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# WALK THE DOG, OR: PRODUCTS OF OPEN BALLS IN THE SPACE OF CONTINUOUS FUNCTIONS

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This paper is dedicated to Susanne Dierolf
who passed away far too early

**Abstract:** Let $C\,[\,0,1\,]$ be the Banach algebra of real valued continuous functions on $[\,0,1\,]$, provided with the supremum norm. For $f,g\in C\,[\,0,1\,]$ and balls $B_f$, $B_g$ with center $f$ and $g$, respectively, it is not necessarily true that $f\cdot g$ is in the interior of $B_f\cdot B_g$. In the present paper we characterize those pairs $f,g$ where this is the case.

The problem is illustrated by using a suitable translation. One studies walks in a landscape with hills and valleys where an accompanying dog can move in a certain prescribed way.

**Keywords:** continuous functions, open sets, Banach algebra.

## 1. Introduction

For $f\in C\,[\,0,1\,]$ and $\varepsilon>0$ we denote by $B_\varepsilon(f)$ the closed ball with center $f$ and radius $\varepsilon$. It has been observed in [1] that, for $f,g\in C\,[\,0,1\,]$, it is in general not true that $f\cdot g$ is an interior point of $B_\varepsilon(f)\cdot B_\varepsilon(g)$ (= the collection of all $\tilde f\cdot\tilde g$ with $\tilde f\in B_\varepsilon(f),\ \tilde g\in B_\varepsilon(g)$).

**Example.** As a simple example consider $f(t)=g(t)=t-0.5$. When $\tilde f$ is close to $f$ and $\tilde g$ close to $g$, then $\tilde f$ and $\tilde g$ must vanish at some point. Consequently also $\tilde f\cdot\tilde g$ is zero somewhere, and therefore no $\tilde f\cdot\tilde g$ is a function which is strictly larger than $f\cdot g$ at every point.

In [1] it has been shown that $B_\varepsilon(f)\cdot B_\varepsilon(g)$ always contains interior points, a result which is generalized to the space of differentiable functions in [3]. In the present paper we characterize the topological properties of $f$ and $g$ which guarantee that $f\cdot g$ itself is always an interior point.

We introduce the following

**Definition 1.1.** *Let $f,g\in C\,[\,0,1\,]$ be given. We say that $f,g$* have property (*) *if for every $\varepsilon>0$ the function $f\cdot g$ is in the interior of $B_\varepsilon(f)\cdot B_\varepsilon(g)$.*

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It will be convenient to translate the problem as follows. We consider a " landscape" where the height above sea level at the point with coordinates $(x, y) \in \mathbb{R}^2$ is $H(x, y) := x \cdot y$. Every two functions $f, g \in C\,[\,0,1\,]$ generate a "walk"

$$t \mapsto \big(f(t), g(t), H(f(t), g(t))\big)$$

in this landscape.

Now suppose that $\varepsilon > 0$ and that the walker is accompanied by a "dog" which is always close to him or her: we assume that the position of the dog at "time" $t$ is

$$\big(f(t) + d_1(t), g(t) + d_2(t), H(f(t) + d_1(t), g(t) + d_2(t))\big),$$

where $|d_1(t)|, |d_2(t)| \leqslant \varepsilon$. The height above sea level of the dog can be larger or smaller than that of the walker, the relative difference is

$$\tau(t) := H\big(D(t)\big) - H(f(t), g(t))\big),$$

where $D$ is the function $(f+d_1, g+d_2)$. (It will be convenient to call both $D(t) \in \mathbb{R}^2$ and $\big(D(t), H(D(t))\big) \in \mathbb{R}^3$ the position of the dog at time $t$.)

It is then obvious that $f, g$ have (*) if and only if the following holds:

*For every $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary $\tau \in C\,[\,0,1\,]$ with $||\tau|| \leqslant \delta$ there is a "walk of the dog" $t \mapsto \big(f(t) + d_1(t), g(t) + d_2(t)\big)$ with $||d_1||, ||d_2|| \leqslant \varepsilon$ such that the height difference at every time $t$ is just $\tau(t)$.*

Fix $f$ and $g$. We will write $\gamma(t) := \big(f(t), g(t)\big)$ for $t \in C\,[\,0,1\,]$: this is the walk seen from above. Denote by $Q^{++}$, $Q^{-+}$,$Q^{--}$,$Q^{+-}$ the four quadrants of the plane, i.e., the sets

$$\{x, y \geqslant 0\}, \qquad \{x \leqslant 0,\ y \geqslant 0\}, \qquad \{x, y \leqslant 0\}, \qquad \{x \geqslant 0,\ y \leqslant 0\}.$$

The height function $H$ is $\geqslant 0$ on $Q^{++}$ and $Q^{--}$, it is $\leqslant 0$ on $Q^{-+}$ and $Q^{+-}$, and $(0, 0)$ is a saddle point of $H$. It will turn out that the behaviour of $\gamma$ close to this saddle point is crucial.

**Definition 1.2.** *Let $t_0 \in \,]\,0,1\,[$. We say that $\gamma$ has* a positive saddle point crossing *at $t_0$ if $\gamma(t_0) = (0, 0)$ and there exists $r > 0$ such that:*

- *$\gamma(t) \in Q^{++} \cup Q^{--}$ for $t \in [\,t_0 - r, t_0 + r\,]$;*
- *there are $t_1 \in [\,t_0 - r, t_0\,]$ and $t_2 \in [\,t_0, t_0 + r\,]$ such that $\gamma(t_1) \in Q^{++} \setminus \{(0, 0)\}$ and $\gamma(t_2) \in Q^{--} \setminus \{(0, 0)\}$ or vice versa.*

*A* negative saddle point crossing *is defined similarly: here $\gamma$ moves from $Q^{-+}$ to $Q^{+-}$ or from $Q^{+-}$ to $Q^{-+}$.*

(Note that this could be formulated equivalently as follows: if $\gamma$ has, e.g., no positive saddle point crossings, then for $t_1, t_2$ with $\gamma(t_1) \in Q^{++} \setminus \{(0, 0)\}$ and $\gamma(t_2) \in Q^{--} \setminus \{(0, 0)\}$ there must be a $t$ between $t_1$ and $t_2$ such that $H\big(\gamma(t)\big) < 0$.)

And here is our characterization (theorem 3.1):

**Theorem.** *$f, g$ have property (*) iff the associated curve $\gamma$ has no positive and no negative saddle point crossings.*

In Section 2 we will prepare our investigations with the definition of some auxiliary functions. Section 3 contains the main result, and in Section 4 one finds some invitations for further study.

## 2. Preparations

We will fix $f, g$ in $C\,[\,0, 1\,]$, and $\gamma = (f, g)$ has the same meaning as above. In the proof of the main theorem we will have to design paths $D = \gamma + d$ of the "dog" such that all $||d(t)||$ are small[1] and $H \circ D - H \circ \gamma$ equals a continuous function $\tau$ which has small norm but which is not otherwise restricted.

We will glue together the desired paths from several pieces: those where $\gamma(t)$ is "sufficiently far away" from the saddle point $(0, 0)$ and those where the path $\gamma$ approaches it.

*The gradient field of H*

Suppose that you are at a particular point $(x, y, H(x, y))$ of our "landscape" and you want to find a position of the dog which is close to you where the altitude over sea level is a little bit larger or smaller than yours. It is natural to go into the direction of the gradient (or the negative gradient) if a higher (or smaller) level is wanted.

This will now be formalized. Fix $(x_0, y_0) \neq (0, 0)$ and consider the initial value problem

$$(x'(s), y'(s)) = (\operatorname{grad} H)\big(x(s), y(s)\big) = \big(y(s), x(s)\big),\ \big(x(0), y(0)\big) = (x_0, y_0).$$

The solution is simple:

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} \frac{x_0+y_0}{2}e^{s} + \frac{x_0-y_0}{2}e^{-s} \\ \frac{x_0+y_0}{2}e^{s} - \frac{x_0-y_0}{2}e^{-s} \end{pmatrix}.$$

It follows that $H\big(x(s), y(s)\big)$ increases with increasing $s$: one has

$$H\big(x(s), y(s)\big) = \frac{x_0^2 + y_0^2}{4}(e^{2s} - e^{-2s}) + \frac{x_0 y_0}{2}(e^{2s} + e^{-2s})$$

and therefore

$$\frac{d}{ds}H\big(x(s), y(s)\big) = \frac{x_0^2 + y_0^2}{2}(e^{2s} + e^{-2s}) + x_0 y_0 (e^{2s} - e^{-2s}).$$

Now let $L, s_0 > 0$ be numbers such that $x_0^2 + y_0^2 \geqslant L$ and $|e^{2s} - e^{-2s}| \leqslant 1/2$ for $|s| \leqslant s_0$. The above derivative has at least the value $L/4$ so that $s \mapsto H\big(x(s), y(s)\big)$ meets all values between $-s_0 L/4$ and $s_0 L/4$ precisely once when $s$ runs through $[\,-s_0, s_0\,]$.

[1] We will work with the maximum norm on $\mathbb{R}^2$: $||(x, y)|| := \max\{|x|, |y|\}$ for $(x, y) \in \mathbb{R}^2$.

This can be summarized as follows:

**Lemma 2.1.** *Let $L > 0$ and $\tilde{\varepsilon} > 0$ be given. Then there are a positive $\tilde{\delta}$ and a continuous function*

$$\Psi : \{(x, y) \mid x^2 + y^2 \geqslant L\} \times \left[-\tilde{\delta}, \tilde{\delta}\right] \to \mathbb{R}^2$$

*such that for all $(x, y)$ with $x^2 + y^2 \geqslant L$ and all $\alpha \in \left[-\tilde{\delta}, \tilde{\delta}\right]$ the following hold:*

- $||(x, y) - \Psi(x, y, \alpha)|| \leqslant \tilde{\varepsilon}$;
- $H\big(\Psi(x, y, \alpha)\big) - H(x, y) = \alpha$.

**Proof.** Above we have shown that $\Psi(x, y, \alpha)$ can be defined for all $x, y, \alpha$ under consideration and that the possible range of $\alpha$ only depends on $L$. That $\Psi$ is continuous (and even $C^\infty$) follows from the implicit function theorem. ∎

*A topological lemma*

It will be important to know that our path $\gamma$ will be "close to" the saddle point only finitely often. To make this precise we prove the following

**Lemma 2.2.** *Let $\varepsilon_0 > 0$ be given. We assume that there exists at least one $t \in [0, 1]$ with $\gamma(t) = (0, 0)$.*

*Suppose that both $\gamma(0)$ and $\gamma(1)$ are different from $(0, 0)$ and that $||\gamma(0)|| > \varepsilon_0$ and $||\gamma(1)|| > \varepsilon_0$.*

*Then there are $0 < a_1 < b_1 < a_2 < b_2 \cdots < a_r < b_r < 1$ such that*

- $||\gamma(t)|| \leqslant \varepsilon_0$ *on* $\bigcup_{i=1,\ldots,r} [a_i, b_i]$;
- *every* $[a_i, b_i]$ *contains at least one zero of* $\gamma$, *and* $\bigcup_{i=1,\ldots,r} [a_i, b_i]$ *contains all of them;*
- $||\gamma(a_i)|| = ||\gamma(b_i)|| = \varepsilon_0$ *for all* $i$.

**Proof.** In a first step choose finitely many closed subintervals of $[0, 1]$ where $||\gamma(t)||$ is bounded by $\varepsilon_0$ such that each interval contains a zero and the union covers all zeros of $\gamma$. Next pass to unions to get disjoint intervals. Enlarge these intervals (if necessary) such that the norm of $\gamma$ at the endpoints is $\varepsilon_0$. And finally it might be that one has to pass to unions again to obtain disjoint intervals. ∎

*The canonical walk*

With the notation of the preceding paragraph let $\Delta$ be the set

$$\Delta = [0, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_r, 1].$$

$\gamma$ has no zero in $\Delta$ so that there is an $L > 0$ such that $||\gamma(t)|| \geqslant L$ for $t \in \Delta$. Let also $\tilde{\varepsilon} > 0$ be given, we choose $\tilde{\delta}$ and $\Psi$ as in lemma 2.1.

We will now construct "walks of the dog" where the times $t$ run through $\Delta$ with prescribed values of $H$. Let a continuous function $\tau : \Delta \to \mathbb{R}$ be given such that $|\tau(t)| \leqslant \tilde{\delta}$ on $\Delta$. For $t \in \Delta$ we put

$$D(t) := \Psi\big(\gamma(t), \tau(t)\big), \qquad d(t) := D(t) - \gamma(t).$$

Then $t \mapsto D(t)$ is continuous, and and for all $t$ one has $||d(t)|| \leqslant \tilde{\varepsilon}$ and $H\big(D(t)\big) = H\big(\gamma(t)\big) + \tau(t)$.

We note in passing that one gets with this construction walks defined on $[\,0,1\,]$ if $\gamma$ never vanishes. Thus $f, g$ have (*) whenever $f$ and $g$ have no common zeros. Since it is easy to perturbate arbitrary $f, g$ such that the perturbations never vanish simultaneously it follows at once that $B_\varepsilon(f) \cdot B_\varepsilon(g)$ always has interior points (this was proved in [1] for continuous and in [3] for differentiable functions).

*Suitable walks in the neighbourhood of* $(0,0)$

The walks close to the saddle point have to be prepared more subtly. First we need some definitions. If $\mu$ is positive we denote by $Q_\mu$ the square in the plane with vertices $(\pm\mu, \pm\mu)$. We will define certain paths in $Q_\mu$. The paths of the first type lead from points in $Q_\mu$ to the edges, those of the second type use only the edges of $Q_\mu$.

*The $P$-walks*

Let $[\,t_1, t_2\,]$ be an interval, $(q_1, q_2) \in Q_\mu$, and $\eta : [\,t_1, t_2\,] \to \mathbb{R}$ a continuous function with $H(q_1, q_2) = q_1 q_2 = \eta(t_1)$. We will define a path which leads between times $t_1$ and $t_2$ from $(q_1, q_2)$ to an edge of $Q_\mu$ in such a way that at every time $t$ the height is just $\eta(t)$.

There will be *four families* of such paths.

(1) ("Walks to the right edge") Suppose that $q_1 > 0$ and that $||\eta|| \leqslant q_1\mu$. We define

$$P^R_{t_1,t_2,\eta,q_1,q_2} : [\,t_1, t_2\,] \to Q_\mu$$

by $t \mapsto \big(N_{q_1,\mu}(t), \eta(t)/N_{q_1,\mu}(t)\big)$, where the $N$-map describes the linear interpolation between two values:

$$N_{\alpha,\beta}(t) := \frac{t - t_1}{t_2 - t_1}\beta + \frac{t_2 - t}{t_2 - t_1}\alpha.$$

This path has the desired properties: it is continuous, it starts at $(q_1, q_2)$, it ends on the right edge, it stays in $Q_\mu$ and the $H$-value at time $t$ is $\eta(t)$.

(2) ("Walks to the top edge") If $q_2 > 0$ and $||\eta|| \leqslant q_2\mu$ we define $P^T_{t_1,t_2,\eta,q_1,q_2} : [\,t_1, t_2\,] \to Q_\mu$ by $t \mapsto \big(\eta(t)/N_{q_2,\mu}(t), N_{q_2,\mu}(t)\big)$.

(3) ("Walks to the left edge") Suppose that $q_1 < 0$ and $||\eta|| \leqslant -q_1\mu$. $P^L_{t_1,t_2,\eta}, q_1, q_2$ is the map $t \mapsto \big(N_{q_1,-\mu}(t), \eta(t)/N_{q_1,-\mu}(t)\big)$.

(4) ("Walks to the bottom edge") Finally, let $q_2 < 0$ and $||\eta|| \leqslant -q_2\mu$. Then $P^B_{t_1,t_2,\eta,q_1,q_2}$ is defined by $t \mapsto \big(\eta(t)/N_{q_2,-\mu}(t), N_{q_2,-\mu}(t)\big)$.

We will also need a version of these paths where *the time is reversed*: they start on the edge and run backwards. This will be indicated by a tilde: for example, $\tilde{P}^R_{t_1,t_2,\eta,q_1,q_2}$ is defined on $[\,t_1, t_2\,]$ by $t \mapsto P^R_{t_1,t_2,\eta,q_1,q_2}(t\,t_1 + (1-t)t_2)$ whenever $q_1 > 0$ and $||\eta|| \leqslant q_1\mu$.

*The E-walks*

This is simpler: we consider $(q_1, q_2)$ on the boundary of $Q_\mu$, a time interval $[\,t_1, t_2\,]$, and a continuous function $\eta : [\,t_1, t_2\,] \to \mathbb{R}$ such that $||\eta|| \leqslant \mu^2$ with $\eta(t_1) = H(q_1, q_2) = q_1 q_2$, and we want to find a path on an edge of $Q_\mu$ such that the $H$-value at time $t$ is $\eta(t)$.

(1) ("Walks on the right edge") Suppose that $q_1 = \mu$.

Then $E^R_{t_1,t_2,\eta,q_1,q_2} : [\,t_1, t_2\,] \to Q_\mu$ is defined by $t \mapsto (\mu, \eta(t)/\mu)$.

(2), (3), (4) The "walks on the top resp. left resp. bottom edge". $E^T, E^L, E^B$ are defined similarly when $q_2 = \mu$ or $q_1 = -\mu$ or $q_2 = -\mu$, respectively:

$$\begin{aligned} E^T_{t_1,t_2,\eta,q_1,q_2}(t) &= (\eta(t)/\mu, \mu), \\ E^L_{t_1,t_2,\eta,q_1,q_2}(t) &= (-\mu, -\eta(t)/\mu), \\ E^B_{t_1,t_2,\eta,q_1,q_2}(t) &= (-\eta(t)/\mu, -\mu). \end{aligned}$$

Now let $\varepsilon_0 > 0$ be fixed. We put $\mu := 2\varepsilon_0$. Suppose that $[\,a, b\,] \subset\, ]\,0, 1\,[$, that $||\gamma(a)|| = ||\gamma(b)|| = \varepsilon_0$ and that $||\gamma(t)|| \leqslant \varepsilon_0$ for $t \in [\,a, b\,]$.

**Definition 2.3.** *We say that $\gamma(a)$ and $\gamma(b)$ are* admissible[1] *if there exist $\varepsilon', \delta' > 0$ with the following properties:*

- $\varepsilon' \leqslant \varepsilon_0$.
- *Suppose that $P_a, P_b \in Q_\mu$ are given such that $||P_a - \gamma(a)||, ||P_b - \gamma(b)|| \leqslant \varepsilon'$ and that $\tau' : [\,a, b\,] \to \mathbb{R}$ is a continuous function with $||\tau'|| \leqslant \delta'$ and $H(P_a) = H\big(\gamma(a)\big) + \tau'(a)$, $H(P_b) = H\big(\gamma(b)\big) + \tau'(b)$. Then there is a continuous walk $D' : [\,a, b\,] \to Q_\mu$ such that $D'(a) = P_a$, $D'(b) = P_b$ and*

$$H\big(D'(t)\big) = H\big(\gamma(t)\big) + \tau'(t)$$

*for all $t \in [\,a, b\,]$.*

The following lemma will be crucial for our investigations:

**Lemma 2.4.** *Suppose that $\gamma$ has no positive and no negative saddle point crossings. Then, under the preceding assumptions, $\gamma(a)$ and $\gamma(b)$ are always admissible.*

**Proof.** $\gamma(a)$ has (maximum) norm $\varepsilon_0$. We may assume that this point lies in the first quadrant $Q^{++}$, the other possibilities can be treated in a similar way. We will consider the following cases:

*Case 1:* $\gamma(a) =: (\alpha_1, \alpha_2)$ lies in the interior of $Q^{++}$.

*Case 1.1:* $\gamma(b) =: (\beta_1, \beta_2)$ lies in the open right halfplane.

We put $\varepsilon' := \min\{\alpha_1/4, \beta_1/4\}$ and $\delta' := \varepsilon_0\alpha_1/4$, and we claim that these numbers have the desired properties.

[1] Note that "admissible" just means that one can find walks of the dog in $Q_\mu$ from $P_a$ to $P_b$ where the relative altitude compared with $H(\gamma)$ can be arbitrarily prescribed provided it is small. The distance between $\gamma(t)$ and $D(t)$ is at most $3\varepsilon_0$ since $||\gamma(t)|| \leqslant \varepsilon_0$ and $D(t) \in Q_\mu$.

Let $P_a, P_b$ and $\tau'$ be given with the properties described in definition 2.3. Our path $D$ will *consist of three parts*: first it moves to the right edge of $Q_\mu$, then it stays there for some time, and finally it moves towards $P_b$.

Choose $t_1 \in \,]\,a, b\,[$ such that $||\gamma(a) - \gamma(t)|| \leqslant \varepsilon'$ for $t \in [\,a, t_1\,]$. We define $D : [\,a, t_1\,] \to \mathbb{R}^2$ by

$$D(t) := P^R_{a,t_1,\eta_1,q_1,q_2}(t),$$

where $(q_1, q_2) = P_a$ and $\eta_1$ is the restriction to $[\,a, t_1\,]$ of

$$t \mapsto \eta(t) := H\big(\gamma(t)\big) + \tau'(t).$$

Note that:

- $\eta_1$ is continuous with $\eta_1(a) = H\big(P(a)\big)$.
- $\gamma(t) \leqslant 5\alpha_1\varepsilon_0$ for all $t \in [\,0, t_1\,]$.
- $q_1 \geqslant 3\alpha_1/4$.

Consequently

$$|\eta_1(t)| = H\big(\gamma(t)\big) + |\tau'(t)| \leqslant 5\varepsilon_0\alpha_1/4 + \varepsilon_0\alpha_1/4 \leqslant \mu q_1;$$

this was a relevant condition in the definition of $P^R_{a,t_1,\eta_1,q_1,q_2}$.

The walk continues. Let $t_2 \in \,]\,t_1, b\,[$ be such that $||\gamma(b) - \gamma(t)|| \leqslant \varepsilon'$ for $t \in [\,t_2, b\,]$. The second part of the walk moves from $(q_1, q_2) := D(t_1) = \big(\mu, \eta(t_1)/\mu\big)$ to $\big(\mu, \eta(t_2)/\mu\big)$ between the times $t_1$ and $t_2$. This is achieved by the map $E^R_{t_1,t_2,\eta_2,q_1,q_2}$ (with $\eta_2 = \eta|_{[\,t_1,t_2\,]}$), and the path stays in $Q_\mu$ since $|\eta_2(t)| \leqslant |H\big(\gamma(t)\big)| + |\tau'(t)| \leqslant \mu^2$.

And finally we use $\tilde{P}^R_{t_2,b,\eta_3,\tilde{q}_1,\tilde{q}_2}$ to come from $D(t_2)$ to $P_B =: (\tilde{q}_1, \tilde{q}_2)$ (with $\eta_3 = \eta|_{[\,t_2,b\,]}$).

*Case 1.2:* $\gamma(b) =: (\beta_1, \beta_2)$ lies in the open upper halfplane.

This case can be treated in a similar way: move to the upper edge in a short time interval $[\,a, t_1\,]$ where $\gamma$ stays close to $\gamma(a)$, stay on this edge for some time, and finally (for $t \in [\,t_2, b\,]$ where $\gamma$ is already close to $\gamma(b)$) move to $P_b$.

*Case 1.3:* It remains to treat the case $\gamma(b) =: (\beta_1, \beta_2) \in Q^{--}$. Here our assumption comes into play.

Since $\gamma$ has no positive saddle point crossings there must exist $t_0 \in [\,a, b\,]$ with $H\big(\gamma(t_0)\big) < 0$. We choose a small interval $[\,t_2, t_3\,]$ with $t_2 < t_0 < t_3$ such that $H\big(\gamma(t)\big) < 0$ for $t$ in this interval. Note that $\eta(t) := H\big(\gamma(t)\big) + \tau'(t)$ will be strictly negative there if $||\tau'||$ is small.

Suppose first that $\beta_1 < 0$. Then $D$ is defined on $[\,a, b\,]$ as follows, at every $t$ the "height" $H\big(D(t)\big)$ of $D(t)$ is $\eta(t)$:

- Move to the upper edge of $Q_\mu$ while $t \in [\,a, t_1\,]$. This part uses the path $P^T$.
- Stay there during $[\,t_1, t_2\,]$. (The explicit formule is given by the $E^T$-path.)
- For the $t \in [\,t_2, t_3\,]$ the dog moves (with $P^L$) from the top edge to the left edge of $Q_\mu$.

- Between $t_3$ and $t_4$ the walk is driven by $E^L$; here $t_4$ is close to $b$ such that $\gamma(t)$ is close to $\gamma(b)$ for $t \in [\,t_4, b\,]$.
- Finally, on $[\,t_4, b\,]$, walk according to $\tilde{P}^L$ to come from the left edge to $P_B$.

If $\beta_1 = 0$ we need a different procedure. (The $t_i$ have the same meaning as before.) First move to the right edge between times $a$ and $t_1$; stay there until $t_2$; move between $t_2$ and $t_3$ to the bottom edge; stay there until $t_4$; turn to $P_b$ between $t_4$ and $b$.

*Case 2:* $\gamma(a) =: (\alpha_1, \alpha_2)$ lies on the boundary of $Q^{++}$. This means that $\gamma(a) = (\varepsilon_0, 0)$ or $\gamma(a) = (0, \varepsilon_0)$. Let us suppose that $\gamma(a) = (0, \varepsilon_0)$.

If $\gamma(b)$ is not in the set $\{(-\varepsilon_0, 0), (0, -\varepsilon_0), (\varepsilon_0, 0)\}$ we are done: one simply has to reverse the roles of $\gamma(a)$ and $\gamma(b)$, to find a path for this situation as in the preceding investigations and then let the time $t$ run backwards. Thus there remain three cases which have to be considered.

*Case 2.1:* $\gamma(b) = (\varepsilon_0, 0)$.

It cannot be the case that all $\gamma(t)$ for $t \in [\,a, b\,]$ are in $Q^{-+} \cup Q^{+-}$ since this would mean that $\gamma$ has a negative saddle point crossing. Therefore there must exist $t_0 \in [\,a, b\,]$ with $H\big(\gamma(t_0)\big) > 0$. Thus $\eta := H \circ \gamma + \tau'$ will be positive on a suitable small interval $[\,t_2, t_3\,]$ around $t_0$ provided that $||\tau'||$ is small.

Here is the construction of $D$ with $H\big(D(t)\big) = \eta(t)$: move, starting at $P_a$, between times $a$ and $t_1$ (with $t_1$ close to $a$) to the top edge; stay there until $t_2$; move between $t_2$ and $t_3$ to the right edge; stay there until $t_4$ (which is close to $b$); turn to $P_b$ between $t_3$ and $b$.

*Case 2.2:* $\gamma(b) = (-\varepsilon_0, 0)$.

This situation can be treated in a similar way.

*Case 2.3:* $\gamma(b) = (0, -\varepsilon_0)$.

This is the most complicated case. We claim that there must be $t_+, t_- \in [\,a, b\,]$ such that

$$H\big(\gamma(t_+)\big) > 0,\ H\big(\gamma(t_+)\big) < 0\ :$$

since if this would not hold $\gamma$ would have a positive or a negative saddle point crossing. Suppose that, without loss of generality, $t_+ < t_-$. We choose $t_1, \ldots, t_6$ with $t_1$ close to $a$, $t_6$ close to $b$, and on $[\,t_2, t_3\,]$ (resp. on $[\,t_4, t_5\,]$) the function $\eta := H \circ \gamma + \tau'$ is strictly positive (resp. strictly negative); as above this can be achieved if $||\tau'||$ is small. This time the dog has to be rather busy. It walks:

- from $P_a$ to the top edge between $a$ and $t_1$;
- on the top edge between $t_1$ and $t_2$;
- from the top edge to the right edge between $t_2$ and $t_3$;
- on the right edge between $t_3$ and $t_4$;
- to the bottom edge between $t_4$ and $t_5$;
- on the bottom edge between $t_5$ and $t_6$;
- from the bottom etge to $P_b$ between $t_6$ and $b$.

And this can be done in such a way that $H \circ D = H \circ \gamma + \tau'$ during the walk. ∎

## 3. The main result

Our main result is

**Theorem 3.1.** *Let $f, g \in C\,[0,1]$ be given and $\gamma = (f, g)$ the associated path in $\mathbb{R}^2$. Then the following assertions are equivalent:*

(1) *$f \cdot g$ is an interior point of $B_\varepsilon(f)B_\varepsilon(g)$ for all $\varepsilon > 0$.*
(2) *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that the functions $f \cdot g + \underline{\delta}$ and $f \cdot g - \underline{\delta}$ are in $B_\varepsilon(f)B_\varepsilon(g)$; here $\underline{\delta}$ (resp. $-\underline{\delta}$) denotes the constant function $\delta$ (resp. $-\delta$).*
(3) *$\gamma$ has no positive and no negative saddle point crossings.*

**Proof.** "1 ⇒ 2:" This is trivially true.

"2 ⇒ 3:" This will be shown by proving that "¬ 3" implies "¬ 2". Let us assume, e.g., that $\gamma$ has a positive saddle point crossing: Suppose, e.g., that $\gamma$ is in $Q^{++}$ on $[\,t_0 - r, t_0\,]$ and in $Q^{--}$ on $[\,t_0, t_0 + r\,]$, and $\gamma(t_1)$ (resp. $\gamma(t_1)$) is in $Q^{++} \setminus \{(0,0)\}$ (resp. $Q^{--} \setminus \{(0,0)\}$) for a suitable $t_1 \in [\,t_0 - r, t_0\,]$ (resp. $t_2 \in [\,t_0, t_0 + r\,]$). Choose $\varepsilon > 0$ such that an $x \in \mathbb{R}^2$ lies necessarily in the upper half plane if $H(x) > 0$ and $||x - \gamma(t_1)|| \leqslant 2\varepsilon$. Also we assume that $y$ lies in the lower halfplane provided that $H(y) > 0$ and $||\gamma(t_2) - y|| \leqslant 2\varepsilon$. This implies that $(\gamma + d)(t_1)$ is in the upper halfplane (resp. $(\gamma + d)(t_2)$ is in the lower halfplane) whenever $||d(t_1)|| \leqslant \varepsilon$ (resp. $||d(t_2)|| \leqslant \varepsilon$) and $H \circ (\gamma + d) > H \circ \gamma$. Thus, necessarily, there exist $t \in [\,t_1, t_2\,]$ where $\gamma + d$ meets the line $\mathbb{R} \times \{0\}$ so that $H\big((\gamma + d)(t)\big) = 0$. It follows that $f \cdot g + \underline{\delta}$ is not contained in $B_\varepsilon(f)B_\varepsilon(g)$ for any positive number $\delta$. (There is no walk of the dog such that it stays close to you and its altitude is always your altitude $+\delta$.)

Similarly it follows that there is an $\varepsilon > 0$ such that $f \cdot g - \underline{\delta}$ is not contained in $B_\varepsilon(f)B_\varepsilon(g)$ if $\gamma$ has a negative saddle point crossing.

"3 ⇒ 1:" This is the most difficult part of the proof. We assume that $\gamma$ has no positive and no negative saddle point crossings, and we have to find for given $\varepsilon > 0$ a $\delta > 0$ such that all $f \cdot g + \tau$ lie in $B_\varepsilon(f)B_\varepsilon(g)$, where $\tau \in C\,[0,1]$ is arbitrary with $||\tau|| \leqslant \delta$.

Let $\varepsilon > 0$ be given. We suppose for the moment that $||\gamma(0)||, ||\gamma(1)|| > \varepsilon$. (The necessary modifications of the proof if this doesn't hold are indicated later.) Our aim is to find a positive $\delta$ such that for every continuous $\tau : [\,0,1\,] \to \mathbb{R}$ with $||\tau|| \leqslant \delta$ there exists a path $\gamma + d$ which is $\varepsilon$-close to $\gamma$ at every moment, and $H\big(\gamma(t) + d(t)\big) = H\big(\gamma(t)\big) + \tau(t)$ is true for all $t$.

First we fix $\varepsilon_0$, a positive number less than $\varepsilon/3$. We find the $[\,a_i, b_i\,]$, $i = 1, \ldots, r$ according to lemma 2.2, and as above $\Delta$ is the set

$$[\,0, a_1\,] \cup [\,b_1, a_2\,] \cup \cdots \cup [\,b_r, 1\,].$$

As before we denote by $L$ the positive number $\min_{t \in \Delta} ||\gamma(t)||$. By lemma 2.4 we may choose positive $\delta', \varepsilon'$ such that the definition of admissibility applies to all $[\,a_i, b_i\,]$. Now find a $\tilde{\delta} > 0$ for $L$ and $\tilde{\varepsilon} := \min\{\varepsilon', \varepsilon\}$ as in lemma 2.1. We claim that $\delta := \min\{\delta', \tilde{\delta}\}$ has the desired properties.

Let $\tau \in C[0,1]$ with $||\tau|| \leqslant \delta$ be given. We define $D : \Delta \to \mathbb{R}^2$ as the canonical walk which is constructed by using the map $\Psi$ associated with $\tilde{\varepsilon}$ and $L$. Thus $||D(t) - \gamma(t)|| \leqslant \tilde{\varepsilon} \leqslant \varepsilon$, and $H\big(D(t)\big) = H\big(\gamma(t)\big) + \tau(t)$ for $t \in \Delta$. And the gaps can be closed with the help of lemma 2.4 which will be used successively for $i = 1, \ldots, r$, the intervals $[a, b] = [a_i, b_i]$ and $P_a = D(a_i), P_b = D(b_i)$. Note that always $||D(t) - \gamma(t)|| \leqslant 3\varepsilon_0 \leqslant \varepsilon$ since $D(t) \in Q_\mu$ and $||\gamma(t)|| \leqslant \varepsilon_0$ for $t \in \bigcup [a_i, b_i]$.

To complete the proof of the theorem we have to discuss the case when $\gamma(a)$ or $\gamma(b)$ or both vanish.

If $\gamma(t) = 0$ for all $t$ it is easy to see that property (*) holds. Given $\varepsilon > 0$ we put $\delta := \varepsilon^2$. Then, if $\tau$ is continuous with $||\tau|| \leqslant \delta$, the walk $D(t) := (\varepsilon, \tau(t)/\varepsilon)$ has the desired property: $||\gamma(t) - D(t)|| \leqslant \varepsilon$, and

$$H\big(D(t)\big) = \tau(t) = H\big(\gamma(t)\big) + \tau(t).$$

Thus suppose that there exists $t'$ with $\gamma(t') \neq (0,0)$. We will restrict ourselves to $\varepsilon$ with $\varepsilon < ||\gamma(t')||$. We start as above by setting $\varepsilon_0 = \varepsilon/3$. Let $\gamma(a) = (0,0)$. Now the topological lemma 2.2 is only true with the following modification: The first interval $[a_1, b_1]$ is of the form $[0, b_1]$, and $||\gamma(b_1)|| = \varepsilon_0$. On the complement of $\bigcup [a_i, b_i]$ the paths $D$ can be defined in the canonical way, but it remains to close the gap between 0 and $b_1$ (and maybe between $a_r$ and 1 if it happens that $\gamma(1) = (0,0)$, too).

Suppose that $||\gamma(t)|| \leqslant \varepsilon_0$ on $[0, b_1]$, that $||\gamma(b_1)|| = \varepsilon_0$ and that, e.g., the $x$-component of $\gamma(b_1)$ equals $\varepsilon_0$ (recall that we work with the maximum norm on $\mathbb{R}^2$). A continuous function $\tau$ is given with sufficiently small norm, and $\eta := H \circ \gamma + \tau$ is written on $\Delta$ as $H \circ D$. Put $P_b := D(b_1)$, and we want to extend the definition of $D$ in a continuous way to $[0, b_1]$ such that $H \circ \gamma + \tau = H \circ D$ holds also on this interval.

The construction is similar to those in the proof of lemma 2.4. We choose $t_1$ close to $b_1$ such that $\gamma(t)$ stays close to $\gamma(b_1)$ on $[t_1, t]$. We start our walk at $D(0) := (\mu, \tau(0)/\mu)$ and continue to stay on the right edge of $Q_\mu$ by setting $D(t) := (\mu, \eta(t)/\mu)$ for $t \in [0, t_1]$; as before $\mu$ stands for $2\varepsilon_0$. And between $t_1$ and $b_1$ we define $D$ with the help of $\tilde{P}^R$ to arrive at $D(b_1) = P_b$.

Depending on $\gamma(b_1)$ it might be necessary to use one of the other edges of $Q_\mu$ (e.g., the top edge if the $y$-component of $\gamma(b_1)$ equals $\varepsilon_0$). And the final part (for the $t$ between $a_r$ and 1) is treated similarly if $\gamma(1) = (0,0)$. ■

We close this section by stating the following immediate corollaries to our theorem:

**Corollary 3.2.** *Let $f, g \in C[0,1]$ be such that $f \cdot g$ doesn't change sign, i.e., all $\gamma(t)$ lie in $Q^{++} \cup Q^{--}$ or all lie in $Q^{+-} \cup Q^{-+}$. Then $f, g$ have (*) iff neither $f$ nor $g$ changes sign on $[0,1]$.*

**Proof.** One only has to note that the condition is equivalent with the fact that $(f, g)$ has no positive and no negative saddle point crossing. ■

**Remark.** Now it is clear why our first example $f(t) = g(t) = t - 0.5$ fails to have (*): here $\gamma$ has a positive saddle point crossing at $t = 0.5$.

On the other hand, if $f(t) = g(t) = |t - 0.5|$, then $\gamma$ meets the saddle point of $H$ at $t = 0.5$. But no saddle point crossing occurs so that $f, g$ have (*) in this case.

**Corollary 3.3.** *Suppose that $O_1, O_2 \subset C\,[0,1]$ are open and that $f, g$ have (*) for arbitrary $f \in O_1, g \in O_2$. Then $O_1 O_2$ is also open.*

## 4. Invitations for further study

The aim of this paper was the study of the obstructions which are responsible for the fact that $f \cdot g$ is sometimes not in the interior of $B_\varepsilon(f)B_\varepsilon(g)$.

It would be interesting to consider the similar problem in other Banach algebras $Y$. As before we say that $f, g \in Y$ have $(*)$ if $f \cdot g$ is an interior point of $B_\varepsilon(f)B_\varepsilon(g)$ for every $\varepsilon > 0$. Here are some first observations:

(1) In $l^\infty$ every two $f, g$ have (*). This is obvious. (More generally it has been shown in [2] that, for compact $X$, each two $f, g \in CX$ have (*) iff $X$ is zero dimensional.)

(2) Let $C_b\mathbb{R}$ be the Banach algebra of bounded continuous real valued functions on $\mathbb{R}$, provided with the supremum norm. Then $f, g$ fail to have (*) if the associated path $\gamma = (f, g)$ has positive or negative saddle point crossings, but the reverse implication doesn't hold. A characterization seems to be difficult.

(3) If we consider $Y$ to be the set of *complex-valued* continuous functions on $[0,1]$, then every two $f, g$ have (*) (so that the product of open subsets of $X$ is always open). This can be veryfied by adapting the methods used in this paper. For the proof it is crucial that the boundary of the unit sphere in $\mathbb{C}$ (in contrast to that of $\mathbb{R}$) is path-connected.

(4) It is completely open how a characterization could look like in the case of the Banach algebra $CX$ of real (or complex) valued continuous functions on an arbitrary compact space $X$.

(5) The Banach algebra $C^N\,[0,1]$ of $N$ times continuously differentiable real functions studied in [3] can be treated in a similar way as $C\,[0,1]$ in the present paper.The characterization of $f, g$ such that $f \cdot g$ have (*) is the same: no positive or negative saddle point crossings. One only has to observe that the constructions above can be modified such that one arrives at differentiable functions. (Rigorous proofs, however, will be rather clumsy.)

Also generalizations of the walk-the-dog illustration could be studied by replacing $H$ by a more general function. Consider continuous functions $\Phi : \mathbb{R}^2 \to \mathbb{R}$ and $\gamma : [0,1] \to \mathbb{R}^2$. We will say that $\gamma$ is $\Phi$-admissible if for every $\varepsilon > 0$ there exists $\delta > 0$ with the following property:

*Whenever $\tau \in C\,[0,1]$ is given with $||\tau|| \leqslant \delta$ there exists a continuous $D : [0,1] \to \mathbb{R}^2$ such that $||\gamma(t) - D(t)|| \leqslant \varepsilon$ and $\Phi\big(D(t)\big) = \Phi\big(\gamma(t)\big) + \tau(t)$ for every $t$.*

This is the same situation as before (where we worked with $\Phi(x,y) = H(x,y) = xy$), but the "landscape" might be more complicated.

In some simple cases one "sees" that all $\gamma$ are $\Phi$-admissible. This is, e.g., the case when $\Phi(x,y) := \max\{x,y\}$, or $\Phi(x,y) := \min\{x,y\}$, or $\Phi(x,y) := x+y$, a fact which corresponds to results in [1] where it has been shown that $(f,g) \mapsto \max\{f,g\}$, $(f,g) \mapsto \min\{f,g\}$, and $(f,g) \mapsto f+g$ (from $C\,[0,1] \times C\,[0,1]$ to $C\,[0,1]$) map open sets to open sets.

It is clear that $\gamma$ cannot be $\Phi$-admissible if $\gamma$ passes through a local maximum or a local minimum of $\Phi$. If $\Phi$ is smooth a characterization of $\Phi$-admissibility will have to use the behaviour of $\gamma$ close to the saddle points of $\Phi$. It has to be expected that the same techniques as in the present paper can be applied successfully.

Even more generally one could study "walks" in higher dimensions, i.e. smooth functions $\Phi : \mathbb{R}^n \to \mathbb{R}$ and continuous $\gamma : \mathbb{R} \to \mathbb{R}^n$. Also then the singular points of $\Phi$ will be of crucial importance for the characterization of $\Phi$-admissible $\gamma$.

For general $\Phi$ which are continuous but not necessarily smooth (that is, for possibly "very rough" landscapes), it is not likely that a simple description of the $\Phi$-admissible $\gamma$ exists.

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