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MINIMAL GENUS ONE CURVES

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Abstract: In this paper we consider genus one equations of degree n, namely a (generalised) binary quartic when $n = 2$, a ternary cubic when $n = 3$, and a pair of quaternary quadrics when $n = 4$. A new definition for the minimality of genus one equations of degree n over local fields is introduced. The advantage of this definition is that it does not depend on invariant theory of genus one curves. We prove that this definition coincides with the classical definition of minimality for all $n \leqslant 4$. As an application, we give a new proof for the existence of global minimal genus one equations over number fields of class number 1.

Keywords: Genus one curves, minimal models of curves, genus one equations of degree n.

1. Introduction

The following definitions can be found in [5]. A *genus one equation of degree n* over a Dedeking domain R is defined as follows:

A genus one equation of degree $n = 1$ is a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad a_i \in R. \tag{1}$$

Two genus one equations of degree 1 with coefficients in R are *R-equivalent* if they are related by the substitutions: $x' = u^2x + r$ and $y' = u^3y + su^2x + t$, where $r, s, t \in R,\ u \in R^*$. We set $\det([u; r, s, t]) = u^{-1}$.

A genus one equation of degree $n = 2$ is a (generalised) binary quartic

$$y^2 + (\alpha_0x^2 + \alpha_1xz + \alpha_2z^2)y = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4. \tag{2}$$

Two genus one equations of degree 2 with coefficients in R are *R-equivalent* if they are related by the substitutions: $x' = m_{11}x + m_{21}z,\ z' = m_{12}x + m_{22}z$ and $y' = \mu^{-1}y + r_0x^2 + r_1xz + r_2z^2$, where $\mu \in R^*,\ r_i \in R$ and $M = (m_{ij}) \in \mathrm{GL}_2(R)$. We set $\det([\mu, (r_i), M]) = \mu \det M$.

A genus one equation of degree $n = 3$ is a ternary cubic

$$\begin{aligned} F(x, y, z) = ax^3 + by^3 + cz^3 + a_2x^2y + a_3x^2z \\ + b_1y^2x + b_3y^2z + c_1z^2x + c_2z^2y + mxyz = 0. \end{aligned} \tag{3}$$

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Two genus one equations of degree 3 with coefficients in R are *R-equivalent* if they are related by multiplying by $\mu \in R^*$, then substituting $x' = m_{11}x + m_{21}y + m_{31}z$, $y' = m_{12}x + m_{22}y + m_{32}z$ and $z' = m_{13}x + m_{23}y + m_{33}z$, where $M = (m_{ij}) \in \mathrm{GL}_3(R)$. Set $\det([\mu, M]) = \mu \det M$.

A genus one equation of degree $n \geqslant 4$ is given by $l = n(n-3)/2$ quadratic forms in n variables. Two genus one equations of degree n with coefficients in R are *R-equivalent* if they are related by the substitutions: $F_i' = m_{i1}F_1 + m_{i2}F_2 + \ldots + m_{il}F_l$, $i = 1, \ldots, l$, for $M = (m_{ij}) \in \mathrm{GL}_l(R)$, and then $x_j' = \sum_{i=1}^n n_{ij}x_i$ for $N = (n_{ij}) \in \mathrm{GL}_n(R)$. When $n = 4$, set $\det([M, N]) = \det M \det N$.

From now on we work throughout over a Henselian discrete valuation field K with ring of integers $\mathcal{O}_K$. We assume moreover that $\mathrm{char}(K) \neq 2, 3$. We fix a uniformiser t, a normalised valuation ν, and write $k = \mathcal{O}_K/t\mathcal{O}_K$. Set $S = \operatorname{Spec} \mathcal{O}_K$.

Let C be a smooth genus one curve over K. Let D be a K-rational divisor on C of degree n. If $n = 1$, then $C(K) \neq \emptyset$. If $n \geqslant 2$, then the divisor class $[D]$ defines a morphism $C \to \mathbb{P}_K^{n-1}$. In fact, the pair $(C, [D])$ is described by a genus one equation of degree n over K.

For $n \leqslant 4$, we associate invariants $c_{4,\phi}$, $c_{6,\phi}$ and the *discriminant* Δ_ϕ to a genus one equation ϕ of degree n such that $\Delta_\phi = (c_{4,\phi}^3 - c_{6,\phi}^2)/1728$. Moreover, ϕ defines a smooth curve of genus one if and only if $\Delta_\phi \neq 0$. The invariants $c_{4,\phi}, c_{6,\phi}$ and Δ_ϕ are of weights $r = 4, 6$ and 12 respectively. In other words, if $F \in \{c_{4,\phi}, c_{6,\phi}, \Delta_\phi\}$, then $F \circ g = (\det g)^r F$ for every K-equivalence g. These invariants have been known since the nineteenth century, and can be found in [1]. We scale these invariants according to [10].

Definition 1.1. *A genus one equation ϕ of degree n defined over the discrete valuation field K, $n \leqslant 4$, with $\Delta_\phi \neq 0$ is*

(a) integral *if the defining polynomials have coefficients in $\mathcal{O}_K$.*
(b) minimal *if it is integral and $\nu(\Delta_\phi)$ is minimal among all the valuations of the discriminants of the integral genus one equations K-equivalent to ϕ.*

Genus one equations of degree n appear when we try to describe elements in the n-Selmer group of an elliptic curve. Producing integral genus one equations with small coefficients representing elements in n-Selmer groups has been a target for investigations. In order to obtain such genus one equations, we need to *reduce* and *minimize* them. By reducing genus one equations, we mean reducing the size of the coefficients while keeping the invariants unchanged. To minimise genus one equations, we need to make the associated invariants smaller. Reduced and minimal genus one equations of degree 2 appear as an essential part of the 2-descent algorithm described by Birch and Swinnerton-Dyer in [2]. More recent treatment for the minimisation of genus one equations of degree 2, 3 and 4 can be found in [15], [9] and [16] respectively. An algorithmic approach for minimising genus one equations of degree n, $n \leqslant 4$, can be found in [6]. This paper is dedicated to explaining the minimisation of genus one equations of degree n, for $n \leqslant 4$, in a geometric sense.

The main obstacle which holds back the existence of a solution to the minimisation question when n is large is the difficulty of describing the rings of invariants associated to genus one equations. In order to overcome this difficulty, we give an alternative definition for the minimality of genus one equations of degree n.

An integral genus one equation ϕ of degree n defines an S-scheme $\mathcal{C}$. We give criteria for $\mathcal{C}$ to be normal, and hence an S-model for its generic fiber $\mathcal{C}_K$, see Definition 2.1 below. We say that ϕ is *geometrically minimal* if the minimal desingularisation of $\mathcal{C}$ is isomorphic to the minimal proper regular model of $\mathcal{C}_K$. This definition does not involve invariant theory of genus one curves. Therefore, it can be generalised to any $n \geqslant 5$ easily. Furthermore, geometric minimality is not hard to check in practice once we know how to produce desingularisations.

We prove that geometric minimality agrees with the classical definition of minimality, see Definition 1.1, when $n \leqslant 4$. Geometric minimality provides a non-explicit way and hence sometimes a more convenient way of identifying classical minimality.

In [14] we use geometric minimality to count the number of minimal genus one equations of degree n, up to $\mathcal{O}_K$-equivalence, in a given K-equivalence class, for $n \leqslant 4$. These counting results are used in [8] to find bounds for heights of rational points on elliptic curves.

Finally, we give a new proof for the following theorem ([6], Theorem 4.17).

Theorem 1.2. *Let F be a number field of class number* 1 *with ring of integers $\mathcal{O}_F$. Let C be a smooth genus one curve defined over F by a genus one equation ϕ of degree n where $n \leqslant 4$. Assume that $C(F_\nu) \neq \emptyset$ for every completion F_ν of F. Let E be the Jacobian elliptic curve of C with minimal discriminant Δ. Then ϕ is F-equivalent to an $\mathcal{O}_F$-integral genus one equation whose discriminant is Δ.*

2. Models of genus one curves

Recall that K is a discrete valuation field with ring of integers $\mathcal{O}_K$, residue field k, and $S = \operatorname{Spec} \mathcal{O}_K$.

Definition 2.1. *An S-*curve *is an integral, projective, flat, normal S-scheme $f : X \to S$ of dimension 2. The generic fiber of $\mathcal{C}$ will be denoted by $\mathcal{C}_K$ and its special fiber by $\mathcal{C}_k$. We define an S-*model *for a smooth curve C over K to be an S-curve $\mathcal{C}$ such that $\mathcal{C}_K$ is isomorphic to C.*

Definition 2.2. *Let $\mathcal{C}$ be an S-curve. Let $(\Gamma_i)_{i\in I}$ be the family of irreducible components of $\mathcal{C}_k$. For a strict subset $J \subset I$, a* contraction *of the components Γ_j, $j \in J$, in $\mathcal{C}$ consists of an S-morphism $u : \mathcal{C} \to \mathcal{C}^J$ of S-schemes such that*

(a) *For each $j \in J$, the image $u(\Gamma_j)$ consists of a single point $x_j \in \mathcal{C}^J$, and*
(b) *u defines an isomorphism $\mathcal{C} - \bigcup_{j\in J} \Gamma_j \xrightarrow{\sim} \mathcal{C}^J - \bigcup_{j\in J} x_j$.*

Since $\mathcal{O}_K$ is Henselian, the contraction $u : \mathcal{C} \to \mathcal{C}^J$ of the components Γ_j, $j \in J$, exists for any strict subset $J \subset I$. Moreover, the morphism u is unique up to unique isomorphism, see ([12], Theorem 8.3.36 and Proposition 8.3.28). The following theorem, Theorem 1 of ([3], §6.7), describes the contraction morphism explicitly.

Theorem 2.3. *Let $\mathcal{C}$ be an S-curve. Let $(\Gamma_i)_{i\in I}$ be the family of irreducible components of $\mathcal{C}_k$. Let D be a non-trivial effective Cartier divisor on $\mathcal{C}$. Let J be the set of all indices $j \in I$ such that* $\operatorname{Supp}(D) \cap \Gamma_j = \emptyset$. *Then the canonical morphism*

$$u : \mathcal{C} \to \mathcal{C}^J := \operatorname{Proj}(\bigoplus_{m=0}^{\infty} H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(mD)))$$

is the contraction of the components $\Gamma_j, j \in J$, and $\mathcal{C}^J$ is an S-curve.

Let C be a smooth genus one curve over K. Assume that $C(K) \neq \emptyset$. Let E be the Jacobian elliptic curve of C with minimal proper regular model E^{min}. Since $C \cong_K E$, the minimal proper regular model of C is E^{min}.

The S-scheme $\mathcal{C}$ defined by an integral genus one equation $\phi : y^2 + g(x,z)y = f(x,z)$ of degree 2 is the scheme obtained by glueing $\{y^2+g(x,1)y = f(x,1)\} \subset \mathbb{A}^2_S$ and $\{v^2 + g(1,u)v = f(1,u)\} \subset \mathbb{A}^2_S$ via $x = 1/u$ and $y = x^2v$. It comes with a natural morphism $\mathcal{C} \to \mathbb{P}^1_S$ given on these affine pieces by $(x,y) \mapsto (x:1)$ and $(u,v) \mapsto (1:u)$.

The S-scheme defined by an integral genus one equation ϕ of degree n, where $n = 1$ or $n \geqslant 3$, is the subscheme $\mathcal{C} \subset \mathbb{P}^m_S$ defined by ϕ, where $m = 2$ when $n = 1$, and $m = n-1$ when $n \geqslant 3$.

Now we give the key definition of this paper.

Definition 2.4. *Let ϕ be an integral genus one equation of degree n, $n \geqslant 1$, defining a normal S-scheme $\mathcal{C}$. Assume moreover that $\mathcal{C}_K$ is smooth and $\mathcal{C}_K(K) \neq \emptyset$. Let E^{min} be the minimal proper regular model of the Jacobian of $\mathcal{C}_K$. Then ϕ is said to be* geometrically minimal *if the minimal desingularisation $\widetilde{\mathcal{C}} \to \mathcal{C}$ satisfies $\widetilde{\mathcal{C}} \cong E^{min}$.*

For the definitions of minimal proper regular models and minimal desingularisations see ([12], §9.3).

3. Normality

In this section we prove that an S-scheme defined by a minimal genus one equation of degree n, $n \leqslant 4$, is normal, and hence is an S-model for its generic fiber. Let $\mathcal{C}$ be an S-scheme defined by an integral genus one equation of degree n where $\mathcal{C}_K$ is smooth. It is known that the normality of $\mathcal{C}$ implies that there are only finitely many non-regular points on $\mathcal{C}$, and all these points are closed points in the special fiber, see p.8 of [4] for details. Moreover, if $n \leqslant 4$, then $\mathcal{C}$ is a complete intersection. It follows that $\mathcal{C}$ is normal if and only if $\mathcal{C}$ is regular at the generic points of $\mathcal{C}_k$. The latter statement is a direct consequence of Serre's criterion for normality, see ([12], Corollary 8.2.24). If $\mathcal{C}_k$ is reduced, then $\mathcal{C}$ is normal, see ([12], Lemma 4.1.18).

Lemma 3.1. *Let $(A, \mathfrak{m})$ be a regular Noetherian local ring.*

(i) *Suppose that $f \in \mathfrak{m}\backslash\{0\}$. Then A/fA is regular if and only if $f \notin \mathfrak{m}^2$.*

(ii) *Suppose that I is a proper ideal of A. Then A/I is regular if and only if I is generated by r elements of $\mathfrak{m}$ which are linearly independent mod $\mathfrak{m}^2$, with $r = \dim A - \dim A/I$.*

Proof. See ([12], Corollaries 4.2.12 and 4.2.15). ∎

Lemma 3.2. *Let K' be a finite extension of K with ring of integers $\mathcal{O}_{K'}$. Let $\mathcal{C}$ be an S-scheme. Set $S' = \operatorname{Spec} \mathcal{O}_{K'}$, and $\mathcal{C}' = \mathcal{C} \times_S S'$. If $\mathcal{C}'$ is S'-normal, then $\mathcal{C}$ is S-normal.*

Proof. That $\mathcal{C}'$ is S'-normal means that $\mathcal{O}_{\mathcal{C}',x}$ is integrally closed in $\operatorname{Frac}(\mathcal{O}_{\mathcal{C}',x})$ for every $x \in \mathcal{C}'$. Now let $x \in \mathcal{C}$, and $\alpha \in \operatorname{Frac}(\mathcal{O}_{\mathcal{C},x})$ satisfy an integral relation for α over $\mathcal{O}_{\mathcal{C},x}$. The S'-normality of $\mathcal{C}'$ implies that $\alpha \in \mathcal{O}_{\mathcal{C}',x}$. Therefore, $\alpha \in \mathcal{O}_{\mathcal{C}',x} \cap \operatorname{Frac}(\mathcal{O}_{\mathcal{C},x}) = \mathcal{O}_{\mathcal{C},x}$. ∎

If $f(x_1, \ldots, x_n) = \sum_{i=1}^m a_i x_1^{l_{1i}} \ldots x_n^{l_{ni}} \in \mathcal{O}_K[x_1, \ldots, x_n]$, then $\tilde{f}(x_1, \ldots, x_n)$ will denote its image in $k[x_1, \ldots, x_n]$. Moreover, $\nu(f) = \min\{\nu(a_i) : 1 \leqslant i \leqslant m\}$.

Let $\mathcal{C}$ be an S-scheme defined by an integral genus one equation ϕ of degree 3 where

$$\phi : by^3 + f_1(x,z)y^2 + f_2(x,z)y + f_3(x,z) = 0, \tag{4}$$

with $f_1(x,z) = b_1x + b_3z$, $f_2(x,z) = a_2x^2 + mxz + c_2z^2$ and $f_3(x,z) = ax^3 + a_3x^2z + c_1z^2x + cz^3$. If $\mathcal{C}_k$ contains an irreducible component of multiplicity m, $m \geqslant 2$, then we can assume without loss of generality that the defining equation of this multiplicity-m component is $y = 0$. This means that $\min\{\nu(f_2), \nu(f_3)\} \geqslant 1, \nu(f_1) = 0$ when $m = 2$, and $\min\{\nu(f_1), \nu(f_2), \nu(f_3)\} \geqslant 1, \nu(b) = 0$ when $m = 3$.

Proposition 3.3. *Let $\mathcal{C}$ be an S-scheme defined by an integral genus one equation ϕ of degree n, $n \leqslant 3$. Assume that $\mathcal{C}_K$ is smooth.*

(i) *If $\mathcal{C}_k$ consists only of multiplicity-1 components, then $\mathcal{C}$ is normal.*

Now assume that $\mathcal{C}_k$ contains an irreducible component of multiplicity greater than 1.

(ii) *If $n = 2$ and $\phi : y^2 + g(x)y = f(x)$, then $\mathcal{C}$ is normal if and only if there exists $R(x) \in \mathcal{O}_K[x]$ such that $\nu(f(x) + g(x)R(x) - R(x)^2) = 1$.*

(iii) *If $n = 3$, $\phi : F(x,y,z) = 0$ is given as in equation (4), and $\mathcal{C}_k$ contains a multiplicity-m component $\Gamma : \{y = 0\}$, $m \geqslant 2$, then $\mathcal{C}$ is normal if and only if $\nu(f_3) = 1$.*

Proof. *(i)* Since $\mathcal{C}_k$ is reduced and $\mathcal{C}_K$ is smooth, the S-scheme $\mathcal{C}$ is normal. *(ii)* This is Lemme 2 (c) of [11]. *(iii)* The maximal ideal corresponding to the generic point ξ of Γ is $\mathfrak{m}_\xi = \langle t, y \rangle$. Now $\mathcal{C}$ is normal if and only if $F(x,y,z) \notin \mathfrak{m}_\xi^2$, see Lemma 3.1 *(i)*. Since $\nu(f_2) \geqslant 1$, we have $y^3, y^2, f_2(x,z)y \in \mathfrak{m}_\xi^2$. Therefore, $F(x,y,z) \notin \mathfrak{m}_\xi^2$ if and only if $\nu(f_3) = 1$. ∎

Now we study the normality of an S-scheme $\mathcal{C}$ defined by an integral genus one equation $\phi : F(x_1, x_2, x_3, x_4) = G(x_1, x_2, x_3, x_4) = 0$ of degree 4. We will assume that $\tilde{F}$ and $\tilde{G}$ are coprime. Moreover, we may assume that ϕ is not $\mathcal{O}_K$-equivalent to an equation whose reduction mod t is given by $x_1^2 = x_2^2 = 0$. These two assumptions are reasonable to make because of the following lemma which can be found in §2.5.1 of [16].

Lemma 3.4.

(i) *If $\tilde{F}$ and $\tilde{G}$ have a common factor, then ϕ is not minimal.*

(ii) *If $\mathcal{C}_k$ is defined by $x_1^2 = x_2^2 = 0$, then either ϕ is not minimal, or $\mathcal{C}_K(K) = \emptyset$.*

We observe that (i) all the irreducible components of $\mathcal{C}_k$ are defined over the residue field of a finite unramified extension of K, (ii) the normality of $\mathcal{C}$ over the ring of integers of a finite extension of K implies the normality of $\mathcal{C}$ over $\mathcal{O}_K$, see Lemma 3.2, and (iii) the minimality of ϕ is stable under unramified base changes, see ([6], Theorem 3.6). Therefore, we will assume that k is algebraically closed when we are finding criteria for the normality of $\mathcal{C}$, see Proposition 3.5, and testing the normality of $\mathcal{C}$ when $\mathcal{C}$ is minimal, see Theorem 3.6.

Since we will be interested in $\mathcal{C}$ when $\mathcal{C}_k$ contains a component of multiplicity m, $m \geqslant 2$, we will write down all the possibilities for such a special fiber, up to $\mathcal{O}_K$-equivalence of ϕ. Again k will be algebraically closed for the remainder of this section. For a complete list for the forms of $\mathcal{C}_k$, which includes special fibers with only multiplicity-1 components, see [7].

$\mathcal{C}_k$	Defining equations
conic + double line	$x_1x_3 = x_2^2 + x_1x_4 = 0$
double conic	$x_1^2 = x_2^2 + x_3x_4 = 0$
double line + two lines	$x_1^2 + x_2^2 = x_1x_3 + \mu x_2x_4 = 0,\ \mu \in \{0, 1\}$
triple line + line	$x_1x_2 = x_1^2 + x_2x_4 = 0$
two double lines	$x_2^2 = x_1x_3 + \mu x_2x_4 = 0,\ \mu \in \{0, 1\}$
quadruple line	$x_1^2 = x_2^2 + x_1x_3 = 0$

Let the quadrics F and G be given by the following two polynomials respectively:

$$\begin{aligned} &a_1x_1^2 + a_2x_1x_2 + a_3x_1x_3 + a_4x_1x_4 + a_5x_2^2 \\ &\qquad\qquad + a_6x_2x_3 + a_7x_2x_4 + a_8x_3^2 + a_9x_3x_4 + a_{10}x_4^2, \\ &b_1x_1^2 + b_2x_1x_2 + b_3x_1x_3 + b_4x_1x_4 + b_5x_2^2 \\ &\qquad\qquad + b_6x_2x_3 + b_7x_2x_4 + b_8x_3^2 + b_9x_3x_4 + b_{10}x_4^2. \end{aligned} \tag{5}$$

Proposition 3.5. *Let $\mathcal{C}$ be the S-scheme defined by the integral equation $\phi : F = G = 0$, where F and G are given in (5). Assume that $\mathcal{C}_K$ is smooth.*

(i) *If $\mathcal{C}_k$ contains a multiplicity-1 component Γ, then $\mathcal{C}$ is normal at Γ.*

(ii) *If* $\tilde{F} = x_1x_3$ *and* $\tilde{G} = x_2^2 + x_1x_4$, *then* $\mathcal{C}$ *is normal if and only if*

$$\nu(x_4F(0,0,x_3,x_4) - x_3G(0,0,x_3,x_4)) = 1.$$

(iii) *If* $\tilde{F} = x_1^2$ *and* $\tilde{G} = x_2^2 + x_3x_4$, *then* $\mathcal{C}$ *is normal unless* $F(0,x_2,x_3,x_4) \equiv \mu(x_2^2 + x_3x_4) \mod t^2$, *for some* $\mu \in \mathcal{O}_K$.

(iv) *Assume that* $\mathcal{C}_k$ *contains a line* $\Gamma : \{x_1 = x_2 = 0\}$ *of multiplicity* m, $m \geqslant 2$, *with* $\tilde{F} = q(x_1,x_2)$ *and* $\tilde{G} = x_1x_3 + \mu x_2x_4 + q'(x_1,x_2)$, $\mu \in \{0,1\}$. *If* $\nu(F(0,0,x_3,x_4)) = 1$, *then* $\mathcal{C}$ *is normal at* Γ.

Proof. (i) Since $\mathcal{C}_k$ is reduced at the generic point ξ of Γ, we see that $\mathcal{C}$ is normal at ξ.

Now we use Lemma 3.1 (ii) to study the normality of $\mathcal{C}$ at components of multiplicity greater than 1. The model $\mathcal{C}$ is normal if and only if $F, G \notin \mathfrak{m}_\xi^2$, and F, G are linearly independent mod $\mathfrak{m}_\xi^2$, for every generic point ξ of $\mathcal{C}_k$. The linear independence condition is: For $\lambda_1, \lambda_2 \in \mathcal{O}_K[x_1,\dots,x_4]_\xi$, if $\lambda_1F + \lambda_2G \in \mathfrak{m}_\xi^2$, then $\lambda_1, \lambda_2 \in \mathfrak{m}_\xi$.

(ii) Let ξ be the generic point of the double line $\{x_1 = x_2 = 0\}$ in $\mathcal{C}_k$, then $\mathfrak{m}_\xi = \langle x_1, x_2, t\rangle$. It is clear that $F, G \notin \mathfrak{m}_\xi^2$. If $\lambda_1F + \lambda_2G \in \mathfrak{m}_\xi^2$, then the fact that x_1 and t are linearly independent mod $\mathfrak{m}_\xi^2$ implies that $\lambda_1x_3 + \lambda_2x_4 \in \mathfrak{m}_\xi$, i.e., $\lambda_1 \equiv \mu x_4 \mod \mathfrak{m}_\xi$ and $\lambda_2 \equiv -\mu x_3 \mod \mathfrak{m}_\xi$ for some $\mu \in \mathcal{O}_K$. Thus $\mathcal{C}$ is normal if and only if $\nu(f) = 1$, where

$$f = x_4(a_8x_3^2 + a_9x_3x_4 + a_{10}x_4^2) - x_3(b_8x_3^2 + b_9x_3x_4 + b_{10}x_4^2).$$

(iii) Let $\mathfrak{m}_\xi = \langle x_1, x_2^2 + x_3x_4, t\rangle$ be the maximal ideal corresponding to the generic point ξ of the conic. We have $G \notin \mathfrak{m}_\xi^2$. If $a_5x_2^2 + a_9x_3x_4 = tu(x_2^2 + x_3x_4), u \in \mathcal{O}_K$, then $F \notin \mathfrak{m}_\xi^2$ if and only if $\nu(a_6x_2x_3 + a_7x_2x_4 + a_8x_3^2 + a_{10}x_4^2) = 1$, otherwise $F \notin \mathfrak{m}_\xi^2$ if and only if $\nu(a_5x_2^2 + a_6x_2x_3 + a_7x_2x_4 + a_8x_3^2 + a_9x_3x_4 + a_{10}x_4^2) = 1$. If $\lambda_1F + \lambda_2G \in \mathfrak{m}_\xi^2$, then $\lambda_2 \in \mathfrak{m}_\xi$ because t and $x_2^2 + x_3x_4$ are linearly independent mod $\mathfrak{m}_\xi^2$. Then the condition we obtained from $F \notin \mathfrak{m}_\xi^2$ implies that $\lambda_1 \in \mathfrak{m}_\xi$.

(iv) Assume that ξ is the generic point of $\Gamma : \{x_1 = x_2 = 0\}$. The ideal $\mathfrak{m}_\xi$ is given by $\langle x_1, x_2, t\rangle$. Since $\tilde{F} = q(x_1,x_2)$ and $\nu(a_8x_3^2 + a_9x_3x_4 + a_{10}x_4^2) = 1$, we have $F \notin \mathfrak{m}_\xi^2$. Since $\tilde{G} = x_1x_3 + \mu x_2x_4 + q'(x_1,x_2)$, we have $G \notin \mathfrak{m}_\xi^2$ because $x_1x_3 \notin \mathfrak{m}_\xi^2$. Assume that $\lambda_1F + \lambda_2G \in \mathfrak{m}_\xi^2$. Since x_1, x_2 and t are linearly independent mod $\mathfrak{m}_\xi^2$, it follows that $\lambda_2 \in \mathfrak{m}_\xi$. Moreover, as $\nu(a_8x_3^2 + a_9x_3x_4 + a_{10}x_4^2) = 1$, we get $\lambda_1 \in \mathfrak{m}_\xi$. ∎

Theorem 3.6. *Let* ϕ *be an integral genus one equation of degree* n, $n \leqslant 4$, *defining an* S*-scheme* $\mathcal{C}$. *Assume that* $\mathcal{C}_K$ *is smooth and* $\mathcal{C}_K(K) \neq \emptyset$. *If* ϕ *is minimal, then* $\mathcal{C}$ *is normal.*

Proof. If $\mathcal{C}_k$ consists only of multiplicity-1 components, then $\mathcal{C}_k$ is reduced and hence $\mathcal{C}$ is normal. So, we only need to assume that ϕ is of degree n, $n \geqslant 2$, and $\mathcal{C}_k$ contains a component of multiplicity greater than 1. Furthermore, we assume that $\mathcal{C}$ is not normal, and hence ϕ does not satisfy the conditions included in

Propositions 3.3 and 3.5. Then we will prove that ϕ is not minimal by finding a genus one equation K-equivalent to ϕ whose discriminant has valuation less than Δ_ϕ. Recall that the discriminant varies by the 12th power of the determinant of the K-equivalence transformation, see §1.

Let $n = 2$ and $\phi : y^2 + g(x)y = f(x)$. Since $\mathcal{C}_k$ is a double line, we can assume that $\tilde{g} = \tilde{f} = 0$. Since ϕ is not normal, we have $\nu(f) \geqslant 2$. The equation ϕ is not minimal because it is K-equivalent to the equation $y^2 + \frac{1}{t}g(x)y = \frac{1}{t^2}f(x)$.

Let $n = 3$ and $\phi : F(x, y, z) = 0$ as in Equation (4). Now $\mathcal{C}_k$ contains a multiplicity-m component $\Gamma : \{y = 0\}$, $m \geqslant 2$. Since $\mathcal{C}$ is not normal, we have $\nu(f_2), \nu(f_3) \geqslant 1$ and $\nu(f_3) \geqslant 2$. Now ϕ is not minimal because it is K-equivalent to the equation $\frac{1}{t^2}F(x, ty, z) = 0$.

Let $n = 4$ and $\phi : F = G = 0$, where F and G are given as in (5). We will go through the different cases of Proposition 3.5.

Assume that $\mathcal{C}_k : \{x_1x_3 = x_2^2 + x_1x_4 = 0\}$ and

$$\nu(x_4F(0, 0, x_3, x_4) - x_3G(0, 0, x_3, x_4)) \geqslant 2.$$

We use a matrix in $\mathrm{GL}_4(\mathcal{O}_K)$ to get rid of the x_1^2, x_1x_2 and x_1x_4-terms in F and the x_1^2, x_1x_2 and x_1x_3-terms in G. We notice that in the equation

$$x_4F(0, 0, x_3, x_4) - x_3G(0, 0, x_3, x_4) = -b_8x_3^3 + (a_8 - b_9)x_3^2x_4 + (a_9 - b_{10})x_3x_4^2 + a_{10}x_4^3,$$

we have $\min\{\nu(b_8), \nu(a_8 - b_9), \nu(a_9 - b_{10}), \nu(a_{10})\} \geqslant 2$. We apply the transformation $x_1 \mapsto x_1 - a_8x_3 - a_9x_4, x_i \mapsto x_i, i = 2, 3, 4$, to get rid of the terms $a_8x_3^2$ and $a_9x_3x_4$. Thereafter, we obtain the genus one equation $\phi' : F' = G' = 0$, where

$$\begin{aligned} F' &= x_1x_3 + a_5x_2^2 + a_6x_2x_3 + a_7x_2x_4 + a_{10}x_4^2, \\ G' &= x_1x_4 + x_2^2 + b_6x_2x_3 + b_7x_2x_4 + b_8x_3^2 + (b_9 - a_8)x_3x_4 + (b_{10} - a_9)x_4^2. \end{aligned}$$

We deduce that ϕ' is not minimal because it is K-equivalent to the equation

$$\frac{1}{t^2}F'(t^2x_1, tx_2, x_3, x_4) = \frac{1}{t^2}G'(t^2x_1, tx_2, x_3, x_4) = 0.$$

Assume that $\mathcal{C}_k : \{x_1^2 = x_2^2 + x_3x_4 = 0\}$, and that $\nu(F(0, x_2, x_3, x_4) - \mu(x_2^2 + x_3x_4)) \geqslant 2$ for some $\mu \in \mathcal{O}_K$. Then ϕ is not minimal because it is K-equivalent to

$$\frac{1}{t^2}(F(tx_1, x_2, x_3, x_4) - \mu G(tx_1, x_2, x_3, x_4)) = G(tx_1, x_2, x_3, x_4) = 0.$$

Now assume that $\mathcal{C}_k$ contains a line $\Gamma : \{x_1 = x_2 = 0\}$ of multiplicity m, $m \geqslant 2$, with $\tilde{F} = q(x_1, x_2)$ and $\tilde{G} = x_1x_3 + \mu x_2x_4 + q'(x_1, x_2)$, where $\mu \in \{0, 1\}$. Assume that $\nu(a_8x_3^2 + a_9x_3x_4 + a_{10}x_4^2) \geqslant 2$. Then ϕ is not minimal because it is K-equivalent to

$$\frac{1}{t^2}F(tx_1, tx_2, x_3, x_4) = \frac{1}{t}G(tx_1, tx_2, x_3, x_4) = 0. \qquad \blacksquare$$

4. Criteria for minimality

We state the first main theorem of this paper.

Theorem 4.1. *Let ϕ be an integral genus one equation of degree n, $n \leqslant 4$. Assume moreover that ϕ defines a normal S-scheme $\mathcal{C}$, and that $\mathcal{C}_K$ is smooth with $\mathcal{C}_K(K) \neq \emptyset$. Then ϕ is minimal, see Definition 1.1, if and only if ϕ is geometrically minimal, see Definition 2.4.*

Theorem 4.1 is known for the case $n = 1$, see ([12], §9.4) or [4]. When $n = 2$, Liu proved that if ϕ is minimal, then ϕ is geometrically minimal, see ([11], Corollaire 5 (a)). We introduce a proof which works for all $n \leqslant 4$.

4.1. Canonical sheaves of S-models

Let E be an elliptic curve with minimal proper regular model E^{min}. If $\mathcal{C}$ is an S-model for E, then the canonical sheaf $\omega_{\mathcal{C}/S}$ of $\mathcal{C}$ satisfies $\omega_{\mathcal{C}/S}|_E = \omega_{E/K}$, moreover the restriction of the canonical sheaf $\omega_{\mathcal{C}/S}$ on E gives a canonical injection $H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) \hookrightarrow H^0(E, \omega_{E/K})$. We have $H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}) = \mathcal{O}_K$. In particular, if $\omega_{\mathcal{C}/S} = \omega\mathcal{O}_\mathcal{C}$, then $H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = \omega\mathcal{O}_K$. In addition, there exists an $\omega_0 \in H^0(E^{min}, \omega_{E^{min}/S})$ such that $\omega_{E^{min}/S} = \omega_0\mathcal{O}_{E^{min}}$, see ([4], Example 7.7).

Lemma 4.2. *Let E be an elliptic curve over K with minimal proper regular model E^{min}. Let $\mathcal{C}$ be a normal S-scheme with $\mathcal{C}_K \cong E$. Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ be the minimal desingularisation of $\mathcal{C}$. Then the following statements hold.*

(i) $H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}) \subseteq H^0(\mathcal{C}, \omega_{\mathcal{C}/S})$ *as subgroups in* $H^0(E, \omega_{E/K})$.

(ii) *If* $\widetilde{\mathcal{C}} \cong E^{min}$, *then* $H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S})$.

Proof. (*i*) The equality holds because $\widetilde{\mathcal{C}}$ and E^{min} are two regular S-models for E, see ([12], Corollary 9.2.25 (b)). The inequality holds because $\mathcal{C}$ is obtained from $\widetilde{\mathcal{C}}$ as a contraction of a finite number of irreducible components, see ([12], Lemma 9.2.17 (a)).

(*ii*) Since $\widetilde{\mathcal{C}} \cong E^{min}$, we have a contraction morphism $f : E^{min} \to \mathcal{C}$. Therefore, $f_*\omega_{E^{min}/S} = \omega_{\mathcal{C}/S}$, see ([12], Corollary 9.4.18 (b)). ■

Proposition 4.3. *Let $\mathcal{C}$ be an S-scheme defined by an integral genus one equation ϕ of degree n. Assume that $\mathcal{C}_K$ is smooth. Then $\omega_{\mathcal{C}/S} = \omega_\phi\mathcal{O}_\mathcal{C}$, where $\omega_\phi \in H^0(\mathcal{C}_K, \omega_{\mathcal{C}_K/K})$ is*

(i) *if* $n = 1$*:* $\omega_\phi = du/(2v + a_1u + a_3)$, *where* $u = x/z, v = y/z \in K(\mathcal{C})$,

(ii) *if* $n = 2$*:* $\omega_\phi = dx/(2y + g(x))$,

(iii) *if* $n = 3$*:* $\omega_\phi = du/(\partial F/\partial v)$, *where* $u = x/z, v = y/z \in K(\mathcal{C})$,

(iv) *if* $n = 4$*:* $\omega_\phi = du_2/(\frac{\partial F_1}{\partial u_4}\frac{\partial F_2}{\partial u_3} - \frac{\partial F_1}{\partial u_3}\frac{\partial F_2}{\partial u_4})$, *where* $u_i = x_i/x_1 \in K(\mathcal{C})$, $i = 2, 3, 4$.

Proof. This is a direct consequence of ([12], Corollary 6.4.14). ■

Lemma 4.4. *Let ϕ_1, ϕ_2 be two K-equivalent integral genus one equations of degree n, $n \leqslant 4$. Assume that ϕ_i defines an S-scheme $\mathcal{C}_i$ with $C_i := (\mathcal{C}_i)_K$ being smooth. If $\omega_{\mathcal{C}_i/S} = \omega_{\phi_i}\mathcal{O}_{\mathcal{C}_i}$, then*

$$\Delta_{\phi_1}\omega_{\phi_1}^{\otimes 12} = \lambda\Delta_{\phi_2}\omega_{\phi_2}^{\otimes 12} \in H^0(C_1, \omega_{C_1/K})^{\otimes 12} = H^0(C_2, \omega_{C_2/K})^{\otimes 12}, \text{ where } \lambda \in \mathcal{O}_K^*.$$

Proof. Assume that $\phi_1 = g.\phi_2$ where g is a transformation defining the K-equivalence. The transformation g defines an isomorphism $\gamma : C_1 \cong C_2$ which satisfies $\gamma^*\omega_{\phi_2} = (\det g)\omega_{\phi_1}$, see ([10], Proposition 5.19). Hence, $\omega_{\phi_2} = \alpha(\det g)\omega_{\phi_1}$ as elements in $H^0(C_i, \omega_{C_i/K})$, where $\alpha \in \mathcal{O}_K^*$. Recall that $\Delta_{\phi_1} = (\det g)^{12}.\Delta_{\phi_2}$. It follows that $\Delta_{\phi_1}\omega_{\phi_1}^{\otimes 12} = \alpha^{12}\Delta_{\phi_2}\omega_{\phi_2}^{\otimes 12}$. ∎

If ϕ_1 is minimal, then we call the integer m such that $\omega_{\phi_2} = ut^{-m}\omega_{\phi_1}, u \in \mathcal{O}_K^*$, the *level* of ϕ_2, and denote it by level(ϕ_2). The above corollary implies that the level of an integral genus one equation of degree n does not depend on the choice of the minimal genus one equation ϕ_1. Notice that $\nu(\Delta_{\phi_2}) = \nu(\Delta_{\phi_1}) + 12\,\mathrm{level}(\phi_2)$.

Lemma 4.5. *Keep the hypotheses of Lemma 4.4. Then we have $H^0(\mathcal{C}_1, \omega_{\mathcal{C}_1/S}) \subseteq H^0(\mathcal{C}_2, \omega_{\mathcal{C}_2/S})$ as sub-$\mathcal{O}_K$-modules of $H^0(C_i, \omega_{C_i/K})$ if and only if $\nu(\Delta_{\phi_1}) \leqslant \nu(\Delta_{\phi_2})$. Moreover, the equality of the two submodules holds if and only if ϕ_1 and ϕ_2 have the same level.*

Proof. The assumption $H^0(\mathcal{C}_1, \omega_{\mathcal{C}_1/S}) \subseteq H^0(\mathcal{C}_2, \omega_{\mathcal{C}_2/S})$ is equivalent to $\omega_{\phi_1} \in \omega_{\phi_2}\mathcal{O}_K$. Thus Lemma 4.4 asserts that $\Delta_{\phi_2} \in \Delta_{\phi_1}\mathcal{O}_K$, i.e., $\nu(\Delta_{\phi_1}) \leqslant \nu(\Delta_{\phi_2})$.

The equality of the sub-$\mathcal{O}_K$-modules $H^0(\mathcal{C}_1, \omega_{\mathcal{C}_1/S}) = H^0(\mathcal{C}_2, \omega_{\mathcal{C}_2/S})$ means that $\omega_{\phi_1}\mathcal{O}_K = \omega_{\phi_2}\mathcal{O}_K$ as $\mathcal{O}_K$-modules, i.e., $\omega_{\phi_1} \in \omega_{\phi_2}\mathcal{O}_K^*$. Hence, ϕ_1 and ϕ_2 have the same level by Lemma 4.4. ∎

4.2. Constructing genus one equations

Let E be an elliptic curve over K with minimal proper regular model E^{min}. Let $P \in E(K)$. We will denote the Zariski closure of $\{P\}$ in E^{min} by $\overline{\{P\}}$.

When $n = 1$, set $D_1 = 3.\overline{\{P\}}$ where $P \in E(K)$. When $n \geqslant 2$, let $\sum(P_i) \in \mathrm{Div}(E)$ be a K-rational divisor on E of degree n. Assume moreover that $\overline{\{P_i\}} \cap E_k^{min}$ is contained in exactly one irreducible component of E_k^{min}. Consider the divisor D_n on E^{min}, and the S-model $\mathcal{C}_n$ for E given by

$$D_n = \sum \overline{\{P_i\}}, \text{ and } \mathcal{C}_n := \mathrm{Proj}(\bigoplus_{m=0}^{\infty} H^0(E^{min}, \mathcal{O}_{E^{min}}(mD_n))).$$

There is a canonical morphism $u_n : E^{min} \to \mathcal{C}_n$ contracting all the irreducible components of E_k^{min} except the ones having nonempty intersection with D_n, see Theorem 2.3.

Lemma 4.6. *Let D_n, $n \in \{1, 2, 3, 4\}$, be as above. Then $H^0(E^{min}, \mathcal{O}_{E^{min}}(mD_n))$, $m \geqslant 1$, is a free $\mathcal{O}_K$-module of rank $3m$ if $n = 1$, and of rank mn if $n \geqslant 2$.*

Proof. It is known that $H^0(E^{min},\mathcal{O}_{E^{min}}(mD_n))\otimes_{\mathcal{O}_K}K\cong H^0(C,\mathcal{O}_C(mD_n|_C))$, see for example ([12], Corollary 5.2.27). By virtue of the Riemann-Roch Theorem, $H^0(C,\mathcal{O}_C(mD_n|_C))$ is a $3m$-dimensional K-vector space when $n=1$, and an mn-dimensional K-vector space when $n\geqslant 2$.

Since $\mathcal{O}_{E^{min}}(mD_n)$ is an invertible sheaf on E^{min}, it follows that $H^0(E^{min},\mathcal{O}_{E^{min}}(mD_n))$ is a flat $\mathcal{O}_K$-module. Therefore, $H^0(E^{min},\mathcal{O}_{E^{min}}(mD_n))$ is a finitely generated flat module over the local ring $\mathcal{O}_K$, hence it is free. ■

Theorem 4.7. *Let $\mathcal{C}_n$ and D_n, $n\leqslant 4$, be as above. Then there exists an integral genus one equation ϕ_n of degree n defining $\mathcal{C}_n$.*

Proof. For $n=1$, see ([12], §9.4). For $n\geqslant 2$, we pick a basis $\{x_1,\dots,x_n\}$ of $H^0(E^{min},\mathcal{O}_{E^{min}}(D_n))$. Consider the morphism $\lambda_n: E^{min}\longrightarrow\mathbb{P}^{n-1}_S$ associated to the basis $\{x_1,\dots,x_n\}$.

For $n=2$, put $x_1=x$ and $x_2=1$. We have $\mathbb{P}^1_S=\operatorname{Spec}\mathcal{O}_K[x]\cup\operatorname{Spec}\mathcal{O}_K[1/x]$. Let $U=\lambda_2^{-1}(\operatorname{Spec}\mathcal{O}_K[x])$ and $V=\lambda_2^{-1}(\operatorname{Spec}\mathcal{O}_K[1/x])$. We have $\mathcal{C}_2=U\cup V$. Taking the integral closure of $\mathcal{O}_K[x]$ in $K(\mathcal{C}_2)$, we have

$$\mathcal{O}_{\mathcal{C}_2}(U)=\mathcal{O}_K[x]\oplus y\mathcal{O}_K[x],\qquad \text{for some } y\in\mathcal{O}_{\mathcal{C}_2}(U),$$

moreover there exist $g(x),f(x)\in\mathcal{O}_K[x]$ such that $\deg g\leqslant 2$, $\deg f\leqslant 4$ and $y^2+g(x)y=f(x)$, see ([11], Lemme 1). Following the same argument for $\mathcal{O}_{\mathcal{C}_2}(V)$, we deduce that $\mathcal{C}_2$ is the union of the two affine open schemes

$$\begin{aligned}U&=\operatorname{Spec}\mathcal{O}_K[x,y]/(y^2+g(x)y-f(x)),\\ V&=\operatorname{Spec}\mathcal{O}_K[w,z]/(z^2+w^2g(1/w)z-w^4f(1/w)),\end{aligned}$$

where $w=1/x$, $z=y/x^2$. Hence, we are done when $n=2$.

Now let $n=3,4$. Let Z_n be the closed subset $\lambda_n(E^{min})\subset\mathbb{P}^{n-1}_S$ endowed with the reduced scheme structure. According to the description of the contraction morphism included in the proof of ([12], Proposition 8.3.30), the morphism $\lambda_n: E^{min}\to Z_n\subseteq\mathbb{P}^{n-1}_S$ factors into $u_n: E^{min}\to\mathcal{C}_n$ followed by $v_n:\mathcal{C}_n\to Z_n$, where v_n is the normalisation morphism. It is understood that v_n is a finite morphism, hence for an irreducible component Γ of E^{min}_k, $\lambda_n(\Gamma)$ is a point if and only if $u_n(\Gamma)$ is a point. In other words, the special fibers of $\mathcal{C}_n$ and Z_n have the same number of irreducible components. We are going to show that Z_n is defined by an integral genus one equation of degree n. Then we show that $\mathcal{C}_n\cong Z_n$.

When $n=3$, the free $\mathcal{O}_K$-module $H^0(E^{min},\mathcal{O}_{E^{min}}(3D_3))$ is of rank 9, see Lemma 4.6, but it contains the 10 elements x_1^3, x_2^3, x_3^3, $x_1^2x_2$, $x_1^2x_3$, $x_2^2x_1$, $x_2^2x_3$, $x_3^2x_1$, $x_3^2x_2$, $x_1x_2x_3$. It follows that there are $a_i\in\mathcal{O}_K$ such that

$$\begin{aligned}F:=a_1x_1^3+a_2x_2^3+a_3x_3^3+a_4x_1^2x_2+a_5x_1^2x_3+a_6x_2^2x_1&\\ +a_7x_2^2x_3+a_8x_3^2x_1+a_9x_3^2x_2+a_{10}x_1x_2x_3&=0.\end{aligned}$$

Rescaling x_1, x_2 and x_3, we can assume that there is at least one $a_i\in\mathcal{O}_K^*$. Now Z_3 is contained in $\operatorname{Proj}\mathcal{O}_K[x_1,x_2,x_3]/(F)$.

When $n=4$, we consider the 10 elements x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2 in the rank-8 free $\mathcal{O}_K$-module $H^0(E^{min}, \mathcal{O}_{E^{min}}(2D_4))$. They satisfy two quadrics Q and R with coefficients in $\mathcal{O}_K$. Moreover, by rescaling x_1, x_2, x_3 and x_4 we can assume that both Q and R have at least one coefficient in $\mathcal{O}_K^*$. Now Z_4 is contained in the intersection of Q and R.

We want to show that $Z_n = \operatorname{Proj} \mathcal{O}_K[x_1, \ldots, x_n]/I_n$, where $I_3 = (F)$ and $I_4 = (Q, R)$. Recall that both schemes are of dimension 2. Since $Z_n \subseteq \operatorname{Proj} \mathcal{O}_K[x_1, \ldots, x_n]/I_n$, we have $\operatorname{Proj} \mathcal{O}_K[x_1, \ldots, x_n]/I_n = Z_n \cup Z_n'$, for some closed subscheme $Z_n' \subset \mathbb{P}_S^{n-1}$, where $Z_n' \neq \operatorname{Proj} \mathcal{O}_K[x_1, \ldots, x_n]/I_n$. Since $\operatorname{Proj}(\mathcal{O}_K[x_1, \ldots, x_n]/I_n \otimes K)$ is irreducible, $\operatorname{Proj} \mathcal{O}_K[x_1, \ldots, x_n]/I_n$ is irreducible itself. It follows from the definition of irreducibility that $Z_n' = \emptyset$, and the closed subscheme Z_n is $\operatorname{Proj} \mathcal{O}_K[x_1, \ldots, x_n]/I_n$.

Now both $\mathcal{C}_n$ and Z_n, $n = 3, 4$, have dimension 2, their generic fibers are isomorphic, and their special fibers have the same number of irreducible components. By virtue of ([12], Remark 8.3.25), $v_n : \mathcal{C}_n \to Z_n$ is an isomorphism. ■

Remark 4.8. Let $n \geqslant 2$. Let ϕ be an integral genus one equation of degree n defined by a hyperplane section H. Assume moreover that ϕ defines a morphism $E \to \mathbb{P}_K^{n-1}$. Then we can choose $D_n = (n-1)\overline{\{P\}} + \overline{\{Q\}}$ where $P, Q \in E(K)$ are such that $(n-1)P + Q \sim H$. It follows that ϕ_n is K-equivalent to ϕ.

4.3. Proof of Theorem 4.1

Proof of Theorem 4.1. We first assume that ϕ is geometrically minimal, i.e., that $\widetilde{\mathcal{C}} \cong E^{min}$. Thus we have a contraction morphism $f : E^{min} \to \mathcal{C}$. We claim that if $\mathcal{C}'$ is a normal S-scheme defined by an integral genus one equation ϕ' which is K-equivalent to ϕ, then $H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) \subseteq H^0(\mathcal{C}', \omega_{\mathcal{C}'/S})$, and hence $\nu(\Delta_\phi) \leqslant \nu(\Delta_{\phi'})$, see Lemma 4.5. Therefore, ϕ is minimal.

Lemma 4.2 (i) shows that $H^0(E^{min}, \omega_{E^{min}/S}) \subseteq H^0(\mathcal{C}', \omega_{\mathcal{C}'/S})$. The fact that $\mathcal{C}$ is obtained from E^{min} by contraction implies the equation $H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = H^0(E^{min}, \omega_{E^{min}/S})$, see Lemma 4.2 (ii). Thus the claim is proved.

Before we proceed with the proof of the second part of the theorem we need the following lemma.

Lemma 4.9. *Let ϕ be a genus one equation of degree n defining a normal S-scheme $\mathcal{C}$ with $\mathcal{C}_K$ being smooth and $\mathcal{C}_K(K) \neq \emptyset$. Let E^{min} be the minimal proper regular model of the Jacobian of $\mathcal{C}_K$. Then $H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S})$ as sub-$\mathcal{O}_K$-modules of $H^0(\mathcal{C}_K, \omega_{\mathcal{C}_K/K})$ if and only if ϕ is minimal.*

Proof. Assume that $H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S})$. Let $\mathcal{C}'$ be a normal S-scheme defined by a genus one equation K-equivalent to ϕ. By virtue of Lemma 4.2 (i), we have $H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) \subseteq H^0(\mathcal{C}', \omega_{\mathcal{C}'/S})$. Therefore, ϕ is minimal by Lemma 4.5.

Now assume that ϕ is minimal. According to Theorem 4.7 and Remark 4.8, there exists an integral genus one equation ϕ' which is K-equivalent to ϕ. Moreover, ϕ' is geometrically minimal because the minimal desingularisation of the

S-scheme $\mathcal{C}'$ defined by ϕ' is isomorphic to E^{min}. Therefore, $H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\mathcal{C}', \omega_{\mathcal{C}'/S})$ by Lemma 4.2 (*ii*). Moreover, ϕ' is minimal by the first part of Theorem 4.1. Since the genus one equations ϕ and ϕ' are both minimal, in particular they have the same level, Lemma 4.5 implies that $H^0(\mathcal{C}', \omega_{\mathcal{C}'/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S})$, and we are done. ∎

Now we conclude the proof of Theorem 4.1.

Assume that ϕ is minimal. We assume on the contrary that $\widetilde{\mathcal{C}} \not\cong E^{min}$, and therefore the strict transform of $\mathcal{C}_k$ in $\widetilde{\mathcal{C}}_k$ contains an exceptional divisor Γ. By ([12], Proposition 9.3.10), we have $\deg \omega_{\widetilde{\mathcal{C}}/S}|_\Gamma < 0$. It follows that $H^0(\Gamma, \omega_{\widetilde{\mathcal{C}}/S}|_\Gamma) = 0$, therefore $\omega_{\widetilde{\mathcal{C}}/S}$ is not generated by its global sections on Γ. But we have

$$\omega_{\widetilde{\mathcal{C}}/S}|_\Gamma = \omega_{\mathcal{C}/S}|_\Gamma = \omega_\phi \mathcal{O}_{\mathcal{C}}|_\Gamma,$$

where ω_ϕ is given as in Proposition 4.3. The global sections of $\omega_{\widetilde{\mathcal{C}}/S}$ are

$$H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}) = H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = \omega_\phi \mathcal{O}_K,$$

where the second equality is justified by $\mathcal{C}$ being minimal, see Lemma 4.9. Therefore, $\omega_{\widetilde{\mathcal{C}}/S}$ is generated by its global sections at every $x \in \Gamma$, which is a contradiction. ∎

Corollary 4.10. *Let $\phi, \mathcal{C}$ and $\widetilde{\mathcal{C}}$ be as in Theorem 4.1, and ω_ϕ as in Proposition 4.3. Then the following are equivalent.*

(i) *ϕ is minimal;*
(ii) *ϕ is geometrically minimal;*
(iii) $\omega_{\widetilde{\mathcal{C}}/S} = \omega_\phi \mathcal{O}_{\widetilde{\mathcal{C}}}$;
(iv) $H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = \omega_\phi \mathcal{O}_K$.

5. Existence of global models

In order to prove the global result included in Theorem 1.2, we need the following local result.

Lemma 5.1. *Let ϕ be a minimal genus one equation of degree n defining a normal S-scheme $\mathcal{C}$ such that $\mathcal{C}_K$ is smooth and $\mathcal{C}_K(K) \neq \emptyset$. Let E be the Jacobian of $\mathcal{C}_K$ with minimal discriminant Δ. Then $\nu(\Delta_\phi) = \nu(\Delta)$.*

Proof. Let $\mathcal{E}$ and E^{min} be the minimal Weierstrass model and minimal proper regular model of E respectively. We identify $\mathcal{C}_K$ and E via an isomorphism $\gamma : \mathcal{C}_K \cong_K E$. For explicit formulae for γ, see ([10], §6) or ([6], §2). Let ω_ϕ and ω be the generators of the canonical sheaves $\omega_{\mathcal{C}/S}$ and $\omega_{\mathcal{E}/S}$ given as in Proposition 4.3.

Now according to Lemma 4.9, we have $H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\mathcal{E}, \omega_{\mathcal{E}/S}) = \omega \mathcal{O}_K$, and $H^0(E^{min}, \omega_{E^{min}/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = \omega_\phi \mathcal{O}_K$ as sub-$\mathcal{O}_K$-modules of $H^0(E, \omega_{E/K})$. Therefore, $\omega_\phi = \lambda\omega$ for some $\lambda \in \mathcal{O}_K^*$. According to Lemma 4.4, we have $\Delta_\phi . \omega_\phi^{\otimes 12} = \Delta . \omega^{\otimes 12}$ up to a unit. Hence, $\Delta_\phi = \Delta$ up to a unit in $\mathcal{O}_K^*$. In other words, $\nu(\Delta_\phi) = \nu(\Delta)$. ∎

Corollary 5.2. *Let ϕ be an integral genus one equation defining a normal S-scheme $\mathcal{C}$ such that $\mathcal{C}_K$ is smooth and $\mathcal{C}_K(K) \neq \emptyset$. Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ be the minimal desingularisation of $\mathcal{C}$. Let E be the Jacobian of $\mathcal{C}_K$ with minimal discriminant Δ. Then*

$$\nu(\Delta_\phi) = \nu(\Delta) + 12\,\mathrm{length}_{\mathcal{O}_K}(H^0(\mathcal{C}, \omega_{\mathcal{C}/S})/H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S})),$$

where $\mathrm{length}_{\mathcal{O}_K}$ *denotes the length of an $\mathcal{O}_K$-module.*

Proof. Theorem 4.7 and Remark 4.8 admit the existence of an integral geometrically minimal genus one equation ϕ' K-equivalent to ϕ. Moreover, Theorem 4.1 implies that ϕ' is minimal. We have $\nu(\Delta_{\phi'}) = \nu(\Delta)$ by Lemma 5.1. We only need to show that $\nu(\Delta_\phi) = \nu(\Delta_{\phi'}) + 12\,\mathrm{length}_{\mathcal{O}_K}(H^0(\mathcal{C}, \omega_{\mathcal{C}/S})/H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}))$. The latter follows immediately from the fact that $\Delta_\phi.\omega_\phi^{\otimes 12} = \Delta_{\phi'}.\omega_{\phi'}^{\otimes 12}$ up to a unit, see Lemma 4.4, and $H^0(\mathcal{C}', \omega_{\mathcal{C}'/S}) = H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}) \subseteq H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = \omega_\phi \mathcal{O}_{\mathcal{C}}$, where $\mathcal{C}'$ is the normal S-scheme defined by ϕ'. ■

We observe that $\mathrm{length}_{\mathcal{O}_K}(H^0(\mathcal{C}, \omega_{\mathcal{C}/S})/H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}))$ in the above corollary is an interpretation for level(ϕ) mentioned after Lemma 4.4. Now we conclude with the proof of Theorem 1.2.

Proof of Theorem 1.2. We write $\mathcal{O}_\nu$ for the ring of integers of the completion F_ν.

By virtue of Theorem 4.7 and Remark 4.8, ϕ is F_ν-equivalent to a geometrically minimal $\mathcal{O}_\nu$-integral genus one equation ϕ_ν. Moreover, ϕ_ν is minimal, see Theorem 4.1. According to Lemma 5.1, $\nu(\Delta_{\phi_\nu}) = \nu(\Delta)$ for every non-archimedean place ν.

Since F is of class number 1, we can use strong approximation to find an $\mathcal{O}_F$-integral genus one equation ϕ' which is F-equivalent to ϕ and $\nu(\Delta_{\phi'}) = \nu(\Delta_{\phi_\nu}) = \nu(\Delta)$, for every non-archimedean place ν. See [9] for details when $n = 2, 3$. The case $n = 4$ is similar. Since the discriminant is of weight 12, we have $\Delta_{\phi'} = \lambda^{12}\Delta$ for some $\lambda \in \mathcal{O}_F^*$. By rescaling the coefficients of the defining polynomials of ϕ', we can assume that $\lambda = 1$. ■

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