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A SUPERCONGRUENCE FOR GENERALIZED DOMB NUMBERS

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Abstract: Using techniques due to Coster, we prove a supercongruence for a generalization of the Domb numbers. This extends a recent result of Chan, Cooper and Sica and confirms a conjectural supercongruence for numbers which are coefficients in one of Zagier's seven "sporadic" solutions to second order Apéry-like differential equations.

Keywords: Domb numbers, supercongruences.

1. Introduction

It is now well-known that the *Apéry numbers*

$$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

play a crucial role in the irrationality proof of $\zeta(3)$, satisfy many interesting congruences and are related to modular forms. For example, Gessel [10] showed that

$$A(np) \equiv A(p) \pmod{p^3} \tag{1}$$

for any prime $p > 3$, while if

$$F(z) = \frac{\eta^7(2z)\eta^7(3z)}{\eta^5(z)\eta^5(6z)} \qquad \text{and} \qquad t(z) = \left(\frac{\eta(6z)\eta(z)}{\eta(2z)\eta(3z)} \right)^{12},$$

then by a result of Peters and Stienstra [16], we have

$$F(z) = \sum_{n=0}^{\infty} A(n)\, t^n(z).$$

Here $\eta(z)$ is the Dedekind eta-function. Since then, there have been several papers which study arithmetic properties of coefficients of power series expansions in t of modular forms where t is a modular function (see [3], [6], [7], [11], [14], [15], [19], [20]).

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Our interest is in the sequence of numbers given by

$$D(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

The first few terms in the sequence of *Domb numbers* $\{D(n)\}_{n\geqslant 0}$ are as follows:

$$1,\ 4,\ 28,\ 256,\ 2716,\ 31504,\ \ldots$$

This ubiquitous sequence (see $A002895$ of Sloane [18]) not only arises in the theory of third order Apéry-like differential equations [1], odd moments of Bessel functions in quantum field theory [2], uniform random walks in the plane [4], new series for $1/\pi$ [5], interacting systems on crystal lattices [9] and the enumeration of abelian squares of length $2n$ over an alphabet with 4 letters [17], but if

$$G(z) = \frac{\eta^4(z)\eta^4(3z)}{\eta^2(2z)\eta^2(6z)} \qquad \text{and} \qquad s(z) = \left(\frac{\eta(2z)\eta(6z)}{\eta(z)\eta(3z)}\right)^6,$$

then (see [5])

$$G(z) = \sum_{n=0}^{\infty} (-1)^n D(n)\, s^n(z).$$

Motivated by (1), Chan, Cooper and Sica [6] recently proved the congruence

$$D(np) \equiv D(p) \pmod{p^3}. \tag{2}$$

The purpose of this short note is to prove a supercongruence for the *generalized Domb numbers*. Recall that the term *supercongruence* refers to congruences that are stronger than those suggested by formal group theory (for recent developments in this area, see [12], [13], [21]). For integers A, B and $C \geqslant 1$, let

$$D(n,A,B,C) := \sum_{k=0}^{n} \binom{n}{k}^A \binom{2k}{k}^B \binom{2(n-k)}{n-k}^C. \tag{3}$$

Our main result is the following.

Theorem 1.1. *Let A, B and C be integers $\geqslant 1$ and $p > 3$ be a prime. For any integers m, $r \geqslant 1$, we have*

$$D(mp^r, A, B, C) \equiv D(mp^{r-1}, A, B, C) \pmod{p^{3r}}$$

if $A \geqslant 2$.

Note that Theorem 1.1 recovers (2) in the case $A = 2$, $B = C = 1$, $r = 1$ and generalizes a numerical observation in Section 3 of [14] (see case (xii) in Table 3). The method of proof for Theorem 1.1 is due to Coster in his influential Ph.D. thesis [8]. Namely, one expresses the summands in (3) as products $g_{AB}(X,k)$

and $g^*_{AB}(X,k)$ (see Section 2), then utilizes the combinatorial features of these products. One then writes (3) as two sums, one for which $p \mid k$ and the other for which $p \nmid k$. In the case $p \nmid k$, the sum vanishes modulo an appropriate power of p while for $p \mid k$, the sum reduces to the required result. This strategy not only leads to a generalization of (1) (see Theorem 4.3.1 in [8]), but can be used to prove supercongruences for other similar sequences [15]. Additionally, a proof similar to that of Theorem 1.1 can be employed to show

$$D(mp^r,1,1,1) \equiv D(mp^{r-1},1,1,1) \pmod{p^{2r}},$$

thereby confirming another conjectural supercongruence in Section 3 of [14] (see case (ix) in Table 2). The details are left to the interested reader. The numbers $D(n,1,1,1)$ are coefficients in one of Zagier's seven "sporadic" solutions (see #10 in Table 1 of [20] or the modular parameterization given by Case E in Table 3 of [20]) to a general family of second order Apéry-like differential equations. Our hope is that the present note will inspire others to further explore the techniques in [8]. In Section 2, we recall the relevant properties of the products $g_{AB}(X,k)$ and $g^*_{AB}(X,k)$ and then prove Theorem 1.1.

2. Proof of Theorem 1.1

We first recall the definition of two products and one sum and list some of their main properties. For more details, see Chapter 4 of [8]. For integers A, $B \geqslant 0$, k, $j \geqslant 1$ and X and for a fixed prime $p > 3$, we define

$$g_{AB}(X,k) = \prod_{i=1}^{k}\left(1-\frac{X}{i}\right)^{A}\left(1+\frac{X}{i}\right)^{B},$$
$$g^*_{AB}(X,k) = \prod_{\substack{i=1\\ p\nmid i}}^{k}\left(1-\frac{X}{i}\right)^{A}\left(1+\frac{X}{i}\right)^{B},$$

and

$$S_j(k) = \sum_{\substack{i=1\\ p\nmid i}}^{k}\frac{1}{i^j}.$$

The following proposition (see Lemmas 4.2.1 and 4.2.5 in [8]) provides some of the main properties of $g_{AB}(X,k)$, $g^*_{AB}(X,k)$ and $S_j(k)$.

Proposition 2.1. *For any integers A, $B \geqslant 0$, $X \in \mathbb{Z}$ and integers m, k, $r \geqslant 1$, we have*

(i) $S_j(mp^r) \equiv 0 \pmod{p^r}$ *for* $j \not\equiv 0 \pmod{p-1}$,
(ii) $S_{2j-1}(mp^r) \equiv 0 \pmod{p^{2r}}$ *for* $j \not\equiv 0 \pmod{\frac{p-1}{2}}$,
(iii) $g_{AB}(pX,k) = g^*_{AB}(pX,k)\, g_{AB}(X,\lfloor \frac{k}{p} \rfloor)$,

(iv) $g^*_{AB}(X,k) \equiv 1 + (B-A)S_1(k)X + \frac{1}{2}\Big((A-B)^2 S_1(k)^2 - (A+B)S_2(k)\Big)X^2 \pmod{X^3}$,

(v) $\binom{n}{k}^A \binom{n+k}{k}^B = (-1)^{Ak}\Big(\frac{n}{n-k}\Big)^A g_{AB}(n,k)$.

We now prove Theorem 1.1.

Proof of Theorem 1.1. We first note that it suffices to prove the result with $p \nmid n$, $p \nmid m$ where $m,\, n \geqslant 1$ are integers and $p > 3$ is a prime. We now assume that $A \geqslant 2$ and $B \geqslant C \geqslant 1$. Recall that for integers $m,\, n,\, r \geqslant 1$ with $p \nmid n$, $p \nmid m$ and $s \geqslant 0$ with $s \leqslant r$, we have

$$ord_p \binom{mp^r}{np^s}^A = A(r-s). \tag{4}$$

Also, by Lemma 2.2 in [15], we have for a prime $p > 3$ and integers $m \geqslant 0$, $r \geqslant 1$

$$\binom{2mp^r}{mp^r} \equiv \binom{2mp^{r-1}}{mp^{r-1}} \pmod{p^{3r}}. \tag{5}$$

Now, taking $j = 2$ in (i), $j = 1$ in (ii) and $X = mp^r$, $k = np^s$ in (iv) of Proposition 2.1, we have

$$g^*_{AB}(mp^r, np^s) \equiv 1 \pmod{p^{r+2s}} \tag{6}$$

for any non-negative integers m, n, r and s with $s \leqslant r$. Letting $n = mp^r$, $k = np^s$, $A \to A+2C$, $B = 0$ in (v) and $X = mp^{r-1}$, $k = np^s$ in (iii) of Proposition 2.1, we have, for $s \geqslant 1$,

$$\begin{aligned}
\binom{mp^r}{np^s}^{A+2C} &= (-1)^{(A+2C)np^s}\left(\frac{mp^r}{mp^r - np^s}\right)^{A+2C} g_{(A+2C)\,0}(mp^r, np^s)\\
&= (-1)^{Anp^{s-1}}\left(\frac{mp^{r-1}}{mp^{r-1} - np^{s-1}}\right)^{A+2C} g^*_{(A+2C)\,0}(mp^r, np^s)\\
&\quad \times g_{(A+2C)\,0}(mp^{r-1}, np^{s-1})\\
&= \binom{mp^{r-1}}{np^{s-1}}^{A+2C} g^*_{(A+2C)\,0}(mp^r, np^s).
\end{aligned} \tag{7}$$

In the last step of (7), we have applied (v) of Proposition 2.1 with $n = mp^{r-1}$, $k = np^{s-1}$, $A \to A+2C$ and $B = 0$. Thus,

$$\begin{aligned}
&\binom{mp^r}{np^s}^{A+2C}\binom{2np^s}{np^s}^{B-C}\binom{2mp^{r-1}}{2np^{s-1}}^{C}\\
&\quad = \binom{mp^{r-1}}{np^{s-1}}^{A+2C} g^*_{(A+2C)\,0}(mp^r, np^s)\binom{2np^s}{np^s}^{B-C}\binom{2mp^{r-1}}{2np^{s-1}}^{C}.
\end{aligned} \tag{8}$$

Similarly, letting $n=2mp^r$, $k=2np^s$, $A=C$, $B=0$ in (v) and $X=2mp^{r-1}$, $k=2np^s$ in (iii) of Proposition 2.1, we have

$$\begin{aligned}\binom{2mp^r}{2np^s}^C &= (-1)^{2Cnp^s}\left(\frac{2mp^r}{2mp^r-2np^s}\right)^C g_{C0}(2mp^r,2np^s)\\ &= \left(\frac{2mp^{r-1}}{2mp^{r-1}-2np^{s-1}}\right)^C g^*_{C0}(2mp^r,2np^s)g_{C0}(2mp^{r-1},2np^{s-1}) \\ &= \binom{2mp^{r-1}}{2np^{s-1}}^C g^*_{C0}(2mp^r,2np^s).\end{aligned} \tag{9}$$

In the last step of (9), we have taken $n=2mp^{r-1}$, $k=2np^{s-1}$, $A=C$ and $B=0$ in (v) of Proposition 2.1. By (5) and (6), we have

$$\binom{2np^s}{np^s}^{B-C} \equiv \binom{2np^{s-1}}{np^{s-1}}^{B-C} \pmod{p^{3s}} \tag{10}$$

and

$$g^*_{(A+2C)\,0}(mp^r,np^s)\equiv g^*_{C0}(2mp^r,2np^s)\equiv 1 \pmod{p^{r+2s}}. \tag{11}$$

For $r\geqslant s$, $A\geqslant 2$ and $C\geqslant 1$, we now claim that

$$\frac{\binom{mp^r}{np^s}^{A+2C}\binom{2np^s}{np^s}^{B-C}}{\binom{2mp^r}{2np^s}^C} \equiv \frac{\binom{mp^{r-1}}{np^{s-1}}^{A+2C}\binom{2np^{s-1}}{np^{s-1}}^{B-C}}{\binom{2mp^{r-1}}{2np^{s-1}}^C} \pmod{p^{3r}}. \tag{12}$$

To see this, we first note that by (9) and (11), (10) and (11), we have

$$\binom{2mp^{r-1}}{2np^{s-1}}^C = \binom{2mp^r}{2np^s}^C - \gamma p^{r+2s}\binom{2mp^{r-1}}{2np^{s-1}}^C, \tag{13}$$

$$\binom{2np^s}{np^s}^{B-C} = \binom{2np^{s-1}}{np^{s-1}}^{B-C} + \alpha p^{3s} \tag{14}$$

and

$$g^*_{(A+2C)\,0}(mp^r,np^s) = 1+\beta p^{r+2s} \tag{15}$$

for some γ, α and $\beta\in\mathbb{Z}$. After substituting (13)–(15) into the right hand side of (8) and multiplying, we consider the following seven terms:

(a) $p^{r+2s}\binom{mp^{r-1}}{np^{s-1}}^{A+2C}\binom{2np^{s-1}}{np^{s-1}}^{B-C}\binom{2mp^{r-1}}{2np^{s-1}}^C$;

(b) $p^{3s}\binom{mp^{r-1}}{np^{s-1}}^{A+2C}\binom{2mp^r}{2np^s}^C$;

(c) $p^{r+5s}\binom{mp^{r-1}}{np^{s-1}}^{A+2C}\binom{2mp^{r-1}}{2np^{s-1}}^C$;

(d) $p^{r+2s}\binom{2mp^r}{2np^s}^{C}\binom{2np^{s-1}}{np^{s-1}}^{B-C}\binom{mp^{r-1}}{np^{s-1}}^{A+2C}$;

(e) $p^{2r+4s}\binom{2np^{s-1}}{np^{s-1}}^{B-C}\binom{mp^{r-1}}{np^{s-1}}^{A+2C}\binom{2mp^{r-1}}{2np^{s-1}}^{C}$;

(f) $p^{r+5s}\binom{2mp^r}{2np^s}^{C}\binom{mp^{r-1}}{np^{s-1}}^{A+2C}$;

(g) $p^{2r+7s}\binom{2mp^{r-1}}{2np^{s-1}}^{C}\binom{mp^{r-1}}{np^{s-1}}^{A+2C}$.

As ord_p is at least $3r + C(r-s)$ in each of the cases (a)–(g) above and we have (4), (12) follows. Now, using the identity

$$\binom{a-b}{c-d}\binom{b}{d} = \frac{\binom{a}{c}\binom{c}{d}\binom{a-c}{b-d}}{\binom{a}{b}},$$

we have

$$D(mp^r, A, B, C) = \binom{2mp^r}{mp^r}^{C}\sum_{k=0}^{mp^r}\frac{\binom{mp^r}{k}^{A+2C}\binom{2k}{k}^{B-C}}{\binom{2mp^r}{2k}^{C}}.$$

We now split $D(mp^r, A, B, C)$ into two sums, namely

$$D(mp^r, A, B, C) = \binom{2mp^r}{mp^r}^{C}\sum_{\substack{k=0\\ p\nmid k}}^{mp^r}\frac{\binom{mp^r}{k}^{A+2C}\binom{2k}{k}^{B-C}}{\binom{2mp^r}{2k}^{C}} + \binom{2mp^r}{mp^r}^{C}\sum_{\substack{k=0\\ p\mid k}}^{mp^r}\frac{\binom{mp^r}{k}^{A+2C}\binom{2k}{k}^{B-C}}{\binom{2mp^r}{2k}^{C}}.$$

Since $A \geqslant 2$, $B \geqslant C \geqslant 1$, the first sum vanishes modulo p^{3r} using (4) and the result then follows from reindexing the second sum and applying (5) and (12). A similar argument holds in the case $A \geqslant 2$, $C > B \geqslant 1$ upon noting that

$$\frac{\binom{mp^r}{k}^{A+2B}\binom{2(mp^r-k)}{mp^r-k}^{C-B}}{\binom{2mp^r}{2k}^{B}} \equiv 0 \pmod{p^{3r}}$$

if $p \nmid k$ and

$$\frac{\binom{mp^r}{np^s}^{A+2B}\binom{2(mp^r-np^s)}{mp^r-np^s}^{C-B}}{\binom{2mp^r}{2np^s}^{B}} \equiv \frac{\binom{mp^{r-1}}{np^{s-1}}^{A+2B}\binom{2(mp^{r-1}-np^{s-1})}{mp^{r-1}-np^{s-1}}^{C-B}}{\binom{2mp^{r-1}}{2np^{s-1}}^{B}} \pmod{p^{3r}}$$

if $p \mid k$. ■

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References

[1] G. Almkvist, D. van Straten and W. Zudilin, *Generalizations of Clausen's formula and algebraic transformations of Calabi-Yau differential equations*, Proc. Edinb. Math. Soc. (2) **54** (2011), no. 2, 273–295.

[2] D. Bailey, J. Borwein, D. Broadhurst and M. Glasser, *Elliptic integral evaluations of Bessel moments and applications*, J. Phys. A **41** (2008) 205203, 46 pp.

[3] F. Beukers, *Some congruences for the Apéry numbers*, J. Number Theory **21** (1985), no. 2, 141–155.

[4] J. Borwein, D. Nuyens, A. Straub and J. Wan, *Some arithmetic properties of short random walk integrals*, Ramanujan J. **26** (2011), no. 1, 109–132.

[5] H. Chan, S. Chan and Z. Liu, *Domb's numbers and Ramanujan-Sato type series for* $1/\pi$, Adv. Math. **186** (2004), no. 2, 396–410.

[6] H. Chan, S. Cooper and F. Sica, *Congruences satisfied by Apéry-like numbers*, Int. J. Number Theory **6** (2010), no. 1, 89–97.

[7] H. Chan, A. Kontogeorgis, C. Krattenthaler and R. Osburn, *Supercongruences satisfied by coefficients of* ${}_2F_1$ *hypergeometric series*, Ann. Sci. Math. Québec **34** (2010), no. 1, 25–36.

[8] M. Coster, *Supercongruences*, Ph.D. thesis, Universiteit Leiden, 1988.

[9] C. Domb, *On the theory of cooperative phenomena in crystals*, Adv. Phys. **9** (1960), 149–361.

[10] I. Gessel, *Some congruences for Apéry numbers*, J. Number Theory **14** (1982), no. 3, 362–368.

[11] F. Jarvis, H. Verrill, *Supercongruences for the Catalan-Larcombe-French numbers*, Ramanujan J. **22** (2010), no. 2, 171–186.

[12] L. Long, *Hypergeometric evaluation identities and supercongruences*, Pacific J. Math. **249** (2011), no. 2, 405–418.

[13] D. McCarthy, *Extending Gaussian hypergeometric series to the p-adic setting*, Int. J. Number Theory **8** (2012), no. 7, 1581–1612.

[14] R. Osburn, B. Sahu, *Congruences via modular forms*, Proc. Amer. Math. Soc. **139** (2011), no. 7, 2375–2381.

[15] R. Osburn, B. Sahu, *Supercongruences for Apéry-like numbers*, Adv. in Appl. Math. **47** (2011), no. 3, 631–638.

[16] C. Peters, J. Stienstra, *A pencil of* $K3$*-surfaces related to Apéry's recurrence for* $\zeta(3)$ *and Fermi surfaces for potential zero*, Arithmetic of complex manifolds (Erlangen, 1988), 110–127, Lecture Notes in Math., 1399, Springer, Berlin, 1989.

[17] L. Richmond, J. Shallit, *Counting abelian squares*, Electron. J. Combin. **16** (2009), no. 1, Research Paper 72, 9pp.
[18] N. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available at http://oeis.org.
[19] H. Verrill, *Congruences related to modular forms*, Int. J. Number Theory **6** (2010), no. 6, 1367–1390.
[20] D. Zagier, *Integral solutions of Apéry-like recurrence equations*, Group and Symmetries, CRM Proc. Lecture Notes **47**, 349–366, Amer. Math. Soc., Providence, RI, 2009.
[21] W. Zudilin, *Ramanujan-type supercongruences*, J. Number Theory **129** (2009), no. 8, 1848–1857.

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