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LOCAL REGULARITY RESULTS FOR SECOND ORDER ELLIPTIC SYSTEMS ON LIPSCHITZ DOMAINS

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Dedicated with great pleasure to Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: For a class of strongly elliptic, second order systems L with rough coefficients on a Lipschitz domain Ω, we show that if $Lu = 0$ on Ω and u vanishes on an open subset of the boundary, then weak a priori hypotheses on the nontangential maximal function of u lead to strong estimates on ∇u, in nontangential and Besov norms, near this subset.

Keywords: Second order elliptic PDE, Sobolev Besov spaces, boundary regularity, Lipschitz domains

1. Introduction

In this paper we are concerned with the regularity of elliptic PDE's under minimal smoothness assumptions. A specific question we address in this context can be phrased as follows: *given a variable coefficient, second order, symmetric, strongly elliptic system L and a Lipschitz domain Ω, how smooth is the solution of the Dirichlet problem for L in Ω away from the support of its boundary datum*? The coefficients of the operator L are only mildly regular and the smoothness of the solution is characterized both in terms of membership to Besov spaces, as well as the size of the associated nontangential maximal function.

In order to be more specific, we need some notation. Consider a second-order, strongly elliptic, formally self-adjoint operator L, acting between sections of vector bundles over a compact, Riemannian manifold M. It is assumed that the metric structures on M and the vector bundles in question have C^1 coefficients and that, in local coordinates $U \subset M$ over which the bundles are trivial,

$$Lu = \sum_{j,k} \partial_j A^{jk}(x)\partial_k u + \sum_j B^j(x)\partial_j u - V(x)u, \tag{1.1}$$

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where A^{jk}, B^j and V are matrix-valued functions with the property that

$$A^{jk} \in C^{1+\theta}, \ \theta > 0, \quad B^j \in H^{1,r}, \quad V \in L^r, \ r > \dim M. \tag{1.2}$$

The metric structures on M and the vector bundles involved are assumed to have C^1 coefficients.

We shall say that L satisfies the *non-singularity hypothesis* provided

$$\forall D (\subseteq M) \text{ Lipschitz domain}, \quad u \in L^2_{1,0}(D), \ Lu = 0 \text{ in } D \Longrightarrow u = 0 \text{ in } D. \tag{1.3}$$

As observed in [MT2], if L is negative-definite on M then (1.3) is automatically verified. Also, if L is strongly elliptic, then $L - \lambda$, $\lambda \in \mathbb{R}$, satisfies the non-singularity hypothesis (1.3) provided λ is sufficiently large. This follows from the Gårding inequality which, as observed in [MMT], holds in our setting (despite the fact that V may be unbounded). Also, if L is strongly elliptic and negative semidefinite, then $L - \lambda$ satisfies (1.3) for any $\lambda > 0$.

For L as above and Ω a Lipschitz subdomain of M, consider the Dirichlet problem

$$Lu = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f \in L^2(\partial\Omega), \quad u^* \in L^2(\partial\Omega). \tag{1.4}$$

Above, u^* denotes the nontangential maximal function of u (see (2.3)). Let $B^{p,q}_s$ denote the Besov scale of spaces (cf. the discussion in Section 2).

Theorem 1.1. *Assume that L is strongly elliptic, formally selfadjoint operator satisfying (1.1)-(1.2), and that $\Omega \subset M$ is Lipschitz. Let Γ_0 be an open subset of $\partial\Omega$. There exist $p = p(\Omega, \Gamma_0, L) > 2 > q = q(\Omega, \Gamma_0, L)$ such that if u solves (1.4) then*

$$\begin{aligned} &f \in L^q(\partial\Omega), \ \operatorname{supp} f \subset \Gamma_0 \Longrightarrow \\ &\qquad (\nabla u)^*\big|_{\partial\Omega\setminus\Gamma_0} \in L^p(\partial\Omega \setminus \Gamma_0) \ \textit{and} \ u \in B^{p,p}_{1+1/p}(\bar{\Omega} \setminus \Gamma_0; \mathrm{loc}). \end{aligned} \tag{1.5}$$

See also Theorem 3.1 for a more refined version. Our main results, Theorems 1.1–3.1, are sharp. Indeed, take $L := \Delta$, the Laplacian, define the Lipschitz domain Ω and the harmonic function $u : \Omega \to \mathbb{R}$ by

$$\begin{aligned} \Omega &:= \big\{ z \in \mathbb{C} = \mathbb{R}^2 : |z| < 1 \text{ and } 0 < \arg z < 2(1-\delta)\pi \big\}, \quad 0 < \delta < 1, \\ &\text{and } u(z) := \operatorname{Im} z^\beta, \quad z \in \Omega, \text{ where } \beta := 1/(2 - 2\delta), \end{aligned} \tag{1.6}$$

and, finally, consider Γ_0 a small neighborhood of $\{e^{i\theta} : 0 \leqslant \theta \leqslant 2(1-\delta)\pi\}$ in $\partial\Omega$ (in the relative topology induced by $\mathbb{R}^2$ on $\partial\Omega$). In this setting, we see that, given any $p_0 > 2$, it is possible to choose $\delta \in (0,1)$ such that *both* conclusions in (1.5) fail for $p = p_0$.

When all structures involved are smooth, these issues are well understood on *all* scales of smoothness, via pseudodifferential operators and Calderón-Zygmund theory (a classical reference is [ADN]). However, as pointed out above (cf. also Remark 3.1) there are natural limitations of the theory which can be developed in the presence of irregularities. The Hölder regularity at the boundary in a general *scalar* context was worked out by Stampacchia [St] (when the coefficients are only bounded and measurable; for related results and more recent progress see also [Ke]). Nonetheless, at the level of *systems* of PDE's, much less is known. The main goal of this note is to shed some light in this regard.

In order to carry out our program we rely primarily on our work in [MMT], [MT2], where the global, well-posedness problem of the Dirichlet problem with boundary data in Sobolev-Besov spaces on Lipschitz domains was addressed. The latter theme is reviewed and further expanded in §2 (cf. especially Theorem 2.1), while the main estimates are derived in §3. Our approach is rather flexible and could, in principle, be adapted to other situations of interest.

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2. Notation and preliminary results

Let M be a smooth, compact, boundaryless, Riemannian manifold of real dimension n, and let L be a second order differential operator (between two Hermitian vector bundles over M). Locally, we assume that L is as in (1.1) and that the (matrix-valued) coefficients are as in (1.2). The metric tensors on M and the vector bundles are assumed to have C^1 coefficients.

First, recall that a domain $\Omega \subset M$ is called Lipschitz provided $\partial\Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions. Also, the Sobolev scale $L^p_s(M)$, $1 < p < \infty$, $s \geqslant 0$, is obtained by lifting $L^p_s(\mathbb{R}^n) := \{(I-\Delta)^{s/2} f : f \in L^p(\mathbb{R}^n)\}$ to M. We denote by $L^p_s(\Omega)$ the restriction of elements in $L^p_s(M)$ to the Lipschitz domain Ω. As is customary, for $s > 0$ and $1 < p, q < \infty$ with $1/p + 1/q = 1$, we set $L^p_{-s}(\Omega) := (L^q_{s,0}(\Omega))^*$, where $L^p_{s,0}(\Omega)$ stands for the space consisting of restrictions to Ω of elements from $L^p_s(M)$ with support contained in $\bar{\Omega}$. Boundary Sobolev spaces $L^p_s(\partial\Omega)$, $1 < p < \infty$, $0 \leqslant s \leqslant 1$, can be introduced by starting with the Euclidean model, $L^p_s(\mathbb{R}^{n-1})$, via a partition of unity and pull-back. These Sobolev scales are then related to Besov spaces via real interpolation. For instance, we have the formulas

$$\left(L^p(\Omega), L^p_k(\Omega)\right)_{s,q} = B^{p;q}_{sk}(\Omega) \tag{2.1}$$

and

$$\left(L^p(\partial\Omega), L^p_1(\partial\Omega)\right)_{s,q} = B^{p,q}_s(\partial\Omega) \tag{2.2}$$

when $0 < s < 1$, $1 < p, q < \infty$ and k is a positive integer. Furthermore, the trace operator Tr is well-defined from $L^p_s(\Omega)$ and $B^{p,p}_s(\Omega)$ onto $B^{p,p}_{s-1/p}(\partial\Omega)$ for each $1 < p < \infty$ and $1/p < s < 1 + 1/p$. For a more detailed exposition, the interested reader is referred to [BL], [JK], [Tr], [MT2].

Next, we let $(\cdot)^*$ stand for the nontangential maximal operator. More specifically, if $\{\gamma(x)\}_{x\in\partial\Omega}$ is a family of nontangential approach regions with "vertices" at boundary points (cf. [JK], [MT1] for more details), and u is defined in Ω then u^*, the nontangential maximal function of u, if defined at boundary points by

$$u^*(x) := \sup\{|u(y)| : y \in \gamma(x)\}, \qquad x \in \partial\Omega. \tag{2.3}$$

We also set

$$u\Big|_{\partial\Omega}(x) := \lim_{\substack{y\to x\\ y\in\gamma(x)}} u(y), \qquad x \in \partial\Omega, \tag{2.4}$$

i.e., for boundary traces taken in the nontangential limit sense, whenever meaningful. Denote by $u = \mathrm{PI}\, f$ the solution operator for the Dirichlet problem (1.4), whenever this is well-posed.

Theorem 2.1. *Assume that L is a strongly elliptic, formally self-adjoint operator satisfying (1.1)-(1.3), and that $\Omega \subset M$ is Lipschitz. Then* PI *is well-defined and there exists $\varepsilon = \varepsilon(\partial\Omega, L) > 0$, $C = C(\partial\Omega, L) > 0$, such that*

$$\|(\mathrm{PI}\, f)^*\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\partial\Omega)}, \quad 2-\varepsilon < p < 2+\varepsilon, \tag{2.5}$$

and

$$\|(\nabla\, \mathrm{PI}\, f)^*\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p_1(\partial\Omega)}, \quad 2-\varepsilon < p < 2+\varepsilon. \tag{2.6}$$

Furthermore, for each $p \in (2-\varepsilon, 2+\varepsilon)$, PI *extends as a bounded operator*

$$\mathrm{PI} : L^p_s(\partial\Omega) \to B^{p,p\vee 2}_{s+1/p}(\Omega), \quad 0 \leqslant s \leqslant 1, \tag{2.7}$$

(*hereafter, $p \vee 2 := \max\{p, 2\}$*), *as well as an isomorphism*

$$\mathrm{PI} : B^{p,p}_s(\partial\Omega) \to L^p_{s+1/p}(\Omega) \cap B^{p,p}_{s+1/p}(\Omega) \cap \ker L, \quad 0 < s < 1. \tag{2.8}$$

Remark 2.1. The fact that PI is well-defined on $L^2(\partial\Omega)$ answers a question raised by E. Fabes on p. 78 of [Fa], in the context of general, variable coefficient systems on Lipschitz domains.

Remark 2.2. If $n = \dim M = 2, 3$ then as a consequence of (2.7) and embedding results,

$$\mathrm{PI} : L^p_1(\partial\Omega) \to C^\alpha(\overline{\Omega}), \quad p > 2, \;\; \alpha = \alpha(p) > 0. \tag{2.9}$$

In particular, the Poisson integral of Lipschitz functions on $\partial\Omega$ is globally Hölder continuous for $n \leqslant 3$ (if L is scalar, then this is true in general). This is related to another question raised in [Fa] (cf. p. 79, *loc. cit.*).

Remark 2.3. It is also implicit in the proof below that any u satisfying $u^* \in L^p(\partial\Omega)$, $2-\varepsilon < p < 2+\varepsilon$, and $Lu = 0$ in Ω, is of the form $u = \mathrm{PI}\, f$ for some $f \in L^p(\partial\Omega)$. Also, if in addition $(\nabla u)^* \in L^p(\partial\Omega)$ then in fact $f \in L^p_1(\partial\Omega)$.

Remark 2.4. In the case when the operator L is scalar, the range of indices s, p for which the conclusions of Theorem 2.1 are valid extends considerably. The interested reader is referred to [MT2] for details.

Proof of Theorem 2.1. Since L satisfies the nonsingularity hypothesis (1.3) the results in §3 of [MMT] imply that PI is well-defined and that (2.5)-(2.6) hold. In fact, an inspection of the arguments in [MMT] reveals that

$$\mathrm{PI}\, f = \mathcal{S}(S^{-1} f) \qquad \text{in } \Omega, \tag{2.10}$$

where $\mathcal{S}$ is the so-called single layer potential operator associated with L in Ω, and S is its boundary trace. More specifically, if $E(x, y)$ is the Schwartz kernel of L^{-1} and if $d\,\sigma$ denotes the natural surface measure on $\partial\Omega$ (induced by the metric on M), then

$$\mathcal{S} f(x) := \int_{\partial\Omega} \langle E(x, y), f(y)\rangle \, d\sigma(y), \qquad x \in \Omega. \tag{2.11}$$

Also, we denote by $Sf := \mathcal{S} f|_{\partial\Omega}$, the boundary version of (2.11). The key ingredient, proved in [MMT], is that

$$S : L^p(\partial\Omega) \to L^p_1(\partial\Omega), \quad 2 - \varepsilon < p < 2 + \varepsilon,$$

is an invertible operator. By duality and complex interpolation we also get that

$$S : L^p_{-s}(\partial\Omega) \to L^p_{1-s}(\partial\Omega), \quad 2 - \varepsilon < p < 2 + \varepsilon, \;\; 0 \leqslant s \leqslant 1,$$

is invertible and, further, by real interpolation, that

$$S : B^{p,p}_{-s}(\partial\Omega) \to B^{p,p}_{1-s}(\partial\Omega), \quad 2 - \varepsilon < p < 2 + \varepsilon, \;\; 0 < s < 1,$$

is invertible. Next, recall from §7 of [MT2] that for each $1 < p < \infty$,

$$\mathcal{S} : L^p_{-s}(\partial\Omega) \longrightarrow B^{p, p\vee 2}_{1-s+1/p}(\Omega), \quad 0 \leqslant s \leqslant 1,$$

and

$$\mathcal{S} : B^{p,p}_{-s}(\partial\Omega) \longrightarrow B^{p,p}_{1-s+1/p}(\Omega), \quad 0 < s < 1,$$

are bounded operators. These, in concert with (2.10), justify the claim made about (2.7) plus the fact that the operator in (2.8) is well-defined and bounded. That the latter operator is also onto, follows from what we have proved so far with the aid of the lemma below (itself, of independent interest). ■

We now turn to the uniqueness result invoked above. To state it properly, we recall a useful approximation result. Concretely, given a Lipschitz domain Ω, it is possible a family $\{\Omega_j\}_{j\in\mathbb{N}}$ of domains in M satisfying the following properties:

(i) Each Ω_j is a Lipschitz domain whose Lipschitz character is bounded uniformly in $j \in \mathbb{N}$;

(ii) One has $\Omega_j \subset \Omega_{j+1} \subset \Omega$ and $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$;

(iii) There exist a family of bi-Lipschitz homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, $j \in \mathbb{N}$, such that $\Lambda_j(x) \to x$ as $j \to \infty$, in a nontangential fashion;

(iv) There exist non-negative, measurable functions ω_j on $\partial\Omega$, $j \in \mathbb{N}$, which are bounded away from zero and infinity uniformly in $j \in \mathbb{N}$, and which have the following properties. First, $\omega_j(x) \to 1$ as $j \to \infty$, for a.e. $x \in \partial\Omega$. Second, for each integrable function $g : \partial\Omega_j \to \mathbb{R}$ the following change of variable formula holds
$$\int_{\partial\Omega_j} g\, d\sigma_j = \int_{\partial\Omega} g \circ \Lambda_j\, \omega_j\, d\sigma,$$
where σ_j is the canonical surface measure on $\partial\Omega_j$.

In other words, the Jacobians of the transformations $x' = \Lambda_j(x)$ are bounded away from zero and infinity, and converge pointwise a.e. to 1. In the Euclidean setting, this has been proved in [Ne], [Ve], and a further adaptation to the manifold setting can be found in Appendix A of [MT1], where other pertinent properties are presented.

Lemma 2.2. *Retain the same hypotheses as in Theorem 2.1, and fix a sequence of approximating domains $\Omega_j \nearrow \Omega$ as discussed above. Then there exists $\varepsilon > 0$ so that for any $u \in C^0_{\mathrm{loc}}(\Omega)$,*
$$Lu = 0 \ \text{ in } \ \Omega, \qquad \lim_j \int_{\partial\Omega_j} |u|^{2-\varepsilon}\, d\sigma_j = 0 \Longrightarrow u \equiv 0. \tag{2.12}$$

Proof. As in the proof of Theorem 2.1, there is no loss of generality assuming that L satisfies the nonsingularity hypothesis (3.3) of [MMT]. Then the conclusion follows from the estimate (3.51) of [MMT]. ∎

Some comments are in order here (they are particularly useful in the various applications of the above lemma). Concretely, consider a sequence of approximating domains $\Omega_j \nearrow \Omega$, as above. Then, so we claim, if Tr and Tr_j denote, respectively, the trace operators on $\partial\Omega$ and on $\partial\Omega_j$,
$$(\mathrm{Tr}_j\, w) \circ \Lambda_j \to \mathrm{Tr}\, w \ \text{ in } \ L^p(\partial\Omega), \qquad \forall\, w \in L^p_{s+1/p}(\Omega), \tag{2.13}$$
whenever $1 < p < \infty$, $s > 0$. Indeed, since the norms of the operators $L^p_{s+1/p}(\Omega) \ni w \mapsto (\mathrm{Tr}_j\, w) \circ \Lambda_j \in L^p(\partial\Omega)$ are bounded uniformly in j, it suffices to check that (2.13) holds for w in a dense subclass of $L^p_{s+1/p}(\Omega)$, such as $C^\infty(\bar{\Omega})$. In this latter case, granted our assumptions, the desired conclusion follows from Lebesgue's Dominated Convergence Theorem. A statement similar to (2.13) holds when $w \in B^{p,p}_{s+1/p}(\Omega)$.

As a consequence, if $u = v + w$ with $w \in L^p_{s+1/p}(\Omega) \cup B^{p,p}_{s+1/p}(\Omega)$, $1 < p < \infty$, $s > 0$, and $v^* \in L^p(\partial\Omega)$ is such that $v|_{\partial\Omega}$ exists in the nontangential sense, then $\int_{\partial\Omega} |u|^p\, d\sigma = \lim_j \int_{\partial\Omega_j} |u|^p\, d\sigma_j$.

The above considerations readily yield the following useful corollary.

Corollary 2.3. *Let L, Ω be as in Theorem 2.1. Then there exists $\varepsilon>0$ such that for $p>2-\varepsilon$ and $s>0$ the following is true. If $Lu=0$, $u=v+w$ with $v^*\in L^p(\partial\Omega)$ and $v|_{\partial\Omega}$ exists in the nontangential sense, while $w\in L^p_{s+1/p}(\Omega)\cup B^{p,p}_{s+1/p}(\Omega)$, then*

$$
\begin{aligned}
&u|_{\partial\Omega}\in B^{q,q}_{\alpha}(\partial\Omega),\ \ 2-\varepsilon<q<2+\varepsilon,\ \ 0<\alpha<1\Longrightarrow\\
&\qquad\qquad\qquad\qquad u\in L^q_{\alpha+1/q}(\Omega)\cap B^{q,q}_{\alpha+1/q}(\Omega)
\end{aligned}
\tag{2.14}
$$

Also, corresponding to $\alpha=0$ and $\alpha=1$ in (2.14), we have

$$
u|_{\partial\Omega}\in L^q(\partial\Omega),\ \ 2-\varepsilon<q<2+\varepsilon\Longrightarrow u^*\in L^q(\partial\Omega)\ \ \textit{and}\ \ u\in B^{q,q\vee 2}_{1/q}(\Omega),
\tag{2.15}
$$

$$
\begin{aligned}
&u|_{\partial\Omega}\in L^q_1(\partial\Omega),\ \ 2-\varepsilon<q<2+\varepsilon\Longrightarrow\\
&\qquad\qquad\qquad (\nabla u)^*\in L^q(\partial\Omega)\ \ \textit{and}\ \ u\in B^{q,q\vee 2}_{1+1/q}(\Omega).
\end{aligned}
\tag{2.16}
$$

3. Main estimates

Retain the notation and assumptions made in §1-2. In particular, the hypotheses of Theorem 1.1 are enforced throughout this section. Also, if $\Omega\subseteq K\subseteq\bar\Omega$, we denote by $B^{p,q}_{\alpha}(K;\mathrm{loc})$ the space of functions u defined in Ω and enjoying the following property. For each $x\in K$, there exits an open set $\mathcal{O}$ containing x and such that $\Omega\cap\mathcal{O}$ is a Lipschitz domain and $u|_{\mathcal{O}}\in B^{p,q}_{\alpha}(\Omega\cap\mathcal{O})$.

We are ready to present the

Proof of Theorem 1.1. There is no loss of generality in assuming that L satisfies (1.3). More specifically, fix some constant $A\in(0,\infty)$ so that, in the current setting, $L-A$ is negative-definite on $L^2(M)$. Also, pick $B>A$ and set $B_j=B_j(x)=B(1-\chi_{\mathcal{O}_j})$ where, for some fixed $p\in\Omega$, $\mathcal{O}_j:=\{x\in M:\mathrm{dist}(x,p)<1/j\}$. Then it has been proved in [MT2] that

$$
L-B_{j_o}\text{ is negative-definite on } M \text{ for some large } j_o.
$$

Now, since both the hypotheses and the results we seek to establish are local in nature, one can assume that Ω is small enough so that $\Omega\subset\mathcal{O}_{j_o}$. This ensures that the alteration B_{j_o} vanishes on Ω, as desired.

Moving on, let $K\subset\subset\Gamma_0$ and assume that $\operatorname{supp} f\subset K$. Pick $\chi\in C_0^\infty(M)$, equal to 1 on a neighborhood of K and to 0 off a slightly larger neighborhood U of K; in particular, χ vanishes on a neighborhood of $\partial\Omega\setminus\Gamma_0$. Consider $v:=\chi u$ on Ω. We have

$$
Lv=Xu=:g\qquad\text{in }\Omega,
\tag{3.1}
$$

where X is a first-order differential operator whose coefficients have smoothness properties dictated by those of L. More specifically, $Xu=\partial_j(a_ju)+b_j\partial_ju+cu$ where $a_j,b_j\in C^{1+\theta}$, $\theta>0$ and $c\in H^{1,r}$, $r>\dim M$. By Theorem 2.1, we have $u\in B^{q,2}_{1/q}(\Omega)\hookrightarrow L^q_{1/q-\delta}(\Omega)$, $\forall\,\delta>0$, which further forces $g\in L^q_{-1+1/q-\delta}(\Omega)$, for any $\delta>0$. In fact, extending u across $\partial\Omega$ as an element of $B^{q,2}_{1/q}(M)$ provides, via (3.1), an extension $\tilde{g} \in L^q_{-1+1/q-\delta}(M)$, $\delta > 0$, of g. We can now produce a neighborhood $\mathcal{O}$ of $\overline{\Omega}$ and $w \in L^q_{1+1/q-\delta}(\mathcal{O})$, $\delta > 0$, such that $Lw = \tilde{g}$ on $\mathcal{O}$. It follows that $(u-v)|_{\partial\Omega} = 0$ on $\partial\Omega$ and $L(u-v+w) = 0$ in Ω. Since $w|_{\partial\Omega} \in B^{q,q}_{1-\delta}(\partial\Omega)$, $\delta > 0$, Corollary 2.3 entails $u - v \in L^q_{1+1/q-\delta}(\Omega)$ for any $\delta > 0$, granted that q is sufficiently close to 2. This is a bit weaker than the conclusion in (1.5), so we proceed.

Consider $\Omega_1 := \Omega \setminus \overline{U}$ which, by picking U appropriately, can be assume to be a Lipschitz domain. Set $U_0 := U \cap \Omega$ and also let $K_1 := \partial U_0 \cap \Omega = \partial U_0 \setminus \Gamma_0$. We have

$$Lu = 0 \ \text{ in } \ \Omega_1, \quad u \in L^q_{1+1/q-\delta}(\Omega_1),$$

and

$$u\big|_{\partial\Omega_1} = f_1, \quad \operatorname{supp} f_1 \subset K_1 \subset\subset \Gamma_1,$$

where $\Gamma_1 := K_1 \cup (\Gamma_0 \setminus \partial U_0)$, so $\partial\Omega_1 \setminus \Gamma_1 = \partial\Omega \setminus \Gamma_0$.

Now pick $\chi_1 \in C_0^\infty(M)$, equal to 1 on a neighborhood of K_1 and equal to 0 off a slightly larger neighborhood U_1 of K_1, in particular on a neighborhood of $\partial\Omega_1 \setminus \Gamma_1$. Consider $v_1 := \chi_1 u$ on Ω_1. We have the analogue of (3.1); i.e., $Lv_1 = X_1 u =: g_1$, with $g_1 \in L^q_{1/q-\delta}(\Omega_1)$ extendible to $\tilde{g}_1 \in L^q_{1/q-\delta}(M)$. We can then pick a neighborhood $\mathcal{O}_1$ of $\overline{\Omega}_1$ and $w_1 \in L^q_{2+1/q-\delta}(\mathcal{O}_1)$ such that $Lw_1 = \tilde{g}_1$ on $\mathcal{O}_1$. Thus,

$$L(u - v_1 + w_1) = 0 \quad \text{in } \ \Omega_1, \qquad u - v_1 + w_1\big|_{\partial\Omega_1} = w_1\big|_{\partial\Omega_1}. \tag{3.2}$$

This time we apply the trace theorem as follows: $(\nabla w_1)|_{\partial\Omega_1} \in B^{q,q}_{1-\delta}(\partial\Omega_1) \hookrightarrow L^p(\partial\Omega_1)$, for some $p > 2$; in particular, $w_1\big|_{\partial\Omega_1} \in L^p_1(\partial\Omega_1)$. Using this back in (3.2), it follows from Corollary 2.3 that we have the non-tangential maximal function estimate

$$(\nabla(u - v_1 + w_1))^* \in L^p(\partial\Omega_1), \quad \text{for some} \quad p = p(\Omega_1, L) > 2. \tag{3.3}$$

Now v_1 vanishes on a neighborhood of $\partial\Omega \setminus \Gamma_0 = \partial\Omega_1 \setminus \Gamma_1$. On the other hand, we can assume $\operatorname{supp} \tilde{g}_1$ is disjoint from $\partial\Omega_1 \setminus \Gamma_1$, so $w_1 \in C^{3-\varepsilon}$ (for all $\varepsilon > 0$) on a neighborhood of $\partial\Omega_1 \setminus \Gamma_1 = \partial\Omega \setminus \Gamma_0$. Thus, the first conclusion in (1.5) follows from (3.3).

Finally, note that $u - v_1 \in B^{p,p}_{1+1/p}(\Omega_1)$, which ultimately accounts for the type of Besov regularity for u indicated in the statement of the theorem. ∎

Remark 3.1. Assume that L is a homogeneous, constant coefficient operator and that $\Omega \subset \mathbb{R}^n$. For each $r \in (0, \operatorname{diam}\Omega]$ and each boundary point x_0, denote by $S_r(x_0)$ the surface ball $B_r(x_0) \cap \partial\Omega$. Then there exist $p > 2 > q$ such that

$$f \in L^q(\partial\Omega) \ \text{ and } \ \operatorname{supp} f \subseteq S_r(x_0) \Longrightarrow$$
$$\Big(\int_{\partial\Omega\setminus S_{2r}(x_0)} [(\nabla \operatorname{PI} f)^*]^p \, d\sigma\Big)^{1/p} \leqslant C r^{-1+(n-1)(1/p-1/q)} \Big(\int_{S_r(x_0)} |f|^q \, d\sigma\Big)^{1/q},$$

where $C = C(\Omega, p) > 0$ is independent of f and r. Indeed, due to the translation and dilation invariant nature of the estimate at hand, matters can be reduced to the case $r = 1$ via a rescaling argument. This is, however, implicit in what we have proved so far.

The following is a refinement of Theorem 1.1, requiring a slightly more elaborate proof.

Theorem 3.1. *Let Ω, L be as in Theorem 1.1. Also, let Γ be a closed subset of $\partial\Omega$, and $\widetilde{\Gamma}$ a neighborhood of Γ in $\partial\Omega$. There exist $p > 2 > q$, depending only on Ω, L, Γ and $\widetilde{\Gamma}$, such that*

$$\begin{aligned} Lu = 0 \ \text{ in } \Omega, \quad u^*\big|_{\widetilde{\Gamma}} \in L^q(\widetilde{\Gamma}), \quad u\big|_{\widetilde{\Gamma}} = 0 \Longrightarrow& \\ (\nabla u)^*\big|_{\Gamma} \in L^p(\Gamma) \ \text{ and } \ u \in B^{p,p}_{1+1/p}(\Omega \cup \Gamma; \mathrm{loc}).& \end{aligned} \tag{3.4}$$

Proof. Select some function $\psi \in C_0^\infty(M)$ such that $\psi = 1$ on a neighborhood V_0 of Γ and $\psi = 0$ off a slightly larger neighborhood V of Γ, in particular equal to 0 on a neighborhood of $\partial\Omega \setminus \widetilde{\Gamma}$. Consider $v := \psi u$ on Ω. Parallel to (3.1) we have $Lv = Xu =: g$ with X a first-order differential operator. This time we do not have $u \in B^{q,2}_{1/q}(\Omega)$, so extra arguments are required.

For starters, $u|_{V\cap\Omega} \in L^q(V\cap\Omega)$, so we can say $g \in L^q_{-1}(\Omega)$ and, extending u, we have an extension $\tilde{g} \in L^q_{-1}(M)$ of g. Take a neighborhood $\mathcal{O}$ of $\overline{\Omega}$ and $w \in L^q_1(\mathcal{O})$ such that $Lw = \tilde{g}$ on $\mathcal{O}$. Now we have $v|_{\partial\Omega} = 0$ and hence

$$L(v - w) = 0 \ \text{ in } \ \Omega, \quad (v - w)\big|_{\partial\Omega} = -w\big|_{\partial\Omega} \in B^{q,q}_{1-1/q}(\partial\Omega).$$

From Corollary 2.3 it follows $v - w \in L^q_1(\Omega)$ and, hence, we have $v \in L^q_1(\Omega)$. Thus, with $\widetilde{\Omega} := \Omega \cap V_0$ (assumed to be Lipschitz), we see that $u \in L^q_1(\widetilde{\Omega})$, which is more than enough to yield $u^*|_{\partial\widetilde{\Omega}} \in L^q(\partial\widetilde{\Omega})$, by Theorem 2.1. At this stage Theorem 1.1 applies (cf. also Remark 2.3), and the desired conclusion follows. ∎

Remark 3.2. The same domain and function constructed in (1.6) can be used to show that the conclusion in Theorem 3.1 is in the nature of best possible.

Remark 3.3. Under the hypotheses of Theorem 3.1 and assuming that $n = 2, 3$, the following holds. There exist $\alpha = \alpha(\Omega, \Gamma, L) > 0$ and $q = q(\Omega, \Gamma, L) < 2$ such that if $Lu = 0$ in Ω, $u^*|_{\widetilde{\Gamma}} \in L^q(\widetilde{\Gamma})$ and $u|_{\widetilde{\Gamma}} = 0$, then u is Hölder continuous of class C^α on a neighborhood of Γ in $\overline{\Omega}$.

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