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ON EXTENDED EULERIAN NUMBERS

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Abstract: In this paper, we will study some properties of the extended Eulerian numbers $H(n,\lambda)$, with a parameter λ. In fact, for any integer n, we investigate the asymptotic behavior, find lower and upper bounds for $H(n,\lambda)$. As application, for a champion number N, we obtain asymptotic formulas, lower and upper bounds of the arithmetic functions $\omega(N)$ and $\Omega(N)$.

Keywords: Kalmar's function, extended Eulerian numbers, Champion numbers, asymptotic formula, Ikehara-Wiener theorem.

1. Introduction and preliminaries

Let λ be a complex number. An extended Eulerian number $H(n,\lambda)$ is defined by means of its Dirichlet series

$$\frac{\lambda-1}{\lambda-\zeta(s)} = \sum_{n\geqslant 1} \frac{H(n,\lambda)}{n^s}, \tag{1.1}$$

where $\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s}$ is the Riemann Zeta function defined for $\Re(s) > 1$. A champion number N for the function H is a number that satisfies

$$n < N \implies H(n,\lambda) < H(N,\lambda).$$

In this paper, we will study several properties of extended Eulerian numbers $H(n,\lambda)$, with a real parameter $\lambda > 1$. We will extend and improve some results obtained by Kalmár [17], Hille [14], Erdös [8], Evans [9], Klazar and Luca [18], Deleglise, Hernane, and Nicolas [13], concerning the maximal order of extended Eulerian numbers. As an application of our results, we investigate the H-champion numbers.

We are motivated by the following important particular cases:

1) If $\lambda = 0$, then $H(n,0) = \mu(n)$ is a Möbius number.

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2) If $n = p_1 p_2 \cdots p_r$ is square-free, then $H(n, \lambda) = H_r(\lambda)$, where $H_r(\lambda)$ is an Eulerian number. In addition, if $\lambda = -1$, thus

$$H(n, -1) = H_r(-1) = E_r$$

is the so-called Euler number. One can notice that an Euler number E_n corresponds to $2^{-n}C_n$ in [21, p.28]. One can also see [3, p.688] and [1, p.354, formula (1.1.5)].

3) If $\lambda = 2$, then $H(n, 2) = K(n)$, where K is the Kalmár arithmetic function which counts the number of ordered factorizations of a positive integer n in factors bigger than 1. Various properties of this function were studied by many mathematicians. In fact, Kalmár found the average order of $K(n)$, for $x \to \infty$

$$\sum_{n \leqslant x} K(n) = -\frac{x^\rho}{\rho \zeta'(\rho)} \{1 + o(1)\}, \tag{1.2}$$

where $\rho = 1.72864\ldots$ is the positive real solution to $\zeta(s) = 2$. On the other hand, this result was improved by Hwang [15]. Moreover, bounds on the maximal order of $K(n)$ were studied by Erdös [8]), Chor, Lemke and Mador [5], Coppersmith and Lewenstein [6], and Hille [14]. Recently, Klazar and Luca [18], Deleglise, Hernane, and Nicolas [13] improved the bounds for the maximal order of $K(n)$.

It is well-known that the extended Eulerian numbers $H(n, \lambda)$ satisfy the following properties:

1) The recurrence formulas are given by

$$\begin{cases} \lambda H(n, \lambda) = \sum_{d|n} H(d, \lambda), & n \geqslant 2,\ \lambda \neq 1, \\ H(1, \lambda) = 1. \end{cases} \tag{1.3}$$

See formula (1.15) in [3].

2) The expression $(\lambda - 1)^{\Omega(n)} H(n,\, \lambda)$ is a polynomial in λ of degree less than $\Omega(n)$, where $\Omega(n) = \sum_{k=1}^{r} k_i$, if $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. In the special case where $n = p_1 p_2 \cdots p_r$, we have

$$(\lambda - 1)^{\Omega(n)} H(n,\, \lambda) = (\lambda - 1)^r H_r(\lambda) = \sum_{t=1}^{r} A_{r,\,t} \lambda^{t-1}, \qquad r \geqslant 1$$

with $A_{r,\,t} = \sum_{j=0}^{t} (-1)^j \binom{r+1}{j} (t - j)^r$. See formulas (5.12), (5.13) in [3].

Writing

$$\frac{\lambda - 1}{\lambda - \zeta(s)} = \frac{\lambda - 1}{\lambda} \frac{1}{1 - \frac{\zeta(s)}{\lambda}} \tag{1.4}$$

and expanding it, we obtain the explicit formula

$$H(n,\, \lambda) = \frac{\lambda - 1}{\lambda} \sum_{k=1}^{\Omega(n)} \frac{d_k(n)}{\lambda^k}, \qquad \text{where } d_k(n) = \sum_{\substack{n_1 n_2 \cdots n_k = n \\ n_1,\, n_2,\, \ldots,\, n_k \geqslant 1}} 1. \tag{1.5}$$

If we rewrite $\frac{\lambda-1}{\lambda-\zeta(s)} = \frac{1}{1-\left(\frac{\zeta(s)-1}{\lambda-1}\right)}$ and expand it, we obtain another explicit formula proved in [10]

$$H(n,\,\lambda) = \sum_{k=1}^{\Omega(n)} \frac{d'_k(n)}{(\lambda-1)^k}, \qquad \text{where } d'_k(n) = \sum_{\substack{n_1 n_2 \cdots n_k = n \\ n_1,\,n_2,\,\ldots,\,n_k \geqslant 2}} 1. \tag{1.6}$$

For $\lambda > 1$, we consider the Mellin inverse integral

$$\frac{1}{2\pi i}\int_{\sigma-i\,\infty}^{\sigma+i\,\infty} \frac{\lambda-1}{\lambda-\zeta(s)} x^s \,\frac{ds}{s} = \sum_{n\geqslant 1} H(n,\,\lambda) \int_{\sigma-i\,\infty}^{\sigma+i\,\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s} \tag{1.7}$$

by Perron's formula, we get

$$\frac{1}{2\pi i}\int_{\sigma-i\,\infty}^{\sigma+i\,\infty} \frac{\lambda-1}{\lambda-\zeta(s)} x^s \,\frac{ds}{s} = \sideset{}{^\star}\sum_{n\leqslant x} H(n,\,\lambda). \tag{1.8}$$

Here, the symbol * on the summation indicates that the last term of the sum must be multiplied by $1/2$ when x is an integer. Now, using Ikehara-Wiener theorem, we obtain the average formula for extended Eulerian numbers

$$\sideset{}{^\star}\sum_{n\leqslant x} H(n,\,\lambda) \sim C(\lambda) x^{\rho(\lambda)}, \tag{1.9}$$

where $\rho(\lambda)$ is the positive real number solution to the equation $\zeta(s)) = \lambda$ and

$$C(\lambda) = \frac{1-\lambda}{\rho(\lambda)\zeta'(\rho(\lambda))}.$$

Formula (1.5) was proved by Evans [9]. Formulas (1.6), (1.8) and (1.9) were also obtained by Grosswald [10].

In this paper, we will study the behavior of the functions $\lambda \mapsto C(\lambda)$ and $\lambda \mapsto \rho(\lambda)$. As an application of this study, we will investigate H-champions numbers, specially their asymptotic, lower, and upper bounds. The size of the exponents of their prime factors will be estimated.

2. Statement of the main results

Let k be a positive integer, $\mathcal{N}_k$ a multiplicative system (including 1) associated to the set of primes numbers, $\mathcal{P}_k = \{p_1, p_2, \ldots, p_k\}$, with $p_1 = 2 < p_2 = 3 < \cdots < p_k$. Define

$$\zeta_k(s) = \prod_{p\in\mathcal{P}_k} (1-p^{-s})^{-1}, \tag{2.1}$$

which is equivalent to

$$\zeta_k(s) = \sum_{n\in\mathcal{N}_k} \frac{1}{n^s}. \tag{2.2}$$

Let $\rho_k(\lambda)$ be the positive real solution to $\zeta_k(s) = \lambda$ and $a_k(\lambda) = -\frac{\lambda}{\zeta_k'(\rho_k(\lambda))}$. For $k = \infty$, we get the Riemann zeta function $\zeta(s)$ and $\rho(\lambda) = \rho_\infty(\lambda)$, $\zeta(\rho(\lambda)) = \lambda$ and $a(\lambda) = -\frac{\lambda}{\zeta'(\rho(\lambda))}$. We state our first main result.

Theorem 1. *Let $\lambda > 1$. Then*

1. *The function $\lambda \mapsto \rho(\lambda)$ is strictly increasing and $\rho(\lambda) > 1$, for any $\lambda > 1$.*
2. *The sequence $(\rho_k(\lambda))_{k\geqslant 1}$ is strictly increasing and bounded. Moreover, for $\Re(s) \geqslant \sigma > 1$, the function $\zeta_k(s)$ uniformly converges to $\zeta(s)$ as k tends to ∞.*
3. *We have*

$$\rho(\lambda) - \rho_k(\lambda) = \frac{\lambda}{(\rho(\lambda)-1)|\zeta'(\rho(\lambda))|k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}} \left\{1 + O\left(\frac{\log\log k}{\log k}\right)\right\}. \tag{2.3}$$

4. *The sequence $(a_k(\lambda))$ is decreasing and converges to $a(\lambda)$, as k tends to ∞. Moreover, there exists a positive constant $\gamma(\lambda)$, depending only on λ, such that for any prime p, we have the inequality*

$$\left|\frac{a_k(\lambda)}{p^{\rho_k(\lambda)}-1} - \frac{a(\lambda)}{p^{\rho(\lambda)}-1}\right| \leqslant \gamma(\lambda)\frac{\log p}{p^{\rho_2(\lambda)}(k\log k)^{\rho_k(\lambda)-1}}. \tag{2.4}$$

To state our remaining main results, we need some notations and definitions. So let $\underline{x} = (x_j)_{1\leqslant j\leqslant\omega}$ be a finite or infinite sequence of positive real numbers such that

$$0 < \Omega(\underline{x}) = \sum_{j=1}^{\omega} x_j < \infty.$$

There exists a unique $C = C(\underline{x}) > 0$ such that

$$\prod_{j=1}^{\omega}\left(1 + \frac{x_j}{C(\underline{x})}\right) = \lambda. \tag{2.5}$$

Put

$$T(\underline{x}) = \sum_{i=1}^{\omega} \frac{x_i}{x_i + C(\underline{x})}, \tag{2.6}$$

$$A(\underline{x}) = \frac{\lambda-1}{\lambda\sqrt{\lambda}}\exp(-\Omega(\underline{x}))\prod_{i=1}^{\omega}\frac{(x_i+C(\underline{x}))^{x_i}}{\Gamma(x_i+1)}, \tag{2.7}$$

and

$$B(\underline{x}) = \sqrt{\frac{2\,C(\underline{x})}{T(\underline{x})}}. \tag{2.8}$$

If $n = q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k}$, $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, $\alpha_i > 0$, $\Omega = \Omega(\underline{\alpha})$, R. Evans [9] proved the following important result

$$H(n,\lambda) = \sqrt{\pi}\, A(\underline{\alpha})B(\underline{\alpha})\left\{1 + O(\Omega^{-F})\right\}, \tag{2.9}$$

when $\Omega(\underline{\alpha})$ tends to ∞, for any F such that $0 < F < \frac{1}{2}$. Next, we state main important inequalities needed in our study.

Theorem 2. *For $\lambda > 1$, we have*

$$\frac{\Omega(\underline{x})}{\lambda-1} < C(\underline{x}) \leqslant \frac{\Omega(\underline{x})}{\log\lambda}, \tag{2.10}$$

$$\frac{\lambda-1}{\lambda} \leqslant T(\underline{x}) \leqslant \lambda-1, \tag{2.11}$$

$$\sqrt{\frac{2}{\lambda-1}\,C(\underline{x})} \leqslant B(\underline{x}) \leqslant \sqrt{\frac{2\lambda}{\lambda-1}\,C(\underline{x})}. \tag{2.12}$$

If $n = 2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k}$, $\alpha_1,\alpha_2,\ldots,\alpha_k \geqslant 1$, then there exist absolute positive constants C_1 and C_2 such that we have

$$C_1\,\frac{\exp(-k)}{\lambda\sqrt{\lambda}}\,\frac{\exp(F(\underline{\alpha}))}{\sqrt{\alpha_1\alpha_2\cdots\alpha_k}} \leqslant H(n,\,\lambda) \leqslant C_2\,\frac{1}{\lambda}\sqrt{\frac{\lambda-1}{\log\lambda}}\,\frac{n^{\rho_k(\lambda)}}{\pi^{k/2}}, \tag{2.13}$$

where

$$F(\underline{x}) = \sum_{j=1}^{k} x_j \log\left(1+\frac{C(\underline{x})}{x_j}\right). \tag{2.14}$$

Put

$$f(\lambda) = C_2\,\frac{1}{\lambda}\sqrt{\frac{\lambda-1}{\log\lambda}},\; C_2>0 \qquad \text{and} \qquad g(\lambda) = C_1\,\frac{1}{\lambda\sqrt{\lambda}}.$$

Theorem 3. *Let n be a positive integer and N an H-champion number.*

1) *For n large, we have*

$$\log H(n,\,\lambda) \leqslant U(n,\lambda), \tag{2.15}$$

where

$$U(n,\lambda) = \rho(\lambda)\log n - \gamma_1(\lambda)\frac{(\log n)^{1/\rho(\lambda)}}{\log\log n}$$

and

$$\gamma_1(\lambda) = \min\left(\frac{1}{2}, \frac{\lambda}{(\rho(\lambda)-1)|\zeta'(\rho(\lambda))|}\right).$$

2) *For a large H-champion number N, there exist a positive constant $\gamma_2(\lambda)$ depending only on λ such that*

$$L(N,\lambda) \leqslant \log H(N,\,\lambda) \leqslant U(N,\lambda), \tag{2.16}$$

where

$$L(N,\lambda) = \rho(\lambda)\log N - \gamma_2(\lambda)\frac{(\log N)^{1/\rho(\lambda)}}{\log\log N} + \log g(\lambda) - \rho(\lambda)\log 2.$$

3) *For a large H-champion number N, there exist $\gamma_3(\lambda),\gamma_4(\lambda) > 0$ such that*

$$\gamma_3(\lambda)\frac{(\log N)^{1/\rho(\lambda)}}{\log\log N} \leqslant \omega(N) \leqslant \gamma_4(\lambda)\frac{(\log N)^{1/\rho(\lambda)}}{\log\log N}.$$

Here is the statement of our last result.

Theorem 4. *Let N be a H-champion number and $\delta(\lambda) = \left(1 + \frac{1}{\rho(\lambda)}\right)/2$. Then, for*

$$N = 2^{\alpha_1} 3^{\alpha_2} \cdots p_k^{\alpha_k}$$

such that N tends to ∞, we have

1. $\Omega(N) = \sum_{i=1}^{k} \alpha_i = a(\lambda) \log N \sum_{i=1}^{k} \frac{1}{p_i^{\rho(\lambda)}-1} + O\left((\log N)^{\delta(\lambda)}\right)$ *and* $\alpha_i = \frac{a(\lambda)}{p_i^{\rho(\lambda)}-1} \log N + O\left(\frac{(\log N)^{\delta(\lambda)}}{\log p_i}\right)$;
2. $C(\underline{\alpha}) = a(\lambda) \log N + O\left((\log N)^{\delta(\lambda)}\right)$;
3. $T(\underline{\alpha}) = \zeta_{\mathcal{P}}(\rho(\lambda)) + O\left((\log N)^{\delta(\lambda)-1}\right)$;
4. $B(\underline{\alpha}) = \sqrt{\frac{2a(\lambda)}{\zeta_{\mathcal{P}}(\rho(\lambda))} \log N} \left\{1 + O\left((\log N)^{\delta(\lambda)-1}\right)\right\}$;
5. $H(N, \lambda) = \sqrt{\frac{2a(\lambda)}{\zeta_{\mathcal{P}}(\rho(\lambda))} \pi \log N}\, A(\underline{\alpha}) \left\{1 + O\left((\log N)^{\delta(\lambda)-1}\right)\right\}$.

The next sections will be devoted to the proofs of the above theorems.

3. Proof of Theorem 1

Let $k \geqslant 2$. There exist unique positive real numbers $\rho_k(\lambda), \rho(\lambda)$ such that $\zeta_k(\rho_k(\lambda)) = \zeta(\rho(\lambda)) = \lambda$. Then, we have

$$\prod_{p \in \mathcal{P}_k} \left(1 - p^{-\rho_k(\lambda)}\right)^{-1} = \prod_{p \in \mathcal{P}} \left(1 - p^{-\rho(\lambda)}\right)^{-1} = \lambda. \tag{3.1}$$

Thus, we obtain

$$\lambda = \prod_{p \in \mathcal{P}_k} \left(1 - p^{-\rho_k(\lambda)}\right)^{-1} = \prod_{p \in \mathcal{P}_k} \left(1 - p^{-\rho_k(\lambda)}\right)^{-1} \prod_{p > p_k} \left(1 - p^{-\rho(\lambda)}\right)^{-1}. \tag{3.2}$$

Taking the logarithm of the above equation, we get

$$\log \zeta_k(\rho_k(\lambda)) - \log \zeta_k(\rho(\lambda)) = -\sum_{p > p_k} \log \left(1 - p^{-\rho(\lambda)}\right). \tag{3.3}$$

For fixed $\lambda > 1$, there exists a positive integer $m(\lambda) \geqslant 2$ such that for all $k \geqslant m(\lambda)$ we have $\rho_k(\lambda) > 1$. By Lagrange's mean-value, we obtain

$$\log \zeta_k(\rho_k(\lambda)) - \log \zeta_k(\rho(\lambda)) = (\rho(\lambda) - \rho_k(\lambda)) \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\sigma_p(\lambda)} - 1}, \tag{3.4}$$

for some $\sigma_p(\lambda) \in (\rho_k(\lambda), \rho(\lambda)) \subset (1, 2)$. We have

$$-\sum_{p > p_k} \log \left(1 - p^{-\rho(\lambda)}\right) = \sum_{p > p_k} \frac{1}{p^{\rho(\lambda)}} + O\left(\sum_{p > p_k} \frac{1}{p^{2\rho(\lambda)}}\right), \tag{3.5}$$

and by standard estimates we get

$$\sum_{p>p_k} \frac{1}{p^{\rho(\lambda)}} = \frac{(\rho(\lambda)-1)^{-1}}{p_k^{\rho(\lambda)-1}\log p_k} + O\left(\frac{1}{p_k^{\rho(\lambda)-1}(\log p_k)^2}\right), \tag{3.6}$$

since $p_k = k\log k + k\log\log k + O(k)$. Therefore, we obtain

$$-\sum_{p>p_k} \log\left(1-p^{-\rho(\lambda)}\right) = \frac{(\rho(\lambda)-1)^{-1}}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}\left\{1+O\left(\frac{\log\log k}{\log k}\right)\right\}. \tag{3.7}$$

On the other hand, we have

$$\frac{\log p}{p^{\sigma_p(\lambda)}-1} = \frac{\log p}{p^{\sigma_p(\lambda)}-1}\left\{1+\frac{p^{\sigma_p(\lambda)}}{p^{\sigma_p(\lambda)}-1}\left(p^{\rho(\lambda)-\sigma_p(\lambda)}-1\right)\right\}. \tag{3.8}$$

Note that

$$1 \leqslant \frac{p^{\sigma_p(\lambda)}}{p^{\sigma_p(\lambda)}-1} \leqslant 2 \tag{3.9}$$

and for $2 \leqslant p \leqslant p_k$, we have

$$\begin{aligned} p^{\rho(\lambda)-\sigma_p(\lambda)}-1 \leqslant p_k^{\rho(\lambda)-\rho_k(\lambda)}-1 &= \exp\left((\rho(\lambda)-\rho_k(\lambda))\log p_k\right)-1 \\ &\ll (\rho(\lambda)-\rho_k(\lambda))\log p_k \\ &\ll \frac{1}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)-1}}. \end{aligned} \tag{3.10}$$

For these inequalities, we used $\exp(x) \ll x$, when x tends to 0 and

$$\rho(\lambda)-\rho_k(\lambda) \ll \frac{1}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}, \tag{3.11}$$

since $p_k = k\log k + k\log\log k + O(k)$. We remark that

$$\begin{aligned} &\log\zeta_k(\rho(\lambda)) - \log\zeta_k(\rho_k(\lambda)) \\ &\qquad = (\rho(\lambda)-\rho_k(\lambda))\sum_{p\in\mathcal{P}_k}\frac{\log p}{p^{\sigma_p(\lambda)}-1} \\ &\qquad = (\rho(\lambda)-\rho_k(\lambda))\left(\frac{\log 2}{2^{\sigma_p(\lambda)}-1}+\frac{\log 3}{3^{\sigma_p(\lambda)}-1}+\cdots+\frac{\log p_k}{p_k^{\sigma_p(\lambda)}-1}\right) \\ &\qquad > (\rho(\lambda)-\rho_k(\lambda))\frac{\log 2}{2^{\sigma_p(\lambda)}-1}, \quad \text{with}\,\sigma_p(\lambda)>1. \end{aligned} \tag{3.12}$$

From (3.3), (3.8), (3.10), and (3.12), we obtain

$$\begin{aligned}(\rho(\lambda)-\rho_k(\lambda))\sum_{p\in\mathcal{P}_k}\frac{\log p}{p^{\sigma_p(\lambda)}-1} &= (\rho(\lambda)-\rho_k(\lambda))\sum_{p\in\mathcal{P}_k}\frac{\log p}{p^{\rho(\lambda)}-1}\\ &\quad\times\left\{1+O\left(k^{1-\rho(\lambda)}(\log k)^{1-\rho(\lambda)}\right)\right\}\\ &= (\rho(\lambda)-\rho_k(\lambda))\sum_{p\in\mathcal{P}_k}\frac{\log p}{p^{\rho(\lambda)}-1}\left\{1+O\left(k^{1-\rho(\lambda)}\right)\right\}.\end{aligned} \tag{3.13}$$

Therefore, from (3.3) and (3.7), we get

$$(\rho(\lambda)-\rho_k(\lambda))\sum_{p\in\mathcal{P}_k}\frac{\log p}{p^{\sigma_p(\lambda)}-1} = \frac{(\rho(\lambda)-1)^{-1}}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}\left\{1+O\left(\frac{\log\log k}{\log k}\right)\right\}. \tag{3.14}$$

On the other hand, we have

$$\frac{|\zeta'(\rho(\lambda))|}{\zeta(\rho(\lambda))} = \sum_{p\in\mathcal{P}}\frac{\log p}{p^{\rho(\lambda)}-1} = \sum_{p\in\mathcal{P}_k}\frac{\log p}{p^{\rho(\lambda)}-1} + \sum_{p>p_k}\frac{\log p}{p^{\rho(\lambda)}-1}. \tag{3.15}$$

From (3.6), we deduce

$$\frac{|\zeta'(\rho(\lambda))|}{\lambda} = \sum_{p\in\mathcal{P}_k}\frac{\log p}{p^{\rho(\lambda)}-1} + O\left(k^{1+\varepsilon-\rho(\lambda)}\right), \tag{3.16}$$

for any small $\varepsilon>0$ and for any integer k such that $k>m(\lambda)$. Thus, we get

$$\begin{aligned}&(\rho(\lambda)-\rho_k(\lambda))\left[\frac{|\zeta'(\rho(\lambda))|}{\lambda}+O\left(k^{1+\varepsilon-\rho(\lambda)}\right)\right]\\ &\qquad = \frac{(\rho(\lambda)-1)^{-1}}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}\left\{1+O\left(\frac{\log\log k}{\log k}\right)\right\},\end{aligned} \tag{3.17}$$

which implies that

$$\rho(\lambda)-\rho_k(\lambda) = \frac{\lambda}{(\rho(\lambda)-1)|\zeta'(\rho(\lambda))|k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}\left\{1+O\left(\frac{\log\log k}{\log k}\right)\right\}, \tag{3.18}$$

$\forall\ k>m(\lambda)$.

Now, we will prove relation (2.4). We apply the mean-value theorem to the function $x \mapsto \frac{1}{p^x-1}$ to obtain

$$\frac{a_k(\lambda)}{p^{\rho_k(\lambda)}-1} - \frac{a(\lambda)}{p^{\rho(\lambda)}-1} = \left(\frac{1}{p^{\rho_k(\lambda)}-1} - \frac{1}{p^{\rho(\lambda)}-1}\right) a_k(\lambda) + \frac{a_k(\lambda)-a(\lambda)}{p^{\rho(\lambda)}-1}$$

$$= (\rho(\lambda)-\rho_k(\lambda)) a_k(\lambda) \frac{p^{x_0}\log p}{(p^{x_0}-1)^2} + \frac{a_k(\lambda)-a(\lambda)}{p^{\rho(\lambda)}-1}, \quad (3.19)$$

with $\rho_k(\lambda) \leqslant x_0 \leqslant \rho(\lambda)$. Moreover, we get

$$\frac{p^{x_0}}{(p^{x_0}-1)^2} \leqslant \frac{p^{\rho_2(\lambda)}}{(p^{\rho_2(\lambda)}-1)^2} \leqslant \frac{C(\lambda)^2}{p^{\rho_2(\lambda)}}, \quad (3.20)$$

$$\frac{1}{p^{x_0}-1} \leqslant \frac{1}{p^{\rho_2(\lambda)}-1} \leqslant \frac{1}{p^{\rho_2(\lambda)}} \cdot C(\lambda) \leqslant \frac{3\rho_3(\lambda) C(\lambda)\log p}{2\log 2} \cdot \frac{1}{p^{\rho_2(\lambda)}}, \quad (3.21)$$

where

$$C(\lambda) = \frac{2^{\rho_2(\lambda)}}{2^{\rho_2(\lambda)}-1}.$$

On the other hand, $(a_k(\lambda))_k$ is a decreasing sequence such that

$$\frac{2^{\rho_2(\lambda)}-1}{\log 3} \leqslant a_2(\lambda) \leqslant \frac{3^{\rho_3(\lambda)}}{2\log 2} \leqslant \frac{3^{\rho_3(\lambda)}}{2} \quad (3.22)$$

and

$$\left|\frac{a_k(\lambda)}{p^{\rho_k(\lambda)}-1} - \frac{a(\lambda)}{p^{\rho(\lambda)}-1}\right|$$

$$\leqslant \frac{3^{\rho_3(\lambda)}\log p}{2p^{\rho_2(\lambda)}} \left[C(\lambda)^2(\rho(\lambda)-\rho_k(\lambda)) + C(\lambda)(a_k(\lambda)-a(\lambda))\right]. \quad (3.23)$$

We have

$$a_k(\lambda) - a(\lambda) = a(\lambda)a_k(\lambda)\left[\frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)}\right] \sim a(\lambda)^2 \left(\frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)}\right), \quad (3.24)$$

and

$$\frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)} = L'(\rho(\lambda)) - L'_k(\rho(\lambda)) + L'_k(\rho(\lambda)) - L'_k(\rho_k(\lambda)), \quad (3.25)$$

where

$$L'(\rho(\lambda)) - L'_k(\rho(\lambda)) = \sum_{i=k+1}^{\infty} \frac{\log p_i}{p_i^{\rho(\lambda)}-1} \sim \left(\frac{1}{\rho(\lambda)-1}\right) \frac{1}{(k\log k)^{\rho(\lambda)-1}}, \quad (3.26)$$

$$L'_k(\rho(\lambda)) - L'_k(\rho_k(\lambda)) = (\rho(\lambda)-\rho_k(\lambda))L''(\rho(\lambda)). \quad (3.27)$$

Using Theorem 1, 3), we obtain

$$L'_k(\rho(\lambda)) - L'_k(\rho_k(\lambda)) \sim \frac{\lambda L''(\rho(\lambda))}{|\zeta'(\rho(\lambda))|(\rho(\lambda)-1)k^{\rho(\lambda)-1}(logk)^{\rho(\lambda)}}. \tag{3.28}$$

Therefore, we have

$$\frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)} \sim \frac{1}{(\rho(\lambda)-1)(k\log k)^{\rho(\lambda)-1}} \tag{3.29}$$

and

$$a_k(\lambda) - a(\lambda) \sim \frac{a^2(\lambda)}{\rho(\lambda)-1} \cdot \frac{1}{(k\log k)^{\rho(\lambda)-1}}. \tag{3.30}$$

From relations (2.3), (3.23), and (3.30), there exists a constant $\gamma(\lambda)$ depending on λ such that

$$\left|\frac{a_k(\lambda)}{p^{\rho_k(\lambda)}-1} - \frac{a(\lambda)}{p^{\rho(\lambda)}-1}\right| \leqslant \gamma(\lambda)\frac{\log p}{(\rho(\lambda)-1)(k\log k)^{\rho_k(\lambda)-1}}. \tag{3.31}$$

Therefore, the proof of Theorem 1 is completed.

4. Proof of Theorem 2

We consider the function

$$t \mapsto H(\underline{x}, t) = \sum_{j=1}^{\omega} \log\left(1 + \frac{x_j}{t}\right). \tag{4.1}$$

We have

$$H(\underline{x}, t) \leqslant \sum_{j=1}^{\omega} \log\frac{x_j}{t}, \qquad \text{for any } x_j > 0. \tag{4.2}$$

The function $t \mapsto H(\underline{x}, t)$ is decreasing from ∞ to 0, when $t \in [0, \infty)$. Thus, there exists a unique $C(\underline{x}) > 0$ such that

$$H(\underline{x}, C(\underline{x})) = \log\lambda, \qquad \lambda > 1.$$

Then, for $t = \frac{\Omega(\underline{x})}{\lambda-1}$, we obtain

$$H(\underline{x}, t) = \log\left(\prod_{j=1}^{\omega}\left(1 + (\lambda-1)\frac{x_j}{\Omega(\underline{x})}\right)\right) \geqslant \log\left(1 + (\lambda-1)\sum_{j=1}^{\omega}\frac{x_j}{\Omega(\underline{x})}\right) = \log\lambda. \tag{4.3}$$

This implies that

$$C(\underline{x}) \geqslant \frac{\Omega(\underline{x})}{\lambda-1}.$$

For the upper bound, we write

$$\log\lambda = \sum_{j=1}^{\omega}\log\left(1+\frac{x_j}{C(\underline{x})}\right) \leqslant \sum_{j=1}^{\omega}\frac{x_j}{C(\underline{x})} = \frac{\Omega(\underline{x})}{C(\underline{x})}. \tag{4.4}$$

Therefore, we get

$$\frac{\Omega(\underline{x})}{\lambda-1} \leqslant C(\underline{x}) \leqslant \frac{\Omega(\underline{x})}{\log\lambda}. \tag{4.5}$$

Now, we will prove relation (2.11). From equation (2.10), we obtain

$$T(\underline{x}) = \sum_{j=1}^{\omega}\frac{x_j}{x_j+C(\underline{x})} \leqslant \sum_{j=1}^{\omega}\frac{x_j}{C(\underline{x})} = \frac{\Omega(\underline{x})}{C(\underline{x})} \leqslant \lambda-1. \tag{4.6}$$

We set $\gamma_j = \frac{x_j}{x_j+C(\underline{x})}$. One can see that $0 \leqslant \gamma_j < 1$ and

$$T(\underline{x}) = \sum_{j=1}^{\omega}\gamma_j \geqslant \prod_{j=1}^{\omega}(1-\gamma_j) = 1-\prod_{j=1}^{\omega}\frac{x_j}{x_j+C(\underline{x})} = 1-\frac{1}{\lambda} = \frac{\lambda-1}{\lambda}. \tag{4.7}$$

Hence, we have

$$\frac{\lambda-1}{\lambda} \leqslant T(\underline{x}) \leqslant \lambda-1. \tag{4.8}$$

Relation (2.12) is immediate from (2.11).

To finish the proof of Theorem 2, first we will use formula (2.9). We take $F = \frac{1}{4}$. So there exist positive constants C_1 and C_2 such that

$$C_1\sqrt{\pi}A(\underline{\alpha})B(\underline{\alpha}) \leqslant H(n,\lambda) \leqslant C_2\sqrt{\pi}A(\underline{\alpha})B(\underline{\alpha}). \tag{4.9}$$

Second, from the expression of $A(\underline{\alpha})$, we have

$$C_1\sqrt{\pi}\sqrt{\frac{2}{\lambda-1}C(\underline{\alpha})}\,\frac{\lambda-1}{\lambda\sqrt{\lambda}}\exp\left(F(\underline{\alpha})\right) \leqslant H(n,\lambda) \tag{4.10}$$

and

$$H(n,\lambda) \leqslant C_2\sqrt{\pi}\sqrt{\frac{2\lambda}{\lambda-1}C(\underline{\alpha})}\,\frac{\lambda-1}{\lambda\sqrt{\lambda}}\exp\left(F(\underline{\alpha})\right)\frac{1}{(2\pi)^{k/2}\sqrt{\alpha_1\dot{\alpha}_2\cdots\alpha_k}}. \tag{4.11}$$

By the Stirling's formula $\Gamma(x+1) = x^x e^{-x}s(x)$, where $\sqrt{2\pi x} \leqslant s(x) \leqslant e\sqrt{x}$, $x \geqslant 1$, the relation $F(\underline{\alpha}) \leqslant \rho_k(\lambda)\log n$ and equations (2.10), (2.11) imply equation (2.13). Notice that the inequality $F(\underline{\alpha}) \leqslant \rho_k(\lambda)\log n$ can be obtained by the use of the result given in [13, Lemma 3.5], with $A = \log n$, $\rho_k = \rho_k(\lambda)$, $a_k = a_k(\lambda)$. This completes the proof of Theorem 2.

5. Proof of Theorem 3

We start by proving inequality (2.15). Put

$$\theta_1(\lambda) = \frac{\lambda}{(\rho(\lambda)-1)|\zeta'(\rho(\lambda))|}. \tag{5.1}$$

From Theorem 2 and inequality (2.13), we have

$$\log H(n,\lambda) \leqslant \log\left(\frac{C_2}{\lambda}\sqrt{\frac{\lambda-1}{\log\lambda}}\right) - \frac{k}{2}\log\pi + \rho_k(\lambda)\log n. \tag{5.2}$$

Write

$$\rho_k(\lambda)\log n = \rho(\lambda)\log n - (\rho(\lambda)-\rho_k(\lambda))\log n.$$

By Theorem 1, there exists a positive constant $\theta_1(\lambda)$ such that

$$\rho(\lambda)-\rho_k(\lambda) \geqslant \frac{\theta_1(\lambda)}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}. \tag{5.3}$$

For $n \geqslant 16$, we have $\log\log n > 1$. Thus, for $2 \leqslant k \leqslant \frac{(\log n)^{1/\rho(\lambda)}}{\log\log n} < \log n$, we obtain

$$\rho(\lambda)-\rho_k(\lambda) \geqslant \frac{\theta_1(\lambda)}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}} \geqslant \theta_1(\lambda)\frac{(\log n)^{\frac{1}{\rho(\lambda)}-1}}{\log\log n} \tag{5.4}$$

and for $k > \frac{(\log n)^{1/\rho(\lambda)}}{\log\log n}$, we get

$$\frac{k}{2}\log\pi > \frac{1}{2}\frac{(\log n)^{1/\rho(\lambda)}}{\log\log n}. \tag{5.5}$$

Therefore, we deduce that

$$\begin{aligned}(\rho(\lambda)-\rho_k(\lambda))\log n + \frac{k}{2}\log\pi &\geqslant \min\left(\frac{1}{2},\,\theta_1(\lambda)\right)\frac{(\log n)^{1/\rho(\lambda)}}{\log\log n}\\ &= \gamma_1(\lambda)\frac{(\log n)^{1/\rho(\lambda)}}{\log\log n}.\end{aligned} \tag{5.6}$$

For a large positive integer n, we deduce inequality (2.15).

Now, we will prove inequalities (2.16). For this, we take a large H-champion number N. We apply [13, Lemme 3.5], with $A = \log n$, $k = \left[\alpha\frac{(\log n)^{1/\rho(\lambda)}}{\log\log n}\right]$, and $0 < \alpha < \rho(\lambda)a(\lambda)^{1/\rho(\lambda)}$. So, for a large H-champion number $n = 2^{\alpha_1}\cdot 3^{\alpha_2}\cdots p_k^{\alpha_k}$, we follow the procedure used in [13] (see pages 1694-1697) to obtain formula (2.16).

By Theorem 2 and inequality (2.13), we have

$$\log H(n,\lambda) \geqslant F(\underline{\alpha}) - k - \frac{1}{2}\sum_{i=1}^{k}\log\alpha_i + \log g(\lambda).$$

On the other hand, we know that

$$F(\underline{\alpha}) \geqslant F(\underline{x}^\star) - \rho(\lambda)\log 2 - k,$$

where $\underline{x}^\star$ satisfies

$$F(\underline{x}^\star) = \rho(\lambda)\log n,$$

see [13, Lemme 3.5]. Therefore, we get

$$\log H(n,\lambda) \geqslant \rho(\lambda)\log n - (\rho(\lambda)-\rho_k(\lambda))\log n - 2k - \frac{1}{2}\sum_{i=1}^{k}\log\alpha_i + \log g(\lambda) - \rho(\lambda)\log 2.$$

Using equation (2.3) of Theorem 1, we obtain

$$(\rho(\lambda)-\rho_k(\lambda))\log n = O\left((\log n)^{\frac{1}{\rho(\lambda)}}/\log\log n\right) = O(k)$$

and

$$\frac{1}{2}\sum_{i=1}^{k}\log\alpha_i = O(k).$$

See [13, Lemme 4.2]. Hence, we deduce that

$$\log H(n,\lambda) \geqslant \rho(\lambda)\log n - \gamma_2(\lambda)\frac{(\log n)^{\frac{1}{\rho(\lambda)}}}{\log\log n} + \log g(\lambda) - \rho(\lambda)\log 2,$$

for some positive constant γ_2. Therefore, we complete the proof of the second part of Theorem 3.

It remains to prove the third part of Theorem 3. So let $k = \omega(N)$. From equation (2.13), we have

$$\log H(N,\lambda) \leqslant \rho_k(\lambda)\log N - \frac{k}{2}\log\pi + \log f(\lambda) \tag{5.7}$$

and

$$\log H(N,\lambda) \leqslant \rho(\lambda)\log N - (\rho(\lambda - \rho_k(\lambda))\log N - \frac{k\log\pi}{2} + \log f(\lambda). \tag{5.8}$$

Thus, we get

$$(\rho(\lambda - \rho_k(\lambda))\log N \leqslant \rho(\lambda)\log N - \log H(N,\lambda) - k\frac{\log\pi}{2} + \log f(\lambda). \tag{5.9}$$

From Theorem 3, 1), we obtain

$$(\rho(\lambda) - \rho_k(\lambda))\log N \leqslant \gamma_1(\lambda)\frac{(\log N)^{1/\rho_\lambda}}{\log\log N} + \log\left(\frac{f(\lambda)}{g(\lambda)}\right). \tag{5.10}$$

One can see that there exists $C > 0$ such that

$$k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)} \geqslant C(\log N)^{1-1/\rho(\lambda)}\log\log N. \tag{5.11}$$

Note that the function $g:\ t\mapsto t^{\rho(\lambda)-1}(\log t)^{\rho(\lambda)}$ tends to infinity when $t\to\infty$ and is increasing, for $t\geqslant 1$. Its inverse function satisfies $g^{-1}(y)\sim\left(\frac{y}{\log(y)^{\rho}}\right)^{\frac{1}{\rho-1}}$ as $y\to\infty$. Then, we get

$$k\geqslant g^{-1}\left(C(\lambda)\frac{(\log N)^{1-1/\rho(\lambda)}}{\log\log N}\right)\sim C'(\lambda)\frac{(\log N)^{1/\rho(\lambda)}}{\log\log N},\qquad C'(\lambda)>0. \tag{5.12}$$

Therefore, we obtain the lower bound for $k=\omega(N)$

$$\omega(N)=k\geqslant\gamma_3(\lambda)\frac{(\log N)^{1/\rho_\lambda}}{\log\log N}. \tag{5.13}$$

Similarly, we obtain the second inequality. This completes the proof of Theorem 3.

6. Proof of Theorem 4

Let $N=2^{\alpha_1}\cdot 3^{\alpha_2}\cdots p_k^{\alpha_k},\ \ \alpha_1,\alpha_2,\dots\alpha_k\geqslant 1,$ where p_k is the k^{th} prime number. We have $\log N=\sum_{i=1}^{k}\alpha_i\log p_i$. We consider the maximization problem for the function

$$F(\underline{x})=\sum_{j=1}^{k}x_j\log\left(1+\frac{C(\underline{x})}{x_j}\right),\qquad \underline{x}=(x_1,x_2,\dots,x_k) \tag{6.1}$$

on the set

$$\mathcal{D}_N=\{\underline{x}\in{\mathbb{R}_+^*}^k:\sum_{j=1}^{k}x_j\log p_j\leqslant\log N\}. \tag{6.2}$$

This problem is equivalent to the following one

$$\begin{cases}\sum_{j=1}^{k}x_j\log p_j=\log N\\ \max F(\underline{x}).\end{cases} \tag{6.3}$$

Referring to [9, Lemma 6] and [13], there exists a unique $\underline{x}^*=(x_1^*,x_2^*,\dots,x_k^*)\in\mathcal{D}_N$ such that

$$\log N=\sum_{i=1}^{k}x_i\log p_i),\qquad C(\underline{x}^*)=a_k(\lambda)\log N, \tag{6.4}$$

and

$$x_i^*=\frac{a_k(\lambda)\log N}{p_i^{\rho_k(\lambda)}-1},\qquad F(\underline{x}^*)=\rho_k(\lambda)\log N. \tag{6.5}$$

Moreover, for $\underline{\beta}=(\beta_1,\beta_2,\dots,\beta_k)\in\mathcal{D}_N$, we have

$$F(\underline{x}^*)\geqslant F(\underline{\beta})+\frac{1}{4\log N\log p_k}\sum_{i=1}^{k}(\beta_i-x_i^*)^2(\log p_i)^2,\qquad\forall\ k\geqslant 2. \tag{6.6}$$

If N is a H-champion number, then we have

$$\log p_k \sim \frac{1}{\rho(\lambda)} \log N \log N, \qquad \text{and} \qquad \sum_{i=1}^{k} (\beta_i - x_i^*)(\log p_i) = O(\log N)^{\delta(\lambda)}. \quad (6.7)$$

Now, we can start the proof of Theorem 4. We set

$$\beta_i = \frac{a(\lambda)}{p_i^{\rho(\lambda)} - 1} \qquad \text{and} \qquad k = \omega(N). \quad (6.8)$$

Then, we have

$$\left| \sum_{i=1}^{k} (\alpha_i - \beta_i \log N) \right| \leqslant \sum_{i=1}^{k} |\alpha_i - \beta_i \log N| \leqslant \sum_{i=1}^{k} |\alpha_i - x_i^*| + \sum_{i=1}^{k} |x_i^* - \beta_i \log N|, \quad (6.9)$$

where

$$\sum_{i=1}^{k} |\alpha_i - x_i^*| \leqslant O(\log N)^{\delta(\lambda)} \quad (6.10)$$

and

$$\sum_{i=1}^{k} |x_i^* - \beta_i \log N| = (\log N) \sum_{i=1}^{k} \left| \frac{a_k(\lambda)}{p_i^{\rho_k(\lambda)} - 1} - \frac{a(\lambda)}{p_i^{\rho(\lambda)} - 1} \right|$$
$$\leqslant \frac{\gamma(\lambda)(\log N)}{(k \log k)^{\rho(\lambda)-1}} \sum_{i=1}^{k} \frac{\log p_i}{p_i^{\rho(\lambda)}} = O\left(\frac{\log N}{k^{\rho(\lambda)-1}} \right). \quad (6.11)$$

By Theorem 3. 2), we have

$$k \geqslant \gamma_3(\lambda) \frac{(\log N)^{1/\rho(\lambda)}}{\log N \log N}. \quad (6.12)$$

Thus, we get

$$\frac{\log N}{k^{\rho(\lambda)-1}} = O\left((\log N)^{1/\rho(\lambda)} (\log \log N)^{\rho(\lambda)-1} \right) = O\left((\log N)^{\delta(\lambda)} \right), \quad (6.13)$$

where $\rho(\lambda) > 1$. One can see that

$$\sum_{i=k+1}^{\infty} \beta_i = \sum_{i=k+1}^{\infty} \frac{a(\lambda)}{p_i^{\rho(\lambda)} - 1} \leqslant 2a(\lambda) \sum_{i=k+1}^{\infty} \frac{1}{p_i^{\rho(\lambda)}} \leqslant \frac{2a(\lambda)}{(\rho(\lambda) - 1) k^{\rho(\lambda)-1}}. \quad (6.14)$$

Hence, we have

$$\left| \Omega(N) - \sum_{i=1}^{\infty} \beta_i \log N \right| \leqslant O(\log N)^{\delta(\lambda)}. \quad (6.15)$$

For $1 \leqslant i \leqslant k$, we obtain

$$|\alpha_i - x_i^*| \ll \frac{(\log N)^{\delta(\lambda)}}{p_i} \tag{6.16}$$

and

$$|x_i^* - \beta_i \log N| \leqslant \gamma(N) \frac{\log p_i}{p_i^{\rho(\lambda)}} \frac{\log N}{(k \log k)^{\rho(\lambda)-1}} = \frac{1}{\log p_i} O\left(\frac{\log N}{k^{\rho(\lambda)-1}}\right), \qquad \forall\, i. \tag{6.17}$$

We deduce that

$$|x_i^* - \beta_i \log N| \ll \frac{(\log N)^{1/\delta(\lambda)}}{\log p_i} \tag{6.18}$$

and

$$|x_i^* - \beta_i \log N| \leqslant O\left(\frac{(\log N)^{\delta(\lambda)}}{\log p_i}\right), \qquad \forall\, 1 \leqslant i \leqslant k. \tag{6.19}$$

This shows the first part of Theorem 4.

One can see that

$$|C(\underline{\alpha}) - C(\underline{\beta}) \log N| \leqslant \sum_{i=1}^{k} |\alpha_i - \beta_i \log N| + \sum_{i=1}^{\infty} \beta_i \log N = O(\log N)^{\delta(\lambda)}, \tag{6.20}$$

$$\lambda = \zeta(\rho(\lambda)) = \prod_{i=1}^{\infty} \left(1 - p_i^{-\rho(\lambda)}\right)^{-1} = \prod_{i=1}^{\infty} \frac{p_i^{\rho(\lambda)}}{p_i^{\rho(\lambda)} - 1} = \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i^{\rho(\lambda)} - 1}\right). \tag{6.21}$$

Therefore, equations

$$C\left(\frac{1}{2^{\rho(\lambda)} - 1}, \dots\right) = 1 \qquad \text{and} \qquad C\left(\frac{a(\lambda)}{2^{\rho(\lambda)} - 1}, \frac{a(\lambda)}{3^{\rho(\lambda)} - 1} \dots\right) = a(\lambda) \tag{6.22}$$

imply that

$$C(\underline{\beta}) = a(\lambda). \tag{6.23}$$

Thus, the second part of Theorem 4 is proved.

From Theorem 4 1) and relation (2.6), we obtain

$$T(\underline{\alpha}) = \zeta_{\mathcal{P}}(\rho(\lambda)) + O((\log N)^{\delta(\lambda)-1}). \tag{6.24}$$

Using Theorem 4. 2), 3) and equation (2.8), we complete the proof of Theorem 4.

7. Comments on numerical computations

In this section, we do some numerical computations to study the behavior of some constants. Let λ be a complex number, k a positive integer, and $\mathcal{P}_k$ the set of primes numbers defined by $\mathcal{P}_k = \{p_1, p_2, \ldots, p_k\}$, with $p_1 = 2 < p_2 = 3 < \cdots < p_k$. We know that

$$\zeta_k(s) = \prod_{p \in \mathcal{P}_k} (1 - p^{-s})^{-1},$$

(see (2.1)), $\rho_k(\lambda)$ is a positive real solution to $\zeta_k(s) = \lambda$ and

$$a_k(\lambda) = -\frac{\lambda}{\zeta_k'(\rho_k(\lambda))}.$$

Moreover, we consider

$$p_k(\lambda) = \frac{\lambda}{(\rho_k(\lambda) - 1)\zeta_k'(\rho_k(\lambda))}$$

the truncated part of the coefficient in equation (2.3) and

$$\delta(\lambda) = \frac{1}{2}\left(1 + \frac{1}{\rho(\lambda)}\right),$$

(see Theorem 4). We wrote a program in Maple that we run under a MacBook Air. We restricted the computations to $3 \leqslant \lambda \leqslant 7$ and $3 \leqslant k \leqslant 20$, because the result is known for $\lambda = 2$ and the computations become extremely slow for higher values of λ and k. Here are some comments on these computations, which confirm the properties that we proved:

1. As we consider $\rho_k(\lambda) > 1$, we notice that the first value of $\rho_k(\lambda)$ is obtained for higher values of k when λ is higher.
2. The values of $\rho_k(\lambda)$ increase when k increases. They are smaller when λ increases. But in all cases, we have $\rho_k(\lambda) < 2$.
3. When k increases, the values of $a_k(\lambda), p_k(\lambda), p_k(\lambda), \delta_k(\rho_k)$ decrease. We notice that as λ increases, the behavior is different. In all cases, we see that

$$a_k(\lambda) < 0.772, \qquad p_k(\lambda) < 125.406, \qquad \delta_k(\rho_k) < 1.$$

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