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# THE LARGE SIEVE WITH QUADRATIC AMPLITUDE

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**Abstract:** We establish a large sieve bound for expressions of the form

$$\sum_{r=1}^{R}\left|\sum_{M<n\leqslant M+N} a_n e\left(\alpha_r f(n)\right)\right|^2,$$

where $f(x)=\alpha x^2+\beta x+\theta\in\mathbb{R}[x]$ is a quadratic polynomial with $\alpha>0$ and $\beta\geqslant 0$. We also consider the case when $f(x)=x^d$ with $d\in\mathbb{N}$, $d\geqslant 3$.
**Keywords:** large sieve, quadratic amplitude, double large sieve, exponential sums.

## 1. Introduction

Throughout this paper, we suppose that $Q,R,M,N$ are integers with $Q\geqslant 1$, $R\geqslant 1$, $N\geqslant 1$ and $M\geqslant 0$. As usual, by $\varepsilon$ we denote a fixed but arbitrary (small) positive real number. Further, we suppose that $(a_n)$ and $(\alpha_r)$ are sequences of complex numbers. We set

$$S(\alpha):=\sum_{M<n\leqslant M+N} a_n e(\alpha n)$$

and

$$Z:=\int\limits_0^1 |S(\alpha)|^2\mathrm{d}\alpha=\sum_{M<n\leqslant M+N}|a_n|^2.$$

By $\|x\|$ we denote the distance of a real number $x$ to its closest integer.

In its modern form, the large sieve is an inequality connecting a discrete and the continuous mean value $Z$ of the trigometrical polynomial $S(\alpha)$, *i.e.* an inequality of the form

$$\sum_{r=1}^{R}\left|S\left(\alpha_r\right)\right|^2\leqslant\Delta(N;\alpha_1,...,\alpha_r)Z. \tag{1}$$

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Montgomery and Vaughan [9] proved that (1) holds with

$$\Delta(N;\alpha_1,...,\alpha_r)=N+\delta^{-1},$$

where

$$\delta:=\min_{\substack{r,s\leqslant R\\ r\neq s}}||\alpha_r-\alpha_s||. \tag{2}$$

In many applications, the sequence $\alpha_1,...,\alpha_R$ consists of Farey fractions. If $\alpha_1,...,\alpha_R$ is the sequence of all fractions $a/q$ with $1\leqslant a\leqslant q$, $(a,q)=1$ and $q\leqslant Q$, then the above results implies that

$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\ (a,q)=1}}^{q}\left|S\left(\frac{a}{q}\right)\right|^2\leqslant (N+Q^2)Z,$$

which is a sharpened version of the classical large sieve inequality of Bombieri [2].

In [11] L. Zhao dealt with sums of the form

$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\ (a,q)=1}}^{q}\left|\sum_{M<n\leqslant M+N}a_ne\left(\frac{af(n)}{q}\right)\right|^2,$$

where

$$f(x)=\alpha x^2+\beta x+\theta\in\mathbb{R}[x]$$

is a *quadratic* polynomial with $\alpha\neq 0$. Without loss of generality, we can assume that $\alpha>0$ (if $\alpha<0$, then we just need to replace $f(x)$ by $-f(x)$), which we suppose from now on.

For the case when $\beta/\alpha$ is rational, Zhao established the following bound (Theorem 2. in [11]): If $\beta/\alpha=u/v$ with $u,v\in\mathbb{Z}$, $v>0$ and $(u,v)=1$, then

$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\ (a,q)=1}}^{q}\left|\sum_{M<n\leqslant M+N}a_ne\left(\frac{af(n)}{q}\right)\right|^2 \tag{3}$$
$$\ll\left(Q^2+Q\sqrt{\alpha N(M+N+u/v)+1}\right)\Pi Z,$$

where

$$\Pi=\left(\frac{v}{\alpha}+1\right)^{1/2+\varepsilon}[Nv(M+N)+|u|+v/\alpha]^{\varepsilon}.$$

We recall that we here suppose $M\geqslant 0$.

Zhao also dealt with the case when $\beta/\alpha$ is a general real number (see Proposition 1 in [11]). However, for irrational $\beta/\alpha$ his result is weaker than (3) unless $\beta/\alpha$ is in a sense well-approximable by rational numbers.

In many applications, the quantity

$$Z^* := N \max_{M<n\leqslant M+N} |a_n|^2 \tag{4}$$

does not exceed the quantity $Z = \sum\limits_{M<n\leqslant M+N} |a_n|^2$ much. In the present paper we are concerned with large sieve inequalities of the form

$$\sum_{r=1}^{R} \left| \sum_{M<n\leqslant M+N} a_n e\left(\alpha_r f(n)\right) \right|^2 \ll \Delta(M,N;\alpha_1,...,\alpha_r) Z^*.$$

To avoid technical complications, we confine ourselves to the case when $\beta \geqslant 0$. Though, our method should lead to the same result for $\beta < 0$. We shall prove

**Theorem 1.** *Define $\delta$ as in (2) and $Z^*$ as in (4). Let $f(x) = \alpha x^2 + \beta x + \theta \in \mathbb{R}[x]$, where $\alpha > 0$ and $\beta \geqslant 0$. Then we have, with an absolute $\ll$-constant,*

$$\sum_{r=1}^{R} \left| \sum_{M<n\leqslant M+N} a_n e\left(\alpha_r f(n)\right) \right|^2 \tag{5}$$
$$\ll (1+\alpha^{-1/2}) R^{1/2} \left( N^{1/2}(M+N)^{1/2} + \delta^{-1/2} \right) Z^* \times \log^{1/2}(2+\alpha^{-1}) \log 2N$$

*if $N > N_0$, where $N_0$ is a non-negative constant which depends only on $\alpha$ and $\beta$.*

An immediate consequence of Theorem 1 is

**Corollary 1.** *Define $Z^*$ as in (4). Let $f(x) = \alpha x^2 + \beta x + \theta \in \mathbb{R}[x]$, where $\alpha > 0$ and $\beta \geqslant 0$. Then we have, with an absolute $\ll$-constant,*

$$\sum_{q\leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{M<n\leqslant M+N} a_n e\left( \frac{af(n)}{q} \right) \right|^2 \tag{6}$$
$$\ll (1+\alpha^{-1/2}) \left( QN^{1/2}(M+N)^{1/2} + Q^2 \right) Z^* \times \log^{1/2}(2+\alpha^{-1}) \ \log 2N$$

*if $N > N_0$, where $N_0$ is a non-negative constant which depends only on $\alpha$ and $\beta$.*

In the following two sections we shall prove Theorem 1. In the last section we shall touch the case of polynomials $f(x)$ of degree $\geqslant 3$.

## 2. Preliminaries

Like Zhao's method in [11], our method relies on the double large sieve of Bombieri and Iwaniec (Lemma 5.2 in [1]). Here we state only the one-dimensional version of the double large sieve.

**Proposition 1.** *Suppose that* $x_1, ..., x_R$ *and* $y_1, ..., y_S$ *are real numbers with*

$$-\frac{X}{2} \leqslant x_r \leqslant \frac{X}{2}, \qquad -\frac{Y}{2} \leqslant y_s \leqslant \frac{Y}{2}$$

*for* $r = 1, ..., R$ *and* $s = 1, ..., S$. *Put* $\Lambda(x) := \max(1 - |x|, 0)$. *Then we have*

$$\left| \sum_{r=1}^{R} \sum_{s=1}^{S} c_r d_s e\left(x_r y_s\right) \right|^2 \leqslant \left(\frac{\pi}{2}\right)^4 AB(XY + 1), \tag{7}$$

*where*

$$A := \sum_{r=1}^{R} \sum_{\rho=1}^{R} c_r c_\rho \Lambda((x_r - x_\rho)Y)$$

*and*

$$B := \sum_{s=1}^{S} \sum_{\sigma=1}^{S} d_s d_\sigma \Lambda((y_s - y_\sigma)X).$$

Using Proposition 1, we shall reduce the problem in question to estimating the number of solutions $k, l, u, v \in \mathbb{Z}$ of a Diophantine inequality of the form

$$|l(v + \gamma) - k(u + \gamma)| \leqslant h, \tag{8}$$

where $h$ and $\gamma$ are fixed real numbers, and the variables $k, l, u, v$ lie in certain intervals. We shall employ the following bound which is essentially due to G. Harman.

**Proposition 2.** *Let* $\gamma \in \mathbb{R}$ *and* $h, K, L, U, V \geqslant 1$ *be given. Then the number of solutions* $k, l, u, v \in \mathbb{Z}$ *with* $K \leqslant k \leqslant 2K$, $L \leqslant l \leqslant 2L$, $U \leqslant u \leqslant 2U$, $V \leqslant v \leqslant 2V$ *of the inequality* (8) *is*

$$\ll \Big(\min\{K, L\} \max\{U, V\}(1 + |\log K/L|) + (K + L)^{3/2+\varepsilon}\Big) \tag{9}$$
$$\times\, h \log 2h \ \log 2(K + L),$$

*where the implied* $\ll$*-constant depends only on* $\varepsilon$.

G. Harman stated and used the bound (9) for $U = V$ in the proof of Lemma 3 in [4] (note that our notations differ from those in [4]). He did not prove this bound in [4] but refered to his paper [3] in which he established a similar bound, Lemma 7, for irrational real $\gamma$'s which satisfy the condition

$$||q\gamma|| > A^{-q}, \qquad \text{all } q \in \mathbb{N}, \tag{10}$$

for some $A$. Proposition 2 can also be established by the method used to prove Lemma 7 in [3]. Instead of the estimate (5.6) in [3] one here uses the slightly weaker estimate $\ll hTl^{-1}$ (see the remark at the beginning of the proof of Lemma 8 in [3]) which is satisfied for *all* real $\gamma$. We also note that the term $h^2$ in (5.3) in [3] can be replaced by $h \log 2h$ (however, for the application in [3] it was sufficient to use (5.3) with $h^2$). The term $h^2$ arose from the crude estimate $1 + \log h \ll h$ at the end of the proof of Lemma 7 in [3].

We shall also need the following slightly modified version of Proposition 2, which can be established by the same method.

**Proposition 3.** *Let* $\gamma \in \mathbb{R}$ *and* $h, K, L, U, Z \geqslant 1$ *be given. Suppose that* $Z \leqslant U$. *Then the number of solutions* $k, l, u, v \in \mathbb{Z}$ *with* $K \leqslant k \leqslant 2K$, $L \leqslant l \leqslant 2L$, $U \leqslant u \leqslant U+Z$, $U \leqslant v \leqslant U+Z$ *of the inequality* (8) *is*

$$\ll \left(\min\{K,L\}Z(1+|\log K/L|)+(K+L)^{3/2+\varepsilon}\right) h\log 2h \ \log(K+L), \tag{11}$$

*where the implied* $\ll$*-constant depends only on* $\varepsilon$.

## 3. Proof of Theorem 1

We are now ready to prove Theorem 1, our main result. As in [11], we begin with applying the double large sieve.

Multiplying out the square, we get

$$\begin{aligned} &\sum_{r=1}^{R} \left| \sum_{M<n\leqslant M+N} a_n e\left(\alpha_r f(n)\right) \right|^2 \\ &= \sum_{r=1}^{R} \sum_{M<m\leqslant M+N} \sum_{M<n\leqslant M+N} a_m \overline{a_n} e\left(\alpha_r (f(m)-f(n))\right) \\ &= \sum_{r=1}^{R} \sum_{M<m\leqslant M+N} \sum_{M<n\leqslant M+N} a_m \overline{a_n} e\left(\alpha_r \alpha (m-n)(m+n+\beta/\alpha)\right). \end{aligned} \tag{12}$$

In the remaining part of this paper, we assume without loss of generality that

$$-1/2 \leqslant \alpha_r \leqslant 1/2$$

for $r = 1, ..., R$, and we put $\gamma := \beta/\alpha$. Then, applying Proposition 1 with

$$(x_r)_{1\leqslant r\leqslant R} = (\alpha\alpha_r)_{1\leqslant r\leqslant R}, \quad (y_s)_{1\leqslant s\leqslant S} = ((m-n)(m+n+\gamma))_{M<m,n\leqslant M+N},$$

$$(c_r) \equiv 1, \quad (d_s)_{1\leqslant s\leqslant S} = (a_m\overline{a_n})_{M<m,n\leqslant M+N}, \quad X=\alpha, \quad Y = 2N(M+N+\gamma),$$

we obtain

$$\begin{aligned} &\left| \sum_{r=1}^{R} \sum_{M<m\leqslant M+N} \sum_{M<n\leqslant M+N} a_m \overline{a_n} e\left(\alpha_r \alpha (m-n)(m+n+\gamma)\right) \right|^2 \\ &\ll AB(\alpha N(M+N+\gamma)+1) \max_{M<n\leqslant M+N} |a_n|^4, \end{aligned} \tag{13}$$

where $A$ is the number of solutions $\alpha_r, \alpha_\rho$ with $1 \leqslant r, \rho \leqslant R$ of the inequality

$$|\alpha_r - \alpha_\rho| \leqslant \frac{1}{2\alpha N(M+N+\gamma)},$$

and $B$ is the number of solutions $m_1, n_1, m_2, n_2 \in \mathbb{Z}$ with $M < m_1, n_1, m_2, n_2 \leqslant M+N$ of the inequality

$$|(m_1-n_1)(m_1+n_1+\gamma)-(m_2-n_2)(m_2+n_2+\gamma)| \leqslant 1/\alpha.$$

Since the sequence $\alpha_1, ..., \alpha_R$ is well-spaced with spacing $\delta$, we have

$$A \leqslant R\left(1+\frac{1}{\delta\alpha N(M+N+\gamma)}\right). \tag{14}$$

Obviously, $B$ is $\leqslant$ the number $B'$ of solutions $k, l, u, v \in \mathbb{Z}$ with

$$-2N \leqslant k, l \leqslant 2N, \quad 2M < u, v \leqslant 2(M+N) \tag{15}$$

of the inequality

$$|l(v+\gamma)-k(u+\gamma)| \leqslant 1+1/\alpha. \tag{16}$$

In the following, we derive an estimate for $B'$. We always suppose that the conditions in (15) are satisfied.

*Case* 1: If $k=0$, then (16) has

$$\ll N \sum_{2M<v\leqslant 2(M+N)} \left(1+\frac{1+\alpha^{-1}}{v+\gamma}\right) \ll N^2+N(1+\alpha^{-1})\log 2N$$

solutions $(l, u, v)$.

*Case* 2: Similarly, if $l=0$, then (16) has $\ll N^2+N(1+\alpha^{-1})\log 2N$ solutions $(k, u, v)$.

*Case* 3: Suppose that $k<0$ and $l>0$. Then a crude bound for the number of solutions $k, l, u, v$ of (16) is

$$\ll \left(\sum_{1\leqslant t\leqslant 1+1/\alpha} d(t)\right)^2 \ll (1+\alpha^{-1})^2 \log^2 2(1+\alpha^{-1}),$$

where $d(t)$ is the number of divisors of $t$.

*Case* 4: Suppose that $k>0$ and $l<0$. Then, like in Case 3, there are $\ll (1+\alpha^{-1})^2\log^2 2(1+\alpha^{-1})$ solutions $k, l, u, v$ of (16).

*Case* 5: Suppose that $k>0$, $l>0$ and $M \geqslant N$. Put $J := [\log_2 N]+1$. Then, by Proposition 3, the number of solutions $k, l, u, v$ of (16) is

$$\begin{aligned} &\ll \sum_{i=0}^{J}\sum_{j=0}^{J}\left(\min\left\{\frac{N}{2^i}, \frac{N}{2^j}\right\} N(1+|\log(2^j/2^i)|)+N^{3/2+\varepsilon}\right) \qquad (17) \\ &\quad \times (1+\alpha^{-1})\log 2(1+\alpha^{-1})\ \log 2N \\ &\ll \left(N^{3/2+2\varepsilon}+N^2\sum_{i=0}^{J}\ \sum_{j=0}^{J}\min\left\{2^{-i}, 2^{-j}\right\}(1+|j-i|)\right) \\ &\quad \times (1+\alpha^{-1})\log 2(1+\alpha^{-1})\ \log 2N. \end{aligned}$$

The double sum in the last line of (17) can be estimated by

$$\begin{aligned}
&\sum_{i=0}^{J}\ \sum_{j=0}^{J} \min\left\{2^{-i},2^{-j}\right\}(1+|j-i|) \qquad (18)\\
&\ll \sum_{i=0}^{J}\ \sum_{j=i}^{J} 2^{-j}(1+j-i)\\
&\ll \sum_{i=0}^{J}\sum_{j=i}^{J}\left(\frac{2}{3}\right)^{j}\\
&\ll \sum_{i=0}^{J}\left(\frac{2}{3}\right)^{i},
\end{aligned}$$

and the sum in the last line of (18) is bounded by a constant. So the number of solutions in question is

$$\ll N^2(1+\alpha^{-1})\log 2(1+\alpha^{-1})\ \log 2N.$$

*Case* 6: Suppose that $k>0$, $l>0$ and $M<N$. Put $J:=[\log_2 N]+1$. Then, by Proposition 2, the number of solutions $k,l,u,v$ of (16) is

$$\ll (1+\alpha^{-1})\log 2(1+\alpha^{-1})\ \log 2N \sum_{i=0}^{J}\sum_{j=0}^{J}\sum_{f=0}^{J+1}\sum_{g=0}^{J+1} \qquad (19)$$
$$\left(\min\left\{\frac{N}{2^i},\frac{N}{2^j}\right\}\max\left\{\frac{M+N}{2^f},\frac{M+N}{2^g}\right\}(1+|\log(2^j/2^i)|)+N^{3/2+\varepsilon}\right).$$

In a similar manner like in Case 5 one proves that the expression in (19) is

$$\ll N^2(1+\alpha^{-1})\log 2(1+\alpha^{-1})\ \log^2 2N.$$

*Case* 7: Suppose that $k<0$, $l<0$ and $M\geqslant N$. Then we get the same bound like in Case 5.

*Case* 8: Suppose that $k<0$, $l<0$ and $M<N$. Then we get the same bound like in Case 6.

Collecting all contributions together, we find that the total number of solutions $k,l,u,v$ of (16) is

$$\ll N^2(1+\alpha^{-1})\log 2(1+\alpha^{-1})\ \log^2 2N \qquad (20)$$

if $N>N_0(\alpha)$, where $N_0(\alpha)$ is a non-negative constant which depends only on $\alpha$.

Now, combining (12), (13), (14) and the bound (20) for the term $B$, we obtain the result of Theorem 1. ■

## 4. Polynomials of higher degree

In this section we deal with the simplest polynomials of higher degree, namely the polynomials $f(x) = x^d$ with $d \geqslant 3$. Our aim is to estimate the expression

$$\sum_{r=1}^{R} \left| \sum_{M<n\leqslant M+N} a_n e\left(\alpha_r n^d\right) \right|^2 .$$

For simplicity, we confine ourselves to the case when $M = 0$. In what follows, we allow the implied $\ll$-constants to depend on $d$ and on some parameter $k$ which we introduce below.

Using Hölder's inequality, we get for $k \geqslant 2$

$$\sum_{r=1}^{R} \left| \sum_{n=1}^{N} a_n e\left(\alpha_r n^d\right) \right|^2 \leqslant R^{1-2/k} \left( \sum_{r=1}^{R} \left| \sum_{n=1}^{N} a_n e\left(\alpha_r n^d\right) \right|^k \right)^{2/k} . \tag{21}$$

If $k \in \mathbb{N}$, then

$$\begin{aligned} &\sum_{r=1}^{R} \left| \sum_{n=1}^{N} a_n e\left(\alpha_r n^d\right) \right|^k \qquad (22)\\ &= \sum_{r=1}^{R} \left| \sum_{n_1=1}^{N} \dots \sum_{n_k=1}^{N} a_{n_1} \cdots a_{n_k} e\left(\alpha_r \left(n_1^d + \dots + n_k^d\right)\right) \right| \\ &= \sum_{r=1}^{R} \sum_{n_1=1}^{N} \dots \sum_{n_k=1}^{N} \epsilon_r a_{n_1} \cdots a_{n_k} e\left(\alpha_r \left(n_1^d + \dots + n_k^d\right)\right) \end{aligned}$$

for suitable complex $\epsilon_r$ with $|\epsilon_r| = 1$.

Applying Proposition 1 with

$$(x_r)_{1\leqslant r\leqslant R} = (\alpha_r)_{1\leqslant r\leqslant R}, \qquad (y_s)_{1\leqslant s\leqslant S} = \left(n_1^d + \dots + n_k^d\right)_{0<n_1,\dots,n_k\leqslant N},$$

$$(c_r)_{1\leqslant r\leqslant R} = (\epsilon_r)_{1\leqslant r\leqslant R}, \qquad (d_s)_{1\leqslant s\leqslant S} = (a_{n_1} \cdots a_{n_k})_{0<n_1,\dots,n_k\leqslant N},$$

$$X = 1, \qquad Y = 2kN^d,$$

we obtain

$$\begin{aligned} &\left| \sum_{r=1}^{R} \sum_{n_1=1}^{N} \dots \sum_{n_k=1}^{N} \epsilon_r a_{n_1} \cdots a_{n_k} e\left(\alpha_r \left(n_1^d + \dots + n_k^d\right)\right) \right|^2 \qquad (23)\\ &\ll ABN^d \max_{n\leqslant N} |a_n|^{2k}, \end{aligned}$$

where $A$ is the number of solutions $\alpha_r, \alpha_\rho$ with $1 \leqslant r, \rho \leqslant R$ of the inequality

$$|\alpha_r - \alpha_\rho| \leqslant \frac{1}{2kN^d},$$

and $B$ is the number of solutions $(m_1, ..., m_k, n_1, ..., n_k) \in \mathbb{N}^{2k}$ with $m_1, ..., m_k, n_1, ..., n_k \leqslant N$ of the equation

$$m_1^d + ... + m_k^d - (n_1^d + ... + n_k^d) = 0.$$

Since the sequence $\alpha_1, ..., \alpha_R$ is well-spaced with spacing $\delta$, we have

$$A \leqslant R\left(1 + \frac{1}{\delta k N^d}\right). \tag{24}$$

Combining (21), (22), (23) and (24), we obtain

**Theorem 2.** *Define $\delta$ as in (2). Suppose that $d, k \in \mathbb{N}$, $d \geqslant 3$ and $k \geqslant 2$. Then we have*

$$\sum_{r=1}^{R} \left| \sum_{n=1}^{N} a_n e\left(\alpha_r n^d\right) \right|^2 \ll R^{1-1/k} \left(N^{d/k} + \delta^{-1/k}\right) B_{d,k}^{1/k}(N) \max_{n \leqslant N} |a_n|^2, \tag{25}$$

*where*

$$B_{d,k}(N) := |\{(m_1, ..., m_k, n_1, ..., n_k) \in \mathbb{N}^{2k} : m_1, ..., m_k, n_1, ..., n_k \leqslant N, \\ m_1^d + ... + m_k^d = n_1^d + ... + n_k^d\}|.$$

The term $B_{d,k}(N)$ can be expressed in the form

$$B_{d,k}(N) = \int_0^1 \left| \sum_{n=1}^{N} e\left(\alpha n^d\right) \right|^{2k} \mathrm{d}\alpha,$$

and this integral can be estimated by using Hua's inequality (see [7]). In particular, for $d = 3 = k$ Hua's inequality yields (see [5])

$$B_{3,3}(N) \ll N^{7/2+\varepsilon}.$$

Hooley [6] and Heath-Brown [5] established independently the much sharper bound

$$B_{3,3}(N) \ll N^{3+\varepsilon}$$

under the Riemann hypothesis for certain Hasse-Weil $L$-functions. Thus, Theorem 2 implies

**Theorem 3.** *Define $\delta$ as in (2) and $Z^*$ as in (4). Then we have*

$$\sum_{r=1}^{R} \left| \sum_{n=1}^{N} a_n e\left(\alpha_r n^3\right) \right|^2 \ll R^{2/3} \left(N + \delta^{-1/3}\right) N^{1/6+\varepsilon} Z^*. \tag{26}$$

*If the Riemann hypothesis for Hasse-Weil $L$-functions holds true, then the left-hand side of (26) is*

$$\ll R^{2/3} \left(N + \delta^{-1/3}\right) N^{\varepsilon} Z^*.$$

In particular, for the special case of Farey fractions we obtain

**Corollary 2.** *Define $Z^*$ as in (4). Then we have*

$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{n=1}^{N}a_n e\left(\frac{an^3}{q}\right)\right|^2 \ll \left(Q^{4/3}N+Q^2\right)N^{1/6+\varepsilon}Z^*. \tag{27}$$

*If the Riemann hypothesis for Hasse-Weil $L$-functions holds true, then the left-hand side of (27) is*

$$\ll \left(Q^{4/3}N+Q^2\right)N^{\varepsilon}Z^*.$$

Heuristicly, one may expect that

$$B_{d,k}(N)\ll N^{\max\{k,2k-d\}+\varepsilon}. \tag{28}$$

If this inequality holds, then for large $N$ the optimal choice of the parameter $k$ in Theorem 4 is $k=d$. In this case ($k=d$) the bound (28) follows from Hooley's hypothesis $K^*$ in Waring's problem (see [6]) which asserts that

$$\sum_{n\leqslant X}R_{d,d}^2(n)\ll X^{1+\varepsilon},$$

where $R_{d,d}(n)$ is the number of solutions $(n_1,...,n_d)\in\mathbb{N}^d$ of the equation

$$n_1^d+...+n_d^d=n.$$

Thus, Theorem 2 implies

**Theorem 4.** *Define $\delta$ as in (2) and $Z^*$ as in (4). Let $d\geqslant 3$ be a natural number. Assume that hypothesis $K^*$ holds. Then we have*

$$\sum_{r=1}^{R}\left|\sum_{n=1}^{N}a_n e\left(\alpha_r n^d\right)\right|^2 \ll R^{1-1/d}\left(N+\delta^{-1/d}\right)N^{\varepsilon}Z^*. \tag{29}$$

In particular, for the special case of Farey fractions we obtain

**Corollary 3.** *Define $Z^*$ as in (4). Let $d\geqslant 3$ be a natural number. Assume that hypothesis $K^*$ holds. Then we have*

$$\sum_{q\leqslant Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{n=1}^{N}a_n e\left(\frac{an^d}{q}\right)\right|^2 \ll \left(Q^{2(1-1/d)}N+Q^2\right)N^{\varepsilon}Z^*. \tag{30}$$

Actually, Hooley [6] and Heath-Brown [5] proved the hypothesis $K^*$ for $d=3$ under the Riemann hypothesis for certain Hasse-Weil $L$-functions.

We note that for $d = 2$ the bounds (29) and (30) with $\log 2N$ in place of $N^\varepsilon$ follow from Theorem 1 and Corollary 1. For $d = 1$ the bounds (29) and (30) with the term $N^\varepsilon$ omitted follow from the ordinary large sieve inequalities given at the beginning of this paper.

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