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# PAIRS OF ADDITIVE FORMS OF DEGREE $p^\tau(p-1)$

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**Abstract:** Let

$$f(x_1,...,x_n) = a_1x_1^k + \cdots + a_nx_n^k$$
$$g(x_1,...,x_n) = b_1x_1^k + \cdots + b_nx_n^k$$

be a pair of additive forms of degree $k = p^\tau(p-1)$. We are interested in finding conditions which guarantee the existence of $p$-adic zeros for this pair of forms. A well-known conjecture due to Emil Artin states that the condition $n > 2k^2$ is sufficient. Here we prove that

$$n > 2\left(\frac{p}{p-1}\right)k^2 - 2k$$

is sufficient, provided that $p > 5$ and $\tau \geqslant \dfrac{p-1}{2}$.

**Keywords:** additive forms, Artin's Conjecture, $p$-adic forms.

## 1. Introduction

We are interested in finding sufficient conditions which guarantee the existence of non-trivial $p$-adic zeros for a pair of additive forms of degree $k$ in $n$ variables

$$\begin{aligned} f &= a_1x_1^k + \cdots + a_nx_n^k \\ g &= b_1x_1^k + \cdots + b_nx_n^k, \end{aligned} \tag{1.1}$$

where the coefficients $a_j$, $b_j$ are rational numbers.

There is a longstanding conjecture of E. Artin stating that, for pairs of additive forms, $n > 2k^2$ could be such a condition. Many efforts have been made to prove Artin's Conjecture, starting with the pioneering works of Davenport and Lewis [2] who confirmed the conjecture for the case of odd degrees. For even degrees, they found that if $n \geqslant 7k^3$ the existence of non-trivial $p$-adic zeros is guaranteed. More recently, J. Brüdern and H. Godinho [1] proved that $n > 2k^2$ does suffice

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to establish $p$-adic solubility of (1.1) unless the degree is basically of the form $k = p^\tau(p-1)$ with $\tau \geqslant 1$. And they also proved that for all degrees $k$ there is always $p$-adic solubility provided $n \geqslant 4k^2$. In a recent paper [3], the authors proved that any pair of additive forms of degree $k = 3^\tau \times 2$ or $k = 5^\tau \times 4$ in $n > 2\left(\frac{p}{p-1}\right)k^2 - 2k$ variables has common $p$-adic zeros. Here we extend these results for other values of $p$, and our main theorem is presented below. For more information on the subject we refer the readers to [1].

**Theorem 1.1.** *Let $f, g$ be a pair of additive forms of degree $k = p^\tau(p-1)$ and rational coefficients, with $p \geqslant 7$. If $n > 2\left(\frac{p}{p-1}\right)k^2 - 2k$ and $\tau \geqslant \frac{p-1}{2}$, then the system* (1.1) *has $q$-adic solutions for all primes $q$.*

The proof follows a combinatorial approach of looking for zero-sum subsequences of the sequence of all column-vectors of the $2 \times n$ matrix of coefficients of the system (1.1). It is important to observe that for $k = p^\tau(p-1)$ only the case $q = p$ need to be considered, since for all other primes $q$, the condition $n > 2k^2$ is sufficient for $q$-adic solubility (see [1]).

We shall start with the Davenport and Lewis $p$-normalization process, an important technique that shall give the starting point for our analysis.

## 2. $p$-normalization

Let us consider a pair of additive forms of degree $k$ in $n$ variables

$$\begin{aligned} f(x_1, ..., x_n) &= a_1 x_1^k + \cdots + a_n x_n^k \\ g(x_1, ..., x_n) &= b_1 x_1^k + \cdots + b_n x_n^k, \end{aligned} \tag{2.1}$$

with rational coefficients $a_j$, $b_j$.

Let $v_1, ..., v_n$ be integers and $\lambda, \delta, \mu, \rho$ be rational numbers such that $\lambda\delta - \mu\rho \neq 0$. Now define the following transformations of the pair (2.1)

$$\begin{aligned} F_1(x_1, ..., x_n) &= f(p^{v_1}x_1, ..., p^{v_n}x_n) \\ G_1(x_1, ..., x_n) &= g(p^{v_1}x_1, ..., p^{v_n}x_n) \end{aligned} \tag{2.2}$$

$$\begin{aligned} F_2(x_1, ..., x_n) &= \lambda f(x_1, ..., x_n) + \mu g(x_1, ..., x_n) \\ G_2(x_1, ..., x_n) &= \rho f(x_1, ..., x_n) + \delta g(x_1, ..., x_n). \end{aligned} \tag{2.3}$$

For each pair of additive forms, define the parameter

$$\vartheta(f, g) = \prod_{i \neq j}(a_i b_j - a_j b_i).$$

Davenport and Lewis[2] proved that

$$\vartheta(F_1, G_1) = p^{2k(n-1)\sum v_i}\vartheta(f, g)$$

and

$$\vartheta(F_2, G_2) = (\lambda\delta - \mu\rho)^{n(n-1)}\vartheta(f, g).$$

Two pairs of additive forms are said to be *p−equivalent* if one can be obtained from another by repeated applications of transformations (2.2) and (2.3). It is an immediate consequence that if the pair $f, g$ has a non-trivial $p$-adic zero, the same holds for any pair of forms in its $p$-equivalence class.

Let $F, G$ be a pair of additive forms with integer coefficients and suppose that $\ell = \nu_p(\vartheta(F,G)) < \infty$, where $\nu_p$ is the $p$-adic valuation (in [2] it is proven that we can always assume $\vartheta(F,G) \neq 0$). The pair $F, G$ is said to be *p-normalized* if $\ell$ is the least power of $p$ diving in $\vartheta(f,g)$, for all pairs $f, g$ having integer coefficients and $p$-equivalent to $F, G$. The most important feature of the $p$-normalization process is the affirmation that it is enough to prove Theorem 1.1 for a pair of $p$-normalized additive forms, with all the nice properties described in the next lemma (see [2] for a proof).

**Lemma 2.1.** *A p-normalized pair of additive forms of degree k can be written (after renumbering variables) as*

$$\begin{aligned} f &= f_0 + pf_1 + \cdots + p^{k-1} f_{k-1} \\ g &= g_0 + pg_1 + \cdots + p^{k-1} g_{k-1}, \end{aligned} \tag{2.4}$$

*where each variable in a pair of subforms $f_i, g_i$ occurs with a coefficient not divisible by p in at least one of these subforms. Moreover, if we denote by $q_0$ the minimum number of variables appearing with coefficients not divisible by p in any form $\lambda f_0 + \mu g_0$, with $\lambda, \mu$ not both divisible by p, and define $m_i$ as the number of variables present in the pair $f_i, g_i$, then*

$$q_0 \geqslant n/2k \qquad \text{and} \qquad m_0 + \cdots + m_j \geqslant (j+1)n/k \qquad \text{for} \ \ 0 \leqslant j \leqslant k-1. \tag{2.5}$$

To guarantee the existence of non-trivial $p$-adic zeros, we are going to use the following Hensel-type lemma also due to Davenport and Lewis[2].

**Lemma 2.2.** *Let $f, g$ be a pair of forms as in* (1.1)*, of degree $k = p^\tau(p-1)$, $p \geqslant 3$, and define $\gamma = \tau + 1$. If the system*

$$\begin{aligned} f &\equiv 0 \pmod{p^\gamma} \\ g &\equiv 0 \pmod{p^\gamma} \end{aligned} \tag{2.6}$$

*has a solution $(x_1, \ldots, x_n)$ for which the matrix*

$$\begin{pmatrix} a_1x_1 & \cdots & a_nx_n \\ b_1x_1 & \cdots & b_nx_n \end{pmatrix} \tag{2.7}$$

*has rank 2 modulo p (i.e., for a pair $i, j$, $(a_ib_j - a_jb_i)x_ix_j \not\equiv 0 \mod p$) then the pair $f, g$ has p-adic zeros.*

## 3. Sequences over abelian groups

Let $G = (G, +)$ be a finite abelian group and $S$ be a sequence of elements of $G$ (repetitions allowed). For each $g \in G$, define $v_g(S)$ as the number of times that the element $g$ occurs in the sequence $S$. We say that $T$ is a subsequence of $S$ if $v_g(T) \leqslant v_g(S)$ for all $g \in G$. Define the *support* of $S$ to be $\mathrm{supp}(S) = \{g \in G;\ v_g(S) \neq 0\}$ and the *length* of $S$ as

$$|S| = \sum_{g \in G} v_g(S).$$

Here all sequences will be written in the multiplicative form, either as $S = g_1 \dots g_r$ or

$$S = \prod_{g \in G} g^{v_g(S)}, \qquad \text{with} \quad \mathrm{supp}(S) = \{g_1, \dots, g_r\}.$$

With $S = g_1 \dots g_r$, define the *sum* of $S$ as

$$\sigma(S) = \sum_{i=1}^{r} g_i \in G,$$

and the set of all non-empty sums of subsequences of $S$ as

$$\Sigma(S) = \left\{ \sum_{i=1}^{r} \varepsilon_i g_i :\ \varepsilon_i \in [0,1] \text{ and } \sum_{i=1}^{r} \varepsilon_i > 0 \right\} \subseteq G.$$

**Definition 3.1.** *Let $S$ be a sequence in a group $G$.*

(i) *If $\sigma(S) = 0$, we say that $S$ is a* zero-sum sequence.
(ii) *If $\sigma(S) = 0$ and $|S| \in \{1, \dots, \exp(G)\}$ (the exponent of $G$) we say that $S$ is a* short zero-sum sequence.
(iii) *If $0 \notin \Sigma(S)$ we say that $S$ is a* zero-sum free sequence.

**Definition 3.2.** *Let $G$ be a finite abelian group.*

(i) *Define $D(G)$ (*the Davenport constant*) as the smallest positive integer $r$ such that every sequence $S$ over $G$ of length $r$ has a non-empty zero-sum subsequence.*
(ii) *Define $\eta(G)$ as the smallest integer $r$ such that every sequence $S$ over $G$ with length $r$ has a short zero-sum subsequence.*

In [4, 5], J. Olson proved:

**Lemma 3.3.** *Let $p$ be a prime number and $C_p$ the cyclic group of order $p$. Then*

(i) $D(C_p \oplus C_p) = 2p - 1$ *and*
(ii) $\eta(C_p \oplus C_p) = 3p - 2$.

**Definition 3.4.** *Let $G$ be a finite abelian group and $t \in \{1, \dots, p(G) - 1\}$, where $p(G)$ is the least prime divisor of $|G|$. We define $\mathfrak{s}_t = \mathfrak{s}_t(G)$ to be the smallest positive integer such that every sequence of nonzero elements of $G$ of length $\mathfrak{s}_t$ has a zero-sum free subsequence of length $t$.*

It is easy to verify that $\mathfrak{s}_1(G) = 1$ and $\mathfrak{s}_2(G) = 3$, for all groups $G$. Less obvious is the fact that

$$\mathfrak{s}_3(G) = 5, \tag{3.1}$$

and its proof can be found in [3].

**Lemma 3.5.** *Let $S$ be a sequence of nonzero elements of $G$ and suppose $|\operatorname{supp}(S)| > t$. If there is $g \in G$ such that $v_g(S) \geqslant t-1$, then $S$ has a zero-sum free subsequence of length $t$.*

**Proof.** Using the additive operation of the group, consider the following equations

$$\ell g + x = 0, \qquad \text{with } \ \ell \in \{1, \ldots, t-1\}.$$

Since each of these equations has a unique solution in $G$ and $|\operatorname{supp}(S)| > t$, there exists $h \in \operatorname{supp}(S)$ which does not satisfy any of these equations. Therefore

$$T = g^{t-1} h$$

is a zero-sum free subsequence of length $t$. ■

**Lemma 3.6.** *Let $p$ be a prime, $p \geqslant 7$. If $3 < t < p-1$, then*

$$\mathfrak{s}_t(C_p) \leqslant (p-1)(t-2) + 1.$$

**Proof.** Let $S$ be a sequence of nonzero elements of $G$ of length $|S| = (p-1)(t-2)+1$. If there is $g \in G$ such that $v_g(S) \geqslant t$, then $T = g^t$ is a zero-sum free subsequence of length $t$, hence we may assume $v_g(S) \leqslant t-1$ for all $g \in G$. On the other hand, the length of $S$ implies that there must exist an element $h \in \operatorname{supp}(S)$ such that $v_h(S) = t-1$. But this fact also gives a zero-sum free subsequence of length $t$, unless $|\operatorname{supp}(S)| \leqslant t$, according to Lemma 3.5. Write then

$$S = g_1^{v_1} \cdots g_r^{v_r} \qquad \text{with } \ v_i \leqslant t-1 \ \text{ and } \ r \leqslant t.$$

If $|\operatorname{supp}(S)| \leqslant t-1$, then

$$|S| - (t-1)^2 = (p-1)(t-2) + 1 - (t-1)^2 = (t-2)(p-t-1) \leqslant 0$$

which is impossible for $t > 3$ and $p > t+1$. Hence $|\operatorname{supp}(S)| = t$ and $|S| \leqslant t(t-1)$. Now,

$$|S| - (t(t-1) - 1) = pt - 2p - t^2 + 4 \geqslant (t+2)(t-2) - t^2 + 4 \geqslant 0$$

since $p \geqslant t+2$. Therefore

$$t(t-1) - 1 \leqslant |S| \leqslant t(t-1).$$

Suppose $|S| = t(t-1)$. Then $(p-1)(t-2)+1 = t(t-1) = (t+1)(t-2)+2$, which implies $(t-2)(p-t-2) = 1$. And this is an impossibility, for $t > 3$ and $p > t+1$. Hence we must have $|S| = t(t-1) - 1$. Let us write

$$S = g_1^{t-1} \cdots g_{t-1}^{t-1} g_t^{t-2},$$

and observe that $(p-1)(t-2) + 1 = t(t-1) - 1 = (t+1)(t-2) + 1$, implies $(t-2)(t+1) = (p-1)(t-2)$, that is, $t = p-2$.

Since $p \geqslant 7$, then $t = p - 2 > \frac{p-1}{2} + 1$, hence the sets $\{g_1, g_2, \ldots, g_{t-1}\}$ and $\{2g_1, 2g_2, \ldots, 2g_t\}$ must intersect, and there should be a $g \in \operatorname{supp}(S)$ such that $v_g(S) = t - 1$ and $2g \in \operatorname{supp}(S)$. It follows that the subsequence

$$g^{t-1}(2g)$$

is zero-sum free, for $(t-1)g + 2g \not\equiv 0 \pmod p$. ■

### 3.1. Sequences over $\mathbb{Z}/p^m\mathbb{Z}$

Let $S = g_1 \cdots g_r$ be a sequence of integer numbers and $p$ a prime number. Considering

$$\pi_i : \mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z} \tag{3.2}$$

to be the canonical epimorphism, define the sequence (the image sequence) $\pi_i(S) = \pi_i(g_1) \cdots \pi_i(g_r)$ in $\mathbb{Z}/p^i\mathbb{Z}$.

**Lemma 3.7.** *Let $S$ be a sequence of integers coprime to $p$ and of length $r \geqslant 3p-2$. Then $S$ has a short subsequence $T$ such that $\pi_1(T)$ is a short zero-sum sequence in $\mathbb{Z}/p\mathbb{Z}$, but $\pi_2(T)$ is not a zero-sum sequence in $\mathbb{Z}/p^2\mathbb{Z}$.*

**Proof.** Let $S = g_1 \cdots g_r$, and write these elements as $g_i = a_i + pb_i$, where $a_i = \pi_1(g_i) \in \{1, \ldots, p-1\}$. By Lemma 3.3(ii), the sequence

$$(a_1, \pi_1(b_1)), (a_2, \pi_1(b_2)), \ldots, (a_r, \pi_1(b_r))$$

has a short zero-sum subsequence over $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, that is, there exists $I \subset \{1, \ldots, r\}$ such that $|I| \leqslant p$ and

$$\sum_{i \in I} a_i \equiv \sum_{i \in I} b_i \equiv 0 \pmod p.$$

Therefore

$$\sum_{i \in I} g_i \equiv \sum_{i \in I} a_i + p \sum_{i \in I} b_i \equiv 0 \pmod p.$$

Since $0 < a_i < p$ for all $i \in \{1, \ldots, r\}$, we have

$$\sum_{i \in I} g_i \not\equiv 0 \pmod{p^2}.$$

It is now clear that the subsequence $T = \prod_{i \in I} g_i$ has the desired properties. ■

The proof of next two lemmas can be found in [3].

**Lemma 3.8.** *Let $S = g_1 g_2 \cdots g_p$ be a sequence of integers coprime to $p$ and choose any $g_i \in \operatorname{supp}(S)$. Then it is always possible to find a subsequence $T$ of $S$ with $g_i \in \operatorname{supp}(T)$ such that $\pi_1(T)$ is a zero-sum in $\mathbb{Z}/p\mathbb{Z}$.*

**Lemma 3.9.** *If $a, m, k \in \mathbb{N}$, define recursively*

$$\left\lfloor \frac{a}{m} \right\rfloor_{(1)} = \left\lfloor \frac{a}{m} \right\rfloor \qquad \text{and} \qquad \left\lfloor \frac{a}{m} \right\rfloor_{(k+1)} = \left\lfloor \frac{\left\lfloor \frac{a}{m} \right\rfloor_{(k)}}{m} \right\rfloor.$$

*Then*

$$\left\lfloor \frac{a}{m} \right\rfloor_{(k)} = \left\lfloor \frac{a}{m^k} \right\rfloor.$$

### 3.2. Sequences over $\mathbb{Z}/p^m\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z}$

Let $f, g$ be a pair of $p$-normalized forms and

$$\mathscr{A} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdots \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

be the sequence of their coefficients (see (2.1)). According to Lemma 2.1, we can rewrite the sequence $\mathscr{A}$ as

$$\mathscr{A} = \mathscr{M}_0 \mathscr{M}_1 \cdots \mathscr{M}_{k-1} \tag{3.3}$$

where an element $\begin{pmatrix} a_j \\ b_j \end{pmatrix}$ of $\mathscr{A}$ is in $\mathscr{M}_i$ if $a_j, b_j$ are coefficients of the pair of subforms $(p^i f_i, p^i g_i)$. The elements of $\mathscr{M}_i$ are said to be at *level* $i$.

Write $\mathbb{F}_p^2 \setminus \{(0,0)\}$ as the disjoint union of the $p+1$ lines $L_0, L_1, \ldots, L_p$,

$$\mathbb{F}_p^2 \setminus \{(0,0)\} = \bigsqcup_{i=0}^{p} L_i, \tag{3.4}$$

where

$$L_0 = \left\{ \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mid \lambda \in \{1, \ldots, p-1\} \right\}$$

and

$$L_i = \left\{ \lambda \begin{pmatrix} i \\ 1 \end{pmatrix} \mid \lambda \in \{1, \ldots, p-1\} \right\}$$

for all $i \in \{1, \ldots, p\}$.

Now, define (see (3.2)) the epimorphism

$$\varphi_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^i\mathbb{Z}$$

as

$$\varphi_i \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \pi_i(a) \\ \pi_i(b) \end{pmatrix}. \tag{3.5}$$

With these in mind, let $S$ be a subsequence of $\mathscr{M}_0$, and for each $j \in \{0, 1, \ldots, p\}$, define the subsequence of $S$

$$I_j(S) = \prod_{g \in B_j} g^{v_g(S)}, \qquad \text{where} \quad B_j = \{g \in \mathscr{M}_0 \ ; \ \varphi_1(g) \in L_j\}.$$

We will say that an element $g$ of $S$ has *color* $\jmath$ if $g \in B_j$. Writing $i_j(S) = |I_j(S)|$ we have that

$$S = \prod_{j=0}^{p} I_j(S), \qquad \text{and} \qquad |S| = \sum_{j=0}^{p} i_j(S). \tag{3.6}$$

In an analogous way, if $S$ is a subsequence of $\mathscr{M}_\ell$, we can also write

$$I_j(S) = \prod_{g \in C_j} g^{v_g(S)}, \qquad \text{where } \ C_j = \{g \in \mathscr{M}_\ell \ ; \ \varphi_1(p^{-\ell}g) \in L_j\}$$

and in this case we also say that $g$ has color $\jmath$ (at level $\ell$) if $g \in C_j$.

It is simple to see that for any fixed subsequence $S$ of $\mathscr{M}_i$, we can always assume

$$i_0(S) \geqslant i_p(S) \geqslant i_j(S) \qquad \text{for all } \ j \in \{1, 2, \ldots, p-1\}, \tag{3.7}$$

and thus define the sequence

$$Q_0(S) = \prod_{j=1}^{p} I_j(S), \qquad \text{with } \ q_0(S) = |Q_0(S)|. \tag{3.8}$$

Hence, for every subsequence $S$ of $\mathscr{M}_i$, we have

$$S = I_0(S) Q_0(S) \qquad \text{and} \qquad |S| = i_0(S) + q_0(S).$$

**Lemma 3.10.** *Let $S$ be a subsequence of $\mathscr{M}_\ell$ and suppose that there exists $j \in \{0, \ldots, p\}$ such that $i_j(S) \geqslant 3p-2$. Then $S$ has a short subsequence $T$ such that $\varphi_{\ell+1}(T)$ is a zero-sum sequence but $\varphi_{\ell+2}(T)$ is not a zero-sum sequence.*

**Proof.** As seen above (see 3.7), we can assume $i_0(S) \geqslant 3p-2$. Let $U$ be a subsequence of $I_0(S)$ of length $3p-2$

$$U = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdots \begin{pmatrix} a_{3p-2} \\ b_{3p-2} \end{pmatrix}.$$

Observe that for $\jmath \in \{1, \ldots, 3p-2\}$ we have

$$p^{-\ell} a_j \not\equiv 0 \pmod{p} \qquad \text{and} \qquad p^{-\ell} b_j \equiv 0 \pmod{p}.$$

Since the sequence of integers $V = p^{-\ell} a_1 \cdots p^{-\ell} a_{3p-2}$ has a short subsequence $T^*$ (see Lemma 3.7) such that $\pi_1(T^*)$ is a short zero-sum sequence but $\pi_2(T^*)$ is not a zero-sum sequence, it is now simple to choose the short subsequence $T$ of $U$ with the desired properties. ∎

**Lemma 3.11.** *Let $S$ be a subsequence of $\mathscr{M}_\ell$. If $i_0(S) \geqslant \mathfrak{s}_u(\mathbb{Z}/p\mathbb{Z})$ and $q_0(S) \geqslant \mathfrak{s}_v(\mathbb{Z}/p\mathbb{Z})$, then $\varphi_{\ell+1}(S)$ has a zero-sum free subsequence of length $u+v$ over $\mathbb{Z}/p^{\ell+1}\mathbb{Z} \oplus \mathbb{Z}/p^{\ell+1}\mathbb{Z}$.*

**Proof.** Let us consider

$$T = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdots \begin{pmatrix} a_r \\ b_r \end{pmatrix} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \cdots \begin{pmatrix} c_s \\ d_s \end{pmatrix}$$

a subsequence of $S$ containing $r = \mathfrak{s}_u(\mathbb{Z}/p\mathbb{Z})$ elements of $I_0(S)$ and $s = \mathfrak{s}_v(\mathbb{Z}/p\mathbb{Z})$ elements of $Q_0(S)$. Then

$$\varphi_1(p^{-\ell}T) = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} A_r \\ 0 \end{pmatrix} \begin{pmatrix} C_1 \\ D_1 \end{pmatrix} \cdots \begin{pmatrix} C_s \\ D_s \end{pmatrix}.$$

It follows from Definition 3.4 that the sequence $A_1 \cdots A_r$ has a zero-free subsequence over $\mathbb{Z}/p\mathbb{Z}$ of length $u$, and in an analogous way, there is also a zero-sum free subsequence of $D_1 \cdots D_s$ of length $v$ over $\mathbb{Z}/p\mathbb{Z}$. Denoting these zero-sum free sequences as $A_1 \cdots A_u$ and $D_1 \cdots D_v$, we have a sequence of length $u+v$ of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin \Sigma\left( \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} A_u \\ 0 \end{pmatrix} \begin{pmatrix} C_1 \\ D_1 \end{pmatrix} \cdots \begin{pmatrix} C_v \\ D_v \end{pmatrix} \right),$$

concluding this proof. ∎

## 4. Proof of Theorem 1.1

From this point on we are considering the pair of forms $f, g$ to be $p$-normalized, with all the properties described in Lemma 2.1. Since $k = p^\tau(p-1)$ and $n > 2\left(\frac{p}{p-1}\right)k^2 - 2k$, inequalities (2.5) give

$$q_0 \geqslant \frac{n}{2k} > k\frac{p}{p-1} - 1 = p^{\tau+1} - 1,$$

and

$$\sum_{i=0}^{\ell} m_i \geqslant (\ell+1)\frac{n}{k} > 2(\ell+1)\left(k\left(\frac{p}{p-1}\right) - 1\right).$$

Hence

$$q_0 \geqslant p^{\tau+1} \qquad \text{and} \qquad \sum_{i=0}^{\ell} m_i > 2(\ell+1)(p^{\tau+1}-1). \tag{4.1}$$

In order to guarantee the existence of $p$-adic zeros for this pair of additive forms, it is sufficient, by Lemma 2.2, to obtain a solution of rank 2 to the system (see (2.7))

$$\begin{aligned} a_1x_1^k + \cdots + a_nx_n^k &\equiv 0 \pmod{p^\gamma} \\ b_1x_1^k + \cdots + b_nx_n^k &\equiv 0 \pmod{p^\gamma}. \end{aligned} \tag{4.2}$$

Since $\gamma = \tau + 1$ and

$$k = p^\tau(p-1) = \varphi(p^\gamma),$$

(where $\varphi$ is the Euler function), the above system is equivalent to the equation

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \varepsilon_1 + \cdots + \begin{pmatrix} a_n \\ b_n \end{pmatrix} \varepsilon_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.3}$$

in $\mathbb{Z}/p^\gamma\mathbb{Z} \oplus \mathbb{Z}/p^\gamma\mathbb{Z}$, with $\varepsilon_i \in \{0, 1\}$.

**Definition 4.1.** *We will say that a subsequence $\mathcal{S}$ of $\mathscr{A} = \mathscr{M}_0 \dots \mathscr{M}_{k-1}$ is a* non-singular zero-sum sequence modulo $p^i$ *if $\varphi_i(\mathcal{S})$ is a zero-sum sequence and* $\operatorname{supp}(\varphi_1(\mathcal{S})) \cap \operatorname{supp}(\varphi_1(\mathscr{M}_0))$ *contains at least two elements of distinct colors (see* (3.6)*).*

The system (4.2) and the equation (4.3) presents a correspondence between the problem of finding solutions of rank 2 for a pair of additive forms $f, g$ and the question of existence of a non-singular zero-sum subsequence sequence modulo $p^\gamma$ of the sequence

$$\mathscr{A} = \mathscr{M}_0\mathscr{M}_1 \cdots \mathscr{M}_{k-1} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdots \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

of the coefficients of the forms $f, g$.

**Definition 4.2.** *Let $\mathcal{S}$ be a subsequence of $\mathscr{A} = \mathscr{M}_0 \dots \mathscr{M}_{k-1}$. If $S$ is a non-singular zero-sum sequence modulo $p^\ell$, we will say that $\sigma(S)$ is a* Primary Element *at level $\ell$ or higher. If $\varphi_\ell(S)$ is a zero-sum sequence but $\varphi_{\ell+1}(S)$ is not a zero-sum sequence, we will say that $\sigma(S)$ is a* Secondary Element *at level $\ell$. We will denote by $\mathcal{P}_\ell$ the sequence of the primary elements at level $\ell$ (or higher), and $\mathscr{S}_\ell$ the sequence of the secondary elements at level $\ell$. Let $\mathcal{S}_\ell$ be the sequence $\mathscr{S}_\ell \cdot \mathscr{M}_\ell$ (it is only natural to considerer the elements of $\mathscr{M}_\ell$ as secondary elements at level $\ell$). Let us denote by*

$$p_\ell = |\mathcal{P}_\ell| \qquad \textit{and} \qquad s_\ell = |\mathcal{S}_\ell|. \tag{4.4}$$

The next theorem, proved in [2], give us a lower bound to the length of $P_1$.

**Theorem 4.3.** *If $f, g$ is a $p$-normalized pair, then*

$$p_1 \geqslant \min\left( \left\lfloor \frac{m_0}{2p-1} \right\rfloor, \left\lfloor \frac{q_0}{p} \right\rfloor \right).$$

It now follows from Theorem 4.3 and (4.1) that the minimum number of primary elements that can be obtained at the first level is (since $m_0 \geqslant 2p^{\tau+1} - 1$)

$$p_1 \geqslant p^\tau. \tag{4.5}$$

As pointed out by the inequality in Theorem 4.3, to produce a primary element at level 1, we can use only zero-sum sequences of length at most $2p$. Hence to produce $p_1$ primary elements at level 1, we are using at most $p^\tau$ sequences of maximum length $2p$ at level zero. Thus we define

$$s_0 = m_0 - 2p \times p^\tau = m_0 - 2p^{\tau+1}. \tag{4.6}$$

**Lemma 4.4.** *If $p_\ell \geqslant p$ and $i_j(S_\ell) \geqslant p-1$, for some $j \in \{0,1,\dots,p\}$, then we can produce a primary element at level $\ell+1$ or higher using at most $p$ primary elements and $p-1$ secondary elements*

**Proof.** Suppose that the $p_\ell$ primary elements are at level $\ell$ (otherwise we already have a primary element at level $\ell+1$ or higher) and $i_0(S_\ell) \geqslant p-1$ (see (3.7)). Let

$$T = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdots \begin{pmatrix} a_p \\ b_p \end{pmatrix} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \cdots \begin{pmatrix} c_{p-1} \\ d_{p-1} \end{pmatrix}$$

be a subsequence of $S_\ell$ containing $p$ primary elements and $p-1$ elements of the sequence $I_0(S_\ell)$. Then, for $j \in \{1,\dots,p-1\}$

$$p^{-\ell} c_j \not\equiv 0 \pmod p \qquad \text{and} \qquad d_j \equiv 0 \pmod{p^{\ell+1}}.$$

If there is $i \in \{1,\dots,p\}$ such that $b_i \equiv 0 \pmod{p^{\ell+1}}$, then we must have $a_i \not\equiv 0 \pmod{p^{\ell+1}}$, and Lemma 3.8 tells us that there exists $J \subseteq \{1,\dots,p-1\}$ such that

$$\sum_{j\in J} p^{-\ell} c_j + p^{-\ell} a_i \equiv 0 \pmod p.$$

Therefore,

$$\varphi_{\ell+1}\left(\begin{pmatrix} a_i \\ b_i \end{pmatrix} + \sum_{j\in J} \begin{pmatrix} c_j \\ d_j \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

giving a primary element at level $\ell+1$. So, let us assume $b_i \not\equiv 0 \pmod{p^{\ell+1}}$ for $i \in \{1,\dots,p\}$. Again by Lemma 3.8, there exists $I \subseteq \{1,\dots,p\}$ such that $\sum_{i\in I} p^{-\ell} b_i \equiv 0 \pmod p$, hence

$$\varphi_{\ell+1}\left(\sum_{i\in I} \begin{pmatrix} a_i \\ b_i \end{pmatrix}\right) = \begin{pmatrix} a \\ 0 \end{pmatrix}.$$

If $a \equiv 0 \pmod{p^{\ell+1}}$, we have a primary element at level $\ell+1$. So assume $p^{-\ell} a \not\equiv 0 \pmod p$, and repeating the arguments above, we can find $J \subseteq \{1,\dots,p-1\}$ such that

$$\varphi_{\ell+1}\left(\sum_{i\in I} \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \sum_{j\in J} \begin{pmatrix} c_j \\ d_j \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

giving a primary element at level $\ell+1$ or higher, and completing the proof. ■

**Lemma 4.5.** *Let $\ell \geqslant 0$. Then*

$$s_{\ell+1} \geqslant m_{\ell+1} + \left\lfloor \frac{s_\ell - 3(p^2-1)}{p} \right\rfloor.$$

**Proof.** This result is a consequence of Lemma 3.10, since a secondary element at level $\ell+1$ can always be produced, unless for every $j \in \{0,1,\dots,p\}$, $i_j(S_\ell) \leqslant 3p-3$. Thus there would remain at most $(3p-3)(p+1) = 3(p^2-1)$ elements at level $\ell$. ■

**Lemma 4.6.** *If $\ell \geqslant 1$, then*

$$s_\ell \geqslant \left\lfloor \frac{\sum_{i=1}^{\ell} p^i m_i + s_0 - 3(p+1)(p^\ell - 1)}{p^\ell} \right\rfloor.$$

**Proof.** (Induction on $\ell$) For $\ell = 1$, Lemma 4.5 gives

$$s_1 \geqslant m_1 + \left\lfloor \frac{s_0 - 3(p^2-1)}{p} \right\rfloor = \left\lfloor \frac{pm_1 + s_0 - 3(p^2-1)}{p} \right\rfloor.$$

Now, from Lemmas 3.9, 4.5 and the induction hypothesis, it follows that

$$s_\ell \geqslant m_\ell + \left\lfloor \frac{\left\lfloor \frac{\sum_{i=1}^{\ell-1} p^i m_i + s_0 - 3(p+1)(p^{\ell-1}-1)}{p^{\ell-1}} \right\rfloor - 3(p^2-1)}{p} \right\rfloor$$

$$s_\ell \geqslant m_\ell + \left\lfloor \frac{\sum_{i=1}^{\ell-1} p^i m_i + s_0 - 3(p+1)(p^{\ell-1}-1) - 3p^{\ell-1}(p^2-1)}{p^\ell} \right\rfloor$$

$$s_\ell \geqslant \left\lfloor \frac{\sum_{i=1}^{\ell} p^i m_i + s_0 - 3(p+1)(p^\ell - 1)}{p^\ell} \right\rfloor. \qquad \blacksquare$$

**Lemma 4.7.** *Suppose $\ell \geqslant 1$ and $p \geqslant 11$. If $p_\ell \geqslant p$ and $s_\ell \geqslant p^2 - 6p + 5$, then we can obtain a primary element at level $\ell + 1$ or higher using at most $p$ primary elements and $p - 1$ secondary elements.*

**Proof.** We have (see (3.7))

$$i_0(S_\ell) \geqslant \left\lceil \frac{s_\ell}{p+1} \right\rceil \geqslant p - 6.$$

By Lemma 4.4, we can also assume $i_0(S_\ell) \leqslant p - 2$. And since $p \geqslant 11$, we have $i_0(S_\ell) \geqslant 5 = \mathfrak{s}_3(\mathbb{Z}/p\mathbb{Z})$, according to (3.1). On the other hand, Lemma 3.6 give us

$$q_0(S_\ell) = s_\ell - i_0(S_\ell) \geqslant (p-1)(p-6) + 1 \geqslant \mathfrak{s}_{p-4}(\mathbb{Z}/p\mathbb{Z}).$$

Now we can use Lemma 3.11 to find a zero-sum free subsequence $T$ of $S_\ell$ over $\mathbb{Z}/p^{\ell+1}\mathbb{Z} \oplus \mathbb{Z}/p^{\ell+1}\mathbb{Z}$ of length $3 + (p-4) = p - 1$.

Observe that the sequence $TP_\ell$ has length $m \geqslant 2p - 1$, and according to Lemma 3.3(i), it has a subsequence $U$ such that $\varphi_{\ell+1}(U)$ is a zero-sum sequence. Since $T$ is zero-sum free sequence, the sequence $U$ must contain at least one element of $P_\ell$, hence $\sigma(U)$ is a primary element at level $\ell + 1$. $\blacksquare$

**Lemma 4.8.** *Suppose $p = 7$ and $\ell \geqslant 1$. If $p_\ell \geqslant 7$ and $s_\ell \geqslant 41$ we can obtain a primary element at level $\ell + 1$ or higher using at most 7 primary elements and 6 secondary elements at level $\ell$.*

**Proof.** Since $i_0(S_\ell) \geqslant \lceil \frac{s_\ell}{8} \rceil \geqslant 6$, the result follows immediately from Lemma 4.4. ■

**Lemma 4.9.** *Suppose, for $\ell \in \{1, \ldots, \tau-1\}$, that*

$$s_\ell \geqslant \begin{cases} p^2 - 5p + 4 & \text{if } p \geqslant 11 \\ 47 & \text{if } p = 7. \end{cases}$$

*Then*

$$p_{\ell+1} \geqslant \left\lfloor \frac{p_\ell}{p} \right\rfloor.$$

**Proof.** Let us assume that $p_\ell \geqslant kp$. If $k \leqslant 2$ we can apply Lemmas 4.7 and 4.8 to obtain $k$ primary elements at level $\ell+1$ or higher, since $p^2 - 5p + 4 - (p-1) = p^2 - 6p + 5$ and $47 - 6 = 41$. Now suppose that, for any $t < k$, if there are $tp$ primary elements at level $\ell$, we can then obtain $t$ primary elements at level $\ell+1$ or higher. Since we are assuming $k \geqslant 3$, we can use Lemma 3.3(ii) to obtain a primary element at level $\ell+1$ or higher and still have left $(k-1)p$ primary elements at level $\ell$. By the induction hypothesis we can obtain other $k-1$ primary elements at level $\ell+1$ or higher, concluding this proof. ■

### 4.1. Conclusion

The final step of this proof is to guarantee the existence of a non-singular zero-sum at level $\gamma$, that is, to prove that $p_\gamma \neq 0$, and this is accomplished in the next lemmas. From this point on, we are assuming the validity of the conditions (4.1),(4.5), (4.6), $p \geqslant 7$ and $\tau \geqslant \frac{p-1}{2}$.

**Lemma 4.10.** *Under the conditions above we have, for $\ell \in \{1, \ldots, \tau-1\}$,*

$$s_\ell \geqslant 2\ell p^2 - 3p - 4, \tag{4.7}$$

*and*

$$s_\tau \geqslant p^2 - 4p - 4. \tag{4.8}$$

**Proof.** The conditions (4.1) and (4.6) give

$$\sum_{i=0}^{\ell} p^i m_i \geqslant \sum_{i=0}^{\ell} m_i > 2(\ell+1)(p^{\tau+1} - 1) \quad \text{and} \quad s_0 = m_0 - 2p^{\tau+1}.$$

Hence Lemma 4.6 gives

$$s_\ell \geqslant \left\lfloor \frac{\sum_{i=0}^{\ell} p^i m_i - 2p^{\tau+1} - 3(p+1)(p^\ell - 1)}{p^\ell} \right\rfloor,$$

thus

$$s_\ell \geqslant \left\lfloor \frac{2(\ell+1)(p^{\tau+1} - 1) - 2p^{\tau+1} - 3(p+1)p^\ell + 3(p+1)}{p^\ell} \right\rfloor,$$

giving

$$s_\ell \geqslant 2\ell p^{\tau-\ell+1} - 3(p+1) + \left\lfloor \frac{3p-2\ell+1}{p^\ell} \right\rfloor.$$

Now, since we are assuming $p \geqslant 7$ we have

$$\left\lfloor \frac{3p-2\ell+1}{p^\ell} \right\rfloor = \begin{cases} 2 & \text{if } \ \ell = 1 \\ 0 & \text{if } \ 3p \geqslant 2\ell - 1 \text{ and } \ell > 1 \\ -1 & \text{if } \ 3p < 2\ell - 1 \text{ and } \ell > 1. \end{cases}$$

Therefore, for $\ell \in \{1, \dots, \tau - 1\}$

$$s_\ell \geqslant 2\ell p^{\tau-\ell+1} - 3p - 4 \geqslant 2\ell p^2 - 3p - 4,$$

and, since we are assuming $\tau \geqslant \dfrac{p-1}{2}$

$$s_\tau \geqslant 2(\frac{p-1}{2})p - 3p - 4 = p^2 - 4p - 4. \qquad \blacksquare$$

**Lemma 4.11.** *Under the conditions above we have*

$$p_\tau \geqslant p.$$

**Proof.** From Lemma 4.10 it follows that $s_\ell \geqslant 2\ell p^2 - 3p - 4$, for $\ell \in \{1, 2, \dots, \tau - 1\}$. Since $2\ell p^2 - 3p - 4 \geqslant p^2 - 5p + 4$ if $p \geqslant 11$, and $2\ell p^2 - 3p - 4 \geqslant 47$ if $p = 7$, we can apply Lemma 4.9 and inequality (4.5) to obtain $p_\tau \geqslant p^{\tau-(\tau-1)} = p$. $\blacksquare$

**Lemma 4.12.** *Under the conditions stated above and for $p \geqslant 11$ we have*

$$p_{\tau+1} = p_\gamma \neq 0.$$

**Proof.** It follows from Lemmas 4.10 and 4.11 that

$$s_\tau \geqslant p^2 - 4p - 4 \qquad \text{and} \qquad p_\tau \geqslant p.$$

Since $p \geqslant 11$, we have $p^2 - 4p - 4 \geqslant p^2 - 6p + 5$, hence we can apply Lemma 4.7 to obtain $p_\gamma \neq 0$. $\blacksquare$

**Lemma 4.13.** *Under the conditions stated above and for $p = 7$ we have*

$$p_{\tau+1} = p_\gamma \neq 0.$$

**Proof.** Again, it follows from Lemmas 4.10 and 4.11 that

$$s_\tau \geqslant p^2 - 4p - 4 = 17 \qquad \text{and} \qquad p_\tau \geqslant 7,$$

hence $i_0(S_\tau) \geqslant \left\lceil \dfrac{s_\tau}{8} \right\rceil \geqslant 3$.

If $i_0(S_\tau) \geqslant 6$, the result follows from Lemma 4.4. If $i_0(S_\tau) = 5$, then we have $q_0(S_\tau) = s_\tau - i_0(S_\tau) \geqslant 12 \geqslant \mathfrak{s}_3(\mathbb{Z}/7\mathbb{Z})$ (see (3.1)) and also $i_0(S_\tau) = \mathfrak{s}_3(\mathbb{Z}/7\mathbb{Z})$. It follows from Lemma 3.11 that we can find a subsequence $S$ of $S_\tau$ such that $\varphi_{\tau+1}(S)$ is a zero-sum free subsequence of length 6 over $\mathbb{Z}/7^{\tau+1}\mathbb{Z}\oplus\mathbb{Z}/7^{\tau+1}\mathbb{Z}$. Observe that the sequence $SP_\tau$ has length $m \geqslant 13$, and according to Lemma 3.3(i), it has a subsequence $U$ such that $\varphi_{\tau+1}(U)$ is a zero-sum sequence. Since $S$ is zero-sum free sequence, the sequence $U$ must contain at least one element of $P_\tau$, hence $\sigma(U)$ is a primary element at level $\tau+1=\gamma$.

Hence we may assume $3 \leqslant i_0(S_\tau) \leqslant 4$. Now we have $q_0(S_\tau) = s_\tau - i_0(S_\tau) \geqslant 13 \geqslant \mathfrak{s}_4(\mathbb{Z}/7\mathbb{Z})$ (see Lemma (3.6)) and also $i_0(S_\tau) \geqslant 3 = \mathfrak{s}_2(\mathbb{Z}/7\mathbb{Z})$ (see (3.1)). Again, it follows from Lemma 3.11 that we can find a subsequence $S$ of $S_\tau$ such that $\varphi_{\tau+1}(S)$ is a zero-sum free subsequence of length 6 over $\mathbb{Z}/7^{\tau+1}\mathbb{Z}\oplus\mathbb{Z}/7^{\tau+1}\mathbb{Z}$. Now, as above, the result follows from Lemma 3.3(i), since the sequence $SP_\tau$ has length $m \geqslant 13$. ■

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