Functiones et Approximatio
XXXIX.1 (2008), 71–95

ON EXPRESSIBLE SETS OF GEOMETRIC SEQUENCES

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Dedicated to Professor Władysław Narkiewicz
at the occasion of his 70th birthday

Abstract: We prove that the expressible sets of geometric sequences are Borel measurable and give lower and upper bounds for their Lebesgue measure.
Keywords: expressible set, Lebesgue measure

1. Introduction

For a given sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ we say that $\mathrm{X}\{a_n\}_{n=1}^{\infty} = \big\{x \in \mathbb{R} \mid \exists\, c_n \in \mathbb{N}$ such that $x = \sum_{n=1}^{\infty} \frac{1}{a_n c_n}\big\}$ is its expressible set[1)]. Erdős [1] proved that the set $\mathrm{X}\big\{2^{2^n}\big\}_{n=1}^{\infty}$ does not contain any rational numbers. In [2] it is shown that if $\lim_{n\to\infty} \frac{1}{n} \log_2 \log_2 a_n < 1$ and $a_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ then $\mathrm{X}\{a_n\}_{n=1}^{\infty}$ contains an interval. In [4] the authors show that $\lambda\big(\mathrm{X}\big\{2^{3^n}\big\}_{n=1}^{\infty}\big) = 0$. Here λ denotes the Lebesgue measure. In [3] it is proved that if $\lim_{n\to\infty} a_n^{1/3^n} = \infty$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ then $\lambda\big(\mathrm{X}\{a_n\}_{n=1}^{\infty}\big) = 0$. (The case of arbitrary reals is left open.) In [2] it is shown that if

$$\frac{1}{2a_n} \le \sum_{j=n+1}^{\infty} \frac{1}{a_j} \tag{1.1}$$

and $a_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ then $\mathrm{X}\{a_n\}_{n=1}^{\infty} = \big(0, \sum_{j=1}^{\infty} \frac{1}{a_j}\big]$. Professor Zbigniew Ciesielski observed that the expressible sets are analytic and hence Lebesgue measurable. It seems that evaluating the Lebesgue measure of the set $\mathrm{X}\{a_n\}_{n=1}^{\infty}$ is not easy if (1.1) does not hold. In this paper we estimate the Lebesgue measure of the set $\mathrm{X}\{A^n\}_{n=1}^{\infty}$ for a real number $A > 3$. We prove the following.

Theorem 1.1. *We have* $\big(0, \frac{1}{6}\big] \subset \mathrm{X}\{4^n\}_{n=1}^{\infty}$ *and* $\lambda\big(\mathrm{X}\{4^n\}_{n=1}^{\infty}\big) = \frac{1}{4}$.

Mathematics Subject Classification: 11K55
Supported by the grants no. 201/04/0381, 201/07/0191 and MSM6198898701.

[1)] In this paper $\mathbb{N}$ is the set of all positive integers. Nowadays $\mathbb{N}$ more often means the set of all non-negative integers.

Let us note that for a real number A with $0 < A \leq 3$ the sequence $\{A^n\}_{n=1}^{\infty}$ satisfies the condition (1.1) hence $\mathrm{X}\{A^n\}_{n=1}^{\infty} = (0,\infty)$ for $0 < A \leq 1$ and $\mathrm{X}\{A^n\}_{n=1}^{\infty} = \bigl(0, \frac{1}{A-1}\bigr]$ for $1 < A \leq 3$.

Other expressions of real numbers can be found in [5], [6], [8], [7].

2. Main results

Theorem 1.1 follows from Theorems 2.1–2.4. In the sequel, for a real number x we use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x and $\lceil x \rceil$ to denote the least integer greater than or equal to x. The following theorem estimates the length of the largest interval with left end point at zero contained in the expressible set of a geometric sequence.

Theorem 2.1. *For $A > 3$, $\mathrm{X}\{A^n\}_{n=1}^{\infty}$ contains the interval*

$$\left(0\ ,\ \frac{1}{(A-1)(\lceil A \rceil - 2)}\right].$$

Theorem 2.2. *The set $\mathrm{X}\{A^n\}_{n=1}^{\infty}$ is Borel measurable.*

The next theorem deals with a lower bound for the Lebesgue measure of the expressible sets of a geometric sequence.

Theorem 2.3. *Let $A > 3$. Put $\omega_1 := 1$ and for $j \geq 2$*

$$\omega_j := j - \left\lceil \frac{j(j-1)}{A-1} \right\rceil + 1\,.$$

Then

$$\lambda\bigl(\mathrm{X}\{A^n\}_{n=1}^{\infty}\bigr) \geq \frac{1}{(\lfloor A \rfloor - 1)A} \prod_{j=1}^{\lfloor A \rfloor - 1} \frac{A - \omega_j + 1}{A - \omega_j}\,.$$

The following theorem presents an upper bound for the Lebesgue measure of the expressible sets of geometric sequences.

Theorem 2.4. *Let $A > 3$. Put*

$$B := \max\left\{ \left\lfloor \frac{1}{2} + \sqrt{A - \frac{3}{4}} \right\rfloor, \left\lfloor -\frac{1}{2} + \sqrt{2A - \frac{7}{4}} \right\rfloor \right\},$$

$$\alpha_1 := 1 \quad \text{and} \quad \alpha_u := \left\lceil \frac{u(u-1)}{A-1} \right\rceil \quad \text{for } u \geq 2\,,$$

$$\beta_u := \begin{cases} B & \text{if } u(u+1) \leq A-1\,, \\ \min\left\{ B, \left\lfloor \dfrac{u(u+1)}{u(u+1)-(A-1)} \right\rfloor \right\} & \text{if } u(u+1) > A-1\,, \end{cases} \tag{2.1}$$

$$\delta_u := \min\left\{\beta_u - 1, \left\lceil -\frac{3}{2} + \sqrt{A - \frac{3}{4}} \right\rceil \right\},$$

$$\varepsilon_u := \frac{1}{\alpha_u} - \frac{1}{\delta_u + 1} - \frac{\delta_u - \alpha_u + 1}{A - 1},$$

$$\sigma_u := \min\{u, \beta_u\} - \alpha_u + 1,$$

$$\zeta_0 := 0, \ \ \zeta_1 := \frac{A}{A-1} \ \ \textit{and} \ \ \zeta_u := \frac{A - \sigma_u + 1}{A - \sigma_u} \zeta_{u-1} \ \ \textit{for } u \geq 2. \tag{2.2}$$

Then

$$\lambda\big(\mathrm{X}\{A^n\}_{n=1}^{\infty}\big) \leq \frac{1}{A-1} - \frac{1}{A} \sum_{u=1}^{B} \varepsilon_u \big(\zeta_u - \zeta_{u-1}\big).$$

Corollary 2.1. *Denote by $L(A)$ and $U(A)$ the lower and upper bounds given by Theorem 2.3 and Theorem 2.4, respectively. For $A \in (3, 9)$ the values $L(A)$ and $U(A)$ are shown in the following table.*

<table>
<tr><th>A</th><th>$L(A)$</th><th>$U(A)$</th></tr>
<tr><td>$A \in (3,4)$</td><td>$\frac{1}{2(A-2)}$</td><td>$\frac{A+1}{2(A-1)^2}$</td></tr>
<tr><td>$A \in [4,5)$</td><td>$\frac{A-1}{3(A-2)^2}$</td><td rowspan="3">$\frac{1}{2(A-2)}$</td></tr>
<tr><td>$A \in [5,6)$</td><td>$\frac{(A-1)^2}{4(A-2)^3}$</td></tr>
<tr><td>$A \in [6,7)$</td><td>$\frac{(A-1)^3}{5(A-2)^4}$</td></tr>
<tr><td>$A \in \left[7, \frac{23}{3}\right)$</td><td>$\frac{(A-1)^2}{6(A-2)(A-3)^2}$</td><td rowspan="4">$\frac{2A-1}{6(A-2)^2}$</td></tr>
<tr><td>$A \in \left[\frac{23}{3}, 8\right)$</td><td>$\frac{(A-1)(A-2)}{6(A-3)^3}$</td></tr>
<tr><td>$A \in \left[8, \frac{17}{2}\right)$</td><td>$\frac{(A-1)^2}{7(A-3)^3}$</td></tr>
<tr><td>$A \in \left[\frac{17}{2}, 9\right)$</td><td>$\frac{(A-1)(A-2)^2}{7(A-3)^4}$</td></tr>
</table>

Example 2.1. Set $A = \pi$. Then from Theorems 2.1–2.4 we obtain that $(0, \frac{1}{2(\pi-1)}] \subset \mathrm{X}\{\pi^n\}_{n=1}^{\infty}$ and $\frac{1}{2(\pi-2)} \leq \lambda\big(\mathrm{X}\{\pi^n\}_{n=1}^{\infty}\big) \leq \frac{\pi+1}{2(\pi-1)^2}$.

Theorem 2.5. *For $A \to \infty$ the lower bound $L(A)$ for the Lebesgue measure satisfies*

$$L(A) \sim \frac{\exp \frac{2\pi}{3\sqrt{3}}}{A^2} \doteq \frac{3.350}{A^2}.$$

Theorem 2.6. *For $A \to \infty$ the upper bound $U(A)$ for the Lebesgue measure satisfies*

$$\frac{0.929}{A^{3/2}} \doteq \Big(\frac{7}{12}\sqrt{6}-\frac{1}{2}\Big)\frac{1}{A^{3/2}}+\mathrm{O}\Big(\frac{1}{A^2}\Big) \le U(A)$$
$$\le \Big(\frac{13}{6}-\frac{1}{2}\sqrt{6}\Big)\frac{1}{A^{3/2}}+\mathrm{O}\Big(\frac{1}{A^2}\Big) \doteq \frac{0.942}{A^{3/2}}\,.$$

Open Problem. It is unknown to the authors how to determine the order of $\lambda(\mathrm{X}\{A^n\}_{n=1}^\infty)$ for A large.

Remark. For a particular A the result of Theorem 2.1 can be improved. For $A=7$ Theorem 2.1 implies that $\big(0,\frac{1}{30}\big] \subseteq \mathrm{X}\{7^n\}_{n=1}^\infty$. Let $\mathcal{I}(\mathbf{s})$ be defined by (3.6). It follows that $\mathcal{I}(\mathbf{s}) \subseteq \mathrm{X}\{7^n\}_{n=1}^\infty$ for every $\mathbf{s}$. Thus

$$\begin{aligned} E &:= \left(0,\frac{1}{30}\right] \cup \mathcal{I}((12,1,3)) \cup \mathcal{I}((5,11,1)) \\ &\quad \cup \mathcal{I}((11,1)) \cup \mathcal{I}((6,2)) \cup \mathcal{I}((10,1)) \cup \mathcal{I}((5,3)) \\ &= \left(0,\frac{1}{30}\right] \cup \left(\frac{137}{4116},\frac{229}{6860}\right] \cup \left(\frac{629}{18865},\frac{757}{22638}\right] \\ &\quad \cup \left(\frac{18}{539},\frac{551}{16170}\right] \cup \left(\frac{5}{147},\frac{17}{490}\right] \cup \left(\frac{17}{490},\frac{26}{735}\right] \cup \left(\frac{26}{735},\frac{53}{1470}\right] \\ &= \left(0,\frac{53}{1470}\right] \subseteq \mathrm{X}\{7^n\}_{n=1}^\infty\,. \end{aligned}$$

Thus $\big(0,\frac{1}{28}\big] \subseteq \big(0,\frac{53}{1470}\big] \subseteq \mathrm{X}\{7^n\}_{n=1}^\infty$. Put

$$I_n := \sum_{i=1}^n \frac{1}{4\cdot 7^i} + \frac{1}{7^n}\left(0,\frac{1}{28}\right] = \left(\frac{1}{24}\Big(1-\frac{1}{7^n}\Big),\frac{1}{24}\Big(1-\frac{1}{7^{n+1}}\Big)\right] \subseteq \mathrm{X}\{7^n\}_{n=1}^\infty\,.$$

It follows that

$$E \cup \bigcup_{n=1}^\infty I_n = \left(0,\frac{1}{24}\right) \subseteq \mathrm{X}\{7^n\}_{n=1}^\infty\,.$$

We have $\frac{1}{24}=\sum_{n=1}^\infty \frac{1}{4\cdot 7^n}$, so the result of Theorem 2.1 can be improved to $\big(0,\frac{1}{24}\big] \subseteq \mathrm{X}\{7^n\}_{n=1}^\infty$.

Remark. For $A=4$ Theorem 2.1 gives $\big(0,\frac{1}{6}\big] \subseteq \mathrm{X}\{4^n\}_{n=1}^\infty$. The value $\frac{1}{6}$ is best possible. The system $\mathcal{T}$ from the proof of Theorem 2.4 can be simplified to the form

$$\mathcal{T} = \big\{(\mathbf{s},1) \;\big|\; \mathbf{s}=(c_1,\dots,c_n)\,,\ \ n\ge 0\,,\ \ c_i\in\{1,2\}\big\}$$

since for $A=4$ we have $B=2$, $\alpha_1=\alpha_2=\delta_1=\delta_2=1$ and $\beta_1=\beta_2=2$. Thus $((2,\dots,2,1),1)\in\mathcal{T}$ and

$$\mathcal{J}\big((\underbrace{2,\dots,2}_{n-1},1),1\big) = \frac{1}{6}+\frac{1}{4^n}\left(\frac{13}{24},\frac{14}{24}\right].$$

The proof of Theorem 2.4 implies that this set is disjoint with $\mathrm{X}\{4^n\}_{n=1}^{\infty}$. On the other hand, the proof of Theorem 2.3 implies that for every n

$$\mathcal{I}\big((\underbrace{2,\dots,2}_{n-1},1)\big) = \frac{1}{6} + \frac{1}{4^n}\left(\frac{8}{24},\frac{12}{24}\right] \subseteq \mathrm{X}\{4^n\}_{n=1}^{\infty}.$$

3. Proofs

Let us define a sum and a product of a real number a and a set $B \subseteq \mathbb{R}$ in an usual way $a+B := \{a+b \mid b \in B\}$ and $aB := \{ab \mid b \in B\}$. As usually $g(x) = \mathrm{O}(f(x))$ if there exists positive real K such that $|g(x)| \le Kf(x)$ for all sufficiently large positive real x.

Suppose that $A > 3$. Set $S := \mathrm{X}\{A^n\}_{n=1}^{\infty}$. Then

$$\begin{aligned}\mathrm{X}\{A^n\}_{n=2}^{\infty} &= \left\{x \in \mathbb{R} \;\middle|\; \exists\{c_n\}_{n=2}^{\infty} \subseteq \mathbb{N} \text{ such that } x = \sum_{n=2}^{\infty}\frac{1}{A^n c_n}\right\}\\ &= \left\{\frac{x}{A} \;\middle|\; \exists\{c_n\}_{n=1}^{\infty} \subseteq \mathbb{N} \text{ such that } x = \sum_{n=1}^{\infty}\frac{1}{A^n c_n}\right\} = \frac{1}{A}S.\end{aligned}$$

The definition of the expressible set implies that

$$\mathrm{X}\{A^n\}_{n=1}^{\infty} = \bigcup_{c_1=1}^{\infty}\left(\frac{1}{A^1 c_1} + \mathrm{X}\{A^n\}_{n=2}^{\infty}\right).$$

Hence for the set S we have the identity

$$S = \bigcup_{n=1}^{\infty}\left(\frac{1}{An} + \frac{1}{A}S\right). \tag{3.1}$$

Lemma 3.1. *For every $x \in \left(0, \frac{A}{(A-1)(\lceil A\rceil-2)}\right]$ there exists a positive integer c such that*

$$0 < x - \frac{1}{c} \le \frac{1}{(A-1)(\lceil A\rceil - 2)}.$$

Proof. Let c be the least integer such that $\frac{1}{c} < x$. If $c \le \lceil A\rceil - 2$ then

$$x - \frac{1}{c} \le \frac{A}{(A-1)(\lceil A\rceil-2)} - \frac{1}{\lceil A\rceil - 2} = \frac{1}{(A-1)(\lceil A\rceil-2)}.$$

If $c \ge \lceil A\rceil - 1$ then since $\frac{1}{c} < x \le \frac{1}{c-1}$ we obtain

$$x - \frac{1}{c} \le \frac{1}{c(c-1)} \le \frac{1}{(\lceil A\rceil-1)(\lceil A\rceil-2)} \le \frac{1}{(A-1)(\lceil A\rceil-2)}. \tag{3.2}$$

■

Proof of Theorem 2.1. It suffices to show that for every k there exists an integer c_k such that

$$0 < x - \sum_{i=1}^{k} \frac{1}{A^i c_i} \le \frac{1}{A^k(A-1)(\lceil A\rceil - 2)}\,.$$

We proceed by induction on k. For $k = 1$ we obtain the statement from Lemma 3.1 on division by A.

Assume now that for $k \ge 2$

$$0 < x - \sum_{i=1}^{k-1} \frac{1}{A^i c_i} \le \frac{1}{A^{k-1}(A-1)(\lceil A\rceil - 2)}\,.$$

Hence

$$0 < A^k\left(x - \sum_{i=1}^{k-1} \frac{1}{A^i c_i}\right) \le \frac{A}{(A-2)(\lceil A\rceil - 2)}$$

and by Lemma 3.1 there exists an integer c_k such that

$$0 < A^k\left(x - \sum_{i=1}^{k-1} \frac{1}{A^i c_i}\right) - \frac{1}{c_k} \le \frac{1}{(A-1)(\lceil A\rceil - 2)}\,.$$

It follows that

$$0 < x - \sum_{i=1}^{k} \frac{1}{A^i c_i} \le \frac{1}{A^k(A-1)(\lceil A\rceil - 2)}$$

and the inductive proof is complete. ■

In the next lemma we use the symbol $\mathrm{X}\{a_n\}_{n=1}^N$ for the set

$$\mathrm{X}\{a_n\}_{n=1}^N := \left\{x \in \mathbb{R} \;\middle|\; \exists c_n \in \mathbb{N} \text{ such that } x = \sum_{n=1}^{N} \frac{1}{a_n c_n}\right\}.$$

We put $\mathrm{X}\{a_n\}_{n=1}^0 := \{0\}$.

Lemma 3.2. *The set* $\mathrm{X}\{A^n\}_{n=1}^\infty \cup \bigcup_{N=0}^{\infty} \mathrm{X}\{A^n\}_{n=1}^N$ *is closed.*

Proof. With the convention $\frac{1}{\infty} = 0$ we can write every element of

$$Z := \mathrm{X}\{A^n\}_{n=1}^\infty \cup \bigcup_{N=0}^{\infty} \mathrm{X}\{A^n\}_{n=1}^N$$

as $\sum_{n=1}^\infty \frac{1}{A^n c_n}$, where $c_n \in \mathbb{N} \cup \{\infty\}$. Conversely,

$$\text{if } c_n \in \mathbb{N} \cup \{\infty\} \text{ then } \sum_{n=1}^{\infty} \frac{1}{A^n c_n} \in Z\,. \tag{3.3}$$

Indeed, if $c_n < \infty$ for all $n < N$, $c_N = \infty$ and $c_n < \infty$ for some $n > N$ then by Theorem 2.1

$$\sum_{n=1}^{\infty} \frac{1}{A^n c_n} \in \sum_{n=1}^{N-1} \frac{1}{A^n c_n} + \left(0 \ , \ \frac{1}{A^N(A-1)}\right]$$
$$\subseteq \sum_{n=1}^{N-1} \frac{1}{A^n c_n} + \left(0 \ , \ \frac{1}{A^{N-1}(A-1)(\lceil A \rceil - 2)}\right] \subseteq \mathrm{X}\{A^n\}_{n=1}^{\infty} ,$$

hence $\sum_{n=1}^{\infty} \frac{1}{A^n c_n} \in Z$.

Let $x_m \in Z$, $\lim_{m\to\infty} x_m = x$, $x_m = \sum_{n=1}^{\infty} \frac{1}{A^n d_{m,n}}$, $d_{m,n} \in \mathbb{N} \cup \{\infty\}$. We set $M_0 := \mathbb{N}$ and if M_{n-1} is already defined and infinite then put

$$M_n^* := \left\{ m \in M_{n-1} \;\middle|\; d_{m,n} = \liminf_{k \in M_{n-1}} d_{k,n} \right\}$$

and $M_n := M_n^*$ if M_n^* is infinite and $M_n := M_{n-1}$ otherwise. We have $M_0 \supseteq M_1 \supseteq \cdots$. Let $c_n := \liminf_{k\in M_{n-1}} d_{k,n}$. Now consider two cases.

1. Suppose that all sets M_n^* are infinite. Then for every N and infinitely many $m \in M_{N-1}$

$$0 \le x_m - \sum_{n=1}^{N} \frac{1}{A^n c_n} \le \frac{1}{A^N(A-1)} ,$$

hence passing to the limit with m

$$0 \le x - \sum_{n=1}^{N} \frac{1}{A^n c_n} \le \frac{1}{A^N(A-1)}$$

and passing to the limit with N

$$x = \sum_{n=1}^{\infty} \frac{1}{A^n c_n} ,$$

hence, by (3.3), $x \in Z$.

2. Let N be the least positive integer such that the set M_N^* is finite. This means that $\lim_{m\in M_{N-1}} d_{m,N} = \infty$ and for $m > m_0$, $d_{m,N} < \infty$. Therefore, for $m \in M_{N-1}$

$$0 \le x_m - \sum_{n=1}^{N-1} \frac{1}{A^n c_n} - \frac{1}{d_{m,N} A^N} \le \frac{1}{A^N(A-1)}$$

and passing to the limit with m we obtain

$$0 \le x - \sum_{n=1}^{N-1} \frac{1}{A^n c_n} \le \frac{1}{A^N(A-1)} .$$

Let N_1 be the least integer less than N such that $c_{N_1} = \infty$ if such integers exist and $N_1 := N$ otherwise. In any case

$$0 \le x - \sum_{n=1}^{N_1-1} \frac{1}{A^n c_n} \le \frac{1}{A^{N_1}(A-1)}\,.$$

If $x = \sum_{n=1}^{N_1-1} \frac{1}{A^n c_n}$ then $x \in \mathrm{X}\{A^n\}_{n=1}^{N_1-1} \subseteq Z$, otherwise $x \in \mathrm{X}\{A^n\}_{n=1}^{\infty} \subseteq Z$ by Theorem 2.1. ■

Proof of Theorem 2.2. Since the set $\bigcup_{N=0}^{\infty} \mathrm{X}\{A^n\}_{n=1}^{N}$ is countable, Lemma 3.2 implies that the set $\mathrm{X}\{A^n\}_{n=1}^{\infty}$ is Borel measurable. ■

Lemma 3.3. *(Cauchy) Let $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ be sequences of real numbers such that for every x with $|x| < R$ the series $F(x) := \sum_{n=0}^{\infty} f_n x^n$ and $G(x) := \sum_{n=0}^{\infty} g_n x^n$ converge. Let $\{h_n\}_{n=0}^{\infty}$ be a sequence defined by $h_n := \sum_{k=0}^{n} f_k g_{n-k}$ for every n. Then for every x with $|x| < R$ the series $H(x) := \sum_{n=0}^{\infty} h_n x^n$ converges and $H(x) = F(x)G(x)$.*

Proof. The result follows from direct computation and from the fact that the series for $F(x)$ and $G(x)$ are absolutely convergent for $|x| < R$.

$$\begin{aligned} F(x)G(x) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_m g_n x^{m+n} &(3.4)\\ &= \sum_{p=0}^{\infty}\sum_{m=0}^{p} f_m g_{p-m} x^p = \sum_{p=0}^{\infty} h_p x^p = H(x)\,. \end{aligned}$$

■

Lemma 3.4. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that for every x with $|x| < R$ the series $F(x) := \sum_{n=0}^{\infty} f_n x^n$ converges. Let $\omega > 0$ and let the sequence $\{h_n\}_{n=0}^{\infty}$ be defined by*

$$h_n := f_n + \sum_{k=1}^{n} f_{k-1}\omega^{n-k}.$$

Then for every x with $|x| < \min\{R, \frac{1}{\omega}\}$ the series $H(x) := \sum_{n=0}^{\infty} h_n x^n$ converges and

$$H(x) = \left(\frac{\omega-1}{\omega} + \frac{1}{\omega(1-\omega x)}\right)F(x)\,.$$

Proof. We have

$$h_n = f_n + \frac{1}{\omega}\sum_{k=0}^{n-1} f_k \omega^{n-k} = \frac{\omega-1}{\omega} f_n + \frac{1}{\omega}\sum_{k=0}^{n} f_k\omega^{n-k}.$$

Then Lemma 3.3 yields

$$H(x) = \frac{\omega-1}{\omega}F(x) + \frac{1}{\omega}F(x)\sum_{n=0}^{\infty}\omega^n x^n = \left(\frac{\omega-1}{\omega} + \frac{1}{\omega(1-\omega x)}\right)F(x)\,. \qquad (3.5)$$

■

Proof of Theorem 2.3. Let $S := \mathrm{X}\{A^n\}_{n=1}^{\infty}$. Consider the system of finite sequences

$$\mathcal{R} := \left\{ \mathbf{s} = (c_1, \dots, c_n) \,\middle|\, n \geq 0\,,\;\; c_i \in \mathbb{N}\,,\;\; 1 \leq c_1 \leq \lfloor A \rfloor - 1\,, \right.$$
$$\left. \max_{i=1,\dots,k-1} \left\lceil \frac{c_i(c_i - 1)}{A - 1} \right\rceil \leq c_k \leq \lfloor A \rfloor - 1\,,\;\; 2 \leq k \leq n \right\}.$$

By Theorem 2.1 we have $\big(0, \frac{1}{(\lfloor A \rfloor - 1)A}\big] \subseteq S$, thus by (3.1) for every $\mathbf{s} = (c_1, \dots$ $\dots, c_n) \in \mathcal{R}$ we obtain

$$\mathcal{I}(\mathbf{s}) := \sum_{i=1}^{n} \frac{1}{A^i c_i} + \frac{1}{A^n} \left(0 \;,\; \frac{1}{(\lfloor A \rfloor - 1)A} \right] \subseteq S\,, \tag{3.6}$$

hence $\bigcup_{\mathbf{s} \in \mathcal{R}} \mathcal{I}(\mathbf{s}) \subseteq S$. For every $\mathbf{s} = (c_1, \dots, c_n) \in \mathcal{R}$ and $1 \leq k < \ell \leq n$ we have

$$c_k(c_k - 1) \leq (A - 1) \left\lceil \frac{c_k(c_k - 1)}{A - 1} \right\rceil \leq (A - 1)c_\ell \leq (A - 1)(\lfloor A \rfloor - 1)\,.$$

The intervals $\mathcal{I}(\mathbf{s})$, $\mathbf{s} \in \mathcal{R}$, are pairwise disjoint. In order to prove this fact consider two cases.

1. Suppose that $\mathbf{s} = (c_1, \dots, c_m) \in \mathcal{R}$, $\mathbf{s}^* = (c_1^*, \dots, c_n^*) \in \mathcal{R}$, $m \geq 1$, $n \geq 1$, $c_i = c_i^*$ for $i < k$ and $c_k < c_k^*$. Then

$$\begin{aligned}
\inf \mathcal{I}(\mathbf{s}) - \inf \mathcal{I}(\mathbf{s}^*) &= \sum_{i=1}^{m} \frac{1}{A^i c_i} - \sum_{i=1}^{n} \frac{1}{A^i c_i^*} \\
&= \frac{1}{A^k} \left(\frac{1}{c_k} - \frac{1}{c_k^*} \right) + \sum_{i=k+1}^{m} \frac{1}{A^i c_i} - \sum_{i=k+1}^{n} \frac{1}{A^i c_i^*} \\
&\geq \frac{1}{A^k c_k^*(c_k^* - 1)} - \sum_{i=k+1}^{n} \frac{1}{A^i \left\lceil \frac{c_k^*(c_k^* - 1)}{A - 1} \right\rceil} \\
&= \frac{1}{A^k c_k^*(c_k^* - 1)} - \frac{1}{A^k (A - 1) \left\lceil \frac{c_k^*(c_k^* - 1)}{A - 1} \right\rceil} + \frac{1}{A^n (A - 1) \left\lceil \frac{c_k^*(c_k^* - 1)}{A - 1} \right\rceil} \\
&\geq \frac{1}{A^n (A - 1) \left\lceil \frac{c_k^*(c_k^* - 1)}{A - 1} \right\rceil} \geq \frac{1}{A^n (A - 1)(\lfloor A \rfloor - 1)} > \frac{1}{A^n} \frac{1}{(\lfloor A \rfloor - 1)A}\,,
\end{aligned}$$

so $\inf \mathcal{I}(\mathbf{s}) > \sup \mathcal{I}(\mathbf{s}^*)$ and $\mathcal{I}(\mathbf{s}) \cap \mathcal{I}(\mathbf{s}^*) = \emptyset$.

2. Suppose that $\mathbf{s} = (c_1, \dots, c_n) \in \mathcal{R}$, $\mathbf{s}^* = (c_1^*, \dots, c_m^*) \in \mathcal{R}$, $0 \leq m < n$, $c_i = c_i^*$ for $i \leq m$. Then

$$\inf \mathcal{I}(\mathbf{s}) - \inf \mathcal{I}(\mathbf{s}^*) = \sum_{i=m+1}^{n} \frac{1}{A^i c_i} \geq \frac{1}{A^{m+1} c_{m+1}} \geq \frac{1}{A^m} \frac{1}{(\lfloor A \rfloor - 1)A}\,,$$

so $\inf \mathcal{I}(\mathbf{s}) \geq \sup \mathcal{I}(\mathbf{s}^*)$ and $\mathcal{I}(\mathbf{s}) \cap \mathcal{I}(\mathbf{s}^*) = \emptyset$.

Now we find $\lambda\big(\bigcup_{\mathbf{s}\in\mathcal{R}} \mathcal{I}(\mathbf{s})\big)$. For $1 \leq u \leq \lfloor A \rfloor - 1$ and $n \geq 0$ denote by $M_u(n)$ the number of sequences $(c_1, \dots, c_n) \in \mathcal{R}$ with $c_i \leq u$,

$$M_u(n) := \#\big\{(c_1, \dots, c_n) \in \mathcal{R} \bigm| c_i \leq u\,, 1 \leq i \leq n\big\}\,.$$

Obviously, $M_1(n) = 1$ for every n. For $u \geq 2$ the number $M_u(n)$ counts 1. sequences with $c_i \leq u - 1$ for every i and 2. sequences such that there exists k with $c_k = u$.

1. The number of sequences with $c_i \leq u - 1$ is $M_{u-1}(n)$.

2. Suppose that $c_i \leq u - 1$ for $i \leq k - 1$ and $c_k = u$. Then $(c_1, \dots, c_n) \in \mathcal{R}$ if $\lceil \frac{u(u-1)}{A-1} \rceil \leq c_j \leq u$ for every $j \geq k + 1$. So the total number of sequences such that there exists k with $c_k = u$ is

$$\sum_{k=1}^{n} M_{u-1}(k-1)\left(u - \left\lceil \frac{u(u-1)}{A-1} \right\rceil + 1\right)^{n-k}.$$

Thus we have for $u \geq 2$

$$M_u(n) = M_{u-1}(n) + \sum_{k=1}^{n} M_{u-1}(k-1)\left(u - \left\lceil \frac{u(u-1)}{A-1} \right\rceil + 1\right)^{n-k}.$$

Recall that $\omega_1 = 1$ and for $j \geq 2$

$$\omega_j = j - \left\lceil \frac{j(j-1)}{A-1} \right\rceil + 1\,.$$

From Lemma 3.4 we obtain that for x with $|x| < \min_{j=1,\dots,u} \frac{1}{\omega_j}$

$$\sum_{n=0}^{\infty} M_u(n) x^n = \prod_{j=1}^{u} \left(\frac{\omega_j - 1}{\omega_j} + \frac{1}{\omega_j (1 - \omega_j x)} \right).$$

We have $\omega_j \leq \lfloor A \rfloor - 1$ for every j. Then

$$\begin{aligned} \lambda(S) \geq \lambda\Big(\bigcup_{\mathbf{s}\in\mathcal{R}} \mathcal{I}(\mathbf{s})\Big) &= \sum_{n=0}^{\infty} \frac{M_{\lfloor A \rfloor - 1}(n)}{A^n} \frac{1}{(\lfloor A \rfloor - 1)A} \\ &= \frac{1}{(\lfloor A \rfloor - 1)A} \prod_{j=1}^{\lfloor A \rfloor - 1} \left(\frac{\omega_j - 1}{\omega_j} + \frac{A}{\omega_j (A - \omega_j)} \right) \\ &= \frac{1}{(\lfloor A \rfloor - 1)A} \prod_{j=1}^{\lfloor A \rfloor - 1} \frac{A - \omega_j + 1}{A - \omega_j}\,. \end{aligned}$$

■

Proof of Theorem 2.4. Let $S := \mathrm{X}\{A^n\}_{n=1}^{\infty}$. Assume for a moment that we do not know the constants B, α_u, β_u and δ_u. Consider the system

$$\mathcal{T} := \left\{ (\mathbf{s}, \ell) \;\middle|\; \begin{array}{c} \mathbf{s} = (c_1, \ldots, c_n)\,, \;\; n \geq 0\,, \;\; c_i \in \mathbb{N}\,, \;\; \ell \in \mathbb{N}\,, \\ 1 \leq c_1 \leq B\,, \\ \max\limits_{i=1,\ldots,k-1} \alpha_{c_i} \leq c_k \leq \min\limits_{i=1,\ldots,k-1} \beta_{c_i}\,, \;\; 2 \leq k \leq n\,, \\ \gamma_1 \leq \;\ell\; \leq \delta_1 \qquad\qquad \text{if } n = 0\,, \\ \max\limits_{i=1,\ldots,n} \gamma_{c_i} \leq \;\ell\; \leq \min\limits_{i=1,\ldots,n} \delta_{c_i} \;\; \text{if } n \geq 1 \end{array} \right\}$$

such that for every $(\mathbf{s}, \ell) \in \mathcal{T}$ and for α_u, β_u, γ_u and δ_u with $u = 1, \ldots, B$

$$\big(A - 1 - u(u+1)\big)\beta_u + u(u+1) \geq 0\,, \tag{3.7}$$

$$(\beta_u - A)\ell + A(\beta_u - 1) \geq 0\,, \tag{3.8}$$

$$A\ell \geq (A-1)\alpha_u\,, \tag{3.9}$$

$$(A-1)\alpha_u \geq u(u-1)\,. \tag{3.10}$$

The constants α_u, β_u, γ_u and δ_u are chosen so that they are positive integers and that α_u and γ_u are minimal and β_u and δ_u are maximal. The constant B is chosen so that $\alpha_u \leq \beta_u$ for $u \leq B$. (In some cases we may obtain that $\gamma_u > \delta_u$ or that $\delta_u = 0$ but these facts do not make problems.)

Inequality (3.10) and the fact that $\alpha_u \in \mathbb{N}$ immediately give

$$\alpha_u = \max\left\{1, \left\lceil \frac{u(u-1)}{A-1} \right\rceil \right\}.$$

From (3.9) we obtain

$$\ell \geq \gamma_u = \left\lceil \frac{A-1}{A}\,\alpha_u \right\rceil = \alpha_u\,.$$

If $u(u+1) \leq A - 1$ then (3.7) is satisfied. Otherwise it implies that

$$\beta_u \leq \left\lfloor \frac{u(u+1)}{u(u+1) - (A-1)} \right\rfloor.$$

From the condition $\beta_u \leq B$ we obtain (2.1). Inequality (3.8) implies that

$$\ell \leq \left\lfloor \frac{A(\beta_u - 1)}{A - \beta_u} \right\rfloor.$$

Another condition on δ_u comes from (3.11). It follows that

$$\delta_u = \min\left\{ \left\lfloor \frac{A(\beta_u - 1)}{A - \beta_u} \right\rfloor, \left\lceil -\frac{3}{2} + \sqrt{A - \frac{3}{4}} \right\rceil \right\}$$

$$= \min\left\{ \beta_u - 1, \left\lceil -\frac{3}{2} + \sqrt{A - \frac{3}{4}} \right\rceil \right\}.$$

Now we find B. Suppose that $\alpha_u \ge 3$. Then

$$u > \left\lfloor \frac{1}{2} + \sqrt{2A - \frac{7}{4}} \right\rfloor$$

and $\beta_u = 1$. This is in contradiction with $\alpha_u \le \beta_u$. Hence $\alpha_u \in \{1, 2\}$ for every u. This fact and the fact that $\beta_u \ge 1$ for every u imply that $\alpha_u \le \beta_u$ is equivalent with

$$(\alpha_u = 1) \text{ or } (\alpha_u = 2 \text{ and } \beta_u \ge 2),$$

hence with

$$u \le \max\left\{ \left\lfloor \frac{1}{2} + \sqrt{A - \frac{3}{4}} \right\rfloor, \left\lfloor -\frac{1}{2} + \sqrt{2A - \frac{7}{4}} \right\rfloor \right\} =: B.$$

It follows that $2 \le B < A$ for every $A > 3$.

Now the system $\mathcal{T}$ is completely determined.

For a positive integer n put $T_n := \frac{1}{An} + \left(0, \frac{1}{A(A-1)}\right]$. From (3.1) we obtain that $S \subseteq \bigcup_{n=1}^{\infty} T_n$. For $\ell \in \mathbb{N}$ put

$$H_\ell := \left(\sup T_{\ell+1}, \inf T_\ell\right] = \left(\frac{1}{A(\ell+1)} + \frac{1}{A(A-1)} \;,\; \frac{1}{A\ell} \right].$$

This set is nonempty if $\sup T_{\ell+1} < \inf T_\ell$. This holds if

$$\ell < -\frac{1}{2} + \sqrt{A - \frac{3}{4}},$$

hence if

$$\ell \le \left\lceil -\frac{3}{2} + \sqrt{A - \frac{3}{4}} \right\rceil. \tag{3.11}$$

Moreover, $H_\ell \cap S = \emptyset$ for every such ℓ.

For every $(\mathbf{s}, \ell) \in \mathcal{T}$, $\mathbf{s} = (c_1, \dots, c_n)$, $n \ge 0$, put

$$\mathcal{J}(\mathbf{s}, \ell) := \sum_{i=1}^{n} \frac{1}{A^i c_i} + \frac{1}{A^n} H_\ell.$$

We have

$$\inf \mathcal{J}(\mathbf{s}, \ell) = \sum_{i=1}^{n} \frac{1}{A^i c_i} + \frac{1}{A^{n+1}} \left(\frac{1}{\ell+1} + \frac{1}{A-1} \right),$$

$$\sup \mathcal{J}(\mathbf{s}, \ell) = \sum_{i=1}^{n} \frac{1}{A^i c_i} + \frac{1}{A^{n+1}\ell}$$

and

$$\lambda\big(\mathcal{J}(\mathbf{s},\ell)\big)=\frac{1}{A^{n+1}}\left(\frac{1}{\ell(\ell+1)}-\frac{1}{A-1}\right).$$

We will now prove that for every $(\mathbf{s},\ell)\in\mathcal{T}$ the set $\mathcal{J}(\mathbf{s},\ell)$ is disjoint with S.

Let $x\in S$ and $(\mathbf{s},\ell)\in\mathcal{T}$, $\mathbf{s}=(c_1,\dots,c_n)$. There exist positive integers c_i^*, $i\in\mathbb{N}$, such that $x=\sum_{i=1}^{\infty}\frac{1}{A^i c_i^*}$. Now consider three cases.

1. Suppose that $c_i=c_i^*$ for $i\le n$. From $\sum_{i=1}^{\infty}\frac{1}{A^i c_{i+n}^*}\in S$ we obtain $\sum_{i=1}^{\infty}\frac{1}{A^i c_{i+n}^*}\notin H_\ell$ and

$$x=\sum_{i=1}^{n}\frac{1}{A^i c_i^*}+\frac{1}{A^n}\sum_{i=1}^{\infty}\frac{1}{A^i c_{i+n}^*}\notin\sum_{i=1}^{n}\frac{1}{A^i c_i^*}+\frac{1}{A^n}H_\ell=\mathcal{J}(\mathbf{s},\ell)\,.$$

2. Suppose that $c_i=c_i^*$ for $i<k$ and $c_k<c_k^*$. From the definition of the system $\mathcal{T}$ we obtain that $c_i\le\beta_{c_k}$ for $i>k$, that

$$\big(A-1-c_k(c_k+1)\big)\beta_{c_k}+c_k(c_k+1)\ge 0$$

and that

$$(\beta_{c_k}-A)\ell+A(\beta_{c_k}-1)\ge 0\,.$$

Then

$$\begin{aligned}
&\inf\mathcal{J}(\mathbf{s},\ell)-x\\
&=\frac{1}{A^k}\left(\frac{1}{c_k}-\frac{1}{c_k^*}\right)+\sum_{i=k+1}^{n}\frac{1}{A^i c_i}+\frac{1}{A^{n+1}}\left(\frac{1}{\ell+1}+\frac{1}{A-1}\right)-\sum_{i=k+1}^{\infty}\frac{1}{A^i c_i^*}\\
&\ge\frac{1}{A^k}\frac{1}{c_k(c_k+1)}+\sum_{i=k+1}^{n}\frac{1}{A^i\beta_{c_k}}+\frac{1}{A^{n+1}}\left(\frac{1}{\ell+1}+\frac{1}{A-1}\right)-\frac{1}{A^k}\frac{1}{A-1}\\
&=\frac{\big(A-1-c_k(c_k+1)\big)\beta_{c_k}+c_k(c_k+1)}{A^k c_k(c_k+1)(A-1)\beta_{c_k}}+\frac{(\beta_{c_k}-A)\ell+A(\beta_{c_k}-1)}{A^{n+1}(A-1)(\ell+1)\beta_{c_k}}\ge 0\,,
\end{aligned}$$

hence $x\notin\mathcal{J}(\mathbf{s},\ell)$.

3. Suppose that $c_i=c_i^*$ for $i<k$ and $c_k>c_k^*$. From the definition of the system $\mathcal{T}$ we obtain that $c_i\ge\alpha_{c_k}$ for $i>k$ and that

$$A\ell\ge(A-1)\alpha_{c_k}\ge c_k(c_k-1)\,.$$

Then

$$\begin{aligned} x-\sup\mathcal{J}(\mathbf{s},\ell) &= \frac{1}{A^k}\left(\frac{1}{c_k^*}-\frac{1}{c_k}\right)+\sum_{i=k+1}^{\infty}\frac{1}{A^i c_i^*}-\sum_{i=k+1}^{n}\frac{1}{A^i c_i}-\frac{1}{A^{n+1}\ell} \\ &> \frac{1}{A^k}\frac{1}{c_k(c_k-1)}-\sum_{i=k+1}^{n}\frac{1}{A^i\alpha_{c_k}}-\frac{1}{A^{n+1}\ell} \\ &= \frac{1}{A^k}\left(\frac{1}{c_k(c_k-1)}-\frac{1}{(A-1)\alpha_{c_k}}\right) \\ &\quad+\frac{1}{A^n}\left(\frac{1}{(A-1)\alpha_{c_k}}-\frac{1}{A\ell}\right)\geq 0\,, \end{aligned}$$

hence $x\notin\mathcal{J}(\mathbf{s},\ell)$.

Now we will prove that for $(\mathbf{s},\ell)\neq(\mathbf{s}^*,\ell^*)$ the sets $\mathcal{J}(\mathbf{s},\ell)$ and $\mathcal{J}(\mathbf{s}^*,\ell^*)$ are disjoint. Again, consider three cases.

1. If $\mathbf{s}=\mathbf{s}^*$ then this fact follows from $H_\ell\cap H_{\ell^*}=\emptyset$.

2. Suppose that $\mathbf{s}=(c_1,\dots,c_m)$, $m\geq 1$, $\mathbf{s}^*=(c_1^*,\dots,c_n^*)$, $n\geq 1$, $c_i=c_i^*$ for $i<k$ and $c_k<c_k^*$. From the definition of the system $\mathcal{T}$ we obtain that $c_i^*\geq\alpha_{c_k^*}$ for $i>k$ and that $\ell^*\geq\alpha_{c_k^*}$. We use the fact that

$$(A-1)\alpha_{c_k^*}\geq c_k^*(c_k^*-1)\,.$$

Then

$$\begin{aligned} &\inf\mathcal{J}(\mathbf{s},\ell)-\sup\mathcal{J}(\mathbf{s}^*,\ell^*) \\ &= \frac{1}{A^k}\left(\frac{1}{c_k}-\frac{1}{c_k^*}\right)+\sum_{i=k+1}^{m}\frac{1}{A^i c_i}+\frac{1}{A^{m+1}}\left(\frac{1}{\ell+1}+\frac{1}{A-1}\right) \\ &\quad-\sum_{i=k+1}^{n}\frac{1}{A^i c_i^*}-\frac{1}{A^{n+1}\ell^*} \\ &\geq \frac{1}{A^k}\frac{1}{c_k^*(c_k^*-1)}-\sum_{i=k+1}^{n}\frac{1}{A^i\alpha_{c_k^*}}-\frac{1}{A^{n+1}\alpha_{c_k^*}} \\ &= \frac{1}{A^k}\left(\frac{1}{c_k^*(c_k^*-1)}-\frac{1}{(A-1)\alpha_{c_k^*}}\right)+\frac{1}{A^n}\left(\frac{1}{(A-1)\alpha_{c_k^*}}-\frac{1}{A\alpha_{c_k^*}}\right)\geq 0\,. \end{aligned}$$

3. Suppose that $\mathbf{s}=(c_1,\dots,c_n)$, $\mathbf{s}^*=(c_1^*,\dots,c_m^*)$, $n>m\geq 0$ and $c_i=c_i^*$ for $i\leq m$.

3a. If $c_{m+1}\leq\ell^*$ then

$$\begin{aligned} &\inf\mathcal{J}(\mathbf{s},\ell)-\sup\mathcal{J}(\mathbf{s}^*,\ell^*) \\ &\qquad= \sum_{i=1}^{n}\frac{1}{A^i c_i}+\frac{1}{A^{n+1}}\left(\frac{1}{\ell+1}+\frac{1}{A-1}\right)-\sum_{i=1}^{m}\frac{1}{A^i c_i^*}-\frac{1}{A^{m+1}\ell^*} \\ &\qquad\geq \frac{1}{A^{m+1}c_{m+1}}-\frac{1}{A^{m+1}\ell^*}\geq 0\,. \end{aligned}$$

3b. If $c_{m+1} \geq \ell^* + 1$ then

$$\begin{aligned}
&\inf \mathcal{J}(\mathbf{s}^*, \ell^*) - \sup \mathcal{J}(\mathbf{s}, \ell) \\
&\qquad = \sum_{i=1}^{m} \frac{1}{A^i c_i^*} + \frac{1}{A^{m+1}} \left(\frac{1}{\ell^* + 1} + \frac{1}{A-1} \right) - \sum_{i=1}^{n} \frac{1}{A^i c_i} - \frac{1}{A^{n+1} \ell} \\
&\qquad > \frac{1}{A^{m+1}} \left(\frac{1}{\ell^* + 1} + \frac{1}{A-1} \right) - \frac{1}{A^{m+1} c_{m+1}} - \sum_{i=m+2}^{\infty} \frac{1}{A^i} \geq 0 \,.
\end{aligned}$$

Hence the sets $\mathcal{J}(\mathbf{s}, \ell)$, $(\mathbf{s}, \ell) \in \mathcal{T}$, are pairwise disjoint.

Now we find $\lambda\big(\bigcup_{(\mathbf{s},\ell)\in\mathcal{T}} \mathcal{J}(\mathbf{s}, \ell)\big)$. For $1 \leq u \leq B$ and $n \geq 0$ denote by $M_u(n)$ the number of sequences $\mathbf{s} = (c_1, \ldots, c_n)$ with $c_i \leq u$, so $M_u(n) := \# L_u(n)$, where

$$\begin{aligned}
L_u(n) := \Big\{ (c_1, \ldots, c_n) \;\Big|\; & c_i \in \mathbb{N}, \;\; 1 \leq c_1 \leq u \,, \\
& \max_{i=1,\ldots,k-1} \alpha_{c_i} \leq c_k \leq \min_{i=1,\ldots,k-1} \{u, \beta_{c_i}\} \,, \;\; 2 \leq k \leq n \Big\} \,.
\end{aligned}$$

Obviously, $M_1(n) = 1$ for every n. For $u \geq 2$ the number $M_u(n)$ counts 1. sequences with $c_i \leq u - 1$ for every i and 2. sequences such that there exists k with $c_k = u$.

1. The number of sequences with $c_i \leq u - 1$ is $M_{u-1}(n)$.

2. Suppose that $c_i \leq u - 1$ for $i \leq k - 1$ and $c_k = u$. Then $(c_1, \ldots, c_n) \in L_u(n)$ if $\alpha_u \leq c_j \leq \min\{u, \beta_u\}$ for every $j \geq k + 1$. From the fact that $\alpha_u \leq \min\{u, \beta_u\}$ for $2 \leq u \leq B$ we obtain that the total number of sequences such that there exists k with $c_k = u$ is

$$\sum_{k=1}^{n} M_{u-1}(k-1) \Big(\min\{u, \beta_u\} - \alpha_u + 1 \Big)^{n-k} = \sum_{k=1}^{n} M_{u-1}(k-1) \sigma_u^{n-k}.$$

Thus we have for $u \geq 2$

$$M_u(n) = M_{u-1}(n) + \sum_{k=1}^{n} M_{u-1}(k-1) \sigma_u^{n-k}.$$

Since $\sigma_u < A$, Lemma 3.4 implies that for $u \geq 1$

$$\zeta_u := \sum_{n=0}^{\infty} \frac{M_u(n)}{A^n} = \prod_{i=1}^{u} \frac{A - \sigma_i + 1}{A - \sigma_i} \,.$$

(Notice that one can use (2.2) to compute ζ_u for $u \geq 1$.)

For $1 \leq u \leq B$ and $n \geq 1$ denote by $M_u^*(n)$ the number of sequences $(c_1, \ldots, c_n)$ with $\max_{i=1,\ldots,n} c_i = u$,

$$M_u^*(n) := \# \Big\{ (c_1, \ldots, c_n) \in L_u(n) \;\Big|\; \max_{i=1,\ldots,n} c_i = u \Big\} \,.$$

If we set $M_0(n) := 0$ then we have

$$M_u^*(n) = M_u(n) - M_{u-1}(n)$$

for every $u \geq 1$. From the fact that $M_u(0) = 1$ for every $u \geq 1$ we obtain that $M_u^*(0) = 0$ for $u \geq 2$.

Now for every $\mathbf{s}_0 \in \bigcup_{u,n} L_u(n)$ evaluate $\lambda\big(\bigcup_{(\mathbf{s}_0,\ell)\in\mathcal{T}} \mathcal{J}(\mathbf{s}_0,\ell)\big)$. If $\mathbf{s}_0$ is an empty sequence then put $v := 1$, else if $\mathbf{s}_0 = (c_1,\dots,c_n)$ put $v := \max_{i=1,\dots,n} c_i$. Then $(\mathbf{s}_0,\ell) \in \mathcal{T}$ if $\alpha_v \leq \ell \leq \delta_v$. From $\alpha_v - 1 \leq \delta_v$ we obtain

$$\begin{aligned}\lambda\Big(\bigcup_{(\mathbf{s}_0,\ell)\in\mathcal{T}} \mathcal{J}(\mathbf{s}_0,\ell)\Big) &= \lambda\Big(\bigcup_{\ell=\alpha_v}^{\delta_v} \mathcal{J}(\mathbf{s}_0,\ell)\Big) = \sum_{\ell=\alpha_v}^{\delta_v} \frac{1}{A^{n+1}}\left(\frac{1}{\ell(\ell+1)} - \frac{1}{A-1}\right) \\ &= \frac{1}{A^{n+1}}\left(\frac{1}{\alpha_v} - \frac{1}{\delta_v+1} - \frac{\delta_v-\alpha_v+1}{A-1}\right) = \frac{\varepsilon_v}{A^{n+1}}\,.\end{aligned}$$

Notice that this formula is correct even in the case that $\delta_v = \alpha_v - 1$.

Then

$$\begin{aligned}\lambda\Big(\bigcup_{(\mathbf{s},\ell)\in\mathcal{T}} \mathcal{J}(\mathbf{s},\ell)\Big) &= \frac{\varepsilon_1}{A} + \sum_{u=1}^{B}\sum_{n=1}^{\infty} \frac{M_u^*(n)\varepsilon_u}{A^{n+1}} = \sum_{u=1}^{B}\sum_{n=0}^{\infty} \frac{M_u^*(n)\varepsilon_u}{A^{n+1}} \\ &= \frac{1}{A}\sum_{u=1}^{B} \varepsilon_u\left(\sum_{n=0}^{\infty}\frac{M_u(n)}{A^n} - \sum_{n=0}^{\infty}\frac{M_{u-1}(n)}{A^n}\right) \\ &= \frac{1}{A}\sum_{u=1}^{B}\varepsilon_u\big(\zeta_u - \zeta_{u-1}\big)\,.\end{aligned}$$

From the facts that $\mathcal{J}(\mathbf{s},\ell) \subseteq \big(0, \frac{1}{A-1}\big]$ and $\mathcal{J}(\mathbf{s},\ell) \cap S = \emptyset$ we obtain

$$\begin{aligned}\lambda\big(\mathrm{X}\{A^n\}_{n=1}^{\infty}\big) &\leq \frac{1}{A-1} - \sum_{(\mathbf{s},\ell)\in\mathcal{T}} \lambda\big(\mathcal{J}(\mathbf{s},\ell)\big) \qquad (3.12) \\ &= \frac{1}{A-1} - \frac{1}{A}\sum_{u=1}^{B}\varepsilon_u\big(\zeta_u - \zeta_{u-1}\big)\,.\end{aligned}$$

■

Lemma 3.5. *Suppose that a function $f(x)$ is twice differentiable on the interval (k,m) and that $|f''(x)| \leq M_j$ for every j and every $x \in (j-1,j)$. Then*

$$\left|\sum_{j=k+1}^{m} f\Big(j-\frac{1}{2}\Big) - \int_k^m f(x)\,\mathrm{d}x\right| \leq \sum_{j=k+1}^{m}\frac{M_j}{8}\,.$$

Proof. Taylor's Theorem implies that for $x \in (j-1,j)$

$$\left|f(x) - f\Big(j-\frac{1}{2}\Big) - \Big(x-j+\frac{1}{2}\Big)f'\Big(j-\frac{1}{2}\Big)\right| \leq \frac{M_j}{8}\,.$$

Then

$$
\begin{aligned}
&\Big|\sum_{j=k+1}^{m} f\Big(j-\frac{1}{2}\Big) - \int_k^m f(x)\,\mathrm{d}x\Big| \\
&\qquad = \Big|\sum_{j=k+1}^{m} \int_{j-1}^{j} \Big(f(x) - f\Big(j-\frac{1}{2}\Big) - \Big(x-j+\frac{1}{2}\Big) f'\Big(j-\frac{1}{2}\Big)\Big)\,\mathrm{d}x\Big| \\
&\qquad \le \sum_{j=k+1}^{m} \frac{M_j}{8}\,.
\end{aligned}
$$

■

Proof of Theorem 2.5. Let

$$f(x) := \frac{A-1}{x^2-(A-1)x+\big(A^2-\frac{3}{2}A+\frac{1}{4}\big)}\,.$$

Then

$$f\Big(j-\frac{1}{2}\Big) = \frac{A-1}{j^2-Aj+(A^2-A)}\,.$$

We use Lemma 3.5 and the fact that if $|c-d| \le 1$ then

$$\frac{1}{A+c} = \frac{1}{A+d}\Big(1+\mathrm{O}\Big(\frac{1}{A}\Big)\Big)\,.$$

Replacing ω_j by $j-\frac{j(j-1)}{A-1}$ we obtain

$$
\begin{aligned}
\prod_{j=1}^{\lfloor A\rfloor-1} \frac{A-\omega_j+1}{A-\omega_j} &= \exp\Big(\sum_{j=1}^{\lfloor A\rfloor-1} \frac{1}{A-\omega_j} + \mathrm{O}\Big(\frac{1}{A}\Big)\Big) \\
&= \exp\bigg(\bigg(\sum_{j=1}^{\lfloor A\rfloor-1} \frac{A-1}{j^2-Aj+(A^2-A)}\bigg)\Big(1+\mathrm{O}\Big(\frac{1}{A}\Big)\Big) + \mathrm{O}\Big(\frac{1}{A}\Big)\bigg) \\
&= \exp\bigg(\bigg(\int_0^{\lfloor A\rfloor-1} f(x)\,\mathrm{d}x + \mathrm{O}\Big(\frac{1}{A^3}\Big)\bigg)\Big(1+\mathrm{O}\Big(\frac{1}{A}\Big)\Big) + \mathrm{O}\Big(\frac{1}{A}\Big)\bigg) \\
&= \exp\bigg(\bigg(\int_0^{A-1} f(x)\,\mathrm{d}x + \mathrm{O}\Big(\frac{1}{A}\Big)\bigg)\Big(1+\mathrm{O}\Big(\frac{1}{A}\Big)\Big) + \mathrm{O}\Big(\frac{1}{A}\Big)\bigg) \\
&= \exp\bigg(\bigg(2\frac{A-1}{\sqrt{\frac{3}{4}A^2-A}} \arctan\frac{A-1}{2\sqrt{\frac{3}{4}A^2-A}} + \mathrm{O}\Big(\frac{1}{A}\Big)\bigg) \\
&\qquad \times \Big(1+\mathrm{O}\Big(\frac{1}{A}\Big)\Big) + \mathrm{O}\Big(\frac{1}{A}\Big)\bigg) \\
&\to \exp\Big(\frac{4}{\sqrt{3}}\arctan\frac{\sqrt{3}}{3}\Big) = \exp\frac{2\pi}{3\sqrt{3}}\,.
\end{aligned}
$$

The result follows from $\frac{1}{(\lfloor A\rfloor-1)A} \sim A^{-2}$. ■

Proof of Theorem 2.6. Let A be a sufficiently large positive real number. The symbol u will always be used for a positive integer not exceeding B. Set

$$\begin{aligned}
B_1 &:= \lfloor \sqrt{A} \rfloor\,,\\
B_2 &:= \left\lfloor \frac{1}{2} + \sqrt{A - \frac{3}{4}} \right\rfloor,\\
B_3 &:= \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + \left(1 + \frac{1}{-\frac{1}{2} + \sqrt{A - \frac{3}{4}}}\right)(A-1)} \right\rfloor,\\
B_4 &:= \left\lfloor -\frac{1}{2} + \sqrt{\frac{3}{2}A - \frac{5}{4}} \right\rfloor.
\end{aligned}$$

Notice that $B_1 \le B_2 \le B_3 \le B_1 + 1$. We have

$$\alpha_u = \begin{cases} 1 & \text{if } u \le B_2\,,\\ 2 & \text{if } u \ge B_2 + 1 \end{cases}$$

and

$$\beta_u = \begin{cases} B & \text{if } u \le B_2 - 1\,,\\ \min\{B, \lfloor \frac{u(u+1)}{u(u+1)-(A-1)} \rfloor\} & \text{if } u \ge B_2\,. \end{cases}$$

From $u \le B$ and from the fact that $u \le \frac{u(u+1)}{u(u+1)-(A-1)}$ if $u \le \lfloor \sqrt{A} \rfloor$ we obtain for $u \ge \lfloor \frac{1}{2} + \sqrt{A - \frac{3}{4}} \rfloor$

$$\begin{aligned}
\min\{u, \beta_u\} &= \min\left\{u, B, \left\lfloor \frac{u(u+1)}{u(u+1)-(A-1)} \right\rfloor \right\}\\
&= \min\left\{u, \left\lfloor \frac{u(u+1)}{u(u+1)-(A-1)} \right\rfloor \right\}\\
&= \left\lfloor \min\left\{u, \frac{u(u+1)}{u(u+1)-(A-1)} \right\} \right\rfloor,
\end{aligned}$$

so

$$\min\{u, \beta_u\} = \begin{cases} u & \text{if } u \le B_1\,,\\ \lfloor \frac{u(u+1)}{u(u+1)-(A-1)} \rfloor & \text{if } u \ge B_1 + 1\,. \end{cases}$$

We modify the definition of δ_u:

$$\delta_u^* := \min\left\{ (\beta_u - 1), \left\lfloor -\frac{1}{2} + \sqrt{A - \frac{3}{4}} \right\rfloor \right\}.$$

This modification will not change the result, since if we replace δ_u by δ_u^* then ε_u and hence $U(A)$ does not change. We use the fact that $\lfloor -\frac{1}{2} + \sqrt{A - \frac{3}{4}} \rfloor < B - 1$. For $u \le B_2 - 1$ we have

$$\delta_u^* = \left\lfloor -\frac{1}{2} + \sqrt{A - \frac{3}{4}} \right\rfloor.$$

For $u \geq B_2$ we use the fact that

$$\frac{A-1}{u(u+1)-(A-1)} \geq T \quad \text{if} \quad u \leq \left\lfloor -\frac{1}{2} + \sqrt{\frac{1}{4} + \Big(1+\frac{1}{T}\Big)(A-1)} \right\rfloor .$$

We have

$$\begin{aligned} \delta_u^* &= \min\Big\{(B-1), \Big\lfloor \frac{A-1}{u(u+1)-(A-1)} \Big\rfloor, \Big\lfloor -\frac{1}{2} + \sqrt{A-\frac{3}{4}} \Big\rfloor \Big\} \\ &= \min\Big\{ \Big\lfloor \frac{A-1}{u(u+1)-(A-1)} \Big\rfloor, \Big\lfloor -\frac{1}{2} + \sqrt{A-\frac{3}{4}} \Big\rfloor \Big\} . \end{aligned}$$

This implies that

$$\delta_u^* = \begin{cases} \lfloor -\frac{1}{2} + \sqrt{A-\frac{3}{4}} \rfloor & \text{if } u \leq B_3 - 1 , \\ \lfloor \frac{A-1}{u(u+1)-(A-1)} \rfloor & \text{if } u \geq B_3 . \end{cases}$$

From the earlier facts we get

$$\varepsilon_u = \begin{cases} 1 - \frac{1}{\lfloor \frac{1}{2} + \sqrt{A-\frac{3}{4}} \rfloor} - \frac{\lfloor -\frac{1}{2} + \sqrt{A-\frac{3}{4}} \rfloor}{A-1} & \text{if } u \leq B_3 - 1 , \\ 1 - \frac{1}{\lfloor \frac{u(u+1)}{u(u+1)-(A-1)} \rfloor} - \frac{\lfloor \frac{A-1}{u(u+1)-(A-1)} \rfloor}{A-1} & \text{if } B_2 = B_3 \text{ and } u = B_2 , \\ \frac{1}{2} - \frac{1}{\lfloor \frac{u(u+1)}{u(u+1)-(A-1)} \rfloor} - \frac{\lfloor \frac{A-1}{u(u+1)-(A-1)} \rfloor - 1}{A-1} & \text{if } B_2 + 1 \leq u \leq B_4 \\ 0 & \text{if } u \geq B_4 + 1 \end{cases}$$

and

$$\sigma_u = \begin{cases} u & \text{if } u \leq B_1 , \\ \lfloor \frac{u(u+1)}{u(u+1)-(A-1)} \rfloor & \text{if } B_2 = B_1 + 1 \text{ and } u = B_2 , \\ \lfloor \frac{A-1}{u(u+1)-(A-1)} \rfloor & \text{if } u \geq B_2 + 1 . \end{cases}$$

For $u \leq B_1$ we have

$$\zeta_u = \prod_{j=1}^{u} \frac{A-j+1}{A-j} = \frac{A}{A-u} .$$

If $B_2 = B_1$ then

$$\zeta_{B_2} = \zeta_{B_1} = \frac{A}{A - \lfloor \sqrt{A} \rfloor} .$$

If $B_2 = B_1 + 1$ then

$$\begin{aligned} \zeta_{B_2} &= \Big(1 + \frac{1}{A - \sigma_{\lfloor \sqrt{A} \rfloor + 1}}\Big) \zeta_{\lfloor \sqrt{A} \rfloor} \\ &= \Big(1 + \frac{1}{A - 1 - \lfloor \frac{A-1}{\lfloor \sqrt{A} \rfloor^2 + 3\lfloor \sqrt{A} \rfloor - A + 3} \rfloor}\Big) \frac{A}{A - \lfloor \sqrt{A} \rfloor} . \end{aligned}$$

In any case,

$$\zeta_{B_2} = 1 + \mathrm{O}\Big(\frac{1}{\sqrt{A}}\Big)\,.$$

For $u \ge B_2 + 1$ we have

$$\zeta_u = \zeta_{B_2} \prod_{j=B_2+1}^{u} \left(1 + \frac{1}{A - \left\lfloor \frac{A-1}{j(j+1)-(A-1)} \right\rfloor}\right).$$

Notice that

$$\zeta_u - \zeta_{u-1} = \frac{\zeta_{u-1}}{A - \sigma_u}\,.$$

Now consider three cases.

1. Suppose that $B_1 = B_2 = B_3$. Then

$$\sum_{u=1}^{B} \varepsilon_u(\zeta_u - \zeta_{u-1}) = \varepsilon_1 \zeta_{\lfloor\sqrt{A}\rfloor - 1} + \sum_{u=\lfloor\sqrt{A}\rfloor}^{\lfloor\sqrt{A}\rfloor+2} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u} + \sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u}\,.$$

We have

$$\varepsilon_1 = 1 - \frac{1}{\left\lfloor \frac{1}{2} + \sqrt{A - \frac{3}{4}} \right\rfloor} - \frac{\left\lfloor -\frac{1}{2} + \sqrt{A - \frac{3}{4}} \right\rfloor}{A-1} = 1 - \frac{2}{\sqrt{A}} + \mathrm{O}\Big(\frac{1}{A}\Big)$$

and

$$\zeta_{\lfloor\sqrt{A}\rfloor - 1} = \frac{A}{A - \lfloor\sqrt{A}\rfloor + 1}\,,$$

so

$$\varepsilon_1 \zeta_{\lfloor\sqrt{A}\rfloor - 1} = 1 - \frac{1}{\sqrt{A}} + \mathrm{O}\Big(\frac{1}{A}\Big)\,.$$

For $\lfloor\sqrt{A}\rfloor \le u \le \lfloor\sqrt{A}\rfloor + 2$ we have $\varepsilon_u = \mathrm{O}(1)$, $\zeta_{u-1} = \mathrm{O}(1)$ and $\sigma_u = \mathrm{O}(\sqrt{A})$, hence

$$\sum_{u=\lfloor\sqrt{A}\rfloor}^{\lfloor\sqrt{A}\rfloor+2} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u} = \mathrm{O}\Big(\frac{1}{A}\Big)$$

and

$$\varepsilon_1 \zeta_{\lfloor\sqrt{A}\rfloor - 1} + \sum_{u=\lfloor\sqrt{A}\rfloor}^{\lfloor\sqrt{A}\rfloor+2} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u} = 1 - \frac{1}{\sqrt{A}} + \mathrm{O}\Big(\frac{1}{A}\Big)\,.$$

2. Suppose that $B_1 = B_2 < B_3 = B_1 + 1$. Then

$$\sum_{u=1}^{B} \varepsilon_u(\zeta_u - \zeta_{u-1}) = \varepsilon_1 \zeta_{\lfloor\sqrt{A}\rfloor} + \sum_{u=\lfloor\sqrt{A}\rfloor+1}^{\lfloor\sqrt{A}\rfloor+2} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u} + \sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u}\,.$$

We have

$$\varepsilon_1 = 1 - \frac{1}{\lfloor \frac{1}{2} + \sqrt{A - \frac{3}{4}} \rfloor} - \frac{\lfloor -\frac{1}{2} + \sqrt{A - \frac{3}{4}} \rfloor}{A-1} = 1 - \frac{2}{\sqrt{A}} + \mathrm{O}\Big(\frac{1}{A}\Big)$$

and

$$\zeta_{\lfloor \sqrt{A} \rfloor} = \frac{A}{A - \lfloor \sqrt{A} \rfloor},$$

so

$$\varepsilon_1 \zeta_{\lfloor \sqrt{A} \rfloor} = 1 - \frac{1}{\sqrt{A}} + \mathrm{O}\Big(\frac{1}{A}\Big).$$

Again, for $\lfloor \sqrt{A} \rfloor + 1 \le u \le \lfloor \sqrt{A} \rfloor + 2$ we have $\varepsilon_u = \mathrm{O}(1)$, $\zeta_{u-1} = \mathrm{O}(1)$ and $\sigma_u = \mathrm{O}(\sqrt{A})$, hence

$$\sum_{u=\lfloor \sqrt{A} \rfloor+1}^{\lfloor \sqrt{A} \rfloor+2} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u} = \mathrm{O}\Big(\frac{1}{A}\Big)$$

and

$$\varepsilon_1 \zeta_{\lfloor \sqrt{A} \rfloor} + \sum_{u=\lfloor \sqrt{A} \rfloor+1}^{\lfloor \sqrt{A} \rfloor+2} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u} = 1 - \frac{1}{\sqrt{A}} + \mathrm{O}\Big(\frac{1}{A}\Big).$$

3. Suppose that $B_1 < B_2 = B_3 = B_1 + 1$. Although the definition of $\zeta_{\lfloor \sqrt{A} \rfloor+1}$ is different from that in Case 2, the result is the same.

Hence, in all three cases we obtain

$$\sum_{u=1}^{B} \varepsilon_u (\zeta_u - \zeta_{u-1}) = 1 - \frac{1}{\sqrt{A}} + \sum_{u=\lfloor \sqrt{A} \rfloor+3}^{B_4} \frac{\varepsilon_u \zeta_{u-1}}{A - \sigma_u} + \mathrm{O}\Big(\frac{1}{A}\Big).$$

For $j \ge B_2 + 1$ we have

$$j(j+1) \ge \left(\frac{1}{2} + \sqrt{A - \frac{3}{4}}\right)\left(\frac{3}{2} + \sqrt{A - \frac{3}{4}}\right) = A + 2\sqrt{A - \frac{3}{4}},$$

hence

$$A - \frac{A-1}{j(j+1) - (A-1)} \ge A - \frac{A-1}{1 + 2\sqrt{A - \frac{3}{4}}} \ge \frac{1}{2} A.$$

Since $u \le B \le \sqrt{2A}$ we obtain

$$\sum_{j=B_2+1}^{u} \frac{1}{A - \frac{A-1}{j(j+1)-(A-1)}} \le 2\sqrt{\frac{2}{A}}$$

and

$$\prod_{j=B_2+1}^{u}\left(1+\frac{1}{A-\left\lfloor\frac{A-1}{j(j+1)-(A-1)}\right\rfloor}\right)=\exp\left(\mathrm{O}\Big(\frac{1}{\sqrt{A}}\Big)\Big(1+\mathrm{O}\Big(\frac{1}{A}\Big)\Big)\right)$$
$$=1+\mathrm{O}\Big(\frac{1}{\sqrt{A}}\Big).$$

So we have for $u\geq B_2+1$

$$\zeta_u=1+\mathrm{O}\Big(\frac{1}{\sqrt{A}}\Big).$$

Hence

$$\sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4}\frac{\varepsilon_u\zeta_{u-1}}{A-\sigma_u}$$
$$=\left(1+\mathrm{O}\Big(\frac{1}{\sqrt{A}}\Big)\right)\sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4}\frac{\frac12-\frac{1}{\left\lfloor\frac{A-1}{u(u+1)-(A-1)}\right\rfloor+1}-\frac{\left\lfloor\frac{A-1}{u(u+1)-(A-1)}\right\rfloor-1}{A-1}}{A-\left\lfloor\frac{A-1}{u(u+1)-(A-1)}\right\rfloor}$$
$$=\left(1+\mathrm{O}\Big(\frac{1}{\sqrt{A}}\Big)\right)\sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4}F\big(\lfloor G(u)\rfloor\big),$$

where

$$F(x)=\frac{\frac12-\frac{1}{x+1}-\frac{x-1}{A-1}}{A-x}=\frac{1}{2(A-1)}\frac{2x^2-(A-1)x+A-3}{x^2-(A-1)x-A}$$

and

$$G(x)=\frac{A-1}{x(x+1)-(A-1)}.$$

The function $F(x)$ is increasing for $\sqrt{A}<x<\sqrt{\frac32A}$. Now the computation splits into two parts.

1. From $\lfloor G(u)\rfloor\leq G(u)$ and from Lemma 3.5 we obtain that

$$\sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4}F\big(\lfloor G(u)\rfloor\big)\leq\sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4}F\big(G(u)\big)\leq\int_{\sqrt{A}+1}^{-\frac12+\sqrt{\frac32A-\frac54}}H_1(u)\,\mathrm{d}u+\mathrm{O}\Big(\frac{1}{A}\Big),$$

where

$$H_1(u)=F\Big(G\Big(u+\frac12\Big)\Big)$$
$$=\frac{3-A}{2A(A-1)}+\frac{2(A-1)}{A+1}\Big(\frac{1}{2u+1}-\frac{1}{2u+3}\Big)$$
$$+\frac{2(A^2+2A-3)}{A(A+1)(4A^2+A-4)}\frac{1}{1-\frac{4A}{4A^2+A-4}(1+u)^2}.$$

The error term $\mathrm{O}\big(\frac{1}{A}\big)$ comes from Lemma 3.5 and from the fact that $H_1''(u) = \mathrm{O}(A^{-3/2})$ for $\sqrt{A} \le u \le \sqrt{\frac{3}{2}A}$. From

$$\int \frac{\mathrm{d}x}{1-\alpha(1+x)^2} = \frac{1}{2\sqrt{\alpha}} \ln \frac{\sqrt{\alpha}(1+x)+1}{\sqrt{\alpha}(1+x)-1}$$

we obtain

$$\begin{aligned}\int H_1(u)\,\mathrm{d}u &= \frac{3-A}{2A(A-1)}u + \frac{A-1}{A+1}\ln\frac{2u+1}{2u+3}\\ &\quad + \frac{A^2+2A-3}{2A^{3/2}(A+1)\sqrt{4A^2+A-4}} \ln \frac{\sqrt{\frac{4A}{4A^2+A-4}}(1+u)+1}{\sqrt{\frac{4A}{4A^2+A-4}}(1+u)-1}\end{aligned}$$

and

$$\int_{\sqrt{A}+1}^{-\frac{1}{2}+\sqrt{\frac{3}{2}A-\frac{5}{4}}} H_1(u)\,\mathrm{d}u = \frac{1}{\sqrt{A}}\Big(\frac{3}{2} - \frac{7}{12}\sqrt{6}\Big) + \mathrm{O}\Big(\frac{1}{A}\Big).$$

Hence

$$\sum_{\lfloor\sqrt{A}\rfloor+3}^{B_4} F\big(\lfloor G(u)\rfloor\big) \le \frac{1}{\sqrt{A}}\Big(\frac{3}{2} - \frac{7}{12}\sqrt{6}\Big) + \mathrm{O}\Big(\frac{1}{A}\Big).$$

This implies that

$$\sum_{u=1}^{B} \varepsilon_u(\zeta_u - \zeta_{u-1}) \le 1 - \frac{1}{\sqrt{A}}\Big(\frac{7}{12}\sqrt{6} - \frac{1}{2}\Big) + \mathrm{O}\Big(\frac{1}{A}\Big)$$

and

$$U(A) = \frac{1}{A-1} - \frac{1}{A}\sum_{u=1}^{B} \varepsilon_u(\zeta_u - \zeta_{u-1}) \ge \frac{1}{A^{3/2}}\Big(\frac{7}{12}\sqrt{6} - \frac{1}{2}\Big) + \mathrm{O}\Big(\frac{1}{A^2}\Big).$$

2. From $\lfloor G(u)\rfloor \ge G(u) - 1$ and from Lemma 3.5 we obtain that

$$\sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4} F\big(\lfloor G(u)\rfloor\big) \ge \sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4} F\big(G(u)-1\big) \ge \int_{\sqrt{A}+2}^{-\frac{3}{2}+\sqrt{\frac{3}{2}A-\frac{5}{4}}} H_2(u)\,\mathrm{d}u + \mathrm{O}\Big(\frac{1}{A}\Big),$$

where

$$\begin{aligned}H_2(u) &= F\Big(G\Big(u+\frac{1}{2}\Big)-1\Big)\\ &= \frac{6A^2+A+3}{4(A-1)(A+1)^2} - \frac{1}{A^2-1}(2u+u^2)\\ &\quad + \frac{2(A^2+2A-3)}{(A+1)^2(4A^2+5A-7)}\frac{1}{1-\frac{4(A+1)}{4A^2+5A+7}(1+u)^2}.\end{aligned}$$

Again, the error term $\mathrm{O}\left(\frac{1}{A}\right)$ comes from Lemma 3.5 and from the fact that $H_2''(u) = \mathrm{O}(A^{-3/2})$ for $\sqrt{A} \le u \le \sqrt{\frac{3}{2}A}$. We have

$$\int H_2(u)\,\mathrm{d}u = \frac{6A^2+A+3}{4(A-1)(A+1)^2}u - \frac{1}{A^2-1}u^2 - \frac{1}{3(A^2-1)}u^3 + \frac{A^2+2A-3}{2(A+1)^{5/2}\sqrt{4A^2+5A-7}} \ln \frac{\sqrt{\frac{4(A+1)}{4A^2+5A+7}}(1+u)+1}{\sqrt{\frac{4(A+1)}{4A^2+5A+7}}(1+u)-1}$$

and

$$\int_{\sqrt{A}+2}^{-\frac{3}{2}+\sqrt{\frac{3}{2}A-\frac{5}{4}}} H_2(u)\,\mathrm{d}u = \frac{1}{\sqrt{A}}\left(\frac{1}{2}\sqrt{6}-\frac{7}{6}\right) + \mathrm{O}\left(\frac{1}{A}\right).$$

Hence

$$\sum_{u=\lfloor\sqrt{A}\rfloor+3}^{B_4} F(\lfloor G(u)\rfloor) \ge \frac{1}{\sqrt{A}}\left(\frac{1}{2}\sqrt{6}-\frac{7}{6}\right) + \mathrm{O}\left(\frac{1}{A}\right).$$

This implies that

$$\sum_{u=1}^{B} \varepsilon_u(\zeta_u - \zeta_{u-1}) \le 1 - \frac{1}{\sqrt{A}}\left(\frac{13}{6}-\frac{1}{2}\sqrt{6}\right) + \mathrm{O}\left(\frac{1}{A}\right)$$

and

$$U(A) = \frac{1}{A-1} - \frac{1}{A}\sum_{u=1}^{B} \varepsilon_u(\zeta_u - \zeta_{u-1}) \le \frac{1}{A^{3/2}}\left(\frac{13}{6}-\frac{1}{2}\sqrt{6}\right) + \mathrm{O}\left(\frac{1}{A^2}\right).$$

This finishes the proof of Theorem 2.6. ■

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Received: 05 April 2007; **revised:** 30 December 2007