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A PARITY PROBLEM ON THE FREE PATH LENGTH OF A BILLIARD IN THE UNIT SQUARE WITH POCKETS

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Dedicated to Professor Eduard Wirsing
on the occasion
of his seventy-fifth birthday

Abstract: We present a result on short intervals about the moments of the free path length of the linear trajectory of a billiard in the unit square with small triangular pockets of size ε removed at the corners, in which the trajectory ends in a specified corner pocket.

Keywords: Billiards, periodic Lorentz gas, free path length, Farey fractions, visible points, Kloosterman sums.

1. Introduction and statement of results

A variety of ergodic and statistical properties of the periodic Lorentz gas and billiards have been intensively studied in the last decades by a number of authors (see for example [8], [9], and [10]). Such systems were introduced by Lorentz [21] in 1905 to investigate the dynamics of electrons in metals. In [4] and [5], a problem on the length of the linear trajectory of a two-dimensional Euclidean billiard generated by the free motion of a single billiard ball subject to elastic reflections on the boundary of the unit square $[0,1]^2 \subset \mathbb{R}^2$ having vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, with small pockets of size ε removed at the four corners, is considered. The billiard problem has the point mass moving from the origin along a geodesic line with constant speed and angle $\theta \in [0,\pi/2]$, until it collides with the boundary. At a smooth boundary point, the billiard ball reflects so that the tangential component of its velocity remains the same, while the normal component changes its sign. The reflection is specular, and the trajectory between two such reflections is rectilinear. The motion ends when the billiard ball reaches one of the corner pockets. In [4], the kth moment of the *free path length* (also called the *first exit time* by some authors) is estimated for any subinterval of the interval $[0,\pi/2]$. The purpose of this paper is to see precisely what can be said about the contribution each corner

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pocket will make to these moments. Our study is carried out within the framework of number theory provided in [4] and [5]. We consider separately the moments over subsets of the trajectories that end in each corner pocket and determine whether each pocket has asymptotically equivalent contribution to these moments. In order to differentiate between trajectories that end in the corner pockets, we will need to consider questions on the distribution of visible points in the coordinate plane, and hence Farey fractions, with parity constraints. The use of Farey fractions is strongly influenced by geometric ideas and further links the billiard problem to the distribution of inverses in residue classes, in which Kloosterman sums play an important role. For work done in this direction, see [15], [20], [14], [2], [3], [17], and [18]. For surveys of Farey fractions, see [12], [16], [11], and [7].

We begin by introducing some notations. Let $l_\varepsilon(\theta)$ denote the length of the trajectory of a particle moving with angle $\theta \in [0, \pi/2]$ from the origin. For $i, j \in \{0, 1\}$, let $A_{i,j}$ be the set of angles θ for which the trajectory ends at the corner (i, j). For any $k \geqslant 1$, we write the kth moment of $l_\varepsilon(\theta)$ over the interval $[0, \pi/2]$ as a sum of four integrals corresponding to the contribution made by each of the pockets at the corners (i, j), that is,

$$\int_0^{\frac{\pi}{2}} l_\varepsilon^k(\theta)\, d\theta = \sum_{i,j \in \{0,1\}} \int_{A_{i,j}} l_\varepsilon^k(\theta)\, d\theta. \tag{1.1}$$

According to Theorem 1.2 in [4] (p. 305), the kth moment of $l_\varepsilon(\theta)$ over any fixed subinterval $[\alpha, \beta] \subseteq [0, \pi/4]$ is asymptotic to a constant depending on k, α, and β only, times ε^{-k}. Thus, for any k, α, β, $\delta > 0$,

$$\int_\alpha^\beta l_\varepsilon^k(\theta)\, d\theta = c_k \varepsilon^{-k} \int_\alpha^\beta \frac{dx}{\cos^k x} + O_{k,\delta}(\varepsilon^{-k+\frac{1}{6}+\delta}), \tag{1.2}$$

where the constant c_k is proved in Theorem 1.1 from [4] (p. 304) to be

$$c_k = \frac{12}{\pi^2} \int_0^{\frac{1}{2}} \left(x(x^{k-1} + (1-x)^{k-1}) + \frac{1-(1-x)^k}{kx(1-x)} - \frac{1-(1-x)^{k+1}}{(k+1)x(1-x)} \right) dx.$$

We will see that the kth moment of $l_\varepsilon(\theta)$ splits up asymptotically into three equivalent parts for each pocket at the corners $(1, 0)$, $(1, 1)$, and $(0, 1)$, while the pocket at the corner $(0, 0)$ makes no contribution at all to these moments. More precisely, we will prove the following result.

Theorem 1.1. *For any subinterval* $[\alpha, \beta] \subseteq [0, \pi/4]$, *any* $k \geqslant 1$, *any* $\delta > 0$, *and any* $\varepsilon > 0$, *we have*

$$\int_{A_{i,j} \cap [\alpha,\beta]} l_\varepsilon^k(\theta)\, d\theta = \begin{cases} 0 & \text{if } (i,j) = (0,0), \\ \frac{c_k \varepsilon^{-k}}{3} \int_\alpha^\beta \frac{dx}{\cos^k x} + O_{k,\delta}(\varepsilon^{-k+\frac{1}{14}-\delta}) & \text{otherwise.} \end{cases}$$

2. Counting inverses with parity constraints

In this section, we prove some lemmas that will be used in the proof of Theorem 1.1 in Section 3. An important tool employed in [1], [4], [5], and [6], to estimate sums over primitive lattice points is the Weil-Salié type [22] inequality

$$|K(m,n;q)| \ll \sigma_0(q)\gcd(m,n,q)^{\frac{1}{2}}q^{\frac{1}{2}}, \tag{2.1}$$

proved in [19] and [13], for complete Kloosterman sums

$$K(m,n;q) = \sum_{\substack{x \pmod q \\ \gcd(x,q)=1}} e\left(\frac{mx+n\bar{x}}{q}\right)$$

in the presence of an integer albeit not necessarily prime modulus q. Here σ_0 is the "number of divisors" function and $\bar{x}$ denotes the multiplicative inverse of x $\pmod q$. The bound (2.1) is used to prove, for a fixed integer $q \geqslant 2$ and any subintervals $I, J \subset [0,q)$, the estimate (see Lemma 1.7 in [1], p. 445)

$$\begin{aligned} N_q(I,J) &:= |\{(x,y) \in I \times J \colon xy \equiv 1 \pmod q\}| \\ &= \frac{\varphi(q)}{q^2}|I||J| + O_\delta(q^{\frac{1}{2}+\delta}), \end{aligned} \tag{2.2}$$

where φ stands for Euler's totient function. Thus, the arithmetic problem concerning the number of solutions of the congruence $xy \equiv 1 \pmod q$, where $(x,y) \in I \times J$, is reduced to the estimate of exponential sums.

We remark that the ordinary incomplete Kloosterman sums

$$K_I(m,n;q) = \sum_{\substack{x \in I \\ \gcd(x,q)=1}} e\left(\frac{mx+n\bar{x}}{q}\right)$$

may be written in terms of the complete Kloosterman sums, so that the inequality (2.1) gives (see Lemma A2 in [6], p. 1823)

$$|K_I(0,n;q)| \ll_\delta \gcd(n,q)^{\frac{1}{2}}q^{\frac{1}{2}+\delta}.$$

This bound is used to prove, for any integer j, the estimate (see Proposition A3 in [6], p. 1823)

$$\begin{aligned} N_{q,j}(I,J) &:= |\{(x,y) \in I \times J \colon \gcd(x,q)=1, xy = j \pmod q\}| \\ &= \frac{\varphi(q)}{q^2}|I||J| + O_\delta(q^{\frac{1}{2}+\delta}\gcd(j,q)^{\frac{1}{2}}). \end{aligned} \tag{2.3}$$

Estimates (2.2) and (2.3) allow us to immediately deduce the following key technical tool.

Lemma 2.1. *Let $i,j\in\{0,1\}$ and $\delta>0$. Assume that $q\geqslant 1$ is an integer such that $q\equiv i \pmod 2$ and $I,J\subset[0,q)$. Denote by $V_{q,j}(I,J)$ the number of pairs of integers $(a,b)\in I\times J$ for which $ab\equiv 1 \pmod q$ and $a\equiv j \pmod 2$. Then*

$$\begin{aligned} V_{q,j}(I,J) &:= |\{(a,b)\in I\times J\colon ab\equiv 1 \pmod q, a\equiv j \pmod 2\}| \\ &= \frac{\eta_{i,j}\varphi(q)}{2q^2}|I||J| + O_\delta(q^{\frac12+\delta}), \end{aligned}$$

where

$$\eta_{i,j} = \begin{cases} 0 & \text{if } (i,j)=(0,0), \\ 1 & \text{if } (i,j)=(0,1), \\ 1/2 & \text{if } (i,j)=(1,0) \text{ or } (1,1). \end{cases}$$

Proof. It is easily seen that $V_{q,0}(I,J)=0$, if $(i,j)=(0,0)$. Since

$$V_{q,0}(I,J)+V_{q,1}(I,J) = |\{(a,b)\in I\times J\colon ab\equiv 1 \pmod q\}|,$$

if $(i,j)=(0,1)$, then by (2.2)

$$V_{q,1}(I,J) = \frac{\varphi(q)}{q^2}|I||J| + O_\delta(q^{\frac12+\delta}).$$

If $(i,j)=(1,0)$, then by (2.3)

$$\begin{aligned} V_{q,0}(I,J) &= |\{(x,y)\in I/2\times J\colon \gcd(x,q)=1, xy\equiv \bar{2} \pmod q\}| \\ &= N_{q,\bar{2}}(I/2,J) = \frac{\varphi(q)}{2q^2}|I||J| + O_\delta(q^{\frac12+\delta}). \end{aligned}$$

Lastly, if $(i,j)=(1,1)$, then by (2.2)

$$V_{q,1}(I,J) = N_q(I,J) - V_{q,0}(I,J) = \frac{\varphi(q)}{2q^2}|I||J| + O_\delta(q^{\frac12+\delta}).$$

This completes the proof of the lemma. ■

Next, we note some important corollaries of Lemma 2.1. We give a detailed proof for Lemma 2.2 below, and indicate the modifications needed for Lemma 2.3. For each subintervals $I,J\subset\mathbb{R}$ and each C^1 function $f\colon I\times J\to\mathbb{R}$, we denote

$$\begin{aligned} \|f\|_\infty &= \sup_{(x,y)\in I\times J} |f(x,y)|, \\ \|Df\|_\infty &= \sup_{(x,y)\in I\times J} \left(\left|\frac{\partial f}{\partial x}(x,y)\right| + \left|\frac{\partial f}{\partial y}(x,y)\right| \right). \end{aligned}$$

Lemma 2.2. *Let $i,j\in\{0,1\}$ and $\delta>0$. Let $\eta_{i,j}$ be as in Lemma 2.1. Assume that $q,T\geqslant 1$ are integers, $q\equiv i \pmod 2$, and $f\colon I\times J\to\mathbb{R}$ is a C^1 function with $I,J\subset[0,q)$. Then*

$$\sum_{\substack{(a,b)\in I\times J \\ ab\equiv 1 \pmod q \\ a\equiv j \pmod 2}} f(a,b) = \frac{\eta_{i,j}\varphi(q)}{2q^2} \iint\limits_{I\times J} f + E_{q,I,J,f,T},$$

where

$$E_{q,I,J,f,T} \ll_\delta T^2 q^{\frac12+\delta}\|f\|_\infty + T\|Df\|_\infty\left(q^{\frac32+\delta} + \frac{|I||J|}{T^2}\right).$$

Proof. With the proof of Lemma 2.2 from [4] (pp. 309–310) as a guide, we observe on the one hand that if $T \geqslant q$, the error is larger than the sum to estimate. Hence, there is nothing to prove.

On the other hand, if $T < q$, we approximate the function $f(x,y)$ by a constant whenever $(x,y) \in I_r \times J_s$ by partitioning the intervals I and J, respectively, into T intervals, $I_1, \ldots, I_T$ and $J_1, \ldots, J_T$, of equal size $|I_r| = |I|/T$ and $|J_s| = |J|/T$. For each pair of indices (r,s), we choose a point $(x_{rs}, y_{rs}) \in I_r \times J_s$ for which

$$\iint\limits_{I_r \times J_s} f = |I_r||J_s| f(x_{rs}, y_{rs}). \tag{2.4}$$

Now for $(x,y) \in I_r \times J_s$, the mean-value theorem gives

$$f(x,y) = f(x_{rs}, y_{rs}) + O((|I_r| + |J_s|)\|Df\|_\infty) = f(x_{rs}, y_{rs}) + O\left(\frac{q}{T}\|Df\|_\infty\right),$$

from which follows

$$\begin{aligned} \sum_{\substack{(a,b)\in I\times J \\ ab\equiv 1 \pmod q \\ a\equiv j \pmod 2}} f(a,b) &= \sum_{r,s=1}^{T} \sum_{\substack{(x,y)\in I_r\times J_s \\ xy\equiv 1 \pmod q \\ x\equiv j \pmod 2}} f(x,y) \\ &= \sum_{r,s=1}^{T} V_{q,j}(I_r, J_s)\left[f(x_{rs}, y_{rs}) + O\left(\frac{q\|Df\|_\infty}{T}\right)\right]. \end{aligned} \tag{2.5}$$

Since $I_r, J_s \subset [0,q)$, Lemma 2.1 applies "infinitesimally" and produces

$$V_{q,j}(I_r, J_s) = \frac{\eta_{i,j}\varphi(q)}{2q^2}|I_r||J_s| + O_\delta(q^{\frac{1}{2}+\delta}).$$

In view of (2.4), the main term on the right side of (2.5) becomes

$$\begin{aligned} &\frac{\eta_{i,j}\varphi(q)}{2q^2} \sum_{r,s=1}^{T} |I_r||J_s| f(x_{rs}, y_{rs}) + O_\delta(T^2 q^{\frac{1}{2}+\delta}\|f\|_\infty) \\ &= \frac{\eta_{i,j}\varphi(q)}{2q^2} \iint\limits_{I\times J} f + O_\delta(T^2 q^{\frac{1}{2}+\delta}\|f\|_\infty), \end{aligned}$$

whereas the error term there will be

$$\ll \frac{q\|Df\|_\infty}{T}\left(T^2 q^{\frac{1}{2}+\delta} + \frac{\varphi(q)}{q^2}|I||J|\right) \ll T\|Df\|_\infty\left(q^{\frac{3}{2}+\delta} + \frac{|I||J|}{T^2}\right).$$

Piecing this together gives the required result. ■

Lemma 2.3. *Let $i, j \in \{0,1\}$ and $\delta > 0$. Let $\eta_{i,j}$ be as in Lemma 2.1. Assume that $q, T_1, T_2 \geqslant 1$ are integers, $q \equiv i \pmod 2$, and $f\colon I \times J \to \mathbb{R}$ is a C^1 function with $I, J \subset [0, q)$. Then*

$$\sum_{\substack{(a,b)\in I\times J\\ ab\equiv 1 \pmod q\\ a\equiv j \pmod 2}} f(a,b) = \frac{\eta_{i,j}\varphi(q)}{2q^2} \iint\limits_{I\times J} f + E_{q,I,J,f,T},$$

where

$$E_{q,I,J,f,T} \ll_\delta T_1T_2 q^{\frac{1}{2}+\delta}\|f\|_\infty + (T_2\|D_x f\|_\infty + T_1\|D_y f\|_\infty)\left(q^{\frac{3}{2}+\delta} + \frac{|I||J|}{T_1T_2}\right).$$

Proof. Only the case $T < q$ is interesting. We approximate the function $f(x,y)$ by a constant whenever $(x,y) \in I_r \times J_s$ by partitioning the intervals I and J, respectively, into T_1 intervals $I_1, \ldots, I_{T_1}$ and T_2 intervals $J_1, \ldots, J_{T_2}$ of equal size $|I_r| = |I|/T_1$ and $|J_s| = |J|/T_2$. For $(x,y) \in I_r \times J_s$, by the mean-value theorem

$$\begin{aligned}\sum_{\substack{(a,b)\in I\times J\\ ab\equiv 1 \pmod q\\ a\equiv j \pmod 2}} f(a,b) &= \sum_{r=1}^{T_1}\sum_{s=1}^{T_2} \sum_{\substack{(x,y)\in I_r\times J_s\\ xy\equiv 1 \pmod q\\ x\equiv j \pmod 2}} f(x,y)\\ &= \sum_{r=1}^{T_1}\sum_{s=1}^{T_2} V_{q,j}(I_r,J_s)\left[f(x_{rs},y_{rs}) + O\left(\left(\frac{1}{T_1}+\frac{1}{T_2}\right)q\|Df\|_\infty\right)\right].\end{aligned}$$

By virtue of Lemma 2.1, the main term is

$$\begin{aligned}&\frac{\eta_{i,j}\varphi(q)}{2q^2}\sum_{r=1}^{T_1}\sum_{s=1}^{T_2}|I_r||J_s|f(x_{rs},y_{rs}) + O_\delta(T_1T_2q^{\frac{1}{2}+\delta}\|f\|_\infty)\\ &\quad= \frac{\eta_{i,j}\varphi(q)}{2q^2}\iint\limits_{I\times J} f + O_\delta(T_1T_2q^{\frac{1}{2}+\delta}\|f\|_\infty),\end{aligned}$$

while the error term is

$$\begin{aligned}&\ll q\|Df\|_\infty\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\left(T_1T_2q^{\frac{1}{2}+\delta} + \frac{\varphi(q)}{q^2}|I||J|\right)\\ &\ll (T_2\|D_xf\|_\infty + T_1\|D_yf\|_\infty)\left(q^{\frac{3}{2}+\delta} + \frac{|I||J|}{T_1T_2}\right).\end{aligned}$$

Hence, the required result follows obviously. ∎

We will need one other key summation formula.

Lemma 2.4. *Let $i \in \{0,1\}$. Assume that $0 < a < b$ are real numbers and f is a continuous piecewise C^1 function on the interval $[a,b]$. Then*

$$\sum_{\substack{a<k\leqslant b\\ k\equiv i \pmod 2}} \frac{\varphi(k)}{k} f(k) = \frac{\varrho_i}{\zeta(2)}\int_a^b f + O\left(\log(2+b)\left(\|f\|_\infty + \int_a^b |f'|\right)\right),$$

where $\varrho_i = (i+1)/3$.

Proof. Let $i=0$. We use the Möbius function and rearrange summations to see that

$$\begin{aligned}\sum_{\substack{a<k\leqslant b\\ k \text{ even}}} \frac{\varphi(k)}{k} f(k) &= \sum_{1\leqslant d\leqslant b} \frac{\mu(d)}{d} \sum_{\substack{a<k\leqslant b\\ k \text{ even}\\ d\mid k}} f(k)\\ &= \sum_{\substack{1\leqslant d\leqslant b\\ d \text{ even}}} \frac{\mu(d)}{d} \sum_{\substack{a<k\leqslant b\\ d\mid k}} f(k) + \sum_{\substack{1\leqslant d\leqslant b\\ d \text{ odd}}} \frac{\mu(d)}{d} \sum_{\substack{a<k\leqslant b\\ 2d\mid k}} f(k)\\ &= \sum_{\substack{1\leqslant d\leqslant b\\ d \text{ even}}} \frac{\mu(d)}{d} \sum_{a/d<m\leqslant b/d} f_d(m) + \sum_{\substack{1\leqslant d\leqslant b\\ d \text{ odd}}} \frac{\mu(d)}{d} \sum_{a/2d<m\leqslant b/2d} f_{2d}(m),\end{aligned}$$

where $f_d(x)=f(dx)$. By Lemma 2.2 from [1] (p. 448) and using the fact that

$$\sum_{\substack{d\geqslant 1\\ (d,2)=1}} \frac{\mu(d)}{d^2} = \frac{4}{3\zeta(2)},$$

the right side above equals

$$\begin{aligned}&\left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \frac{1}{2} \sum_{\substack{d=1\\ d \text{ odd}}}^{\infty} \frac{\mu(d)}{d^2}\right) \int_a^b f + O\left(\log(2+b)\left(\|f\|_\infty + \int_a^b |f'|\right)\right)\\ &\quad = \frac{1}{3\zeta(2)} \int_a^b f + O\left(\log(2+b)\left(\|f\|_\infty + \int_a^b |f'|\right)\right).\end{aligned}$$

Now let $i=1$. By Lemma 2.3 from [1] (p. 449), we have

$$\begin{aligned}\sum_{\substack{a<k\leqslant b\\ k \text{ odd}}} \frac{\varphi(k)}{k} f(k) &= \sum_{a<k\leqslant b} \frac{\varphi(k)}{k} f(k) - \sum_{\substack{a<k\leqslant b\\ k \text{ even}}} \frac{\varphi(k)}{k} f(k)\\ &= \frac{2}{3\zeta(2)} \int_a^b f + O\left(\log(2+b)\left(\|f\|_\infty + \int_a^b |f'|\right)\right).\end{aligned}$$

This completes the proof of the lemma. ∎

3. Proof of Theorem 1.1

As in [4] and [5], we work with the equivalent formulation of the billiard problem in the plane. We pave the unit lattice by placing around each integer point (n,m)

the square with vertices $(n+\varepsilon, m)$, $(n, m+\varepsilon)$, $(n-\varepsilon, m)$, and $(n, m-\varepsilon)$. For each angle $\theta \in [0, \pi/2]$, we consider the trajectory that starts from the origin and ends at one of these squares. The length of this trajectory equals the length of the trajectory in the original billiard problem. Further, this trajectory never ends at a square around a point (n, m) that is not visible from the origin. We remark that a necessary and sufficient condition for the point (n, m) to be *visible* is that the integers m and n be relatively prime. Thus, no trajectory in the plane can end at the square around an integer point with both coordinates being even. Hence, the set $A_{0,0}$ is empty and the pocket at the corner $(0, 0)$ makes no contribution to the kth moments of $l_\varepsilon(\theta)$.

Let $[x]$ be the integer part of a real number x. It is proved in Lemma 3.1 from [4] (p. 311), for $Q = [1/\varepsilon]$ with $0 < \varepsilon < 1/2$ fixed, that for any angle $\theta \in [0, \pi/4]$ the corresponding trajectory always ends on the square around a visible point that lies inside or on the sides of the triangle with vertices $(0, 0)$, $(Q, 0)$, and (Q, Q). The slopes of the straight lines from the origin through these visible points are Farey fractions $a/q \in (0, 1]$ with $1 \leqslant a \leqslant q \leqslant Q$ and $\gcd(a, q) = 1$ in the *Farey sequence* $\mathcal{F}_Q$ *of order* Q. We remark that, given an arbitrary angle $\theta \in [0, \pi/4]$, the slope $\tan\theta \in [0, 1]$ will lie between two consecutive Farey fractions. Thus, the trajectory from the origin at angle θ will end at the boundary of the square around one of the corresponding visible points.

Now let $a''/q'' < a/q < a'/q'$ be three consecutive fractions in $\mathcal{F}_Q$. Using Lemmas 2.2 and 2.3 from [5] (pp. 60–61), we distinguish between the following four cases. For brevity's sake, we will let

$$J_{q,a,\varepsilon} = \begin{cases} \left[\arctan \frac{a-\varepsilon}{q}, \arctan \frac{a+\varepsilon}{q}\right] & \text{if } q < \min(q', q''), \\ \left[\arctan \frac{a''+\varepsilon}{q''}, \arctan \frac{a'-\varepsilon}{q'}\right] & \text{if } q > \max(q', q''), \\ \left[\arctan \frac{a-\varepsilon}{q}, \arctan \frac{a'-\varepsilon}{q'}\right] & \text{if } q' < q < q'', \\ \left[\arctan \frac{a-\varepsilon}{q}, \arctan \frac{a''-\varepsilon}{q''}\right] & \text{if } q'' < q < q'. \end{cases}$$

Here we note that $|J_{q,a,\varepsilon}|$ equals

$$\begin{cases} \frac{2\varepsilon q}{q^2+a^2} + O\left(\frac{\varepsilon^2}{q^2+a^2}\right) & \text{if } q < \min(q', q''), \\ \left(\frac{1}{qq'} + \frac{1}{qq''} - \frac{\varepsilon}{q'} - \frac{\varepsilon}{q''}\right)\frac{q^2}{q^2+a^2} + O\left(\varepsilon^2 \max\left(\frac{1}{q'}, \frac{1}{q''}\right)\right) & \text{if } q > \max(q', q''), \\ \left(\frac{1}{qq'} - \frac{\varepsilon}{q'} + \frac{\varepsilon}{q}\right)\frac{q^2}{q^2+a^2} + O\left(\varepsilon^2 \max\left(\frac{1}{q}, \frac{1}{q'}\right)\right) & \text{if } q' < q < q'', \\ \left(\frac{1}{qq''} - \frac{\varepsilon}{q''} + \frac{\varepsilon}{q}\right)\frac{q^2}{q^2+a^2} + O\left(\varepsilon^2 \max\left(\frac{1}{q}, \frac{1}{q''}\right)\right) & \text{if } q'' < q < q'. \end{cases}$$

We remark that every slope $\tan\theta \in [0,1]$ through the origin will necessarily intersect the sets $A_{i,j}$. If we put

$$A_{i,j} = \bigcup_{\substack{a/q\in\mathcal{F}_Q\\ (q,a)\equiv(i,j) \pmod 2}} J_{q,a,\varepsilon},$$

where $J_{q,a,\varepsilon}$ is the set of angles $\theta \in [0,\pi/4]$ for which the trajectory starts at angle θ from the origin and ends at the square around the point (q,a) with $a/q \in \mathcal{F}_Q$, then

$$\int_{A_{i,j}\cap[\alpha,\beta]} l_\varepsilon^k(\theta)\,d\theta = \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j) \pmod 2}} \int_{J_{q,a,\varepsilon}} l_\varepsilon^k(\theta)\,d\theta.$$

For any angle $\theta \in J_{q,a,\varepsilon}$, we have

$$l_\varepsilon(\theta) = (q^2+a^2)^{\frac12}\left[1+O\left(\frac{\varepsilon}{(q^2+a^2)^{\frac12}}\right)\right],$$

so that

$$l_\varepsilon^k(\theta) = (q^2+a^2)^{\frac{k}{2}} + O_k(Q^{k-2}).$$

Using this and the fact that $(q^2+a^2)^{\frac{k}{2}}|J_{q,a,\varepsilon}|$ equals

$$\begin{cases} 2\varepsilon q(q^2+a^2)^{\frac{k}{2}-1}+O(Q^{k-4}) & \text{if } q<\min(q',q''),\\ \left(\frac{1}{qq'}+\frac{1}{qq''}-\frac{\varepsilon}{q'}-\frac{\varepsilon}{q''}\right)q^2(q^2+a^2)^{\frac{k}{2}-1} & \\ \quad +O\left(Q^{k-2}\max\left(\frac{1}{q'},\frac{1}{q''}\right)\right) & \text{if } q>\max(q',q''),\\ \left(\frac{1}{qq'}-\frac{\varepsilon}{q'}+\frac{\varepsilon}{q}\right)q^2(q^2+a^2)^{\frac{k}{2}-1}+O\left(\frac{Q^{k-2}}{q'}\right) & \text{if } q'<q<q'',\\ \left(\frac{1}{qq''}-\frac{\varepsilon}{q''}+\frac{\varepsilon}{q}\right)q^2(q^2+a^2)^{\frac{k}{2}-1}+O\left(\frac{Q^{k-2}}{q''}\right) & \text{if } q''<q<q', \end{cases}$$

and recognizing that the cardinality $\Phi(Q) = \phi(1)+\ldots+\phi(Q)$ of $\mathcal{F}_Q$ grows quadratically in Q (see Theorem 330, p. 268 in [16]), we have

$$
\begin{aligned}
&\int_{A_{i,j}\cap[\alpha,\beta]} l_\varepsilon^k(\theta)\,d\theta \\
&= \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j)\pmod 2}} \int_{J_{q,a,\varepsilon}} ((q^2+a^2)^{\frac{k}{2}} + O_k(Q^{k-2}))\,d\theta \\
&= \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j)\pmod 2}} \left(\int_{J_{q,a,\varepsilon}} (q^2+a^2)^{\frac{k}{2}}\,d\theta + O_k(Q^{k-2}|J_{q,a,\varepsilon}|)\right) \\
&= \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j)\pmod 2}} (q^2+a^2)^{\frac{k}{2}}|J_{q,a,\varepsilon}| + O_k\left(Q^{k-2}\sum_{a/q\in\mathcal{F}_Q}|J_{q,a,\varepsilon}|\right) \\
&= S_{i,j,1}+S_{i,j,2}+S_{i,j,3}+S_{i,j,4}+O_k(Q^{k-1}),
\end{aligned}
\tag{3.1}
$$

where

$$
S_{i,j,1} = \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j)\pmod 2\\ q<\min(q',q'')}} 2\varepsilon q(q^2+a^2)^{\frac{k}{2}-1},
$$

$$
S_{i,j,2} = \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j)\pmod 2\\ q>\max(q',q'')}} \left(\frac{1}{qq'}+\frac{1}{qq''}-\frac{\varepsilon}{q'}-\frac{\varepsilon}{q''}\right)q^2(q^2+a^2)^{\frac{k}{2}-1},
$$

$$
S_{i,j,3} = \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j)\pmod 2\\ q'<q<q''}} \left(\frac{1}{qq'}-\frac{\varepsilon}{q'}+\frac{\varepsilon}{q}\right)q^2(q^2+a^2)^{\frac{k}{2}-1},
$$

$$
S_{i,j,4} = \sum_{\substack{a/q\in\mathcal{F}_Q\\ \tan\alpha\leqslant a/q\leqslant\tan\beta\\ (q,a)\equiv(i,j)\pmod 2\\ q''<q<q'}} \left(\frac{1}{qq''}-\frac{\varepsilon}{q''}+\frac{\varepsilon}{q}\right)q^2(q^2+a^2)^{\frac{k}{2}-1}.
$$

We now focus our attention on the sum $S_{i,j,1}$. First, we use the definition and properties of consecutive Farey fractions to express the condition $q<\min(q',q'')$ in terms of a and q only. Recall that if $a/q<a'/q'$ are two consecutive fractions in $\mathcal{F}_Q$, then $a'q-aq'=1$ and $q+q'>Q$. Conversely, if $q,q'\in[1,Q]$ and $q+q'>Q$, then there are integers $a \in [1, q-1]$ and $a' \in [1, q'-1]$ such that $a/q < a'/q'$ are consecutive fractions in $\mathcal{F}_Q$ (see Theorems 28 and 29, p. 24 in [16]). Thus, we have $a' = (aq'+1)/q$ since $a'q - aq' = 1$. Since $q + q' > Q$, there are q consecutive integers in the interval $Q - q < q' \leqslant Q$. The fact that $a'q - aq' = 1$ implies $-aq' \equiv 1 \pmod{q}$. In addition, since $\gcd(a,q) = 1$ we have $q' \equiv -\bar{a} \pmod{q}$, where $\bar{a}$ is the multiplicative inverse of $a \pmod{q}$. This residue class is uniquely determined and exactly one integer in the interval $Q - q < q' \leqslant Q$ satisfies this congruence. Now, $Q - q + \bar{a} < q' + \bar{a} \leqslant Q + \bar{a}$ and we have

$$\frac{Q+\bar{a}}{q} - 1 < \frac{q'+\bar{a}}{q} \leqslant \frac{Q+\bar{a}}{q}.$$

Because $q \mid (q'+\bar{a})$, we have $(q'+\bar{a})/q = [(Q+\bar{a})/q]$. Hence,

$$q' = \left[\frac{Q+\bar{a}}{q}\right] q - \bar{a}.$$

By the same argument as above, we get

$$q'' = \left[\frac{Q-\bar{a}}{q}\right] q + \bar{a}.$$

Let us observe that

$$\begin{aligned}
q < \min(q', q'') \quad &\Longleftrightarrow \quad q < \left[\tfrac{Q+\bar{a}}{q}\right] q - \bar{a} \text{ and } q < \left[\tfrac{Q-\bar{a}}{q}\right] q + \bar{a} \\
&\Longleftrightarrow \quad \left[\tfrac{Q+\bar{a}}{q}\right] \geqslant 2 \text{ and } \left[\tfrac{Q-\bar{a}}{q}\right] \geqslant 1 \\
&\Longleftrightarrow \quad Q + \bar{a} \geqslant 2q \text{ and } Q - \bar{a} \geqslant q \\
&\Longleftrightarrow \quad \bar{a} \in [\max(2q - Q, 0), \min(Q - q, q)] \text{ with } q \leqslant 2Q/3,
\end{aligned}$$

and denote

$$J_{q,1} = \begin{cases} [0, Q-q] \cap [0, q] = [0, q] & \text{if } 1 \leqslant q \leqslant Q/2, \\ [2q - Q, Q - q] & \text{if } Q/2 \leqslant q < 2Q/3, \\ \emptyset & \text{if } q \geqslant 2Q/3. \end{cases}$$

We write

$$\begin{aligned}
S_{i,j,1} &= 2\varepsilon \sum_{\substack{1 \leqslant q \leqslant 2Q/3 \\ q \equiv i \pmod{2}}} q \sum_{\substack{q \tan\alpha \leqslant a \leqslant q \tan\beta \\ \gcd(a,q)=1 \\ a \equiv j \pmod{2} \\ \bar{a} \in J_{q,1}}} (q^2 + a^2)^{\frac{k}{2}-1} \\
&= \frac{2}{Q} \sum_{\substack{1 \leqslant q \leqslant 2Q/3 \\ q \equiv i \pmod{2}}} q B_{j,1},
\end{aligned}$$

where

$$B_{j,1} = \sum_{\substack{(a,\bar{a})\in I_q\times J_{q,1} \\ a\equiv j \pmod 2}} f_{q,1}(a,\bar{a}) \quad \text{and} \quad f_{q,1}(x,y) = (q^2+x^2)^{\frac{k}{2}-1},$$

with $(x,y)\in I_q\times J_{q,1} = qI_0\times[\max(2q-Q,0),\min(Q-q,q)]$ and $I_0=[\tan\alpha,\tan\beta]\subseteq[0,1]$ fixed. We take $R=[Q^a]$ and $S=[Q^b]$, with $0<a,b<1$ to be precisely chosen later. The trivial estimate

$$qB_{j,1} \ll q^2Q^{k-2} \leqslant Q^k$$

gives

$$S_{i,j,1} = \frac{2}{Q}\sum_{\substack{1\leqslant q\leqslant 2Q/3-R \\ q\equiv i \pmod 2}} qB_{j,1} + O(Q^{k-1+a}). \tag{3.2}$$

In the range of summation for q from (3.2), we have

$$\frac{\partial f_{q,1}}{\partial x} = x(k-2)(q^2+x^2)^{\frac{k}{2}-2} \quad\text{and}\quad \frac{\partial f_{q,1}}{\partial y} = 0,$$

so that

$$\|f_{q,1}\|_\infty \ll Q^{k-2}, \quad \|Df_{q,1}\|_\infty \ll Q^{k-3}.$$

According to Lemma 2.2 and to (3.2), this gives

$$S_{i,j,1} = T_{i,j,1} + R_{\delta,1}, \tag{3.3}$$

where, taking $a=5/6$ and $b=1/6$,

$$\begin{aligned} R_{\delta,1} &\ll_\delta Q^{k-1}R + Q\left[S^2Q^{\frac{1}{2}+\delta}\cdot Q^{k-2} + S\cdot Q^{k-3}\left(Q^{\frac{3}{2}+\delta}+\frac{Q^2}{S^2}\right)\right] \\ &\ll_\delta Q^{k-1+\delta+\max(a,\frac{1}{2}+2b,1-b)} \ll_\delta Q^{k-\frac{1}{6}+\delta} \end{aligned}$$

and

$$\begin{aligned} T_{i,j,1} = {} & \frac{2}{Q}\sum_{\substack{1\leqslant q\leqslant Q/2 \\ q\equiv i \pmod 2}} q\cdot\frac{\eta_{i,j}\varphi(q)}{2q^2}\int_{qI_0}(q^2+x^2)^{\frac{k}{2}-1}\,dx\int_0^q dy \\ & + \frac{2}{Q}\sum_{\substack{Q/2\leqslant q\leqslant 2Q/3-R \\ q\equiv i \pmod 2}} q\cdot\frac{\eta_{i,j}\varphi(q)}{2q^2}\int_{qI_0}(q^2+x^2)^{\frac{k}{2}-1}\,dx\int_{2q-Q}^{Q-q} dy \\ = {} & \frac{1}{Q}\sum_{\substack{1\leqslant q\leqslant Q/2 \\ q\equiv i \pmod 2}} \frac{\eta_{i,j}\varphi(q)}{q}\cdot q^{k-1}\int_{I_0}(1+u^2)^{\frac{k}{2}-1}\,du\cdot q \\ & + \frac{1}{Q}\sum_{\substack{Q/2\leqslant q\leqslant 2Q/3-R \\ q\equiv i \pmod 2}} \frac{\eta_{i,j}\varphi(q)}{q}\cdot q^{k-1}\int_{I_0}(1+u^2)^{\frac{k}{2}-1}\,du\cdot(2Q-3q) \\ = {} & \eta_{i,j}b_k\sum_{\substack{1\leqslant q\leqslant Q/2 \\ q\equiv i \pmod 2}} \varphi(q)q^{k-1}\frac{1}{Q} + \eta_{i,j}b_k\sum_{\substack{Q/2\leqslant q\leqslant 2Q/3-R \\ q\equiv i \pmod 2}} \varphi(q)q^{k-2}\left(2-\frac{3q}{Q}\right), \end{aligned}$$

where
$$b_k = \int_\alpha^\beta \frac{dx}{\cos^k x}.$$

Since the function
$$g_{k,1}(t) = t^{k-1} \times \begin{cases} 1 & \text{if } t \in [0,1/2], \\ 2-3t & \text{if } t \in [1/2,2/3), \end{cases}$$
is bounded, and since
$$\int_0^{\frac{2}{3}} |g'_{k,1}(t)|\, dt < \infty,$$
Lemma 2.4 applies and yields
$$T_{i,j,1} = \frac{\eta_{i,j}\varrho_i b_k Q^k}{\zeta(2)} \int_{\frac{1}{Q}}^{\frac{2}{3}-\frac{R}{Q}} g_{k,1}(t)\, dt + O_k(Q^{k-1}\log Q).$$
Inserting this into (3.3), we obtain
$$S_{i,j,1} = \frac{\eta_{i,j}\varrho_i b_k c_{k,1} Q^k}{\zeta(2)} + O_\delta(Q^{k-\frac{1}{6}+\delta}), \quad c_{k,1} = \int_0^{\frac{2}{3}} g_{k,1}(t)\, dt.$$

The analysis of the sum $S_{i,j,2}$ is slightly more intricate. We have
$$\begin{aligned} q > \max(q', q'') \quad &\iff \quad \left[\tfrac{Q+\bar a}{q}\right] \leqslant 1 \text{ and } \left[\tfrac{Q-\bar a}{q}\right] = 0 \\ &\iff \quad Q + \bar a < 2q \text{ and } Q - \bar a < q \\ &\iff \quad \bar a \in (Q-q, 2q-Q) \text{ with } q > 2Q/3. \end{aligned}$$
Let
$$J_{q,2} = \begin{cases} (Q-q, 2q-Q) & \text{if } q > 2Q/3, \\ \emptyset & \text{if } q \leqslant 2Q/3, \end{cases}$$
and observe that

(i) $q'' < q \iff \left[\dfrac{Q-\bar a}{q}\right] = 0$, so that $q'' = \bar a$;

(ii) $q' < q \iff \left[\dfrac{Q+\bar a}{q}\right] = 1$, so that $q' = q - \bar a$.

Thus, we have
$$\begin{aligned} S_{i,j,2} &= \sum_{\substack{2Q/3<q\leqslant Q \\ q\equiv i \pmod 2}} q^2 \sum_{\substack{q\tan\alpha\leqslant a\leqslant q\tan\beta \\ \gcd(a,q)=1 \\ a\equiv j \pmod 2 \\ \bar a\in J_{q,2}}} \left(\frac{1}{q(q-\bar a)} + \frac{1}{q\bar a} - \frac{\varepsilon}{q-\bar a} - \frac{\varepsilon}{\bar a}\right)(q^2+a^2)^{\frac{k}{2}-1} \\ &= \sum_{\substack{2Q/3<q\leqslant Q \\ q\equiv i \pmod 2}} q(1-q\varepsilon) \sum_{\substack{q\tan\alpha\leqslant a\leqslant q\tan\beta \\ \gcd(a,q)=1 \\ a\equiv j \pmod 2 \\ \bar a\in J_{q,2}}} \left(\frac{1}{\bar a} + \frac{1}{q-\bar a}\right)(q^2+a^2)^{\frac{k}{2}-1} \\ &= \sum_{\substack{2Q/3<q\leqslant Q \\ q\equiv i \pmod 2}} q\left(1-\frac{q}{Q}\right) B_{j,2}, \end{aligned}$$

where

$$B_{j,2} = \sum_{\substack{(a,\bar{a}) \in I_q \times J_{q,2} \\ a \equiv j \pmod 2}} f_{q,2}(a,\bar{a}), \quad f_{q,2}(x,y) = (q^2+x^2)^{\frac{k}{2}-1}\left(\frac{1}{y} + \frac{1}{q-y}\right),$$

with $(x,y) \in I_q \times J_{q,2} = qI_0 \times (Q-q, 2q-Q)$. Putting $R = [Q^a]$, $S = [Q^b]$, and $T = [Q^c]$, with $0 < a, b, c < 1$ to be precisely chosen later, we see that the trivial estimate

$$q\left(1 - \frac{q}{Q}\right) B_{j,2} \ll qQ^{k-2} \log q \leqslant Q^{k-1} \log Q$$

gives

$$S_{i,j,2} = \sum_{\substack{2Q/3 < q \leqslant Q-R \\ q \equiv i \pmod 2}} q\left(1 - \frac{q}{Q}\right) B_{j,2} + O_\delta(Q^{k-1+a+\delta}). \tag{3.4}$$

In the range of summation for q from (3.4), we have

$$\begin{aligned} \frac{\partial f_{q,2}}{\partial x} &= x(k-2)(q^2+x^2)^{\frac{k}{2}-2}\left(\frac{1}{y} + \frac{1}{q-y}\right), \\ \frac{\partial f_{q,2}}{\partial y} &= (q^2+x^2)^{\frac{k}{2}-1}\left(-\frac{1}{y^2} + \frac{1}{(q-y)^2}\right), \end{aligned}$$

so that

$$\begin{aligned} \|f_{q,2}\|_\infty &\ll \frac{Q^{k-2}}{Q-q} \leqslant \frac{Q^{k-2}}{R} \ll Q^{k-2-a}, \\ \|D_x f_{q,2}\|_\infty &\ll \frac{Q^{k-3}}{Q-q} \leqslant \frac{Q^{k-3}}{R} \ll Q^{k-3-a}, \\ \|D_y f_{q,2}\|_\infty &\ll \frac{Q^{k-2}}{(Q-q)^2} \leqslant \frac{Q^{k-2}}{R^2} \ll Q^{k-2-2a}. \end{aligned}$$

According to Lemma 2.3 and to (3.4), this gives

$$S_{i,j,2} = T_{i,j,2} + R_{\delta,2}, \tag{3.5}$$

where $R_{\delta,2}$ is

$$\begin{aligned} &\ll_\delta Q^{k-1+\delta} R + Q^2 \left[STQ^{\frac{1}{2}+\delta} \cdot \frac{Q^{k-2}}{R} + \left(T \cdot \frac{Q^{k-3}}{R} + S \cdot \frac{Q^{k-2}}{R^2}\right)\left(Q^{\frac{3}{2}+\delta} + \frac{Q^2}{ST}\right)\right] \\ &\ll_\delta Q^{k-1+\delta+\max(a, \frac{3}{2}+b+c-a, \frac{5}{2}+b-2a, 2-a-b, 3-c-2a)} \ll_\delta Q^{k-\frac{1}{14}+\delta}, \end{aligned}$$

by taking $a = 13/14$, $b = 1/7$, and $c = 3/14$, and $T_{i,j,2}$ equals

$$\sum_{\substack{2Q/3<q\leqslant Q-R\\ q\equiv i \pmod 2}} q\left(1-\frac{q}{Q}\right)\frac{\eta_{i,j}\varphi(q)}{2q^2}\int_{qI_0}(q^2+x^2)^{\frac{k}{2}-1}\,dx\int_{Q-q}^{2q-Q}\left(\frac{1}{y}+\frac{1}{q-y}\right)dy$$

$$=\eta_{i,j}\sum_{\substack{2Q/3<q\leqslant Q-R\\ q\equiv i \pmod 2}} q\left(1-\frac{q}{Q}\right)\frac{\varphi(q)}{2q^2}\cdot q^{k-1}\int_{I_0}(1+u^2)^{\frac{k}{2}-1}\,du\cdot 2\log\frac{2q-Q}{Q-q}$$

$$=\eta_{i,j}b_k\sum_{\substack{2Q/3<q\leqslant Q-R\\ q\equiv i \pmod 2}} \varphi(q)q^{k-2}\left(1-\frac{q}{Q}\right)\log\frac{2q-Q}{Q-q}.$$

The function

$$g_{k,2}(t)=t^{k-1}(1-t)\log\frac{2t-1}{1-t},\quad t\in[2/3,1),$$

is bounded and

$$\int_{\frac{2}{3}}^{1}|g'_{k,2}(t)|\,dt<\infty.$$

Thus, Lemma 2.4 applies and yields

$$T_{i,j,2}=\frac{\eta_{i,j}\varrho_i b_k Q^k}{\zeta(2)}\int_{\frac{2}{3}}^{1-\frac{R}{Q}} g_{k,2}(t)\,dt+O_k(Q^{k-1}\log Q). \tag{3.6}$$

In view of (3.5), (3.6), and the estimate

$$\int_0^{\xi} u|\log u|\,du\ll\xi,$$

we conclude that

$$S_{i,j,2}=\frac{\eta_{i,j}\varrho_i b_k c_{k,2} Q^k}{\zeta(2)}+O_{k,\delta}(Q^{k-\frac{1}{14}+\delta}),\quad c_{k,2}=\int_{\frac{2}{3}}^{1} g_{k,2}(t)\,dt.$$

By an analysis similar to the one used above (with $q'=\bar{a}$, $q''=q-\bar{a}$, and $q\geqslant Q/2$), we have

$$S_{i,j,3}=\sum_{\substack{Q/2\leqslant q\leqslant Q\\ q\equiv i \pmod 2}} q \sum_{\substack{q\tan\alpha\leqslant a\leqslant q\tan\beta\\ \gcd(a,q)=1\\ a\equiv j \pmod 2\\ \bar{a}\in[\max(2q-Q,Q-q),q]}} \left[\frac{1}{\bar{a}}\left(1-\frac{q}{Q}\right)+\frac{1}{Q}\right](q^2+a^2)^{\frac{k}{2}-1}$$

$$=\frac{\eta_{i,j}b_k}{2}\sum_{\substack{Q/2\leqslant q\leqslant 2Q/3\\ q\equiv i \pmod 2}}\varphi(q)q^{k-2}\left[\left(1-\frac{q}{Q}\right)\log\frac{q}{Q-q}+\frac{2q}{Q}-1\right]$$

$$+\frac{\eta_{i,j}b_k}{2}\sum_{\substack{2Q/3\leqslant q\leqslant Q-R\\ q\equiv i \pmod 2}}\varphi(q)q^{k-2}\left[\left(1-\frac{q}{Q}\right)\log\frac{q}{2q-Q}-\frac{q}{Q}+1\right]$$

and

$$\begin{aligned}
S_{i,j,4} &= \sum_{\substack{Q/2\leqslant q\leqslant Q\\ q\equiv i \pmod 2}} q \sum_{\substack{q\tan\alpha\leqslant a\leqslant q\tan\beta\\ \gcd(a,q)=1\\ a\equiv j \pmod 2\\ \bar{a}\in[0,\min(2q-Q,Q-q)]}} \frac{Q-\bar{a}}{Q(q-\bar{a})}(q^2+a^2)^{\frac{k}{2}-1}\\
&= \frac{\eta_{i,j}b_k}{2}\sum_{\substack{Q/2\leqslant q\leqslant 2Q/3\\ q\equiv i \pmod 2}} \varphi(q)q^{k-2}\left\{\frac{q}{Q}-\left(1-\frac{q}{Q}\right)\left[1+\log\left(\frac{Q}{q}-1\right)\right]\right\}\\
&\quad+\frac{\eta_{i,j}b_k}{2}\sum_{\substack{2Q/3\leqslant q\leqslant Q-R\\ q\equiv i \pmod 2}} \varphi(q)q^{k-2}\left(1-\frac{q}{Q}\right)\left[1-\log\left(2-\frac{Q}{q}\right)\right].
\end{aligned}$$

Therefore,

$$S_{i,j,h} = \frac{\eta_{i,j}\varrho_i b_k c_{k,h} Q^k}{2\zeta(2)} + O_{k,\delta}(Q^{k-\frac{1}{14}+\delta}), \quad c_{k,h} = \int_{\frac{1}{2}}^{1} g_{k,h}(t)\,dt, \quad h=3,4,$$

where

$$g_{k,3}(t) = t^{k-1}\times\begin{cases}(1-t)\log\frac{t}{1-t}+2t-1 & \text{if } t\in[1/2,2/3],\\ (1-t)\log\frac{t}{2t-1}-t+1 & \text{if } t\in[2/3,1),\end{cases}$$

and

$$g_{k,4}(t) = t^{k-1}\times\begin{cases}t-(1-t)\left(1-\log\frac{t}{1-t}\right) & \text{if } t\in[1/2,2/3],\\ (1-t)\left(1+\log\frac{t}{2t-1}\right) & \text{if } t\in[2/3,1).\end{cases}$$

Collecting all estimates in (3.1), we record

$$\int_{A_{i,j}\cap[\alpha,\beta]} l_\varepsilon^k(\theta)\,d\theta = \eta_{i,j}\varrho_i b_k d_k Q^k + O_{k,\delta}(Q^{k-\frac{1}{14}+\delta}),$$

where

$$d_k = \frac{2c_{k,1}+2c_{k,2}+c_{k,3}+c_{k,4}}{2\zeta(2)}.$$

Since $\eta_{i,j}\varrho_i = 0$ if $(i,j)=(0,0)$ and $\eta_{i,j}\varrho_i = 1/3$ otherwise, and since the sum of all four moments equals $b_k c_k Q^k + O_{k,\delta}(Q^{k-\frac{1}{6}+\delta})$ by (1.1) and (1.2), it follows that each of the three moments with $(i,j)\neq(0,0)$ satisfies the asymptotic formula given in the theorem. Thus our theorem is completely proved.

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