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ON THE REDUCED LENGTH OF A POLYNOMIAL WITH REAL COEFFICIENTS

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To Professor Eduard Wirsing
with best wishes
for his 75th birthday

Abstract: The length $L(P)$ of a polynomial P is the sum of the absolute values of the coefficients. For $P \in \mathbb{R}[x]$ the properties of $l(P)$ are studied, where $l(P)$ is the infimum of $L(PG)$ for G running through monic polynomials over $\mathbb{R}$.
Keywords: length of a polynomial, unit circle.

We shall consider only polynomials with real coefficients. For such a polynomial $P = \sum_{i=0}^{d} a_i x^{d-i}$ the length $L(P)$ is defined by the formula

$$L(P) = \sum_{i=0}^{d} |a_i| \,.$$

A. Dubickas [1] has introduced the reduced length by the formula

$$\widehat{l}\,(P) = \inf_{G \in \widehat{\Gamma}} L(PG),$$

where

$$\widehat{\Gamma} = \left\{ \sum_{i=0}^{n} b_i x^{n-i} \in \mathbb{R}[x],\ \text{where } b_0 = 1 \text{ or } b_n = 1 \right\}.$$

It follows, see [1], p. 3, that

$$\widehat{l}\,(P) = \min \left\{ l_0(P),\, l_0\left(P^*\right) \right\},$$

where

$$l_0(P) = \inf_{G \in \Gamma_0} L(PG),\ \Gamma_0 = \left\{ \sum_{i=0}^{n} b_i x^{n-i} \in \mathbb{R}[x],\, b_n = 1 \right\},\ P^* = x^{\deg P} P\left(x^{-1}\right).$$

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Since polynomials with the leading coefficient 1 have a name (monic) and polynomials with the constant term 1 have no name, I prefer to work with

$$l(P) = l_0\,(P^*) = \inf_{G\in\Gamma} L(PG),\ \Gamma = \left\{\sum_{i=0}^{n} b_i x^{n-i} \in \mathbb{R}[x],\, b_0 = 1\right\}$$

Dubickas's results about l_0 translated in the language of l give the following

Proposition A. (Dubickas [1]) *Suppose that* $\omega, \eta, \psi \in \mathbb{R}$, $\nu \in \mathbb{C}$, $\overline{\nu}$ *is the complex conjugate to* ν, $|\omega| \geqslant 1$, $|\eta| < 1$, $|\nu| < 1$, *then for every* $Q \in \mathbb{R}[x]$

(i) $l(\psi Q) = |\psi| l(Q)$,
(ii) $l(x+\omega) = 1 + |\omega|$,
(iii) *if* $T(x) = Q(x)(x-\eta)$, *then* $l(T) = l(Q)$,
(iv) *if* $T(x) = Q(x)(x-\nu)(x-\overline{\nu})$, *then* $l(T) = l(Q)$.

We shall prove the following

Proposition. *For all monic polynomials* P, Q *in* $\mathbb{R}[x]$ *and all positive integers* k

(i) $\max\{l(P), l(Q)\} \leqslant l(PQ) \leqslant l(P) l(Q)$,
(ii) $M(P) \leqslant l(P)$, *where* M *is the Mahler measure,*
(iii) $l(P(-x)) = l(P(x))$,
(iv) $l(P(x^k)) = l(P(x))$.

Theorem 1. *If* $P \in \mathbb{R}[x]$ *is monic of degree* d *with* $P(0) \neq 0$, *then* $l(P) = \inf_{Q\in S_d(P)} L(Q)$, *where* $S_d(P)$ *is the set of all monic polynomials* Q *over* $\mathbb{R}$ *divisible by* P *with* $Q(0) \neq 0$ *and with at most* $d+1$ *non-zero coefficients, all belonging to the field* $K(P)$, *generated by the coefficients of* P.

Theorem 2. *If* $P \in \mathbb{R}[x]$ *has all zeros outside the unit circle, then* $l(P)$ *is attained and effectively computable, moreover* $l(P) \in K(P)$ *($l(P)$ is attained means that* $l(P) = L(Q)$, *where* $Q/P \in \Gamma$*).*

Corollary 1. *If* $P \in \mathbb{R}[x]$ *has no zeros on the unit circle, then* $l(P)$ *is effectively computable.*

Theorem 3. *Let* $P, Q \in \mathbb{R}[x]$, Q *be monic and have all zeros on the unit circle. Then for all* $m \in \mathbb{N}$

$$l\,(PQ^m) = l(PQ).$$

Theorem 4. *If* $P \in \mathbb{R}[x]$ *is monic and has all zeros on the unit circle, then* $\widehat{l}\,(P) = l(P) = 2$, *with* $l(P)$ *attained, if and only if all zeros are roots of unity and simple.*

Theorem 5. *Let* $P(x) = P_0(x)(x-\varepsilon)^e$, *where* $P_0 \in \mathbb{R}[x]$, $\varepsilon = \pm 1$, $e \in \mathbb{N}$ *and all zeros of* P_0 *are outside the unit circle. Assume that the set* Z *of zeros of* P_0 *has a subset* Z_0, *possibly empty, such that its elements are real of the same sign and the elements of* $Z \setminus Z_0$ *are algebraically independent over* $\mathbb{Q}(Z_0)$. *Then* $l(P)$ *can be effectively computed. Moreover, if* $\deg P_0 = d_0$, *then*

$$l(P) \leqslant \inf_{Q\in S_{d_0}(P_0)} \Big\{ L(Q) + \big|Q(\varepsilon)\big| \Big\}.$$

For quadratic polynomials P Theorems 2, 4 and 5 together with Proposition A (iii) and (iv) exhaust all possibilities, so that $l(P)$ can be effectively computed. A more precise information is given by the following

Theorem 6. *If $P(x) = (x-\alpha)(x-\beta)$, where $|\alpha| \geqslant |\beta| \geqslant 1$, then*

$$l(P) \geqslant 2|\alpha|$$

with the equality attained, if and only if $|\beta| = 1$.

Corollary 2. *If $P \in \mathbb{R}[x]$ is of degree at most two with no zeros inside the unit circle, then*

$$l(P) \in K(P).$$

Corollary 3. *If $P(x) = (x-\alpha)(x-\beta)$, where $|\alpha| \geqslant |\beta| \geqslant 0$, then*

$$\widehat{l}\,(P) = \begin{cases} |\alpha\beta|, & \text{if } \ |\beta| > 1, \\ 2|\alpha|, & \text{if } \ |\beta| = 1, \\ |\alpha| + \min\{1, |\alpha\beta|\}, & \text{if } \ |\alpha| > 1 > |\beta|, \\ 2, & \text{if } \ |\alpha| = 1, \\ 1, & \text{if } \ |\alpha| < 1. \end{cases}$$

Corollary 4. *The function $\widehat{l}$ is not submultiplicative.*

The last corollary is of interest, because of Proposition, part (i).

The problem of computing $l(P)$ for cubic polynomials remains open already for $P = 2x^3 + 3x^2 + 4$. Another open question is whether $l(P) \in K(P)$ for all $P \in \mathbb{R}[x]$ with no zeros inside the unit circle.

We begin with

Proof of Proposition. We have by definition for all monic polynomials R, S in $\mathbb{R}[x]$

$$l(P) \leqslant L(PQR),\ l(PQ) \leqslant L(PQRS) \leqslant L(PR)L(QS)$$

hence

$$l(P) \leqslant \inf_{R\in\Gamma} L(PQR) = l(PQ),$$

$$l(PQ) \leqslant \inf_{R\in\Gamma} L(PR) \inf_{S\in\Gamma} L(QS) = l(P)l(Q).$$

This proves (i). As to (ii) we have for every R in $\mathbb{R}[x]$

$$M(R) \leqslant L(R)$$

(see [4]), hence

$$M(P) \leqslant M(PQ) \leqslant L(PQ),$$

thus

$$M(P) \leqslant \inf_{Q\in\Gamma} L(PQ) = l(P)$$

and (ii) holds. The statement (iii) follows from

$$L\Big(P(-x)\Big) \leqslant L\Big(P(-x)Q(-x)(-1)^{\deg Q}\Big) = L(PQ),$$

whence

$$l\Big(P(-x)\Big) \leqslant \inf_{Q\in\Gamma} L(PQ) = l(P).$$

Similarly,

$$l\Big(P\left(x^k\right)\Big) \leqslant L\Big(P\left(x^k\right)Q\left(x^k\right)\Big) = L(PQ),$$

whence

$$l\Big(P\left(x^k\right)\Big) \leqslant \inf_{Q\in\Gamma} L(PQ) = l(P). \tag{1}$$

Finally, if

$$P\left(x^k\right)Q(x) = \sum_{i=0}^{k-1} x^i A_i\left(x^k\right), \text{ where } A_i \in \mathbb{R}[x], \tag{2}$$

let $A_i = Q_i P + R_i$, where $Q_i, R_i \in \mathbb{R}[x]$ and $\deg R_i < \deg P$. It follows that

$$P\left(x^k\right) \mid \sum_{i=0}^{k-1} x^i R_i\left(x^k\right)$$

and since the degree of the sum is less than that of $P\left(x^k\right)$, $R_i = 0$ $(0 \leqslant i < k)$. Let i be chosen so that $\deg x^i A_i\left(x^k\right)$ is the greatest. It follows from (2) that Q_i is monic. Hence, by (2)

$$L\Big(P\left(x^k\right)Q(x)\Big) \geqslant L\left(A_i\right) = L\left(PQ_i\right) \geqslant l(P),$$

thus $l(P(x^k)) \geqslant l(P)$, which together with (1) implies (iv).

Remark. The above proof of (iv), simpler than author's original proof, has been kindly suggested by A. Dubickas.

For the proof of Theorem 2 we need two lemmas

Lemma 1. *Let* $k \geqslant n$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $L_i(\boldsymbol{x})$ *for* $i \leqslant k$ *be linear forms over* $\mathbb{R}$; $L_1, \ldots, L_n$ *linearly independent,* $a_i \in \mathbb{R}$ $(1 \leqslant i \leqslant k)$. *Then*

$$S(\boldsymbol{x}) = \sum_{i=1}^{k} |L_i(\boldsymbol{x}) + a_i|$$

attains its infimum.

Proof. Let $L_i(\boldsymbol{x}) = \sum\limits_{j=1}^{n} a_{ij}x_j \ \ (1 \leqslant i \leqslant k),\ A = \max\limits_{i,j\leqslant n} |a_{ij}|,$

$$D = \left|\det\left(a_{ij}\right)_{i,j\leqslant n}\right|, \quad s = \sum_{i=1}^{k} |a_i|\,.$$

Let s_0 be the infimum of $S(\boldsymbol{x})$ in the hypercube (degenerated if $s = 0$)

$$H : \max_{1\leqslant i\leqslant n} |x_i| \leqslant \frac{2n^{\frac{n-1}{2}} s A^{n-1}}{D}.$$

Since H is compact, there exists $\boldsymbol{x}_0 \in H$ such $S(\boldsymbol{x}_0) = s_0$. We shall show that $s_0 = \inf\limits_{\boldsymbol{x}\in\mathbb{R}^n} S(\boldsymbol{x})$. Indeed, if for some $\boldsymbol{x}_1 \in \mathbb{R}^n$

$$S\left(\boldsymbol{x}_1\right) < s_0, \tag{3}$$

then

$$\sum_{i=1}^{n} \left|L_i\left(\boldsymbol{x}_1\right)\right| < s_0 + s \leqslant 2s.$$

Solving the system $L_i(\boldsymbol{x}) = L_i(\boldsymbol{x}_1)$ $(1 \leqslant i \leqslant n)$ by means of Cramer's formulae and using Hadamard's inequality to estimate the relevant determinants we obtain

$$\max_{1\leqslant i\leqslant n} |x_{1i}| < \frac{2n^{\frac{n-1}{2}} s A^{n-1}}{D},$$

hence $\boldsymbol{x}_1 \in H$, a contradiction with (3) and the definition of s_0.

Lemma 2. *Let $k \geqslant n$, $\boldsymbol{x} \in \mathbb{R}^n$, K be a subfield of $\mathbb{R}$, $L_1(\boldsymbol{x}), \ldots, L_k(\boldsymbol{x})$ be linear forms over K, n of them linearly independent, $a_i \in K$. There exists a point $\boldsymbol{x}_0 \in K^n$ in which $S(\boldsymbol{x}) = \sum\limits_{i=1}^{k} |L_i(\boldsymbol{x}) + a_i|$ attains its infimum over $\mathbb{R}^n$ and $L_i(\boldsymbol{x}_0) + a_i = 0$, for n indices $i = i_1, i_2, \ldots, i_n$ such that $L_{i_1}, L_{i_2}, \ldots, L_{i_n}$ are linearly independent.*

Proof by induction on $\boldsymbol{k}$. If $k = 1$ we have $n = 1$ and the assertion is trivial. Assume it is true for $k-1$ forms and consider the case of k forms, $k \geqslant 2$. If one of them, say L_k is identically 0, then among $L_1, \ldots, L_{k-1}$ there are n linearly independent, hence $k - 1 \geqslant n$ and applying the inductive assumption to $L_1, \ldots, L_{k-1}$ we obtain the assertion. Therefore, we assume that all forms $L_1, \ldots, L_k$ are non-zero. Suppose that $\inf S(\boldsymbol{x}) = S(\boldsymbol{x}_1)$ and $L_i(\boldsymbol{x}_1) + a_i \neq 0$ for all $i \leqslant k$. Then there is an $\varepsilon > 0$ such that $|\boldsymbol{x} - \boldsymbol{x}_1| < \varepsilon$ implies $\operatorname{sgn}(L_i(\boldsymbol{x}) + a_i) =: \varepsilon_i$ for all $i \leqslant k$. We have

$$S(\boldsymbol{x}) = \sum_{i=1}^{k} \varepsilon_i \left(L_i(\boldsymbol{x}) + a_i\right) = M\left(\boldsymbol{x} - \boldsymbol{x}_1\right) + S\left(\boldsymbol{x}_1\right),$$

where

$$M(\boldsymbol{x})=\sum_{i=1}^{k}\varepsilon_i L_i(\boldsymbol{x}).$$

If $M\neq 0$, then there exists a point $\boldsymbol{x}_0$ with $|\boldsymbol{x}_0|<\varepsilon$ and $M(\boldsymbol{x})<0$, hence taking $\boldsymbol{x}_2=\boldsymbol{x}_1+\boldsymbol{x}_0$ we obtain $S(\boldsymbol{x}_2)<S(\boldsymbol{x}_1)$, a contradiction. Thus either $L_{i_1}(\boldsymbol{x}_1)+a_{i_1}=0$ for a certain i_1, or $M=0$. In the latter case we take the point $\boldsymbol{x}_2$ nearest to $\boldsymbol{x}_1$ (or one of these) with $L_{i_2}(\boldsymbol{x}_2)+a_{i_2}=0$ for a certain i_2. Since the hyperplanes $L_i(x)+a_i=0$ either are disjoint with the ball $|\boldsymbol{x}-\boldsymbol{x}_1|\leqslant|\boldsymbol{x}_2-\boldsymbol{x}_1|$, or are tangent to it, taking $\langle\boldsymbol{x}_3,i_3\rangle$ equal either to $\langle\boldsymbol{x}_1,i_1\rangle$ or to $\langle\boldsymbol{x}_2,i_2\rangle$ we obtain $S(\boldsymbol{x}_3)=S(\boldsymbol{x}_1)$ and $L_{i_3}(\boldsymbol{x}_3)+a_{i_3}=0$. Without loss of generality we may assume that $i_3=k$ and L_k is of positive degree in x_n. The equation $L_k(\boldsymbol{x})+a_k=0$ is equivalent to $x_n=C(x_1,\ldots,x_{n-1})+c$, where C is a linear form over K and $c\in K$. We now apply the inductive assumption to the forms $L_i'=L_i(x_1,\ldots,x_{n-1},C(x_1,\ldots,x_{n-1}))$ and numbers $a_i'=a_i+L_i(0,\ldots,0,c)$ $(1\leqslant i\leqslant k-1)$. By virtue of the theorem about the rank of the product of matrices, the number of linearly independent among forms L_i' is $n-1$. By the inductive assumption there exists a point $\boldsymbol{x}_0'\in K^{n-1}$ such that $\sum\limits_{i=1}^{k-1}|L_i'(\boldsymbol{x}')+a_i'|=S'(\boldsymbol{x}')$ attains at $\boldsymbol{x}_0'$ its infimum over $\mathbb{R}^{n-1}$ and $L_i'(\boldsymbol{x}_0')+a_i'=0$ for $n-1$ indices $i=i_1',\ldots,i_{n-1}'$ such that $L'_{i_1'},\ldots,L'_{i_{n-1}'}$ are linearly independent. By the definition of L_i' and a_i' we have

$$S\left(\boldsymbol{x}_3\right)=S'\left(x_{3,1},\ldots,x_{3,n-1}\right)\geqslant\inf_{\boldsymbol{x}'\in\mathbb{R}^{n-1}}S'(\boldsymbol{x}')\geqslant\inf_{\boldsymbol{x}\in\mathbb{R}^n}S(\boldsymbol{x})=S\left(\boldsymbol{x}_3\right),$$

hence

$$S'(\boldsymbol{x}_0')=\inf_{\boldsymbol{x}'\in\mathbb{R}^{n-1}}S'(\boldsymbol{x}')=\inf_{\boldsymbol{x}\in\mathbb{R}^n}S(\boldsymbol{x}).$$

Moreover, $L'_{i_j'}(\boldsymbol{x}_0')+a'_{i_j'}=0$ implies

$$L'_{i_j'}\left(x'_{0\,1},\ldots,x'_{0\,n-1},C\left(\boldsymbol{x}_0'\right)\right)+a_{ij}=0$$

and the linear independence of $L'_{i_1'},\ldots,L'_{i_{n-1}'}$ implies the linear independence of the forms $L_{i_1'},\ldots,L_{i_{n-1}'}$. The latter forms are also linearly independent with L_k since identity

$$L_1\left(x_1,\ldots,x_n\right)=\sum_{j=1}^{n-1}c_jL_{i_j'}\left(x_1,\ldots,x_n\right),\quad c_j\in\mathbb{R}$$

gives on substitution $x_n=C(x_1,\ldots,x_{n-1})$

$$0=\sum_{j=1}^{n-1}c_jL'_{i_j}\left(x_1,\ldots,x_n\right),\quad\text{hence } c_j=0\ (1\leqslant j<n).$$

Taking $\boldsymbol{x}_0=(x'_{0\,1},\ldots,x'_{0\,n-1},C(x_0'))$, $i_j=i_j'$ $(1\leqslant j<n)$, $i_n=k$ we obtain the inductive assertion.

Proof of Theorem 1. We have by definition

$$l(P)=\inf L(PG),$$

where G runs through all monic polynomials. Let $P=x^d+\sum\limits_{i=1}^{d}a_ix^{d-i}$, $G=x^n+\sum\limits_{i=1}^{n}x_ix^{n-i}$. We have

$$PG=x^{n+d}+\sum_{i=1}^{n+d}b_ix^{n+d-i},$$

where, with $a_0=1$ for $i\leqslant d$

$$b_i=a_i+\sum_{j=1}^{\min\{i,n\}}a_{i-j}x_j,$$

for $i>d$

$$b_i=\sum_{j=i-d}^{\min\{i,n\}}a_{i-j}x_j.$$

Therefore,

$$l(P)=1+\inf_{n,\,\boldsymbol{x}\in\mathbb{R}^n}\left\{\sum_{i=1}^{d}|L_i(\boldsymbol{x})+a_i|+\sum_{i=d+1}^{d+n}|L_i(\boldsymbol{x})|\right\},$$

where

$$L_i(\boldsymbol{x})=\sum_{j=\max\{1,i-d\}}^{\min\{i,n\}}a_{i-j}x_j.$$

The forms L_i satisfy the assumptions of Lemma 2. Indeed, the n forms $L_{d+1},\dots,$ L_{d+n} are linearly independent, since $L_{d+1}(\boldsymbol{x})=\ldots=L_{d+n}(\boldsymbol{x})=0$ gives $PG\equiv 0(\operatorname{mod}x^n)$, hence $G\equiv 0(\operatorname{mod}x^n)$, i.e. $x_1=\ldots=x_n=0$. Applying Lemma 2 and Proposition A (iii) with $\eta=0$ we obtain that for a given n, PG with the minimal length occurs in $S_d(P)$.

For the proof of Theorem 2 we need

Definition 1. Let $P=\prod\limits_{s=1}^{r}(x-\alpha_s)^{m_s}$, where α_s are distinct and non-zero, $m_s\in\mathbb{N}$ $(1\leqslant s\leqslant r)$, $m_1+\ldots+m_r=d$, $n_0>n_1>\ldots>n_{d-1}>n_d\geqslant 0$ be integers. If $s\geqslant 1$, $1\leqslant i\leqslant d$, $0\leqslant j\leqslant d$, then i can be written in the form $i=m_1+\ldots+m_{s-1}+g$ for some $1\leqslant s\leqslant r$ and $1\leqslant g\leqslant m_s$. We put

$$c_{ij}=\alpha_s^{n_j}\prod_{f=0}^{g-2}(n_j-n_f)\,,\ \text{where the empty product is 1}$$

and for $\nu=0,1$

$$C\left(P;n_0,\dots,n_d\right)=(c_{ij})_{\substack{1\leqslant i\leqslant d\\0\leqslant j\leqslant d}}\,,\quad C_\nu\left(P;n_\nu,\dots,n_{d-1+\nu}\right)=(c_{ij})_{\substack{1\leqslant i\leqslant d\\\nu\leqslant j<d+\nu}}\,.$$

Definition 2. $T_d(P) = \Big\{Q \in S_d(P) : Q = x^{n_0} + \sum_{i=1}^{d} b_i x^{n_i}$, where $n_0 > n_1 > \ldots > n_d = 0$, $|C_0(P; n_0, \ldots, n_{d-1})| \neq 0 \neq |C_1(P; n_1, \ldots, n_d)|$, $L(Q) \leqslant L(P)\Big\}$.

Lemma 3. *We have for* $x_j \in \mathbb{C}$

$$\sum_{j=0}^{d} x_j x^{n_j} \equiv 0 (\operatorname{mod} P) \tag{4_1}$$

if and only if

$$\sum_{j=0}^{d} c_{ij} x_j = 0 \quad (1 \leqslant i \leqslant d). \tag{4_2}$$

Proof. Clearly the condition (4_1) is equivalent to

$$\sum_{j=0}^{d} x_j \binom{n_j}{g-1} \alpha_s^{n_j} = 0 (1 \leqslant g \leqslant m_s,\, 1 \leqslant s \leqslant r),$$

that is to the vector equation

$$M\boldsymbol{x} = 0, \tag{5}$$

where $\boldsymbol{x} = (x_0, x_1, \ldots, x_d)^t$, $M = (m_{ij})_{\substack{1 \leqslant i \leqslant d \\ 0 \leqslant j \leqslant d}}$ and if $i = m_1 + \ldots + m_{s-1} + g$, $1 \leqslant g \leqslant m_s$, then

$$m_{ij} = \binom{n_j}{g-1} \alpha_s^{n_j}. \tag{6}$$

Now define the numbers b_{gh} by the equation

$$\prod_{f=0}^{g-2} (x - n_f) = \sum_{h=1}^{g} b_{gh} \binom{x}{h-1} \tag{7}$$

and put for $i = m_1 + \ldots + m_{s-1} + g$, $1 \leqslant g \leqslant m_s$, $1 \leqslant j \leqslant d$

$$a_{ij} = \begin{cases} b_{gh} & \text{if } j = m_1 + \ldots + m_{s-1} + h,\ 1 \leqslant h \leqslant g, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

$$A = (a_{ij})_{1 \leqslant i,j \leqslant d}\,. \tag{9}$$

The matrix A is lower triangular and non-singular, since $b_{gg} = (g-1)!$. Hence the equation (5) is equivalent to

$$AM\boldsymbol{x} = 0. \tag{10}$$

However, by (6)–(9) the element in i-th row $(1 \leqslant i \leqslant d)$ and j-th column $(0 \leqslant j \leqslant d)$ of AM for $i = m_1 + \ldots + m_{s-1} + g$, $1 \leqslant g \leqslant m_s$ is

$$\sum_{t=1}^{d} a_{it} m_{tj} = \sum_{h=1}^{g} b_{gh} \binom{n_j}{h-1} \alpha_s^{n_j} = \alpha_s^{n_j} \prod_{f=0}^{g-2} (n_j - n_f) = c_{ij},$$

hence (4_1) is equivalent to (4_2).

Lemma 4. *We have* $\inf\limits_{Q\in S_d(P)} L(Q) = \inf\limits_{Q\in T_d(P)} L(Q)$.

Proof. Let P be as in Definition 1. We shall prove by induction with respect to n that

$$\inf_{\substack{Q\in S_d(P)\\ \deg Q\leqslant n+d}} L(Q) = \inf_{\substack{Q\in T_d(P)\\ \deg Q\leqslant n+d}} L(Q). \tag{11}$$

If $n = 0$ then $Q \in S_d(P)$, $\deg Q \leqslant n + d$ implies $Q = P$. We shall show that $P \in T_d(P)$. Otherwise $P = x^{n_0} + \sum\limits_{i=1}^{d} a_i x^{n_i}$ $(n_i = d - i)$ and either $|C_0(P; n_0, \dots, n_{d-1})| = 0$, or $|C_1(P; n_1, \dots, n_d)| = 0$. In the former case there exists $[d_0, \dots, d_{d-1}] \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ such that

$$\sum_{j=0}^{d-1} c_{ij} d_j = 0 \quad (1 \leqslant i \leqslant d). \tag{12}$$

By Lemma 3

$$\sum_{j=0}^{d-1} d_j x^{n_j} \equiv 0 (\operatorname{mod} P) \tag{13}$$

and, since $n_0 = d$, $\sum\limits_{j=0}^{d-1} d_j x^{n_j} = d_0 P$; $d_0 \neq 0$, $P(0) = 0$, a contradiction. In the latter case, similarly, there exists $[e_1, \dots, e_d] \in \mathbb{C}^d \setminus \{\mathbf{0}\}$ such that

$$\sum_{j=1}^{d} c_{ij} e_j = 0 \quad (1 \leqslant i \leqslant d). \tag{14}$$

By Lemma 3

$$\sum_{j=1}^{d} e_j x^{n_j} \equiv 0 (\operatorname{mod} P), \tag{15}$$

which is impossible since $n_1 < n_0 = d = \deg P$.

Assume now that the equality (11) holds with n replaced by $n - 1$ and suppose that

$$\inf_{\substack{Q\in S_d(P)\\ \deg Q\leqslant n+d}} L(Q) = L(Q_0) \text{, where } Q_0 = x^{n_0} + \sum_{j=1}^{d} b_j x^{n_j} \in K(P)[x], \qquad n_0 > n_1 > \ldots > n_d \geqslant 0. \tag{16}$$

Clearly $L(Q_0) \leqslant L(P)$.

If $n_0 < n + d$ the inductive assertion follows immediately from the inductive assumption. If $n_0 = n + d$, let $L_i(x)$ be the linear forms defined in the proof of

Theorem 1 and a_i have the meaning of that proof, if $i \leqslant d$, $a_i = 0$ otherwise. We have

$$L\,(Q_0) = \inf_{\boldsymbol{x} \in \mathbb{R}^n} \sum_{i=1}^{n+d} |L_i(\boldsymbol{x}) + a_i|\,,$$

hence, by Lemma 2, the above infimum is attained in a point $\boldsymbol{x}_0$, such that for n indices $i_1, \ldots, i_n$ simultaneously $L_{i_j}(\boldsymbol{x}_0) + a_{ij} = 0$ and $L_{i_1}, \ldots, L_{i_n}$ are linearly independent. Since the system of equations $L_{i_j}(\boldsymbol{x}_0) + a_{i_j} = 0$ $(1 \leqslant j \leqslant n)$ determines $\boldsymbol{x}_0$ uniquely the coefficients of x^{n+d-i} in Q, where $i \neq i_1, \ldots, i_n$ (hence $n + d - i = n_1, n_2, \ldots, n_d$) are uniquely determined by the condition $Q \equiv 0(\operatorname{mod} P)$, Q monic in $\mathbb{C}[x]$. On the other hand, if $|C_0(P; n_0, \ldots, n_{d-1})| = 0$, then there exists $[d_0, \ldots, d_{d-1}] \in \mathbb{C}^d \smallsetminus \{\mathbf{0}\}$ such that (12) and (13) hold again. If $d_0 = 0$, then

$$Q_1 := Q_0 + \sum_{i=1}^{d-1} d_j x^{n_j} \equiv 0(\operatorname{mod} P),$$

where the Q_1 is again monic, contrary to the uniqueness property. If $d_0 \neq 0$, then by the uniqueness property

$$Q_0 = d_0^{-1} \sum_{j=1}^{d-1} d_j x^{n_j} = x^{n_{d-1}} Q_2, \quad Q_2 \in K(P)[x],$$

hence

$$L\,(Q_0) = L\,(Q_2)\,, \quad \deg Q_2 < n + d$$

and, by the inductive assumption

$$L\,(Q_0) = \inf_{\substack{Q \in S_d(P) \\ \deg Q < n+d}} L(Q) = \inf_{\substack{Q \in T_d(P) \\ \deg Q < n+d}} L(Q) \geqslant \inf_{\substack{Q \in T_d(P) \\ \deg Q \leqslant n+d}} L(Q).$$

By (16) this gives (11).

If $|C_1(P; n_1, \ldots, n_d)| = 0$, then there exists $[e_1, \ldots, e_d] \in \mathbb{C}^d \smallsetminus \{\mathbf{0}\}$ such that (14) and (15) hold again. We have

$$Q_3 := Q_0 + \sum_{j=1}^{d} e_j x^{n_j} \equiv 0(\operatorname{mod} P)$$

and Q_3 is again monic, contrary to the uniqueness property.

In the remaining case

$$|C_0\,(P; n_0, \ldots, n_{d-1})| \neq 0 \neq |C_1\,(P; n_1, \ldots, n_d)|$$

we have $Q_0 \in T_d(P)$, hence (11) holds again.

Lemma 5. *Let in the notation of Definition* 1, $i = m_1 + \ldots + m_{s(i)-1} + g(i)$, $1 \leqslant g(i) \leqslant m_{s(i)}$. *Then for every* $j \geqslant h \geqslant g(i) - 1$

$$|c_{ij}| \leqslant |c_{ih}| \max\left\{1, \frac{g(i)-1}{\log\left|\alpha_{s(i)}\right|}\right\}^{g(i)-1}. \tag{17}$$

Proof. For the sake of brevity, put $s(i) = s$, $g(i) = g$. For $g = 1$ we have $|c_{ij}| = |\alpha_s^{n_j}| \leqslant |\alpha_s^{n_h}| = |c_{ih}|$. Assume $g > 1$. For every $f \leqslant g - 2$ the function

$$\varphi(x) = \max\left\{1, \frac{g-1}{\log|\alpha_s|}\right\} |\alpha_s|^{\frac{n_h - x}{g-1}} - \frac{n_f - x}{n_f - n_h}$$

satisfies $\varphi(n_h) \geqslant 0$, $\varphi'(x) \leqslant 0$ for $x \leqslant n_h$. Hence $\varphi(n_j) \geqslant 0$,

$$\max\left\{1, \frac{g-1}{\log|\alpha_s|}\right\} |\alpha_s|^{\frac{n_h}{g-1}} (n_f - n_h) \geqslant |\alpha_s|^{\frac{n_j}{g-1}} (n_f - n_j)$$

and (17) follows on taking products over f from 0 to $g-2$.

Lemma 6. *Let* $a, b, c, x \in \mathbb{R}$, $a > 1$, $b \geqslant 0$, $c > 0$, $x > 0$. *If*

$$a^x / x^b \leqslant c, \tag{18}$$

then

$$x \leqslant \left(\frac{2b}{e \log a} + \sqrt{\frac{b^2}{e^2 (\log a)^2} + \frac{\log c}{\log a}}\right)^2 =: \psi(a, b, c). \tag{19}$$

The function ψ *is decreasing in* a, *increasing in* b *and* c.

Proof. Put $x = y^2$, $y > 0$. It follows from (18) that

$$y^2 \log a - 2b \log y \leqslant \log c$$

and, since $\log y \leqslant y/e$

$$y^2 \log a - \frac{2b}{e} y \leqslant \log c.$$

Solving this inequality for y and squaring we obtain (19).

Lemma 7. *For every subset* I *of* $\{1, \ldots, d\}$ *of cardinality* h *we have*

$$\left|\det\left(c_{ij}\right)_{\substack{i \in I \\ 1 \leqslant j \leqslant h}}\right| \leqslant h^{\frac{h}{2}} \prod_{i \in I} \left|\alpha_{s(i)}\right|^{n_{\max\{1, g(i)-1\}}} \prod_{f=0}^{g(i)-2} (n_f - n_h) \tag{20}$$

and

$$\left|\det\left(c_{ij}\right)_{\substack{i \in I \\ 0 \leqslant j < h}}\right| \leqslant h^{\frac{h}{2}} \prod_{i \in I} \left|\alpha_{s(i)}\right|^{n_{g(i)-1}} \prod_{f=0}^{g(i)-2} (n_f - n_{h-1}). \tag{21}$$

Proof. For all $i \in I$ and $j \leqslant g(i)-2$ we have $c_{ij}=0$, while for $j > g(i)-2$

$$|c_{ij}| = \left|\alpha_{s(i)}\right|^{n_j} \prod_{f=0}^{g(i)-2}(n_f - n_h) \leqslant \begin{cases} \left|\alpha_{s(i)}\right|^{n_{\max\{1,g(i)-1\}}} \prod_{f=0}^{g(i)-2}(n_f - n_h) & \text{if } 1 \leqslant j \leqslant h, \\ \left|\alpha_{s(i)}\right|^{n_{g(i)-1}} \prod_{f=0}^{g(i)-2}(n_f - n_{h-1}) & \text{if } 0 \leqslant j < h, \end{cases}$$

hence (20) and (21) follow by Hadamard's inequality. Note that if $g(i) > h+1$ or $g(i) > h$ for $i \in I$, then both sides of (20) or (21), respectively, are zero.

Definition 3. In the notation of Definition 1 and of Lemma 5 put for a positive integer $h < d$, positive integers $e_1, \dots, e_h$ and a subset J of $\{1, \dots, d\}$ of cardinality $h+1$ such that $\max_{i\in J} g(i) \leqslant h+1$

$$D(J; e_1, \dots, e_h) = \left|\det\left(\alpha_{s(i)}^{\sum_{\mu=j+1}^{h} e_\mu} \prod_{f=0}^{g(i)-2} \sum_{\nu=f+1}^{j} e_\nu\right)_{\substack{i\in J\\ 0\leqslant j\leqslant h}}\right| \\ \times \prod_{i\in J}\left|\alpha_{s(i)}\right|^{-\sum_{\mu=\max\{2,g(i)\}}^{h} e_\mu} \prod_{i\in J}\prod_{f=0}^{g(i)-2}\left(\sum_{\nu=f+1}^{h} e_\nu\right)^{-1}.$$

Definition 4. $D(e_1, \dots, e_h) = \max D(J; e_1, \dots, e_h)$, where the maximum is taken over all subsets of $\{1, \dots, d\}$ of cardinality $h+1$ such that $\max_{i\in J} g(i) \leqslant h+1$.

Remark. The definition is meaningful, since always there exists a subset J of $\{1, \dots, d\}$ with the required property. If for all $i \leqslant d$ we have $g(i) \leqslant h+1$ this is clear and if for some $i_0 : g(i_0) > h+1$ we take

$$J = \left\{i : m_1 + \ldots + m_{s(i_0)-1} < i \leqslant m_1 + \ldots + m_{s(i_0)-1} + h + 1\right\}.$$

Proof of Theorem 2. Using the notation of Definition 1 we define the sequence $d_1, \dots, d_d$ inductively as follows.

$$d_1 = \frac{\log\bigl(L(P)-1\bigr)}{\log|\alpha_1|} \tag{22}$$

and, if $d_1, \dots, d_h$ $(d > h \geqslant 1)$ are already defined, put

$$D_{h+1} = (h+1)^{-1} h^{\frac{h}{2}} \min\left\{D(e_1, \dots, e_h) : 1 \leqslant e_i \leqslant d_i, D(e_1, \dots, e_h) > 0\right\} \tag{23}$$

(the minimum over an empty set being ∞), $m = \max_{1 \leqslant s \leqslant r} m_s$,

$$d_{h+1} = \begin{cases} \max\Bigg\{ d_1 + \ldots + d_h, \\ \psi\left(|\alpha_r|, m-1, \left(\max\left\{2, \dfrac{2(m-1)}{\log|\alpha_r|}\right\}\right)^{m-1} D_{h+1}^{-1}\big(L(P)-1\big)\right)\Bigg\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } D_{h+1} \neq \infty, \\ 0 \text{ otherwise.} \end{cases} \tag{24}$$

We shall show that if $Q \in T_d(P)$, $Q = x^{n_0} + \sum_{j=1}^{d} b_j x^{n_j}$, then

$$n_{j-1} - n_j \leqslant d_j \quad (1 \leqslant j \leqslant d). \tag{25}$$

We proceed by induction on j. Since $Q \in T_d(P)$ the equation

$$\alpha_1^{n_0} + \sum_{j=1}^{d} b_j \alpha_1^{n_j} = 0$$

implies

$$|\alpha_1|^{n_0} \leqslant |\alpha_1|^{n_1} \sum_{j=1}^{d} |b_j| \leqslant |\alpha_1|^{n_1} \big(L(Q)-1\big) \leqslant |\alpha_1|^{n_1} \big(L(P)-1\big),$$

which, in view of (22) gives (25) for $j = 1$. Assume now that (25) holds for all $j \leqslant h$ $(h < d)$ and consider the matrix $(c_{ij})_{\substack{1 \leqslant i \leqslant d \\ 0 \leqslant j \leqslant h}}$ for c_{ij} defined in Definition 1. Since $Q \in T_d(P)$ we have

$$\operatorname{rank}\ (c_{ij})_{\substack{1 \leqslant i \leqslant d \\ 0 \leqslant j \leqslant h}} = h + 1,$$

hence also

$$\operatorname{rank}\ \left(c_{ij} \alpha_{s(i)}^{-n_h}\right)_{\substack{1 \leqslant i \leqslant d \\ 0 \leqslant j \leqslant h}} = h + 1.$$

Therefore, there exists a subset J of $\{1, \ldots, d\}$ of cardinality $h+1$ such that

$$\Delta(J) = \det\left(c_{ij} \alpha_{s(i)}^{-n_h}\right)_{\substack{i \in J \\ 0 \leqslant j \leqslant h}} \neq 0. \tag{26}$$

For every subset J with the above property consider

$$M(J) = \max_{i \in J} \left| \left(c_{i0} + \sum_{j=1}^{h} c_{ij} b_j \right) \alpha_{s(i)}^{-n_h} \right|.$$

Solving the system of equations

$$\left(c_{i0}x_0+\sum_{j=1}^{h}c_{ij}x_j\right)\alpha_{s(i)}^{-n_h}=\left(c_{i0}+\sum_{j=1}^{h}c_{ij}b_j\right)\alpha_{s(i)}^{-n_h}\quad (i\in J)$$

by means of Cramer's formulae and developing the numerator according to the first column we obtain

$$1\leqslant\frac{(h+1)M(J)\max\left|\det\left(c_{ij}\alpha_{s(i)}^{-n_h}\right)_{\substack{i\in I\\1\leqslant j\leqslant h}}\right|}{\left|\Delta(J)\right|},$$

where the maximum is taken over all subsets I of J of cardinality h. Now, by Lemma 7, since $|\alpha_{s(i)}|\geqslant 1$

$$\max\left|\det\left(c_{ij}\alpha_{s(i)}^{-n_h}\right)_{\substack{i\in I\\1\leqslant j\leqslant h}}\right|\geqslant h^{\frac{-h}{2}}\prod_{i\in J}\left|\alpha_{s(i)}\right|^{n_{\max\{1,g(i)-1\}}}\prod_{f=0}^{g(i)-2}\left(n_f-n_h\right).$$

This gives, by Definitions 3 and 4, for every J satisfying (26)

$$M(J)\geqslant(h+1)^{-1}h^{-\frac{h}{2}}D\left(J;n_0-n_1,\dots,n_{h-1}-n_h\right)>0$$

and, since such J exist

$$\max{}^{*}M(J)\geqslant(h+1)^{-1}h^{-\frac{h}{2}}D\left(J;n_0-n_1,\dots,n_{h-1}-n_h\right)<\infty,$$

where $\max^*$ is taken over all subsets J of $\{1,\dots,d\}$ such that $\operatorname{card}J=h+1$ and $\max\limits_{i\in J}g(i)\leqslant h+1$.

By the inductive assumption and (23)

$$\max{}^{*}M(J)\geqslant D_{h+1}>0,$$

thus there exists a set $J_0\subset\{1,\dots,d\}$ such that

$$\operatorname{card}J_0=h+1,\quad \max_{i\in J_0}g(i)\leqslant h+1\quad\text{and}$$

$$M\left(J_0\right)\geqslant D_{h+1}. \tag{27}$$

On the other hand, by Lemma 3

$$c_{i0}+\sum_{j=1}^{d}c_{ij}b_j=0\quad\left(i\in J_0\right),$$

hence

$$\left|\left(c_{i0}+\sum_{j=1}^{h} c_{ij}b_j\right)\alpha_{s(i)}^{-n_h}\right|\cdot\left|\alpha_{s(i)}\right|^{n_h}=\left|\sum_{j=h+1}^{d} c_{ij}b_j\right|. \tag{28}$$

By (27) for a certain $i_0\in J_0$ the left-hand side is at least $D_{h+1}|\alpha_{s(i)}^{n_h}|$. As to the right-hand side, replacing in Lemma 5 h by $h+1$, we obtain

$$\begin{aligned}\left|\sum_{j=h+1}^{d} c_{i_0 j}b_j\right| &\leqslant |c_{i_0,h+1}|\left(\max\left\{1,\frac{g(i_0)-1}{\log\left|\alpha_{s(i_0)}\right|}\right\}\right)^{g(i_0)-1}\sum_{j=h+1}^{d}|b_j|\\ &\leqslant |c_{i_0,h+1}|\left(\max\left\{1,\frac{m_{s(i_0)}-1}{\log\left|\alpha_{s(i_0)}\right|}\right\}\right)^{m_{s(i_0)}-1}\big(L(P)-1\big).\end{aligned} \tag{29}$$

If $n_h-n_{h+1}\leqslant n_0-n_h$, we obtain $n_h-n_{h+1}\leqslant d_1+\ldots+d_h\leqslant d_{h+1}$, hence the inductive assertion holds. If $n_h-n_{h+1}>n_0-n_h$, then

$$\begin{aligned}|c_{i_0,h+1}| &= \left|\alpha_{s(i_0)}\right|^{n_{h+1}}\prod_{f=0}^{g(i_0)-2}(n_f-n_{h+1})\\ &\leqslant \left|\alpha_{s(i_0)}\right|^{n_{h+1}}\Big(2\,(n_h-n_{h+1})\Big)^{g(i_0)-1}\\ &\leqslant \left|\alpha_{s(i_0)}\right|^{n_{h+1}}2^{m_{s(i_0)}-1}\,(n_h-n_{h+1})^{m_{s(i_0)}-1}.\end{aligned} \tag{30}$$

Combining this inequality with (28) and (29) we obtain

$$\frac{D_{h+1}\left|\alpha_{s(i_0)}\right|^{n_h-n_{h+1}}}{(n_h-n_{h+1})^{m_{s(i_0)}-1}}\leqslant\left(\max\left\{2,\frac{2\left(m_{s(i_0)}-1\right)}{\log\left|\alpha_{s(i_0)}\right|}\right\}\right)^{m_{s(i_0)}-1}\big(L(P)-1\big), \tag{31}$$

hence, by Lemma 6,

$$\begin{aligned}&n_h-n_{h+1}\\ &\leqslant\max_{1\leqslant s\leqslant r}\psi\left(|\alpha_s|\,,\,m_s-1,\left(\max\left\{2,\frac{2\,(m_s-1)}{\log|\alpha_s|}\right\}\right)^{m_s-1}D_{h+1}^{-1}\big(L(P)-1\big)\right)\\ &\leqslant\psi\left(|\alpha_r|\,,\,m-1,\left(\max\left\{2,\frac{2\,(m-1)}{\log|\alpha_r|}\right\}\right)^{m-1}D_{h+1}^{-1}\big(L(P)-1\big)\right)\leqslant d_{h+1}.\end{aligned}$$

The inductive assertion being proved, it follows that

$$n_0-n_d\leqslant\sum_{h=1}^{d}d_h.$$

However, $n_d = 0$, hence
$$l(P) = \inf_{Q\in U_d(P)} L(Q),$$
where $U_d(P) = \left\{Q \in T_d(P) : \deg Q \leqslant \sum_{h=1}^{d} d_h\right\}$.

The set $U_d(P)$ is finite and effectively computable, since for $Q = x^{n_0} + \sum_{j=1}^{d} b_j x^{n_j} \in U_d(P)$ there are only finitely many choices for $\langle n_0, \dots, n_d\rangle$ and for each choice the coefficients b_j are determined uniquely and are effectively computable. Moreover, $Q \in T_d(P)$ implies $Q \in K(P)[x]$, hence $L(Q) \in K(P)$. The theorem follows.

Proof of Corollary 1. If $P(x) = a_0 \prod_{i=1}^{c}(x-\alpha_i) \prod_{i=c+1}^{d}(x-\alpha_i)$, where $|\alpha_i| > 1$ for $i \leqslant c$, $|\alpha_i| < 1$ for $i > c$, then by Proposition A
$$l(P) = |a_0|\, l\left(\prod_{i=1}^{c}(x-\alpha_i)\right)$$
and the right hand side is effectively computable by Theorem 2.

For the proof of Theorem 3 we need two lemmas.

Lemma 8. *If* $P_n \in \mathbb{R}[x]$, $p_n, q_n \in \mathbb{N}\cup\{0\}$ $(n = 0, 1, \dots)$ *and*
$$\liminf_{n\to\infty} L\left(P_n(x) - P_0\left(x^{p_n}\right)x^{q_n}\right) = 0, \tag{32}$$
then
$$\liminf_{n\to\infty} l\left(P_n\right) \leqslant l\left(P_0\right). \tag{33}$$

Proof. By definition of $l(P_0)$ for every n there exists G_n monic such that
$$L\left(P_0 G_n\right) \leqslant l\left(P_0\right) + \frac{1}{n}.$$
By (32) there exists $k_n \in \mathbb{N}$ such that $k_n > n$ and
$$L\left(P_{k_n}(x) - P_0\left(x^{p_{k_n}}\right)x^{q_{k_n}}\right) \leqslant \frac{1}{nL\left(G_n\right)}.$$
Hence
$$\begin{aligned} L\left(P_{k_n}(x)G_n\left(x^{p_{k_n}}\right)\right) &\leqslant L\left(P_0\left(x^{p_{k_n}}\right)x^{q_{k_n}}G_n\left(x^{p_{k_n}}\right)\right)\\ &\quad + L\left(\left(P_{k_n}(x) - P_0\left(x^{p_{k_n}}\right)x^{q_{k_n}}\right)G_n\left(x^{p_{k_n}}\right)\right) \leqslant L\left(P_0G_n\right)\\ &\quad + L\left(P_{k_n}(x) - P_0\left(x^{p_{k_n}}\right)x^{q_{k_n}}\right)L\left(G_n\right) \leqslant l\left(P_0\right) + \frac{2}{n}, \end{aligned}$$
thus
$$l\left(P_{k_n}\right) \leqslant l\left(P_0\right) + \frac{2}{n}.$$
This implies (33).

Remark. The equality $\lim\limits_{n\to\infty} L(P_n - P_0) = 0$ does not imply $\liminf\limits_{n\to\infty} l(P_n) = l(P_0)$, as is shown by the example $P_n = x - \frac{n-1}{n}$, $P_0 = x - 1$, see Proposition A, (ii) and (iii).

Lemma 9. *Let Q be a monic polynomial, irreducible over $\mathbb{R}$ of degree $d \leqslant 2$ with the zeros on the unit circle. There exists a sequence of monic polynomials R_n such that*

$$Q^2 \mid R_n \tag{34}$$

and

$$\lim_{n\to\infty} L\left(R_n - x^{dn}Q\right) = 0. \tag{35}$$

Proof. It suffices to take

$$R_n = x^{x+1} - \varepsilon\left(1 + \frac{1}{n}\right)x^n + \frac{\varepsilon^{n+1}}{n}, \quad \text{if} \quad d = 1, Q = x - \varepsilon$$

and

$$R_n = \left(x^{n+1} - \zeta\left(1 + \frac{1}{n}\right)x^n + \frac{\zeta^{n+1}}{n}\right)\left(x^{n+1} - \overline{\zeta}\left(1 + \frac{1}{n}\right)x^n + \frac{\overline{\zeta}^{n+1}}{n}\right),$$
$$\text{if} \quad d = 2, \quad Q = (x - \zeta)(x - \overline{\zeta}).$$

Indeed, we have for every ε

$$(x - \varepsilon)^2 \mid x^{n+1} - \varepsilon\left(1 + \frac{1}{n}\right)x^n + \frac{\varepsilon^{n+1}}{n},$$

which implies (34) and

$$L\left(R_n - x^{dn}Q\right) \leqslant \begin{cases} 2/n & \text{if } d = 1 \\ (8n+4)/n^2 & \text{if } d = 2, \end{cases}$$

which implies (35).

Proof of Theorem 3. We proceed by induction with respect to the number N of irreducible factors of Q^{m-1} counted with multiplicities. If $N = 1$, then $m = 2$, Q is irreducible and by Lemma 9 we have

$$PQ^2 \mid PR_n \tag{36}$$

and

$$\lim_{n\to\infty} L\left(PR_n - x^{dn}PQ\right) = 0. \tag{37}$$

By (37) and Lemma 8 we have

$$\liminf_{n\to\infty} l\left(PR_n\right) \leqslant l(PQ).$$

However, by Proposition (i) and (36)

$$l(PQ) \leqslant l\left(PQ^2\right) \leqslant l\left(PR_n\right),$$

hence

$$l(PQ) \leqslant l\left(PQ^2\right) \leqslant l\left(PQ\right),$$

which gives the theorem for $N+1$.

Assume now that the number of irreducible factors of Q^{m-1} is $N>1$ and the theorem is true for the number of irreducible factors less than N. If $m>2$ then the number of irreducible factors of Q^{m-2} and of Q is less than N, hence applying the inductive assumption with P replaced first by PQ we obtain

$$l\left(PQ^m\right) = l\left(PQ^2\right) = l(PQ).$$

If $m=2$ and the number of irreducible factors of Q^{m-1} is $N>1$, then Q is reducible, $Q=Q_1Q_2$, where $\deg Q_i > 1$ $(i=1,2)$. The number of irreducible factors of Q_i is less than N, hence applying the inductive assumption with P replaced first by PQ_1^2 and then by PQ_2 we obtain

$$l\left(PQ^2\right) = l\left(PQ_1^2Q_2^2\right) = l\left(PQ_1^2Q_2\right) = l\left(PQ_1Q_2\right) = l(PQ).$$

The inductive proof is complete.

Proof of Theorem 4. By Theorem 3 and Proposition A (ii) we have for $d \in \mathbb{N}$

$$l\left((x-1)^d\right) = l(x-1) = 2. \tag{38}$$

Now, let

$$P(x) = \prod_{j=1}^{d}\left(x - \exp 2\pi i r_j\right), \quad \text{where } r_j \in \mathbb{R}.$$

By Dirichlet's approximation theorem for every positive integer n there exists a positive integer p_n such that

$$\|p_n r_j\| \leqslant \frac{1}{2\pi n} \quad (1 \leqslant j \leqslant d),$$

hence

$$|\exp 2\pi i p_n r_j - 1| < \frac{1}{n}.$$

It follows that the polynomial

$$Q_n(x) = \prod_{j=1}^{d}\left(x^{p_n} - \exp 2\pi i p_n r_j\right)$$

satisfies

$$P \mid Q_n \tag{39}$$

and

$$L\left(Q_n - (x^{p_n}-1)^d\right) \leqslant \left(2+\frac{1}{n}\right)^d - 2^d. \tag{40}$$

Now, (39) implies by Proposition (i)

$$l(P) \leqslant \liminf_{n\to\infty} l\left(Q_n\right),$$

while (40), Lemma 8 and (38) imply

$$\liminf_{n\to\infty} l\left(Q_n\right) \leqslant l\left((x-1)^d\right) = 2.$$

Hence $l(P) \leqslant 2$. On the other hand, if $P \mid Q$, $Q = x^n + \sum\limits_{j=1}^{n} b_j x^{n-j}$, then for a zero α of P we have

$$1 = |\alpha|^n = \left|\sum_{j=1}^{n} b_j x^{n-j}\right| \leqslant \sum_{j=1}^{n} |b_j| = L(Q) - 1,$$

hence $L(Q) \geqslant 2$; so

$$l(P) \geqslant 2,$$

which gives the first part of the theorem. In order to prove the second part assume that $P \mid Q$, Q monic and $L(Q) = 2$. Let

$$Q = x^n + \sum_{j=1}^{n} b_j x^{n-j}, \quad b_n \neq 0.$$

For every zero α of P we have

$$\alpha^n + \sum_{j=1}^{n} b_j \alpha^{n-j} = 0, \tag{41}$$

hence

$$\left|\sum_{i=1}^{n} b_i \alpha^{n-i}\right| = |\alpha^n| = 1 = \sum_{i=1}^{n} |b_i|.$$

It follows that for every j with $b_j \neq 0$

$$\arg b_j \alpha^{n-j} = \arg b_n.$$

Since $\arg b_i = 0$ or π, either α is a root of unity, or $b_j = 0$ for all $j < n$. However the latter case, by virtue of (41) leads to the former. Suppose now that α is a multiple zero of P, hence also of Q. Then

$$n\alpha^{n-1} + \sum_{j=1}^{n-1} b_j (n-j) \alpha^{n-j-1} = 0,$$

hence

$$\left|\sum_{j=1}^{n-1} b_j(n-j)\alpha^{n-j-1}\right| = |n\alpha^n| = n > \sum_{j=1}^{n-1} |b_j|\, n,$$

which is impossible, since for each $j < n$, $|b_j(n-j)\alpha^{n-j-1}| \leqslant |b_j|n$. Thus all zeros of P are roots of unity and simple. If this condition is satisfied, then $P \mid x^m - 1$, where m is the least common multiple of orders of the roots of unity in question and

$$L\left(x^m - 1\right) = 2.$$

For the proof of Theorem 5 we need seven lemmas.

Lemma 10. *Let $d > 2$, I be a subset of $\{1, 2, \ldots, d-1\}$ and J a subset of $\{0, \ldots, d-2\}$ both of cardinality $d-2$. Then*

$$\left|\det\left(c_{ij}\right)_{\substack{i\in I\\ j\in J}}\right| \leqslant (d-2)^{\frac{d-2}{2}} \prod_{i\in I} \left|\alpha_{s(i)}\right|^{n_{g(i)-1}} \prod_{f=0}^{g(i)-2} \left(n_f - n_{d-2}\right).$$

Proof. is similar to the proof of Lemma 7.

Lemma 11. *Under the assumptions of Theorem 5 we have, in the notation of Definition 1, for every $h \leqslant d-e$ and $\nu = 0, 1$*

$$D_{h\nu} = \det\left(c_{ij}\right)_{\substack{1\leqslant i\leqslant h\\ \nu\leqslant j<h+\nu}} \neq 0. \tag{42}$$

Proof. In the notation of Definition 1 we have $|\alpha_s| > 1$ for $s < r$, $\alpha_r = \varepsilon$, $m_r = e$. Assume that $\alpha_s \in Z_0$, if and only if $s \in S_0$. In the notation of Lemma 5

$$h = m_1 + \ldots + m_{s(h)-1} + g(h), \quad 1 \leqslant g(h) \leqslant m_{s(h)}.$$

If $\nu = 0$ or 1, $D_{h\nu} = 0$ and $\alpha_s \in Z_0$ for all $s \leqslant s(h)$, the system of equations

$$\sum_{j=\nu}^{h-1+\nu} c_{ij}x_j = 0 \qquad (1 \leqslant i \leqslant h)$$

has a solution $\langle x_\nu, \ldots, x_{h-1+\nu}\rangle \in \mathbb{R}^h \setminus \{\mathbf{0}\}$. It follows by Lemma 3 that

$$\sum_{j=\nu}^{h-1+\nu} x_j x^{n_j} \equiv 0 \left(\bmod \prod_{s=1}^{s(h)-1} \left(x-\alpha_s\right)^{m_s} \left(x-\alpha_{s(h)}\right)^{g(h)}\right)$$

hence the polynomial $\sum\limits_{j=\nu}^{h-1+\nu} x_j x^{n_j} \in \mathbb{R}[x]$ has h zeros of the same sign, counted with multiplicities. This, however, contradicts the Descartes rule of signs (see [5], Satz 12), hence, if $\alpha_s \in Z_0$ $(s \leqslant s(h))$ (42) holds. It also shows that $D_{h\nu}$ as a polynomial in α_s $(s \notin S_0)$ is not identically zero for any fixed $\alpha_s \in Z_0$. Since the coefficients of the polynomial in question belong to $\mathbb{Q}(Z_0)$, the algebraic independence of α_s $(s \notin S_0)$ over $\mathbb{Q}(Z_0)$ implies $D_{h\nu} \neq 0$.

Lemma 12. *Under the assumptions of Theorem* 5 *let* P_0 *be of degree* $d-e$. *For all positive integers* $e_1,\dots,e_{d-e}$ *there exists a unique polynomial* $Q=Q(P_0;e_1,\dots,e_{d-e})$ *such that*

$$Q = x^{\sum\limits_{\mu=1}^{d-e} e_\mu} + \sum_{j=1}^{d-e} b_j x^{\sum\limits_{\mu=j+1}^{d-e} e_\mu}$$

and

$$Q \equiv 0 \pmod{P_0}. \tag{43}$$

Moreover, $Q\in\mathbb{R}[x]$.

Proof. For $j=0,\dots,d-e$ put $n_j=\sum\limits_{\nu=j+1}^{d-e} e_\nu$ and for $i\leqslant d-e$, let c_{ij} be defined by Definition 1 with P replaced by P_0. By Lemma 3 the congruence (43) is equivalent to

$$\sum_{j=1}^{d-e} c_{ij} b_j = -c_{i0} \quad (1\leqslant i\leqslant d-e).$$

By Lemma 11 with $h=d-e$ and $\nu=1$ the determinant of this system is non-zero, hence b_j are uniquely determined. If we replace c_{ij} by $\overline{c}_{ij}$ we obtain the same system of equations, hence $b_j\in\mathbb{R}$.

Lemma 13. *For every positive integer* $h<d-e$ *and all positive integers* $e_1,\dots,e_h$ *we have in the notation of Definition* 3

$$D\left(\{1,\dots,h+1\},e_1,\dots,e_h\right)>0.$$

Proof. We have

$$\max_{i\leqslant h+1} g(i) \leqslant \max_{i\leqslant h+1} i = h+1,$$

hence $D(\{1,\dots,h+1\},e_1,\dots,e_h)$ is defined. Its only factor, which could possibly vanish is

$$\det\left(\alpha_{s(i)}^{\sum\limits_{\mu=j+1}^{h} e_\mu} \prod_{f=0}^{g(i)-2} \sum_{\nu=f+1}^{j} e_\nu\right)_{\substack{1\leqslant i\leqslant h+1\\ 0\leqslant j\leqslant h}} = \det\left(c_{ij}\alpha_{s(i)}^{-n_h}\right)_{\substack{1\leqslant i\leqslant h+1\\ 0\leqslant j\leqslant h}},$$

where $n_j=\sum\limits_{\mu=j+1}^{d} e_\mu$. By (42) with $\nu=0$ the above determinant is non-zero.

Definition 5. Let the sequence $d_i (1 \leqslant i \leqslant d-e)$ be defined inductively as follows.

$$d_1 = \frac{\log\bigl(L(P)-1\bigr)}{\log|\alpha_1|} \tag{44}$$

and if $d_1, \ldots, d_h \ (h < d-e)$ are already defined

$$D_{h+1} = (h+1)^{-1} h^{\frac{-h}{2}} \min\{D\left(\{1,\ldots,h+1\}, e_1,\ldots,e_h\right) : 1 \leqslant e_i \leqslant d_i\}\,, \tag{45}$$

$$d_{h+1} = \max\Biggl\{ d_1, \ldots, d_h, \tag{46}$$

$$\psi\left(\left|\alpha_{s(i)}\right|, m-1, \left(\max\left\{2, \frac{2(m-1)}{\log|\alpha_s(h+1)|}\right\}\right)^{m-1}\right) D_{h+1}^{-1}\bigl(L(P)-1\bigr)\Biggr\}.$$

Lemma 14. *For every* $Q \in T_d(P)$, $Q = x^{n_0} + \sum\limits_{j=1}^{d} b_j x^{n_j}$ *we have*

$$n_{j-1} - n_j \leqslant d_j \quad (1 \leqslant j < d). \tag{47}$$

Proof is by induction on j. Since $Q \in T_d(P)$ the equation

$$\alpha_1^{n_0} + \sum_{j=1}^{d} b_j \alpha_1^{n_j} = 0$$

implies

$$|\alpha_1|^{n_0} \leqslant |\alpha_1|^{n_1} \sum_{j=1}^{d} |b_j| \leqslant |\alpha_1|^{n_1} \bigl(L(Q)-1\bigr) \leqslant |\alpha_1|^{n_1} \bigl(L(P)-1\bigr),$$

which, in view of (44) gives (47) for $j=1$. Assume now that (47) holds for all $j \leqslant h \ (h < d-1)$. By Lemma 11 we have

$$\Delta = \det\left(c_{ij}\alpha_{s(i)}^{-n_h}\right)_{\substack{1\leqslant i\leqslant h+1\\ 0\leqslant j\leqslant h}} \neq 0.$$

Let

$$M = \max_{1\leqslant i\leqslant h+1} \left|\left(c_{i0} + \sum_{j=1}^{h} c_{ij} b_j\right) \alpha_{s(i)}^{-n_h}\right|.$$

Solving the system of equations

$$\left(c_{i0}x_0 + \sum_{j=1}^{h} c_{ij}x_j\right)\alpha_{s(i)}^{-n_h} = \left(c_{i0} + \sum_{j=1}^{h} c_{ij}b_j\right)\alpha_{s(i)}^{-n_h} \qquad (1 \leqslant i \leqslant h+1)$$

by means of Cramer's formulae and developing the numerator according to the first column we obtain

$$1 \leqslant \frac{(h+1)M \max \left|\det\left(c_{ij}\alpha_{s(i)}^{-n_h}\right)_{\substack{i\in I\\ 1\leqslant j\leqslant h}}\right|}{|\Delta|},$$

where the maximum is taken over all subsets I of $\{1,\ldots,h+1\}$ of cardinality h. Now, by Lemma 7, since $|\alpha_{s(i)}|>1$, we have

$$\max \left|\det\left(c_{ij}\alpha_{s(i)}^{-n_h}\right)_{\substack{i\in I\\ 1\leqslant j\leqslant h}}\right| \leqslant h^{\frac{h}{2}} \prod_{i=1}^{h+1} \left|\alpha_{s(i)}\right|^{n_{\max\{1,g(i)-1\}}} \prod_{f=0}^{g(i)-2} (n_f - n_h)\,.$$

This gives, by Definition 3,

$$m \geqslant (h+1)^{-1} h^{-\frac{1}{2}} D\left(\{1,\ldots,h+1\}, n_0-n_1,\ldots,n_{h-1}-n_h\right)$$

and by the inductive assumption and (45)

$$M \geqslant D_{h+1}. \tag{48}$$

On the other hand, by Lemma 3

$$c_{i0} + \sum_{j=1}^{d} c_{ij} b_j = 0 \qquad (1 \leqslant i \leqslant h+1),$$

hence

$$\left|\left(c_{i0} + \sum_{j=1}^{h} c_{ij} b_j\right) \alpha_{s(i)}^{-n_h}\right| \cdot \left|\alpha_{s(i)}\right|^{n_h} = \left|\sum_{j=1}^{h} c_{ij} b_j\right|. \tag{49}$$

By (48) for a certain $i_0 \leqslant h+1$ the left-hand side is at least $D_{h+1}|\alpha_{s(i)}|^{n_h}$. As to the right-hand side, by (29) we obtain

$$\left|\sum_{j=1}^{d} c_{i_0 j} b_j\right| \leqslant |c_{i_0,h+1}| \left(\max\left\{1, \frac{m_{s(i_0)-1}}{\log\left|\alpha_{s(i_0)}\right|}\right\}\right)^{m_{s(i_0)}-1} \bigl(L(P)-1\bigr). \tag{50}$$

If $n_h - n_{h+1} \leqslant n_0 - n_h$, we obtain $n_h - n_{h+1} \leqslant d_1 + \ldots + d_h \leqslant d_{h+1}$, hence the inductive assumption holds. If $n_h - n_{h+1} > n_0 - n_h$, then by (30)

$$|c_{i_0,h+1}| \leqslant \left|\alpha_{s(i_0)}\right|^{n_{h+1}} 2^{m_{s(i_0)}-1} \left(n_h - n_{h+1}\right)^{m_{s(i_0)}-1}.$$

Combining the inequality with (49) and (50) we obtain (31), where, however, D_{h+1} has the new meaning given by (45).

It follows, by Lemma 6

$$\begin{aligned} &n_h - n_{h+1} \\ &\leqslant \max_{1\leqslant s\leqslant s(h+1)} \psi\left(|\alpha_s|,\, m_s-1,\, \left(\max\left\{2, \frac{2(m_s-1)}{\log|\alpha_s|}\right\}\right)^{m_s-1} \times D_{h+1}^{-1}\bigl(L(P)-1\bigr)\right) \\ &\leqslant \psi\left(\alpha_{s(h+1)},\, m-1,\, \left(\max\left\{2, \frac{2(m-1)}{\log\left|\alpha_{s(h+1)}\right|}\right\}\right)^{m-1} D_{h+1}^{-1}\bigl(L(P)-1\bigr)\right) \leqslant d_{h+1} \end{aligned}$$

and the inductive proof is complete.

Definition 6. Assume that, under the assumptions of Theorem 5, $e=1$. Put for positive integers $n_1 > \ldots > n_{d-2} > n_{d-1}$

$$\left(c'_{ij}\right)_{\substack{1\leqslant i<d\\ 1\leqslant j\leqslant d}} = C\left(P_0; n_1, \ldots, n_{d-1}, 0\right), \quad c'_{dj} = 1 \quad (1 \leqslant j \leqslant d),$$

$$\begin{aligned} &E\left(P_0; n_1 - n_{d-1}, \ldots, n_{d-2} - n_{d-1}\right) \\ &= \left|\det\left(c'_{ij}\alpha_{s(i)}^{-n_{d-1}}\right)_{1\leqslant i,j<d}\right|^{-1} \prod_{i=1}^{d-1} \left|\alpha_{s(i)}\right|^{n_{g(i)}-n_{d-1}} \prod_{f=1}^{g(i)-1} (n_f - n_{d-1}). \end{aligned} \tag{51}$$

Remark. $E(P_0; n_1 - n_{d-1}, \ldots, n_{d-2} - n_{d-1})$ is well defined since $\det(c'_{ij}\alpha_{s(i)}^{-n_{d-1}})$ is non-zero by Lemma 11 with $h = d-1$, $\nu = 0$. Moreover the right-hand side of (51) depends only on P_0 and the differences $n_j - n_{d-1}$ $(1 \leqslant j \leqslant d-2)$.

Lemma 15. *Assume that, under the assumptions of Theorem* 5 *and in the notation of Definition* 1, $e=1$. *If for positive integers* $n_1 > \ldots > n_{d-1}$ *and for* $n > 1$, $a \in \mathbb{R}$

$$\begin{aligned} n_{d-1} > \max\Bigg\{ & n_1 - n_{d-1}, \psi\Bigg(\left|\alpha_{r-1}\right|, m-1, d(d-1)^{\frac{d+1}{2}} 2^{m-1} n \\ & \times E\left(P_0; n_1 - n_{d-1}, \ldots, n_{d-2} - n_{d-1}\right) \max\left\{1, \frac{nd|a|}{nd-1}\right\}\Bigg)\Bigg\}, \end{aligned} \tag{52}$$

then there exists a polynomial $R \in \mathbb{R}[x]$ *of degree at most* n_1 *such that*

$$P_0 \mid R(x) - a, \tag{53}$$

$$x - 1 \mid R(x), \tag{54}$$

$$L(R) < \frac{1}{n}. \tag{55}$$

Proof. Put

$$R(x) = \sum_{j=1}^{d-1} r_j x^{n_j} + r_d, \quad r_j \in \mathbb{C}.$$

By Lemma 3 the conditions (53) and (54) are equivalent to the following system of linear equations for r_j

$$\begin{aligned} &\sum_{j=1}^{d} c'_{ij} r_j = c'_{id} a \quad (1 \leqslant i < d) \\ &\sum_{j=1}^{d} c'_{dj} r_j = 0. \end{aligned} \tag{56}$$

The determinant of this system equals

$$\Delta_0 = \prod_{i=1}^{d} \alpha_{s(i)}^{n_{d-1}} \det\left(c'_{ij}\alpha_{s(i)}^{-n_{d-1}}\right)_{1\leqslant i,j\leqslant d}.$$

Developing the last determinant according to the last column we obtain

$$\begin{aligned}&\det\left(c'_{ij}\alpha_{s(i)}^{-n_{d-1}}\right)_{1\leqslant i,j\leqslant d}\\&= \det\left(c'_{ij}\alpha_{s(i)}^{-n_{d-1}}\right)_{1\leqslant i,j<d} + \sum_{k=1}^{d-1}(-1)^{k+d}c'_{kd}\alpha_{s(k)}^{-n_{d-1}} \det\left(c'_{ij}\alpha_{s(i)}^{-n_{d-1}}\right)_{\substack{i\neq k\\ j<d}},\end{aligned}$$

hence, by (21) with $h = d-1$, $I = \{1,\dots,d\}\smallsetminus\{k\}$ and by the condition $\alpha_r = e = 1$

$$\begin{aligned}&\left|\Delta_0 \prod_{i=1}^{d} \alpha_{s(i)}^{-n_{d-1}} - \det\left(c'_{ij}\alpha_{s(i)}^{-n_{d-1}}\right)_{1\leqslant i,j<d}\right| \qquad (57)\\&< (d-1)^{\frac{d+1}{2}} |\alpha_{r-1}|^{-n_{d-1}} \left(\prod_{i=1}^{d-1}\left|\alpha_{s(i)}\right|^{n_{g(i)}-n_{d-1}} \prod_{f=1}^{g(i)-1}(n_f - n_{d-1})\right) \max_{1\leqslant k<d}|c'_{kd}|.\end{aligned}$$

Since, by (52) $n_{d-1} > n_1 - n_{d-1}$, we have

$$\max_{1\leqslant k<d}|c'_{kd}| \leqslant \prod_{f=1}^{m-1} n_f \leqslant (2n_{d-1})^{m-1}. \qquad (58)$$

In view of Definition 6 the right-hand side of (57) does not exceed

$$\begin{aligned}&(d-1)^{\frac{d+1}{2}} |\alpha_{r-1}|^{-n_{d-1}} \left|\det\left(c'_{ij}\alpha_{s(i)}^{-n_{d-1}}\right)_{1\leqslant i,j<d}\right|\\&\quad\times E\left(P_0; n_1 - n_{d-1},\dots,n_{d-2}-n_{d-1},0\right) 2^{m-1} n_{d-1}^{m-1}.\end{aligned}$$

Since, by (52),

$$\begin{aligned}n_{d-1} > \psi\Big(&|\alpha_{r-1}|, m-1, d(d-1)^{\frac{d+1}{2}} 2^{m-1} n\\&\times E\left(P_0; n_1 - n_{d-1},\dots,n_{d-2}-n_{d-1},0\right)\Big)\end{aligned}$$

we have by Lemma 6 and (57)

$$|\Delta_0| > \left(1 - \frac{1}{dn}\right)\left|\det\left(c'_{ij}\right)_{1\leqslant i,j<d}\right|, \qquad (59)$$

hence by the Remark after Definition 6, $\Delta_0 \neq 0$. Thus the system (56) is uniquely solvable and since on replacing c'_{ij} by $\overline{c}'_{ij}$ we obtain the same system, r_j are real $(1 \leqslant j \leqslant d)$.

The determinant Δ_k obtained by substituting in $(c'_{ij})_{1\leqslant i,j\leqslant d}$ for the k-th column the column

$$\left[c'_{1d}, \dots, c'_{d-1\,d}, 0\right]^t a$$

satisfies for $k < d$

$$\Delta_k = \pm\left(\det c'_{ij}\right)_{\substack{i<d\\ j\neq k}} a,$$

hence developing the last determinant according to the last column, using Lemma 10, Definition 6 and (58) we obtain

$$\begin{aligned}|\Delta_k| &\leqslant |a| \sum_{l=1}^{d-1} |c'_{ld}|\,(d-2)^{\frac{d-2}{2}} \prod_{\substack{i=1\\ i\neq l}}^{d-1} \left|\alpha_{s(i)}\right|^{n_{g(i)}} \prod_{f=1}^{g(i)-1} (n_f - n_{d-1})\\ &\leqslant |a|(d-1)(d-2)^{\frac{d-2}{2}} \left(2n_{d-1}\right)^{m-1} |\alpha_{r-1}|^{-n_{d-1}} \left|\det\left(c'_{ij}\right)_{1\leqslant i,j<d}\right|\\ &\quad\times E\left(P_0; n_1 - n_{d-1}, \dots, n_{d-2} - n_{d-1}\right),\end{aligned}$$

where $(d-2)^{\frac{d-2}{2}} = 1$ for $d = 2$.

Since $(d-1)(d-2)^{\frac{d-2}{2}} < (d-1)^{\frac{d+1}{2}}$ we obtain, by virtue of (52),

$$|\Delta_k| < \frac{dn-1}{d^2n^2} \left|\det\left(c'_{ij}\right)_{1\leqslant i,j<d}\right|$$

hence, by (59), $r_k = \Delta_k/\Delta_0$ satisfies

$$|r_k| < \frac{1}{dn} \quad (1 \leqslant k < d). \tag{60}$$

It remains to consider $k = d$. In this case developing Δ_d according to the last column we obtain

$$|\Delta_d| \leqslant |a| \sum_{l=1}^{d-1} |c'_{ld}| \left|\det\left(c'_{ij}\right)_{\substack{i\neq l\\ j<d}}\right|.$$

Using (21) with $h = d-1$, $I = \{1, \dots, d\} \setminus \{l\}$, the condition $\alpha_r = e = 1$ and (58) we obtain

$$\begin{aligned}|\Delta_d| &\leqslant |a|(d-1)^{\frac{d+1}{2}} \left(2n_{d-1}\right)^{m-1} |\alpha_{r-1}|^{-n_{d-1}}\\ &\quad\times \prod_{i=1}^{d-1} \left|\alpha_{s(i)}\right|^{n_{g(i)}} \prod_{f=1}^{g(i)-1} (n_f - n_{d-1}) \leqslant (d-1)^{\frac{d+1}{2}} 2^{m-1} n_{d-1}^{m-1} |\alpha_{r-1}|^{-n_{d-1}}\\ &\quad\times \left|\det\left(c'_{ij}\right)_{1\leqslant i,j<d}\right| E\left(P_0; n_1 - n_{d-1}, \dots, n_{d-2} - n_{d-1}\right).\end{aligned}$$

Again, by virtue of (52) and of Lemma 6,

$$|\Delta_d| < \frac{dn-1}{d^2n^2}\left|\det\left(c'_{ij}\right)_{1\leqslant i,j<d}\right|,$$

hence $r_d = \Delta_d/\Delta_0$ satisfies

$$|r_d| < \frac{1}{dn}.$$

It follows now from (60) that

$$L(R) = \sum_{k=1}^{d} |r_k| < \frac{1}{n},$$

which proves (55).

Lemma 16. *Assume, under the assumptions of Theorem* 5, *that* $\varepsilon = e = 1$. *Then*

$$l(P) \leqslant \inf_{Q\in S_{d-1}(P_0)}\left\{L(Q)+\left|Q(1)\right|\right\}.$$

Proof. Let

$$Q = x^{q_0} + \sum_{j=1}^{d-1} b_j x^{q_j},$$

where $q_0 > q_1 > \ldots > q_{d-1}$ and we may assume $q_{d-1} = 0$. By Lemma 15 with $a = Q(1)$, $n_j = n_{d-1} + q_j$ $(1 \leqslant j < d)$, if

$$n_{d-1} > \max\Bigg\{q_1, \psi\Bigg(|\alpha_{r-1}|, m-1, d(d-1)^{\frac{d+1}{2}}2^{m-1}n \\ \times E\left(P_0; q_1,\ldots,q_{d-2}\right)\max\left\{1, \frac{nd}{nd-1}\left|Q(1)\right|\right\}\Bigg)\Bigg\}$$

there exists a polynomial $R \in \mathbb{R}[x]$ of degree at most n_1 satisfying (53)–(55). We consider the polynomial

$$S(x) = Q(x)x^{n_{d-1}} + R(x) - Q(1).$$

It follows from (53)–(54) that

$$P_0 \mid S, \quad x-1 \mid S, \quad \text{thus } P \mid S$$

and since S is monic

$$l(P) \leqslant L(S).$$

On the other hand, by (55)

$$L(S) \leqslant L(Q) + |Q(1)| + \frac{1}{n}.$$

Since n is arbitrary, the lemma follows.

Proof of Theorem 5. Since, by Proposition (iii), $l(P(-x)) = l(P(x))$, we may assume that $\varepsilon = 1$ and, by virtue of Theorem 3, that $e = 1$. Thus Lemmas 15 and 16 are applicable. The second part of the theorem follows from Lemma 16. In order to prove the first part we shall show that for every $n > 1$

$$\begin{aligned} 0 \geqslant l(P) - \overset{*}{\min}\min\{&L\left(Q\left(P; n_0, \ldots, n_{d-1}, 0\right)\right),\ L\left(Q\left(P_0; n_0 - n_{d-1}, \ldots,\right.\right.\\ &\left.\left.n_{d-2} - n_{d-1}, 0\right)\right) + \left|Q\left(P_0; n_0 - n_{d-1}, \ldots, n_{d-2} - n_{d-1}, 0\right)(1)\right|\} > -\frac{1}{n}, \end{aligned} \tag{61}$$

where the $\min^*$ is taken over all integers $n_0 > \ldots > n_{d-1} > 0$ such that

$$n_{j-1} - n_j \leqslant d_j \quad (1 \leqslant j < d) \tag{62}$$

$$\begin{aligned} n_{d-1} \leqslant \psi\Bigg(&|\alpha_{r-1}|, m-1, d(d-1)^{\frac{d+1}{2}} 2^{m-1} n\\ &\times E\left(P_0; n_1 - n_{d-1}, \ldots, n_{d-2} - n_d\right)\\ &\times \max\left\{1, \frac{nd}{nd-1}\left|Q\left(P_0; n_0 - n_{d-1}, \ldots, n_{d-2} - n_{d-1}, 0\right)(1)\right|\right\}\Bigg) \end{aligned} \tag{63}$$

and, in the notation of Definition 1

$$\left|C_1\left(P; n_1, \ldots, n_{d-1}, 0\right)\right| \neq 0. \tag{64}$$

The condition (64) implies that there is a unique polynomial

$$Q = x^{n_0} + \sum_{j=1}^{d-1} b_j x^{n_j} + b_d$$

divisible by P, denoted in (61) by $Q(P; n_0, \ldots, n_{d-1}, 0)$. Similarly $Q\left(P_0; n_0 - n_{d-1}, \ldots, n_{d-2} - n_{d-1}, 0\right)$ is the unique polynomial

$$Q = x^{n_0 - n_{d-1}} + \sum_{j=1}^{d-1} b_j x^{n_j - n_{d-1}}$$

divisible by P_0. The inequality

$$l(P) \leqslant \min{}^* L\left(Q\left(P; n_0, \ldots, n_{d-1}, 0\right)\right)$$

is clear and the inequality

$$\begin{aligned} l(P) \leqslant \min{}^* \{&L\left(Q\left(P_0; n_0 - n_{d-1}, \ldots, n_{d-2} - n_{d-1}, 0\right)\right)\\ &+ \left|Q\left(P_0; n_0 - n_{d-1}, \ldots, n_{d-2} - n_{d-1}, 0\right)(1)\right|\} \end{aligned}$$

follows from Lemma 16. This shows the first of inequalities (61). In order to prove the second one we notice that by Lemmas 4 and 14

$$l(P) = \inf L\left(Q\left(P; n_0, \dots, n_{d-1}, 0\right)\right), \tag{65}$$

where $\langle n_0, \dots, n_{d-1}\rangle$ runs through all strictly decreasing sequences of d positive integers satisfying (62) and (64). If (63) is satisfied then, clearly

$$L\left(Q\left(P; n_0, \dots, n_{d-1}, 0\right)\right) \geqslant \min{}^{*} L\left(Q\left(P; n_0, \dots, n_{d-1}, 0\right)\right) \tag{66}$$

and, if not, then by Lemma 15 there exists a polynomial $R \in \mathbb{R}[x]$ of degree at most n_1 such that (53)–(55) hold with

$$a = Q\left(P_0; n_0 - n_{d-1}, \dots, n_{d-2} - n_{d-1}, 0\right)(1).$$

Then the polynomial

$$\begin{aligned} S(x) = {} & Q\left(P_0; n_0 - n_{d-1}, \dots, n_{d-2} - n_{d-1}, 0\right) x^{n_{d-1}} + R(x) \\ & - Q\left(P_0; n_0 - n_{d-1}, \dots, n_{d-2} - n_{d-1}, 0\right)(1) \end{aligned}$$

is monic, satisfies

$$P \mid S(x)$$

and, by (64),

$$S(x) = Q\left(P; n_0, \dots, n_{d-1}, 0\right). \tag{67}$$

By (55)

$$\begin{aligned} L(S) > {} & L\left(Q\left(P_0; n_0 - n_{d-1}, \dots, n_{d-2} - n_{d-1}, 0\right)\right) \\ & + \left|Q\left(P_0; n_0 - n_{d-1}, \dots, n_{d-2} - n_{d-1}, 0\right)(1)\right| - \frac{1}{n}. \end{aligned} \tag{68}$$

The formulae (66)–(68) imply

$$\begin{aligned} & L\left(Q\left(P; n_0, \dots, n_{d-1}, 0\right)\right) \\ \geqslant & \min{}^{*} \min\Big\{ L\left(Q\left(P; n_0, \dots, n_{d-1}, 0\right)\right), L\left(Q\left(P_0; n_0 - n_{d-1}, \dots, n_{d-2} - n_{d-1}, 0\right)\right) \\ & + \left|Q\left(P_0; n_0 - n_{d-1}, \dots, n_{d-2} - n_{d-1}, 0\right)(1)\right| \Big\} - \frac{1}{n} \end{aligned}$$

for all sequences $\langle n_0, \dots, n_{d-1}\rangle$ satisfying (64), hence by (65) the second of the inequalities (61) follows. The conditions (62) and (63) are for a given n satisfied by only finitely many sequences $\langle n_0, \dots, n_{d-1}\rangle$, since

$$n_j - n_{d-1} \leqslant \sum_{\mu=j+1}^{d-1} d_j$$

and all such sequences can be effectively determined, hence $l(P)$ can be effectively computed.

For the proof of Theorem 6 we need

Definition 7. For α, β in $\mathbb{C}$ and $n > m > 0$

$$Q_n(\alpha,\beta) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } \alpha \neq \beta, \\ n\alpha^{n-1} & \text{if } \alpha = \beta, \end{cases}$$

$$E_{n,m}(\alpha\beta) = \left| \frac{Q_n(\alpha,\beta)}{Q_m(\alpha,\beta)} \right| + |\alpha\beta|^m \left| \frac{Q_{n-m}(\alpha,\beta)}{Q_m(\alpha,\beta)} \right|,$$

$$F_{n,m}(x,\beta) = x^n - \beta^n + |\beta|^m x^m (x^{n-m} - \beta^{n-m}) - (2x-1)(x^m - \beta^m).$$

Lemma 17. *In the notation of Definition 2, if $P(x) = (x-\alpha)(x-\beta)$, $\alpha\beta \neq 0$, then all elements of $T_2(P)$ are of the form*

$$x^n - \frac{Q_n(\alpha,\beta)}{Q_m(\alpha,\beta)} x^m + (\alpha\beta)^m \frac{Q_{n-m}(\alpha,\beta)}{Q_m(\alpha,\beta)} = F_{n,m}(x;\alpha,\beta), \tag{69}$$

where $n > m > 0$, $Q_m(\alpha,\beta)Q_{n-m}(\alpha,\beta) \neq 0$.

Proof. Let an element Q of $T_2(P)$ be $x^n + Ax^m + B$, where $n > m > 0$. By Lemma 3 the condition $Q \equiv 0 (\operatorname{mod} P)$ is equivalent to

$$c_{i0} + c_{i1}A + c_{i2}B = 0 \quad (i = 1, 2), \tag{70}$$

where c_{ij} are given in Definition 1 for $\alpha_1 = \alpha$, $\alpha_2 = \beta$, hence

$$\begin{aligned} &c_{10} = \alpha^n,\ c_{11} = \alpha^m,\ c_{12} = 1; \\ &c_{20} = \beta^n,\ c_{21} = \beta^m,\ c_{22} = 1,\ \text{if } \beta \neq \alpha; \\ &c_{20} = 0,\ c_{21} = (m-n)\beta^m,\ c_{22} = -n,\ \text{if } \beta = \alpha. \end{aligned}$$

Since $Q \in T_2(P)$ we have

$$|C_0(P; n, m)| \neq 0 \neq |C_1(P; n, m)|,$$

hence $Q_m(\alpha,\beta)Q_{n-m}(\alpha,\beta) \neq 0$. Solving the system (70) we obtain for Q the form (69).

Lemma 18. *If $\beta \in \mathbb{R}$, $|\beta| \geqslant 1$, then for all positive integers $n > m$ and all integers $k \geqslant 0$ we have*

$$G_{n,m,k}(\beta) = \frac{1}{k!} \frac{d^k}{dx^k} F_{n,m}(x,\beta)|_{x=|\beta|} \geqslant 0$$

and if $|\beta| > 1$ for $k = 0$ or 1

$$\inf_{n>m} G_{n,m,k}(\beta) > 0.$$

Proof. Consider first the case $\beta > 0$. For $k = 0$ we have $G_{n,m,k}(\beta) = 0$. For $k \geqslant 1$ we have

$$\begin{aligned} G_{n,m,k}(\beta) &= \binom{n}{k}\beta^{n-k} + \binom{n}{k}\beta^{n+m-k} - \binom{m}{k}\beta^{n+m-k} \\ &\quad - 2\binom{m+1}{k}\beta^{m-k+1} + \binom{m}{k}\beta^{m-k} + 2\binom{1}{k}\beta^{m-k+1} \\ &= \beta^{m-k}\Bigg(\binom{n}{k}\beta^{n-m} + \binom{n}{k}\beta^{n} - \binom{m}{k}\beta^{n} \\ &\quad - 2\binom{m+1}{k}\beta + \binom{m}{k} + 2\binom{1}{k}\beta\Bigg). \end{aligned}$$

The expression in the parenthesis is non-negative, since for $\beta = 1$ it is equal to

$$2\binom{n}{k} - 2\binom{m+1}{k} + 2\binom{1}{k} \geqslant 0$$

and its derivative with respect to β is

$$\begin{aligned} &\binom{n}{k}(n-m)\beta^{n-m-1} + \left(\binom{n}{k} - \binom{m}{k}\right)n\beta^{n-1} - 2\binom{m+1}{k} + 2\binom{1}{k} \\ &\geqslant \binom{m+1}{k} + \left(\binom{m+1}{k} - \binom{m}{k}\right)(m+1) - 2\binom{m+1}{k} + 2\binom{1}{k} \\ &= (k-1)\binom{m+1}{k} + 2\binom{1}{k} \geqslant 2\binom{1}{k}. \end{aligned}$$

It follows that

$$G_{n,m,k}(\beta) \geqslant 2(\beta - 1)$$

and the obtained lower bound, independent of n, m is positive for $\beta > 1$. Consider now the case $\beta < 0$. We distinguish four cases according to the parity of n, m.

If $n \equiv m \equiv 0 (\operatorname{mod} 2)$, then

$$G_{n,m,k}(\beta) = G_{n,m,k}\big(|\beta|\big)$$

and the case reduces to the former.

If $n \equiv 0$, $m \equiv 1 (\operatorname{mod} 2)$, then

$$G_{n,m,0}(\beta) = 2\big(|\beta|\big)^m \left(|\beta|^n - 2|\beta| + 1\right) \geqslant 2\big(|\beta| - 1\big)^2$$

and the obtained lower bound, independent of n, m is positive for $|\beta| > 1$. Further, for $k \geqslant 1$

$$\begin{aligned} G_{n,m,k}(\beta) &= \binom{n}{k}|\beta|^{n-k} + \binom{n}{k}|\beta|^{n+m-k} + \binom{m}{k}|\beta|^{n+m-k} \\ &\quad - 2\binom{m+1}{k}|\beta|^{m-k+1} + \binom{m}{k}|\beta|^{m-k} - 2\binom{1}{k}|\beta|^{m-k+1} \\ &= |\beta|^{m-k}\left(\binom{n}{k}|\beta|^{n-m} + \binom{n}{k}|\beta|^{n} + \binom{m}{k}|\beta|^{n} - 2\binom{m+1}{k}|\beta| \right. \\ &\quad \left. + \binom{m}{k} - 2\binom{1}{k}|\beta|\right). \end{aligned}$$

The expression in the parenthesis is non-negative, since

$$\binom{n}{k}|\beta|^{n-m}+\binom{n}{k}|\beta|^{n}\geqslant 2\binom{m+1}{k}|\beta|$$

and

$$\binom{m}{k}|\beta|^{n}+\binom{m}{k}\geqslant\binom{m}{k}\left(|\beta|^{2}+1\right)\geqslant 2\binom{1}{k}|\beta|.$$

If $n\equiv 1,\ m\equiv 0(\operatorname{mod}2)$, then

$$G_{n,m,k}(\beta)\geqslant G_{n,m,k}(|\beta|)$$

and the case reduces to the already considered one.

Finally, if $n\equiv m\equiv 1\operatorname{mod}2$, then

$$G_{n,m,0}(\beta)=2|\beta|^{m}\left(|\beta|^{n-m}-2|\beta|+1\right)\geqslant 2\left(|\beta|-1\right)^{2}$$

and the obtained lower bound, independent of n,m is positive for $|\beta|>1$.

Further, for $k\geqslant 1$

$$\begin{aligned}G_{n,m,k}(\beta)&=\binom{n}{k}|\beta|^{n-k}+\binom{n}{k}|\beta|^{n+m-k}-\binom{m}{k}|\beta|^{n+m-k}\\&\quad-2\binom{m+1}{k}|\beta|^{m-k+1}+\binom{m}{k}|\beta|^{m-k}-2\binom{1}{k}|\beta|^{m-k+1}\\&=|\beta|^{m-k}\left(\binom{n}{k}|\beta|^{n-m}+\binom{n}{k}|\beta|^{n}-\binom{m}{k}|\beta|^{n}-2\binom{m+1}{k}|\beta|\right.\\&\quad\left.+\binom{m}{k}-2\binom{1}{k}|\beta|\right).\end{aligned}$$

The expression in the parenthesis is non-negative, since for $|\beta|=1$ it is equal to

$$2\binom{n}{k}-2\binom{m+1}{k}-2\binom{1}{k}\geqslant 2\binom{m+2}{k}-2\binom{m+1}{k}-2\binom{1}{k}\geqslant 0$$

and its derivative with respect to $|\beta|$ is

$$\begin{aligned}&\binom{n}{k}(n-m)|\beta|^{n-m-1}+\binom{n}{k}n-|\beta|^{n-1}-\binom{m}{k}n|\beta|^{n-1}-2\binom{m+1}{k}-2\binom{1}{k}\\&\geqslant\binom{n}{k}(n-m)+\binom{n}{k}n-\binom{m}{k}n-2\binom{m+1}{k}-2\binom{1}{k}\\&\geqslant 2\binom{m+2}{k}+\left(\binom{m+2}{k}-\binom{m}{k}\right)(m+2)-2\binom{m+1}{k}-2\binom{1}{k}\geqslant 0.\end{aligned}$$

Proof of Theorem 6. Consider first the case of α,β real. Since, by Proposition (iii), $l(P(-x))=l(P(x))$, we may assume that $\alpha>0$, hence $\alpha\geqslant|\beta|$.

By the Taylor formula we have in the notation of Lemma 18

$$(\alpha^m - \beta^m)\,(E_{n,m}(\alpha,\beta) - 2\alpha + 1) = \sum_{k=0}^{n} G_{n,m,k}(\beta)\big(\alpha - |\beta|\big)^k,$$

hence, by the said lemma,

$$(\alpha^m - \beta^m)\,(E_{n,m}(\alpha,\beta) - 2\alpha + 1) \geqslant 0 \tag{71}$$

and, if $\alpha > |\beta| > 1$

$$\inf_{n>m}\,(\alpha^m - \beta^m)\,(E_{n,m}(\alpha,\beta) - 2\alpha + 1) > 0. \tag{72}$$

If $\alpha \neq \pm\beta$ then (71) gives

$$E_{n,m}(\alpha,\beta) \geqslant 2\alpha - 1,$$

hence by Lemma 17

$$\inf_{Q\in T_2(P)} L(Q) \geqslant 2\alpha$$

and by Lemma 4,

$$l(P) \geqslant 2\alpha. \tag{73}$$

Now, if $\beta = -1$, then $L(P) = 2\alpha$, hence $l(P) \leqslant 2\alpha$ and, by (73), $l(P) = 2\alpha$. If $\beta = 1$, then by Theorem 5 with $P_0 = x - \alpha$

$$l(P) \leqslant L\,(P_0) + |P_0(1)| = 1 + \alpha + \alpha - 1 = 2\alpha$$

and, by (73), $l(P) = 2\alpha$ again.

If $\alpha > |\beta| > 1$, then by (72)

$$\inf_{\substack{n>m\\ m<m_0}} E_{n,m}(\alpha,\beta) > 2\alpha - 1 \tag{74}$$

for every m_0. Choose now

$$m_0 = \frac{\log 4\alpha - \log\big(\alpha - |\beta|\big)}{\log|\beta|}.$$

Then for $m \geqslant m_0 : E_{n,m}(\alpha,\beta) \geqslant |\alpha\beta|^m \frac{\alpha-|\beta|}{2\alpha^m} \geqslant 2\alpha$ and, by (74)

$$\inf_{n>m} E_{n,m}(\alpha,\beta) > 2\alpha - 1.$$

Using, as above, Lemmas 17 and 4 we obtain

$$l(P) > 2\alpha.$$

If $\alpha = -\beta$, then $P(x) = x^2 - \alpha^2$ and by Proposition (iv) and Proposition A (ii)

$$l(P) = l\left(x - \alpha^2\right) = 1 + \alpha^2 \begin{cases} = 2\alpha & \text{if } \alpha = 1, \\ > 2\alpha, & \text{otherwise.} \end{cases}$$

If $\alpha = \beta$, then

$$E_{n,m}(\alpha,\beta) - 2\alpha + 1 = \frac{n\alpha^{n-m} + (n-m)\alpha^n}{m} - 2\alpha + 1.$$

The right-hand side is equal to $2(n-m)/m > 0$ for $\alpha = 1$ and its derivative with respect to α is

$$\frac{n(n-m)}{m}\left(\alpha^{n-m-1} + \alpha^{n-m}\right) - 2 > \alpha - 1.$$

For $\alpha = \beta = 1$, $l(P) = 2 = 2\alpha$, by Theorem 4; otherwise

$$\inf_{n>m} E_{n,m}(\alpha,\alpha) > 2\alpha - 1$$

and, by Lemmas 17 and 4, $l(P) > 2\alpha$.

Consider now the case, where α, β are complex conjugate:

$$\alpha = |\alpha| e^{2i\varphi}, \quad \beta = |\alpha| e^{-2i\varphi}, \quad \varphi \in \left(0, \frac{\pi}{2}\right), \quad |\alpha| > 1$$

(the case $|\alpha| = 1$ is settled by Theorem 4). Then

$$E_{n,m}(\alpha,\beta) = |\alpha|^{n-m}\left|\frac{\sin n\varphi}{\sin m\varphi}\right| + |\alpha|^n \left|\frac{\sin(n-m)\varphi}{\sin m\varphi}\right|,$$

where, by virtue of the condition $Q_m(\alpha,\beta) \neq 0$ we have $\sin m\varphi \neq 0$. Since

$$|\sin m\varphi| \leqslant |\sin n\varphi| + |\sin(n-m)\varphi| \tag{75}$$

we have

$$E_{n,m}(\alpha,\beta) \geqslant |\alpha|^{n-m} \geqslant |\alpha|^2$$

unless $n - m = 1$. In this final case we have, by (75)

$$\left|\frac{\sin n\varphi}{\sin m\varphi}\right| \geqslant 1 - \left|\frac{\sin\varphi}{\sin m\varphi}\right|$$

and by the well known inequality

$$\left|\frac{\sin\varphi}{\sin m\varphi}\right| \geqslant \frac{1}{m}.$$

Hence

$$E_{n,m}(\alpha,\beta) \geqslant |\alpha| \left(1 - \left|\frac{\sin\varphi}{\sin m\varphi}\right|\right) + |\alpha|^{m+1} \left|\frac{\sin\varphi}{\sin m\varphi}\right|$$
$$\geqslant |\alpha| + \frac{|\alpha|^{m+1} - |\alpha|}{m} \geqslant |\alpha| + |\alpha|\big(|\alpha| - 1\big) = |\alpha|^2,$$

where in the middle we have used Bernoulli's inequality. It follows, by Lemma 17, that $L(Q) \geqslant 1 + |\alpha|^2$ for every $Q \in T_2(P)$, hence, by Lemma 4,

$$l(P) \geqslant 1 + |\alpha|^2 > 2|\alpha|.$$

Proof of Corollary 2. If $\deg P = 1$, then $l(P) \in K(P)$ follows from Proposition A. If $P = a(x-\alpha)(x-\beta)$, where $|\alpha| \geqslant |\beta| > 1$, then, by Theorem 2, $l(P)$ is attained and by Theorem 1, $l(P) \in K(P)$. If $P = a(x-\alpha)(x-\beta)$, where $|\beta| = 1$, then, by Theorem 6, $l(P) = 2|a\alpha|$. Since either $|\alpha| = 1$ or $\alpha \in \mathbb{R}$, $l(P) \in K(P)$ follows.

Proof of Corollary 3. If, in the notation of the Corollary, $|\beta| > 1$, then, by Proposition A, $l(P^*) = |\alpha\beta|$ and, by Proposition (ii) $l(P) \geqslant |\alpha\beta|$, thus $\widehat{l}(P) = |\alpha\beta|$. If $|\alpha| > 1 = |\beta|$, then, by Proposition (iii) and Theorem 6, $l(P^*) = 2|\alpha| = l(P)$, thus $\widehat{l}(P) = 2|\alpha|$. If $|\alpha| > 1 > |\beta|$, then, by Proposition A, $l(P^*) = 1 + |\alpha|$, $l(P) = |\alpha\beta|(1 + |\beta|^{-1})$, hence $\widehat{l}(P) = |\alpha| + \min\{1, |\alpha\beta|\}$. If $|\alpha| = 1 = |\beta|$, then by Theorem 6, $l(P) = l(P^*) = 2$. If $|\alpha| = 1 > |\beta|$, then, by Proposition A, $l(P) = 2$, by Theorem 6, $l(P^*) = |\alpha\beta|2|\beta|^{-1} = 2$, thus $\widehat{l}(P) = 2$. Finally, if $|\alpha| < 1$, then by Proposition A, $l(P) = 1$, by Proposition (ii) $l(P^*) \geqslant 1$, thus $\widehat{l}(P) = 1$.

Proof of Corollary 4. If $|\alpha| > 1 > |\beta| > 0$ we have $\widehat{l}(x-\alpha) = |\alpha|$, $\widehat{l}(x-\beta) = 1$,

$$\widehat{l}((x-\alpha)(x-\beta)) = |\alpha| + \min\{1, |\alpha\beta|\} > |\alpha|.$$

Note added in proof. An apparently similar problem has been considered in [2] and [3]. However, the restriction of G in the definition of $l(P)$ to polynomials with integer coefficients makes a great difference, shown by the fact, clear from Lemma 17 above, that no analogue of Lemma 2 of [2] or Lemma 3 of [3] holds in our case.

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