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ON THE DIOPHANTINE EQUATION $(x+1)^2+(x+2)^2+...+(x+d)^2=y^n$

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Abstract: In this paper, we give all the integer solutions of the equation $(x+1)^2+(x+2)^2+...+(2x)^2=y^n$.

Keywords: diophantine equations, binomial Thue equations.

1. Introduction

The Diophantine equation

$$1^k+2^k+...+x^k=y^n$$

was studied by Lucas[4] for $(k,n)=(2,2)$ and Schäffer[6] for the general situation. There are many results on this equation (see [2],[3] and [5]). Further, we can consider the more general equation

$$(x+1)^k+(x+2)^k+...+(x+d)^k=y^n.$$

In this paper, we discuss it for $k=2$. Since

$$(x+1)^2+(x+2)^2+...+(x+d)^2=dx^2+d(d+1)x+\frac{d(d+1)(2d+1)}{6},$$

we only need to deal with the equation

$$dx^2+d(d+1)x+\frac{d(d+1)(2d+1)}{6}=y^n,$$

that is

$$d(6x^2+6(d+1)x+(d+1)(2d+1))=6y^n. \tag{1}$$

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If $d = x$, then equation (1) can be written as

$$x(2x+1)(7x+1) = 6y^n. \tag{2}$$

In this paper we prove the following two results.

Theorem 1.1. *The integer solutions of equation (2) such that $n > 1$ are $(x, y) = (0, 0)$, $(x, y, n) = (1, \pm 2, 2)$, $(2, \pm 5, 2)$, $(24, \pm 182, 2)$ or $(x, y) = (-1, -1)$ with $2 \nmid n$.*

From the result of Lucas[4] and Theorem 1.1, we obtain the following interesting fact:

$$1^2 + 2^2 + \dots + 24^2 = 70^2,$$
$$(24+1)^2 + (24+2)^2 + \dots + (24+24)^2 = 182^2.$$

Theorem 1.2. *Let p be a prime and $p \equiv \pm 5 \pmod{12}$. If $p|d$ and $v_p(d) \not\equiv 0 \pmod{n}$, then equation (1) has no integer solution (x, y).*

2. Some preliminary result

In this section we present a lemma of A. Baszsó, A. Bérczes, K. Győry and Á. Pintér [1] which will be used to prove Theorem 1.1.

Lemma 2.1. *Let $B > A \geqslant 1$ be integers such that $\gcd(A, B) = 1$ and $\max\{A, B\} \leqslant 50$, then all integer solutions (x, y, n) to equation*

$$Ax^n - By^n = \pm 1$$

with $|xy| > 1, n \geqslant 3$ and $(A, B, n) \neq (21, 38, 17)$, $(26, 41, 17)$, $(22, 43, 17)$, $(17, 46, 17)$, $(31, 46, 17)$, $(21, 38, 19)$ are given by

$$\begin{aligned}
n = 3, \qquad (A, B, x, y) = {} & (1, 7, \pm(2, 1)), (1, 9, \pm(2, 1)), (1, 17, \pm(18, 7)), \\
& (1, 19, \pm(8, 3)), (1, 20, \pm(19, 7)), (1, 26, \pm(3, 1)), \\
& (2, 15, \pm(2, 1)), (12, 17, \pm(2, 1)), (3, 10, \pm(3, 2)), \\
& (5, 13, \pm(11, 8)), (5, 17, \pm(3, 2)), (8, 17, \pm(9, 7)), \\
& (8, 19, \pm(4, 3)), (11, 19, \pm(6, 5)), \\
n = 4, \qquad (A, B, x, y) = {} & (1, 5, \pm 3, \pm 2), (1, 15, \pm 2, \pm 1), \\
& (1, 17, \pm 2, \pm 1), (1, 39, \pm 5, \pm 2).
\end{aligned}$$

3. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. First we assume $n \geqslant 3$ and $2 \nmid n$ in equation (2). Since $\gcd(x, 2x+1) = 1, \gcd(x, 7x+1) = 1$ and $\gcd(2x+1, 7x+1) = \gcd(2x+1, x-2) = \gcd(x-2, 5) \in \{1, 5\}$, one has

$$x = 2^{\alpha_1} \cdot 3^{\beta_1} y_1^n, \qquad 2x+1 = 3^{\beta_2} \cdot 5^{\gamma_2} y_2^n, \qquad 7x+1 = 2^{\alpha_3} \cdot 3^{\beta_3} \cdot 5^{\gamma_3} y_3^n$$

with

$$\alpha_i, \qquad \beta_i=0,1, \qquad \gamma_j=0,1,n-1$$

and

$$\alpha_1+\alpha_3=1, \qquad \beta_1+\beta_2+\beta_3=1, \qquad \gamma_2+\gamma_3=0 \quad \text{or} \quad n.$$

Now we have

$$3^{\beta_2}\cdot 5^{\gamma_2}y_2^n-2^{\alpha_1+1}\cdot 3^{\beta_1}y_1^n=1 \tag{3}$$

and

$$2^{\alpha_3}\cdot 3^{\beta_3}\cdot 5^{\gamma_3}y_3^n-2^{\alpha_1}\cdot 3^{\beta_1}\cdot 7y_1^n=1. \tag{4}$$

In the discussion we can distinguish two cases.

Case 1: $\gamma_2=0$. In this case, one has

$$3^{\beta_2}y_2^n-2^{\alpha_1+1}\cdot 3^{\beta_1}y_1^n=1$$

from equation (3). Let $A=3^{\beta_2}, B=2^{\alpha_1+1}\cdot 3^{\beta_1}$, then $(A,B)=(3,2),(3,4)$, $(1,2)$, $(1,4)$, $(1,6)$, $(1,12)$. By Lemma 2.1, we obtain

$$(A,B,y_1,y_2)=(3,2,1,1),(3,4,-1,-1),(1,2,-1,-1)$$

and n arbitrary. Then we get $x=1,-2,-1$ and only $x=-1$, $y=-1$ is an integer solution of equation (2).

Case 2: $\gamma_2>0$. In this case, one has $\gamma_2=1$ or $\gamma_3=1$.

- $(\gamma_2=1)$ From equation (3) we have

  $$3^{\beta_2}\cdot 5y_2^n-2^{\alpha_1+1}\cdot 3^{\beta_1}y_1^n=1.$$

  Let $A=3^{\beta_2}\cdot 5, B=2^{\alpha_1+1}\cdot 3^{\beta_1}$, then $(A,B)=(15,2),(15,4)$, $(5,2)$, $(5,4)$, $(5,6)$, $(5,12)$. By Lemma 2.1, we include

  $$(A,B,y_1,y_2)=(15,2,-2,-1),(5,4,1,1),(5,6,-1,-1)$$

  and n arbitrary, which leads to $x=-8,2,-3$. These values yields no integer solution to equation (2).
- $(\gamma_3=1)$ From equation (4) we get

  $$2^{\alpha_3}\cdot 3^{\beta_3}\cdot 5y_3^n-2^{\alpha_1}\cdot 3^{\beta_1}\cdot 7y_1^n=1.$$

  Let $A=2^{\alpha_3}\cdot 3^{\beta_3}\cdot 5, B=2^{\alpha_1}\cdot 3^{\beta_1}\cdot 7$, then we have $(A,B)=(30,7),(15,7)$, $(15,14)$, $(10,7)$, $(10,21)$, $(5,7)$, $(5,14)$, $(5,21)$, $(5,42)$. By Lemma 2.1, we obtain $(A,B,y_1,y_3)=(15,14,1,1)$ and n arbitrary. Then we get $x=2$ and it does not yield to an integer solution of equation (2) since $n\geqslant 3$.

We proceed to consider the situation $n=2$. Since

$$x(2x+1)(7x+1)=6y^2 \tag{5}$$

is an elliptic curve, we only need to find all the integer points on it. Let $u = 84x, v = 504y$, then (5) can be written as

$$v^2 = u^3 + 54u^2 + 504u. \tag{6}$$

Using Magma we get

$$\begin{aligned}(u,v) = {}& (-42,0), (-36,\pm 72), (-32,\pm 80), (-24,\pm 72), (-21,\pm 63), (-14,\pm 28),\\ & (-12,0), (0,0), (3,\pm 45), (6,\pm 72), (18,\pm 180), (28,\pm 280),\\ & (84,\pm 1008), (150,\pm 2160), (168,\pm 2520), (363,\pm 7425), (1458,\pm 56700),\\ & (2016,\pm 91728), (67228,\pm 17438120),\end{aligned}$$

then

$$(x,y) = (0,0), (1,\pm 2), (2,\pm 5), (24,\pm 182).$$

This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.2. Since $p \equiv \pm 5 \pmod{12}$, one has $p \geqslant 5$, together with $p|d$ and $v_p(d) \not\equiv 0 \pmod n$ yields

$$p|6x^2 + 6(d+1)x + (d+1)(2d+1).$$

Then we have

$$p|36x^2 + 36(d+1)x + 6(d+1)(2d+1),$$

that is

$$p|(6x+3(d+1))^2 + (d+1)(3d-3),$$

which is a contradiction to

$$\left(\frac{-(d+1)(3d-3)}{p}\right) = \left(\frac{3}{p}\right) = -1.$$ ■

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