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MULTIPLE POLYLOGARITHMS AND MULTI-POLY-BERNOULLI POLYNOMIALS

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Abstract: In this paper we introduce special generalized Bernoulli polynomials which generalize poly-Bernoulli polynomials and numbers. We call them *multi-poly-Bernoulli polynomials* and *numbers*. We prove a collection of important and fundamental identities satisfied by our multi-poly-Bernoulli polynomials and numbers.

Keywords: multiple polylogarithms, zeta function, multi-poly-Bernoulli numbers and polynomials.

1. Introduction and known results

Let us briefly review poly-Bernoulli polynomials. For details, we refer to [2], [7]. For an integer $k \in \mathbb{Z}$, put

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

which is the k-th polylogarithm if $k \geqslant 1$, and a rational function if $k \leqslant 0$. One knows that $\mathrm{Li}_1(z) = -\log(1-z)$. The formal power series $\mathrm{Li}_k(z)$ can be used to define poly-Bernoulli polynomials. The polynomials $B_n^{(k)}(x)$ $(n = 0, 1, 2, \ldots)$ are said to be *poly-Bernoulli polynomials* if they satisfy

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}.$$

For any $n \geqslant 0$, we have

$$(-1)^n B_n^{(1)}(-x) = B_n(x),$$

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2010 Mathematics Subject Classification: primary: 11B68; secondary: 11G55, 11M32 the classical Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

We proved in [3] the following formulae.

Theorem 1.1 (Explicit formula). *For $k\in\mathbb{Z}$ and $n\geqslant 0$,*

$$B_n^{(k)}(x)=\sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(x-j)^n. \tag{1}$$

Theorem 1.2 (Recurrence formula 1). *For $k\in\mathbb{Z}$ and $n\geqslant 2$,*

$$B_n^{(k)}(x)=\frac{1}{n+1}\left\{B_n^{(k-1)}(x)+xB_0^{(k)}(x)-\sum_{m=1}^{n-1}\left(\binom{n}{m-1}-\binom{n}{m}x\right)B_m^{(k)}(x)\right\}, \tag{2}$$

and

$$\begin{aligned}B_0^{(k)}(x)&=1,\\ B_1^{(k)}(x)&=\frac{1}{2}\left(B_1^{(k-1)}(x)+xB_0^{(k)}(x)\right).\end{aligned}$$

Theorem 1.3 (Recurrence formula 2). *For all $k\geqslant 0$, $n\geqslant 0$,*

$$B_n^{(k)}(x)=\sum_{m=0}^{n}(-1)^m\binom{n}{m}B_{n-m}^{(k-1)}\sum_{l=0}^{m}\frac{(-1)^l}{n-l+1}\binom{m}{l}B_l(x).$$

Theorem 1.4 (Appell sequence). *For $k\in\mathbb{Z}$, $n\geqslant 0$,*

$$\frac{d}{dx}B_{n+1}^{(k)}(x)=(n+1)B_n^{(k)}(x). \tag{3}$$

Theorem 1.5 (Addition formula). *For $k\in\mathbb{Z}$, $n\geqslant 0$,*

$$B_n^{(k)}(x+y)=\sum_{m=0}^{n}\binom{n}{m}B_m^{(k)}(x)y^{n-m}. \tag{4}$$

For $m,n\geqslant 0$, set

$$C_n^{(-m)}(x,y)=\sum_{k=0}^{m}\binom{m}{k}B_n^{(-k)}(x)y^{m-k}.$$

Then we have the following result:

Theorem 1.6 (Symmetric formula).

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_m^{(-n)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} \tag{5}$$

$$= \frac{e^{xt+yu+t+u}}{e^t+e^y-e^{t+u}}. \tag{6}$$

Theorem 1.7 (Duality). *For $m, n \geqslant 0$, we have*

$$C_n^{(-m)}(x,y) = C_m^{(-n)}(y,x). \tag{7}$$

Theorem 1.8 (Inversion formula). *For $m, n \geqslant 0$,*

$$B_n^{(-m)}(x) = \sum_{k=0}^{m} C_n^{(-k)}(x,y)(-y)^{m-k}. \tag{8}$$

Theorem 1.9 (Closed formula). *For $m, n \geqslant 0$,*

$$C_n^{(-m)}(x,y) = \sum_{j=0}^{\infty}(j!)^2\left(\sum_{a=0}^{n}(x+1)^{n-a}\binom{n}{a}\left\{ {a \atop j} \right\}\right) \times\left(\sum_{b=0}^{m}(y+1)^{m-b}\binom{m}{b}\left\{ {b \atop j} \right\}\right), \tag{9}$$

where $\left\{ {n \atop m} \right\}$ are the Stirling numbers of the second kind.

We now introduce certain zeta functions in terms of the Laplace-Mellin integral. Let $k \in \mathbb{Z}$. Define

$$Z_k(s,x) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-xt}t^{s-1}dt.$$

It is defined for $\mathrm{Re}(s) > 0$ and $x > 0$ if $k \geqslant 1$, and for $\mathrm{Re}(s) > 0$ and $x > |k|+1$ if $k \leqslant 0$. In particular, $Z_k(s,1)$ is called the *Arakawa-Kaneko zeta function* defined in [1].

The next result was independently proved by Coppo-Candelpergher [4] and the authors.

Theorem 1.10 (Interpolation formula). *The function $s \mapsto Z_k(s,x)$ is analytically continued to an entire function on the complex s-plane and for $n \geqslant 0$ and $x > 0$,*

$$Z_k(-n,x) = (-1)^n B_n^{(k)}(-x) \tag{10}$$

is satisfied. In addition, this zeta function can be rewritten as follows: For $k \in \mathbb{Z}$, we have

$$Z_k(s,x) = \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{1}{(x+j)^s}. \tag{11}$$

We investigated in [5], [6] generalized poly-Bernoulli numbers, which are called *multi-poly-Bernoulli numbers*, according to a suggestion mentioned in [1], [8]. In this paper, we introduce generalized poly-Bernoulli polynomials, which are called *multi-poly-Bernoulli polynomials* in our paper. The constant terms of these polynomials are *multi-poly-Bernoulli numbers.* We prove a collection of formulae which generalize the above ones. Specializing $x = 0$, our some results are reduced to the results about multi-poly-Bernoulli numbers proved in [5], [6].

2. Multi-poly-Bernoulli polynomials and numbers

Firstly, we recall a generalization of $\mathrm{Li}_k(z)$. For $k_1, \dots, k_r \in \mathbb{Z}$, define the *multiple polylogarithm* by

$$\mathrm{Li}_{k_1,\dots,k_r}(z) = \sum_{\substack{m_1,\dots,m_r \in \mathbb{Z} \\ 0<m_1<\cdots<m_r}} \frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}.$$

The following result will be used in the next section.

Lemma 2.1.

$$\frac{d}{dz}\mathrm{Li}_{k_1,\dots,k_r}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_{k_1,\dots,k_{r-1},k_r-1}(z) & (k_r > 1) \\ \frac{1}{1-z}\mathrm{Li}_{k_1,\dots,k_{r-1}}(z) & (k_r = 1) \end{cases}.$$

Next, using multiple polylogarithms, let us introduce a generalization of poly-Bernoulli polynomials.

Definition 2.2. The multi-poly-Bernoulli polynomials $B_n^{(k_1,\dots,k_r)}(x)$, *where* $n = 0, 1, 2, \dots,$ *are defined for each integer* $k_1, \dots, k_r$ *by the generating series*

$$\frac{Li_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}. \tag{12}$$

We call $B_n^{(k_1,\dots,k_r)} := B_n^{(k_1,\dots,k_r)}(0)$ $(n = 0, 1, 2, \dots)$ *multi-poly-Bernoulli numbers*, which were investigated in [5] and [6]. The multi-poly-Bernoulli numbers satisfy the following recurrence formulae ([6]).

Theorem 2.3 (Recurrence formula 1).

(1) *If* $k_r > 1$ *and* $n \geqslant 2$, *then*

$$B_n^{(k_1,\dots,k_r)} = \frac{1}{n+r}\left[B_n^{(k_1,\dots,k_r-1)}(x) - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{(k_1,\dots,k_r)}\right].$$

(2) *If* $k_r = 1$ *and* $n \geqslant 2$, *then*

$$B_n^{(k_1,\dots,k_{r-1},1)} = \frac{1}{n+r}\left[B_n^{(k_1,\dots,k_{r-1})} - \sum_{m=0}^{n-1}(-1)^{n-m}\left\{r\binom{n}{m}+\binom{n}{m-1}\right\}B_m^{(k_1,\dots,k_{r-1},1)}\right].$$

Theorem 2.4 (Recurrence formula 2).

(1) *If $k_r > 1$, then for any $n \geqslant 0$,*

$$B_n^{(k_1,\dots,k_r)} = (-1)^{r-1}\frac{n!}{(n+r-1)!}$$
$$\times \sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_{r-1}!}\right.$$
$$\left.\times\binom{n-m+r-1}{p}B_p^{(k_1,\dots,k_{r-1},k_r-1)}\right\}$$
$$\times(-1)^m\binom{n+r-1}{m}\sum_{l=0}^{m}\frac{(-1)^l}{n-l+r}\binom{m}{l}\sum_{\substack{i_1+\cdots+i_r\\=l}}B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{l!}{i_1!\cdots i_r!}.$$

(2) *If $k_r = 1$, then for any $n \geqslant 0$,*

$$B_n^{(k_1,\dots,k_r)} = (-1)^{r-1}\frac{n!}{(n+r-1)!}$$
$$\times \sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_r!}\right.$$
$$\left.\times\binom{n-m+r-1}{p}B_p^{(k_1,\dots,k_{r-1})}\right\}$$
$$\times\binom{n+r-1}{m}\frac{1}{n-m+r}\sum_{\substack{i_1+\cdots i_r\\=m}}B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{m!}{i_1!\cdots i_r!}.$$

Remark 2.5. If $k_1 = \cdots = k_r = 1$, then the above defining equation becomes

$$\frac{1}{r!}\left(\frac{-t}{e^{-t}-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(1,\dots,1)}(x)\frac{t^n}{n!},$$

which gives the definition of higher order Bernoulli polynomials.

3. Some formulae

In this section, we will prove some fundamental identities which generalize ones mentioned in the previous sections.

Theorem 3.1 (Explicit formula).

$$B_n^{(k_1,\dots,k_r)}(x) = \sum_{0<m_1<\cdots<m_r\leqslant n+r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}(x-j)^n. \quad (13)$$

Proof. Since

$$\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{0<m_1<\cdots<m_r} \frac{(1-e^{-t})^{m_r-r}}{m_1^{k_1}\cdots m_r^{k_r}}$$
$$= \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \sum_{j=0}^{m_r-r} (-1)^j \binom{m_r-r}{j} e^{-jt},$$

we have

$$\frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{xt} = \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \sum_{j=0}^{m_r-r} (-1)^j \binom{m_r-r}{j} e^{(x-j)t}$$
$$= \sum_{0<m_1<\cdots<m_r} \sum_{n=0}^{\infty} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \sum_{j=0}^{m_r-r} (-1)^j \binom{m_r-r}{j} (x-j)^n \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{0<m_1<\cdots<m_r\leqslant n+r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \sum_{j=0}^{m_r-r} (-1)^j \binom{m_r-r}{j} (x-j)^n \frac{t^n}{n!}.$$

This proves the theorem. ∎

Theorem 3.2 (Recurrence formula 1).

(1) *If $k_r>1$ and $n\geqslant 2$, then*

$$B_n^{(k_1,\dots,k_r)}(x) = \frac{1}{n+r}\Bigg[B_n^{(k_1,\dots,k_r-1)}(x) + xB_0^{(k_1,\dots,k_r)}(x) - \sum_{m=1}^{n-1}\left\{\binom{n}{m-1} - x\binom{n}{m}\right\} B_m^{(k_1,\dots,k_r)}(x)\Bigg].$$

(2) *If $k_r=1$ and $n\geqslant 2$, then*

$$B_n^{(k_1,\dots,k_{r-1},1)}(x) = \frac{1}{n+r}\Bigg[B_n^{(k_1,\dots,k_{r-1})}(x) - \sum_{m=0}^{n-1}(-1)^{n-m}\left\{(x+r)\binom{n}{m} + \binom{n}{m-1}\right\} B_m^{(k_1,\dots,k_{r-1},1)}(x)\Bigg].$$

Proof. Put $z=1-e^{-t}$.

(1) The identity (12) leads

$$\mathrm{Li}_{k_1,\dots,k_r}(z)e^{x\mathrm{Li}_1(z)} = z^r\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!}. \tag{14}$$

Differentiation by z, and division by z^{r-1} gives

$$\frac{\mathrm{Li}_{k_1,\dots,k_r-1}(z)}{z^r}e^{x\mathrm{Li}_1(z)} + \frac{xz}{1-z}\cdot\frac{\mathrm{Li}_{k_1,\dots,k_r}(z)}{z^r}e^{x\mathrm{Li}_1(z)}$$
$$= r\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!} + \frac{z}{1-z}\sum_{n=1}^{\infty} B_n^{(k_1,\dots,k_r)}\frac{\mathrm{Li}_1(z)^{n-1}}{(n-1)!}.$$

Using $z/(1-z)=e^t-1$,

$$\begin{aligned}\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r-1)}(x)\frac{t^n}{n!}-x\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}\\ -r\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}B_{n+1}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}\\ =e^t\sum_{n=0}^{\infty}B_{n+1}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}-e^t\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Comparing both sides of the above identity,

$$\begin{aligned}&B_n^{(k_1,\ldots,k_r-1)}(x)-xB_n^{(k_1,\ldots,k_r)}-rB_n^{(k_1,\ldots,k_r)}+B_{n+1}^{(k_1,\ldots,k_r)}\\ &\qquad=\sum_{m=0}^{n}\binom{n}{m}B_{m+1}^{(k_1,\ldots,k_r)}(x)-x\sum_{m=0}^{n}\binom{n}{m}B_m^{(k_1,\ldots,k_r)}(x).\end{aligned}$$

From this, we have

$$\begin{aligned}&B_n^{(k_1,\ldots,k_r-1)}(x)-(n+r)B_n^{(k_1,\ldots,k_r)}(x)\\ &\qquad=\sum_{m=1}^{n-1}\left\{\binom{n}{m-1}-x\binom{n}{m}\right\}B_m^{(k_1,\ldots,k_r)}(x)-xB_0^{(k_1,\ldots,k_r)}(x).\end{aligned}$$

(2) Differentiating (14) by z, and then dividing by z^{r-1}, we obtain

$$\begin{aligned}&\frac{1}{1-z}\cdot\frac{\mathrm{Li}_{k_1,\ldots,k_{r-1}}(z)}{z^{r-1}}e^{x\mathrm{Li}_1(z)}+\frac{xz}{1-z}\cdot\frac{\mathrm{Li}_{k_1,\ldots,k_r}(z)}{z^r}e^{x\mathrm{Li}_1(z)}\\ &\qquad=r\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!}+\frac{z}{1-z}\sum_{n=0}^{\infty}B_{n+1}^{(k_1,\ldots k_r)}(x)\frac{\mathrm{Li}_1(z)^n}{n!}.\end{aligned}$$

Using $z/(1-z)=e^t-1$ and $1/(1-z)=e^t$,

$$\begin{aligned}&\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_{r-1})}(x)\frac{t^n}{n!}+x\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}-\sum_{n=0}^{\infty}B_{n+1}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}\\ &=e^{-t}\sum_{n=0}^{\infty}xB_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}+e^{-t}\sum_{n=0}^{\infty}rB_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}-e^{-t}\sum_{n=0}^{\infty}rB_{n+1}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Comparing both sides of this identity, we get

$$\begin{aligned}
&B_n^{(k_1,\dots,k_{r-1})}(x) + xB_n^{(k_1,\dots,k_r)}(x) - B_{n+1}^{(k_1,\dots,k_r)}(x)\\
&= \sum_{m=0}^{n}(-1)^{n-m}x\binom{n}{m}B_m^{(k_1,\dots,k_r)}(x) + \sum_{m=0}^{n}(-1)^{n-m}r\binom{n}{m}B_m^{(k_1,\dots,k_r)}(x)\\
&\quad - \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}B_{m+1}^{(k_1,\dots,k_r)}(x)\\
&= xB_n^{(k_1,\dots,k_r)}(x) - B_{n+1}^{(k_1,\dots,k_r)}(x) + (n+r)B_n^{(k_1,\dots,k_r)}(x)\\
&\quad + \sum_{m=0}^{n-1}(-1)^{n-m}\left\{(x+r)\binom{n}{m} + \binom{n}{m-1}\right\}B_m^{(k_1,\dots,k_r)}(x).
\end{aligned}$$

This leads our formula. ■

Theorem 3.3 (Recurrence formula 2).

(1) *If $k_r > 1$, then for any $n \geqslant 0$,*

$$\begin{aligned}
B_n^{(k_1,\dots,k_r)}(rx) &= (-1)^{r-1}\frac{n!}{(n+r-1)!}\\
&\times \sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_{r-1}!}\right.\\
&\qquad\left.\times\binom{n-m+r-1}{p}B_p^{(k_1,\dots,k_{r-1},k_r-1)}\right\}\\
&\times(-1)^m\binom{n+r-1}{m}\\
&\times\sum_{l=0}^{m}\frac{(-1)^l}{n-l+r}\binom{m}{l}\sum_{\substack{i_1+\cdots+i_r\\=l}}B_{i_1}^{(1)}(x)\cdots B_{i_r}^{(1)}(x)\frac{l!}{i_1!\cdots i_r!}.
\end{aligned}$$

(2) *If $k_r = 1$, then for any $n \geqslant 0$,*

$$\begin{aligned}
B_n^{(k_1,\dots,k_r)}(rx) &= (-1)^{r-1}\frac{n!}{(n+r-1)!}\\
&\times \sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_r!}\right.\\
&\qquad\left.\times\binom{n-m+r-1}{p}B_p^{(k_1,\dots,k_r)}\right\}\\
&\times\binom{n+r-1}{m}\frac{1}{n-m+r}\sum_{\substack{i_1+\cdots i_r\\=m}}B_{i_1}^{(1)}(x)\cdots B_{i_r}^{(1)}(x)\frac{m!}{i_1!\cdots i_r!}.
\end{aligned}$$

Proof. (1) By Lemma 2.1, we get

$$\mathrm{Li}_{k_1,\dots k_r}(1-e^{-t}) = \int_0^t \frac{e^{-s}}{1-e^{-s}} \mathrm{Li}_{k_1,\dots k_{r-1},k_r-1}(1-e^{-s})ds.$$

Using this, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(rx)\frac{t^n}{n!} \\
&\quad= \frac{e^{rxt}}{(1-e^{-t})^r}\int_0^t \frac{e^{-s}}{1-e^{-s}}\mathrm{Li}_{k_1,\dots,k_{r-1},k_r-1}(1-e^{-s})ds \\
&\quad= \left(\frac{e^{xt}}{1-e^{-t}}\right)^r \int_0^t e^{-s}(1-e^{-s})^{r-1}\frac{\mathrm{Li}_{k_1,\dots,k_{r-1},k_r-1}(1-e^{-s})}{(1-e^{-s})^r}ds \\
&\quad= \left(\sum_{n=0}^{\infty} B_n^{(1)}(x)\frac{t^{n-1}}{n!}\right)^r \\
&\qquad\times \int_0^t \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\left(-\sum_{n=1}^{\infty}\frac{(-s)^n}{n!}\right)^{r-1}\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_{r-1},k_r-1)}\frac{s^n}{n!}ds.
\end{aligned}$$

Rewriting the last expression as in the proof of Theorem 7 (1) in [6] leads the result.

(2) By Lemma 2.1,

$$\mathrm{Li}_{k_1,\dots,k_{r-1},1}(1-e^{-t}) = \int_0^t \mathrm{Li}_{k_1,\dots,k_{r-1}}(1-e^{-s})ds.$$

From this identity, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_{r-1},1)}(rx)\frac{t^n}{n!} \\
&\quad= \left(\frac{e^{xt}}{1-e^{-t}}\right)^r \int_0^t (1-e^{-s})^{r-1}\frac{\mathrm{Li}_{k_1,\dots,k_{r-1}}(1-e^{-s})}{(1-e^{-s})^{r-1}}ds \\
&\quad= \left(\sum_{n=0}^{\infty} B_n^{(1)}(x)\frac{t^{n-1}}{n!}\right)^r \int_0^t \left(-\sum_{n=1}^{\infty}\frac{(-s)^n}{n!}\right)^{r-1}\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_{r-1})}\frac{s^n}{n!}ds.
\end{aligned}$$

Rewriting this as in the proof of Theorem 7 (2) in [6], we obtain the result. ■

Theorem 3.4 (Appell sequence). *For any $n \geqslant 0$,*

$$\frac{d}{dx}B_{n+1}^{(k_1,\dots,k_r)}(x) = (n+1)B_n^{(k_1,\dots,k_r)}(x). \tag{15}$$

Proof. Differentiation of both sides of (12) by x gives

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}B_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!} &= \frac{t\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{xt}\\ &= \sum_{n=1}^{\infty} nB_{n-1}^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!},\end{aligned}$$

which yields the result. ∎

Theorem 3.5 (Addition formula). *For any $n \geqslant 0$,*

$$B_n^{(k_1,\ldots,k_r)}(x+y) = \sum_{m=0}^{n}\binom{n}{m}B_m^{(k_1,\ldots,k_r)}(x)y^{n-m}. \tag{16}$$

In particular, we have

$$B_n^{(k_1,\ldots,k_r)}(x) = \sum_{m=0}^{n}\binom{n}{m}B_m^{(k_1,\ldots,k_r)}x^{n-m}.$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x+y)\frac{t^n}{n!} &= \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{(x+y)t}\\ &= \left(\sum_{m=0}^{\infty}B_m^{(k_1,\ldots,k_r)}(x)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{y^l t^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_m^{(k_1,\ldots,k_r)}(x)y^{n-m}\right)\frac{t^n}{n!}.\end{aligned}$$

This proves the theorem. ∎

4. Special multi-poly-Bernoulli polynomials

In this section we consider multi-poly-Bernoulli polynomials of special type. For a positive integer r, we define

$$B[r]_n^{(k)}(x) := B_n^{(\overbrace{0,\ldots,0}^{r-1},k)}(x).$$

It is easy to see $B[1]_n^{(k)}(x) = B_n^{(k)}(x)$. Also, we easily verify that $B[r]_n^{(k)}(0) = B[r]_n^{(k)}$, which is defined by

$$B[r]_n^{(k)} := B_n^{(\overbrace{0,\ldots,0}^{r-1},k)}.$$

One sees in [5] that the generating function of these numbers is written as

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B[r]_n^{(-k)} \frac{t^n}{n!}\frac{u^k}{k!} = \left(\frac{e^{t+u}}{e^t + e^u - e^{t+u}}\right)^r. \tag{17}$$

From this, for $n, k \geqslant 0$, we have

$$B[r]_n^{(-k)} = B[r]_k^{(-n)}. \tag{18}$$

For $m, n \geqslant 0$, set

$$C[r]_n^{(-m)}(x,y) = \sum_{k=0}^{m} \binom{m}{k} B[r]_n^{(-k)}(x) y^{m-k}.$$

Then we have

Theorem 4.1 (Symmetric formula).

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_n^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_m^{(-n)}(y,x)\frac{t^n}{n!}\frac{u^m}{m!} \\ &= e^{xt+yu}\left(\frac{e^{t+u}}{e^t + e^u - e^{t+u}}\right)^r.\end{aligned}$$

Proof. By definition of $B[r]_n^{(-k)}(x)$ we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_n^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} B[r]_n^{(-k)}(x) y^l \frac{t^n}{n!}\frac{u^k}{k!}\frac{u^l}{l!} \\ &= e^{yu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B[r]_n^{(-k)}(x)\frac{t^n}{n!}\frac{u^k}{k!} \\ &= e^{yu}\sum_{k=0}^{\infty}\left(e^{xt}\sum_{n=0}^{\infty} B[r]_n^{(-k)}\frac{t^n}{n!}\right)\frac{u^k}{k!} \\ &= e^{xt+yu}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} B[r]_n^{(-k)}\frac{t^n}{n!}\frac{u^k}{k!}.\end{aligned}$$

Using (17), the last identity yields the claim. ∎

As a corollary, we have the following result:

Theorem 4.2 (Duality). *For all $n, m \geqslant 0$,*

$$C[r]_n^{(-m)}(x,y) = C[r]_m^{(-n)}(y,x). \tag{19}$$

In particular, for $x = y = 0$ we obtain (18), the duality property of special multi-poly-Bernoulli numbers. Furthermore, for $r = 1$ we have

$$B_n^{(-m)} = B_m^{(-n)},$$

the duality property of poly-Bernoulli numbers.

Theorem 4.3 (Inversion formula). *For $m, n \geqslant 0$, we have*

$$B[r]_n^{(-m)}(x) = \sum_{k=0}^{m} \binom{m}{k} C[r]_n^{(-k)}(x,y)(-y)^{m-k}. \tag{20}$$

Proof. Putting $j = m - k$ and using Theorem 4.1 gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}C[r]_n^{(-k)}(x,y)(-y)^{m-k}\right)\frac{t^n}{n!}\frac{u^m}{m!} \\
&\qquad = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j+k=m} C[r]_n^{(-k)}(x,y)\frac{(-uy)^j}{j!}\frac{t^n}{n!}\frac{u^k}{k!} \\
&\qquad = e^{-yu}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_n^{(-k)}(x,y)\frac{t^n}{n!}\frac{u^k}{k!} \\
&\qquad = e^{xt}\left(\frac{e^{t+u}}{e^t + e^u - e^{t+u}}\right)^r.
\end{aligned}$$

One sees that right hand side is the generating series of $B[r]_n^{(-m)}(x)$ $(n, m \geqslant 0)$. ■

Theorem 4.4 (Closed formula). *For $m, n \geqslant 0$, we have*

$$\begin{aligned}
C[r]_n^{(-m)}(x,y) = &\sum_{\substack{n=n_1+\cdots n_r \\ n_1,\ldots,n_r\geqslant 0}}\ \sum_{\substack{m=m_1+\cdots m_r \\ m_1,\ldots,m_r\geqslant 0}} \frac{n!m!}{n_1!\cdots n_r!m_1!\cdots m_r!} \\
&\times \sum_{j_1=0}^{\min(n_1,m_1)} \cdots \sum_{j_r=0}^{\min(n_r,m_r)} (j_1!\cdots j_r!)^2 \\
&\times \sum_{i=1}^{r}\left(\sum_{a_i=0}^{n_i}(\frac{x}{r}+1)^{n_i-a_i}\binom{n_i}{a_i}\left\{{a_i \atop j_i}\right\}\right) \\
&\times \left(\sum_{b_i=0}^{m_i}(\frac{y}{r}+1)^{m_i-b_i}\binom{m_i}{b_i}\left\{{b_i \atop j_i}\right\}\right).
\end{aligned}$$

Proof. We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C[r]_n^{(-m)}(x,y)\frac{t^n}{n!}\frac{u^m}{m!} = e^{xt+yu}\left(\frac{e^{t+u}}{e^t+e^u-e^{t+u}}\right)^r$$
$$= \frac{e^{(x+r)t+(y+r)u}}{(1-(e^t-1)(e^u-1))^r}$$
$$= e^{(x+r)t+(y+r)u}\left(\sum_{j=0}^{\infty}(e^t-1)^j(e^u-1)^j\right)^r$$
$$= \left(\sum_{j=0}^{\infty} e^{(x/r+1)t}(e^t-1)^j e^{(y/r+1)u}(e^u-1)^j\right)^r.$$

Here making use of

$$\sum_{n=0}^{\infty}\left\{ \begin{matrix} n \\ k \end{matrix} \right\}\frac{u^n}{n!} = \frac{(e^u-1)^k}{k},$$

the right hand side of the last expression becomes

$$\left[\sum_{j=0}^{\infty}\left(j!\sum_{n=0}^{\infty}\frac{(\frac{x}{r}+1)^n t^n}{n!}\sum_{m=0}^{\infty}\left\{ \begin{matrix} m \\ j \end{matrix} \right\}\frac{t^m}{m!}\right)\right.$$
$$\left.\times\left(j!\sum_{n=0}^{\infty}\frac{(\frac{y}{r}+1)^n u^n}{n!}\sum_{m=0}^{\infty}\left\{ \begin{matrix} m \\ j \end{matrix} \right\}\frac{u^m}{m!}\right)\right]^r$$
$$= \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^n}{n!}\frac{u^m}{m!}\sum_{j=0}^{\min(n,m)}(j!)^2\left(\sum_{a=0}^{n}\left(\frac{x}{r}+1\right)^{n-a}\binom{n}{a}\left\{ \begin{matrix} a \\ j \end{matrix} \right\}\right)\right.$$
$$\left.\times\left(\sum_{b=0}^{m}\left(\frac{y}{r}+1\right)^{m-b}\binom{m}{b}\left\{ \begin{matrix} b \\ j \end{matrix} \right\}\right)\right]^r.$$

This gives the result. ■

5. Generalized Arakawa-Kaneko zeta functions

Let $k_1, k_2, \ldots, k_r \in \mathbb{Z}$. We consider

$$Z_{k_1,k_2,\ldots,k_r}(s,x) := \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{Li}_{k_1,k_2,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r}e^{-xt}t^{s-1}dt,$$

the Laplace-Mellin integral. When all k_i are positive, this function is defined for $\mathrm{Re}(s) > 0$ and $x > 0$. We call it the *generalized Arakawa-Kaneko zeta function.*

Theorem 5.1 (Interpolation formula). *The function $s \mapsto Z_{k_1,\ldots,k_r}(s,x)$ is analytically continued to the whole complex s-plane and for $n \geqslant 0$ and $x > 0$,*

$$Z_{k_1,\ldots,k_r}(-n,x) = (-1)^n B_n^{(k_1,\ldots,k_r)}(-x) \tag{21}$$

is satisfied. Moreover, $Z_{k_1,\ldots,k_r}(s,x)$ is written as follows:

$$\begin{aligned} &Z_{k_1,\ldots,k_r}(s,x) \\ &= \sum_{0\leqslant m_1<\cdots<m_r} \frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}} \sum_{j=0}^{m_r+1-r} (-1)^j \binom{m_r+1-r}{j} \frac{1}{(x+j)^s}. \end{aligned} \tag{22}$$

Proof. We split up $Z_{k_1,\ldots k_r}(s,x)$ as the sum of two integrals:

$$\begin{aligned} Z_{k_1,\ldots,k_r}(s,x) := &\frac{1}{\Gamma(s)} \int_0^1 \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{-xt} t^{s-1} dt \\ &+ \frac{1}{\Gamma(s)} \int_1^\infty \frac{\mathrm{Li}_{k_1,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} e^{-xt} t^{s-1} dt. \end{aligned}$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and $x > 0$ and cancels at negative integers because $1/\Gamma(s)$ so does. If $\mathrm{Re}(s) > 0$, then the first integral is expressed as

$$\frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n^{(k_1,\ldots,k_r)}(-x)}{n!} \cdot \frac{1}{n+s}.$$

From this, for a nonnegative integer n,

$$\begin{aligned} Z_{k_1,\ldots,k_r}(-n,x) &= \left(\lim_{s\to -n} \frac{1}{\Gamma(s)(n+s)} \right) \frac{B_n^{(k_1,\ldots,k_r)}(-x)}{n!} \\ &= (-1)^n B_n^{(k_1,\ldots,k_r)}(-x). \end{aligned}$$

As for the latter part, we calculate

$$\begin{aligned}
Z_{k_1,\dots,k_r}(s,x) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-xt}\sum_{0<m_1<\cdots<m_r}\frac{(1-e^{-t})^{m_r-r}}{m_1^{k_1}\cdots m_r^{k_r}}dt \\
&= \frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\int_0^\infty (1-e^{-t})^{m_r-r}t^{s-1}e^{-xt}dt \\
&= \frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \\
&\quad\times\int_0^\infty\left(\sum_{j=0}^{m_r-r}\binom{m_r-r}{j}(-1)^j e^{-jt}\right)t^{s-1}e^{-xt}dt \\
&= \frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \\
&\quad\times\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}\int_0^\infty t^{s-1}e^{-(x+j)t}dt \\
&= \sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\sum_{j=0}^{m_r-r}(-1)^j\binom{m_r-r}{j}\frac{1}{(x+j)^s} \\
&= \sum_{0\leqslant m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}} \\
&\quad\times\sum_{j=0}^{m_r+1-r}(-1)^j\binom{m_r+1-r}{j}\frac{1}{(x+j)^s}.
\end{aligned}$$

■

Remark 5.2. In the case $x=r$, we have

$$\begin{aligned}
Z_{k_1,\dots,k_r}(s,r) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(e^t-1)^r}t^{s-1}dt \\
&=: \zeta_n(k_1,\dots,k_r;s),
\end{aligned}$$

which are investigated in [9]. The above theorem is an extension of a result in [9].

Theorem 5.3 (Difference formula). *We have*

$$\begin{aligned}
&Z_{k_1,\dots,k_r}(s,x+1)-Z_{k_1,\dots,k_r}(s,x) \\
&= \sum_{0\leqslant m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}\sum_{j=0}^{m_r-r+2}(-1)^{j+1}\binom{m_r-r+2}{j}\frac{1}{(x+j)^s}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\text{L.H.S.} &= -\frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^{r-1}}e^{-xt}t^{s-1}dt \\
&= -\frac{1}{\Gamma(s)}\sum_{0<m_1<\cdots<m_r}\int_0^\infty \frac{(1-e^{-t})^{m_r-r+1}}{m_1^{k_1}\cdots m_r^{k_r}}e^{-xt}t^{s-1}dt \\
&= -\frac{1}{\Gamma(s)}\sum_{0\leqslant m_1<\cdots<m_r}\int_0^\infty \frac{(1-e^{-t})^{m_r-r+2}}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}}e^{-xt}t^{s-1}dt \\
&= \sum_{0\leqslant m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}} \\
&\quad \times \sum_{j=0}^{m_r-r+2}(-1)^{j+1}\binom{m_r-r+2}{j}\frac{1}{\Gamma(s)}\int_0^\infty e^{(x+j)t}t^{s-1}dt,
\end{aligned}$$

which yields the result. ■

Theorem 5.4 (Raabe type formulae).

(1) *If* $s\neq 1$, *then*

$$\begin{aligned}
\int_0^1 Z_{k_1,\dots,k_r}(s,x+w)dw = \frac{1}{s-1}&\sum_{0\leqslant m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}} \\
&\times\sum_{j=0}^{m_r-r+2}(-1)^j\binom{m_r-r+2}{j}\frac{1}{(x+j)^{s-1}}.
\end{aligned}$$

(2) *We have*

$$\begin{aligned}
\int_0^1 B_n^{(k_1,\dots,k_r)}(x-w)dw = \frac{1}{n+1}&\sum_{0\leqslant m_1<\cdots<m_r}\frac{1}{(m_1+1)^{k_1}\cdots(m_r+1)^{k_r}} \\
&\times\sum_{j=0}^{m_r-r+2}(-1)^j\binom{m_r-r+2}{j}(x+j)^{n+1}.
\end{aligned}$$

Proof. (1)

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^{r}}e^{-xt}t^{s-1}\int_0^1 e^{-wt}dwdt \\
&= \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})}{(1-e^{-t})^{r}}e^{-xt}t^{s-2}(1-e^{-t})dt \\
&= \frac{\Gamma(s-1)}{\Gamma(s)}\left(Z_{k_1,\dots,k_r}(s-1,x)-Z_{k_1,\dots,k_r}(s-1,x+1)\right).
\end{aligned}$$

Applying the last theorem to this expression, we get the result.

(2) By $B_n^{(k_1,\dots,k_r)}(x) = (-1)^n Z_{k_1,\dots,k_r}(-n,-x)$, the left hand side becomes

$$(-1)^n \int_0^1 Z_{k_1,\dots,k_r}(-n,-x+w)dw.$$

From the above result (1), we complete the proof. ■

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