Markov Operators on the Solvable Baumslag–Solitar Groups

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CONTENTS

We consider the solvable Baumslag–Solitar group

$$\mathrm{BS}_n = \langle a, b \mid aba^{-1} = b^n \rangle,$$

for $n \geq 2$, and try to compute the spectrum of the associated Markov operators M_S, either for the oriented Cayley graph ($S = \{a, b\}$), or for the usual Cayley graph ($S = \{a^{\pm 1}, b^{\pm 1}\}$). We show in both cases that $\mathrm{Sp}\, M_S$ is connected.

For $S = \{a, b\}$ (nonsymmetric case), we show that the intersection of $\mathrm{Sp}\, M_S$ with the unit circle is the set C_{n-1} of $(n-1)$-st roots of 1, and that $\mathrm{Sp}\, M_S$ contains the $n-1$ circles

$$\{z \in \mathbb{C} : |z - \tfrac{1}{2}\omega| = \tfrac{1}{2}\}, \quad \text{for } \omega \in C_{n-1},$$

together with the $n+1$ curves given by

$$\left(\tfrac{1}{2}w^k - \lambda\right)\left(\tfrac{1}{2}w^{-k} - \lambda\right) - \tfrac{1}{4}\exp 4\pi i\theta = 0,$$

where $w \in C_{n+1}$, $\theta \in [0, 1]$.

Conditional on the Generalized Riemann Hypothesis (actually on Artin's conjecture), we show that $\mathrm{Sp}\, M_S$ also contains the circle $\{z \in \mathbb{C} : |z| = \frac{1}{2}\}$. This is confirmed by numerical computations for $n = 2, 3, 5$.

For $S = \{a^{\pm 1}, b^{\pm 1}\}$ (symmetric case), we show that $\mathrm{Sp}\, M_S = [-1, 1]$ for n odd, and $\mathrm{Sp}\, M_S = [-\frac{3}{4}, 1]$ for $n = 2$. For n even, at least 4, we only get $\mathrm{Sp}\, M_S = [r_n, 1]$, with

$$-1 < r_n \leq -\sin^2 \frac{\pi n}{2(n+1)}.$$

We also give a potential application of our computations to the theory of wavelets.

1. INTRODUCTION

Let Γ be a finitely generated group. Fix a finite, generating, not necessarily symmetric subset S in Γ. To these data we associate the *oriented Cayley graph* (or *Cayley digraph*) $\mathcal{G}(\Gamma, S)$, whose vertex set is Γ, and whose set of oriented edges is $\{(x, xs) : x \in \Gamma,\, s \in S\}$. If S is symmetric ($S = S^{-1}$), it is customary to replace a pair of opposite edges by a single, nonoriented edge.

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On $\mathcal{G}(\Gamma, S)$, consider the simple random walk, in which a particle jumps from x to xs with a probability of $\frac{1}{|S|}$. Denote by $p^{(n)}(x,y)$ the probability of a transition in n steps from x to y. It can be expressed by means of the Markov operator M_S on $\ell^2(\Gamma)$, defined by

$$(M_S\xi)(x) = \frac{1}{|S|}\sum_{s\in S}\xi(xs), \quad \text{for } \xi \in \ell^2(\Gamma),\ x\in\Gamma.$$

Indeed $p^{(n)}(x,y) = \langle M_S^n\delta_x|\delta_y\rangle$, where $(\delta_x)_{x\in\Gamma}$ is the canonical basis of $\ell^2(\Gamma)$.

The operator M_S is a contraction ($\|M_S\| \le 1$) on $\ell^2(\Gamma)$; its spectrum $\operatorname{Sp} M_S$ is therefore a closed nonempty subset of the unit disk in $\mathbb{C}$. A number of results, going back to Kesten [1959] in the symmetric case, and to Day [1964] in the general case, exhibit a close relationship between the properties of $\operatorname{Sp} M_S$ and those of the pair (Γ, S). Here is a sample; we denote by $\mathbb{T}$ the multiplicative group of complex numbers of modulus 1, and by C_n the group of n-th roots of 1 in $\mathbb{C}$.

Theorem 1.1. 1. *Γ is amenable if and only if $1 \in \operatorname{Sp} M_S$.*

2. *If Γ is amenable, then $(\operatorname{Sp} M_S)\cap\mathbb{T}$ is a closed subgroup of $\mathbb{T}$; moreover, for $z\in\mathbb{T}$, the following statements are equivalent*:
   a. *$z \in \operatorname{Sp} M_S$;*
   b. *$\operatorname{Sp} M_S$ is invariant under multiplication by z;*
   c. *there exists a homomorphism $\chi : \Gamma \to \mathbb{T}$ such that $\chi(S) = \{z\}$.*
3. *$\operatorname{Sp} M_S = C_n$ if and only if $\Gamma \simeq \mathbb{Z}/n\mathbb{Z}$ and $|S| = 1$; and $\operatorname{Sp} M_S = \mathbb{T}$ if and only if $\Gamma \simeq \mathbb{Z}$ and $|S| = 1$.*

For proofs, see respectively [Day 1964; de la Harpe et al. 1993, Proposition III; de la Harpe et al. 1994, Proposition 3].

Gromov has asked which properties of $\operatorname{Sp}(M_S)$ are invariant under quasi-isometries, pointing out that the Kesten–Day result mentioned above provides an example (since amenability is a quasi-isometry invariant). To attack Gromov's question, one difficulty lies with the lack of examples of explicitly computed spectra of Markov operators. The aim of this paper is to present a class of solvable, non virtually nilpotent groups, for which some explicit calculations are possible.

In this paper we deal with the *solvable Baumslag–Solitar groups*, a family of one-relator groups defined by the presentations

$$\mathrm{BS}_n = \langle a, b \mid aba^{-1} = b^n\rangle,$$

for $n \ge 2$. These groups belong to a two-parameter family of one-relator groups introduced by Baumslag and Solitar [1962]. There has been recent activity around the groups BS_n; for example, here is a remarkable result by B. Farb and L. Mosher [1998]:

Theorem 1.2. *The following statements are equivalent*:

1. *The groups BS_m and BS_n are quasi-isometric.*
2. *The groups BS_m and BS_n are commensurable.*
3. *There exists an integer $r \ge 2$ such that m and n are powers of r.*

C. Pittet and L. Saloff-Coste [1999] have determined the isoperimetric profile and the rate of decay of the heat kernel on BS_n.

Our goal is to study the spectrum of the Markov operator M_S, where we take either $S = \{a, b\}$ or $S = \{a^{\pm1}, b^{\pm1}\}$ as generating subset of BS_n. Our results are as follows:

- We show in Section 2 that $\operatorname{Sp} M_S$ is always connected.
- For $S = \{a, b\}$ (nonsymmetric case), we show in Section 3 that $(\operatorname{Sp} M_S)\cap\mathbb{T} = C_{n-1}$, and that $\operatorname{Sp} M_S$ contains the $n-1$ circles
$$\{z\in\mathbb{C} : |z - \tfrac{1}{2}\omega| = \tfrac{1}{2}\}, \quad \text{for } \omega \in C_{n-1},$$
together with the $n+1$ curves given by
$$\left(\tfrac{1}{2}w^k - \lambda\right)\left(\tfrac{1}{2}w^{-k} - \lambda\right) - \tfrac{1}{4}\exp 4\pi i\theta = 0,$$
where $w \in C_{n+1}$, $\theta \in [0,1]$ (and also their images under the action of the symmetry group of $\operatorname{Sp} M_S$, which turns out to be the dihedral group D_{n-1} of order $2n-2$). Assuming the Generalized Riemann Hypothesis — or just Artin's conjecture [Murty 1988] — we show that $\operatorname{Sp} M_S$ also contains the circle $\{z \in \mathbb{C} : |z| = \frac{1}{2}\}$. This is confirmed by numerical computations for $n = 2, 3, 5$.
- For $S = \{a^{\pm1}, b^{\pm1}\}$ (symmetric case), we show in Section 4 that
$$\operatorname{Sp} M_S = \begin{cases} [-1,1] & \text{if } n \text{ is odd,}\\ [r_n, 1] & \text{if } n \text{ is even,}\end{cases}$$

where

$$-1 < r_n \leq -\sin^2 \frac{\pi n}{2(n+1)}$$

(notice that $\lim_{n\to\infty} r_n = -1$). For $n = 2$, we get the exact value $r_2 = -\frac{3}{4}$.

- In Section 5, we give a potential application of our study to the theory of wavelets (see, for instance, [Daubechies 1992]); to explain the link, notice that BS_2 is isomorphic to the subgroup of the affine group of the real line, generated by translation by 1 and dilation by 2: these are exactly the two transformations used in multiresolution analysis.

All of our computations rest on the following lemma, useful in constructing points of $\mathrm{Sp}\, M_S$.

Lemma 1.3. *Let Γ be a finitely generated amenable group. Define*

$$h_S = \frac{1}{|S|} \sum_{s\in S} s \in \mathbb{C}\Gamma.$$

For every unitary representation π of Γ, one has $\mathrm{Sp}\,\pi(h_S) \subset \mathrm{Sp}\, M_S$.

Proof. Denoting by ρ the right regular representation of Γ on $\ell^2(\Gamma)$, one has $M_S = \rho(h_S)$. For a group Γ, we shall denote by $C_r^*\Gamma$ the *reduced C*-algebra* of Γ, i.e. the C*-algebra generated by $\rho(\Gamma)$ (notice that $M_S \in C_r^*\Gamma$). When Γ is amenable, any unitary representation π of Γ extends to $C_r^*\Gamma$, hence defines a quotient of $C_r^*\Gamma$. But passing to a quotient only decreases the spectrum. □

For this reason we are interested in finding families of representations of $C_r^*\Gamma$, and especially separating families. Our interest in separating families lies in the fact that, at least for self-adjoint elements in $C_r^*\Gamma$, they "approximate" well the spectrum (see formula (3–1) and the proof of Theorem 4.4). We construct several of them in this paper.

2. CONNECTEDNESS OF SPECTRA

We begin by realizing BS_n in a more concrete way.

For a commutative, unital ring A, we denote by $\mathrm{Aff}_1(A)$ the affine group of A (or "$ax + b$" group): this is the semidirect product of the additive group of A by the multiplicative group. It is easy to check that BS_n can be identified with the subgroup of $\mathrm{Aff}_1(\mathbb{Q})$ generated by the dilation $a : x \mapsto nx$ and the translation $b : x \mapsto x + 1$.

Lemma 2.1. *Every element of $C_r^*\,\mathrm{BS}_n$ has a connected spectrum.*

Proof. The proof is in the same spirit as that of [Béguin et al. 1997, Proposition 1]; the statement to be proved is equivalent to the conjecture of idempotents for $C_r^*\,\mathrm{BS}_n$. Recall that, for a torsion-free group Γ (notice that BS_n is a torsion-free group), the conjecture of idempotents for Γ says that the only idempotents in $C_r^*(\Gamma)$ are 0 and 1. This in turn is a consequence of the Baum–Connes conjecture for Γ, which says that the analytical assembly map (or index map) $\mu_0^\Gamma : RK_0(\mathbb{B}\Gamma) \to K_0(C_r^*(\Gamma))$ is an isomorphism; here $RK_0(\mathbb{B}\Gamma)$ denotes the K-homology with compact support of the classifying space $\mathbb{B}\Gamma$, and $K_0(C_r^*(\Gamma))$ denotes the Grothendieck group of finite type projective modules over $C_r^*(\Gamma)$ (see [Baum et al. 1994; Valette 1989]). There are at least 3 different proofs of the Baum–Connes conjecture for BS_n.

- Kasparov et Skandalis [1991] have shown that the Baum–Connes conjecture is true for every torsion-free discrete subgroup of $\mathrm{Aff}_1(K_1) \times \cdots \times \mathrm{Aff}_1(K_m)$, where the K_i's are local fields. Then one may appeal to an arithmetic realization of BS_n: if $p_1, \ldots, p_k$ is the list of prime divisors of n, the diagonal embedding
$$\mathrm{BS}_n \hookrightarrow \mathrm{Aff}_1(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{Aff}_1(\mathbb{Q}_{p_k}) \times \mathrm{Aff}_1(\mathbb{R})$$
has discrete image.
- The Baum–Connes conjecture has been proved for torsion-free one-relator groups [Béguin et al. 1999].
- Higson and Kasparov [1997] proved the Baum–Connes conjecture for all torsion-free amenable groups, in particular for all torsion-free solvable groups. □

3. THE NONSYMMETRIC CASE

In this section, we set $S = \{a, b\}$. We may already deduce some qualitative informations about the spectrum of M_S.

Theorem 3.1. $\mathrm{Sp}\, M_S$ *is a connected subset of the closed unit disk of $\mathbb{C}$, such that:*

1. $(\operatorname{Sp} M_S) \cap \mathbb{T} = C_{n-1}$;
2. *the symmetry group of* $\operatorname{Sp} M_S$ *is* D_{n-1}, *the dihedral group of order* $2n-2$;
3. $\operatorname{Sp} M_S$ *contains the* $n-1$ *circles*

$$\{z \in \mathbb{C} : |z - \tfrac{1}{2}\omega| = \tfrac{1}{2}\}, \quad \textit{for } \omega \in C_{n-1}.$$

4. $\operatorname{Sp} M_S$ *contains the* $n+1$ *curves given by*

$$\left(\tfrac{1}{2}w^k - \lambda\right)\left(\tfrac{1}{2}w^{-k} - \lambda\right) - \tfrac{1}{4}\exp 4\pi i\theta = 0,$$

where $w \in C_{n+1}$, $\theta \in [0,1]$.

Proof.

1. From the presentation $\mathrm{BS}_n = \langle a, b \mid aba^{-1} = b^n\rangle$, it is clear that any homomorphism β from BS_n to an abelian group satisfies $\beta(b)^{n-1} = 1$; on the other hand, for any $\omega \in C_{n-1}$ one may define a homomorphism $\beta_\omega : \mathrm{BS}_n \to C_{n-1}$ by $\beta_\omega(a) = \beta_\omega(b) = \omega$. The result then follows from Theorem 1.1.2.

2. For every group Γ and every finite generating subset S, the spectrum of M_S is symmetric with respect to the real axis in $\mathbb{C}$ [de la Harpe et al. 1994, Proposition 4(ii)]. So the result is a consequence of part 1 and Theorem 1.1.2.

3. Define an epimorphism

$$\beta_{ab} : \mathrm{BS}_n \to \mathbb{Z} \times \mathbb{Z}/(n-1)\mathbb{Z}$$

by $\beta_{ab}(a) = (1,0)$ and $\beta_{ab}(b) = (0,1)$. The Pontryagin dual of $\mathbb{Z} \times \mathbb{Z}/(n-1)\mathbb{Z}$ is $\mathbb{T} \times C_{n-1}$, and $C_r^*(\mathbb{Z} \times \mathbb{Z}/(n-1)\mathbb{Z})$ is identified via the Fourier transform with the algebra of continuous functions on $\mathbb{T} \times C_{n-1}$. The Fourier transform of $\beta_{ab}(h_S)$ is the function

$$\mathbb{T} \times C_{n-1} \to \mathbb{C} : (z, \omega) \mapsto \frac{z + \omega}{2}.$$

The spectrum of $\beta_{ab}(h_S)$ is the range of this function, i.e. the union of $n-1$ circles appearing in the theorem. By Lemma 1.3, these circles are contained in $\operatorname{Sp} M_S$.

4. Consider the epimorphism of BS_n onto $\langle a, b \mid aba^{-1} = b^{-1}, b^{n+1} = 1\rangle$. This group can be identified with the semidirect product $\mathbb{Z}/(n+1)\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ (where the action is given by $\alpha_m(k) = (-1)^m k$). It contains the normal subgroup $\mathbb{Z}/(n+1)\mathbb{Z} \times 2\mathbb{Z}$. So by Mackey theory, the irreducible representations of the semidirect product, which are obtained by inducing the characters of the normal abelian subgroup, are given by

$$\pi_{k,\theta}(a) = \begin{pmatrix} 0 & e^{2\pi i\theta} \\ e^{2\pi i\theta} & 0 \end{pmatrix}$$

and

$$\pi_{k,\theta}(b) = \begin{pmatrix} w^k & 0 \\ 0 & w^{-k} \end{pmatrix},$$

where $w = e^{\frac{2\pi i}{n+1}}$; $k = 1, \ldots, n+1$; $\theta \in [0,1]$. Now, by computing the spectra of $\pi_{k,\theta}(h_S)$, we obtain the curves appearing in the theorem, and again we conclude by Lemma 1.3. □

In fact, part 3 follows from 2 and 4. Indeed,

$$\bigcup_{\theta \in [0,1]} \operatorname{Sp}(\pi_{n+1,\theta}(h_S)) = \left\{z \in \mathbb{C} : \left|z - \tfrac{1}{2}\right| = \tfrac{1}{2}\right\},$$

and by using part 2 of the theorem, we get part 3. It is easy to check that β_{ab} is just the abelianization homomorphism of BS_n (that is, $\ker \beta_{ab}$ is the commutator subgroup of BS_n). In other words, Theorem 3.1.3 describes the contribution to $\operatorname{Sp} M_S$ of the abelianized group of BS_n.

To proceed, we clearly need other representations which do not factor through the abelianization of BS_n and through the group

$$\langle a, b \mid aba^{-1} = b^{-1}, b^{n+1} = 1\rangle.$$

We now construct a family of such representations, viewing BS_n as an "arithmetic" group.

For a prime p, we denote by $\mathbb{F}_p$ the field with p elements, and by $\ell_0^2(\mathbb{F}_p)$ the orthogonal complement of the constants in $\ell^2(\mathbb{F}_p)$. We begin by recalling the representation theory of the finite group $\mathrm{Aff}_1(\mathbb{F}_p)$.

Lemma 3.2. *The group* $\mathrm{Aff}_1(\mathbb{F}_p)$ *has* p *irreducible representations, namely*:

- *the* $p-1$ *characters* $\chi_0, \ldots, \chi_{p-2}$ *coming from the epimorphism* $\mathrm{Aff}_1(\mathbb{F}_p) \to \mathbb{F}_p^\times$;
- *one representation* π_p *of degree* $p-1$, *on the space* $\ell_0^2(\mathbb{F}_p)$, *associated with the action of* $\mathrm{Aff}_1(\mathbb{F}_p)$ *on* $\mathbb{F}_p$.

Proof. A standard exercise in the representation theory of finite groups; see, for example, [Robert 1983, pp. 159–160]. □

Let us come back to the group BS_n. From the description as a subgroup of $\mathrm{Aff}_1(\mathbb{Q})$, it follows that BS_n is in fact a subgroup of $\mathrm{Aff}_1(\mathbb{Z}\left[\frac{1}{n}\right])$, where $\mathbb{Z}\left[\frac{1}{n}\right]$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{n}$.

If p is a prime not dividing n, reduction modulo p from $\mathbb{Z}\left[\frac{1}{n}\right]$ onto $\mathbb{F}_p$, induces a homomorphism

$$\alpha_p : \mathrm{BS}_n \to \mathrm{Aff}_1(\mathbb{F}_p).$$

Denote by ρ_p the regular representation of $\mathrm{Aff}_1(\mathbb{F}_p)$.

Proposition 3.3. *For every infinite set S of primes not dividing n, the family of representations $(\rho_p \circ \alpha_p)_{p\in S}$ is separating for $C_r^* \mathrm{BS}_n$.*

Proof. We show that the regular representation ρ_{BS_n} of BS_n is weakly equivalent (in the sense of [Dixmier 1977, 3.4.5]) to $\bigoplus_{p\in S} \rho_p \circ \alpha_p$. The latter is weakly contained in ρ_{BS_n}, because of amenability of BS_n. For the converse, for $p \in S$ let us denote by δ_p the characteristic function of the identity of $\mathrm{Aff}_1(\mathbb{F}_p)$, viewed as a vector in $\ell^2(\mathrm{Aff}_1(\mathbb{F}_p))$; consider the positive definite function φ_p on BS_n, defined by

$$\varphi_p(g) = \langle \rho_p(\alpha_p(g))\delta_p | \delta_p\rangle,$$

with $g \in \mathrm{BS}_n$. Clearly

$$\varphi_p(g) = \begin{cases} 1 & \text{if } g \in \ker \alpha_p, \\ 0 & \text{otherwise.} \end{cases}$$

Since any element $g \in \mathrm{BS}_n - \{e\}$ belongs to a finite number of subgroups $\ker \alpha_p$, we have

$$\lim_{p\to\infty,\, p\in P_n} \varphi_p(g) = \delta_{g,e} = \langle \rho(g)\delta_e | \delta_e\rangle;$$

by [Dixmier 1977, 18.1.4], this shows that ρ_{BS_n} is weakly contained in $\bigoplus_{p\in S} \rho_p \circ \alpha_p$, and concludes the proof. □

For a fixed prime p, the homomorphism α_p is onto if and only if n is a primitive root modulo p, i.e. is a generator of the multiplicative group of $\mathbb{F}_p$. Set $a_p = \alpha_p(a)$ and $b_p = \alpha_p(b)$.

Proposition 3.4. *Let p be an odd prime. If n is a primitive root modulo p, then $\mathrm{Sp}\, \pi_p(a_p + b_p)$ consists of 0 and the $(p-1)$-st roots of 1, distinct from 1.*

Proof. In fact we shall determine $\mathrm{Sp}\, \pi_p(a_p^{-1} + b_p^{-1})$, which is the image of the desired spectrum under complex conjugation, and which will turn out to be invariant under complex conjugation. Set $\omega = e^{\frac{2\pi i}{p}}$. We work in the basis of characters of $\ell_0^2(\mathbb{F}_p)$:

$$e_i(j) = \omega^{ij} \quad (i = 1, \ldots, p-1;\ j \in \mathbb{F}_p).$$

Clearly e_i is an eigenvector of $\pi_p(b_p^{-1})$, with eigenvalue ω^i. On the other hand $\pi_p(a_p^{-1})(e_i) = e_{ni}$. The assumption allows us to re-arrange the basis of e_i's according to powers of n; thus we work in the basis $e_1, e_n, e_{n^2}, \ldots, e_{n^{p-2}}$. Then:

$$\pi_p(a_p^{-1}+b_p^{-1}) = \begin{pmatrix} \omega & 0 & 0 & \cdots & 0 & 1 \\ 1 & \omega^n & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \omega^{n^2} & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \omega^{n^{p-3}} & 0 \\ 0 & \cdots & \cdots & 0 & 1 & \omega^{n^{p-2}} \end{pmatrix}.$$

To compute the characteristic polynomial of this matrix, we develop the determinant according to the first row, and get (since p is odd):

$$\begin{aligned} \det(\pi_p(a_p^{-1} + b_p^{-1}) - \lambda) &= \prod_{i=0}^{p-2} (\omega^{n^i} - \lambda) - 1 \\ &= \prod_{j=1}^{p-1} (\omega^j - \lambda) - 1 \\ &= \frac{1-\lambda^p}{1-\lambda} - 1 = \frac{\lambda(1-\lambda^{p-1})}{1-\lambda}, \end{aligned}$$

where the second equality follows from the assumption on n. The result is now clear. □

The preceding proposition is false when n is not a primitive root modulo p; this is clearly visible on Figure 1, which shows, for $n = 2, 3, 5$, the union of the sets $\mathrm{Sp}\, \pi_p\big(\frac{1}{2}(a_p + b_p)\big)$ for p running over the first 300 primes.

Corollary 3.5. *Let p be an odd prime. If n is a primitive root modulo p, the spectrum of $\rho_p(\alpha_p(h_S))$ consists of:*

- 0 *with multiplicity* p;
- $\frac{1}{2}\big(1 + \exp \frac{2\pi i j}{p-1}\big)$ *with multiplicity* 1, *for* $j = 0, \ldots, p-2$ *distinct from* $\frac{1}{2}(p-1)$;
- $\frac{1}{2} \exp \frac{2\pi i k}{p-1}$ *with multiplicity* $p-1$, *for* $k = 1, \ldots, p-2$.

Proof. Notice that $\alpha_p(h_S) = \frac{1}{2}(a_p + b_p)$. Let $\chi_0, \ldots, \chi_{p-2}$ be the characters of $\mathrm{Aff}_1(\mathbb{F}_p)$ (see Lemma 3.2). In view of the assumption, χ_j is determined by its value on a_p, and we may assume $\chi_j(a_p) = \exp(\frac{2\pi i j}{p-1})$, so that

$$\chi_j\big(\tfrac{1}{2}(a_p + b_p)\big) = \tfrac{1}{2}\big(1 + \exp \tfrac{2\pi i j}{p-1}\big).$$

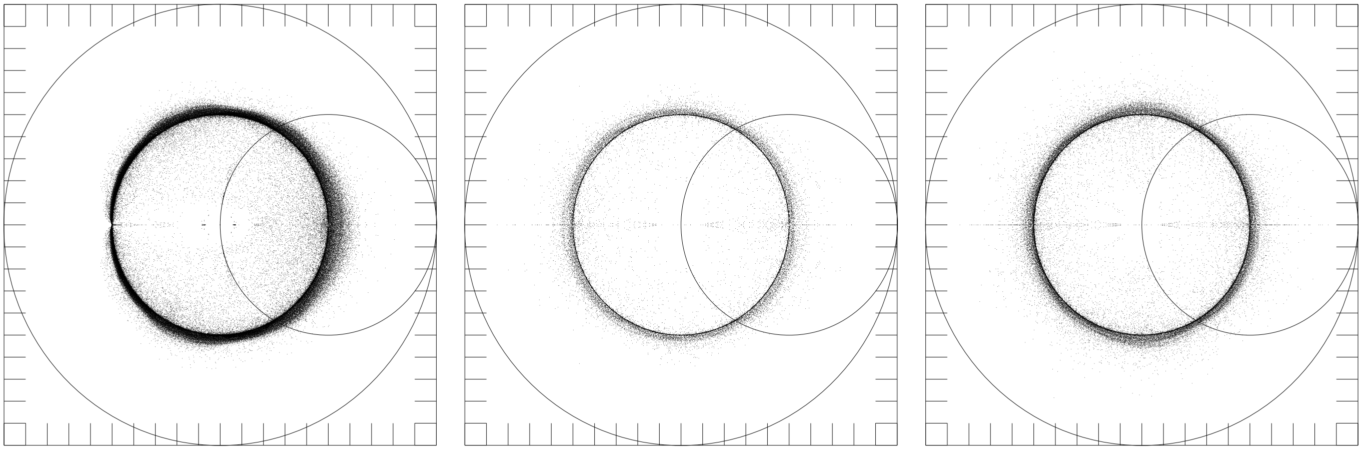


FIGURE 1. For $n = 2$ (left), $n = 3$ (middle) and $n = 5$ (right), the graphs show the union of $\mathrm{Sp}\,\pi_p\big(\frac{1}{2}(a_p + b_p)\big)$ for p running over the first 300 primes.

The regular representation ρ_p decomposes into π_p (with multiplicity $p-1$) and the χ_j's (each with multiplicity 1). The result follows from this and the previous proposition. □

Remark. For $p = 7$, there are some coincidences between eigenvalues of the second and third kind in Corollary 3.5. The reader will have no difficulty to compute the correct multiplicities.

The previous proposition and corollary raise a natural question: how many primes p are there, such that n is a primitive root modulo p? It turns out that this is an open problem in number theory! For $n \geq 2$, denote by P_n the set of primes p such that n is a primitive root modulo p. *Artin's conjecture for the integer* n is the following statement:

Conjecture 3.6. *The set* P_n *is infinite.*

For an excellent introduction to Artin's conjecture, see [Murty 1988]. We collect in the theorem below some striking results on Artin's conjecture. Part 1 is due to Hooley [1967], and parts 2 and 3 to Heath-Brown [1986].

Theorem 3.7.
1. *Artin's conjecture for any* n *follows from the Generalized Riemann Hypothesis* (*the statement that the Dedekind* ζ*-function of any number field* K *satisfies the Riemann Hypothesis*).
2. *Artin's conjecture holds for prime* n, *with at most two possible exceptions.*
3. *Artin's conjecture holds for square-free* n, *with at most three possible exceptions.*

Since spectra are closed subsets of $\mathbb{C}$, it follows immediately from Corollary 3.5 (together with Lemma 1.3):

Theorem 3.8. *If Artin's conjecture holds for* n, *then*

$$\mathrm{Sp}\,M_S \supset \{z \in \mathbb{C} : |z| = \tfrac{1}{2}\}.$$

We will apply Proposition 3.3 and Theorem 3.7 to a classical problem in operator theory, namely the description of the spectrum of a direct sum of operators. Indeed, if A is a C*-algebra and $(\pi_i)_{i\in I}$ is a separating family of representations of A, the equality

$$\mathrm{Sp}\,x = \overline{\bigcup_{i\in I} \mathrm{Sp}\,\pi_i(x)} \qquad (3\text{–}1)$$

holds provided that x is a *normal* element in A (e.g., a self-adjoint element). The classical example, showing that this equality fails in general, is given in [Halmos 1967, solution to Problem 81]. Here we get new examples of the same situation.

Corollary 3.9. *For at least one* $n \in \{2, 3, 5\}$, *the family* $(\rho_p \circ \alpha_p)_{p\in P_n}$ *is separating for* $C^*_r\,\mathrm{BS}_n$, *but the inclusion*

$$\overline{\bigcup_{p\in P_n} \mathrm{Sp}(\rho_p \circ \alpha_p)(h_S)} \subset \mathrm{Sp}\,M_S$$

is strict.

Proof. By Theorem 3.7.2, the set P_n is infinite for at least one $n \in \{2, 3, 5\}$; for this n, the family of representations $(\rho_p\circ\alpha_p)_{p\in P_n}$ is separating for $C_r^*\,\mathrm{BS}_n$ (Proposition 3.3). By Corollary 3.5, we have

$$\overline{\bigcup_{p\in P_n}\mathrm{Sp}(\rho_p\circ\alpha_p)(h_S)} = \big\{z\in\mathbb{C} : |z| = \tfrac12\big\}\cup\big\{z\in\mathbb{C} : |z-\tfrac12| = \tfrac12\big\}.$$

But a glance at Figure 1 shows that, in every case, $\mathrm{Sp}\,M_S$ contains points outside of the union of these two circles.[1] For $n = 3,\ 5$, we may also appeal to the fact that $\mathrm{Sp}\,M_S$ contains -1 (by Theorem 3.1.3). □

4. THE SYMMETRIC CASE

In this section we set $S = \{a^{\pm1}, b^{\pm1}\}$. The following result is analogous to Theorem 3.1.

Theorem 4.1.

$$\mathrm{Sp}\,M_S = \begin{cases}[-1,1] & \text{if } n \text{ is odd},\\ [r_n,1] & \text{if } n \text{ is even},\end{cases}$$

where

$$-1 < r_n \le -\sin^2\frac{\pi n}{2(n+1)}.$$

Proof. Amenability guarantees that the spectrum is $[r_n,1]$ for some $r_n\ge -1$ (Theorem 1.1.1 and Lemma 2.1).

The case of n odd is trivial. Indeed $\mathrm{Sp}(M_S) = [-1,1]$ if and only if there is a homomorphism $\beta : \mathrm{BS}_n\to C_2 = \{1,-1\}$ mapping a and b to -1 (Theorem 1.1.2); and such a homomorphism exists if and only if the relation in the group is of even length (i.e. n is odd).

Now assume that n is even. The preceeding remark shows immediatly that $r_n > -1$. To get the upper bound on r_n, we use the representations $\pi_{k,\theta}$ defined in the proof of Theorem 3.1.4. The spectrum of $\pi_{k,\theta}(h_S)$ is $\frac12(\cos\frac{2\pi k}{n+1}\pm\cos 2\pi\theta)$ and is contained in $\mathrm{Sp}(M_S)$ (Lemma 1.3). For $k = \frac n2$ and $\theta = 0$, we get the minimal value

$$\frac12\Big(\cos\frac{\pi n}{n+1} - 1\Big) = -\sin^2\frac{\pi n}{2(n+1)}.\qquad\square$$

Note that the contribution to $\mathrm{Sp}\,M_S$ of the abelianized group of BS_n (by considering $\beta_{ab}(h_S)$) does not improve the upper bound for r_n.

[1] The subtlety here is that formula (3–1) does not hold for every element in a C*-algebra. Pretending that it does leads to a quick disproof of the Generalized Riemann Hypothesis, just by glancing at Figure 1; the second author used this as the basis of an April fool's joke (à la Bombieri).

The description of BS_n as a subgroup of $\mathrm{Aff}_1\,\mathbb{Z}\big[\frac1n\big]$ makes it clear that BS_n is actually a semidirect product:

$$\mathrm{BS}_n = \mathbb{Z}[\tfrac1n]\rtimes_a\mathbb{Z}.$$

We are going to consider representations of BS_n induced from characters of the normal subgroup $\mathbb{Z}\big[\frac1n\big]$. For $\theta\in\mathbb{R}$, we denote by χ_θ the character of the real line defined by $\chi_\theta(x) = e^{2\pi i\theta x}$, for $x\in\mathbb{R}$.

Lemma 4.2. *The family of representations*

$$\big(\mathrm{Ind}_{\mathbb{Z}[1/n]}^{\mathrm{BS}_n}\mathrm{Rest}_{\mathbb{R}}^{\mathbb{Z}[1/n]}\chi_\theta\big)_{\theta\in\mathbb{R}}$$

is separating on $C_r^*\,\mathrm{BS}_n$.

Proof. Since the dual group of $\mathbb{R}$ is dense in the dual group of $\mathbb{Z}\big[\frac1n\big]$, the family of characters

$$\big(\mathrm{Rest}_{\mathbb{R}}^{\mathbb{Z}[1/n]}\chi_\theta\big)_{\theta\in\mathbb{R}}$$

is weakly equivalent to the regular representation $\rho_{\mathbb{Z}[1/n]}$ of $\mathbb{Z}\big[\frac1n\big]$. By continuity of weak containment with respect to induction, the family

$$\big(\mathrm{Ind}_{\mathbb{Z}[1/n]}^{\mathrm{BS}_n}\mathrm{Rest}_{\mathbb{R}}^{\mathbb{Z}[1/n]}\chi_\theta\big)_{\theta\in\mathbb{R}}$$

is weakly equivalent to $\mathrm{Ind}_{\mathbb{Z}[1/n]}^{\mathrm{BS}_n}\rho_{\mathbb{Z}[1/n]}\simeq\rho_{\mathrm{BS}_n}$. □

Using the semidirect product decomposition of BS_n, we see that the representation

$$\pi_\theta =: \mathrm{Ind}_{\mathbb{Z}[1/n]}^{\mathrm{BS}_n}\mathrm{Rest}_{\mathbb{R}}^{\mathbb{Z}[1/n]}\chi_\theta$$

is canonically realized on $\ell^2(\mathbb{Z})$; in that picture, the generator a acts by the bilateral shift on $\ell^2(\mathbb{Z})$, while the generator b acts by

$$(\pi_\theta(b)\xi)(k) = e^{2\pi i\theta n^{-k}}\xi(k),$$

for $\ell^2(\mathbb{Z})$, $k\in\mathbb{Z}$. Therefore $\pi_\theta(h_S)$ is a tridiagonal operator:

$$(\pi_\theta(h_S)\xi)(k) = \tfrac14\big(\xi(k-1)+\xi(k+1)+2\cos(2\pi\theta n^{-k})\xi(k)\big).$$

To estimate the spectrum of a tridiagonal operator, one may appeal to the following remarkable result by R. Szwarc [1998]:

Proposition 4.3. *Let J be the operator on $\ell^2(\mathbb{Z})$ defined by*

$$J\xi(k) = \lambda_{k+1}\xi(k+1)+\beta_k\xi(k)+\lambda_k\xi(k-1),$$

where $(\beta_k)_{k\in\mathbb{Z}}, (\lambda_k)_{k\in\mathbb{Z}}$ are real, bounded sequences, with $\lambda_k > 0$ for all k. Let m be such that $m <$

$\inf_{k\in\mathbb{Z}} \beta_k$. *Assume there exists a sequence* $(h_k)_{k\in\mathbb{Z}}$ *in* $]0,1[$ *such that*

$$\frac{\lambda_k^2}{(m-\beta_{k-1})(m-\beta_k)} \le h_k(1-h_{k-1})$$

for every $k\in\mathbb{Z}$. *Then* $\operatorname{Sp} J \subset [m,+\infty[$.

From this we deduce:

Theorem 4.4. *For* $n=2$ *and* $S=\{a^{\pm1},b^{\pm1}\}$, *the spectrum of the Markov operator* M_S *on* BS_2 *is* $\operatorname{Sp} M_S = [-\frac{3}{4},1]$.

Proof. We begin by showing that $-\frac{3}{4}$ belongs to the spectrum of M_S. For this, we consider the prime $p=3$ and the representation π_3, of degree 2, appearing in Lemma 3.2. From the formulae in the proof of Proposition 3.4, it is clear that

$$\pi_3(\alpha_3(h_S)) = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

The spectrum of this 2×2 matrix is $\{-\frac{3}{4},\frac{1}{4}\}$, and it is contained in $\operatorname{Sp} M_S$ by Lemma 1.3. To show the converse inclusion, we find a sequence $(h_n)_{n\in\mathbb{Z}}\subset\ ]0,1[$ satisfying

$$\frac{1}{(3+2\cos 2^{-(n-1)}\varphi)(3+2\cos 2^{-n}\varphi)} \le h_n(1-h_{n-1}),$$

where $\varphi=2\pi\theta$. By Proposition 4.3, this will imply that $\operatorname{Sp}(4\pi_\theta(h_S))\subset[-3,\infty[$, for all $\theta\in\mathbb{R}$.

If we define

$$h_n = \frac{1}{2} + \frac{\alpha_n}{3+2\cos 2^{-n}\varphi}$$

for all $n\in\mathbb{Z}$, we have to search for a sequence $(\alpha_n)_{n\in\mathbb{Z}}$ such that

$$\left(\tfrac{3}{2}+\cos 2^{-(n-1)}\varphi-\alpha_{n-1}\right)\left(\tfrac{3}{2}+\cos 2^{-n}\varphi+\alpha_n\right)\ge 1$$

and

$$-\tfrac{3}{2}-\cos 2^{-n}\varphi < \alpha_n < \tfrac{3}{2}+\cos 2^{-n}\varphi.$$

A candidate is $\alpha_n = f(2^{-n}\varphi)$, where f is defined on $[0,2\pi]$ by

$$f(x)=\begin{cases} 0 & \text{if } x\in[0,\frac{\pi}{3}]\cup[\frac{5\pi}{3},2\pi],\\ u_1(x) & \text{if } x\in\,]\frac{\pi}{3},\frac{2\pi}{3}[\,\cup\,]\frac{4\pi}{3},\frac{5\pi}{3}[,\\ u_2(x) & \text{if } x\in[\frac{2\pi}{3},\frac{4\pi}{3}],\end{cases}$$

with

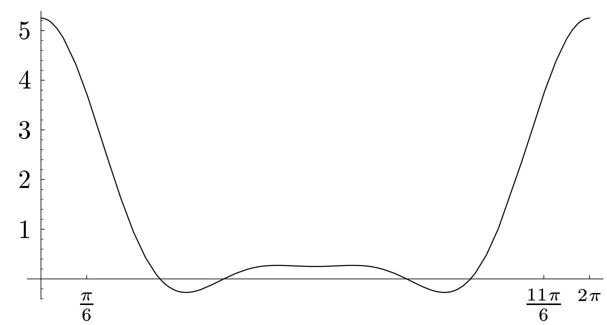
$$u_1(x) := -(\tfrac{3}{2}+\cos x)+\left(\tfrac{3}{2}+\cos\tfrac{x}{2}\right)^{-1},$$

$$u_2(x) := -(\tfrac{3}{2}+\cos x)+\left(3+2\cos\tfrac{x}{2}-\left(\tfrac{3}{2}+\cos\tfrac{x}{4}\right)^{-1}\right)^{-1}.$$

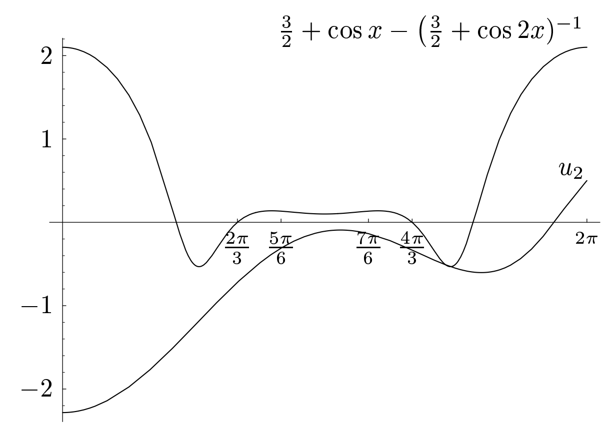
Next extend f to be periodic of period 2π.

To show that $\left(\frac{3}{2}+\cos x-f(x)\right)\left(\frac{3}{2}+\cos 2x+f(2x)\right)\ge 1$ we must verify several conditions:

- $\left(\frac{3}{2}+\cos x\right)\left(\frac{3}{2}+\cos 2x\right)-1\ge 0$ for $x\in\left[0,\frac{\pi}{6}\right]\cup\left[\frac{11\pi}{6},2\pi\right]$. This follows from simple trigonometry estimates and is clear from the graph of the function on the left-hand side of the inequality:

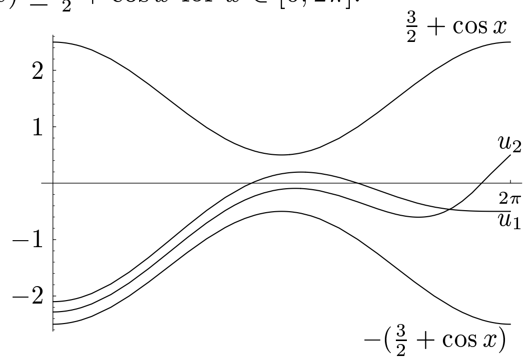


- $\left(\frac{3}{2}+\cos x\right)\left(\frac{3}{2}+\cos 2x+u_1(2x)\right)-1\ge 0$ for $x\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]\cup\left[\frac{5\pi}{3},\frac{11\pi}{6}\right]$. The definition of u_1 was cooked up exactly so that this is satisfied.
- $\left(\frac{3}{2}+\cos x-u_1(x)\right)\left(\frac{3}{2}+\cos 2x+u_2(2x)\right)-1\ge 0$ for $x\in\left[\frac{\pi}{3},\frac{2\pi}{3}\right]\cup\left[\frac{4\pi}{3},\frac{5\pi}{3}\right]$. Again, this comes from the definition of u_2.
- $\left(\frac{3}{2}+\cos x-u_2(x)\right)\left(\frac{3}{2}+\cos 2x+u_1(2x)\right)-1\ge 0$ for $x\in\left[\frac{2\pi}{3},\frac{5\pi}{6}\right]\cup\left[\frac{7\pi}{6},\frac{4\pi}{3}\right]$. A slightly tedious computation shows that this is equivalent to $u_2(x)\le 0$ for the same range of x, and again this is clear from the graph of u_2:



- $\left(\frac{3}{2}+\cos x-u_2(x)\right)\left(\frac{3}{2}+\cos 2x\right)-1\ge 0$ for $x\in\left[\frac{5\pi}{6},\frac{7\pi}{6}\right]$. This is equivalent to $u_2(x)\le\frac{3}{2}+\cos x-\left(\frac{3}{2}+\cos 2x\right)^{-1}$ for the same range of x (see preceding graph).

Finally, each u_i, for $i = 1, 2$, satisfies $-\frac{3}{2} - \cos x \le u_i(x) \le \frac{3}{2} + \cos x$ for $x \in [0, 2\pi]$:



This implies that $-\frac{3}{2} - \cos 2^{-n}\varphi < \alpha_n < \frac{3}{2} + \cos 2^{-n}\varphi$, and the result follows. □

The value $-\frac{3}{4}$ in Theorem 7 was discovered experimentally, by computing numerically the spectrum of $\pi_p\big(\frac{1}{4}(a_p + a_p^{-1} + b_p + b_p^{-1})\big)$ for small primes p.

For n larger than 2 and less than 28, we also approximated the smallest value of

$$\mathrm{Sp}\big(\pi_p\big(\tfrac{1}{4}(a_p + a_p^{-1} + b_p + b_p^{-1})\big)\big)$$

by numerical computations for p running over the first 300 primes, but that does not improve the upper bound in Theorem 6.

5. AN APPLICATION TO WAVELET THEORY

As observed, the connection with wavelet theory comes from the fact that BS_2 is isomorphic to the subgroup of $\mathrm{Aff}_1(\mathbb{R})$ generated by translation by 1 and dilation by 2. These are exactly the two transformations used in multiresolution analysis [Daubechies 1992; Bultheel 1995]; for this reason, we think that BS_2 deserves to be called the *wavelet group*.

We recall some notations from wavelet theory. On $L^2(\mathbb{R})$, define the unitary operators

$$(T_r\xi)(x) = \xi(x - r),$$

for $r \in \mathbb{R}$, $\xi \in L^2(\mathbb{R})$, and

$$(D_s\xi)(x) = \frac{1}{\sqrt{s}}\xi\Big(\frac{x}{s}\Big),$$

for $s > 0$, $\xi \in L^2(\mathbb{R})$. Setting $\pi(a) = D_n$ and $\pi(b) = T_1$ then defines a unitary representation π of BS_n on $L^2(\mathbb{R})$.

Theorem 5.1. *The map π extends to a faithful representation of $C_r^*\,\mathrm{BS}_n$.*

Proof. We have to show that π is weakly equivalent to ρ_{BS_n}. Once more, weak containment of π in ρ_{BS_n} follows from amenability of BS_n. To prove the converse, define a function $\psi \in L^2(\mathbb{R})$ by

$$\psi(x) = \begin{cases} \sqrt{\frac{n}{2}} & \text{on } [0, \frac{1}{n}[, \\ -\sqrt{\frac{n}{2}} & \text{on } [\frac{1}{n}, \frac{2}{n}[, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\|\psi\|_2 = 1$; note that, for $n = 2$, the function ψ is just the *Haar wavelet*. For $k, m \in \mathbb{Z}$, set

$$\psi_{k,m}(x) = (D_{n^k}T_m\psi)(x) = n^{-\frac{k}{2}}\psi(n^{-k}x - m).$$

The $\psi_{k,m}$'s are orthonormal (but not a basis for $n > 2$): indeed, considerations of supports show that two $\psi_{k,m}$'s of the same scale (same value of k) never overlap; on the other hand, if $k < k'$, then the support of $\psi_{k,m}$ lies totally in a region where $\psi_{k',m'}$ is constant, so that $\langle\psi_{k,m}|\psi_{k',m'}\rangle = 0$. For $g \in \mathrm{BS}_n$, the operator $\pi(g)$ can be written uniquely $\pi(g) = D_{n^j}T_r$, with $j \in \mathbb{Z}$ and $r \in \mathbb{Z}\big[\frac{1}{n}\big]$. For $k \in \mathbb{N}$,

$$\begin{aligned}\langle\pi(g)\psi_{-k,0}|\psi_{-k,0}\rangle &= \langle D_{n^j}D_{n^k}T_rD_{n^{-k}}\psi|\psi\rangle \\ &= \langle D_{n^j}T_{rn^k}\psi|\psi\rangle.\end{aligned}$$

But, for k big enough, rn^k is an integer N, so that

$$\langle\pi(g)\psi_{-k,0}|\psi_{-k,0}\rangle = \langle\psi_{j,N}|\psi\rangle = \delta_{e,g},$$

by orthonormality of the $\psi_{k,m}$'s. This shows that ρ_{BS_n} is weakly contained in π, so the proof is complete. □

Remark. In the case $n = 2$, we used in the above proof the Haar wavelet, but any wavelet basis would do as well.

From this result and the connectedness of spectra in $C_r^*\,\mathrm{BS}_n$ (see Lemma 2.1), we immediately deduce:

Corollary 5.2. *On $L^2(\mathbb{R})$, operators of the form*

$$\sum_{k,m\in\mathbb{Z}} c_{k,m}D_{n^k}T_m$$

(*with $c_{k,m} \in \mathbb{C}$, only finitely many nonzero $c_{k,m}$'s*), *have connected spectra.*

In particular, this applies to the operators

$$\sum_{m\in\mathbb{Z}} c_mD_{\frac{1}{2}}T_m$$

appearing in the *two-scale relation* (or *dilation equation*) in multiresolution analysis [Bultheel 1995, § 5].

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