MAHLER AND KOKSMA CLASSIFICATION OF POINTS IN $\mathbb{R}^n$ AND $\mathbb{C}^n$

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Dedicated to Professor Eduard Wirsing
on the occasion of his 75th birthday

Abstract: K. Yu [16] had generalized Mahler's classification to several variables. Here we will deal with generalizations of both the Mahler and Koksma classification. We will discuss the relationship between these classifications, formulate a metric theorem, and pose a number of open problems.

Keywords: Generalized Mahler Classification, Measures of algebraic independence.

1. Introduction

For a point $\xi \in \mathbb{C}^n$ write $|\xi|$ for the maximum modulus of its coordinates, and for a polynomial $P \in \mathbb{C}[\mathbf{X}] = \mathbb{C}[X_1, \dots, X_n]$ write $|P|$ for the maximum modulus of its coefficients. Given natural d , let $\mathcal{P}_d$ be the set of nonzero polynomials $P \in \mathbb{Z}[\mathbf{X}]$ of total degree $\leq d$.

When ξ is a point in $\mathbb{C}^n$, let $\psi_d(\xi)$ be the supremum (possibly ∞) of the numbers ψ for which there are infinitely many polynomials $P \in \mathcal{P}_d$ with

$$0 < |P(\xi)|/|P| < |P|^{-\psi}.$$

Further write $\psi_d^*(\xi)$ for the supremum of the numbers ψ^* for which there are infinitely many $P \in \mathcal{P}_d$ with $P(\xi) \neq 0$ which have a zero $\alpha \in \mathbb{C}^n$ such that

$$|\xi - \alpha| < |P|^{-\psi^*}.$$

In the case $n = 1$, Mahler [8] and Koksma [6] had introduced quantities $\omega_d(\xi)$ and $\omega_d^*(\xi)$, and our $\psi_d(\xi) = \omega_d(\xi) + 1$, $\psi_d^*(\xi) = \omega_d^*(\xi) + 1$. A detailed exposition of these quantities may also be found in [4]. It appears to the author that results can be stated in a cleaner way in terms of ψ_d, ψ_d^* .

When $\xi \in \mathbb{R}^n$, it is natural to introduce $\psi_d^{\mathbb{R}}(\xi)$, which is defined as $\psi_d^*(\xi)$, except that one restricts to approximations $\alpha \in \mathbb{R}^n$. Clearly $\psi_d^{\mathbb{R}}(\xi) \leq \psi_d^*(\xi)$. But

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it was pointed out in [13] (see Section 4, especially Proposition 4.4) that when $\alpha \in \mathbb{C}^n$ is a zero of $P \in \mathcal{P}_d$, then there is a partial derivative $P^{(\mathbf{i})} \neq 0$ having a zero $\beta \in \mathbb{R}^n$ with $|\xi - \beta| \leq c_1(n, d)|\xi - \alpha|$. Since $|P^{(\mathbf{i})}| \leq c_2(d)|P|$, we may infer that in fact $\psi_d^{\mathbb{R}}(\xi) = \psi_d^*(\xi)$.

Clearly $\psi_1(\xi) \leq \psi_2(\xi) \leq \dots$ and $\psi_1^*(\xi) \leq \psi_2^*(\xi) \leq \dots$. It may happen that $\psi_d(\xi) = \infty$ for some d , and then $\psi_j(\xi) = \infty$ for $j \geq d$. The same can be said of the sequence $\{\psi_d^*(\xi)\}_{d \in \mathbb{N}}$. When $P \in \mathbb{Z}[\mathbf{X}]$ has $P(\xi) \neq 0$ and a zero α with $|\xi - \alpha| < 1$, then

$$0 < |P(\xi)| = |P(\xi) - P(\alpha)| \leq c_3(d, \xi)|\xi - \alpha||P|,$$

and therefore

$$\psi_d(\xi) \geq \psi_d^*(\xi). \quad (1.1)$$

Of course $\psi_d(\xi), \psi_d^*(\xi)$ can be defined for $\xi \in \mathbb{C}^\ell$ with any ℓ . Suppose $\ell > n$ and $\xi = (\xi_1, \dots, \xi_n, \dots, \xi_\ell)$. A polynomial $P(X_1, \dots, X_n)$ may be interpreted as a polynomial $\hat{P}$ in $X_1, \dots, X_n, \dots, X_\ell$, and when $\alpha = (\alpha_1, \dots, \alpha_n)$ is a zero of P , then $\hat{\alpha} = (\alpha_1, \dots, \alpha_n, \xi_{n+1}, \dots, \xi_\ell)$ is a zero of $\hat{P}$. Therefore

$$\psi_d(\xi_1, \dots, \xi_n, \dots, \xi_\ell) \geq \psi_d(\xi_1, \dots, \xi_n), \quad (1.2)$$

$$\psi_d^*(\xi_1, \dots, \xi_n, \dots, \xi_\ell) \geq \psi_d^*(\xi_1, \dots, \xi_n). \quad (1.3)$$

Suppose ξ is algebraic, so that the field $\mathbb{Q}(\xi) = \mathbb{Q}(\xi_1, \dots, \xi_n)$ is of finite degree e over $\mathbb{Q}$. For some $a \in \mathbb{N}$, the coordinates of $a\xi$ are algebraic integers, so that $a^d P(\xi)$ is an algebraic integer when $P \in \mathcal{P}_d$. Then $P(\xi) \neq 0$ implies

$$|\mathcal{N}(P(\xi))| = a^{-de} |\mathcal{N}(a^d P(\xi))| \geq a^{-de},$$

where $\mathcal{N}$ denotes the norm from $\mathbb{Q}(\xi)$ to $\mathbb{Q}$. On the other hand,

$$|\mathcal{N}(P(\xi))| \leq c_4(d, \xi)|P(\xi)||P|^{e-1},$$

and therefore $|P(\xi)|/|P| \geq (c_4(d, \xi)a^{de})^{-1}|P|^{-e}$, so that in view of (1.1),

$$\psi_d^*(\xi) \leq \psi_d(\xi) \leq e \quad (d = 1, 2, \dots). \quad (1.4)$$

It will be convenient to write $\mathbb{C}_1 = \mathbb{R}$, $\mathbb{C}_2 = \mathbb{C}$. Throughout, q will be 1 or 2, so that $\mathbb{C}_q$ will be $\mathbb{R}$ or $\mathbb{C}$. Since polynomials in $\mathcal{P}_d$ are linear combinations of

$$m = m_{n,d} = \binom{d+n}{n}$$

monomials, it follows from Dirichlet's box principle, and is also deduced in Section 2 of [16], that given $\xi \in \mathbb{C}_q^n$ and $N \geq 1$, there is a $P \in \mathcal{P}_d$ with $|P| \leq N$ and

$$|P(\xi)| < c_5(d, \xi)N^{1-m/q}.$$

Therefore when $\xi \in \mathbb{C}_q^n$ has algebraically independent components, and when $m > q$, i.e., unless $q = 2$, $d = n = 1$, there are infinitely many $P \in \mathcal{P}_d$ with $0 < |P(\xi)| < c_5(d, \xi)|P|^{1-m/q}$, so that

$$\psi_d(\xi) \geq m/q, \quad (1.5)$$

as already noted in Theorem 1 of [16].

When ξ has transcendence degree t , i.e., when the field $\mathbb{Q}(\xi)$ has transcendence degree t , then it follows from (1.2) (with t, n in place of n, ℓ) that $\psi_d(\xi) \geq m_{t,d}/q$, unless $q = 2$, $d = t = 1$. We now generalize Mahler's classification [8] of transcendental numbers as follows.

When $\xi \in \mathbb{C}^n$ is of transcendence degree $t > 0$, we will call it an:

- *S-point* if each $\psi_d(\xi)$ is finite and $\psi_d(\xi)/m_{t,d}$ remains bounded as $d \rightarrow \infty$,
- a *T-point* if each $\psi_d(\xi)$ is finite, but $\psi_d(\xi)/m_{t,d}$ does not remain bounded,
- a *U-point* if some $\psi_d(\xi) = \infty$.

Since $m_{t,d} \sim d^t/t!$ as $d \rightarrow \infty$, the quotient $\psi_d(\xi)/m_{t,d}$ in this definition may be replaced by $\psi_d(\xi)/d^t$. Our classification coincides with the one given by Yu [16]. In fact a point $\xi \in \mathbb{C}_q^n$ of transcendence degree t which we call an *S, T*, or *U-point* is called an S_t^n, T_t^n , or U_t^n -point, respectively, by Yu.

Generalizing Koksma's classification [6], we define S^*, T^* and U^* -points just as the *S, T* and *U*-points, but involving the quantities ψ_d^* instead of ψ_d . In order to compare these classifications we need to combine (1.1) with a lower bound of $\psi_d^*(\xi)$ in terms of $\psi_d(\xi)$.

For $P \in \mathcal{P}_d$ there is a partial derivative $P^{(\mathbf{i})}$ of some order ℓ , $0 \leq \ell \leq d$, with $|P| \leq c_6(d, \xi)|P^{(\mathbf{i})}(\xi)|$. An inequality $|P(\xi)|/|P| < |P|^{-\psi}$ then yields $(|P(\xi)|/|P^{(\mathbf{i})}(\xi)|)^{1/\ell} \leq c_7(d, \xi)|P|^{-\psi/\ell}$. By [13, Lemma 4.3] there is a partial derivative $P^{(\mathbf{j})} \neq 0$ having a zero α with $|\xi - \alpha| \leq c_8(d, \xi)|P|^{-\psi/\ell} \leq c_8(d, \xi)|P|^{-\psi/d}$. Since $|P^{(\mathbf{j})}| \leq c_9(d)|P|$, this yields

$$|\xi - \alpha| \leq c_{10}(d, \xi)|P^{(\mathbf{j})}|^{-\psi/d}.$$

Incidentally, Brownawell [3], as a very special case of a more general result, gave a good bound for the distance from ξ to the nearest zero α of P itself (rather than of some $P^{(\mathbf{j})}$). See also the Comparison Theorem of Amoroso [1].

In order to be able to draw a conclusion on $\psi_d^*(\xi)$ we need to know that $P^{(\mathbf{j})}(\xi) \neq 0$. At any rate, we see that

$$\psi_d^*(\xi) \geq \psi_d(\xi)/d \quad (1.6)$$

when ξ has algebraically independent components.

When $n = 1$, Wirsing [15] had established $\psi_d^*(\xi) \geq \frac{1}{2}\psi_d(\xi) + 1$. This, together with (1.1), implies that *S, T, U*-numbers¹ are S^*, T^*, U^* -numbers respectively. I

¹ *S, T* or *U*-points in dimension $n = 1$ are called *S, T* or *U*-numbers.

don't know whether for $n > 1$ there is a relation $\psi_d^*(\xi) \geq c_{11}\psi_d(\xi)$ with a positive constant $c_{11} = c_{11}(n)$ or $c_{11} = c_{11}(n, \xi)$ independent of d . If this holds, then S, T, U -points are S^*, T^*, U^* -points respectively. As it is, we can infer from (1.1), (1.6) only that the class of U -points coincides with the class of U^* -points, and S -points are S^* -points, T -points are S^* or T^* -points, whereas T^* -points are T -points, S^* -points are S or T -points. This leads to open

Problem A. *When $n > 1$, are there T -points which are S^* -points? (I conjecture that there are.)*

Mahler proved that if ξ, η are algebraically dependent (over $\mathbb{Q}$) transcendental numbers, then they belong to the same class S, T or U . Now ξ, η are algebraically dependent precisely if the algebraic closures $\overline{\mathbb{Q}(\xi)}, \overline{\mathbb{Q}(\eta)}$ of $\mathbb{Q}(\xi), \mathbb{Q}(\eta)$ coincide. We therefore will call points $\xi \in \mathbb{C}^n$, $\eta \in \mathbb{C}^\ell$ of transcendence degree $t > 0$ *algebraically related* if $\overline{\mathbb{Q}(\xi)} = \overline{\mathbb{Q}(\eta)}$.

Yu [16] had proved

Theorem 1. *Suppose $\xi \in \mathbb{C}_q^n$, $\eta \in \mathbb{C}_q^\ell$ are algebraically related points of transcendence degree $t > 0$. Then they belong to the same class S, T or U .*

For Koksma's classification we have

Theorem 2. *Suppose ξ as well as η in $\mathbb{C}^n$ have algebraically independent components, and they are algebraically related. Then they belong to the same class S^*, T^* or U^* .*

Observe that the hypothesis here is stronger than in Theorem 1. I am not even able to show that for transcendental ξ , the points $\xi \in \mathbb{C}$ and $\eta = (\xi, \xi) \in \mathbb{C}^2$ belong to the same class S^*, T^* or U^* .

Problem B. *Are there S^* -numbers $\xi \in \mathbb{C}$ for which $\eta = (\xi, \xi) \in \mathbb{C}^2$ is a T^* -point? (I believe that there are.)*

Finally some remarks on the existence of points in various classes. It is not hard to see that almost every point (in the sense of Lebesgue measure) $\xi \in \mathbb{C}_q^n$ has $\psi_d^*(\xi) \leq m_{n,d}/q$, so that almost every point is an S^* -point. But in fact it was proved by Bernik, Kleinbock and Margulis [2] in the real case $q = 1$, and by Kleinbock [5] in the complex case $q = 2$, that

$$\psi_d(\xi) = m_{n,d}/q \quad (1.7)$$

for almost every $\xi \in \mathbb{C}_q^n$, so that almost every point is an S -point, hence an S^* -point. As pointed out in [16], the Siegel-Shidlovsky theory of E-functions furnishes explicit examples of points with (1.7).

Theorem 3. *Suppose $j \in \mathbb{N}$, and $\xi \in \mathbb{C}_q$ with finite $\psi_j(\xi)$. Then for almost all $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{C}_q^{n-1}$,*

$$\psi_j(\xi) \leq \psi_j(\xi, \xi_2, \dots, \xi_n) \leq \psi_j(\xi) + jm_{n,j}/q. \quad (1.8)$$

Problem C. *Is the factor j on the right hand side of this relation needed?*

A point $\boldsymbol{\xi} \in \mathbb{C}_q^n$, will be called a U_d -point if $\psi_d(\boldsymbol{\xi}) = \infty$, but $\psi_j(\boldsymbol{\xi})$ is finite for $1 \leq j < d$. Since by (1.1), (1.6) $\psi_d(\boldsymbol{\xi}) \geq \psi_d^*(\boldsymbol{\xi}) \geq \psi_d(\boldsymbol{\xi})/d$ for each d , the U_d -points coincide with the U_d^* -points defined in terms of ψ^* in place of ψ . For $n = q = 1$, the existence of U_d -numbers and T -numbers had been established by LeVeque [7] and Schmidt [11] respectively. They are also U_d or T -numbers in $\mathbb{C}_2 = \mathbb{C}$. When ξ is a U_1 -number, then $(\xi, \xi_2, \dots, \xi_n)$ is a U_1 -point for any $(\xi_2, \dots, \xi_n)$ in view of (1.2). Theorem 3 has the immediate

Corollary 3.1. *Suppose $n > 1$, $d > 1$. When $\xi \in \mathbb{C}_q$ is a U_d -number, then $\boldsymbol{\xi} = (\xi, \xi_2, \dots, \xi_n)$ for almost every $\boldsymbol{\xi}' = (\xi_2, \dots, \xi_n) \in \mathbb{C}_q^{n-1}$ is a U_d -point with*

$$\psi_j(\xi) \leq \psi_j(\boldsymbol{\xi}) \leq \psi_j(\xi) + jm_{n,j}/q \quad (j = 1, \dots, d-1).$$

In Theorem 7.5 of [4], U_d -numbers with $\psi_j^*(\xi) \leq d+j$ for $1 \leq j < d$ were constructed. (Observe that our $\psi_j^* = \omega_j^* + 1$, with ω_j^* as in [4].) In Section 6 we will indicate how to modify the argument in [4] to obtain real U_d -numbers ξ with

$$\psi_j(\xi) \leq d+j \quad (j = 1, \dots, d-1). \quad (1.9)$$

One may also construct (but this will not be carried out here) complex U_d -numbers ξ having $\psi_j(\xi) \leq (d+j)/2$ ($j = 1, \dots, d-1$). According to Sprindžuk [14] (later generalized by Bernik, Kleinbock and Margulis as already noted), almost every $\xi \in \mathbb{C}_q$ has $\psi_j(\xi) = (1+j)/q$ ($j = 1, 2, \dots$). We may thus pose

Problem D. *Given $d > 1$, are there U_d -numbers $\xi \in \mathbb{C}_q$ with $\psi_j(\xi) = (1+j)/q$ for $j = 1, \dots, d-1$?*

Suppose $(\chi_d)_{d \in \mathbb{N}}$ is a non-decreasing sequence with

$$\chi_d > d^3 + 2d^2 + 4d + 4 \quad (d \in \mathbb{N}). \quad (1.10)$$

By Theorem 7.1 of [4], there are real T -numbers ξ with $\psi_d^*(\xi) = \psi_d^*(\xi) = \chi_d$ for each $d \in \mathbb{N}$. We thus have

Corollary 3.2. *Given χ_d with (1.10), there are points $\boldsymbol{\xi} \in \mathbb{R}^n$ with*

$$\chi_d \leq \psi_d^*(\boldsymbol{\xi}) \leq \psi_d(\boldsymbol{\xi}) \leq \chi_d + dm_{n,d} \quad (d \in \mathbb{N}).$$

When χ_d/d^n is unbounded, $\boldsymbol{\xi}$ will be a T - as well as a T^* -point.

This supersedes Theorem 4 of [16], where for given $\alpha > n$, points $\boldsymbol{\xi} \in \mathbb{R}^n$ were constructed having

$$d^\alpha \leq \psi_d(\boldsymbol{\xi}) \ll d^{2^{n-1}(\alpha+1)-1+\epsilon} \quad (d \in \mathbb{N})$$

for each $\epsilon > 0$.

We finally remark that alternative classifications of transcendental numbers due to Sprindžuk [14], Mahler [9] and Philippon [10] could also be generalized to classifications of points in $\mathbb{R}^n$ and $\mathbb{C}^n$.

2. Refinement of Theorems 1 and 2

Let $\Psi = \{\psi_d\}_{d \geq 1}$ be a non-decreasing sequence of nonnegative reals, where however we allow ψ_d to be infinite from some point on, say for $d \geq d_0$. If $\Psi' = \{\psi'_d\}_{d \geq 1}$ is another such sequence, we will write $\Psi \gtrsim \Psi'$ if there are natural numbers u, v with

$$\psi_{ud} \geq \psi'_d/v \quad (d = 1, 2, \dots).$$

In particular $\psi_{ud} = \infty$ if $\psi'_d = \infty$. The relation $\gtrsim$ is transitive, i.e., $\Psi \gtrsim \Psi'$ and $\Psi' \gtrsim \Psi''$ yield $\Psi \gtrsim \Psi''$. We now write $\Psi \sim \Psi'$ if both $\Psi \gtrsim \Psi'$ and $\Psi' \gtrsim \Psi$, and note that $\sim$ is an equivalence relation. In particular, sequences Ψ having $c_1 d^t \leq \psi_d \leq c_2 d^t$ for positive constants² c_1, c_2 form an equivalence class $\mathcal{C}_t$, and sequences having $\psi_d = \infty$ for some d form a class $\mathcal{C}_\infty$.

Given $\xi \in \mathbb{C}^n$, set $\Psi(\xi) = \{\psi_d(\xi)\}_{d \geq 1}$ and $\Psi^*(\xi) = \{\psi_d^*(\xi)\}_{d \geq 1}$. When $\mathcal{C}$ is a class of sequences, write $\mathbb{C}(\mathcal{C})$ (resp. $\mathbb{C}^*(\mathcal{C})$) for the set of points ξ with $\Psi(\xi) \in \mathcal{C}$ (resp. $\Psi^*(\xi) \in \mathcal{C}$). Clearly the S -points of transcendence degree $t > 0$ make up $\mathbb{C}(\mathcal{C}_t)$, and the U -points make up $\mathbb{C}(\mathcal{C}_\infty)$. However, in view of Corollary 3.2, the T -points of given transcendence degree $t > 0$ make up infinitely many classes $\mathbb{C}(\mathcal{C})$ with $\mathcal{C} \neq \mathcal{C}_t, \mathcal{C}_\infty$. Therefore the classification of points into classes $\mathbb{C}(\mathcal{C})$ is a refinement of the classification into S, T and U -points. The situation is analogous for the classes $\mathbb{C}^*(\mathcal{C})$ and S^*, T^*, U^* . In the case $n = 1$, these classifications had been introduced in [12].

Given $\alpha > 0$, the sequences Ψ where α is the infimum of the numbers β with $\psi_d \ll d^\beta$ make up a certain union $\mathcal{T}(\alpha)$ of classes. Yu [16] had defined a set $\mathcal{T}(\alpha)$ which consists of T -points with $\Psi(\xi) \in \mathcal{T}(\alpha)$.

Theorem 1A. *Suppose $\xi \in \mathbb{C}^n$, $\eta \in \mathbb{C}^\ell$ of positive transcendence degree are algebraically related. Then they lie in the same class $\mathbb{C}(\mathcal{C})$ for some $\mathcal{C}$.*

This is in fact contained in Theorem 2 of [16].

Theorem 2A. *Suppose ξ, η in $\mathbb{C}^n$ each have algebraically independent coordinates, and they are algebraically related. Then they lie in the same class $\mathbb{C}^*(\mathcal{C})$ for some $\mathcal{C}$.*

3. On resultants

Let k be a field, and $P(Y), Q(Y)$ in $k[Y]$ of respective degrees $a > 0, b > 0$. Their resultant $R = R(P, Q)$ lies in k , and is nonzero if P, Q have no common zero

² We begin the numbering of constants anew in each section.

in the algebraic closure of k . Furthermore, R is a polynomial in the coefficients of P and Q , with coefficients in $\mathbb{Z}$. In fact R is homogeneous of degree b in the coefficients of P , and homogeneous of degree a in the coefficients of Q . There are polynomials $A(Y), B(Y)$ in $k[Y]$ with

$$A(Y)P(Y) + B(Y)Q(Y) = R,$$

and $\deg A = b - 1$, $\deg B = a - 1$. The coefficients of A, B are polynomials in the coefficients of P, Q , with coefficients in $\mathbb{Z}$. In fact the coefficients of A are homogeneous of degree $b - 1, a$ in the coefficients of P, Q , respectively, whereas the coefficients of B are homogeneous of degree $b, a - 1$, respectively. All this is well known.

Now suppose $P(X_1, \dots, X_n, Y)$ and $Q(X_1, \dots, X_n, Y)$ are in $\mathbb{Z}[X_1, \dots, X_n, Y]$, hence may be interpreted to be in $k[Y]$ with $k = \mathbb{Q}(X_1, \dots, X_n)$. Set

$$a = \deg_Y P, \quad r = \deg_{\mathbf{X}} P, \quad b = \deg_Y Q, \quad s = \deg_{\mathbf{X}} Q,$$

where $\deg_{\mathbf{X}}$ refers to the total degree in $X_1, \dots, X_n$. The resultant of P, Q as polynomials in $k[Y]$ is a polynomial $R(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n]$. We have

$$\deg_{\mathbf{X}} R \leq br + as \leq c_1(Q)d \quad (3.1)$$

where d is the total degree of P in $X_1, \dots, X_n, Y$. Also,

$$|R| \ll |P|^b |Q|^a \leq c_2(d, Q) |P|^b, \quad (3.2)$$

where the constant implicit in $\ll$ may depend on n, a, b, r, s . In this context, $A(Y) = A(X_1, \dots, X_n, Y)$ with

$$\deg_Y A = b - 1, \quad \deg_{\mathbf{X}} A \leq (b - 1)r + as < c_1(Q)d,$$

$$|A| \ll |P|^{b-1} |Q|^a \leq c_2(d, Q) |P|^{b-1}.$$

When $\xi_1, \dots, \xi_n, \eta$ are complex numbers of modulus under some bound K , then

$$|A(\xi_1, \dots, \xi_n, \eta)| \leq c_3(d, Q, K) |P|^{b-1}.$$

Assuming that $|P(\xi_1, \dots, \xi_n, \eta)| \leq \epsilon$ and

$$Q(\xi_1, \dots, \xi_n, \eta) = 0, \quad (3.3)$$

we obtain

$$\begin{aligned} |R(\boldsymbol{\xi})| &= |R(\xi_1, \dots, \xi_n)| = |A(\xi_1, \dots, \xi_n, \eta)P(\xi_1, \dots, \xi_n, \eta)| \\ &\leq c_3(d, Q, K) |P|^{b-1} \epsilon. \end{aligned} \quad (3.4)$$

If moreover η has degree b over $\mathbb{Q}(\boldsymbol{\xi}) = \mathbb{Q}(\xi_1, \dots, \xi_n)$, then the roots of $Q(\xi_1, \dots, \xi_n, Y)$ are the conjugates $\eta^{(1)}, \dots, \eta^{(b)}$ of η over $\mathbb{Q}(\boldsymbol{\xi})$. Therefore when $P(\xi_1, \dots, \xi_n, \eta) \neq 0$, then each $P(\xi_1, \dots, \xi_n, \eta^{(i)}) \neq 0$, so that $P(\xi_1, \dots, \xi_n, Y)$ and $Q(\xi_1, \dots, \xi_n, Y)$ have no common root, and the resultant $R(\xi_1, \dots, \xi_n) \neq 0$.

4. Proof of Theorems 1A and 2A

Since little extra effort is required, we include a proof of Theorem 1A, already established in [16]. By symmetry it will suffice to show that if every coordinate of $\boldsymbol{\eta} = (\eta_1, \dots, \eta_\ell)$ is algebraic over $\mathbb{Q}(\boldsymbol{\xi}) = \mathbb{Q}(\xi_1, \dots, \xi_n)$, then $\Psi(\boldsymbol{\xi}) \gtrsim \Psi(\boldsymbol{\eta})$. Since $\Psi(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_\ell) \gtrsim \Psi(\eta_1, \dots, \eta_\ell) = \Psi(\boldsymbol{\eta})$ by (1.2), it will be enough to show that $\Psi(\xi_1, \dots, \xi_n) \gtrsim \Psi(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_\ell)$. By transitivity, and using induction on ℓ , we need only show that

$$\Psi(\xi_1, \dots, \xi_n) \gtrsim \Psi(\eta_1, \dots, \xi_n, \eta). \quad (4.1)$$

if $\mathbb{Q}(\boldsymbol{\xi})$ is of positive transcendence degree, and η is algebraic over $\mathbb{Q}(\boldsymbol{\xi})$.

Suppose now that η is of degree b over $\mathbb{Q}(\boldsymbol{\xi})$, so that it satisfies an equation (3.3) with $\deg_Y Q = b$. Then $\xi_1, \dots, \xi_n, \eta$ have a modulus $\leq K = K(\boldsymbol{\xi}, Q)$. Further suppose

$$0 < |P(\xi_1, \dots, \xi_n, \eta)| < |P|^{1-\psi}. \quad (4.2)$$

Then by (3.4) with $\epsilon = |P|^{1-\psi}$,

$$0 < |R(\boldsymbol{\xi})| < c_1(d, Q, \boldsymbol{\xi})|P|^{b-1+1-\psi} = c_1(d, Q, \boldsymbol{\xi})|P|^{b-\psi}.$$

If $\psi > b$ we obtain from (3.2)

$$0 < |R(\boldsymbol{\xi})|/|R| < c_1(d, Q, \boldsymbol{\xi})|R|^{(b-\psi)/b}/|R| = c_1(d, Q, \boldsymbol{\xi})|R|^{-\psi/b}. \quad (4.3)$$

When $\psi_d(\xi_1, \dots, \xi_n, \eta) > b$ and $b < \psi < \psi_d(\xi_1, \dots, \xi_n, \eta)$, there are infinitely many polynomials P of degree $\leq d$ with (4.2), hence infinitely many polynomials $R(X_1, \dots, X_n)$ of total degree $\leq c_2(Q)d^3$ with (4.3). Since ψ may be chosen arbitrarily close to $\psi_d(\xi_1, \dots, \xi_n, \eta)$, we obtain

$$\psi_{ud}(\xi_1, \dots, \xi_n) \geq \psi_d(\xi_1, \dots, \xi_n, \eta)/b \quad (4.4)$$

with $u = c_2(Q)$. But (4.4) is also true when $\psi_d(\xi_1, \dots, \xi_n, \eta) \leq b$, for the left hand side is ≥ 1 by (1.5). The relation (4.1), hence Theorem 1A follows. ■

Next, suppose $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$, $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ are as in Theorem 2A. By symmetry it will suffice to prove

$$\Psi^*(\boldsymbol{\xi}) \gtrsim \Psi^*(\boldsymbol{\eta}). \quad (4.5)$$

³ Recall that the numbering of constants starts anew in each section.

Since ξ_1 is transcendental and $\boldsymbol{\eta}$ has transcendence degree n , there will be $n-1$ numbers among $\eta_1, \dots, \eta_n$, say $\eta_2, \dots, \eta_n$, such that $\xi_1, \eta_2, \dots, \eta_n$ are algebraically independent. Since ξ_2 is transcendental over $\mathbb{Q}(\xi_1)$, and $\eta_2, \dots, \eta_n$ are algebraically independent over $\mathbb{Q}(\xi_1)$, there will be $n-2$ elements among $\eta_2, \dots, \eta_n$, say $\eta_3, \dots, \eta_n$, such that $\xi_2, \eta_3, \dots, \eta_n$ are independent over $\mathbb{Q}(\xi_1)$. Then $\xi_1, \xi_2, \eta_3, \dots, \eta_n$ are algebraically independent. Continuing in this way we see that after suitable renumbering of $\eta_1, \dots, \eta_n$, the points

$$\boldsymbol{\zeta}_i = (\xi_1, \dots, \xi_i, \eta_{i+1}, \dots, \eta_n)$$

have independent coordinates. They also have $\overline{\mathbb{Q}(\boldsymbol{\zeta}_i)} = \overline{\mathbb{Q}(\boldsymbol{\xi})}$ ($i = 0, \dots, n$).

Since $\psi_d^*(\boldsymbol{\eta})$ is independent of the ordering of the coordinates of $\boldsymbol{\eta}$, and by transitivity, it will suffice to show that $\Psi^*(\boldsymbol{\zeta}_{i+1}) \gtrsim \Psi^*(\boldsymbol{\zeta}_i)$ ($i = 0, \dots, n-1$). Changing our notation, it will suffice to show that when

$$\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-1}, \xi) \quad \text{and} \quad \boldsymbol{\eta} = (\xi_1, \dots, \xi_{n-1}, \eta)$$

have transcendence degree n and are algebraically related, then (4.5) holds. To do so we will generalize an argument given in [12].

We have $Q(\xi_1, \dots, \xi_{n-1}, \xi, \eta) = 0$ for some irreducible polynomial $Q \in \mathbb{Q}[X_1, \dots, X_{n-1}, X, Y]$. Since $\mathbb{Q}(\xi_1, \dots, \xi_{n-1}, \xi, \eta)$ has transcendence degree n , Q is unique except for a factor in $\mathbb{Q}^\times$. We may choose Q to have coefficients in $\mathbb{Z}$. Since both $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ have transcendence degree n , Q is of positive degree in X as well as in Y , and moreover the partial derivatives $\partial Q/\partial X$ and $\partial Q/\partial Y$ do not vanish at $(\xi_1, \dots, \xi_{n-1}, \xi, \eta)$.

There is thus a neighborhood $\mathcal{U}$ of $\boldsymbol{\eta} = (\xi_1, \dots, \xi_{n-1}, \eta)$ on which there is defined a regular function $z(X_1, \dots, X_{n-1}, Y)$ having $z(\boldsymbol{\eta}) = z(\xi_1, \dots, \xi_{n-1}, \eta) = \xi$ and $Q(X_1, \dots, X_{n-1}, z(X_1, \dots, X_{n-1}, Y), Y) = 0$. If $\mathcal{U}$ is chosen sufficiently small, then

$$|z(\boldsymbol{\beta}) - \xi| = |z(\boldsymbol{\beta}) - z(\boldsymbol{\eta})| \leq c_3 |\boldsymbol{\beta} - \boldsymbol{\eta}|$$

for $\boldsymbol{\beta} \in \mathcal{U}$, with a constant c_3 depending on $\boldsymbol{\xi}, \boldsymbol{\eta}$.

Suppose now that $P \in \mathcal{P}_d$ has a zero $\boldsymbol{\beta}$ with $|\boldsymbol{\eta} - \boldsymbol{\beta}| < |P|^{-\psi}$. Say $\boldsymbol{\beta} = (\alpha_1, \dots, \alpha_{n-1}, \beta)$. When $\psi > 0$ and $|P|$ is large, then $\boldsymbol{\beta} \in \mathcal{U}$, and $\alpha_n = z(\boldsymbol{\beta})$ has $|\alpha_n - \xi| < c_3 |P|^{-\psi}$, hence $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{n-1}, \alpha_n)$ has

$$|\boldsymbol{\alpha} - \boldsymbol{\xi}| < c_3 |P|^{-\psi},$$

provided c_3 was chosen to be ≥ 1 .

Also,

$$\begin{aligned} Q(\alpha_1, \dots, \alpha_{n-1}, \alpha_n, \beta) &= 0, \\ P(\alpha_1, \dots, \alpha_{n-1}, \beta) &= 0. \end{aligned}$$

We may write P as $P(X_1, \dots, X_{n-1}, Y) \in \mathbb{Z}[X_1, \dots, X_{n-1}, X_n, Y]$. We now apply the conclusions of Section 3 with $\alpha_1, \dots, \alpha_{n-1}, \alpha_n, \beta$ in place of $\xi_1, \dots, \xi_n, \eta$, and

with $\epsilon = 0$. Accordingly, there is a nonzero polynomial $R \in \mathbb{Z}[X_1, \dots, X_n]$ with (3.1), (3.2), and $R(\alpha_1, \dots, \alpha_n) = 0$ by (3.4) with $\epsilon = 0$. We obtain

$$|\alpha - \xi| < c_4(d, Q, \xi, \eta)|R|^{-\psi/b}$$

where $b = \deg_Y Q$.

When $\psi < \psi_d^*(\eta)$, there will be infinitely many $P \in \mathcal{P}_d$ having a zero β with $|\eta - \beta| < |P|^{-\psi}$. Since $R(\xi) \neq 0$, we may conclude that

$$\psi_{ud}^*(\xi) \geq \psi_d^*(\eta)/b$$

where again $u = c_2(Q)$. Now (4.5) follows. $\blacksquare$

5. Proof of Theorem 3

The lower bound in (1.8) follows from (1.2). The upper bound is an immediate consequence of the following

Lemma. *Let $j \in \mathbb{N}$ be given, and suppose for some $\omega > 1$ and some $\xi \in \mathbb{C}_q$, we have*

$$|Q(\xi)|/|Q| > c_1|Q|^{-\omega} > 0 \quad (5.1)$$

for every nonzero polynomial $Q(X) \in \mathbb{Z}[X]$ for degree $\leq j$. Then for $\epsilon > 0$, and for almost every $(\xi_2, \dots, \xi_n) \in \mathbb{C}_q^{n-1}$, the inequality

$$|P(\xi, \xi_2, \dots, \xi_n)||P|^{-1} < |P|^{-\omega - (jm_{n,j}/q) - \epsilon} \quad (5.2)$$

holds for at most finitely many polynomials $P \in \mathcal{P}_j$.

Proof. It will be enough to consider points $\xi' = (\xi_2, \dots, \xi_n)$ with $|\xi'| \leq K$, where K is arbitrary but fixed.

Set $R(\mathbf{X}') = R(X_2, \dots, X_n) = P(\xi, X_2, \dots, X_n)$, and let r be the total degree of R . By our hypothesis on ξ , $R \neq 0$. There is then some nonzero partial derivative $R^{(\mathbf{r})}$ of order r which is a nonzero “constant”. It is of the form $R^{(\mathbf{r})} = Q(\xi)$ where $Q \in \mathbb{Z}[X]$, $Q \neq 0$, $\deg Q \leq j$, and $|Q| \leq r!|P| \leq j!|P|$. Therefore $|R^{(\mathbf{r})}| = |Q(\xi)| > c_1|Q|^{1-\omega}$, so that (5.2) yields

$$|R(\xi')|/|R^{(\mathbf{r})}(\xi')| < c_2|P|^{-(jm_{n,j}/q) - \epsilon}.$$

The constant c_2 , as well as subsequent constants $c_3, \dots$ may depend on c_1, ϵ, n, j, K .

By [13, Lemma 4.3] there is a partial derivative $R^{(\ell)} \neq 0$ having a zero $\alpha' = (\alpha_2, \dots, \alpha_n) \in \mathbb{C}_q^{n-1}$ with

$$\begin{aligned} |\xi' - \alpha'| &< c_3|R(\xi')/R^{(\mathbf{r})}(\xi')|^{1/r} \\ &< c_4|P|^{-(jm_{n,j}/q + \epsilon)/r} \\ &< c_4|P|^{-(m_{n,j}/q + \epsilon/r)} = \Delta(P) = \Delta, \end{aligned}$$

say. Thus ξ' has distance (with respect to the maximum norm) less than Δ from the set of zeros of $R^{(\ell)}$ in $\mathbb{C}_q^{n-1}$. Since $|\xi'| \leq K$, we see that ξ' lies in a set of measure $< c_5 \Delta^q$. Taking the sum over the nonzero partial derivatives of R , we get a set $\mathcal{S}(P)$ of measure

$$\mu(\mathcal{S}(P)) < c_6 \Delta^q < c_7 |P|^{-m_{n,j} - \epsilon q/r}.$$

There are $\leq c_8 h^{m_{n,j}-1}$ polynomials $P \in \mathcal{P}_j$ with $|P| = h$. Therefore

$$\sum_{P \in \mathcal{P}_j} \mu(\mathcal{S}(P)) < c_7 c_8 \sum_{h=1}^{\infty} h^{-1 - \epsilon q/r} < \infty.$$

The desired conclusion now follows from the Borel–Cantelli Lemma. $\blacksquare$

6. On U_d -numbers

We will sketch the construction of U_d -numbers with (1.9). Our construction will involve trivial modifications of the argument in [4], which we assume to be available to the reader. To conform with the notations of that exposition, we will exhibit U_m -numbers ξ with $m > 1$, having

$$\psi_n(\xi) \leq m + n \quad \text{for } n = 1, \dots, m-1. \quad (6.1)$$

When F_n is a nonzero polynomial in $\mathbb{Z}[X]$ of degree $n < m$, and both $|\xi'| < 4$, $|\xi''| < 4$, then $|F_n(\xi') - F_n(\xi'')| < c_0 |F_n| |\xi' - \xi''|$ with a constant c_0 depending only on m . Setting

$$F_n \langle \xi \rangle = F_n(\xi) / (c_0 |F_n|),$$

we then have

$$|F_n \langle \xi' \rangle - F_n \langle \xi'' \rangle| \leq |\xi' - \xi''|. \quad (6.2)$$

As in [4], our construction will involve a sequence of real algebraic numbers $\xi_1, \xi_2, \dots$ of degree m . These numbers will have modulus $|\xi_i| < 4$. $H(\xi_i)$ will denote the Height of ξ_i , i.e., $H(\xi_i) = |P_i|$ where P_i is the defining polynomial of ξ_i over $\mathbb{Z}$. Throughout, F_n will play the role of the algebraic number β_n of [4], and $|F_n|$ will play the role of $H(\beta_n)$. All the lower bounds for quantities $|\xi_i - \beta_n|$ will be replaced by lower bounds for $|F_n \langle \xi_i \rangle|$. For ξ_i of degree m and F_n of degree $n < m$,

$$|F_n \langle \xi_i \rangle| > c_1^{-1} H(\xi_i)^{-n} |F_n|^{-m}$$

by the case $t = 1$ of Theorem A.1 of [4], where c_1 depends only on m .

We start with ξ_1 of degree m of modulus $|\xi_1| < 1$. The inequality (7.35) of [4] now becomes

$$|F_n \langle \xi_1 \rangle| > c_1^{-1} H(\xi_1)^{-n} |F_n|^{-m} \geq c_1^{-1} H(\xi_1)^{-m} |F_n|^{-m}.$$

As in (7.36) of [4] we obtain infinitely many numbers ξ_2 of degree m with

$$0 < |\xi_1 - \xi_2| < \kappa(\xi_1)H(\xi_2)^{-m}.$$

In view of (6.2),

$$\begin{aligned} |F_n\langle\xi_2\rangle| &\geq |F_n\langle\xi_1\rangle| - |\xi_1 - \xi_2| \\ &> c_1^{-1}H(\xi_1)^{-m}|F_n|^{-m} - \kappa(\xi_1)H(\xi_2)^{-m}. \end{aligned}$$

Arguing as for (7.37) of [4], we obtain

$$|F_n\langle\xi_2\rangle| \geq (c_2 + 1/2)^{-1}|F_n|^{-m}$$

with $c_2 = c_1H(\xi_1)^m$, provided $|F_n| \leq A$.

Continuing the construction by appropriately modifying the argument in [4], we obtain $\xi_1, \xi_2, \dots$ with (7.40) of that treatise, so that indeed each $|\xi_i| < 4$, and with

$$|F_n\langle\xi_{k+1}\rangle| \geq (c^*)^{-1}|F_n|^{-n-m}(\log 3|F_n|)^{-1} \quad (n = 1, \dots, m-1)$$

for $k = 1, 2, \dots$ and $c^* = c_2 + \sum_{i=1}^{\infty} 2^{-i}$. This relation remains true if ξ_{k+1} is replaced by $\xi = \lim \xi_k$. Thus (6.1) holds. Moreover, $\psi_m^*(\xi) = \infty$ as in [4], so that ξ is a U_m -number. ■

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