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THE MATHEMATICAL WORK OF EDUARD WIRSING

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The papers of Eduard Wirsing can be divided into following subjects.

1. Elementary number theory.
2. Sequences and sests.
3. Polynomials and matrices.
4. Diophantine approximation.
5. Metric theory of algorithms.
6. Multiplicative number theory.
7. Additive number theory.
8. Subjects outside number theory.

We shall comment on his papers in the indicated order.

1. Here belong two papers [5] and [9] on the distribution of perfect and multiply perfect numbers, the first written together with B. Hornfeck. The estimate for the number $N_\kappa(x)$ of solutions $n < x$ of the equation $\sigma(n) = \kappa n$ given in [11]:

$$N_\kappa(x) = O\left(e^{c\frac{\log x}{\log\log x}}\right), \quad c \text{ a number independent of } \kappa,$$

is for $\kappa = 2$ the best upper bound known at present for the number of perfect numbers less then x.

2. In his first paper [1] Wirsing proves that almost all sets of positive integers are totally primitive (also called asymptotically indecomposable), i.e. cannot be represented in the form $((A+B)\cup C)\setminus D$, where $|A| > 1, |B| > 1, |C| < \infty, |D| < \infty$.

In [4] multiplicative bases for the set of positive integers are studied. A sequence of integers $1 \leqslant a_1 < a_2 < \dots$ is called a multiplicative basis of order k, if every integer is the product of k or fewer a's. Denote by $A(x)$ the counting function of a_i. The author proves that for every k

$$\liminf x^{-1}A(x)\log x > 1$$

and that for every ε and every k there exists a multiplicative basis of order k satisfying

$$\liminf X^{-1}A(x)\log x < 1+\varepsilon.$$

In [3] Stöhr and Wirsing give examples of an essential component that is not an additive basis, simpler than the first such exampe due to Linnik (1942). This topic is pursued further in [27], where for every $\varepsilon > 0$ an essential component is constructed with the counting function

$$W(x) = O\left(e^{\varepsilon\sqrt{\log x}\log\log x}\right).$$

This construction was superseded only 11 years later by Ruzsa (1987) who obtained essential components with $W(x) = O((\log x)^{1+\varepsilon})$ for every $\varepsilon > 0$.

Finally in [29] Wirsing considers additive bases of order k for the set of integers and shows that from every such basis satisfying some technical conditions on can choose a subbasis of order k with the counting function $O((x\log x)^{1/k})$. This applies in particular to the sequence of squares, which forms a basis of order 4 and improves upon a result of Zöllner (1985).

3. Here belong paper [40], [42] and [43]. Let $\Phi_n(z)$ be the n-th cyclotomic polynomial, $A(n)$ the maximum of the absolute values of its coefficients and $\omega(n)$ the number of distinct prime factor of n. The main result of [40] is that for each $\varepsilon > 0$

$$\sup_{|z|=1} \log|\Phi_n(z)| \geqslant 2^{\left(\frac{1}{2}-\varepsilon\right)\omega(n)} \quad \text{if} \quad \omega(n) \geqslant k_0(\varepsilon).$$

This implies that for every $C > 2/\log 2$ the inequality

$$A(n) > \exp\left((\log n)^{C\frac{\log 2}{2}-\varepsilon}\right)$$

holds for all n with $\omega(n) \geqslant C\log\log n$ and $n \geqslant n_0(\varepsilon)$.

The main result of [42] says the following. Let $A = (a_{ij})$ be a complex 3×3 matrix. If for sufficiently small $\delta > 0$ we have $1-\delta \leqslant |a_{ij}| \leqslant 1+\delta$ for all i,j and $|\det A| \leqslant \delta^{3/2}$, then there are either two rows or two columns in A such that all three subdeterminants built from these rows or columns are $\ll \delta^{1/2}$. It follows that if $|a_{ij}| = 1$ and det $A = 0$, then either two rows or two columns of A are linearly dependent.

The results of [43] and [44] are more technical.

4. The paper [10] deals with a question of central importance in Diophantine approximation. Wirsing proves that given a real algebraic number ξ and given $d \in \mathbb{N}$, there are for any $\varepsilon > 0$ only finitely many algebraic numbers α of degree d with

$$|\xi - \alpha| < H(\alpha)^{-2d-\varepsilon}, \tag{1}$$

where $H(\alpha)$ is the height of α, i.e. the maximum modulus of the coefficients of the defining polynomial of α over $\mathbb{Z}$. This generalizes a famous theorem of Roth (1955) which deals with the case $d = 1$. The proof combines Roth's arguments with certain inequalities coming from probability theory. The $-2d$ in the exponent can now be improved to $-d-1$ (see Schmidt (1971)), but on the other hand, as pointed out by Vojta, the exponent $-2d$ is the right exponent for certain generalizations.

In [19] Wirsing again takes up approximation by algebraic numbers of degree at most d. Following Mahler (1932) and Koksma (1939), define $\omega_d = \omega_d(\xi)$ as the supremum of the numbers ω such that there are infinitely many polynomials $f \in \mathbb{Z}[X]$ of degree at most d with $0 < |f(\xi)| < H(f)^{-\omega}$ where $H(f)$ is the maximum modulus of the coefficients of f, and $\omega_d^* = \omega_d^*(\xi)$ in the supremum of the numbers ω^* such that there are infinitely many α of degree $\leqslant d$ with $|\xi - \alpha| < H(\alpha)^{-1-\omega^*}$. It is easily seen that unless ξ is algebraic $\omega_d \geqslant d$, $\omega_d \geqslant \omega_d^*$, and in this terminology, Wirsing's result on (1) says that $\omega_d^* \leqslant 2d-1$ when ξ is algebraic. In [19] it is shown that $\omega_d^* \geqslant \frac{1}{2}(\omega_d + 1)$, which shown that the classifications of Mahler and Koksma of transcendental numbers coincide.

One often attributes to Wirsing the conjecture that for transcendental real $\xi, \omega_d^*(\xi) = d$, i.e. that given $\varepsilon > 0$ there are infinitely many α of degree $\leqslant d$ with

$$|\xi - \alpha| < H(\alpha)^{-d-1+\varepsilon}.$$

This attribution may not be fair (especially since the conjecture may well be false), for Wirsing only says that the above is *vielleicht* (i.e. perhaps) true. In fact Wirsing establishes $\omega_d^*(\xi) \geqslant \frac{1}{4}(d+2+\sqrt{d^2+4d-4}) = \frac{d}{2}+\nu(d)$, where $\nu(d) \geqslant \frac{1}{2}, \nu(d) \to 1$. His proofs uses estimates on resultants of polynomials. A slight improvement was given by Bernik and Tishchenko (1993/94) with $\omega_d^*(\xi) \geqslant \frac{d}{2} + \nu'(d)$ where $\nu'(d) \to 2$. On the other hand the conjecture is true for $d = 2$ according to Davenport and Schmidt (1967). There are related papers on approximation by algebraic numbers of exact degree d, as well as by algebraic integers.

In [23] Wirsing and coauthors show that when $f : \mathbb{Z} \to A$, where A is the field of algebraic numbers in $\mathbb{C}$ is periodic with period q and has

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0 \tag{i}$$

and if moreover (ii) $f(r) = 0$ for r with $1 < (r,q) < q$, (iii) the cyclotomic polynomial Φ_q is irrducible over the field $\mathbb{Q}(f(1), \ldots, f(q))$, then $f = 0$. This implies a generalized version of a conjecture of Chowla. A crucial lemma in the proof says that the set G_q of functions $g : \mathbb{Z} \to A$ with period q having

$$\sum_{s=1}^{q-1} g(s) \log(1 - \xi^s) = 0,$$

where $\xi = \exp(2\pi i/q)$ and the logarithms are the principal values is invariant under automorphisms σ of A, i.e. invariant under replacing $g(s)$ by $\sigma g(s)$. This is a consequence of Baker work on linear forms in logarithms, in particular of the fact that the logarithms of algebraic numbers which are linearly independent over $\mathbb{Q}$ are linearly independent over A.

Moreover, it is shown that functions $f:\mathbb{Z}\to A$ with period q and (i), (ii) are necesarily odd and the odd functions $f:\mathbb{Z}\to A$ with period q and (i) are determined. In particular, they form a vector space of dimension $[(q-3)/2]$.

In [26] Wirsing and coauthors generalize a rezult of Newman (1976). Let T be the circle group consisting of z in $\mathbb{C}$ with $|z|=1$, and U_n the group of n-th roots of unity. Then if $\rho_1,\dots,\rho_n$ are distinct elements of T and $\varepsilon_1,\dots,\varepsilon_n$ are in T with

$$\sum_{i=1}^{n}\varepsilon_i\rho_i^k \geqslant 0 \quad \text{for} \quad k=1,2,\dots,$$

then $\{\rho_1,\dots,\rho_n\}=U_n$ and $\rho_i\to\varepsilon_i$ is a group homomorphism. Diophantine approximation enters the picture in a deduction of this result from the special case, when $\rho_1,\dots,\rho_n$ are roots of unity. Kronecker's theorem on the density of sequences $k\beta_1,\dots,k\beta_n, k=1,2,\dots$ modulo 1, where $\beta_1,\dots,\beta_n$ are linearly independent over Q, is used.

A number of further results are established. For instance, when $\chi_1,\dots,\chi_n$ are distinct characters of a compact abelian group C, and $\varepsilon_1,\dots,\varepsilon_n$ are in T, having

$$\alpha(t)=\sum_{i=1}^{n}\varepsilon_i\chi_i(t)\geqslant 0 \quad \text{for} \quad t\in C,$$

then $\{\chi(1),\dots,\chi(n)\}$ makes a subgroup of the character group, and $\chi_i\to\varepsilon_i$ is a character of the latter group.

In [34], written jointly with Schlickewei, estimates on heights are given extending some estimates of S. Zhang (1992) and Zagier (1993), these estimates being useful for subsequent work on diophantine equations $ax+by=1$. More general estimates were later derived by Bombieri and Zannier (1995), Schmidt (1996), David and Philippon (1999), and led to a conjecture of Bogomolov which was proved in increasing generality by S. Zhang, Ulmo and Zhang, and David and Philippon.

5. In [20] Wirsing reports on his new work on the Gauss-Kusmin Theorem. In the later presentation [22] he brings question left open by the theorem to definite conclusion. When irrational α in $0<\alpha<1$ has the regular continued fraction expansion $\alpha=[0;a_1,a_2,\dots]$, set $\alpha_n=\alpha_n(\alpha)=[0,a_n,a_{n+1},\dots]$. Let $M_n(\alpha)$ be the measure of the set of α in $0<\alpha<1$ with $\alpha_n\leqslant x$. Gauss in a letter in 1812 had asserted that $m_n(x)\sim(\log(1+x))/\log 2$ as $n\to\infty$, and Kusmin (1929) showed that in fact $m_k(x)=(\log(1+x))/\log 2+\tau_n(x)$ where $\tau_n(x)\ll q^{\sqrt{n}}$ for some $q, 0<q<1$. This was later improved to $\ll q^n$ by Lévy (1929) and Szüsz (1961). Efforts were made to find small values of q where this holds. Wirsing now shows that $\tau_n(x)=(-x)^n\Psi(x)+O(x(1-x)\mu^n)$ with constants $0<\mu<\lambda=0,303663\dots$ and ψ is a smooth function with $\psi(x)>0$ for $0<x<1$. Thus the best value for q is λ. The sophisticated proof begins with a functional equation on the $m_n(x)$ and proceeds via a new result on the spectrum of positive linear operators, generalizing an old result of Frobenius (1908). This information is of indenendent interest and gives information on Ψ on a holomorphic function.

6. In [2] Wirsing proves that if T is a set of primes and $\sum\limits_{\substack{p\in T\\ p\leqslant x}} \frac{1}{p} = \tau \log\log x + c' + o(1)$, then the counting function of integers composed of primes in T is asymptotic to $cx/(\log x)^{1-\tau}$ with c a constant. This generalizes a result of Landau (1909) in which T is a union arithmetic progressions.

In his habilitation thesis [13] Wirsing obtains, under very general assumptions on a multiplicative function $\lambda(n)$, an asymptotic formula for $\sum\limits_{n\leqslant x} \lambda(n)$. This result is further extended in the brilliant paper [16] in the following manner. Let p be a typical prime and λ be a nonnegative multiplicative function satisfying

$$\sum_{p\leqslant x} \frac{\log p}{p}\lambda(p) \sim \tau \log x \quad (\tau > 0), \tag{2}$$

$$\lambda(p) \ll 1, \tag{3}$$

$$\sum_{p,\nu\geqslant 2} \frac{\lambda(p^\nu)}{p^\nu} < \infty \tag{4}$$

and if $\tau \leqslant 1$

$$\sum_{\substack{p,\nu\geqslant 2\\ p^\nu\leqslant x}} \lambda(p^\nu) \ll x/\log x. \tag{5}$$

Then

$$\sum_{n\leqslant \lambda} \lambda(n) \sim (e^{\gamma\tau}/\Gamma(\tau)) \frac{x}{\log x} \prod_{p\leqslant x} \left(1 + \frac{\lambda(p)}{p} + \frac{\lambda(p^2)}{p^2} + \cdots\right),$$

where γ is Euler's constant.

The same conclusions holds for a complex-valued multiplicative function λ, if (2) holds, (3), (4) and (5) (in case $\tau \leqslant 1$) hold with $|\lambda|$ in place of λ and if, besides,

$$\sum_p p^{-1}(|\lambda(p)| - \mathrm{Re}\lambda(p)) < \infty.$$

The author obtains also as a special case of a more general theorem the following solution to a difficult problem of Wintner (1944).

If λ is a real multiplicative function satisfying $|\lambda(n)| \leqslant 1$, then

$$\lim_{x\to\infty} x^{-1} \sum_{n\leqslant x} \lambda(n) = \lim_{x\to\infty} \prod_{p\leqslant x} \left(1 - \frac{1}{p}\right) \left(1 + \frac{\lambda(p)}{p} + \frac{\lambda(p^2)}{p^2} + \cdots\right)$$

(Wintner's problem concerned the case $\lambda(n) = \pm 1$).

The proofs of the above theorems are elementary, by complicated and the author asks for analytic proofs, which may be simpler and may help to solve some outstanding problems. Such proof have been supplied by Halász (1968), who has, indeed, solved one of the problems. Taking for $\lambda(n)$ the Möbius function $\mu(n)$ one obtains from (6)

$$M(x) = \sum_{n \leqslant x} \mu(n) = o(x),$$

which has long been known to be equivalent elementarily to the prime number theorem (Landau 1911). Thus [16] contains implicitly an elementary proof of the prime number theorem.

Several years earlier, in [12] and [13], Wirsing provided elementary proofs of the prime number theorem with an error term, namely

$$\psi(x) = x + O(x/\log^m x)$$

where $m = 3/4$ in [12] and m arbitrary in [13]. The proofs are based on Selberg's formula. The same error term (m arbitrary) has been obtained about the same time by Bombieri (1962), who first generalized Selberg's formula. The present record due to Diamond and Steinig (1970) is

$$\psi(x) = x + O(x \exp(-c(\log x)^{1/6})), \quad c \text{ a constant.}$$

Another group of Wirsing's papers deals with a circle of problems originating in the paper of Erdös (1946) and concerning a characterization of logarithm as an additive function. In [17] Wirsing proves that if an additive function $f(n)$ satisfies $f(n+1) = f(n) + O(1)$, then $f(n) = c \log n + O(1)$. Another result in the same direction proved in [19] is the following: Let $f(n)$ be an additive function. If there is a constant $\gamma > 1$ and a sequence of numbers $x_i \to \infty$ such that, as $i \to \infty$

$$\sum_{x_i \leqslant n < \gamma x_i} |f(n+1) - f(n)| = o(x_i),$$

then $f(n) = c \log n$. Taking $\gamma = 2$ and $x_i = i/2$ one obtains the following conjecture of Erdös proved independently by Kátai (1970).

If $f(n)$ is an additive function and if

$$\sum_{n \leqslant x} |f(n+1) - f(n)| = o(x),$$

then $f(n) = c \log n$ with a constant c.

In the paper [27] it is proved that if a completely additive function satisfies $f(n+1) - f(n) = o(\log n)$, then $f(n) = c \log n$ and in [33] that if f is additive and $\|f(n+1) - f(n)\| = o(1)$, then there exists c such that $\|f(n) - c \log n\| = 0$ ($\|x\|$ is the distance of x to the nearest integer). Additive functions are studied also in [25] and [28]. One of the results of [25] is that if f is additive and real and $\lim_{p \to \infty} f(p+1) = 0$, then $f = 0$. In [27] it is shown that if an additive function f satisfies

$$f(n+1) - f(n) \ll \log^\alpha n, \quad 1 < \alpha \leqslant 6/5,$$

then

$$f(n) \ll \log^{\alpha} n.$$

In [28] the same implication is shown to hold for $\alpha \geqslant 3$. It is an interesting question, whether the implication holds for all $\alpha > 1$.

The papers [38], [39] and [41] deal with multiplicatiive functions. One of the results of [38] is that if $f : \mathbb{N} \to \mathbb{C}$ is multiplicative and $f(n+1) - f(n) \to 0$, then either $f(n) = n^s$ with $s \in \mathbb{C}, 0 \leqslant \operatorname{Re} s < 1$ or else $f(n) \to 0$ as $n \to \infty$.

The results of [39] are rather technical, but the main result of [40] is simple and general: if f is a multiplicative function from $\mathbb{N}$ to any multiplicative abelian group and the set of limit points of $f(n+1)/f(n)$ is finite, then the set of limit points of $f(n)$ is also finite.

A completely different type of problems is considered in [37] A. Selberg (1991) has introduced a class of function, now called S, central to analytic number theory and proposed several conjectures concerning these functions. A function $f \in S$ is called primitive if an equation $f = f_1 f_2, f_i \in S$ implies $f_1 = 1$ or $f_2 = 1$ Selberg's orthonormality conjecture in the weak form asserts that for primitive functions

$$P(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \text{and} \quad P'(s) = \sum_{n=1}^{\infty} \frac{a'(n)}{n^s}$$

we have

$$\sum_{\substack{p \leqslant x \\ p \text{prime}}} \frac{a(p)\overline{a'(p)}}{p} = \delta(P, P') \log\log x + o(\log\log x)$$

where $\delta(P, P') = 1$ if $P = P'$ and $\delta(P, P') = 0$, otherwise.

Two functions $F \in S$ and $F' \in S$ are shifted with respect to each other, if $F' = F(s + i\vartheta), \vartheta \in \mathbb{R}$. Countability conjecture says that there are only countably many shift classes of primitive functions in S.

A continuous family of functions in S (resp. primitive functions in S) on an interval $I \subset \mathbb{R}$ is a set of functions $P(s,\xi), \xi \in I$, where for every $\xi, P(s,\xi) \in S$ (resp. $P(s,\xi)$ is primitive) and $P(s,\xi)$ is a continuous function of ξ. Sarnak's Rigidity Conjecture says that

(i) any continuous primitive family on an interval $I \subset \mathbb{R}$ is of the form $P(s,\xi) = P(s + ih(\xi))$, where P is primitive and $h : I \to \mathbb{C}$ is continuous.

(ii) any continuous family can be factored into primitive continuous families.

In [37] the authors show that the weak Orthonormality Conjecture and Countability Conjecture imply Sarnak's Rigidity Conjecture. Their proof is based on a topological result of Sierpiński (1918). Earlier a proof of the theorem based on a different principle was published by Kaczorowski and Perelli (2000).

7. Wirsing [10] proves that if f is a non-constant integer-valued polynomial with the leading coefficient positive then the density of the integers of the form $p + f(q), p, q$ primes is positive. For $f(x) = x^k$ one obtains a result of Romanov (1934).

The paper [30] is concerned with $\omega\left(\prod_{m=1}^{N} p(m)\right)$, where $\omega(n)$ is the number of distinct prime factors of n and $p(m)$ is the number of partitions of m. It is proved that

$$\omega\left(\prod_{m=1}^{N} p(m)\right) > (1-\varepsilon)\frac{\log N}{\log 2} \quad \text{if} \quad N > N_0(\varepsilon).$$

The papers [35] and [36] together with a paper of Vorhauer (1999) present a simpler and cleaner proof of the Chen theorem in the circle problem, namely $\sum_{n_1^2+n_2^2 \leqslant x} 1 = \pi x + O(x^{12/37+\varepsilon})$. Moreover, denoting by $r_2(n)$ the number of solutions of $n_1^2 + n_2^2 = n$ the authors prove that

$$\frac{1}{\Gamma(\kappa+1)} \sum_{n \leqslant x} r_2(n) \log^{\kappa}\left(\frac{x}{n}\right) = \pi x + O\left(x^{\frac{12}{37} - \frac{25}{37}\kappa + \varepsilon}\right)$$

for every non-negative $\kappa \leqslant 7/30$ and every $\varepsilon > 0$.

8. In an early paper [8] Wirsing considers convex curves with two equichordal points and proves that a boundary of such a curve is regular. It has since been proved by Rychlik (1997) that there are no such curves. The two papers [14] and [15] are concerned with the zeros of infinite series $\sum_{n=0}^{\infty} (n+1)^k z^n$, k real, and extend some results of Peyerimhoff (1966).

The paper [31] studies convergence properties of algorithms similar to the "regula falsi".

The highly original paper [32] studies those meromorphic functions w such that all residues of w and w^{-1} equal zero.

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