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THE WORK OF WŁADYSŁAW NARKIEWICZ IN NUMBER THEORY AND RELATED AREAS

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All but four Narkiewicz's research papers and monographs concerning number theory deal with one of the following five topics

1. polynomial mappings,
2. arithmetical functions,
3. additive problems,
4. factorization in algebraic number fields,
5. Artin's conjecture in algebraic number fields and related topics.

We shall consider these topics successively, then deal with the four papers out of the above classification and finally consider the four big books by the author.

1. Here belong papers [3], [5], [10], [12], [13], [68], [70]–[73], [77], [80]–[82], [85], [86], [90], [93], [94] and the book [78]. For a field k a polynomial mapping $F : k^n \to k^n$ defined by

$$[x_1, \dots, x_n] \mapsto [f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)]$$

is called admissible, if none of the polynomials $f_1, \dots, f_n$ is linear and their leading forms do not have any non-trivial common zero in the algebraic closure of k. A field k is said to have the property (SP), if for every n and every admissible polynomial mapping $F : k^n \to k^n$ the conditions $X \subset k^n$, $F(X) = X$ imply the finiteness of X. If this implication holds in the case $n = 1$, then k has property (P). Further, k has property (R), if the conditions $X \subset k$, X infinite, $f \in k(T)$ and $f(X) = X$ imply

$$f(T) = \frac{\alpha + \beta\, T}{\gamma + \delta T}; \quad \alpha, \beta, \gamma, \delta \in k.$$

Finally, k has property (K), if the following is true.

Let $\Phi : k^n \to k^n$ be an admissible polynomial mapping and let $\Psi : k^n \to k^n$ be another polynomial mapping. Denote by d the minimum of degrees of polynomials defining Φ and D the maximum of degrees of polynomials defining Ψ. If $d > D$, A

is a subset of k^n satisfying $\Psi(A) \subset \Phi(A)$ and the restriction of Ψ to A is injective, then A is finite.

In [3], his doctorate thesis and in [4] Narkiewicz proved that if k has property (P), then $k(X)$ has it also, where X is a set of elements algebraically independent over k of arbitrary cardinality. In [13] he proved that any algebraic number field has property (SP) and in [73] together with F. Halter-Koch that this property as well as property (K) is preserved under every finite extension and every purely transcendental extension. Earlier, Lewis (1972) and Liardet (1971) proved that all finite extensions of the rationals have property (K) and Liardet proved that all finitely generated fields have property (R), the fact established by Narkiewicz [5] for $k = \mathbb{Q}$. In some cases one can relax the condition of admissibility and still obtain finiteness of sets X such that $F(X) = X$ as shown in [12] and [71]. Finite sets X such that $f(X) = X$ (f a polynomial) have been studied by Narkiewicz in ten papers. More exactly, if f is a polynomial, X a set such that $f(X) \subset X$ and $x_0 \in X$, then the orbit $O_f(x_0) = \{x_0, f(x_0), f_2(x_0), \ldots\}$, where f_m denotes the m-th iterate of f. If the orbit $O_f(x_0) = \{x_0, x_1, \ldots\}$ is finite, $x_{i+1} = f(x_i)$ and k, l are the least integers such that $k < l$ and $x_k = x_l$, the sequence $x_0, x_1, \ldots, x_{l-1}$ consists of two parts: the sequence $x_0, x_1, \ldots, x_{k-1}$ called a precycle and the sequence $x_k, x_{k+1}, \ldots, x_{l-1}$ called a cycle. In [77] Halter-Koch and Narkiewicz proved that in any commutative domain R of zero characteristic, which is finitely generated as a ring, all polynomial cycles have their length uniformly bounded by a constant $B(R)$. For R being the ring $\mathbb{Z}_K$ of integers of an algebraic number field K of degree n, $B(R)$ is bounded by a function $C(n)$ [68]. Further, it has been proved in [81] that in $\mathbb{Z}_K$ there are only finitely many polynomial cycles starting from $0, 1$. In [82] Narkiewicz and Pezda deduced from the result of [68] that also the cardinality of orbits (l in the above notation) is uniformly bounded by a constant $D(n)$. Two sequences $\overline{x} = \{x_1, \ldots, x_n\}$ and $\overline{y} = \{y_1, \ldots, y_n\}$ of distinct elements of a domain R are called equivalent, if there exists an element $a \in R$ and a unit $\varepsilon \in R$ such that for $i = 1, \ldots, n$ one has $y_i = a + \varepsilon x_i$. A finite orbit is called non-linear, if the cycle contained in it cannot be realized as a cycle of a linear polynomial. It was proved in [85] that if F is a finitely generated domain which is moreover a finite factorization domain (i.e. every non-zero element of R is contained in finitely many principal ideals), then there are only finitely many pairwise inequivalent finite non-linear polynomial orbits included in R. In [90] Narkiewicz calculated the lengths of all polynomial cycles for $R = \mathbb{Z}\left[\frac{1}{N}\right]$, where N is odd or twice a prime. The lengths of all polynomial cycles in the ring of integers R_d of the quadratic field $\mathbb{Q}(\sqrt{d})$ being calculated earlier, Narkiewicz in [93] classified all finite orbit, in R_d. Finally, in [94] he calculated possible lengths of polynomial cycles in the ring of integers of a cubic field with negative discriminant.

The work described above published before 1996 has been presented in Part B of [78]. Part A has treated rings of integer-valued polynomials and described results of Nagell, Pólya, Rédei & Szele, Skolem and many others. To this topic belongs the paper [70] in which the following theorem was established. Let R be a Noetherian domain of characteristic zero K its ring of quotients and $\mathrm{Int}(R)$ the ring of univariate polynomials mapping R into R. Then the following conditions are equivalent

(i) $\mathrm{Int}(R)$ is generated by $\binom{X}{i}$ $(i = 0, 1, \ldots)$

(ii) For every rational prime p which is not invertible in R the principal ideal generated by p is a product of distinct maximal ideals of index p in R.

2. Here belong papers [7], [8], [18], [22], [24], [27], [38]–[41], [51], [54], [56]–[58], [88] and the book [62]. Two early papers [7] and [8] concern convolutions of arithmetical functions. In [7] Narkiewicz defined a convolution h of two functions f and g by the formula

$$h(n) = \sum_{d \in A_n} f(d) g\left(\frac{n}{d}\right),$$

where A_n is a certain set of divisors of n and considered the ring R_A of function with ordinary addition and with the above convolution as multiplication. He called a convolution regular, if it preserves multiplicativity, the ring R_A is commutative, associative and has a unit element and, moreover, the inverse function of $f(n) \equiv 1$ assumes for prime powers only the values 0 and -1. He proved a necessary and sufficient condition for sets A_n in order that the convolution defined by them be regular. A problem proposed in [7] concerning isomorphism of rings R_A has been solved by H. Scheid (1969).

Fourteen papers and the book [62] deal with distribution of values of multiplicative functions in residue classes. A sequence $\{a_n\}$ is called weakly uniformly distributed, briefly $WUD(\bmod N)$, if the following two conditions are satisfied

(i) The set $\{n : (a_n, N) = 1\}$ is infinite,

(ii) For every j prime to N one has

$$\lim_{x \to \infty} \frac{\#\{n \le x : a_n \equiv j (\bmod N)\}}{\#\{n \le x : (a_n, N) = 1\}} = \frac{1}{\varphi(N)}.$$

In [18] Narkiewicz proved the following. Let $f(N)$ be a multiplicative function, which is polynomial-like, i.e. for every $j = 1, 2, \ldots$ there exists a polynomial $V_j \in \mathbb{Z}[x]$ such that for all primes p one has $f(p^j) = V_j(p)$. Denote by R_j the set

$$\{V_j(x) : (xV_j(x), N) = 1\}$$

and let Λ_j be the subgroup of $G(N)$, the multiplicative group of restricted residue classes $(\bmod N)$, generated by R_j. If not all sets R_j are empty and m is the least index such that R_m is non-empty, then the sequence $f(1), f(2), \ldots$ is $WUD(\bmod N)$, if and only if for every non-principal character $\chi(\bmod N)$ which is trivial on Λ_m there exists a prime p such that

$$1 + \sum_{j=1}^{\infty} \frac{\chi\left(f\left(p^j\right)\right)}{p^{j/m}} = 0.$$

This implies that if $\Lambda_m = G(N)$, then the sequence $\{f(n)\}$ is $WUD(\bmod N)$, which for $m = 1$ was already observed by Wirsing (1967). Using the above criterion Narkiewicz found in [18] all integers N for which the Euler function and the divisor function are $WUD(\bmod N)$. In [36] he did this together with F. Rayner for the function σ_2 and in [58] for the function σ_k ($k \geq 3$). If the polynomial V_1 is not a perfect power in $\mathbb{C}[x]$, then there is an integer D given explicitly in terms of V_1 such that if $(N, D) = 1$, then $\{f(n)\}$ is $WUD(\bmod N)$, [57]. An analogous result holds also for systems of multiplicative functions, provided one adapts appropriately the notion of $WUD(\bmod N)$. In [51] Narkiewicz obtained such a result for the joint distribution of values of $\varphi(n)$ and $\sigma(n)$.

In his book [62] he considered besides the above topic also distribution mod N of polynomial sequences, of linear recurrent sequences and of the values of an additive function.

A little apart are papers [22], [24], [27] dealing with the counting function of the set of n's for which a given d is a unitary divisor of the value $f(n)$ of a polynomial-like multiplicative function.

Two more papers [32] and [36] concern arithmetical functions but not their values mod N. In [32] Narkiewicz generalized some results of Levin and Faĭnleĭb (1970) and of Mirsky (1949) concerning the counting function of the set of solutions of the equation $f(n) = k$, where f is a multiplicative functions and k an integer.

In [36] he proposed the following conjecture. If a function $f(n) = \sum_{\substack{p \mid n \\ p \text{ primes}}} f(p)$ has a non-decreasing normal order, $f(p)$ is nonnegative and non-decreasing, then

$$f(p) = O\left((\log p)^{1+\varepsilon}\right) \text{ for every } \varepsilon > 0.$$

The conjecture has been proved independently by Elliott (1976) and Kátai (1977).

3. Here belong papers [1] [43], [50], [60], [61] and [74].

In [74] Narkiewicz together with Deshouillers, Granville and Pomerance proved that 210 is the largest positive integer such that every prime in $\left(\frac{n}{2}, n\right)$ occurs in a Goldbach decomposition of n.

4. This topic is treated in the papers [11], [14], [15], [17], [19], [21], [23], [29], [30], [33], [34], [42], [46], [49], [52], [55], [76] which include Narkiewicz's habilitation thesis and in the last chapter of the book [35].

In [11] and [15] Narkiewicz proved that if $h(K)$, the class number of an algebraic number field K is greater than 1, then almost all integers of K have a non-unique factorization and if K is normal, almost all rational integers have a non-unique factorization. Moreover, if $h(K) \geq 3$, then almost all integers of K have factorizations of distinct lengths and if K is normal, almost all rational integers have factorizations of distinct lengths. The assumption of normality has been removed in [33]. In [17] and [21] Narkiewicz gave an asymptotic formula for the number of positive rational integers $n \leq x$ in a given arithmetical progression which have a unique factorization in a given quadratic field. In [29] he gave an asymptotic formula for the number of non-associated integers of a field K whose norms do not exceed x in absolute value and which have in K a unique factorization into irreducibles. If $h(K) \geq 3$ the function $C(K) \log\log n$ with a certain position $C(K)$ serves as a normal order for the number of factorizations of distinct lengths of a rational integer n in K [42]. If $h(K) \geq 2$ and $f(n)$ is the number of factorizations of n into irreducibles, then $\log f(n)$ has the normal order $C_1(K) \log n \cdot \log\log n$ [50].

The papers [46], [52], [55] and [76] deal with problems in finite abelian groups related to the factorization problems in algebraic number field. The relevant group is the class group of a field. [46] and [55] have been the beginning of a large theory on the border of number theory, group theory and combinatorics expounded in the monograph Geroldinger and Halter-Koch (2006).

5. Here belong papers [64]–[67] and [96]. In [64] Narkiewicz adapts the method used by Heath-Brown (1996) for primitive roots in Abelian fields.

Let k be an abelian field and let L be a cyclotomic field containing it. Assume that if L is generated by the f-th roots of unity, then f is divisible by 16. Identifying the Galois group of L with the multiplicative group $G(f)$ of restricted residue classes $(\operatorname{mod} f)$ assume that the intersection H' of the subgroup H of $G(f)$ corresponding to K with $\{1 \operatorname{mod} 8\}$ is not contained in the union

$$\bigcup_{p|f} H_p,$$

where H_p denotes for odd primes p dividing f the subgroup of $G(f)$ consisting of residue classes congruent to unity $(\operatorname{mod} p)$ and H_2 denotes the subgroup of residue classes congruent to unity $(\operatorname{mod} 16)$.

If now a_1, a_2, a_3 are integers of K which are multiplicately independent and satisfy the following conditions

(i) $(N(a_j), 3f) = 1$ for $j = 1, 2, 3$
(ii) none of the norms of $a_j, a_i a_j, a_1 a_2 a_3$ $(i, j = 1, 2, 3;\ i < j)$ is a square of a rational integer,

then at least one of the a_i's is a primitive root for infinitely many prime ideals of K of the first degree.

Using essentially the same ideas the author proves in [66] the following theorem about units.

If $K \neq \mathbb{Q}$ is a real abelian algebraic number field, then there exist infinitely many prime ideals P of first degree in K such that every non-zero residue class $\operatorname{mod} P$ contains infinitely many units with the exception of at most two such fields. If such exceptional fields exist at all, then either there is only one of them which is of degree 3, or they are all quadratic. In [96] Narkiewicz deduced from this result, by a slight modification of the argument of Harper and Ram Murty (2004), that if K is a real quadratic field or a cubic field with a negative discriminant, then K is Euclidean (not norm - Euclidean) with at most two exceptions. This has been known earlier only for quadratic fields with discriminant less than 100.

6. The four papers not fitting in the above classification are [25], [26], [89] and [95]. The first two deal with the relative different of a number field. In [89] Narkiewicz and Pezda deduced from a classical conjecture of Dickson (1904) that if $f(x) = (ax+b)(cx+d)$ is a polynomial with rational integral coefficients, satisfying $a > 0$, $c > 0$ and $ad-bc \neq 0$, then for every natural r there exists an integer N such that $f(x)/N$ represents at least r distinct primes. In [95] Jarden and Narkiewicz proved that if R is a finitely generated integral domain of zero characteristic, then for every n there exist elements of R which are not sums of at most n units.

7. Besides research papers and monographs Narkiewicz wrote six survey papers [20], [37], [45], [75], [79], [83], a popular booklet [31], a textbook [44] in Polish, translated into English as [59] and three big monographs [35], [63] and [84].

The textbook which had three Polish editions is characterized by a variety of topics treated and methods used. It treats congruences, diophantine equations, arithmetical functions, primes, sieve methods, geometry of numbers, additive number theory, probabilistic number theory, diophantine approximation and uniform distribution mod 1, algebraic numbers and p-adic numbers in sufficient detail to give the reader the flavor of the subject.

Among the monographs the chief place is occupied by [35] *Elementary and Analytic Theory of Algebraic Number Fields*, which is a real encyclopedia of algebraic number theory the class-field theory excepted. The bibliography of over 3700 items enhances the value of the book, which has had three editions.

The book [63] *Classical Problems in Number Theory* gives an information on the state of knowledge up to 1986 concerning several problems. In research on primitive roots, on Catalan's problem, on Waring's problem and in smaller degree on the class number problem there has been a progress during the last twenty years, thus a revised version of the book would be welcome.

The third monograph [84] *The development of Prime Number Theory from Euclid to Hardy and Littlewood* presents prime number theory in chronological order. It has six chapters (Early Times, Dirichlet's Theorem an Primes in Arithmetic Progressions, Čebyšev's Theorem, Riemann's Zeta-function and Dirichlet Series, The Prime Number Theorem, The Turn of the Century) and is very rich in historical detail.

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