Functiones et Approximatio
XXXV (2006), 195–207

ZERO MULTIPLICITY AND LOWER BOUND ESTIMATES OF $|\zeta(S)|$

Anatolij A. Karatsuba

Dedicated to Professor Eduard Wirsing
on the occasion of his 75th birthday

Abstract: We give an improved lower bound for $\max_{|T-t|\leqslant H}|\zeta(\frac{1}{2}+it)|$ when $2\leqslant \alpha H\leqslant \log\log T-c$, $1\leqslant\alpha<\pi$. Our theorem slightly refines the result in [11]. We also prove a theorem about an upper bound for the multiplicities of zeros of $\zeta(s)$ conditionally, assuming some lower bound for $\max_{|s-s_1|\leqslant\Delta}|\zeta(s)|$.

Keywords: Riemann zeta-function, zero multiplicity.

1. Introduction

One of the interesting problems in the theory of the function $\zeta(s)$ is the question of multiple zeros of $\zeta(s)$. There are several conjectures about how large the multiplicity of such a zero may be: zeros may be simple, of bounded multiplicity, of unbounded multiplicity. Let $\varkappa(T)$ be the largest multiplicity of a zero of $\zeta(s)$ in the rectangle $0<\operatorname{Re} s<1$, $0<\operatorname{Im} s\leqslant T$. Then the above-mentioned conjectures may be stated as:

Conjecture 1. $\varkappa(T)=1$, $T>0$.

Conjecture 2. $\varkappa(T)\leqslant c$, c *being a constant,* $T>0$.

Conjecture 3. $\varkappa(T)\to+\infty$ *as* $T\to+\infty$.

A simple theorem about nontrivial zeros ρ of $\zeta(s)$, namely the relationship

$$\sum_{\rho}\frac{1}{1+(T-\operatorname{Im}\rho)^2}=\mathrm{O}(\log T),\qquad T\geqslant 2,$$

implies that $\varkappa(T)=\mathrm{O}(\log T)$ (cf. [7, p. 39], or [12, p. 209], or [8, p. 24]). The Riemann Hypothesis implies that

$$\varkappa(T)=\mathrm{O}\left(\frac{\log T}{\log\log T}\right),$$

2001 Mathematics Subject Classification: 11M26, 11N25.

cf. [12, pp. 209 and 346]. Finally, the weaker Mertens hypothesis, i.e. the relationship

$$\int_1^X \left(\frac{1}{x}\sum_{n\leqslant x}\mu(n)\right)^2 dx = \mathrm{O}(\log X),$$

implies that $\varkappa(T)=1$, $T>0$ (cf. [12, p. 374]).

The universality of $\zeta(s)$ (cf. [13], [14]) should include the inequality $\varkappa(T)>1$ and, moreover, the property $\varkappa(T)\to+\infty$ as $T\to+\infty$. However, all these are merely surmises (cf. also [8, p. 137]).

The problems related to the multiplicity of a zero of $\zeta(s)$ were considered by A. Ivič [5], [6]. In particular, these papers provided new upper bounds for the multiplicity of a zero in point s in the left neighbourhood of the line $\mathrm{Re}\, s=1$.

Lower bounds for $|\zeta(s)|$ in small regions of the critical strip allow for the upper bound estimation of $\varkappa(T)$. Such lower bounds are also interesting in their own right. Therefore in Section 2 we show results about lower bound estimates of $|\zeta(s)|$ on short intervals of the critical line, in Section 3 we show results about lower bound estimates of $|\zeta(s)|$ in small regions of the critical strip, and finally, in Section 4 we prove a theorem about an upper bound for $\varkappa(T)$.

Everywhere below A, c, c_1, c_2, $\dots$, T_1 denote positive absolute constants, generally different in different furmulae; $\zeta(s)$ — the Riemann zeta function; RH — the Riemann Hypothesis about zeros of $\zeta(s)$; $T\geqslant T_1$; $\cosh\alpha=\frac{1}{2}(e^{\alpha}+e^{-\alpha})$, $\sinh\alpha=\frac{1}{2}(e^{\alpha}-e^{-\alpha})$; $\mu(n)$ — the Möbius function; $\tau(n)$ — the number of divisors of a natural number n; $\Gamma(s)$ — Euler's gamma function; $s=\sigma+it$, $i^2=-1$, $\sigma=\mathrm{Re}\, s$, $t=\mathrm{Im}\, s$.

2. Lower bounds for the Riemann zeta function on short intervals of the critical line

We have satisfactory knowledge about the behaviour of $\zeta(s)$ and the related quantities when $s=\sigma+it$ and t varies along a large interval, i.e.

$$T<t<T+H,$$

$H=H(T)\to+\infty$ as $T\to+\infty$. It is described by the theorems about the behaviour of $\max|\zeta(s)|$, the mean values of $|\zeta(s)|^{2k}$, $0<k\leqslant 2$, $\arg\zeta(s)$, theorems about the number of zeros of $\zeta(\frac{1}{2}+it)$ and others. At the same time little is known about the answers to similar questions when t varies in a short intervals, for example when $H=H(T)=\mathrm{const}$, or when $H(T)\to 0$, $T\to+\infty$. Of course we assume $0<\sigma<1$, i.e. the points s are inside the critical strip. Define the function $F(T;H)$ as:

$$F(T;H)=\max_{|T-t|\leqslant H}|\zeta(\frac{1}{2}+it)|.$$

If $H = H(T)$ is large, more precisely if

$$c \log\log T \leqslant H \leqslant \frac{1}{10} T,$$

then the following estimate due to Balasubramanian [1] holds for $F(T;H)$:

$$F(T;H) \geqslant \exp\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right).$$

If H is small, $0 < H = \Delta < 1$, then the theorem of Valiron-Landau-Hoheisel [12, p. 217] leads to the estimate:

$$F(T;\Delta) = F(T;H) \geqslant \exp\left(-A\frac{1}{\Delta}\log T\right) \tag{1}$$

It was also noted there that small values of $F(T;\Delta)$ are located in the neighbourhoods of zeros of $\zeta(s)$. Moreover, a theorem shown in [12, pp. 355–358] implies that if RH is true, then the inequality

$$F(T;\Delta) \geqslant \exp\left(-A\frac{1}{\Delta}\log T\frac{\log\log\log T}{\log\log T}\right) \tag{2}$$

holds for $0 < \Delta < 1$. Since RH also implies that the mean distance between subsequent zeros of $\zeta(\frac{1}{2}+it)$ on the interval $T < t < 2T$ is of the order $(\log T)^{-1}$, it is interesting, first of all, what is the lower bound for $F(T;\Delta)$ for $\Delta \leqslant (\log T)^{-1}$. In 2001 the author [10] has shown that for $0 < \Delta \leqslant (\log T)^{-1}$ the following inequality holds:

$$F(T;\Delta) \geqslant \exp\left(-A(\log\frac{1}{\Delta})\log T\right). \tag{3}$$

If $\Delta = (\log T)^{-1}$, the exponents in the right-hand sides of (1), (2), and (3) equal, respectively:

$$-A\log^2 T, \qquad -A(\log^2 T)\left(\frac{\log\log\log T}{\log\log T}\right), \qquad -A(\log T)(\log\log T).$$

The discrepancy in the estimates (1) − (3) is even greater when $\Delta = T^{-1}$, as the corresponding exponents are equal to

$$-AT\log T, \qquad -AT\frac{\log T}{\log\log T}\log\log\log T, \qquad -A\log^2 T$$

respectively in that case. Three conjectures were stated in [10] (each subsequent one stronger than the previous).

Conjecture 1F. *There exists a function* $\Delta = \Delta(T) \to 0$ *as* $T \to +\infty$ *such that the following estimate holds:*

$$F(T;\Delta) \geqslant \exp(-A\log T).$$

Conjecture 2F. *Conjecture 1F is true with*

$$\Delta = (\log\log T)^{-1}.$$

Conjecture 3F. *Conjecture 1F is true with*

$$\Delta = (\log T)^{-1}.$$

We note that (2) implies Conjecture 1F with

$$\Delta = \frac{\log\log\log T}{\log\log T}.$$

During the last three years new results were obtained in this direction. M.Z. Garaev [4] has shown (3) for

$$(\log T)^{-1} \leqslant \Delta \leqslant \frac{1}{3},$$

and has proved Conjecture 3F assuming RH.

Shao-Ji Feng [3] has proved Conjecture 1F assuming the Lindelöf hypothesis. M.E. Changa [2] has obtained the new proof of (3) for $0 < \Delta \leqslant \frac{1}{3}$.

We note that for slightly higher values of $H = H(T)$, namely for

$$10 \leqslant H \leqslant c\log\log T,$$

where $c \geqslant 100$ is the constant in the theorem of Balasubramanian, little is known about lower bound estimates of $F(T;H)$. Of course, trivially we have

$$\exp(-A\log T) \leqslant F(T;\frac{1}{3}) \leqslant F(T;10) \leqslant F(T;H).$$

Moreover, the following unpublished estimate was proved by the author about 1980:

$$F(T;H) \geqslant \exp\left(-\frac{1}{H^2}\log T\right).$$

Finally, in [11] the author has shown that there exists an absolute constant $c > 0$ such that for $T \geqslant T_1 > 0$ and $2 \leqslant H \leqslant \log\log T - c$ the following inequality is satisfied:

$$F(T;H) \geqslant \frac{1}{8}\exp\left(-\frac{1}{2(\cosh H - 1)}\log T\right). \tag{4}$$

This implies, in particular, that for $H \geqslant \log\log T$ the following estimate holds:

$$F(T;H) \geqslant c_1 > 0.$$

A similar result based on the principle of maximum was obtained by M.E. Changa [2]:

$$F(T;H) \geqslant \exp\left(-\frac{1}{\exp(\frac{1}{10}H)}\log T\right),$$

where $40 \leqslant H \leqslant \log\log T$. We note that it is an interesting unsolved problem to prove, for example, an inequality like this:

$$F(T;10) \geqslant \exp(-\epsilon(T)\log T),$$

where $\epsilon(T) \to 0$ as $T \to +\infty$. It would be just as interesting to prove the inequality:

$$F(T;H) \geqslant 1, \qquad H \geqslant \log\log\log T.$$

Below we show a theorem that slightly refines (4).

Theorem 1. *For every α such that $1 \leqslant \alpha < \pi$ there exist absolute positive constants c and T_1 such that for $T \geqslant T_1$ and*

$$2 \leqslant \alpha H \leqslant \log\log T - c$$

the following estimate holds:

$$F(T;H) \geqslant \frac{1}{16}\exp\left(-\frac{5\log T}{6(\frac{\pi}{\alpha}-1)(\cosh\alpha H - 1)}\right).$$

Proof. We follow the argument in [11].

1. We use a simple approximation of $\zeta(s)$ (cf. [12, p. 80]): for $\pi x \geqslant t \geqslant 2\pi$, $\frac{1}{10} \leqslant \sigma \leqslant 2$, $s = \sigma + it$ we have:

$$\zeta(s) = \sum_{n \leqslant x} n^{-s} + \frac{x^{1-s}}{s-1} + \mathrm{O}(x^{-\sigma}).$$

Taking $\frac{1}{2}T \leqslant t \leqslant T$, $T \geqslant 10$, $x = T$, $\sigma = \frac{1}{2}$, $s = \frac{1}{2} + it$, we obtain:

$$\zeta(s) = \sum_{n \leqslant T} n^{-s} + \mathrm{O}(T^{-0.5}).$$

Let $1 \leqslant X \leqslant \sqrt{T}$, $s = \frac{1}{2} + it$,

$$M_X(s) = \sum_{n \leqslant X} \mu(n) n^{-s}.$$

Obviously

$$|M_X(s)| \leqslant \sum_{n \leqslant X} n^{-0.5} \leqslant 2\sqrt{X}.$$

Consequently the product $\zeta(s)M_X(s)$ satisfies the formula:

$$\zeta(s)M_X(s) = \sum_{n\leqslant XT} a(n)n^{-s} + \mathrm{O}(T^{-0.25}), \tag{5}$$

where

$$a(n) = {\sum_{d|n}}' \mu(d),$$

and the $'$ in the sum means that $d \leqslant X$ and $n \leqslant dT$. If $n = 1$ then $a(n) = a(1) = 1$. If $1 < n \leqslant X$, then by the well known property of the Möbius function we have

$$a(n) = \sum_{d|n} \mu(d) = 0.$$

Moreover we always have $|a(n)| \leqslant \tau(n)$. Consequently the equality (5) may be written like this:

$$\zeta(s)M_X(s) = 1 + \sum_{X<n\leqslant XT} a(n)n^{-s} + \mathrm{O}(T^{-0.25}). \tag{6}$$

2. Consider the integral j,

$$j = \int_{-H}^{H} e^{-z\cosh\alpha t}\zeta(s+it)M_X(s+it)\,dt, \tag{7}$$

where $s = \frac{1}{2} + iT$, $1 \leqslant \alpha < \pi$, $z \geqslant 1$. We put

$$j(z) = \int_{-\infty}^{\infty} e^{-z\cosh t}\,dt,$$

and find, by (7),

$$|j| \leqslant 2F(T;H)\sqrt{X}\int_{-\infty}^{\infty} e^{-z\cosh\alpha t}\,dt = 4\alpha^{-1}F(T;H)\sqrt{X}j(z). \tag{8}$$

3. On the other hand the integral j satisfies the following equality:

$$j = \int_{-\infty}^{\infty} e^{-z\cosh\alpha t}\zeta(s+it)M_X(s+it)\,dt + R, \tag{9}$$

where

$$|R| \leqslant \int_{H}^{\infty} e^{-z\cosh\alpha t}|\zeta(s+it)||M_X(s+it)|\,dt + \int_{H}^{\infty} e^{-z\cosh\alpha t}|\zeta(s-it)||M_X(s-it)|\,dt.$$

Since

$$M_X(s\pm it)=\mathrm{O}(\sqrt{X}),\qquad \zeta(s\pm it)=\mathrm{O}((T+t)^{\frac{1}{6}}),$$

we obtain an estimate for R:

$$\begin{aligned}R&=\mathrm{O}\Big(\sqrt{X}\int_H^\infty e^{-z\cosh\alpha t}(T+t)^{\frac{1}{6}}\,dt\Big)\\&=\mathrm{O}\Big(\sqrt{X}T^{\frac{1}{6}}\int_H^\infty e^{-z\cosh\alpha t}\,dt\Big)\\&=\mathrm{O}\Big(\sqrt{X}T^{\frac{1}{6}}e^{-z\cosh\alpha H}(z\sinh\alpha H)^{-1}\Big).\end{aligned}\tag{10}$$

Therefore, by (6), (9), and (10) we find:

$$\begin{aligned}j&=\int_{-\infty}^\infty e^{-z\cosh\alpha t}\,dt+\sum_{X<n\leqslant XT}a(n)n^{-s}\int_{-\infty}^\infty e^{-z\cosh\alpha t}e^{-it\log n}\,dt\\&\quad+\mathrm{O}(j(z)T^{-0.25})+\mathrm{O}\Big(\sqrt{X}T^{\frac{1}{6}}e^{-z\cosh\alpha H}(z\sinh\alpha H)^{-1}\Big)\\&=\alpha^{-1}j(z)+\alpha^{-1}\sum_{X<n\leqslant XT}a(n)n^{-s}\int_{-\infty}^\infty e^{-z\cosh t}e^{-it\frac{\log n}{\alpha}}\,dt\\&\quad+\mathrm{O}(j(z)T^{-0.25})+\mathrm{O}\Big(T^{\frac{5}{12}}e^{-z\cosh\alpha H}(z\sinh H)^{-1}\Big).\end{aligned}\tag{11}$$

4. We estimate the integral in (11) using Basset's formula:

$$K_\nu(z)=\int_0^\infty e^{-z\cosh t}\cosh(\nu t)\,dt=\frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\sqrt{\pi}}\int_0^\infty\frac{\cos t\,dt}{(t^2+z^2)^{\nu+\frac{1}{2}}},$$

where $z>0$, ν is a complex number, $\operatorname{Re}\nu>-\frac{1}{2}$ (cf. [15, p. 191]). In our case $\nu=i\frac{\log n}{\alpha}$, hence

$$\begin{aligned}\Big|\int_{-\infty}^\infty e^{-z\cosh t}e^{-it\frac{\log n}{\alpha}}\,dt\Big|&=2\Big|\int_0^\infty e^{-z\cosh t}\cos\Big(\frac{\log n}{\alpha}t\Big)dt\Big|\\&\leqslant\frac{2\Big|\Gamma(i\frac{\log n}{\alpha}+\frac{1}{2})\Big|}{\sqrt{\pi}}\Big|\int_0^\infty\frac{\cos t\,dt}{(t^2+z^2)^{\nu+\frac{1}{2}}}\Big|.\end{aligned}$$

Then, integrating once by parts we obtain:

$$\begin{aligned}\Big|\int_0^\infty\frac{\cos t\,dt}{(t^2+z^2)^{\nu+\frac{1}{2}}}\Big|&=\Big|\int_0^\infty\frac{d\sin t}{(t^2+z^2)^{\nu+\frac{1}{2}}}\Big|\\&=\Big|(\nu+\frac{1}{2})\int_0^\infty\frac{2t\sin t\,dt}{(t^2+z^2)^{\nu+\frac{3}{2}}}\Big|\\&\leqslant\sqrt{\frac{1}{4}+\frac{\log^2 n}{\alpha^2}}\int_0^\infty\frac{du}{(u+z^2)^{\frac{3}{2}}}\\&\leqslant z^{-1}\sqrt{1+4\log^2 n}.\end{aligned}$$

The following asymptotic formula holds for $\Gamma(\sigma+it)$:

$$\Gamma(\sigma+it) = t^{\sigma-\frac{1}{2}+it} e^{-\frac{\pi}{2}t-it+i\frac{\pi}{2}(\sigma-\frac{1}{2})}\sqrt{2\pi}\big(1+\mathrm{O}\big(\tfrac{1}{t}\big)\big),$$

where $-10 \leqslant \sigma \leqslant 10$, $t \geqslant 2$. In our case we have:

$$\left|\Gamma\left(\frac{1}{2}+i\frac{\log n}{\alpha}\right)\right| = \mathrm{O}\left(e^{-\frac{\pi}{2}\frac{\log n}{\alpha}}\right) = \mathrm{O}\big(n^{-\frac{\pi}{2\alpha}}\big).$$

This way we obtain:

$$z\int_{-\infty}^{\infty} e^{-z\cosh t} e^{-it\frac{\log n}{\alpha}}\, dt = \mathrm{O}\big(n^{-\frac{\pi}{2\alpha}}\log n\big).$$

5. We bring together the estimates found so far and obtain the following for the sum over n in (11):

$$\begin{aligned}
&\left|\sum_{X<n\leqslant XT} a(n)n^{-s}\int_{-\infty}^{\infty} e^{-z\cosh t} e^{-it\frac{\log n}{\alpha}}\, dt\right| \\
&= \mathrm{O}\left(z^{-1}\sum_{X<n\leqslant XT}\tau(n)n^{-\frac{1}{2}-\frac{\pi}{2\alpha}}\log n\right) \\
&= \mathrm{O}\Big(z^{-1}X^{\frac{1}{2}-\frac{\pi}{2\alpha}}\log^2 X\Big).
\end{aligned}$$

Therefore the integral j satisfies the following asymptotic formula:

$$\begin{aligned}
j = \alpha^{-1}j(z) &+ \mathrm{O}\Big(z^{-1}X^{\frac{1}{2}-\frac{\pi}{2\alpha}}\log^2 X\Big) \\
&+ \mathrm{O}\big(j(z)T^{-0.25}\big) + \mathrm{O}\Big(T^{\frac{5}{12}}e^{-z\cosh\alpha H}(z\sinh H)^{-1}\Big).
\end{aligned} \tag{12}$$

6. Using (8) and (12) we find:

$$\begin{aligned}
4F(T;H)\sqrt{X} \geqslant 1 &- \mathrm{O}\Big((j(z))^{-1}z^{-1}X^{\frac{1}{2}-\frac{\pi}{2\alpha}}\log^2 X\Big) - \mathrm{O}\big(T^{-0.25}\big) \\
&- \mathrm{O}\Big((j(z))^{-1}T^{\frac{5}{12}}e^{-z\cosh\alpha H}(z\sinh H)^{-1}\Big)
\end{aligned} \tag{13}$$

The lower bound for $j(z)$ may be found easily:

$$\begin{aligned}
j(z) &= \int_{-\infty}^{\infty} e^{-z\cosh t}\, dt = 2\int_{0}^{\infty} e^{-z\cosh t}\, dt = 2\int_{1}^{\infty} e^{-zu}\frac{du}{\sqrt{u^2-1}} \\
&= 2e^{-z}\int_0^{\infty}\frac{e^{-zw}\, dw}{\sqrt{w(w+2)}} \geqslant 2e^{-z}\int_0^{z^{-1}}\frac{e^{-zw}\, dw}{\sqrt{w(w+2)}} \\
&\geqslant \frac{2}{\sqrt{3}}e^{-z-1}\int_0^{z^{-1}}\frac{dw}{\sqrt{w}} = \frac{4}{\sqrt{3}}z^{-\frac{1}{2}}e^{-z-1}.
\end{aligned}$$

Therefore by (13) we obtain:

$$4F(T;H)\sqrt{X} \geqslant 1 - c_1 z^{-\frac{1}{2}} e^z X^{\frac{1}{2}-\frac{\pi}{2\alpha}} \log^2 X - c_2 T^{-0.25} \\ - c_3 z^{-\frac{1}{2}} T^{\frac{5}{12}} e^{z - z\cosh\alpha H} (\sinh H)^{-1}. \tag{14}$$

7. Now we can fix the parameters z and X with equations:

$$z = \frac{5\log T}{12(\cosh\alpha H - 1)} = \frac{1}{4}\Big(\frac{\pi}{\alpha} - 1\Big)\log X. \tag{15}$$

By (14) and (15) we have:

$$4F(T;H) \geqslant 1 - c_1 z^{-\frac{1}{2}} X^{-\frac{1}{4}(\frac{\pi}{\alpha}-1)} \log^2 X - c_2 T^{-0.25} \\ - c_3 z^{-\frac{1}{2}} (\sinh H)^{-1}.$$

Since H satisfies the inequalities

$$2 \leqslant \alpha H \leqslant \log\log T - c,$$

we have

$$\cosh\alpha H \leqslant e^{\alpha H} \leqslant e^{-c}\log T,$$

i.e., the following lower bound holds for z:

$$z \geqslant \frac{5}{12}e^c.$$

We choose $c = c(\alpha) \geqslant 1$ large enough, so that for $z \geqslant \frac{5}{12}e^c$ the following inequalities hold:

$$c_1 z^{-\frac{1}{2}} X^{-\frac{1}{4}(\frac{\pi}{\alpha}-1)} \log^2 X \leqslant \frac{1}{4},$$
$$c_3 z^{-\frac{1}{2}} (\sinh H)^{-1} \leqslant \frac{1}{4}.$$

Next we choose $T_1 = T_1(\alpha) > 0$ large enough, so that for $T \geqslant T_1$ the following inequalities hold:

$$2 \leqslant \log\log T - c,$$
$$c_2 T^{-0.25} \leqslant \frac{1}{4}.$$

This way, with the selected c and T_1, and for $T \geqslant T_1$, $2 \leqslant \alpha H \leqslant \log\log T - c$, we obtain the inequality:

$$4F(T;H) \geqslant \frac{1}{4}X^{-\frac{1}{2}},$$

i.e.

$$F(T;H) \geqslant \frac{1}{16}\exp\left(-\frac{5\log T}{6(\frac{\pi}{\alpha}-1)(\cosh\alpha H - 1)}\right). \qquad \blacksquare$$

Corollary 1. *Taking* $\alpha H = \log\log T - c$ *in the theorem we have*

$$F(T;H) \geqslant \frac{1}{16}\exp\left(-\frac{5}{6(\frac{\pi}{\alpha}-1)}e^c\right) = c_4 > 0.$$

Hence, for any α *in the interval* $1 \leqslant \alpha < \pi$ *there exists* $T_1 = T_1(\alpha) > 0$ *such that for* $T \geqslant T_1$ *and* $H \geqslant \frac{1}{\alpha}\log\log T$ *we have:*

$$F(T;H) \geqslant c_4(\alpha) > 0.$$

3. Lower bounds for $|\zeta(s)|$ in small regions of the critical strip

A more general problem than that of estimating $F(T;\Delta)$ from below is to estimate $G(s_1;\Delta)$ from below, where, by definition,

$$G(s_1;\Delta) = \max_{|s-s_1|\leqslant\Delta} |\zeta(s)|,$$

$s_1 = \sigma_1 + it_1$, $\frac{1}{2} \leqslant \sigma_1 \leqslant 1$, $t_1 \geqslant 4$, $0 < \Delta \leqslant \frac{1}{3}$. Obviously, for $\sigma_1 = \frac{1}{2}$, $t_1 = T$, we have

$$G(s_1;\Delta) \geqslant F(T;\Delta).$$

In [9] the author has shown that for $t_1 \geqslant c_1 > 0$

$$G(s_1;\Delta) \geqslant \exp\left(-6\left(\log\frac{1}{\Delta}\right)(\log|s_1|)\right).$$

The same paper proposes three conjectures about lower bounds for $G(s_1;\Delta)$, equal to those in Conjectures 1F–3F.

Conjecture 1G. *There exists a function* $\Delta = \Delta(s_1) \to 0$ *as* $|s_1| \to +\infty$ *such that the following estimate holds:*

$$G(s_1;\Delta) \geqslant \exp(-A\log|s_1|).$$

Conjecture 2G. *Conjecture 1G is true with*

$$\Delta = (\log\log|s_1|)^{-1}.$$

Conjecture 3G. *Conjecture 1G is true with*

$$\Delta = (\log|s_1|)^{-1}.$$

The above-mentioned works of M.Z. Garaev [4] and M.E. Changa [2] establish a link between the bounds of $F(T;\Delta)$ and $G(s_1;\Delta)$ and, in particular, demonstrate the equivalence of the F and G conjectures, $s_1 = \frac{1}{2} + iT$.

4. The multiplicity of a zero of $\zeta(s)$ and lower bounds for $G(s_1;\Delta)$

Lower bounds for $G(s_1;\Delta)$ make it possible to obtain upper bounds for $\varkappa(T)$.

Theorem 2. *Suppose for some Δ and A such that $0<\Delta\leqslant\frac{1}{3}$, $A\geqslant 1$, we have*

$$G(s_1;\Delta)\geqslant\exp(-A\log|s_1|).$$

Then the following upper bound holds for $\varkappa(T)$:

$$\varkappa(T)\leqslant 1+\frac{A+4}{\log\frac{1}{\Delta}}\log T.$$

Proof. Let $s_1=\sigma_1+iT$, $T\geqslant T_1$, $\frac{1}{2}\leqslant\sigma_1\leqslant 1$, $K+1=\varkappa(T)$, and

$$\zeta(s_1)=\zeta^{(1)}(s_1)=\ldots=\zeta^{(K)}(s_1)=0. \tag{16}$$

Further let s_2 be such that $|s_2-s_1|=\Delta$ and

$$|\zeta(s_2)|=\max_{|s-s_1|\leqslant\Delta}|\zeta(s)|.$$

We have the equality:

$$\zeta(s_2)=\frac{1}{2\pi i}\int_{|s-s_2|=2}\frac{\zeta(s)}{s-s_2}\,ds. \tag{17}$$

Moreover we find:

$$\begin{aligned}\frac{1}{s-s_2}&=\frac{1}{s-s_1+s_1-s_2}=\frac{1}{s-s_1}\left(1+\frac{s_1-s_2}{s-s_1}\right)^{-1}\\&=\frac{1}{s-s_1}\sum_{\nu=0}^{\infty}(-1)^{\nu}\left(\frac{s_1-s_2}{s-s_1}\right)^{\nu}.\end{aligned}\tag{18}$$

Substituting (18) in (17) and using (16) we subsequently obtain

$$\begin{aligned}\zeta(s_2)&=\sum_{\nu=0}^{\infty}(-1)^{\nu}(s_1-s_2)^{\nu}\frac{1}{2\pi i}\int_{|s-s_2|=2}\frac{\zeta(s)}{(s-s_1)^{\nu+1}}\,ds\\&=\sum_{\nu=K+1}^{\infty}(-1)^{\nu}(s_1-s_2)^{\nu}\frac{1}{2\pi i}\int_{|s-s_2|=2}\frac{\zeta(s)}{(s-s_1)^{\nu+1}}\,ds.\end{aligned}\tag{19}$$

Obviously:

$$2=|s-s_2|=|s-s_1+s_1-s_2|\leqslant|s-s_1|+|s_1-s_2|=|s-s_1|+\Delta,$$

$$|s - s_1| \geqslant 2 - \Delta \geqslant \frac{5}{3}.$$

By assumption we have

$$|\zeta(s_2)| \geqslant \exp\Big(-A\log\sqrt{T^2+1}\Big). \tag{20}$$

Moreover, the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)$$

leads to the following inequality for $|s - s_2| = 2$:

$$|\zeta(s)| \leqslant T^3. \tag{21}$$

Using the relations (19), (20), and (21) we obtain

$$\begin{aligned}
\exp\Big(-A\log\sqrt{T^2+1}\Big) &\leqslant 2\sum_{\nu=K+1}^{\infty}\Delta^{\nu}T^3\Big(\frac{3}{5}\Big)^{\nu} \\
&= T^3\Big(\frac{3\Delta}{5}\Big)^{K+1}\cdot\frac{5}{2} \leqslant T^3\Big(\frac{3\Delta}{5}\Big)^{K}, \\
\Big(\frac{5}{3\Delta}\Big)^{K} &\leqslant T^3\exp\Big(A\log\sqrt{T^2+1}\Big), \\
K &\leqslant \frac{1}{\log\frac{5}{3\Delta}}\Big(A\log\sqrt{T^2+1} + 3\log T\Big).
\end{aligned}$$

The assertion follows. ■

References

[1] R. Balasubramanian, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$*. IV, Hardy-Ramanujan J., **9**, (1986), 1–10.

[2] M.E. Changa, *On lower bounds for the modulus of the Riemann zeta function on the critical line*, Mat. Zametki **76**, (2004), no. 6, 922–927.

[3] Shao-Ji Feng, *On Karatsuba conjecture and the Lindelöf hypothesis*, Acta Arith. **114**, (2004), no. 3, 295–300.

[4] M.Z. Garaev, *Concerning the Karatsuba conjectures*, Taiwanese J. Math. **6**, (2002), no. 4, 573–580

[5] A. Ivić, *On the multiplicity of zeros of the zeta-function*, Bull. Cl. Sci. Math. Nat. Sci. Math., no. 24, (1999), 119–132.

[6] A. Ivić, *The distribution of zeros of the zeta-function*, Bull. Cl. Sci. Math. Nat. Sci. Math., no. 40,(2003), 77–91.

[7] A.A. Karatsuba, *Osnovy analiticheskoi teorii chisel*, Izdat. "Nauka", Moscow, 1975.

[8] A.A. Karatsuba, *Complex analysis in number theory*, CRC Press, Boca Raton, FL, 1995.

[9] A.A. Karatsuba, *Lower bounds for the maximum modulus of $\zeta(s)$ in small domains of the critical strip*, Mat. Zametki, **70**, no. 5, (2001), 796–797.

[10] A.A. Karatsuba, *On lower bounds for the Riemann zeta function*, Dokl. Akad. Nauk **376**, no. 1, (2001), 15–16.

[11] A.A. Karatsuba, *On lower bounds for the maximum modulus of the Riemann zeta function on short intervals of the critical line*, Izv. Ross. Akad. Nauk Ser. Mat. **68**, no. 6, (2004), 99–104.

[12] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford, at the Clarendon Press, 1951.

[13] S.M. Voronin, *The differential independence of ζ-functions*, Dokl. Akad. Nauk SSSR, **209**, (1973), 1264–1266.

[14] S.M. Voronin, *A theorem on the "universality" of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat., **39**, (1975), no. 3, 475–486.

[15] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England, 1944.

Address: Steklov Institute of Mathematics RAS, 8, Gubkina str., 119991, Moscow, Russia
E-mail: `karatsuba@mi.ras.ru`
Received: 10 July 2005; **revised:** 15 March 2006