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POWER SERIES WITH THE VON MANGOLDT FUNCTION

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Abstract: We study the analytic behavior of a power series with coefficients containing the von Mangoldt function. In particular, we extend an explicit formula of Hardy and Littlewood for related functions and derive further representation formulas in the unit disk that reveal logarithmic singularities on a dense subset of the unit circle. As an essential tool for proving the square integrability of occurring limit functions together with respective error estimates we contribute a new proof of a Ramanujan-like expansion of an arithmetic function consisting of the von Mangoldt function and the Euler function.

Keywords: trigonometric series over primes, explicit formulas, arithmetic functions, Ramanujan sums, Hardy spaces.

1. Introduction

In 1916 Hardy and Littlewood [6] investigated series of holomorphic functions with arithmetic coefficients and their natural boundaries. In particular, with the von Mangoldt function $\Lambda\colon \mathbb{N} \to \mathbb{R}$ defined by $\Lambda(n) = \log p$ for $n = p^\nu$ with p prime, $\nu \in \mathbb{N}$, and $\Lambda(n) = 0$ otherwise, they used the residue calculus to derive the representation

$$\sum_{n=1}^{\infty} \Lambda(n)\, e^{-nz} = 1/z + \big(\cosh(z) - 1\big)\log z + T(z) - \sum_{\rho} \Gamma(\rho)\, z^{-\rho} \tag{1.1}$$

valid for $\operatorname{Re} z > 0$. Here T is an entire function and the right-hand series is taken over the non-trivial zeros ρ of the Riemann zeta function according to their multiplicity.

Denote by $U = \{w \in \mathbb{C} : |w| < 1\}$ the open unit disc. In order to analyze the analytic behavior of (1.1) near the natural boundary $\operatorname{Re} z = 0$ we consider the power series

$$F(w) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}\, w^n \tag{1.2}$$

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for $w \in U$. The derivative $\frac{d}{dz} F(e^{-z})$ coincides with the series in (1.1) apart from a minus sign. For w approaching the boundary of U the 1-periodic $L^2(0,1)$-function

$$f(t) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \, e^{2\pi i n t} \qquad (t \in \mathbb{R}) \tag{1.3}$$

satisfying $\lim_{r \to 1-} F(re^{2\pi i t}) = f(t)$ for almost all $t \in \mathbb{R}$ is crucial.

In this paper we aim to investigate the series (1.2) and (1.3).

By using the theory of L-functions and the estimation of certain exponential sums with the von Mangoldt function, the imaginary part of (1.3) was also investigated by Conrey and Myerson in [2], and a similar series by Lucht and Wolke in [17]. They study problems with pointwise convergence, which is different from our approach presented in Section 4, dealing with special Ramanujan series and representation formulas converging in appropriate Hardy spaces.

In Section 2 we use a more convenient direct method compared to that of Hardy and Littlewood to derive explicit formulas for the series (1.1) and for $F(e^{-z})$. In particular, we obtain a closed form for the entire function T on the right-hand side of (1.1). These formulas describe the behaviour of the corresponding series in any cone $\{z : |\arg(z)| \leqslant \phi\}$ with $0 < \phi < \frac{\pi}{2}$.

Knowledge of special Ramanujan series provided in Section 3 is used in Section 4 for suitably representing $F(w)$ with $|w| < 1$ and for studying the analytic behavior of $F(w)$ on $|w| = 1$. The main result is Theorem 4.1 which reveals logarithmic singularities of $f(t)$ at the reduced rational numbers $t = \frac{a}{q}$ with square-free denominator $q \in \mathbb{N}$. Moreover, Theorem 3.3 and Lemma 3.5 afford quantitative remainder term estimates for Theorem 4.1 in the H^2 and $L^2(0,1)$-norm, respectively. In the context of Theorem 4.1 we also discuss the results of Conrey and Myerson [2].

2. Explicit formulas

The holomorphic function $F\colon U \to \mathbb{C}$ in (1.2) is closely related to the entire function

$$G(z) = \sum_{n=1}^{\infty} \Lambda(n) \left(e^{-z/n} - 1 + \frac{z}{n} \right) \qquad (z \in \mathbb{C})$$

with the power series representation

$$G(z) = -\sum_{m=2}^{\infty} \frac{\zeta'(m)}{\zeta(m)} \cdot \frac{(-z)^m}{m!} \qquad (z \in \mathbb{C}).$$

We start with an explicit formula valid in $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

Theorem 2.1. *For $z \in \mathbb{H}$ all series and integrals in*

$$
\begin{aligned}
F(e^{-z}) + G(z) = z \log z - z + \log(2\pi) + \frac{1}{2}\, e^{-z} \, \log\Big(\frac{z}{4\pi^2 e^{\gamma}}\Big) \\
+ \Big(\frac{1}{2} + e^{-z}\Big) \int_0^\infty \frac{e^{-t}}{z+t}\, dt - \frac{1}{2} \int_0^1 \frac{e^{-uz} - e^{-z}}{1-u}\, du \\
- \sum_{\rho} \Gamma(-\rho)\, z^{\rho}
\end{aligned}
\tag{2.1}
$$

converge absolutely. Here γ denotes the Euler constant,[1] *and the last series is taken over the non-trivial zeros ρ of the Riemann zeta function according to their multiplicity.*

Proof. The proof is divided into four steps:

Step 1. The series defining $F(e^{-z})$, $G(z)$ and the integrals in (2.1) converge absolutely for $z \in \mathbb{H}$. Let $\rho = \sigma + it$ with $0 < \sigma < 1$ and $t \in \mathbb{R}$ be any non-trivial zero of the zeta function, and write $\log z = \log|z| + i \arg z$ with $|\arg z| < \frac{\pi}{2}$. Then, for all sufficiently large $|t|$, there is some constant $C > 0$ such that the inequality

$$
\begin{aligned}
|z^{\rho}\, \Gamma(-\rho)| &= \frac{|z|^{\sigma}}{\sqrt{\sigma^2 + t^2}} \exp(-t \arg z) \cdot |\Gamma(1 - \sigma - it)| \\
&\leqslant C\, \frac{|z|^{\sigma}\, |t|^{1/2-\sigma}}{\sqrt{\sigma^2+t^2}} \cdot \exp\big(-\tfrac{\pi}{2}\,|t| - t \arg z\big)
\end{aligned}
\tag{2.2}
$$

holds (cf. Andrews, Askey and Roy [1, Corollary 1.4.4]). Combined with the Riemann-von Mangoldt Theorem on the vertical distribution of the non-trivial zeros of the zeta function, the estimate (2.2) guarantees the absolute convergence of the last series in (2.1) (cf., for instance, Edwards [4, Chapter 6.7] or Karatsuba and Voronin [10, Theorem 1, Chapter I.8]).

Step 2. As usual denote by ψ the summatory function[2] of the von Mangoldt function Λ and recall the well-known explicit formula

$$
\psi(x) = x - \lim_{T\to\infty} \sum_{\substack{\rho \\ |\operatorname{Im}\rho| \leqslant T}} \frac{x^{\rho}}{\rho} + \frac{1}{2} \log \frac{x^2}{x^2-1} - \log(2\pi) \qquad (1 < x \notin \mathbb{N})
$$

(cf., for instance, Edwards [4, Chapter 4] or Ingham [9, Chapter IV]). For $x > 1$ let

$$
\eta(x) = \psi(x) - \Big(x + \frac{1}{2} \log \frac{x^2}{x^2-1} - \log(2\pi)\Big). \tag{2.3}
$$

Then η satisfies

$$
\eta(x) = -\lim_{T\to\infty} \sum_{\substack{\rho \\ |\operatorname{Im}\rho| \leqslant T}} \frac{x^{\rho}}{\rho} \qquad (1 < x \notin \mathbb{N}) \tag{2.4}
$$

---

[1] $\gamma = \lim\limits_{n\to\infty} \Big(\sum\limits_{1\leqslant\nu\leqslant n} \frac{1}{\nu} - \log n \Big)$

[2] $\psi(x) = \sum\limits_{n \leqslant x} \Lambda(n)$ for $x > 0$

and is integrable on $(1,\infty)$. The absolutely convergent series

$$g(x) = -\sum_{\rho} \frac{x^{\rho}}{\rho\,(1-\rho)}$$

represents a continuous function for $x > 0$. To evaluate g, we distinguish the cases $x \geqslant 1$ and $0 < x < 1$ (cf. [13]).

For $x \geqslant 1$ we obtain from (2.4) by integration that

$$g(x) = x\,g(1) - x \int_1^x \frac{\eta(u)}{u^2}\,du. \tag{2.5}$$

For $0 < x < 1$ we use the symmetry of the non-trivial zeros of $\zeta(s)$ with respect to the critical line $\operatorname{Re} s = \frac{1}{2}$, from which we infer that

$$g'(x) = -\lim_{T\to\infty} \sum_{\substack{\rho \\ |\operatorname{Im}\rho|\leqslant T}} \frac{x^{\rho-1}}{1-\rho} = -\lim_{T\to\infty} \sum_{\substack{\rho \\ |\operatorname{Im}\rho|\leqslant T}} \frac{\left(\frac{1}{x}\right)^{1-\rho}}{1-\rho} = \eta\Big(\frac{1}{x}\Big)$$

and, by integration,

$$g(x) = g(1) + \int_1^x \eta\Big(\frac{1}{u}\Big)\,du \qquad (0 < x < 1). \tag{2.6}$$

In particular, for $x = 1$ it is known (cf. Edwards [4, Chapter 3.8]) that

$$g(1) = -\sum_{\rho} \frac{1}{\rho\,(1-\rho)} = 2\,\big(\log(2\pi) - 1\big) - (\gamma + \log\pi).$$

Step 3. For $s \in \mathbb{C}$ fixed with $0 < \operatorname{Re} s < 1$ and $x > 0$ we have

$$-\Gamma(-s)\,x^s = \frac{\Gamma(2-s)\,x^s}{s\,(1-s)} = \int_0^\infty \frac{\left(\frac{x}{t}\right)^s}{s\,(1-s)}\,t\,e^{-t}\,dt,$$

and summation over all ρ leads to

$$\sum_{\rho} \Gamma(-\rho)\,x^{\rho} = \int_0^\infty g\Big(\frac{x}{t}\Big)\,t\,e^{-t}\,dt =: J_-(x) + J_+(x) \qquad (x > 0) \tag{2.7}$$

with the integrals $J_-(x)$ and $J_+(x)$ running over $(0,x]$ and $[x,\infty)$, respectively. We show that

$$\begin{aligned} J_-(x) &= x\,g(1)\,(1-e^{-x}) + \int_1^\infty \eta(u)\,\frac{d}{du}\Big(e^{-x/u} - 1 + \frac{x}{u}\Big)\,du,\\ J_+(x) &= (x+1)\,g(1)\,e^{-x} + \int_1^\infty \eta(u)\,\frac{d}{du}\Big(\frac{e^{-xu}}{u}\Big)\,du. \end{aligned} \tag{2.8}$$

From (2.5) we obtain

$$\begin{aligned}
J_-(x) &= x \int_0^x \left(g(1) - \int_1^{x/t} \frac{\eta(u)}{u^2}\, du \right) e^{-t}\, dt \\
&= x\, g(1)(1-e^{-x}) - x \int_0^x \frac{d}{dt}\left(1-e^{-t}\right) \int_1^{x/t} \frac{\eta(u)}{u^2}\, du\, dt \\
&= x\, g(1)(1-e^{-x}) - \int_0^x (1-e^{-t})\, \eta\Big(\frac{x}{t}\Big)\, dt \\
&= x\, g(1)\,(1-e^{-x}) + \int_1^\infty \eta(u)\, \frac{d}{du}\left(e^{-x/u} - 1 + \frac{x}{u}\right) du,
\end{aligned}$$

and from (2.6) similarly

$$\begin{aligned}
J_+(x) &= \int_x^\infty \left(g(1) + \int_1^{x/t} \eta\Big(\frac{1}{u}\Big)\, du \right) te^{-t}\, dt \\
&= (x+1)\, g(1)\, e^{-x} - \int_x^\infty \frac{d}{dt}\left((1+t)\, e^{-t}\right) \int_1^{x/t} \eta\Big(\frac{1}{u}\Big)\, du\, dt \\
&= (x+1)\, g(1)\, e^{-x} + \int_x^\infty (1+t)\, \frac{(-x)}{t^2}\, e^{-t}\, \eta\Big(\frac{t}{x}\Big)\, dt \\
&= (x+1)\, g(1)\, e^{-x} + \int_1^\infty \eta(u)\, \frac{d}{du}\Big(\frac{e^{-xu}}{u}\Big)\, du.
\end{aligned}$$

This establishes (2.8).

Step 4. Partial integration applied to the right-hand side of (2.8) combined with (2.3) and (2.7) entails

$$\sum_\rho \Gamma(-\rho)\, x^\rho + F(e^{-x}) + G(x) = g(1)\,(x + e^{-x}) + I(x),$$

where

$$I(x) = \int_1^\infty \left(u + \frac{1}{2} \log \frac{u^2}{u^2-1} - \log(2\pi) \right) \left(x\, \frac{1-e^{-x/u}}{u^2} + \frac{1+xu}{u^2} e^{-xu} \right) du$$

for all $x > 0$. An elementary, but lengthy conversion of $I(x)$ using

$$\log \frac{b}{a} = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt \qquad (a, b > 0)$$

yields (2.1) for real $z = x > 0$. Since both sides of (2.1) represent holomorphic functions for $z \in \mathbb{H}$, the assertion follows from applying the identity theorem. ■

Define the regular part $\mathrm{Ei}_0 \colon \mathbb{C} \to \mathbb{C}$ of the exponential integral function by

$$\mathrm{Ei}_0(z) = \sum_{n=1}^\infty \frac{z^n}{n \cdot n!} = \int_0^1 \frac{e^{uz} - 1}{u}\, du.$$

As a consequence of Theorem 2.1 we obtain

Theorem 2.2. *The entire function T in the Hardy-Littlewood relation* (1.1) *is given for all $z \in \mathbb{H}$ by*

$$\begin{aligned} T(z) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \left(1 - e^{-\frac{z}{n}}\right) - \log(2\pi)\, e^{-z} \\ + \gamma \sinh(z) + \frac{e^z}{2} \operatorname{Ei}_0(-z) - \frac{e^{-z}}{2} \operatorname{Ei}_0(z). \end{aligned} \tag{2.9}$$

Proof. Use Ei_0 to rewrite (2.1) for $z \in \mathbb{H}$ as

$$\begin{aligned} F(e^{-z}) + G(z) = z \log z - z + \log(2\pi)\left(1 - e^{-z}\right) - \left(\operatorname{Ei}_0(-z) + \log z + \gamma\right) \\ - \frac{1}{2} e^{z} \left(\operatorname{Ei}_0(-z) + \log z + \gamma\right) - \frac{1}{2} e^{-z} \left(\operatorname{Ei}_0(z) - \log z + \gamma\right) \\ - \sum_{\rho} \Gamma(-\rho)\, z^{\rho}, \end{aligned}$$

and verify (2.9) by differentiating. Observe here that both ρ and $1-\rho$ are zeros of the zeta function and that

$$\frac{d}{dz}\left(\sum_{\rho} \Gamma(-\rho)\, z^{\rho}\right) = -\sum_{\rho} \Gamma(1-\rho)\, z^{\rho-1} = -\sum_{\rho} \Gamma(\rho)\, z^{-\rho}$$

by the functional equation of the Gamma function. ■

Hardy and Littlewood [6] have used the explicit formula (1.1) for an equivalent characterization of the Riemann hypothesis (RH). It follows immediately from Theorem 2.2 and (1.1) that RH is equivalent to

$$\sum_{n=1}^{\infty} \Lambda(n)\, e^{-nx} = \frac{1}{x} + \mathrm{O}\left(x^{-\frac{1}{2}}\right)$$

for $x \to 0+$.

In the same way we may use the explicit formula of Theorem 2.1 to obtain

Remark 2.3. Each of the following two statements is equivalent to RH.

a) $F(e^{-x}) = -\log x - 2\gamma + \mathrm{O}\left(\sqrt{x}\right)$ for $x \to 0+$.

b) $G(z) = z \log z - z + \mathrm{O}\left(\sqrt{z}\right)$, uniformly for $z \in \mathbb{H}$ in each cone $\{z : |\arg z| \leqslant \phi\}$ with $0 < \phi < \frac{\pi}{2}$ (recall that G is entire with $G(0) = 0$).

Theorems 2.1 and 2.2 offer a remarkably efficient calculation of the series (1.1) and of $F(e^{-z})$ even in the problematic case of very small positive $z = x > 0$. Namely, the coefficients $\Gamma(\rho)$ in (1.1) or $\Gamma(-\rho)$ in (2.1) are very small and exponentially decreasing with $|\operatorname{Im}\rho|$. For example, we have $|\Gamma(-\rho)| < 10^{-10}$ for the first non-trivial zero $\rho = \frac{1}{2} + i \cdot 14.134\ldots$

However, the use of the explicit formula for a numerical approximation of $F(e^{-z}) + G(z)$ becomes inefficient for $z = \delta + it$ with very small $\delta > 0$ and $0 \neq t \in \mathbb{R}$. To overcome this difficulty and to comment on the singular structure of F, we derive a new representation formula for F in Section 4. For this purpose we need some deeper knowledge of special Ramanujan series that are also of interest by itself.

3. Series with Ramanujan sums

Let $\mathscr{F} = \{g\colon \mathbb{N} \to \mathbb{C}\}$ be the commutative unital algebra of arithmetic functions under the usual linear operations and the Dirichlet convolution defined by

$$(g * h)(n) = \sum_{md=n} g(m)\, h(d) \qquad (n \in \mathbb{N})$$

for $g, h \in \mathscr{F}$. The unity $\varepsilon \in \mathscr{F}$ is given by $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $1 < n \in \mathbb{N}$. The multiplicative functions g satisfying $g(1) = 1$ and $g(mn) = g(m)\, g(n)$ for all coprime $m, n \in \mathbb{N}$ form a subgroup $\mathscr{M} \subset \mathscr{F}$ under $*$. In particular, the constant function 1 with value 1, the identity I, the Möbius function μ and the Euler function φ belong to $\mathscr{M}$. A function $g \in \mathscr{M}$ is called completely multiplicative, if $g(mn) = g(m)\, g(n)$ for all $m, n \in \mathbb{N}$. Note that ε, 1 and I are completely multiplicative. The multiplicative group of $\mathscr{F}$ under $*$, i.e. the group of invertible functions under $*$, is $\mathscr{F}^* = \{g \in \mathscr{F} : g(1) \neq 0\}$. Denote the inverse of $g \in \mathscr{F}^*$ by g^{-1}. In particular, we have $1{*}\mu = I{*}(\mu I) = \varepsilon$ and $1{*}\varphi = I$ so that $1^{-1} = \mu$, $I^{-1} = \mu I$ and $\varphi = \mu{*}I$. With the natural logarithm $\log \in \mathscr{F}$ considered as an arithmetic function the von Mangoldt function $\Lambda \in \mathscr{F}$ may be defined by $\Lambda = \mu * \log$. By choosing $g = 1$ and $h = \mu$ in the identity $(g * h) \log = (g \log) * h + g * (h \log)$ we obtain $\Lambda = -1 * (\mu \log)$. The values of Λ are given by $\Lambda(n) = \log p$ for $n = p^\nu$ with p prime, $\nu \in \mathbb{N}$ and $\Lambda(n) = 0$ if $n \notin \mathbb{P}^* = \{p^\nu : p \text{ prime}, \nu \in \mathbb{N}\}$.

The following theorem on parameter dependent summatory functions of $\frac{\mu^2}{\varphi} \in \mathscr{M}$ uses non-negative real constants $\vartheta, \vartheta_a, d_a$ defined for $a \in \mathbb{N}$ by

$$\vartheta = \sum_p \frac{\log p}{p\,(p-1)}\,, \quad \vartheta_a = \sum_{p|a} \frac{\log p}{p}\,, \quad d_a = \prod_{p|a} \Big(1 + \frac{1}{\sqrt{p}}\Big) \tag{3.1}$$

and is due to Hildebrand [7, Hilfssatz 2] (see also Schwarz and Spilker [20, Lemma 3.1] and, for a weaker version, Karatsuba and Voronin [10, Chapter V.5, Lemma 1]). Here we outline a slightly simpler proof.

Theorem 3.1 (Hildebrand, 1984). *With the above constants,*

$$\sum_{\substack{n \leqslant x \\ (n,a)=1}} \frac{\mu^2(n)}{\varphi(n)} = \frac{\varphi(a)}{a} \big(\log x + \gamma + \vartheta + \vartheta_a\big) + \mathrm{O}\Big(\frac{d_a}{\sqrt{x}}\Big) \tag{3.2}$$

uniformly for $a \in \mathbb{N}$ and $x \geqslant 1$.

Proof. The proof is divided into five steps:

Step 1. Let χ_0 denote the principal character mod a. We replace the summation condition $(a,n) = 1$ with the factor χ_0, define $g = \frac{\chi_0 \mu^2 I}{\varphi} \in \mathscr{M}$ and write $g = 1 * h$ with $h = \mu * g \in \mathscr{M}$. Then, for prime powers $p^\nu \in \mathbb{P}^*$,

$$h(p^\nu) = \begin{cases} \frac{1}{p-1} & \text{for } \nu = 1,\ p \nmid a \\ -\frac{p}{p-1} & \text{for } \nu = 2,\ p \nmid a \\ -1 & \text{for } \nu = 1,\ p \mid a \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet series $\widetilde{h}(s)$ of h has the Euler product representation

$$\widetilde{h}(s) = \prod_{p \mid a} \Big(1 - \frac{1}{p^s}\Big) \prod_{p \nmid a} \Big(1 + \frac{1}{(p-1)\,p^s} - \frac{p}{(p-1)\,p^{2s}}\Big)$$

converging absolutely for $\operatorname{Re} s > \frac{1}{2}$. In particular, we obtain

$$\widetilde{h}(1) = \frac{\varphi(a)}{a}, \quad \widetilde{h}'(1) = (\vartheta + \vartheta_a)\,\frac{\varphi(a)}{a}\,. \tag{3.3}$$

Insertion of

$$\sum_{m \leqslant x} \frac{1}{m} = \log x + \gamma + \mathrm{O}\Big(\frac{1}{x}\Big)$$

leads to

$$\begin{aligned} \sum_{\substack{n \leqslant x \\ (n,a)=1}} \frac{\mu^2(n)}{\varphi(n)} &= \sum_{n \leqslant x} \frac{g(n)}{n} = \sum_{dm \leqslant x} \frac{h(d)}{dm} = \sum_{d \leqslant x} \frac{h(d)}{d} \sum_{m \leqslant \frac{x}{d}} \frac{1}{m} \\ &= \sum_{d \leqslant x} \frac{h(d)}{d} \Big(\log \frac{x}{d} + \gamma + \mathrm{O}\Big(\frac{d}{x}\Big)\Big) \\ &= \widetilde{h}(1)\,(\log x + \gamma) + \widetilde{h}'(1) + \varrho(x,a) \end{aligned}$$

with

$$\varrho(x,a) \ll \frac{1}{x} \sum_{d \leqslant x} |h(d)| + \sum_{d > x} \frac{|h(d)|}{d} + \sum_{d > x} \frac{|h(d)|}{d} \log \frac{d}{x}\,.$$

Now (3.2) follows from inserting (3.3) and proving that[3]

$$S_1 := \sum_{d \leqslant x} |h(d)| \ll d_a \sqrt{x}\,, \tag{3.4}$$

$$S_2 := \sum_{d > x} \frac{|h(d)|}{d} \ll \frac{d_a}{\sqrt{x}}\,, \tag{3.5}$$

$$S_3 := \sum_{d > x} \frac{|h(d)|}{d} \log \frac{d}{x} \ll \frac{d_a}{\sqrt{x}} \tag{3.6}$$

uniformly for $a \in \mathbb{N}$ and $x \geqslant 1$.

[3]For $f\colon D \to \mathbb{C}$ and $g\colon D \to (0,\infty)$ we write $f \ll g$ with the Vinogradov symbol $\ll$ for $f = \mathrm{O}(g)$ on D.

Step 2. We prepare the proof of (3.4) by providing the rough estimate

$$\sum_{n\leqslant x} \frac{n}{\varphi(n)} \ll x, \tag{3.7}$$

which follows from approximating $f = \frac{I}{\varphi} \in \mathscr{M}$ by the constant function 1. Namely, write $\frac{I}{\varphi} = 1 * b$ with $b = \mu * \frac{I}{\varphi} \in \mathscr{M}$. Now $b(p^\nu) = f(p^\nu) - f(p^{\nu-1})$ entails $b(p) = \frac{1}{p-1}$ and $b(p^\nu) = 0$ for p prime and $\nu \geqslant 2$. Then (3.7) results from

$$\sum_{n\leqslant x} \frac{n}{\varphi(n)} = \sum_{dm\leqslant x} b(d) \leqslant x \sum_{d\leqslant x} \frac{b(d)}{d} \leqslant x \prod_p \Big(1 + \frac{1}{p\,(p-1)}\Big) \ll x.$$

Step 3. We separate three parts of $|h|$ according to their impact on the convergence behavior of $\widetilde{h}(s)$. Define $h_1, h_2, h_3 \in \mathscr{M}$ by

$$h_1(n) = \begin{cases} \mu^2(n) & \text{if } n \mid a \\ 0 & \text{else} \end{cases}, \qquad h_2(n) = \frac{\mu^2(n)}{\varphi(n)}, \qquad h_3(n) = \begin{cases} \frac{m}{\varphi(m)} & \text{if } n = m^2 \\ 0 & \text{else} \end{cases}.$$

By comparing the Euler product factors of $|h|^{\sim}(s)$ with those of the product $\widetilde{h}_1(s)\,\widetilde{h}_2(s)\,\widetilde{h}_3(s)$ we obtain $|h| \leqslant h_1 * h_2 * h_3$. Note that $\widetilde{h}_1(s)$ is finite for every $s \in \mathbb{C}$, while $\widetilde{h}_2(s)$ converges absolutely for $\operatorname{Re} s > 0$ and $\widetilde{h}_3(s)$ converges absolutely for $\operatorname{Re} s > \frac{1}{2}$. Insertion gives

$$\begin{aligned} S_1 &\ll \sum_{n_1 n_2 n_3 \leqslant x} h_1(n_1)\, h_2(n_2)\, h_3(n_3) \\ &= \sum_{n_1 \mid a} \mu^2(n_1) \sum_{n_2 \leqslant \frac{x}{n_1}} \frac{\mu^2(n_2)}{\varphi(n_2)} \sum_{m \leqslant \sqrt{\frac{x}{n_1 n_2}}} \frac{m}{\varphi(m)}, \end{aligned}$$

and (3.7) implies

$$S_1 \ll \sqrt{x} \sum_{n_1 \mid a} \frac{\mu^2(n_1)}{\sqrt{n_1}} \sum_{n_2 \leqslant \frac{x}{n_1}} \frac{\mu^2(n_2)}{\varphi(n_2)\sqrt{n_2}} \ll \sqrt{x} \sum_{n \mid a} \frac{\mu^2(n)}{\sqrt{n}},$$

the latter because of the absolute convergence of $\widetilde{h}_2\big(\frac{1}{2}\big)$. Since the last sum equals d_a, (3.4) follows.

Step 4. Partial summation combined with (3.4) yields

$$\sum_{x<n\leqslant y} \frac{|h(n)|}{n} = \frac{1}{y} \sum_{x<n\leqslant y} |h(n)| + \int_x^y \Big(\sum_{x<n\leqslant t} |h(n)| \Big) \frac{dt}{t^2} \ll \frac{d_a}{\sqrt{x}}.$$

Letting $y \to \infty$, (3.5) follows uniformly for $a \in \mathbb{N}$ and $x \geqslant 1$.

Step 5. As in Step 4 partial summation and (3.5) imply that

$$\begin{aligned}\sum_{x<n\leqslant y}\frac{|h(n)|}{n}\log\frac{n}{x} &= \sum_{x<n\leqslant y}\frac{|h(n)|}{n}\int_x^n\frac{dt}{t}\\ &= \sum_{x<n\leqslant y}\frac{|h(n)|}{n}\int_x^y\frac{dt}{t}-\int_x^y\Bigg(\sum_{x<n\leqslant t}\frac{|h(n)|}{n}\Bigg)\frac{dt}{t}\\ &= \int_x^y\Bigg(\sum_{t<n\leqslant y}\frac{|h(n)|}{n}\Bigg)\frac{dt}{t}\ll d_a\int_x^\infty\frac{dt}{t^{3/2}}\ll\frac{d_a}{\sqrt{x}}\end{aligned}$$

uniformly for $a\in\mathbb{N}$ and $y\geqslant x\geqslant 1$. Letting $y\to\infty$ establishes (3.6) and completes the proof of Theorem 3.1. ■

For later purposes we provide the following simple, but useful estimate.

Remark 3.2. Uniformly for $a\in\mathbb{N}$ and $x\geqslant 1$ we have

$$\sum_{\substack{n\leqslant x\\(n,a)=1}}\frac{\mu^2(n)}{\varphi(n)}\leqslant\prod_{p\leqslant x}\Big(1+\frac{1}{p-1}\Big)\leqslant\exp\Bigg(\sum_{p\leqslant x}\frac{1}{p-1}\Bigg)\ll 1+\log x.$$

For $n\in\mathbb{N}$ and $a\in\mathbb{Z}$ the Ramanujan sum $c_n(a)$ defined by

$$c_n(a)=\sum_{\substack{1\leqslant\nu\leqslant n\\(\nu,n)=1}}e^{2\pi i\frac{\nu}{n}a}$$

is the sum of the ath powers of the nth primitive roots of unity. Note that

$$(1*c_\bullet(a))(n)=\eta_a(n):=\begin{cases}n & \text{for } n\mid a,\\ 0 & \text{for } n\nmid a.\end{cases}$$

Convolving with the Möbius function μ leads to the relation

$$c_\bullet(a)=\mu*\eta_a\qquad(a\in\mathbb{N})\tag{3.8}$$

so that all $c_n(a)$ are real and, in particular, $c_n(a)=\mu(n)$ for $(n,a)=1$. Since the support $\operatorname{supp}\eta_a=\{n\in\mathbb{N}:n\mid a\}$ is a finite set for any $a\in\mathbb{N}$, (3.8) implies that every $c_\bullet(a)$ is related to μ (cf. [16]). With $n'=n/(n,a)$ the "closed" evaluation

$$c_n(a)=\frac{\mu(n')\,\varphi(n)}{\varphi(n')}\tag{3.9}$$

is known as Hölder's relation. For a proof based on (3.8) see, for instance, Knopfmacher [11, Chapter 7, Lemma 2.5].

Recall that $\varepsilon(1) = 1$ and $\varepsilon(a) = 0$ for $1 < a \in \mathbb{N}$.

Theorem 3.3. *The Ramanujan series*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{\varphi(n)}\, c_n(a) \tag{3.10}$$

diverges for $a = 1$ and converges for $a > 1$. More precisely, for $x \geqslant a$,

$$\sum_{n \leqslant x} \frac{\mu(n)}{\varphi(n)}\, c_n(a) = (\log x + \gamma + \vartheta)\, \varepsilon(a) + \frac{\varphi(a)}{a}\, \Lambda(a) + \mathrm{O}\Big(\frac{\kappa_a}{\sqrt{x}}\Big) \tag{3.11}$$

with

$$\kappa_a = 2^{\omega(a)} \sqrt{a} \prod_{p \mid a} \Big(1 - \frac{1}{\sqrt{p}}\Big)^{-1},$$

where $\omega(a)$ denotes the number of prime divisors of $a \in \mathbb{N}$.

Proof. It suffices to verify (3.11). Inserting (3.8) and reordering gives

$$\sum_{n \leqslant x} \frac{\mu(n)}{\varphi(n)}\, c_n(a) = \sum_{\substack{dm \leqslant x \\ m \mid a}} \frac{\mu(dm)}{\varphi(dm)}\, \mu(d)\, m = \sum_{\substack{m \leqslant x \\ m \mid a}} \frac{m\, \mu(m)}{\varphi(m)} \sum_{\substack{d \leqslant \frac{x}{m} \\ (d,m)=1}} \frac{\mu^2(d)}{\varphi(d)}$$

which for $x \geqslant a$ entails

$$\sum_{n \leqslant x} \frac{\mu(n)}{\varphi(n)}\, c_n(a) = \sum_{m \mid a} \frac{m\, \mu(m)}{\varphi(m)} \sum_{\substack{d \leqslant \frac{x}{m} \\ (d,m)=1}} \frac{\mu^2(d)}{\varphi(d)}\,. \tag{3.12}$$

By applying Theorem 3.1 we obtain for $x \geqslant a$ that

$$\begin{aligned} \sum_{n \leqslant x} \frac{\mu(n)}{\varphi(n)}\, c_n(a) &= \sum_{m \mid a} \mu(m) \Big(\log \frac{x}{m} + \gamma + \vartheta + \vartheta_m + r(a,x) \Big) \\ &= (\log x + \gamma + \vartheta)\, \varepsilon(a) + \Lambda(a) + (1 * (\mu\, \vartheta_{\bullet}))(a) + r(a,x) \end{aligned}$$

with the remainder term

$$r(a,x) \ll \frac{1}{\sqrt{x}} \sum_{m \mid a} \frac{|\mu(m)|\, m \sqrt{m}\, d_m}{\varphi(m)}\,.$$

Because of $d_m \leqslant d_a$ and $\sqrt{m} \leqslant \sqrt{a}$ for $m \mid a$ we have

$$\begin{aligned} r(a,x) &\ll \frac{d_a \sqrt{a}}{\sqrt{x}} \sum_{m \mid a} \frac{\mu^2(m)\, m}{\varphi(m)} = \frac{d_a \sqrt{a}}{\sqrt{x}} \prod_{p \mid a} \Big(2 + \frac{1}{p-1}\Big) \\ &\leqslant \frac{d_a\, 2^{\omega(a)} \sqrt{a}}{\sqrt{x}} \prod_{p \mid a} \Big(1 + \frac{1}{p-1}\Big). \end{aligned}$$

Combined with the definition of d_a in (3.1) it follows that

$$r(a,x) \ll \frac{2^{\omega(a)}\sqrt{a}}{\sqrt{x}} \prod_{p|a} \Big(1+\frac{1}{\sqrt{p}}\Big)\Big(1+\frac{1}{p-1}\Big)$$
$$= \frac{2^{\omega(a)}\sqrt{a}}{\sqrt{x}} \prod_{p|a} \Big(1-\frac{1}{\sqrt{p}}\Big)^{-1} = \frac{\kappa_a}{\sqrt{x}}\,.$$

Finally, with $\vartheta_{\bullet}$ from (3.1), observe the identity

$$(1*(\mu\,\vartheta_{\bullet}))(a) = \sum_{m|a} \mu(m) \sum_{p|m} \frac{\log p}{p} = \begin{cases} -\frac{\log p}{p} & \text{for } a = p^\nu \in \mathbb{P}^* \\ 0 & \text{otherwise} \end{cases}$$

covered by the next slightly more general Lemma 3.4, which completes the proof. ∎

Lemma 3.4. *For any $f \in \mathscr{F}$,*

$$\sum_{m|a} \mu(m) \sum_{p|m} f(p) = \begin{cases} -f(p) & \text{for } a = p^\nu \in \mathbb{P}^* \\ 0 & \text{otherwise} \end{cases} \qquad (a \in \mathbb{N}).$$

Proof. With $m = pd$ reordering yields

$$\sum_{m|a} \mu(m) \sum_{p|m} f(p) = \sum_{pd|a} \mu(pd)\, f(p)$$
$$= \sum_{\substack{pd|a \\ p\nmid d}} \mu(p)\, f(p)\, \mu(d) = \sum_{p^\nu \| a} \mu(p)\, f(p)\, \varepsilon\Big(\frac{a}{p^\nu}\Big),$$

where $p^\nu \| a$ stands for $p^\nu \mid a$ and $p^{\nu+1} \nmid a$. For $a = p^\nu \in \mathbb{P}^*$ we obtain the sum $-f(p)$, while the sum vanishes otherwise. ∎

Finally we close the gap in Theorem 3.3, regarding x small compared to a, by the following global estimate

Lemma 3.5. *Uniformly for $1 < a \in \mathbb{N}$ and $x \geqslant 1$ we have*

$$\sum_{n\leqslant x} \frac{\mu(n)}{\varphi(n)}\, c_n(a) \ll e^{\omega(a)} \log a.$$

Proof. First note that the growth rate of $\omega(a)$ is rather small, namely

$$\omega(a) \ll \frac{\log a}{\log\log a} \qquad (a \geqslant e^e). \tag{3.13}$$

Since $\frac{\log a}{\log\log a}$ is increasing for $a \geqslant e^e$, it suffices to show this for the *least square-free number* $a \geqslant e^e$ and $r := \omega(a) > 2$. Denoting the rth prime by p_r we see from the prime number theorem that $\psi(p_r) \sim p_r \sim r \log r$ (cf., for instance, Ingham [9, Theorem 13]). Thus

$$\log a = \sum_{p|a} \log p \sim r \log r \qquad (r \to \infty),$$

and (3.13) follows by inserting $\log r \sim \log\log a$.

For $x \leqslant a$ we see from (3.12) and Remark 3.2 that

$$\begin{aligned}\sum_{n \leqslant x} \frac{\mu(n)}{\varphi(n)}\, c_n(a) &= \sum_{\substack{m|a \\ m \leqslant x}} \mu(m)\, \frac{m}{\varphi(m)} \sum_{\substack{d \leqslant \frac{x}{m} \\ (d,m)=1}} \frac{\mu^2(d)}{\varphi(d)} \\ &\ll \log a \sum_{m|a} \mu^2(m)\, \frac{m}{\varphi(m)} = \log a \prod_{p|a} \Big(2 + \frac{1}{p-1}\Big) \\ &\leqslant 3\, e^{\omega(a)} \log a \ll e^{\omega(a)} \log a.\end{aligned}$$

For $x \geqslant a > 1$ we similarly obtain from (3.11) that

$$\sum_{n \leqslant x} \frac{\mu(n)}{\varphi(n)}\, c_n(a) \ll \log a + 2^{\omega(a)} \prod_{p|a} \Big(1 - \frac{1}{\sqrt{p}}\Big)^{-1} \ll e^{\omega(a)} \log a,$$

which completes the proof. ■

We remark that the factor $\log a$ in Lemma 3.5 cannot be omitted for numbers $a \in \mathbb{N}$ composed of few primes only. But the above proof shows that the factor $e^{\omega(a)}$ can be replaced with $\theta^{\omega(a)}$ for any fixed $\theta > 2$.

4. Boundary behavior of *F*

Let $q \geqslant 1$ and recall that the Hardy space H^q consists of all holomorphic functions $g\colon U \to \mathbb{C}$ satisfying

$$\|g\|_{H^q} := \Big(\sup_{0<r<1} \int_0^1 |g(r\, e^{2\pi i t})|^q \, dt\Big)^{\frac{1}{q}} < \infty.$$

The expression $\|g\|_{H^q}$ defines a norm on H^q. It is well-known that the integral is an increasing function of r (cf. Rudin [19, Theorem 17.5]) so that the supremum can be replaced by the limit $r \to 1-$. The space H^2 is of particular interest for our study, as the H^2-norm can be calculated directly from the coefficients a_n of the Taylor expansion of $g \in H^2$ at $w = 0$, namely

$$\|g\|^2_{H^2} = \sum_{n=0}^{\infty} |a_n|^2.$$

In order to study the function F defined in (1.2) and its $L^2(0,1)$-boundary function f in (1.3) we note that (3.11) for $a > 1$ and $x \to \infty$ takes the form of a pointwise convergent Ramanujan series of $\frac{\varphi(a)}{a}\,\Lambda(a)$ given by

$$\frac{\varphi(a)}{a}\,\Lambda(a) = \sum_{n=1}^{\infty} \frac{\mu(n)}{\varphi(n)}\, c_n(a) \qquad (1 < a \in \mathbb{N}). \tag{4.1}$$

This formula goes back to Hardy [5, Equation 7.3] with a proof based on generating series. Inserting $\Lambda(a) = \Lambda(a)\frac{\varphi(a)}{a} + \Lambda(a)\bigl(1 - \frac{\varphi(a)}{a}\bigr)$ into the power series representation of $F(w)$ for $w \in U$ and replacing the term $\frac{\varphi(a)}{a}\,\Lambda(a)$ with (4.1) yields

$$F(w) = F_1(w) + F_0(w)\,,$$

where

$$F_1(w) := \sum_{a=2}^{\infty} \frac{w^a}{a} \sum_{n=1}^{\infty} \frac{\mu(n)}{\varphi(n)}\, c_n(a)\,, \qquad F_0(w) := \sum_{p^k > 1} \frac{\log p}{p^{k+1}}\, w^{p^k}. \tag{4.2}$$

By Lemma 3.5, $F_1(w)$ converges compactly for $w \in U$ and defines a function $F_1 \in H^2$ on U, while $F_0(w)$ converges uniformly for $w \in \overline{U}$ and represents a bounded holomorphic function $F_0 \in H^\infty$ on U which is continuous on ∂U.

Using the definition of the Ramanujan sums $c_n(a) \in \mathbb{R}$ we may rewrite $F_1(w)$ as

$$F_1(w) = \sum_{a=2}^{\infty} \frac{w^a}{a} \sum_{n=1}^{\infty} \frac{\mu(n)}{\varphi(n)} \sum_{\substack{1 \leqslant k \leqslant n \\ (k,n)=1}} e^{-2\pi i a \frac{k}{n}}$$

and introduce the partial sums

$$\begin{aligned} S_N(w) &:= \sum_{a=2}^{\infty} \frac{w^a}{a} \sum_{n \leqslant N} \frac{\mu(n)}{\varphi(n)} \sum_{\substack{1 \leqslant k \leqslant n \\ (k,n)=1}} e^{-2\pi i a \frac{k}{n}} \\ &= \sum_{n \leqslant N} \frac{\mu(n)}{\varphi(n)} \sum_{\substack{1 \leqslant k \leqslant n \\ (k,n)=1}} \sum_{a=2}^{\infty} \frac{(w\, e^{-2\pi i \frac{k}{n}})^a}{a}\,. \end{aligned}$$

The power series representation of $\log(1 - z)$ for $1 \neq z \in \overline{U}$ gives

$$S_N(w) = -\sum_{n \leqslant N} \frac{\mu(n)}{\varphi(n)} \sum_{\substack{1 \leqslant k \leqslant n \\ (k,n)=1}} \mathcal{L}(w\, e^{-2\pi i \frac{k}{n}}) \qquad (w \neq e^{2\pi i t} \text{ with } t \in \mathbb{Q}), \tag{4.3}$$

where $\mathcal{L}(z) = \log(1 - z) + z$. In order to study $S_N(w)$ for $w = e^{2\pi i t} \in \overline{U}$ we use the 1-periodic sawtooth function $\beta\colon \mathbb{R} \to \mathbb{R}$ with $\beta(t) = t - \frac{1}{2}$ for $0 < t < 1$ and $\beta(0) = 0$. For $t \in \mathbb{R} \setminus \mathbb{Z}$ the Fourier expansion of β is given by

$$\beta(t) = \frac{1}{\pi}\, \operatorname{Im} \log(1 - e^{2\pi i t}) = -\frac{1}{\pi} \sum_{a=1}^{\infty} \frac{\sin(2\pi a t)}{a}\,.$$

Let $\mathcal{B}(t) = \beta(t) + \frac{1}{\pi}\sin(2\pi t)$ and $s_N(t) = \operatorname{Im} S_N(e^{2\pi i t})$. Then (4.3) yields

$$s_N(t) = -\pi \sum_{n\leqslant N} \frac{\mu(n)}{\varphi(n)} \sum_{\substack{1\leqslant k\leqslant n\\(k,n)=1}} \mathcal{B}(t-\tfrac{k}{n}) \qquad (t\in\mathbb{R}\setminus\mathbb{Q}).$$

Theorem 4.1. *For $1\leqslant q\leqslant 2$ the functions $F\in H^q$ and $\operatorname{Im} f\in L^q(0,1)$ are represented*

a) *for $w\in U$ by the H^q-convergent series*

$$F(w) = -\sum_{n=1}^{\infty} \frac{\mu(n)}{\varphi(n)} \sum_{\substack{k=1\\(k,n)=1}}^{n} \mathcal{L}(we^{-2\pi i\frac{k}{n}}) + \sum_{p^k>1} \frac{\log p}{p^{k+1}}\, w^{p^k}, \tag{4.4}$$

b) *for $t\in\mathbb{R}$ by the $L^q(0,1)$-convergent series*

$$\operatorname{Im} f(t) = -\pi \sum_{n=1}^{\infty} \frac{\mu(n)}{\varphi(n)} \sum_{\substack{k=1\\(k,n)=1}}^{n} \mathcal{B}\big(t-\tfrac{k}{n}\big) + \sum_{p^k>1} \frac{\log p}{p^{k+1}} \sin(2\pi p^k t). \tag{4.5}$$

Proof. Since $H^2\subset H^q$ and $L^2\subset L^q$ for $1\leqslant q<2$ by Hölder's inequality applied to the corresponding norms, it suffices to prove the assertions for $q=2$. Further any $g\in H^2$ leads to an L^2-limit function $g_*\colon (0,1)\to\mathbb{C}$ defined by

$$\lim_{r\to 1-} \|g_* - g(re^{2\pi i\,\bullet})\|_2 = 0.$$

If $g(0)=0$, then g can be reconstructed from $\operatorname{Re} g_*$ or $\operatorname{Im} g_*$ via

$$g(w) = \int_0^1 \operatorname{Re} g_*(t)\, \frac{e^{2\pi i t}+w}{e^{2\pi i t}-w}\, dt = i\int_0^1 \operatorname{Im} g_*(t)\, \frac{e^{2\pi i t}+w}{e^{2\pi i t}-w}\, dt \qquad (w\in U).$$

The norms are connected by $\|g\|_{H^2} = \|g_*\|_2 = \sqrt{2}\,\|\operatorname{Re} g_*\|_2 = \sqrt{2}\,\|\operatorname{Im} g_*\|_2$ (cf., for instance, the monographs of Koosis [12, Chapter I, Theorem E 4; Chapter V, Theorem A] and, especially for the a.e. existence of radial limits, Hoffman [8, Chapter 3, second Corollary from Fatou's Theorem]). This reduces the proof to verifying b) for $q=2$, since $F, F_1, F_0\in H^2$ with $F(0)=F_1(0)=F_0(0)=0$.

Let $f_0(t) = \operatorname{Im} F_0(e^{2\pi i t})$ for $t\in\mathbb{R}$. Then $f_0, s_N\in L^2(0,1)$, and we have to show that $\|s_N - (\operatorname{Im} f - f_0)\|_2$ tends to zero as $N\to\infty$. In fact, we prove for $0<\delta<1$ that

$$\|s_N - (\operatorname{Im} f - f_0)\|_2^2 = \frac{1}{2}\sum_{a=2}^{\infty} \frac{1}{a^2}\Bigg(\sum_{n=N+1}^{\infty} \frac{\mu(n)}{\varphi(n)}\, c_n(a)\Bigg)^2 \ll \frac{1}{N^{1-\delta}}\,. \tag{4.6}$$

With $\lambda_a = e^{\omega(a)}\log a$ Lemma 3.5 implies that

$$\Bigg|\sum_{n=N+1}^{\infty} \frac{\mu(n)}{\varphi(n)}\, c_n(a)\Bigg| \ll \lambda_a$$

holds uniformly for $N \geqslant 1$ and $a \geqslant 2$, and (3.13) gives

$$\sum_{a=N+1}^{\infty} \frac{\lambda_a^2}{a^2} \ll \sum_{a \geqslant N+1} \frac{1}{a^{2-\delta}} \leqslant \frac{1}{N^{1-\delta}} .$$

From these estimates combined with Theorem 3.3 we obtain

$$\begin{aligned}\sum_{a=2}^{\infty} \frac{1}{a^2} \Big(\sum_{n=N+1}^{\infty} \frac{\mu(n)}{\varphi(n)} c_n(a) \Big)^2 &\ll \sum_{a=2}^{N} \frac{1}{a^2} \Big(\sum_{n=N+1}^{\infty} \frac{\mu(n)}{\varphi(n)} c_n(a) \Big)^2 + \frac{1}{N^{1-\delta}} \\ &\ll \frac{1}{N^{1-\delta}} ,\end{aligned}$$

which gives the desired conclusion (4.6). ■

We add the following remarks.

Remark 4.2. H^2-convergence implies compact convergence of both series on the right-hand side of (4.4), which is generally valid for a series of H^2-convergent functions on U.

Remark 4.3. The proof of Theorem 4.1 shows that the terms $\mathcal{L}(w\, e^{-2\pi i \frac{k}{n}})$ and $\mathcal{B}(t - \frac{k}{n})$ in the representation formulas (4.4) and (4.5) can be replaced with $\mathcal{L}(w\, e^{2\pi i \frac{k}{n}})$ and $\mathcal{B}(t + \frac{k}{n})$, respectively.

Remark 4.4. Recall the H^2-functions F_0 in (4.2), S_N in (4.3) and their $L^2(0,1)$-boundary functions $f_0 = \operatorname{Im} F_0(e^{2\pi i \bullet})$, $s_N = \operatorname{Im} S_N(e^{2\pi i \bullet})$. The proof of Theorem 4.1 affords quantitative remainder term estimates, namely for every $\delta > 0$,

$$\begin{aligned}\|S_N - (F - F_0)\|_{H^2} &\ll_\delta N^{-\frac{1}{2}(1-\delta)}, \\ \|s_N - (\operatorname{Im} f - f_0)\|_2 &\ll_\delta N^{-\frac{1}{2}(1-\delta)}.\end{aligned}$$

Theorem 4.1 refines the representation

$$F(w) = \sum_{n=1}^{\infty} \mu(n) \frac{\log n}{n} \log(1 - w^n) \qquad (w \in U)$$

obtained from inserting $\Lambda = -1 * (\mu \log)$ into the compactly convergent series (1.2), and predicts the presence of logarithmic singularities of $f\big(\frac{a}{q}\big)$ with $(a,q) = 1$ and q square-free. In fact, the partial sums of the trigonometric series of f and $\operatorname{Re} f$ diverge at $t = \frac{a}{q}$ because of

Proposition 4.5. *Let $a, q \in \mathbb{N}$ be coprime and $x \geqslant 1$. Then*

$$\sum_{n \leqslant x} \frac{\Lambda(n)}{n} e^{2\pi i \frac{a}{q} n} = \frac{\mu(q)}{\varphi(q)} \log x + \mathrm{O}_q(1).$$

Proof. For $q \in \mathbb{N}$ there are $\varphi(q)$ characters χ mod q. Let, in particular, χ_0 denote the principal character $\bmod q$. Then

$$\sum_{n \leqslant x} \frac{\chi(n)\,\Lambda(n)}{n} = \begin{cases} \log x + \mathrm{O}_q(1) & \text{for } \chi = \chi_0, \\ \mathrm{O}_q(1) & \text{for } \chi \neq \chi_0. \end{cases}$$

For $b, q \in \mathbb{N}$ coprime the orthogonality relations for characters give

$$\sum_{\substack{n \leqslant x \\ n \equiv b \bmod q}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(b) \sum_{n \leqslant x} \frac{\chi(n)\,\Lambda(n)}{n},$$

while for $(b, q) > 1$ the left-hand sum is $\ll_q 1$. Insertion yields the quantitative version

$$\sum_{\substack{n \leqslant x \\ n \equiv b \bmod q}} \frac{\Lambda(n)}{n} = \begin{cases} \dfrac{\log x}{\varphi(q)} + \mathrm{O}_q(1) & \text{for } (b, q) = 1, \\ \mathrm{O}_q(1) & \text{for } (b, q) > 1 \end{cases}$$

of Dirichlet's theorem on primes in arithmetic progressions. For $a, q \in \mathbb{N}$ coprime we obtain

$$\sum_{n \leqslant x} \frac{\Lambda(n)}{n}\, e^{2\pi i\,\frac{a}{q}\,n} = \sum_{b=1}^{q} e^{2\pi i\,\frac{a}{q}\,b} \sum_{\substack{n \leqslant x \\ n \equiv b \bmod q}} \frac{\Lambda(n)}{n} = \frac{\log x}{\varphi(q)} \sum_{\substack{1 \leqslant b \leqslant q \\ (b,q)=1}} e^{2\pi i\,\frac{b}{q}\,a} + \mathrm{O}_q(1).$$

The last sum equals $c_q(a) = \mu(q)$ by (3.8) or (3.9), which completes the proof. ∎

Remark 4.6. Proposition 4.5 implies that F is singular at all points $e^{2\pi i \frac{a}{q}}$ with $\mu(q) \neq 0$. For a simple proof via an abelian argument cf., for instance, [15, Theorem 1, Corollary 1, and Example 1].[4] Hence ∂U is the natural boundary of F.

Since the remainder term $\mathrm{O}_q(1)$ in Proposition 4.5 might depend on q, Proposition 4.5 does not suffice to derive the uniform boundedness of $\operatorname{Im} f(t)$ for rational $t \in (0, 1)$.

Recently Conrey and Myerson [2, Theorem 2] have shown that $\operatorname{Im} f$ is indeed bounded and that the sine series for $\operatorname{Im} f$ converges on $\mathbb{R}$. In particular, their result entails $\operatorname{Im} f \in L^\infty(0, 1)$, which implies $\operatorname{Im} f \in L^q(0, 1)$ for $2 < q < \infty$. Therefore $f \in L^q(0, 1)$ by the L^q-mapping property of the Hilbert transform in

$$\operatorname{Re} f(t) = \lim_{\delta \to 0+} \int_{\delta}^{\frac{1}{2}} \big(\operatorname{Im} f(t+u) - \operatorname{Im} f(t-u)\big) \cot(\pi u)\, du$$

[4] A more general argumentation may be based on Fabry's gap theorem (cf., for instance, Landau and Gaier [14, Anhang II, Section 2, § 2]), or on the Tauberian theorem of Fatou and Riesz (cf., for instance, Remmert [18, Chapter 11]).

(cf. Koosis [12, Chapter V.B]). We conclude that $F \in H^q$ for all $q \geqslant 1$. But $F \notin H^\infty$, since F is unbounded.

Conrey and Myerson [2] have conjectured that $\operatorname{Im} f$ is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and has jumps of altitude $\pi \frac{\mu(n)}{\varphi(n)}$ at each rational number $t = \frac{k}{n}$ with $(k, n) = 1$. This assumption is strongly supported by Theorem 4.1 b), since each term $\mathcal{B}\big(t - \frac{k}{n}\big)$ in (4.5) gives a jump of height -1 at the rational number $t = \frac{k}{n}$. It is an interesting open question whether the right hand side of (4.5) converges uniformly for $t \in \mathbb{R}$, which would positively answer the conjecture of Conrey and Myerson and verify the convergence in H^q and $L^q(0,1)$ in Theorem 4.1 for $1 \leqslant q < \infty$ (since $\operatorname{Im} f$ then is a regulated function in the sense of Dieudonné [3, Section 7.6]).

References

[1] G. Andrews, R. Askey and R. Roy, *Special functions,* Cambridge University Press, 2000.

[2] J.B. Conrey and G. Myerson, *On the Balazard-Saias criterion for the Riemann hypothesis,* Preprint, arXiv:math.NT/0002254 v1, 2000.

[3] J. Dieudonné, *Foundations of Modern Analysis, Vol. 1,* 2nd ed., Academic Press, New York, 1969.

[4] H.M. Edwards, *Riemann's zeta function,* Dover Publications, Mineola, New York, 2001.

[5] G.H. Hardy, *Note on Ramanujan's trigonometrical function* $c_q(n)$*, and certain series of arithmetical functions,* Proc. Cambridge Philos. Soc. **20** (1921), 263–271.

[6] G.H. Hardy and J.E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes,* Acta Math. **41** (1916), 119–196.

[7] A. Hildebrand, *Über die punktweise Konvergenz von Ramanujan-Entwicklungen zahlentheoretischer Funktionen,* Acta Arith. **44** (1984), 109–140.

[8] K. Hoffman, *Banach spaces of analytic functions,* Dover Publication, 2007.

[9] A.E. Ingham, *The distribution of prime numbers,* Cambridge University Press, Cambridge, 1990.

[10] A.A. Karatsuba and S.M. Voronin, *The Riemann zeta-function,* Walter de Gruyter, Berlin, New York, 1992.

[11] J. Knopfmacher, *Abstract Analytic Number Theory,* Dover Publ., New York, 1975 (1990).

[12] P. Koosis, *Introduction to* H_p*-spaces,* second edition, Cambridge University Press, Cambridge, 1998.

[13] M. Kunik, *On the formulas of* $\pi(x)$ *and* $\psi(x)$ *of Riemann and von-Mangoldt,* Preprint Nr. 09/2005, Otto-von-Guericke Universität Magdeburg, Fakultät für Mathematik, 2005 (http://www-ian.math.uni-magdeburg.de/ kunik/).

[14] E. Landau und D. Gaier, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie,* Springer-Verlag, Berlin, Heidelberg, 1986.

[15] L. Lucht, *Power series with multiplicative coefficients,* Math. Z. **177** (1981), 359–374.

[16] L.G. Lucht, *A survey of Ramanujan expansions,* Int. J. Number Theory **6** (2010), 1785–1799.

[17] L. Lucht und D. Wolke, *Über trigonometrische Reihen mit zahlentheoretischen Koeffizienten,* Archiv Math. **57** (1991), 571–580.

[18] R. Remmert, *Funktionentheorie 2*, Springer, Berlin, Heidelberg, New York, 1995.

[19] W. Rudin, *Real and Complex Analysis,* McGraw-Hill, London, 1970.

[20] W. Schwarz and J. Spilker, *Arithmetical Functions,* Lecture Note Series 184, Cambridge Univ. Press, Cambridge, 1994.

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