i\neq m} t_i} t_m^2\left(1-\sum_{i=1}^k t_i\right)^{2\ell-2} dt_m d\underline{t}^{(m)}\\ &= \frac{2\ell^2}{2\ell-1}\int_0^1\cdots\int_0^{1-\sum_{i\neq m} t_i} t_m\left(1-\sum_{i=1}^k t_i\right)^{2\ell-1} dt_m d\underline{t}^{(m)}\\ &= \frac{2\ell^2}{2\ell-1}\int_{\Delta_k(1)} t_m\left(1-\sum_{i=1}^k t_i\right)^{2\ell-1} d\underline{t}\end{aligned}$$

Thus,

$$\boldsymbol{\beta}_1^* = \frac{2\ell^2}{2\ell-1}J_{2\ell-1,k} = 2k\ell^2\frac{(2\ell-2)!}{(2\ell+k)!} \tag{5.13}$$

Plugging obtained values for the required integrals into Lemma 5.11 yields the following result.

Theorem 5.12.

$$S(N,\rho) = (1+o(1))\frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}\frac{(2\ell)!}{(2\ell+k)!}\cdot\mathcal{S}_{k,\ell,\theta},$$

where

$$\mathcal{S}_{k,\ell,\theta} = \left(\rho - \frac{k\ell(2\ell+k)}{(2\ell-1)(\theta-2\delta)} + \frac{k\ell}{(2\ell-1)} - 2k\right)$$

5.3. Improvement of a result of Heath-Brown

We obtain the following improvement of a result of Heath-Brown [5], who obtained the theorem below with

$$\rho_k = \frac{3k^2}{2} + 4k.$$

Theorem 5.13. *There exists ρ_k such that there are $\gg x(\log\log x)^{-1}(\log x)^{-k}$ integers $n \leqslant x$ satisfying: the product $\prod_{i=1}^{k}(n+h_i)$ is square-free and*

$$\sum_{i=1}^{k} \tau(n+h_i) \leqslant \lfloor \rho_k \rfloor .$$

For large k, we have $\rho_k \sim \frac{3}{4}k^2$.

Proof. It can be seen from the expression for $S(N,\rho)$ in Theorem 5.12, that

$$\rho > \frac{k\ell(2\ell+k)}{(2\ell-1)(\theta-2\delta)} - \frac{k\ell}{(2\ell-1)} + 2k, \tag{5.14}$$

implies that $\mathcal{S}_{k,\ell,\theta} > 0$ and hence $S(N,\rho) > 0$ for all large N. Thus, with $\lambda_{\underline{d}}$'s chosen as in (3.2), we have that the quantity

$$\begin{aligned} \rho S_1 - \sum_{m=1}^{k} S_2^{(m)} &= \sum_{\substack{n \sim N \\ n \equiv b \pmod W}} \left(\rho - \sum_{j=1}^{k}\tau(n+h_i)\right)\left(\sum_{d_j | n+h_j \forall j} \lambda_{\underline{d}}\right)^2 \\ &= (1+o(1))\frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}\frac{(2\ell)!}{(2\ell+k)!}\cdot \mathcal{S}_{k,\ell,\theta} \end{aligned} \tag{5.15}$$

is positive for all large N. As explained in Proposition 5.1, this means that the inequality

$$\sum_{j=1}^{k} \tau(n+h_j) < \rho, \tag{5.16}$$

holds infinitely often. In fact, one can estimate the number of integers $n \leqslant x$ for which (5.16) holds as follows. Let all $n \sim N, n \equiv b \pmod W$ for which (5.16) holds be denoted by n'. Then the positive contribution to the first term in parenthesis in (5.15) can come only from these integers. As a consequence,

$$\sum_{\substack{n' \sim N \\ n' \equiv b \pmod W}} \left(\rho - \sum_{j=1}^{k}\tau(n'+h_j)\right)\left(\sum_{d_j | n'+h_j \forall j} \lambda_{\underline{d}}\right)^2 \gg \frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k}. \tag{5.17}$$

For each n', the absolute value of the first term in parenthesis is bounded above by ρ. With the $\lambda_{\underline{d}}$'s as chosen in (3.2), we have that each $\lambda_{\underline{d}}$ is bounded in absolute value. Moreover, arguing using the W-trick as done in the proof of Lemma 4.2 of [17] shows that the $n'+h_j$'s are pairwise co-prime. Thus, for some constant c, independent of n,

$$\sum_{d_j | n'+h_j \forall j} |\lambda_{\underline{d}}| \leqslant c \sum_{d_j | n'+h_j \forall j} 1 = c\,\tau\left(\prod_{j=1}^{k}(n'+h_j)\right) = c\prod_{j=1}^{k}\tau(n'+h_j).$$

However, by construction, all the n' satisfy (5.16); in particular, each $n'+h_j$ can have at most ρ divisors. Hence the above product of the divisor functions is bounded above by some constant M (say) independent of n'. This gives

$$\sum_{\substack{n'\sim N\\ n'\equiv b \pmod W}} \left(\rho - \sum_{j=1}^{k} \tau(n'+h_i)\right)\Big(\sum_{d_j|n'+h_j\forall j} \lambda_{\underline{d}}\Big)^2 \ll \sum_{\substack{n'\sim N\\ n'\equiv b \pmod W}} 1, \quad (5.18)$$

with both summations running only over integers n' satisfying (5.16). Combining (5.17) and (5.18), we get

$$\sum_{\substack{n'\sim N\\ n'\equiv b \pmod W}} 1 \gg \frac{W^{k-1}}{\phi(W)^k}\frac{N}{(\log R)^k},$$

which means that the number of integers $\leqslant x$ that satisfy the conditions of the theorem is $\gg x(\log\log x)^{-1}(\log x)^{-k}$.

We now wish to optimize the ρ in (5.16) as a function of k. Let $(\theta-2\delta)=2/3-\epsilon$ in (5.14). Putting $\ell=1$ gives the theorem with

$$\rho_k = \frac{k}{2}(3k+8).$$

Notice that this is precisely Theorem 1 of Heath-Brown [5]. Letting $\ell=2$ gives the theorem with $\rho_k = k^2+16k/3$, which is an improvement for $k>2$. In order to optimize ρ_k, the right hand side of (5.14) should be viewed as a function $h(\ell)$ and minimized with respect to ℓ. Then we obtain

$$h(\ell) = k\frac{\ell(2\ell+k)-(\theta-2\delta)\ell}{(\theta-2\delta)(2\ell-1)} + 2k$$

Differentiating this with respect to ℓ, we have

$$h'(\ell) = \frac{k}{(2\ell-1)^2}\left(\frac{4\ell^2-4\ell-k}{\theta-2\delta}+1\right).$$

From this and $h''(\ell)$, it can be seen that the $h(\ell)$ is minimum when

$$4\ell(\ell-1) = k-(\theta-2\delta).$$

For large k, this means $\ell\sim\sqrt{k}/2$. Plugging this into $h(\ell)$ gives

$$h(\ell)\sim\frac{k^2}{2(\theta-2\delta)}$$

Thus, for large k, $\mathcal{S}_{k,\ell,\theta}$ can be made positive with $\ell\sim\sqrt{k}/2$, and $\rho > \frac{k^2}{2(\theta-2\delta)}$. Choosing $\theta-2\delta=\frac{2}{3}-\epsilon$ completes the proof. ■

We remark that Heath-Brown discusses what seems to be a more general case, with $n + h_i$ replaced by linear functions of the type $L_i(n) = a_i n + h_i$ satisfying certain hypotheses on the a_i's, namely that each a_i is composed of the same set of primes. It is easy to see that our method can be adapted to the same setting, though in the interest of simplicity and elegance, we do not do so here.

Remark. Our quantitative result is only slightly weaker than that of Heath-Brown (by a factor of $(\log\log x)^{-1}$). This can be improved further by a finer estimate of the 'constants' depending on W. The above proof gives that for n' satisfying the conditions of the theorem,

$$\sum_{\substack{n' \sim N \\ n' \equiv b \pmod W}} 1 \gg \frac{1}{W}\left(\frac{W}{\phi(W)}\right)^k \frac{N}{(\log R)^k}.$$

By Merten's theorem,

$$\frac{W}{\phi(W)} = \prod_{p<D_0}\left(1-\frac{1}{p}\right)^{-1} \sim e^{\gamma}\log D_0,$$

where γ is Euler's constant. This gives

$$\sum_{\substack{n' \leqslant x \\ n' \equiv b \pmod W}} 1 \gg \frac{(\log\log\log\log x)^k}{\log\log x}\frac{x}{(\log x)^k},$$

thereby improving the quantitative estimate of Theorem 5.13.

It is also worth noting that if we assume the divisor function analogue of the Elliott-Halberstam conjecture, more precisely, if we assume that the divisor function has a level of distribution $\theta = 1$, then the statement of Theorem 5.13 holds with $\rho_k \sim \frac{k^2}{2}$.

Interestingly, the above theorem leads to some bounds on the number of prime factors of $\prod_{i=1}^k (n + h_i)$ as well as the number of prime factors of each $n + h_i$ individually. We state these results as corollaries. It is worth noting that better bounds have been obtained through different means in the literature. The first corollary is as stated in [5] and follows immediately from the fact that $\tau(n + h_i)$ is at least 2 for each i.

Corollary 5.14. *There exists R_k such that there are $\gg x(\log\log x)^{-1}(\log x)^{-k}$ integers $n \leqslant x$ satisfying: the product $\prod_{i=1}^k (n + h_i)$ is square-free and*

$$\max_{i\in\{1,\ldots,k\}} \omega(n+h_i) \leqslant R_k = \lfloor \log_2\left(\lfloor \rho_k \rfloor - 2(k-1)\right)\rfloor .$$

For large k, we have $R_k \sim 2\frac{\log k}{\log 2}$.

To control the number of prime factors of the product $\prod_{i=1}^{k}(n+h_i)$, observe that

$$\sum_{i=1}^{k}\omega(n+h_i) \leqslant k\log_2\left(\frac{\sum_{i=1}^{k}2^{\omega(n+h_i)}}{k}\right) \leqslant k\log_2\left(\frac{\lfloor\rho_k\rfloor}{k}\right), \tag{5.19}$$

from the AM-GM inequality and Theorem 5.13. Equality occurs everywhere iff all the $\omega(n+h_i)$'s are equal to the quantity $\log_2\left(\frac{\lfloor\rho_k\rfloor}{k}\right)$, which we denote as c_k. However c_k may not be an integer, in which case this maximum is not attained in the context of our problem. We optimize by taking the $\omega(n+h_i)$'s to be 'close' to $\lfloor c_k\rfloor$ and to each other. More precisely, let $k-m$ (where $m\leqslant k$) of the $\omega(n+h_i)$'s be equal to $\lfloor c_k\rfloor$ and the remaining m of them be $\lfloor c_k\rfloor+1$. Then one has

$$\sum_{i=1}^{k}2^{\omega(n+h_i)} = 2^{\lfloor c_k\rfloor}(k+m) \leqslant \lfloor\rho_k\rfloor,$$

from Theorem 5.13, giving

$$m \leqslant \left\lfloor\frac{\lfloor\rho_k\rfloor}{2^{\lfloor c_k\rfloor}}\right\rfloor - k$$

As $\sum_{i=1}^{k}\omega(n+h_i) = k\lfloor c_k\rfloor + m$, we have the following bound on the number of distinct prime factors of the product $\prod_{i=1}^{k}(n+h_i)$.

Corollary 5.15. *There exists r_k such that there are $\gg x(\log\log x)^{-1}(\log x)^{-k}$ integers $n\leqslant x$ satisfying: the product $\prod_{i=1}^{k}(n+h_i)$ is square-free and*

$$\sum_{i=1}^{k}\omega(n+h_i) \leqslant r_k = k(\lfloor c_k\rfloor - 1) + \left\lfloor\frac{\lfloor\rho_k\rfloor}{2^{\lfloor c_k\rfloor}}\right\rfloor.$$

For large k, we have $\rho_k \sim \frac{3}{4}k^2$ and $c_k = \log_2\left(\frac{\lfloor\rho_k\rfloor}{k}\right)$, giving $r_k \sim k\frac{\log k}{\log 2}$.

6. Concluding remarks

We remark that the quantitative results of this paper are by no means optimal and can be improved by various means. For instance, one natural device that suggests itself is to optimize the choice of W.

Acknowledgements. We would like to thank V. Kumar Murty and John Friedlander for some helpful discussion and inputs. We also thank the referee for detailed comments on a previous version of this paper

References

[1] J.-R. Chen, *On the representation of a large even integer as the sum of a prime and the product of at most two primes*, Scientia Sinica **16** (1973), 1–70.

[2] E. Fouvry and H. Iwaniec, *The divisor function over arithmetic progressions, with an appendix by Nicholas Katz*, Acta Arith. **61** (1992), no. 3, 271–287.

[3] J. Friedlander and H. Iwaniec, *Opera de Cribro*, vol. 57, Colloquium Publications, American Math. Society, Providence, Rhode Island, 2010.

[4] D.A. Goldston, J. Pintz and C. Y. Yıldırım, *Primes in tuples. I*, Ann. of Math. (2) **170** (2009), no. 2, 819–862.

[5] D.R. Heath-Brown, *Almost-prime k-tuples*, Mathematika **44** (1997), no. 2, 245–266.

[6] K-H. Ho, K-M, Tsang, *On almost prime k-tuples*, J. Number Theory **120** (2006), no. 1, 33–46.

[7] C. Hooley, *An asymptotic formula in the theory of numbers*, Proc. London Math. Soc. (3) **7** (1957), 396–413.

[8] H. Li and H. Pan, *Bounded gaps between primes of the special form*, International Mathematics Research Notices, (2015), doi:10.1093/imrn/rnv049.

[9] J. Maynard, *Small gaps between primes*, Ann. of Math. (2) **181** (2015), no. 1, 383–413.

[10] J. Maynard, *Bounded length intervals containing two primes and an almost-prime II*, J. Number Theory **154** (2015), 1–15.

[11] M. Ram Murty, V. Kumar Murty, *A variant of the Bombieri-Vinogradov theorem*, in Number theory (Montreal, Que. 1985), CMS Conf. Proc., Amer. Math. Soc., Providence, RI, vol.7 (1987), 243–272.

[12] M. Ram Murty, *Problems in Analytic number theory*, 2nd Edition, Graduate Texts in Mathematics, Springer, New York, 2008.

[13] M. Ram Murty, *Some Remarks on the Riemann Hypothesis, in Cohomology of arithmetic groups, L-functions and automorphic forms*, Tata Inst. Fund. Res. Stud. Math., vol.15 (1998/1999), 180–196.

[14] Polymath D.H.J, *Variants of the Selberg sieve, and bounded intervals containing many primes*, Research in the Mathematical sciences **1** (2014), no 1, 1–83, available at http://dx.doi.org/10.1186/s40687-014-0012-7.

[15] A. Selberg, *Lectures on Sieves*, Collected Papers, vol. II, Springer-Verlag, Berlin, 1991.

[16] A. Selberg, *Sieve Methods*, Proc. Symp. Pure Math. (SUNY, Stony Brook, N.Y. 1969), vol. XX, 311-351, Amer. Math. Soc., Providence, R. I., 1971.

[17] A. Vatwani, *A higher rank Selberg sieve and applications*, Czechoslovak Mathematical Journal, to appear.

[18] Y. Zhang, *Bounded gaps between primes*, Ann. of Math.(2) **179** (2014), no. 3, 1121–1174.

Addresses: M. Ram Murty and Akshaa Vatwani: Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L3N6, Canada.

E-mail: murty@mast.queensu.ca, akshaa@mast.queensu.ca

Received: 22 August 2016; **revised:** 28 January 2017