Functiones et Approximatio
XXXVII.1 (2007), 149–157

GAUSSIAN SEQUENCES IN ARITHMETIC PROGRESSIONS

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Dedicated to Jean-Marc Deshouillers
on the occasion of his sixtieth birthday

Abstract: We prove an optimal 'level of distribution' result sequences of integers of the type $X^2 + Y^{2r}$.
Keywords: distribution, polynomial sequences.

1. Introduction

This paper is motivated by problems in sieve theory. Let $\mathcal{A} = \big(a_n\big)$ be a sequence of non-negative reals. The most ambitious goal for the sieve is the evaluation of the sum

$$S(x) = \sum_{n \leqslant x} a_n \, \Lambda(n) \ , \tag{1}$$

where Λ denotes the von Mangoldt function.

Before one can count primes using the sieve one has to be able to count the multiples of a given integer. More precisely, we need a good asymptotic formula for the congruence sum

$$A_d(x) = \sum_{\substack{n \leqslant x \\ n \equiv 0 (\mathrm{mod}\, d)}} a_n \ . \tag{2}$$

Here the main point is not how sharp the error term, but rather, how wide the range of uniformity in the modulus d. Actually, for the application one needs this only on average over d, say $d < D$, and we want $D = D(x)$ to be as large as possible.

In this paper we are going to consider Gaussian sequences, including some which are quite lacunary. Lacunary sequences present serious challenges, not only for counting primes, but even for addressing issues of divisibility. We are fortunate

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2000 Mathematics Subject Classification: 11N32, 11N69.
* Research partially supported by NSERC grant A5123.
** Research partially supported by NSF grant DMS-03-01168.

that some of the very difficult technical obstacles concerning divisibility can be resolved by means of Landreau's inequality, see Proposition 1.

By a Gaussian sequence we mean a sequence $\mathcal{A} = \big(a_n\big)$ supported on integers n which can be written as the sum of two squares, that is on norms of Gaussian integers.

The method we present works for quite general sequences

$$a_n = \sum_{\substack{\ell^2+m^2=n \\ (\ell,m)=1}} \gamma_\ell \tag{3}$$

where γ_ℓ are any complex numbers with $|\gamma_\ell| \leqslant 1$. The problem is more attractive when γ_ℓ is a lacunary sequence, but a certain spacing property is helpful. Therefore, to control the spacing we fix a positive integer r and assume that γ_ℓ is supported on r–th powers

$$\gamma_\ell = 0 \qquad \text{if} \;\; \ell \neq k^r \tag{4}$$

$$|\gamma_\ell| \leqslant 1 \qquad \text{if} \;\; \ell = k^r. \tag{5}$$

In this case we have

$$\sum_{n \leqslant x} a_n = X + O\Big(x^{\frac{1}{2r}} \log x\Big) \tag{6}$$

where

$$X = \sum_{\ell < \sqrt{x}} \gamma_\ell \, \frac{\varphi(\ell)}{\ell} \, \sqrt{x - \ell^2} \; . \tag{7}$$

Since $X \ll x^{\frac{1}{2}+\frac{1}{2r}}$, the best level of distribution one can hope for is

$$D(x) = x^{\frac{1}{2}+\frac{1}{2r}} \big(\log x\big)^{-A} \tag{8}$$

and we are going to achieve this.

For any d not too large we expect that $A_d(x)$ is well approximated by

$$M_d(x) = g(d) \sum_{\substack{\ell < \sqrt{x} \\ (\ell,d)=1}} \gamma_\ell \, \frac{\varphi(\ell)}{\ell} \, \sqrt{x - \ell^2} \tag{9}$$

where $g(d) = \rho(d)/d$ and $\rho(d)$ is the number of solutions of the congruence

$$\alpha^2 + 1 \equiv 0 \,(\operatorname{mod} d) \; . \tag{10}$$

Theorem. *Let γ_ℓ be supported on r–th powers and $|\gamma_\ell| \leqslant 1$. Then, for*

$$x^{\frac{1}{2}} \leqslant D \leqslant x^{\frac{r+1}{2r}} \tag{11}$$

we have

$$\sum_{d\leqslant D}\left|A_d(x)-M_d(x)\right| \ll D^{\frac{1}{4}}x^{\frac{3(r+1)}{8r}}\big(\log x\big)^{130}\ , \tag{12}$$

where the implied constant depends only on r.

Particular cases of our theorem have been considered and applied before. Examples are the sequence (3) with γ_ℓ the characteristic function on primes [2], on squares [3], in both of which cases the sum (1) has been successfully evaluated, and on cubes [4] in which case the goal for primes remains open.

The main ingredient, aside from Proposition 1, is a large sieve type inequality for the roots of the congruence (10), see Proposition 2. A result of this type was first considered in [2].

2. Basic tools

In this section we state the Landreau inequality and the large sieve inequality which are used in the proof of the theorem.

Proposition 1. *Let* $k\geqslant 2$ *be an integer. For all* $n\geqslant 1$ *we have*

$$\tau(n)\leqslant C\sum_{\substack{d|n\\ d\leqslant n^{1/k}}}\Big(2^{\omega(d)}\tau(d)\Big)^k \tag{13}$$

where C *is a constant.*

We shall need this for $k=4$, that is

$$\tau(n)\ll\sum_{\substack{d|n\\ d\leqslant n^{1/4}}}\big(\tau(d)\big)^8\ . \tag{14}$$

The proof of Proposition 1 is given in [6], and indeed the result there is much more general. See also [1].

Proposition 2. *For any complex numbers* β_n *we have*

$$\sum_{d\leqslant D}\ \sum_{\alpha^2+1\equiv 0 (\mathrm{mod}\, d)}\left|\sum_{n\leqslant N}\beta_n e\Big(\frac{\alpha n}{d}\Big)\right| \ll D^{\frac{1}{2}}\big(D+N\big)^{\frac{1}{2}}\Bigg(\sum_{n\leqslant N}|\beta_n|^2\Bigg)^{\frac{1}{2}}\ , \tag{15}$$

where the implied constant is absolute.

Proof. See (3.6) of [3]. ■

3. Proof of the theorem

Now we proceed to the proof of the theorem. As usual, before applying harmonic analysis it helps to introduce some smoothing factors. Let $f(t)$ be a smooth function on $[0,\infty]$ such that

$$\begin{aligned} &f(t)=1 && \text{if} \quad 0\leqslant t\leqslant (1-\eta)x\\ &f(t)=0 && \text{if} \quad t\geqslant x,\\ &f^{(j)}\ll (\eta x)^{-j} && \text{for} \quad j=0,1,2, \end{aligned}$$

where $x^{-\frac{1}{4r}}\leqslant \eta\leqslant 1$ will be chosen later.

We replace $A_d(x), M_d(x)$ by their smooth counterparts

$$A_d(f)=\sum_{n\equiv 0(d)} a_n\, f(n)\ , \tag{16}$$

$$M_d(f)=g(d)\sum_{(\ell,d)=1}\gamma_\ell\,\frac{\varphi(d)}{d}\int_0^\infty f(\ell^2+t^2)\,dt. \tag{17}$$

We estimate the corrections resulting from this modification by elementary arguments as follows:

$$\sum_{d\leqslant D}\big|A_d(x)-A_d(f)\big|\leqslant \sideset{}{'}\sum_{\substack{(1-\eta)x<\ell^2+m^2\leqslant x\\ (\ell,m)=1}}|\gamma_\ell|\tau(\ell^2+m^2)+O\big(\sqrt{x}\,\log x\big)\ .$$

Here Σ' means that the terms with a value of ℓ which is nearest to $\sqrt{x}$ are omitted. For the remaining points we apply the inequality of Landreau, Proposition 1, getting the bound

$$\sideset{}{'}\sum_{\ell<\sqrt{x}}|\gamma_\ell|\sum_{\substack{d\leqslant x^{\frac14}\\ (d,\ell)=1}}\tau(d)^8\sum_{\substack{(1-\eta)x<\ell^2+m^2\leqslant x\\ \ell^2+m^2\equiv 0(d)}}1\ .$$

Note that m runs over an interval of length $O\left(\eta x/\sqrt{x-\ell^2}\right)$. Splitting into residue classes $m\equiv\alpha\ell(\operatorname{mod} d)$ with α running over the roots of (10) we estimate the above sum by

$$\ll \eta x\Big(\sum_d\Big)\Big(\sideset{}{'}\sum_\ell\Big)\ +\ x^{\frac14+\frac{1}{2r}}\left(\log x\right)^{256},$$

where the first sum is

$$\sum_{d\leqslant x^{\frac14}}\tau^8(d)\rho(d)d^{-1}\ll\big(\log x\big)^{256}\ ,$$

and the second sum is

$$\sideset{}{'}\sum_{\ell<\sqrt{x}} |\gamma_\ell|\big(x-\ell^2\big)^{-\frac12} \leqslant \sideset{}{'}\sum_{k<x^{\frac{1}{2r}}} \big(x-k^{2r}\big)^{-\frac12}$$
$$\ll x^{\frac{1-2r}{4r}} \sideset{}{'}\sum_{k<x^{\frac{1}{2r}}} \Big(x^{\frac{1}{2r}}-k\Big)^{-\frac12} \ll x^{\frac{1-r}{2r}} \ .$$

Hence we obtain

$$\sum_{d\leqslant D} \big|A_d(x)-A_d(f)\big| \ll \eta x^{\frac{r+1}{2r}} \big(\log x\big)^{256} \ . \tag{18}$$

Similarly (actually much easier), we show that

$$\sum_{d\leqslant D} \big|M_d(x)-M_d(f)\big| \ll \eta x^{\frac{r+1}{2r}} \log x \ . \tag{19}$$

Next we decompose $A_d(f)$ as follows (expand m to all $\mathbb{Z}$):

$$A_d(f)=\frac12 \mathop{\sum\sum}_{\substack{\ell^2+m^2\equiv 0(d)\\ (\ell,m)=1}} \gamma_\ell\, f(\ell^2+m^2) = \frac12 \sum_{\alpha^2+1\equiv 0(d)} \sum_\ell \gamma_\ell \sum_{\substack{(m,\ell)=1\\ m\equiv\alpha\ell(d)}} f(\ell^2+m^2)$$
$$=\frac12 \sum_{\alpha^2+1\equiv 0(d)} \sum_a \mu(a) \sum_\ell \gamma_{a\ell} \sum_{m\equiv a\ell(d/(a,d))} f\big(a^2(\ell^2+m^2)\big) \ .$$

To the inner sum we apply Poisson's formula

$$\sum_m = \frac{(a,d)}{d} \sum_{h\in\mathbb{Z}} e\Big(\alpha h\ell\frac{(a,d)}{d}\Big) F_{a\ell}\Big(\frac{h(a,d)}{d}\Big) \ ,$$

where $F_{a\ell}(v)$ is the Fourier integral

$$F_{a\ell}(v)=\int_{-\infty}^{\infty} f\big(a^2(\ell^2+t^2)\big)e(-vt)dt = 2\int_0^\infty f\big(a^2(\ell^2+t^2)\big)\cos(2\pi vt)dt \ . \tag{20}$$

Integrating (20) by parts twice we get an alternative expression

$$F_{a\ell}(v)=\Big(\frac{a}{\pi v}\Big)^2 \int_0^\infty \big(f'+2a^2t^2f''\big)\big(a^2(\ell^2+t^2)\big)\cos(2\pi vt)dt \ . \tag{21}$$

The zero frequency $h=0$ yields exactly $M_d(f)$, so we have

$$\big|A_d(f)-M_d(f)\big| \leqslant \frac1d \sideset{}{^\flat}\sum_a \sum_{\substack{bc=d\\ b|a}} \rho(b)b \sum_{\substack{\alpha(\mathrm{mod}\, c)\\ \alpha^2+1\equiv 0(c)}} \big|W_a(c,\alpha)\big|$$

where

$$W_a(c,\alpha) = \sum_{h>0} \sum_{\ell} \gamma_{a\ell}\, e\Big(\frac{\alpha h \ell}{c}\Big) F_{a\ell}\Big(\frac{h}{c}\Big) \tag{22}$$

and $\Sigma^\flat$ denotes a sum over squarefree integers. Summing over the moduli d in a dyadic segment we get

$$\sum_{D<d\leqslant 2D} \big|A_d(f) - M_d(f)\big| \leqslant \frac{1}{D} \sum_{a}{}^{\flat} \sum_{b|a} \rho(b) b V_a(D/b) \ , \tag{23}$$

where

$$V_a(C) = \sum_{C<c\leqslant 2C} \sum_{\alpha^2+1\equiv 0(c)} \big|W_a(c,\alpha)\big| \ . \tag{24}$$

Next we split the outer summation in (22) into dyadic segments $H \leqslant h < 2H$ and we shall treat these partial sums separately. By (24) we obtain

$$V_a(C) \leqslant \sum_{H} V_a(C,H) \tag{25}$$

where

$$V_a(C,H) = \sum_{C<c\leqslant 2C} \sum_{\alpha^2+1\equiv 0(c)} \big|W_a(H;c,\alpha)\big| \tag{26}$$

and

$$W_a(H;c,\alpha) = \sum_{H\leqslant h<2H} \sum_{\ell} \gamma_{a\ell}\, e\Big(\frac{\alpha h \ell}{c}\Big) F_{a\ell}\Big(\frac{h}{c}\Big) \ . \tag{27}$$

We wish to separate the modulus c from the variables h,ℓ in the Fourier integral $F_{a\ell}\left(\frac{h}{c}\right)$. This can be easily achieved by changing t into tH/h in the integrals (20) or (21), and then holding t fixed. Note that the trivial integration over t in (20) or (21) gives

$$F_{a\ell}(v) \ll \frac{\sqrt{x}}{a} \ ,$$
$$F_{a\ell}(v) \ll \frac{\sqrt{x}}{a}\Big(\frac{a}{\eta v \sqrt{x}}\Big)^2 \ ,$$

respectively, because $f \ll 1$, $f' \ll (\eta x)^{-1}$, $f'' \ll (\eta x)^{-2}$. Hence $F_{a\ell}(h/c) \ll G_a(C,H)$, where

$$G_a(C,H) = \frac{\sqrt{x}}{a} \min\Big\{1, \Big(\frac{aC}{\eta H \sqrt{x}}\Big)^2\Big\} \ . \tag{28}$$

By (26), (27), we obtain

$$V_a(C,H) \ll G_a(C,H) U_a(C,H) \ , \tag{29}$$

where

$$U_a(C,H) = \sum_{C<c\leqslant 2C}\ \sum_{\alpha^2+1\equiv 0(c)} \Big| \sum_{H\leqslant h<2H}\ \sum_{\ell} \gamma_{a\ell}\,\xi_{h\ell}\, e\Big(\frac{\alpha h\ell}{c}\Big)\Big| \tag{30}$$

with some coefficients $\xi_{h\ell}$ which do not depend on c,α and which satisfy $|\xi_{h\ell}| \leqslant 1$.

Now we are ready to apply the large sieve for quadratic roots, Proposition 2, which yields

$$U_a(C,H) \ll C^{\frac12}\big(C + H\sqrt{x/a}\big)^{\frac12} E^{\frac12} \tag{31}$$

where

$$E = \sum_n \Big(\sum_{\substack{h\ell=n \\ H\leqslant h<2H}} |\gamma_{a\ell}| \Big)^2 .$$

Because a is squarefree and $a\ell$ is an r-th power it follows that $\ell = a^{r-1}m^r$ with $m \leqslant a^{-1}x^{\frac{1}{2r}} = M$, say. Therefore we see that E is bounded by the number of solutions of

$$h_1 m_1^r = h_2 m_2^r$$

with $H \leqslant h_1, h_2 < 2H$ and $m_1, m_2 \leqslant M$. The solutions are given explicitly by $m_1 = st_1, m_2 = st_2$ with $(t_1,t_2) = 1$, $st_1, st_2 \leqslant M$ and $h_1 = kt_2^r, h_2 = kt_1^r$ with $k \leqslant 4H(t_1^r + t_2^r)^{-1}$. Hence

$$E \leqslant 8HM \sum\sum_{t_1,t_2\leqslant M} (t_1^r + t_2^r)^{-1}(t_1+t_2)^{-1} \leqslant 16HM \sum_{t\leqslant M} t^{-r} .$$

The worst case is $r = 1$, giving $E \ll Ha^{-1}x^{\frac{1}{2r}}\log x$, and (31) yields

$$U_a(C,H) \ll C^{\frac12}\big(C + H\sqrt{x}\,/\,a\big)^{\frac12} H^{\frac12} a^{-\frac12} x^{\frac{1}{4r}} \big(\log x\big)^{\frac12} . \tag{32}$$

Introducing (32) to (29) we see by (28) that the series (25) over the dyadic endpoints H converges and the largest contribution is at

$$H \asymp aC/\eta\sqrt{x} . \tag{33}$$

Therefore we obtain

$$V_a(C) \ll \big(\eta a\big)^{-1} C^{\frac32} x^{\frac{r+1}{4r}} \big(\log x\big)^{\frac32} . \tag{34}$$

Inserting this to (23) we arrive at

$$\sum_{D<d\leqslant 2D} \big|A_d(f) - M_d(f)\big| \ll \eta^{-1} D^{\frac12} x^{\frac{r+1}{4r}} \big(\log x\big)^{\frac52} . \tag{35}$$

This bound remains the same for the sum over all $d \leqslant D$. Finally, combining this with (18), (19) we conclude (12) by choosing

$$\eta = D^{\frac14} x^{-\frac{r+1}{8r}} \big(\log x\big)^{\frac54 - 128} , \tag{36}$$

proving the theorem.

4. A main term computation

Write

$$X_d = \sum_{\substack{\ell<\sqrt{x} \\ (\ell,d)=1}} \gamma_\ell \frac{\varphi(\ell)}{\ell} \sqrt{x-\ell^2} \ . \tag{37}$$

By Mellin inversion we have

$$\sqrt{x-y} = \frac{\sqrt{x}}{2\pi i} \int_{(\sigma)} \mathcal{B}(s) \Big(\frac{x}{y}\Big)^s ds$$

where, say, $\sigma = 1$ and

$$\mathcal{B}(s) = \int_0^1 \sqrt{1-y}\, y^{s-1} dy = B\left(s, \frac{3}{2}\right) = \frac{\Gamma(s)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(s+\frac{3}{2}\right)} \ .$$

Therefore,

$$X_d = \frac{\sqrt{x}}{2\pi i} \int_{(\sigma)} \mathcal{B}(s) x^s Z(2s) ds \ ,$$

where

$$Z(s) = \sum_{(\ell,d)=1} \gamma_\ell \frac{\varphi(\ell)}{\ell} \ell^{-s} \ .$$

Now specialize to γ_ℓ, the characteristic function of r–th powers, in which case $Z(s)$ is given by

$$\begin{aligned} Z(s) &= \sum_{(k,d)=1} \prod_{p|k} \Big(1-\frac{1}{p}\Big) k^{-rs} = \prod_{p\nmid d} \Big(1-\frac{1}{p^{rs}}\Big)^{-1} \Big(1-\frac{1}{p^{rs+1}}\Big) \\ &= \frac{\zeta(rs)}{\zeta(rs+1)} \prod_{p|d} \Big(1-\frac{1}{p^{rs}}\Big)\Big(1-\frac{1}{p^{rs+1}}\Big)^{-1}. \end{aligned}$$

Moving the contour to $\sigma = \varepsilon$ we encounter a simple pole of $Z(2s)$ at $s = \frac{1}{2r}$ with residue

$$\operatorname*{res}_{s=\frac{1}{2r}} Z(2s) = \frac{1}{2r\zeta(2)} h(d)$$

where

$$h(d) = \prod_{p|d} \Big(1+\frac{1}{p}\Big)^{-1} \ . \tag{38}$$

Because $\mathcal{B}(s) \ll |s|^{-\frac{3}{2}}$ on the line $s = \varepsilon$, the integral converges absolutely so that

$$X_d = x^{\frac{r+1}{2r}} \mathcal{B}\Big(\frac{1}{2r}\Big) \operatorname*{res}_{s=\frac{1}{2r}} Z(2s) + O\left(\tau(d) x^{\frac{1}{2}+\varepsilon}\right) \ .$$

Hence:

Proposition 3. *Let γ_ℓ be the characteristic function of r–th powers. We have, for X_d given by (37),*

$$X_d = h(d)\,\frac{B\left(\frac{1}{2r},\frac{3}{2}\right)}{2\zeta(2)}\,x^{\frac{1}{2}+\frac{1}{2r}} + O\Big(\tau(d)\,x^{\frac{1}{2}+\varepsilon}\Big)\ , \tag{39}$$

where $h(d)$ is given by (38), B is the beta function, and the implied constant depends on r and ε.

Remark. In this case the sequence $\mathcal{A} = \big(a_n\big)$ satisfies the usual linear sieve axioms with the density function

$$g(d) = \rho(d)h(d)d^{-1}\ . \tag{40}$$

An immediate application, using the weighted sieve of Laborde [5], gives the following.

Corollary 1. *For each $r \leqslant 13$, there are infinitely many numbers $\ell^{2r} + m^2$ which have at most two distinct prime factors.*

Of course this is known, even for the sequence $1 + m^2$, however Laborde's sieve leads to the present result much more quickly.

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Received: 20 September 2006