Mahler's Measure and Special Values of L-functions

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CONTENTS

If $P(x_1, \dots, x_n)$ is a polynomial with integer coefficients, the Mahler measure $M(P)$ of P is defined to be the geometric mean of $|P|$ over the n-torus $\mathbb{T}^n$. For $n = 1$, $M(P)$ is an algebraic integer, but for $n > 1$, there is reason to believe that $M(P)$ is usually transcendental. For example, Smyth showed that $\log M(1 + x + y) = L'(\chi_{-3}, -1)$, where χ_{-3} is the odd Dirichlet character of conductor 3. Here we will describe some examples for which it appears that $\log M(P(x, y)) = r L'(E, 0)$, where E is an elliptic curve and r is a rational number, often either an integer or the reciprocal of an integer. Most of the formulas we discover have been verified numerically to high accuracy but not rigorously proved.

1. INTRODUCTION

The aim of this paper is to describe an attempt to understand and generalize a recent formula of Deninger [1997] by means of systematic numerical experiment. This conjectural formula,

$$m\Big(x + \frac{1}{x} + y + \frac{1}{y} + 1\Big) \stackrel{?}{=} L'(E,0),$$

gives the value of the logarithmic Mahler measure $m(P)$ of the Laurent polynomial $P = x + 1/x + y + 1/y + 1$ as a rational multiple of $L'(E, 0)$, where E is the elliptic curve of conductor 15 that is the projective closure of the curve $x+1/x+y+1/y+1 = 0$, and $L(E, s)$ is the L-function of that curve. In fact, numerically the multiple is exactly 1 to at least 50 decimal places. As we explain in more detail later in this introduction, Deninger was led to his formula by the Bloch–Beilinson conjectures on special values of L-series.

Our goal is to try to determine conditions under which such a formula should hold for a polynomial $P(x, y)$ with integer coefficients. An optimistic guess, given Deninger's result, would be

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that a formula of this type should hold if the curve $P(x,y)=0$ is an elliptic curve. However, for the polynomial $P(x,y)=y^2-x^3-k$, we have

$$m(y^2-x^3-k)=m(y+x+k)=\log|k|$$

if $|k|\geq 2$, and

$$m(y^2-x^3\pm 1)=m(y+x+1)=L'(\chi_{-3},-1),$$

by results of Smyth [1981b]. Clearly, these expressions have nothing to do with the L-series of the curve

$$E: \quad y^2=x^3+k.$$

But surely, for a formula of this type, shouldn't it be *necessary* for the curve $P(x,y)=0$ be an elliptic curve, or at least a singular model of such a curve? Again the answer is no. There are polynomials for which the curve $Z=\{P(x,y)=0\}$ is of genus 2 and yet $m(P)$ is (numerically) a rational multiple of $L'(E,0)$ for a certain elliptic curve E. (But in this case, E does have something to do with Z; it is a factor of the Jacobian variety of Z).

The main news of the paper is that there are many infinite families of polynomials $P_k(x,y)$ for which $m(P_k)$ does seem to be given by a rational multiple of an appropriate $L'(E_k,0)$, at least to high numerical accuracy. Furthermore, Fernando Rodríguez Villegas [1996] has used the theory of modular forms to show that, for some of these families, the formulas discovered would follow from the conjectures of Bloch and Beilinson. For certain values of the parameter k, the curves E_k have complex multiplication and in these cases, the formulas can be proved rigorously.

At least in the case where Z has genus 1, we now have a good idea about some of the conditions that P must satisfy in order that such a formula be true. One of the conditions, (A), requires that the "faces" P_F of P (defined in terms of the Newton polygon of P) should be cyclotomic, that is, satisfy $m(P_F)=0$. (It may be that the condition can be relaxed to require only that $M(P_F)=\exp(m(P_F))$ should be an integer for each face, but only if the "interior" coefficients of P satisfy further arithmetic conditions). The apparent necessity of this condition was deduced from the examination of many examples and initially we had no theoretical understanding of why it should be required except that it seemed to be an "arithmetic" condition. Since the first version of this paper was circulated in preprint form, Rodríguez Villegas and Hubert Bornhorn have independently shown that (A) is a natural condition from the point of view of K-theory, as we discuss below. We have opted to leave the statement of the conjectures in the form in which they were originally circulated but have added a short section at the end of the paper to indicate some of the progress that has been made on the conjectures since then.

The second condition, (B), is an analytic condition on the algebraic function $y(x)$ defined by $P(x,y)=0$. For polynomials that do not vanish on the torus, it is expressible as a geometric condition (G) that has to do with how the zero set of $P(x,y)$, thought of as a surface in $\mathbb{C}^2$, "links" the torus $\mathbb{T}^2=\{|x|=1\}\times\{|y|=1\}$.

We must emphasize that, even when $P(x,y)=0$ is a model of an elliptic curve E, $m(P)$ is a property of the *polynomial* $P(x,y)$ and not of the curve E. This will be evident in the many examples in Section 2, but we borrow an example from that section to make this clear now. Consider the polynomial $P=y^2-6xy+y-x^3$, so $P(x,y)=0$ is a (minimal) model for the elliptic curve $E: y^2-6xy+y=x^3$. We will find that

$$m(P)=3L'(E,0).$$

If we write $y=Y+3x$, we obtain $Q=Y^2+Y-x^3-9x^2+3x$, so $Q=0$ is another minimal model for E; but now,

$$m(Q)=\log\left(\frac{9+\sqrt{93}}{2}\right).$$

If we now write $x=X-3$ to obtain $R=Y^2+Y-X^3+30X-63$, we obtain the reduced minimal model for E, and recognize it as the curve 27A4 of conductor 27 in [Cremona 1992]. Now

$$m(R) = \log 63.$$

In terms of our general setup, the difference between P, Q and R is that in P, the "interior" term $6xy$ is dominant, while in Q the face $-x^3 - 9x^2 + 3x$ is dominant, and in R, the face -63 is dominant. The point here is that the changes of variable that preserve $m(P)$, basically those of the form $P(\pm x^a y^b, \pm x^c y^d)$, are quite different from the birational mappings that preserve the isomorphism class of the curve E. This is really what makes Deninger's result so striking.

The organization of this paper is as follows. We begin with some basic facts about Mahler's measure of polynomials in several variables. We then discuss some examples due to Smyth, Ray, and the author, of polynomials in two variables for which the measure can be expressed in terms of the value of Dirichlet L-series evaluated at 2. All these polynomials are linear in one of the variables or a product of such polynomials. Next we give some examples discovered by Mossinghoff and the author of polynomials in two variables that have the smallest known measure for such polynomials. These polynomials are quadratic in one of the variables. We then discuss Deninger's formula mentioned above, which expresses the measure of one of the latter polynomials in terms of the L-series of an elliptic curve evaluated at 2. This formula has not yet been proved but was derived on the basis of a conjecture of Bloch and Beilinson and has been verified numerically to many decimal places. We finish the section by describing some early experiments that produced a number of similar formulas and suggested the systematic experiments to be discussed in the remainder of the paper.

In Section 2, we begin with a general discussion of the conditions (A), (B) and (G) mentioned above. We then specialize to a discussion of families of polynomials of the form

$$P_k(x, y) = A(x)y^2 + (B(x) + kx)y + C(x),$$

for which the zero set $Z_k = \{P_k(x, y) = 0\}$ is (generically) of genus 1. We experimentally determine conditions that seem to insure that the measure of P_k can be expressed in terms of L-series of elliptic curves evaluated at 2. We discuss the recent work of Rodríguez Villegas [1996] that shows that the measure of such a family can in many cases be expressed as a nonholomorphic modular form. This has enabled him to prove some of the formulas discovered numerically in case the elliptic curve in question has complex multiplication.

In Section 3, we discuss some similar families of polynomials of the form

$$P_k(x, y) = A(x)y^2 + (B(x) + kE(x))y + C(x)$$

for which Z_k is (generically) of genus 2. If in addition the polynomial $P_k(x, y)$ is reciprocal then the Jacobian $J(Z_k)$ splits into the product of two elliptic curves $E_k \times F_k$. We give two classes of examples depending on the choice of $E(x)$. In the first, the measure seems to be a rational multiple of $L'(E_k, 0)$ for all values of k for which the discriminant is nonzero. The value of $L'(F_k, 0)$ does not appear to be related to the measure. For the second class, the measure is only given by such a formula for half of the values of k. In all cases, the results have been verified numerically to high accuracy but not proved.

In Section 4, we discuss some of the formulas involving Dirichlet characters that occur as degenerate cases of the examples discussed in Sections 2 and 3. They provide a small amount of further evidence for Chinburg's conjecture.

Section 5 concluded the first version of this paper. In Section 6, we mention briefly some results that have been found since the first version of this paper was circulated as a preprint.

Tables summarizing some of our computational results are interspersed with the discussion at the relevant places. More complete tables can be obtained by anonymous ftp; see the section Electronic Availability on page 79.

1A. Mahler's Measure

If $P(x_1, \dots, x_n)$ is a polynomial with complex coefficients, then the *logarithmic Mahler measure* of P is defined by

$$m(P) = \int_0^1 \cdots \int_0^1 \log \left| P(e(t_1), \dots, e(t_n)) \right| dt_1 \cdots dt_n, \tag{1–1}$$

where $e(t) = \exp(2\pi i t)$. The *Mahler measure* of P is then defined as $M(P) = \exp(m(P))$. Thus $M(P)$ is the geometric mean of $|P|$ over the n-torus. This was introduced by Mahler [1962] in order to give a simple proof of the "Gel'fond–Mahler inequality". In this paper it will be more convenient to deal directly with $m(P)$ rather than $M(P)$.

For $n = 1$, if $P(x) = a_0 \prod_{j=1}^{d} (x - \alpha_j)$, Jensen's formula shows that

$$m(P(x)) = \log |a_0| + \sum_{j=1}^{d} \log^+ |\alpha_j|, \tag{1–2}$$

where

$$\log^+ v = \begin{cases} \max(\log v, 0) & \text{if } v > 0, \\ 0 & \text{if } v = 0. \end{cases}$$

For polynomials with integer coefficients, clearly $m(P) \geq 0$ with $m(P) = 0$ only if P is monic and has all its zeros inside the unit circle, and hence is a product of a monomial x^a and a cyclotomic polynomial, by Kronecker's theorem.

Our interest in the Mahler measure of several variable polynomials arose in connection with a question of Lehmer concerning the Mahler measure of single variable polynomials [Boyd 1981b]. We will briefly describe this since it explains why we had numerically computed many examples of polynomials with $m(P(x,y))$ small and why our initial experiments concentrated on reciprocal polynomials. The main focus of this paper, however, is on explicit formulas for measures and most of the polynomials we consider do not have particularly small measure.

Lehmer [1933] noted that $m(P(x))$ measures the growth rate of the sequence $\Delta_n = \prod_{j=1}^{d} (\alpha_j^n - 1)$, and asked whether $m(P)$ can be arbitrarily small but positive for $P(x) \in \mathbb{Z}[x]$. The smallest value he was able to find was

$$\begin{aligned} m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \\ = \log(1.17628081\ldots) = .16235761\ldots. \end{aligned} \tag{1–3}$$

This still stands as the smallest known positive value of $m(P(x))$, in spite of extensive computations [Boyd 1980; 1989; Mossinghoff 1995; $\geq$ 1998].

In connection with Lehmer's question, definition (1–1) is the natural generalization of $m(P)$ from one-variable polynomials to n-variable polynomials. This follows from the following limit formula [Boyd 1981b], and its generalizations to n variables proved in [Boyd 1981a; Lawton 1983].

Theorem. *As* $N \to \infty$, $m(P(x, x^N)) \to m(P(x,y))$.

These limit theorems enable one to extend many results proved for one variable to many variables. For example, Smyth [1971] proved that if $P(x)$ is a nonreciprocal polynomial with integer coefficients then

$$\begin{aligned} m(P) \geq m(x^3 - x - 1) = \log \theta_0 = \log(1.32471795\ldots) \\ = .28119957\ldots. \end{aligned} \tag{1–4}$$

A one-variable polynomial is said to be *reciprocal* if its coefficients form a palindromic sequence, that is, if $x^d P(1/x) = P(x)$ for some integer d. Otherwise the polynomial is *nonreciprocal*. (Notice that Lehmer's polynomial in (1–3) is reciprocal). A polynomial P in n variables is *reciprocal* if

$$\frac{P(x_1, \dots, x_n)}{P(1/x_1, \dots, 1/x_n)}$$

is a monomial $x_1^{b_1} \cdots x_n^{b_n}$, and *nonreciprocal* otherwise. Using the limit theorems, Smyth's theorem extends immediately to $m(P(x_1, \dots, x_n)) \geq \log \theta_0$ for nonreciprocal P.

In [Boyd 1981a], we used this method to characterize the $P(x_1, \dots, x_n)$ for which $m(P) = 0$ as products of cyclotomic polynomials in monomials, that is, $\Phi(x_1^{b_1} \cdots x_n^{b_n})$, where the Φ are cyclotomic. Smyth [1981a] then gave a more direct proof of this result.

In addition to their role as limit points of the $m(P(x))$, measures of polynomials in several variables have an intrinsic interest in ergodic theory, according to a theorem of Lind, Schmidt and Ward [Lind et al. 1990; Schmidt 1995], which proves that $m(P(x_1,\ldots,x_n))$ is the entropy of a certain $\mathbb{Z}^n$-action on $\mathbb{T}^{\mathbb{Z}^n}$. The measure $m(P(x_1,\ldots,x_n))$ also occurs in the definition of the canonical height of hypersurfaces in toric varieties [Maillot 1997].

1B. Explicit Formulas Involving Dirichlet L-Series

The following formula of Smyth, proved in an Appendix of [Boyd 1981b], was the inspiration for most subsequent investigations into special values of $m(P(x))$ for polynomials in many variables:

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2). \tag{1–5}$$

Here $\chi_{-f}(n) = \left(\frac{-f}{n}\right)$ is the real odd Dirichlet character of conductor f, so

$$L(\chi_{-3}, 2) = 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \cdots.$$

Ray [1987] observed that (1–4) is given a nicer appearance if one uses the functional equation for $L(\chi_{-3}, s)$:

$$m(1+x+y) = L'(\chi_{-3}, -1). \tag{1–6}$$

The proof of (1–5) is worth noting here. Since $1+x+y$ is a linear function of y, Jensen's formula applied to one of the integrals in (1–1) shows that

$$\begin{aligned} m(1+x+y) &= m(1-x+y) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |e^{it}-1|\, dt \\ &= \frac{1}{\pi}\int_0^{2\pi/3} \log |e^{it}-1|\, dt. \end{aligned}$$

Thus $m(1+x+y)$ is given by a special value of the Clausen integral [Lewin 1981]

$$\mathrm{Cl}_2(\theta) = -\int_0^{\theta} \log|e^{it}-1|\, dt = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2},$$

and the result follows.

A similar computation applies to many polynomials $P(x,y) = A(x)y + B(x)$, if $A(x)$ and $B(x)$ are cyclotomic and if the solutions of $|A(x)| = |B(x)|$ on $|x| = 1$ are roots of unity. For example,

$$m(1+x+y-xy) = \frac{2}{\pi} L(\chi_{-4}, 2) = L'(\chi_{-4}, -1), \tag{1–7}$$

where

$$L'(\chi_{-4}, 2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots = G$$

is Catalan's constant,

$$m(1+x+x^2+y) = \tfrac{2}{3} L'(\chi_{-4}, -1), \tag{1–8}$$

$$m(1+x+y+x^2y) = \tfrac{3}{2} L'(\chi_{-3}, -1). \tag{1–9}$$

The expressions $L'(\chi_{-f}, -1)$ occur so often in these computations that we will write

$$d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2).$$

Chinburg [Ray 1987, p. 697] conjectured that, given any odd Dirichlet character $\chi_{-f}(n)$, there should be a polynomial with integer coefficients $P_f(x,y)$ for which $d_f/m(P_f)$ is a rational number. His conjecture was apparently based on considerations from K-theory, but the paper cited by Ray has not been published. Ray was able to construct polynomials $P_f(x,y)$ for which $m(P_f) = r_f d_f$ for certain rationals r_f for $f = 3$, 4, 7, 8, 20, and 24. For $f = 7$ his proof required him to prove some new multivariable identities for dilogarithms. (Formulas for the P_f are given explicitly in Section 4 for $f = 7$, 8, 20, and 24.)

The following formulas of Smyth [1981b] for the measure of certain polynomials of the form $P(x+y)$ are of interest since they involve a combination of terms of different "weights":

$$m((x+y)^2+2) = \tfrac{1}{2}\log 2 + d_4 \tag{1–10}$$

$$m((x+y)^2+3) = \tfrac{2}{3}\log 3 + \tfrac{4}{3} d_3. \tag{1–11}$$

They are really generalizations of his result (1–6), being equivalent to formulas for $m(x+y+\sqrt{2})$ and $m(x+y+\sqrt{3})$.

Recently Maillot and Cassaigne [Maillot 1997] have derived a formula for $m(a_0+a_1x+a_2y)$ for arbitrary complex a_j. If $|a_0|, |a_1|$ and $|a_2|$ are the lengths of the sides of a planar triangle opposite the angles α_0, α_1, α_2, then

$$m(a_0+a_1x+a_2y) = \frac{\alpha_0}{\pi}\log|a_0| + \frac{\alpha_1}{\pi}\log|a_1| + \frac{\alpha_2}{\pi}\log|a_2| + \frac{1}{\pi}\mathcal{D}\left(\frac{|a_1|}{|a_0|}e^{i\alpha_2}\right),$$

where $\mathcal{D}$ is the Bloch–Wigner dilogarithm, $\mathcal{D}(z) = \operatorname{Im}(\operatorname{li}_2(z)+\log|z|\log(1-z))$. In the alternative case,

$$m(a_0+a_1x+a_2y) = \log\max(|a_0|, |a_1|, |a_2|).$$

In this paper, motivated by a formula of Deninger [1997], whose discovery we describe below, we are interested in an analogous question for L-functions of elliptic curves. Here, the counterpart to $d_f = f^{3/2}L(\chi_{-f}, 2)/(4\pi) = L'(\chi_{-f}, -1)$ is

$$b_E = NL(E,2)/(2\pi)^2 = L'(E,0),$$

where N is the conductor of the elliptic curve E, and where the second equation is only valid if E is a modular curve (see below). We are interested in obtaining polynomials $P_E(x,y) \in \mathbb{Z}[x,y]$ for which $b_E/m(P_E)$ is a rational number. We will give some motivation below for expecting such formulas to exist, at least for some E. We are not ready to conjecture that such a polynomial should exist for every elliptic curve E over the rationals but we do make a conjecture about one sort of polynomial for which we expect such a formula to exist. It will be convenient to adopt the following terminology: for a polynomial $P(x,y)$, r a rational number, and c an algebraic integer, we will say that a formula of the form $m(P) = r\log|c|$, $m(P) = rL'(\chi_f, -1)$ or $m(P) = rL'(E,0)$ is of *type* C, D or E, respectively. (The letters come from *logarithmiC*, *Dirichlet* and *Elliptic*. The use of L for logarithmic might suggest instead *L-series*.) Formulas like (1–10) and (1–11) will be said to be of type CD. We will see some formulas of type CE in Section 2 but we have not yet encountered formulas of type DE or CDE for irreducible P. (See Section 6 for an update on this remark). Of course, it is not difficult to construct formulas of mixed type for reducible P.

Our main interest here is in formulas of type E and we have not systematically studied Chinburg's conjecture about formulas of type D. However, in certain degenerate cases the families of polynomials we have studied produce formulas of type D. From these, we have examples (Equations (4–2)–(4–8) on page 76) apparently giving d_{15}, d_{39} and d_{55}, and simpler examples than Ray's for d_7 and d_{24}, so this provides a small amount of additional evidence for Chinburg's conjecture. It must be emphasized that these formulas have only been verified numerically and are not yet proved.

1C. Mahler's Measure and Elliptic Curves

The examples (1–6)–(1–11) are nonreciprocal polynomials and so, by the extension of Smyth's theorem mentioned above, have $m(P) \geq \log\theta_0$. In our numerical studies of Lehmer's conjecture, Mossinghoff [1995; ≥ 1998] and the author [Boyd 1977; 1980; 1989] have discovered a number of reciprocal polynomials with fairly small measure, and in particular those shown in the box on the next page, which are the only known limit points of Mahler's measure smaller than $\log\theta_0 = .28119957\ldots$. The examples (1–12), (1–13), and (1–15) are mentioned in [Boyd 1980; 1981b], and (1–14) was recently found by Mossinghoff [1995]. All were found by searching for patterns in extensive tables of measures of one-variable polynomials.

The polynomials have been written in the given form in order to point out that they are quadratic in y. By multiplication by monomials and by changes of variable of the form $P(\pm x^a y^b, \pm x^c y^d)$, which do not affect $m(P)$, one may obtain a variety of more symmetric presentations; for example, $x+1/x+y+1/y+1$ or $(x+1/x)(y+1/y)+1$ for the polynomial in (1–13).

The computation of the values of $m(P(x,y))$ is a matter of numerical integration. Since one can

$$m_1 = m((x+1)y^2 + (x^2+x+1)y + x(x+1)) = 0.22748122\ldots \quad (1\text{–}12)$$
$$m_2 = m(y^2 + (x^2+x+1)y + x^2) = 0.25133043\ldots \quad (1\text{–}13)$$
$$m_3 = m((x+1)y^2 + (x^4-x^2+1)y + x^3(x+1)) = 0.26933864\ldots \quad (1\text{–}14)$$
$$m_4 = m((x^2+x+1)y^2 + (x^4+x^3+x^2+x+1)y + x^2(x^2+x+1)) = 0.27436329\ldots \quad (1\text{–}15)$$

Some reciprocal polynomials with small measure.

use Jensen's formula (1–2) to take care of one of the integrals, the computation reduces to a single-variable numerical integration. In fact, if $P(x,y)$ does not vanish on the torus $\mathbb{T}^2$, the resulting integrand is a smooth periodic function for which even the trapezoidal rule produces accurate results as is well known to numerical analysts and easily proved by expressing the error in terms of Fourier coefficients [Hamming 1962, p. 284]. When $P(x,y)$ vanishes on the torus, the integrand will have singularities but these are not difficult to handle, as explained below.

For example, if $P = x + 1/x + y + 1/y + 1$, let $x = \exp(it)$ and treat $P(x,y)$ as a polynomial in y to see that

$$|P(x,y)| = |y^2 + y(2\cos t + 1) + 1| = |(y - y_1(t))(y - y_2(t))|,$$

where $y_1(t) = -b - (b^2-1)^{1/2}$, writing $b(t) = \cos t + 1/2$. Thus

$$m(P) = \frac{1}{\pi}\int_0^\pi \log^+ |y_1(t)|\, dt.$$

Since the product of the roots is 1, we will have $|y_1(t)| > 1 > |y_2(t)|$ exactly when the roots are real and unequal, that is, when $\cos t > 1/2$, so $|t| < \pi/3$. Thus

$$m_2 = m\Big(x + \frac{1}{x} + y + \frac{1}{y} + 1\Big) = \frac{1}{\pi}\int_0^{\pi/3} \log(b + \sqrt{b^2-1})\, dt, \quad (1\text{–}16)$$

which can now be integrated numerically. (Most integrals mentioned in this paper were computed using either the intnum procedure of PARI [Batut, Bernardi, Cohen et Olivier 1995], when P does not vanish on the torus, or the numerical integration routine of Maple version V, release 3, otherwise. The latter handles logarithmic singularities quite well, but occasionally has difficulty with square root singularities occuring at an endpoint t_1 of an interval, such as at $t = \pi/3$ in (1–16). In this case the change of variable $v = |\cos(t) - \cos(t_1)|^{1/2}$ was sufficient to make the integral easily tractable.)

Obviously one would like to have a formula more like (1–5) for the integrals in (1–12)–(1–15). Deninger [1997] recently showed that there is a connection between $m(P)$ and higher K-theory provided $P(x_1, \ldots, x_n)$ does not vanish on $\mathbb{T}^n$. Moreover, under this condition $m(P)$ is a Deligne period of a certain mixed motive. We will say a few more words about the meaning of Deninger's result at the end of this section.

Deninger's result does not apply directly to any of the examples above since in each case $P(x,y)$ vanishes on $\mathbb{T}^2$, but even in this case Deninger has been able to show, under some extra assumptions, that $m(P(x,y)) - m(P(x,0))$ can be given a cohomological interpretation. Using this he was able to evaluate m_2 of (1–13) as an Eisenstein–Kronecker series of a certain elliptic curve, and then assuming a conjecture of Beilinson, to conjecture that one should have

$$m\Big(x + \frac{1}{x} + y + \frac{1}{y} + 1\Big) = r\frac{15}{(2\pi)^2}L(E,2) = rL'(E,0), \quad (1\text{–}17)$$

where E is the elliptic curve of conductor 15 given by $x+1/x+y+1/y+1=0$, and r is a rational number unspecified by Beilinson's conjecture. Using a 20 decimal place value of m_2, which had been computed earlier using Maple, and using PARI to compute $L(E,2)$, we found that

$$r = 1.00000000000000000000,$$

and later that

$$r = 1.000000000000000000000000000000000000000000000000000,$$

so presumably

$$m\Big(x+\frac{1}{x}+y+\frac{1}{y}+1\Big) \stackrel{?}{=} L'(E,0). \qquad (1\text{–}18)$$

Here, and in the rest of this paper, the notation $A \stackrel{?}{=} B$, read as "A is conjectured to be equal to B", will mean that A and B are equal to at least 25 decimal places (or PARI precision, which is by default 28 decimal places).

An elliptic curve over $\mathbb{Q}$ is a smooth cubic curve that has a rational point [Cassels 1991; Silverman 1986]. For the purpose of this exposition, we can think of an elliptic curve as given by an equation

$$E : y^2 = h(x), \qquad (1\text{–}19)$$

where $h(x)$ is a monic cubic polynomial with integer coefficients and with distinct roots. If N_p is the number of solutions of (1–19) modulo the prime p then let $a_p = p+1-(N_p+1)$. Associated with E is its *conductor*, an integer N that is the product to various powers of those primes p for which E mod p has multiple points. Using the local data a_p, one constructs the L-function [Silverman 1986; Cohen 1993, p. 382]:

$$\begin{aligned} L(E,s) &= \prod_{p\nmid N}(1-a_p p^{-s}+p^{1-2s})^{-1}\prod_{p\mid N}(1-a_p p^{-s})^{-1} \\ &= \sum_{n=1} \frac{a_n}{n^s}, \end{aligned}$$

where the coefficients a_n are obtained by formally expanding the infinite product. It can be shown that the series converges absolutely if $\mathrm{Re}(s) > \frac{3}{2}$.

The widely believed Shimura–Taniyama–Weil conjecture says that all elliptic curves over $\mathbb{Q}$ are modular. That is, if $L(E,s) = \sum_{n=1}^{\infty} a_n n^{-s}$, and if $q = \exp(2\pi i z)$ then $f(z) = \sum_{n=1}^{\infty} a_n q^n$ is a cusp form for the modular group $\Gamma_0(n)$. This has recently been proved for semistable elliptic curves, that is, those for which N is square-free [Wiles 1995; Taylor and Wiles 1995], and for elliptic curves over $\mathbb{Q}$ with semistable reduction at 3 and 5 [Diamond 1996]. If E is modular and if one defines $\Lambda(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$, then Λ extends to an entire function of s that satisfies the functional equation

$$\Lambda(2-s) = \varepsilon\Lambda(s), \qquad (1\text{–}20)$$

where $\varepsilon = \pm 1$ depends on E. Since $\Gamma(s)$ has a simple pole at $s=0$, it follows that $L(E,0)=0$ and (1–20) gives

$$\Lambda(2) = \frac{N}{(2\pi)^2}L(E,2) = \varepsilon\Lambda(0) = \varepsilon L'(E,0).$$

For example, the curve E that occurs in (1–18) is modular: the cusp form in question is

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} a_n q^n \\ &= q\prod_{n=1}^{\infty}(1-q^n)(1-q^{3n})(1-q^{5n})(1-q^{15n}), \end{aligned} \qquad (1\text{–}21)$$

where $q = \exp(2\pi i z)$ [Ligozat 1975]. Thus the presumed identity (1–18) is a completely explicit formula:

$$\int_0^{2\pi}\!\!\int_0^{2\pi} \log(1+2\cos(s)+2\cos(t))\,ds\,dt \stackrel{?}{=} 15\sum_{n=1}^{\infty}\frac{a_n}{n^2}, \qquad (1\text{–}22)$$

with the a_n given by (1–21). It is a challenge to find a proof of (1–22) that Euler would have understood.

Since the series $L(E,s)$ converges absolutely at the point $s=2$, which occurs in the formulas we consider, it might be thought that modularity does not play an important role here, and that we are really just using $L'(E,0)$ as an abbreviation for $\varepsilon NL(E,2)/(2\pi)^2$. However, the series $\sum_{n=1}^{\infty} a_n n^{-2}$ converges so slowly that it would require on the order of 10^{56} terms to achieve the accuracy we demand here. On the other hand, if E is assumed to be modular then $L(E,2)$ can be computed to high accuracy using the method of [Buhler et al. 1985]. This is implemented in PARI with a variation that allows one to determine the (usually unknown) sign of the functional equation [Cohen 1993, p. 406].

1D. Some Early Experiments

Given (1–17), it was natural to compute $L'(E,0)$ for other curves with small conductor as listed in the tables of [Cremona 1992] and to see if any of these appeared among the small values of $m(P)$ previously computed. Note that $L(E,s)$ depends only on the isogeny class of E so can be specified by indicating the conductor N and the label of the isogeny class as in [Cremona 1992]. For example, for conductor 37 there are two isogeny classes, which Cremona labels A and B. For these we will write $L'(E,0) = b_{37A}$ and $L'(E,0) = b_{37B}$, respectively. If there is only one isogeny class (as for $N = 15$) or if N is beyond the limit of Cremona's tables, we may write b_N. (A Weierstrass equation for the curve will always be specified in these cases).

There are two conductors smaller than 15; they are 14 and 11. We find from PARI that

$$b_{14} = 0.2274812230123511078949823146,$$

which looks suspiciously like the value of m_1 in (1–12). Computing m_1 to 28 decimal places using Maple, we find that

$$m_1 = 0.2274812230123511078949823145,$$

so we feel confident in predicting that

$$m((x+1)y^2+(x^2+x+1)y+(x^2+x)) \stackrel{?}{=} b_{14}. \quad (1–23)$$

Continuing with $N = 11$, we find that

$$b_{11} = 0.1521471417259180494862272969,$$

so

$$\exp(b_{11}) = 1.16433154\ldots < 1.17628081\ldots,$$

where the latter is Lehmer's number (1–3), the smallest known value of Mahler's measure.

The existence of a polynomial with $m(P(x,y)) = b_{11}$ would thus provide an infinite set of counterexamples to the conjecture that Lehmer's number is the smallest Mahler measure. There is no particular reason to believe that there will be any polynomial with $m(P(x,y)) = b_{11}$. We have, however, constructed some examples with $m(P(x,y)) \stackrel{?}{=} rb_{11}$ with rational r. Indeed in all our examples, r is an integer:

$$m(y^2 + (x^2 + 2x - 1)y + x^3) \stackrel{?}{=} 5b_{11}, \quad (1–24)$$

$$m((x+1)y^2 + (x^2 + 4x + 2)y + (x+1)^2) \stackrel{?}{=} 7b_{11}, \quad (1–25)$$

$$m((x-1)^2y^2+(x^3+7x^2+7x+1)y+x(x-1)^2) \stackrel{?}{=} 13b_{11}. \quad (1–26)$$

The order of discovery of these examples was opposite to the order in which they are listed. They are discussed on pages 66, 59, and 70, repectively.

Continuing on in the same vein, we computed a list of 41 values of $L'(E,0)$ using PARI and compared this with a list of 18 small measures we had previously computed. In this way, we recognized:

$$m((x+1)y^2+(x^2-x+1)y+(x^2+x)) \stackrel{?}{=} b_{30}, \quad (1–27)$$

$$m((x+1)y^2+(x^2+1)y+(x^2+x)) \stackrel{?}{=} \tfrac{1}{2}b_{36}, \quad (1–28)$$

$$m((x^2+x+1)y^2+xy+(x^2+x+1)) \stackrel{?}{=} \tfrac{1}{12}b_{105}, \quad (1–29)$$

$$m((x^2+x+1)y^2+(x^2+x)y+(x^3+x^2+x)) \stackrel{?}{=} \tfrac{1}{3}b_{34}. \quad (1–30)$$

It was clear at this point that a more systematic study was needed. Observe that (1–23), (1–27) and (1–28) are all members of the following one parameter family of polynomials:

$$P_k(x,y) = (x+1)y^2 + (x^2 + kx + 1)y + (x^2 + x). \quad (1–31)$$

This suggested a study of $m(P_k)$ for this family and for similar families related to the other polynomials listed above, e.g.

$$P_k(x,y) = y^2 + (x^2 + kx + 1)y + x^2. \qquad (1\text{–}32)$$

Part of the motivation was that, for sufficiently large k, such polynomials do not vanish on the torus $\mathbb{T}^2$, so the theory from [Deninger 1997] works more smoothly. Since, for either (1–31) or (1–32), or any of the other families we study, we have $m(P_k) \sim \log|k|$ as $|k| \to \infty$, it is clear that we are not directly seeking answers to Lehmer's question.

Our goal is to see if $m(P)$ satisfies a formula of type E for all members of such a family. First, one must determine which is the appropriate candidate for E in each case. In Deninger's derivation of (1–17) E is the projective closure of the curve $Z = \{P(x,y) = 0\}$. This is the natural candidate for E in case Z is an elliptic curve, as it is (generically) for the families in (1–31) and (1–32). However, for the example (1–30) Z has genus 2, and for (1–29) Z is of genus 1 but it is not obvious that it is elliptic since it may have no rational point. Given the cohomological nature of Deninger's theory, it is clear that E must be related to $J(Z)$, the Jacobian of Z [Cassels 1991, p. 95; Poonen 1996]. In the case of families of curves of genus 1, $E = J(Z)$. For the curve (1–30), $J(Z) \simeq E \times F$, where E and F are elliptic curves of conductors 34 and 17, respectively. For reasons as yet unknown, $m(P)$ picks out the curve E while ignoring F.

It should be pointed out that, before [Deninger 1997], there was no particular reason to expect geometric aspects of the curve Z to play a role in a formula for $m(P)$. After all, in the formula (1–6), the curve $Z = \{1+x+y = 0\}$ is just the projective line. The quantity $L'(\chi_{-3}, -1)$ appears in (1–6) because of the way Z intersects the torus $\mathbb{T}^2$. This can be seen in Deninger's discussion of Smyth's examples (1–10) and (1–11) [Deninger 1997]. Conversely, for any positive integers m and n,

$$m(y^m - x^n - 2) = m(x + y + 2) = \log 2,$$

but the curves $y^m = x^n + 2$ can have any genus one wishes. At the beginning of Section 2 we discuss some of the properties that seem to be necessary if $m(P(x,y))$ is to be expressible as a formula of type E or type CE. These were discovered by systematic experiments in directions suggested by the work of Deninger [1997] and Rodríguez Villegas [1996].

Deninger's work depends on the theory of higher regulators. The notion of a higher regulator is a generalization of the classical Dirichlet regulator thought of as a homomorphism r of the unit group of an algebraic number field into the product of a suitable number of copies of the reals; for background see, for example, [Beilinson 1980; 1984; Bloch and Grayson 1986; Deninger and Scholl 1991; Deninger and Wingberg 1988; Mestre and Schappacher 1991; Nekovář 1994; Rolshausen 1996].

The determinant $R = \det(r)$ (the classical regulator) appears in the classical Dirichlet class number formula. Bloch defined a regulator r for elliptic curves E as a map from the K-group $K_2(E)$ into a suitable cohomology group and conjectured that its determinant was given by a rational multiple of $L'(E,0)$, a fact that he proved for curves having complex multiplication. Beilinson [1980] then gave a different treatment of this case.

For elliptic curves, Bloch and Beilinson showed how to express values of this regulator in terms of Eisenstein–Kronecker series, that is, series of the form

$$\operatorname{Im}(\tau) \sum_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}} \frac{\chi(\lambda)}{\lambda^2\bar{\lambda}},$$

where the sum is over the lattice $\Lambda = \mathbb{Z}+\tau\mathbb{Z} \subset \mathbb{C}$ of periods of E, where a fundamental parallelogram Λ has area πA and where

$$\chi(\lambda) = \exp((\bar{\xi}\lambda - \xi\bar{\lambda})/A),$$

for some $\xi \in \mathbb{C}$ (that is, a character of the compact group $\mathbb{C}/\Lambda \simeq E$).

Calculations by Bloch and Grayson [1986] have shown the necessity for certain "integrality conditions" in formulating the Bloch–Beilinson conjectures. These conditions arise since the $\mathbb{Q}$ vector space of Eisenstein–Kronecker series attached to torsion points of an elliptic curve may have dimension greater than 1, as the experiments in [Bloch and Grayson 1986] suggest. These conditions were incorporated into [Beilinson 1984] where Beilinson formulated a generalization of his regulator as a map from "motivic cohomology" to Deligne cohomology and again formulated a conjecture about the relationship of $\det(r)$ to special values of L-functions. All of the above is very clearly explained in [Nekovář 1994]. The recent thesis of Rolshausen [1996] contains a useful summary of this theory and some interesting numerical experiments.

The Beilinson conjectures were reformulated in terms of mixed motives by Scholl [1994]. The Deligne periods [1979] of the mixed motive $\mathcal{M}$ are real numbers obtained by integrating certain differential forms over topological cycles. One then can form a matrix of Deligne periods in much the same way as one forms the matrix of Abel–Jacobi periods in the classical theory of complex projective curves [Griffiths and Harris 1978, p. 228]. Under suitable conditions on $\mathcal{M}$, the determinant $c^+(\mathcal{M})$ of the matrix of Deligne periods is predicted to be related to an L-value. One of Deninger's results [1997] is that $m(P)$ is a Deligne period if P does not vanish on the torus. In this case the mixed motive in question sits in the cohomology of the complement of the zero set of P and the cycle in question is the torus $\mathbb{T}^n$ with its usual orientation. So, in some sense $m(P)$ measures the "linking" of the zero set of P with the fixed n-torus $\mathbb{T}^n$.

Notice that $m(P)$ is only one element of a matrix whose determinant is conjectured to be a rational multiple of a special value of the L-function of the motive. Thus one would only expect formulas of the type $m(P) = rL'(E,0)$ if this matrix is one-dimensional. However, the genus 2 examples of Section 3 occur in a case where the matrix is at least 2×2. It is interesting that Beilinson [1984, p. 2057] raises the question of whether the individual entries in this matrix and not only the determinant could be determined by values of L-functions. The examples of Section 3 perhaps have some relationship to this question.

Even assuming that one could reduce the question of whether $m(P) = rL'(E,0)$ to the conjectures of Bloch and Beilinson, these conjectures express no opinion about the value of the rational number, except that $r \neq 0$. Even a *proof* of these conjectures would not reduce the question to one of computation unless the proof were to give an explicit estimate on the size of the denominator of r. However, Bloch and Kato [1990] have formulated a more complete theory about Tamagawa numbers of motives that presumably would predict the exact value of r. Our experience with many thousands of examples suggests that when the conductor of the curve E is sufficiently large $1/r$ is an integer. It would be extremely interesting to see if the computational results we have obtained could be predicted from the Bloch–Kato conjectures.

2. FAMILIES OF CURVES OF GENUS 1

In this section and the next, we consider families of polynomials of the form

$$P(x,y) = P_k(x,y) = A(x)y^2 + B_k(x)y + C(x), \tag{2–1}$$

where $B_k(x)$ depends linearly on the parameter k. We will write $m(P_k) = m_k$. We denote the curve $\{P_k(x,y) = 0\}$ by Z_k. By writing $Y = 2A(x)y + B_k(x)$, we see that Z_k is birationally equivalent to the hyperelliptic equation

$$Y^2 = D_k(x) = B_k(x)^2 - 4A(x)C(x). \tag{2–2}$$

In this section we take $B_k(x) = B(x) + kx$, so

$$P(x,y) = P_k(x,y) = A(x)y^2 + (B(x)+kx)y + C(x), \tag{2–3}$$

where the degrees of $A(x)$, $B(x)$ and $C(x)$ are each at most 2. In particular, $\deg(D_k) \leq 4$, and hence Z_k is generically of genus 1.

The way in which the parameter k enters the polynomial $P_k(x,y)$ is signifigant if one hopes to obtain a formula of the form $m(P_k) = r_k L'(E_k,0)$

for some rational r_k. For example, if one considers the family of polynomials $y^2 - x^3 - k$, then

$$m(y^2 - x^3 - k) = m(y + x + k) = \log|k|, \quad \text{if } |k| \geq 2, \qquad (2\text{–}4)$$

[Smyth 1981b]. Thus $m(y^2 - x^3 - k)$ has no relationship with the L-function of the elliptic curve $E_k : y^2 = x^3 + k$. Notice that, in this case, if we take $|x| = 1$ and consider $y^2 = x^3 + k$ as a quadratic in y, then both roots of the quadratic lie in $|y| > 1$. Hence applying Jensen's formula to evaluate the integral over y one obtains

$$m(y^2 - x^3 - k) = \int_{|x|=1} \log|k| \frac{dx}{2\pi i x} = \log|k|.$$

A similar calculation applies to any $P(x, y)$ having the property that for $|x| = 1$, all roots of $P(x, y) = 0$ lie in $|y| > 1$ (or equally well, if all roots lie in $|y| < 1$). The measures of such polynomials thus have no relationship to the L-function of the curve $\{P(x, y) = 0\}$.

We can generalize this observation by introducing the Newton polytope (or exponent polytope) of $P(x, y)$ as in [Smyth 1981a], where here we do not assume P to be of the form (2–1). The Newton polygon $N(P)$ of P is the convex hull in $\mathbb{R}^2$ of the set of lattice points (i, j) for which the monomial $x^i y^j$ appears as a term in $P(x, y)$. A *face* F of $N(P)$ is the intersection of $N(P)$ with a support line to $N(P)$ and a *face* P_F of P is the sum of the monomials making up P over all lattice points in F. For example, for the $P = P_k$ of (1–31), $N(P)$ is a hexagon. There are six one-dimensional faces, namely x^2+x, $x+y$, y^2+y, y^2+xy^2, xy^2+x^2y and x^2+x^2y, and six zero-dimensional faces, namely x, x^2, x^2y, xy^2, y^2 and y. Notice that each of these faces has $m(P_F) = 0$. On the other hand, for the example considered in (2–4), the Newton polygon is a triangle, and the faces $y^2 - k$, $-x^3 - k$ and $-k$ all have measure $\log|k|$.

As shown in [Smyth 1981a], given a face P_F of P, there is a change of variable of the form $x = u^{m_1}v^{n_1}$, $y = u^{m_2}v^{n_2}$ with $m_1n_2 \neq m_2n_1$, so that $P_F(x, y)$ is of the form $u^a v^b Q_F(u)$, (that is, a polynomial in one variable), and so that if $P(x, y) = Q(u, v)$ is considered as a polynomial in v then the highest degree term has coefficient $Q_F(u)$. Now suppose that we consider polynomials $P(x, y)$ for which k is the coefficient of a single term in P. If this term appears in a face F of P, then by the above change of variable, this term appears in $Q_F(u)$. For sufficiently large $|k|$, Rouché's theorem shows that, for $|u| = 1$, all zeros of $Q(u, v)$ considered as a polynomial in v lie inside the unit circle, and thus $m(Q) = m(Q_F)$ by Jensen's formula, so $m(Q)$ is the logarithm of an algebraic integer. Under such a change of variable, $m(P) = m(Q)$ so the formula for $m(P)$ is of type C. Thus, the only way to obtain a formula of type E for large k for such a family P_k is to have k be the coefficient of a term in the interior of $N(P)$. This partially explains why k occurs where it does in (2–3).

There is a condition also expressible in terms of the Newton polytope that appears to be necessary for a formula of pure type E to hold for any P, namely

(A) *all faces of P must satisfy $m(P_F) = 0$.*

In other words, we conjecture that, for any P, if $m(P) = rL'(E, 0)$ for some rational r and elliptic curve E, then P must satisfy condition (A). Our limited experiments suggest that a necessary but not sufficient condition for a formula of type CE to hold is that the measure of each face of P must be the logarithm of a rational integer. So it appears that the condition (A) is an "arithmetic" condition, perhaps related to the integrality conditions discovered by Bloch and Grayson [1986] in the formulation of Bloch's conjecture. [1]

A second ingredient of our conjecture is an analytic condition (B). We first present a special case

[1] Since this paragraph was first written, the nature of the condition (A) has been clarified, as explained in Section 6. It is best regarded as an *algebraic* condition and is not related to the Bloch–Grayson conditions. Presumably our assumption that the parameter k is an integer is connected with these latter conditions.

where the condition reduces to a more easily understood "geometric" condition (G). Let us ignore the parameter k and simply consider a polynomial $P(x,y) = A(x)y^2 + B(x)y + C(x)$, quadratic in y and let $D(x) = B(x)^2 - 4A(x)C(x)$. Let $y_1(x)$ and $y_2(x)$ be the two roots of $P(x,y) = 0$ that is, the two branches of the algebraic function $y(x)$ defined by $P(x,y) = 0$. If P does not vanish on the torus $\mathbb{T}^2 = \{|x| = 1\} \times \{|y| = 1\}$, then we say that P satisfies the condition (G) if

(G)(i) *exactly one of the roots* $y_1(x)$ *lies strictly outside the circle* $|y| = 1$ *for all* $|x| = 1$, and

(G)(ii) *exactly two of the roots of* $D(x)$ *lie strictly inside* $|x| = 1$.

Notice that (G) obviously holds if P is a *reciprocal* polynomial that does not vanish on the torus. We regard (G) as a description of the way the set $\{P(x,y) = 0\}$, regarded as a surface in $\mathbb{C}^2$, links the torus $\mathbb{T}^2$.

Proposition. *Let* $P(x,y) = A(x)y^2 + B(x)y + C(x)$, *and suppose that the degree of* $D(x) = B(x)^2 - 4A(x)C(x)$ *is* 3 *or* 4. *Suppose that* P *does not vanish on the torus. If* P *does not satisfy condition* (*G*) *then* $m(P)$ *satisfies a formula of type* C.

Proof. Since P does not vanish on the torus neither of the roots $y_1(x)$ or $y_2(x)$ vanish on $|y| = 1$, and since they are continuous functions of x, we see that if $\nu(x)$ denotes the number of j for which $|y_j(x)| < 1$, then $\nu(x) = \nu$ is independent of x. If condition G(i) does not hold then $\nu = 0$ or 2. If $\nu = 2$ then

$$\int_{|y|=1} \log|P(x,y)| \frac{dy}{2\pi i y} = \log|A(x)y_1(x)y_2(x)| = \log|C(x)|,$$

by Jensen's formula, so $m(P(x,y)) = m(C(x))$ is a formula of type C, that is, $m(P)$ is the logarithm of an algebraic number. Similarly, if $\nu = 0$, $m(P(x,y)) = m(A(x))$, a formula of type C.

If $\nu = 1$, assume that $|y_1(x)| > 1 > |y_2(x)|$, and now Jensen's formula gives

$$m(P(x,y)) = m(A(x)) + \int_{|x|=1} \log|y_1(x)| \frac{dx}{2\pi i x}. \tag{2–5}$$

Each $y_j(x)$ has poles at some of the zeros of $A(x)$ and zeros at some of the zeros of $C(x)$. So we can write $A(x) = A_1(x)A_2(x)$ and $C(x) = C_1(x)C_2(x)$ so that $Y_j(x) = A_j(x)y_j(x)/C_j(x)$ has no zeros or poles. Then we may write (2–5) in the form

$$m(P(x,y)) = m(A_2(x)C_1(x)) + \int_{|x|=1} \log|Y_1(x)| \frac{dx}{2\pi i x}. \tag{2–6}$$

Now, assuming $\nu = 1$, consider μ, the number of roots of $D(x)$ in $|x| < 1$. These are the branch points of the $y_j(x)$. We add a branch point at ∞ if $\deg(D) = 3$. First observe that there are no roots of $D(x)$ on $|x| = 1$ since at a branch point we have $y_1(x) = y_2(x)$, contradicting $|y_1(x)| > 1 > |y_2(x)|$. We claim first that μ must be even. Otherwise, if we introduce branch cuts between two pairs of roots then one of these cuts must cross the circle $|x| = 1$. But, by the implicit function theorem, $Y_1(x)$ and $Y_2(x)$ are holomorphic in a neighbourhood of $|x| = 1$ since

$$P_y(x,y) = 2A(x)y + B(x) = Y = \sqrt{D(x)},$$

which is nonzero on $|x| = 1$. Thus a branch cut cannot cross $|x| = 1$ since a circuit of $|x| = 1$ crossing this branch cut would interchange y_1 and y_2, again contradicting $|y_1(x)| > 1 > |y_2(x)|$ for all $|x| = 1$.

Thus $\mu \in \{0, 2, 4\}$. If (G)(ii) does not hold then $\mu = 0$ or 4. If $\mu = 0$, then there are no branch points in $|x| \le 1$, so $Y_1(x)$ is holomorphic in $|x| \le 1$ and hence $\log|Y_1(x)|$ is harmonic there. But then (2–6) shows that $m(P) = \log|Y_1(0)|$, a formula of type C. The case $\mu = 4$ reduces to the case $\mu = 0$ by considering the reciprocal polynomial $P^*(x,y) = x^a y^b P(1/x, 1/y)$.

Thus, if $P(x,y)$ doesn't vanish on the torus and condition (G) does not hold then $m(P)$ is given by a formula of type C. □

For polynomials of the type considered in the Proposition, it thus follows that condition (G) is necessary for formulas of type D or E or formulas of mixed type.

Now we turn to the formulation of our condition (B) for general polynomials. This condition can be gleaned from the proof of the Proposition above. We will formulate this condition under the extra assumption that $m(A) = m(C) = 0$. Note that this is implied by condition (A) since $A(x)y^2$ and $C(x)$ are faces of $P(x,y)$. Since $m(A) = m(C) = 0$ implies that $A(x)$ and $C(x)$ have all their zeros on $|x| = 1$, equation (2–5) shows that if (G) holds then $m(P)$ is expressible as the integral over the circle $|x| = 1$ of a function $y_1(x)$ that is meromorphic in a neighbourhood of $|x| \le 1$ with the exception of a branch cut γ between the two branch points in $|x| < 1$. All poles of $y_1(x)$ lie on $|x| = 1$. That is, $y_1(x)$ is a branch of $y(x)$, the algebraic function defined by $P(x,y) = 0$. This suggests applying Green's formula

$$\int_{\partial\Omega}\Bigl(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Bigr)\,ds = \iint\limits_{\Omega}(u\Delta v - v\Delta u)\,dA,$$

where ds and dA denote elements of arclength and area, to the region Ω with boundary $|x| = 1$ and γ, where $u = \log|y_1(x)|$ and $v = \log|x|$. Since both u and v are harmonic in Ω and $v = 0$ on $|x| = 1$, we see that this expresses the integral in (2–5) as an integral of the form

$$\omega = \frac{1}{2\pi}\bigl(\log|y|\,d\log|x| - \log|x|\,d\log|y|\bigr)$$

along both sides of the branch cut γ in opposite directions (briefly, "around the branch cut γ").

We believe that this integral should be a rational multiple of $L'(E,0)$, or perhaps $L'(\chi,-1)$, provided some arithmetic conditions hold. This seems to be a reasonable guess based on an analogy with similar integrals that appear in [Beilinson 1980; 1984] and in the derivation of (1–17) found in [Deninger 1997]. Thus we can generalize (G) by simply requiring that

(B) *Let $P(x,y)$ be a polynomial and let $y(x)$ be the algebraic function defined by $P(x,y) = 0$ Then (B) holds if $m(P(x,y))$ is expressible as a rational multiple of the integral of a branch of ω around a branch cut between a pair of branch points of $y(x)$.*

For example, if $P(x,y) = A(x)y^2 + B(x)y + C(x)$ is a reciprocal polynomial of the sort considered in the Proposition but allowed to vanish on the torus, then the zeros of $D(x)$ are symmetric with respect to the unit circle and there may be 2 or 4 branch points on $|x| = 1$. We assume $m(A) = m(C) = 0$. By considering the sign of the real number $x^{-2}D(x)$ on $|x| = 1$, we see that if there are two branch points a and $\bar a$ on $|x| = 1$ then, as in the derivation of (1–16), $m(P)$ is expressed as the integral of $\log|y_1(x)|$ over the circle $|x| = 1$ between the branch points a and $\bar a$. If we take a branch cut along the circle between a and $\bar a$ and regard $\int_a^{\bar a}\log|y_1|$ as the integral along the outside of the cut, then along the inside of the cut, the integral is $\int_{\bar a}^{a} -\log|y_1|$—exactly the same, since crossing the cut changes the sign of $x^{-2}D(x)$ and hence changes y_1 to the other root y_2 of $P(x,y) = 0$, which has $|y_2| = 1/|y_1|$. Thus $m(P)$ is half the integral of ω around the branch cut. So (B) holds here. If there are 4 branch points on $|x| = 1$ then (B) would not hold but we have no examples of this type to present.

In some cases, it may be possible to express $m(P)$ as a sum of integrals over different branch cuts. Then it seems possible that one could obtain a formula involving a sum of terms of the form $L'(E,0)$ and $L'(\chi,-1)$. The only example we have that resembles this situation is (3–12), which appears in Section 3B; but in that example, the integral can be expressed as an integral around a single branch cut.

It seems that conditions (A) and (B) are necessary for a formula of type E (or D) to hold for

$m(P)$. Of course, if $P(x,y)=0$ is not of genus 1, then we must assume that E is a factor of the Jacobian of the curve. However these conditions are not sufficient to insure that $m(P)=rL'(E,0)$, even in the case where $P(x,y)=0$ is of genus 1, as we see by the example (4–4). This is a polynomial $P_3(x,y)$ for which $P_3(x,y)=0$ is a curve of genus 1 but $m(P_3)=\frac{1}{6}d_{15}$, a formula of type D. In the examples (4–2), (4–5), (4–7) and (4–8), we have a similar situation. However, we feel there is ample evidence for the following more modest conjecture.

Conjecture. *Let $P(x,y)=A(x)y^2+B(x)y+C(x)$, and suppose that the degree of $D(x)=B(x)^2-4A(x)C(x)$ is 3 or 4. We conjecture that if*

$$P(x,y)=A(x)y^2+B(x)y+C(x)$$

is a polynomial with integer coefficients that satisfies conditions (A) and (B) and for which the equation $P(x,y)=0$ defines an elliptic curve E, then $m(P)$ should be a rational multiple of $L'(E,0)$. If $P(x,y)=0$ is a rational curve then $m(P)$ should satisfy a formula of type D.

Now we consider how this applies to families of polynomials of the form (2–3). As in the discusssion in [Rodríguez Villegas 1996], let K denote the set of k for which $P_k(x,y)$ vanishes on the torus. Thus K is the range of $(A(x)y^2+B(x)y+C(x))/(-xy)$ for $(x,y)\in\mathbb{T}^2$ and thus is a compact subset of $\mathbb{C}$. If $P_k(x,y)$ is a reciprocal polynomial then it is easy to see that K is a subset of the reals. In the complement $G=\mathbb{C}\setminus K$, neither root $y_j(x)$ lies on the circle $|y|=1$ for any $|x|=1$ and by continuity in k, $\nu_k=\nu$ is constant on each connected component of G. In those components for which $\nu=1$, as above, we see that $D_k(x)$ does not vanish on $|x|=1$ and that $\mu_k=\mu$ is independent of k and satisfies $\mu\in\{0,2,4\}$.

We claim that $\nu=1$ and $\mu=2$ on the unbounded component G_∞ of G. For, if $|k|$ is sufficiently large, and $|x|=1$, then the term kxy in $P_k(x,y)$ is dominant and hence by Rouché's theorem, exactly one of the two roots $y_j(x)$ will lie in $|y|<1$, so $\nu=1$. Similarly, the term k^2x^2 is dominant in $D_k(x)$ for sufficiently large $|k|$ and hence $\mu=2$. By continuity, this holds for all $k\in G_\infty$. That is, P_k satisfies condition (G) for all $k\in G_\infty$. There may, of course, be other components of G in which P_k satisfies (G), but we have not run into any such example. We will see an example on page 62 (the family B), where G has a bounded component in which (G)(i) but not (G)(ii) holds.

We thus conjecture that for polynomials P_k of the form (2–3) satisfying condition (A), a formula of type E holds for all integer $k\in G_\infty$ and that a formula of type E or D (if the discriminant vanishes) holds for all integer $k\in\partial G_\infty$. In particular, for reciprocal polynomials since $\bar{G}_\infty=\mathbb{C}$, we are conjecturing that a formula of type E or D holds for all integer k. This formulation of the conjecture in terms of the set K has been strongly influenced by conversations with Rodríguez Villegas about his method.

An additional conjecture, based simply on empirical evidence, is that $1/r_k=L'(E_k,0)/m_k$ is an integer for all sufficiently large $|k|$.

In the early part of this study, our computations concentrated on families of reciprocal polynomials. This was natural, given the motivating examples (1–27)–(1–32). In addition, a few experiments with nonreciprocal polynomials $P(x,y)$ for which $P(x,y)=0$ is elliptic had failed to produce formulas of the type E (or CE or CDE). Rodríguez Villegas explained the likely reason for this to me in terms of the set K that he introduced in [Rodríguez Villegas 1996].

In the cases that his method is able to handle, he shows that $m(P_k)$ is given by a modular form for $k\in G_\infty$ and then extends this by continuity to $\bar{G}_\infty$. Since for reciprocal polynomials, $K\subset\mathbb{R}$ so $\bar{G}=\bar{G}_\infty=\mathbb{C}$, and hence we might expect such formulas to hold for *all* k, provided some arithmetic condition on k holds, e.g. that k be an integer or perhaps that k^2 be an integer.

On the other hand, for nonreciprocal polynomials, K has nonempty interior and there is no reason to expect formulas of type E to hold for $k\in\operatorname{int}(K)$, even if k is an integer. And, as discussed above, it is only in the unbounded component G_∞ that we can have any reason to expect a formula of type E. By continuity, this should extend to ∂G_∞, the outer boundary of K.

In the genus 2 examples presented in Section 3, it seems that conditions (A) and (B) are both necessary. Of course, one also needs the condition that the Jacobian of the curve split into the product of two elliptic curves. We expect that the fact that $m(P)$ picks out just one of these curves can be explained by a change of variable in the integral between branch points that (B) requires but we do not yet have a convincing demonstration of this.

It seems that if (A) is not satisfied but that the measure of each face of P is the logarithm of a rational integer, then $m(P)$ may satisfy a formula of type CE but only if the coefficients of P in the interior of the Newton polygon are divisible by a certain integer related to the measures of the faces. We discuss some examples of this starting on page 61, but we do not have enough examples to make an educated guess about the correct formulation of the divisibilty condition.

We now turn to the discussion of some of the examples that led to the formulation of these conjectures.

2A. Families of Reciprocal Polynomials, Genus 1

We begin the discussion with some families of reciprocal polynomials suggested by the examples we found in our initial exploratory experiments.

We begin with the family (1–32), $P_k(x,y) = y^2 + (x^2+kx+1)y + x^2$. We have $m_k = m(P_k) = m_{-k}$, since $P_k(-x,y) = P_{-k}(x,y)$, so it is only necessary to study $k \geq 0$. The curve $P_k = 0$ has discriminant $k^2(k-4)(k+4)$ so Z_k is elliptic for $k \neq 0, -4, 4$. A Weierstrass form is

$$E_k : y^2 = x^3 + (k^2-8)x^2 + 16x. \qquad (2\text{–}7)$$

Computing m_k and $L'(E_k, 0)$ for $k \leq 100$ to 28 decimal places we find that, for $k \neq 0, 4$, we have $m_k \stackrel{?}{=} r_k L'(E_k, 0)$ for a rational r_k that is either an integer or the reciprocal of an integer. The results of this computation for $1 \leq k \leq 40$ appear in Table 1. (The coefficients in the last five columns refer to the *reduced* minimal model, which satisfies $a_1, a_3 \in \{0,1\}$ and $a_2 \in \{-1,0,1\}$, and hence uniquely identifies E_k up to isomorphism over $\mathbb{Q}$ [Cremona 1992, p. 46]. The name "Family 1.3" is explained on page 56.)

From the table we see that $m_1 \stackrel{?}{=} b_{15}$, $m_5 \stackrel{?}{=} 6b_{15}$ and $m_{16} \stackrel{?}{=} 11b_{15}$, which suggests some nonobvious identities such as

$$M\Big(x + \frac{1}{x} + y + \frac{1}{y} + 5\Big) \stackrel{?}{=} M\Big(x + \frac{1}{x} + y + \frac{1}{y} + 1\Big)^6.$$

Note that there are 8 nonisomorphic curves over $\mathbb{Q}$ with conductor 15 [Cremona 1992], all in a single isogeny class, so the curves E_1, E_5, E_{16} are isogenous, but they are not isomorphic as can be seen from the last column of Table 1.

The largest conductor encountered in this range is $N = 911121 = 3 \cdot 31 \cdot 97 \cdot 101$ for $k = 97$. We compute

$$m_{97} = 4.574498314321773339468384070$$

and

$$b_{911121} = -14492.01065977137793943584071,$$

so

$$b_{911121}/m_{97} = -3167.999999999999999999999995 \stackrel{?}{=} -3168 = -2^5 \cdot 3^2 \cdot 11.$$

For $k = 0$, $P_0(x,y) = (y + x^2)(y+1)$, so $m_0 = 0$. For $k = 4$, the curve $P_4 = 0$ has genus 0, and the nonvanishing factor of the discriminant $k^2(k-4) \times (k+4)$ is $4^2 \times 8$ so we suspect that m_4 might be given by a Dirichlet L-function with conductor a power of 2. Indeed, numerically

$$m_4 = 2d_4 = \frac{4G}{\pi}, \qquad (2\text{–}8)$$

k	s	N	a_1	a_2	a_3	a_4	a_6
1	1	15	1	1	1	0	0
2	1	24	0	−1	0	1	0
3	1/2	21	1	0	0	1	0
4			($m = 2d_4,\ g = 0$)				
5	1/6	15	1	1	1	−5	2
6	2	120	0	1	0	−15	18
7	2	231	1	1	1	−34	62
8	1/4	24	0	−1	0	−64	220
9	2	195	1	0	0	−110	435
10	−8	840	0	−1	0	−175	952
11	−8	1155	1	1	1	−265	1550
12	1/2	48	0	1	0	−384	2772
13	−4	663	1	1	1	−539	4592
14	8	840	0	−1	0	−735	7920
15	−24	3135	1	0	0	−980	11727
16	1/11	15	1	1	1	−80	242
17	−24	4641	1	1	1	−1644	24972
18	−16	1848	0	1	0	−2079	35802
19	−40	6555	1	1	1	−2595	49800
20	2	240	0	−1	0	−3200	70752
21	−12	1785	1	0	0	−3905	93600
22	24	3432	0	−1	0	−4719	126360
23	6	1311	1	1	1	−5654	161282
24	8	840	0	1	0	−6720	209808
25	−16	3045	1	1	1	−7930	268502
26	128	17160	0	−1	0	−9295	348040
27	12	2139	1	0	0	−10829	432840
28	−2	336	0	−1	0	−12544	544960
29	−24	4785	1	1	1	−14455	662900
30	240	26520	0	1	0	−16575	815850
31	−16	3255	1	1	1	−18920	993800
32	1/3	42	1	1	1	−1344	18405
33	−204	35409	1	0	0	−24344	1459935
34	256	38760	0	−1	0	−27455	1760160
35	224	42315	1	1	1	−30855	2073252
36	2	240	0	1	0	−34560	2461428
37	−208	50061	1	1	1	−38589	2901642
38	288	54264	0	−1	0	−42959	3441480
39	336	58695	1	0	0	−47690	4004595
40	−8	1320	0	−1	0	−52800	4687452

TABLE 1. Data for curves E_k with equation (2–7) (Family 1.3). The second column gives $s = 1/r_k = L'(E_k, 0)/m(P_k)$ with P_k as in (1–32), the third gives the conductor N, and the remaining columns show the coefficients of the reduced minimal model, $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, of E_k.

where G is Catalan's constant. In this case, we can prove the identity. First notice that $m(P_k) = m(Q_k)$, where

$$\begin{aligned} Q_k(x,y) &= P_k(xy, y^2)/(xy) \\ &= (x^2+1)(y^2+1) + kxy. \end{aligned} \tag{2–9}$$

Making the change of variable $x \to \pm ix$, $y \to \pm iy$ in (1–7), we have

$$m(1 \pm ix \pm iy + xy) = d_4,$$

and

$$\begin{aligned} &(1+ix+iy+xy)(1-ix-iy+xy) \\ &\qquad = 1 + x^2 + y^2 + 4xy + x^2y^2 = Q_4(x,y). \end{aligned}$$

Thus $m(P_4) = m(Q_4) = 2m(1+ix+iy+xy) = 2d_4$.

Using a result of Lind, Schmidt and Ward [Lind et al. 1990], we have thus shown that the entropy of the discrete Laplacian on $\mathbb{T}^{\mathbb{Z}^2}$ is $2d_4 = 4G/\pi$. The quantity G/π plays a role in many problems involving the integer lattice $\mathbb{Z}^2$. For example $2G/\pi$ is the entropy of the "dimer packing problem" [Fisher 1961; Kasteleyn 1961], and the metric entropy of Asmus Schmidt's Gaussian integer continued fractions [Schmidt 1993; Nakada 1990]. The quantity also appears as the best constant in a sharp version of the Gel'fond–Mahler inequality (via (1–7)) [Boyd 1992]. The quantity $m(P_4)$ appears in a study of Sarnak [1982] of quasi-periodic potentials for the Schrödinger equation, and for a similar reason in [Thouless 1990].

For this family, P_k does not vanish on the torus for $k \notin [-4, 4]$. After seeing a table of the results mentioned above, Fernando Rodríguez Villegas [1996] was inspired to show that, for *all* complex k, not necessarily an integer, $m(P_k)$ is equal to an Eisenstein–Kronecker series for the appropriate curve. Assuming that the Bloch–Grayson conditions hold for P_k this reduces the numerically deduced formulas for $m(P_k)$ to an application of the Bloch–Beilinson conjectures. The basis for his method is his observation that the family E_k can be identified with the modular elliptic surface associated to the group $\Gamma_0(4)$. He shows in fact that if τ is a point in the upper half plane that parameterizes the curve E_k as an elliptic curve over $\mathbb{C}$, then

$$m(P_k) = \mathrm{Re}\biggl(-\pi i\tau + 2\sum_{n=1}^{\infty}\sum_{d|n}\Bigl(\frac{-4}{d}\Bigr)d^2\frac{q^n}{n}\biggr),$$

where $q = \exp(2\pi i\tau)$. It can then be shown that the Fourier series of this modular form is an Eisenstein–Kronecker series. If the integrality conditions can be shown to hold, then $m(P_k)$ is conjecturally a rational multiple of $L'(E_k, 0)$ provided $k^2 \in \mathbb{Z}$. The set K in this case is the real interval $[-4, 4]$ so $\bar{G}_\infty = \mathbb{C}$. For some values of k, such as $k = 2\sqrt{2}$, the curve E_k has complex multiplication, and in such cases Rodríguez Villegas is able to give proofs of the numerically determined formulas, for example $m(P_{2\sqrt{2}}) = b_{32}$ and $m(P_{4i}) = 2b_{32}$, using the fact that all elliptic curves of conductor 32 have complex multiplication. The fact that the Eisenstein–Kronecker series is an explicit multiple of $L'(E_k, 0)$ in these cases is due to Deuring; see [Deninger and Wingberg 1988].

Neither of the polynomials just mentioned have integer coefficients but can be used to construct examples of polynomials with integer coefficients having measures provably equal to b_{32} and $2b_{32}$. Observe that, with Q_k as in (2–9),

$$m(P_kP_{-k}) = m(Q_kQ_{-k}) = m(S_{k^2}(x^2, y^2)) = m(S_{k^2}),$$

where

$$S_n(x, y) = (x + 1)^2(y + 1)^2 + nxy. \qquad (2\text{–}10)$$

So

$$m(P_k) = \tfrac{1}{2}m(S_{-k^2}). \qquad (2\text{–}11)$$

Also, if

$$R_k(x, y) = x - 1/x + y - 1/y + k,$$

then

$$-iR_k(ix, iy) = x + 1/x + y + 1/y - ik = P_{-ik}(x, y),$$

so

$$m(R_k) = \tfrac{1}{2}m(S_{k^2}).$$

The curve $S_n = 0$ has discriminant $n^7(n + 16)$ and hence has genus 1 unless $n = 0$ or -16. A Weierstrass equation is

$$F_n : y^2 = x^3 + n(n + 8)x^2 + 16n^2x.$$

The relationship between this F_n and the E_k of (2–7) is that F_{-k^2} is a quadratic twist of E_k by $-k$. For $|n| \le 100$, one has $m(S_n) \stackrel{?}{=} r_nL'(F_n, 0)$. In particular, by (2–11), $m(R_k) = m(P_{ik})$ is numerically a rational multiple of the appropriate $L'(E, 0)$. Using the results of Rodríguez Villegas for $m(P_k)$, we thus have

$$m(x - 1/x + y - 1/y + 4) = m(P_{4i}) = 2b_{32}. \qquad (2\text{–}12)$$

and

$$m(S_{-8}) = 2m(P_{2\sqrt{2}}) = 2b_{32},$$

both formulas being rigorously true by Rodríguez Villegas' result. Deninger has (privately) reported proving (2–12) as well, using the method of [Deninger 1997]. Notice that R_4 does not vanish on the torus but S_{-8} does vanish there.

Similar computational results hold for the family (1–31) for $|k| \le 100$. Here the discriminant is $(k - 2)^3(k - 3)^2(k + 6)$ and a Weierstrass equation is

$$y^2 = x^3 + (k^2 - 12)x^2 - 16(k - 3)x. \qquad (2\text{–}13)$$

We have already mentioned $m_{-1} \stackrel{?}{=} b_{30}$, $m_0 \stackrel{?}{=} \frac{1}{2}b_{36}$ and $m_1 \stackrel{?}{=} b_{14}$. Some other numerically verified values are $m_{-2} \stackrel{?}{=} \frac{1}{2}b_{20}$, $m_{-5} \stackrel{?}{=} \frac{1}{6}b_{14}$, $m_4 \stackrel{?}{=} \frac{1}{3}b_{20}$ and $m_6 \stackrel{?}{=} \frac{1}{2}b_{36}$. Table 2 contains some data on this family for $|k| \le 40$ in the same format as for Table 1.

Rodríguez Villegas' method applies to this family as well. Here the appropriate modular group is $\Gamma_0(6)$. In particular, since elliptic curves of conductor 36 have complex multiplication, the formulas for m_0 and m_6 have thus been rigorously proved.

For the degenerate cases $2, 3$ and -6, Z is a rational curve. Since $P_2 = (x + y)(y + 1)(x + 1)$, we have $m_2 = 0$ and since $P_3 = (1 + x + y)(x + y + xy)$, we have $m_3 = 2d_3$ by (1–6). In case $k = -6$, we verified numerically that $m_{-6} \stackrel{?}{=} 5d_3$, and this can

k	s	N	a_1	a_2	a_3	a_4	a_6	k	s	N	a_1	a_2	a_3	a_4	a_6
0	2	36	0	0	0	0	1								
1	1	14	1	0	1	−1	0	−1	1	30	1	0	1	1	2
2			$(m=0,\ g=0)$					−2	1/2	20	0	1	0	4	4
3			$(m=2d_3,\ g=0)$					−3	2	90	1	−1	0	6	0
4	1/3	20	0	1	0	−1	0	−4	2	84	0	1	0	7	0
5	1	66	1	0	1	−6	4	−5	1/6	14	1	0	1	4	−6
6	1/2	36	0	0	0	−15	22	−6			$(m=5d_3,\ g=0)$				
7	−1	130	1	0	1	−33	68	−7	1/3	30	1	0	1	−19	26
8	6	420	0	1	0	−61	164	−8	−2	220	0	1	0	−45	100
9	6	630	1	−1	0	−105	441	−9	2	198	1	−1	0	−87	333
10	1/10	14	1	0	1	−11	12	−10	2	156	0	1	0	−148	644
11	1	102	1	0	1	−256	1550	−11	−6	910	1	0	1	−234	1352
12	2	180	0	0	0	−372	2761	−12	−12	1260	0	0	0	−348	2497
13	−12	2090	1	0	1	−524	4566	−13	2	210	1	0	1	−498	4228
14	−6	660	0	1	0	−716	7140	−14	1/3	34	1	0	0	−43	105
15	−12	1638	1	−1	0	−957	11637	−15	−2	306	1	−1	0	−927	11097
16	30	4004	0	1	0	−1253	16660	−16	12	1140	0	1	0	−1221	16020
17	42	4830	1	0	1	−1613	24788	−17	12	2090	1	0	1	−1579	24006
18	1	90	1	−1	1	−128	587	−18	12	1260	0	0	0	−2007	34606
19	1	170	1	0	1	−2554	49452	−19	48	6006	1	0	1	−2516	48350
20	24	2652	0	1	0	−3153	67104	−20	48	7084	0	1	0	−3113	65824
21	−2	342	1	−1	0	−3852	92988	−21	12	2070	1	−1	0	−3810	91476
22	−18	2660	0	1	0	−4660	120900	−22	1/5	30	1	0	1	−289	1862
23	−42	6090	1	0	1	−5589	160336	−23	−12	2210	1	0	1	−5543	158358
24	108	13860	0	0	0	−6648	208633	−24	4	468	0	0	0	−6600	206377
25	−84	15686	1	0	1	−7851	267074	−25	6	798	1	0	1	−7801	264524
26	1	138	1	0	1	−576	5266	−26	−24	4060	0	1	0	−9156	334180
27	6	990	1	−1	0	−10734	430740	−27	−108	18270	1	−1	0	−10680	427500
28	−24	4420	0	1	0	−12441	529984	−28	−168	20460	0	1	0	−12385	526400
29	18	2730	1	0	1	−14344	660002	−29	6	1426	1	0	1	−14286	656000
30	2	252	0	0	0	−16455	812446	−30	2	198	1	−1	1	−1025	12881
31	66	15022	1	0	1	−18791	989850	−31	−36	5610	1	0	1	−18729	984952
32	−288	33060	0	1	0	−21365	1194900	−32	−180	30940	0	1	0	−21301	1189524
33	−216	36270	1	−1	0	−24195	1454625	−33	4	630	1	−1	0	−24129	1448685
34	−2	310	1	0	0	−1706	26980	−34	−24	3108	0	1	0	−27228	1720260
35	−16	2706	1	0	1	−30686	2066384	−35	−174	40774	1	0	1	−30616	2059314
36	324	47124	0	0	0	−34380	2453617	−36	288	44460	0	0	0	−34308	2445913
37	246	51170	1	0	1	−38398	2892828	−37	72	12090	1	0	1	−38324	2884466
38	36	4620	0	1	0	−42756	3388644	−38	2	410	1	0	1	−2668	52806
39	−18	3330	1	−1	0	−47475	3993381	−39	−324	56826	1	−1	0	−47397	3983553
40	342	64676	0	1	0	−52573	4622244	−40	−480	61404	0	1	0	−52493	4611684

TABLE 2. Data for the family 2.3, defined by (1–31) and (2–13).

now be proved due to the result of Rodríguez Villegas for this family.

As a test for the necessity of condition (A), we next consider two families of reciprocal polynomials of the form (2–3) for which the measures of some faces are nonzero. For the first example, take $A(x) = x^2 + x - 1$, $C(x) = A^*(x) = -x^2 + x + 1$ and $B_k(x) = kx$, so the measure of the face $A(x)$ and its opposite $C(x)$ are each $\log(\varphi)$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The discriminant here is $(k^2-4)^2(k^2-20)^2$ so the curve $P_k(x,y) = 0$ has genus 1 provided $(k^2-4)(k^2-20) \neq 0$. Its Jacobian has the equation

$$E : y^2 = x(x - (k^2-4))(x - (k^2-20)).$$

It is easy to prove that $m_k = \log(\varphi)$ for $k = 0, 1$ and 2. So the formulas for m_0 and m_1 are of type C and not of type E. For $k = 2$, we might have expected a formula of type D, not of type C. For $3 \le k \le 20$, using 50 decimal place values of m_k, we were not able to represent m_k as rational linear combinations of $L'(E,0)$, $\log(\varphi)$ and other plausible terms. We could look at this example in another way by interchanging x and y so that now $A(x) = x^2 - 1$, $B_k(x) = x^2 + kx + 1$ and $C(x) = 1 - x^2$. Notice that now $m(A) = m(B) = 0$ so the condition (A) that each face have measure 0 cannot be simplified to the assumption that the coefficient of y^2 in (2–3) have $m(A(x)) = 0$.

Similarly, if we take $A(x) = x^2 + x - 1$ and $C(x) = -x^2 + x + 1$ as in the previous paragraph, but $B_k(x) = x^2 + kx + 1$, the discriminant is $k^2 \times (k^2-16)(k^2-25)^2$ so $P_k = 0$ has genus 1 provided $k(k^2-16)(k^2-25) \neq 0$. Here the Jacobian has the equation

$$E : y^2 = x^3 + (k^2-40)x^2 - 16(k^2-25)x.$$

Again, for the nondegenerate cases, it seems that m_k is not related to $L'(E,0)$, at least for $k \le 20$. For the degenerate case $k = 0$, one can prove easily that $m_0 = \log(\varphi)$, and for the degenerate cases $k = 4$ and 5, to 50 decimal place accuracy, one has the following equations $m_4 \stackrel{?}{=} 2\log(\varphi)$ and

$$m_5 \stackrel{?}{=} \tfrac{2}{3}\log(\varphi) + \tfrac{1}{6}L'(\chi_{-15}, -1). \tag{2–14}$$

The latter equation can be reduced to a dilogarithm identity since $P_5(x,y)$ factors into linear factors over $\mathbb{Q}(\sqrt{5})$ but has not yet been proved. Neither has the apparently more elementary formula for m_4.

Extensive computations have been done for the families of reciprocal polynomials of the form (2–3) satisfying condition (A). So $A(x)$ is a cyclotomic polynomial of degree at most 2. By making use of the symmetry $(x,y) \to (y,x)$ and changes of sign, we can take $A(x)$ to be one of 1, $x+1$, x^2+x+1, or $(x+1)^2$ and $C(x) = x^2A(1/x)$, while $B(x)$ can be chosen to be 0, x^2+1 or (if the degree of $A(x)$ is 2), $2(x^2+1)$. We denote the various families by a.b, where a is 1, 2, 3, 3s, respectively, for the four choices listed for A above, and b is 1, 3, 3s, respectively, for the three choices listed for B. Some of these families can be eliminated from consideration by symmetry, and the family 1.1 is not of genus 1, so there are in fact 7 families of this type that have been considered: 1.3, 2.3, 3.1, 3.3, 3s.1, 3s.3 and 3s.3s. For example, the families (1–31), (1–32) and (2–10) considered above have the names 2.3, 1.3 and 3s.3s, respectively. (The family 1.3 is equivalent to one in which $A(x) = x^2 + 1$, by (2–9).) For families with b = 1 and in a few other cases, one can see by a change of variable that $m_k = m_{-k}$, so only $k \ge 0$ need be considered. See also the section on electronic availability on page 79.

In all cases, we considered at least all integer $|k| \le 40$ for which the conductor $N \le 40{,}000{,}000$. (This is the practical limit on N for the computation of $L(E,2)$ by PARI on a machine with 48 Mbytes of RAM). In a few cases, we extended the computation to $|k| \le 100$. For sufficiently large $|k|$, P does not vanish on the torus and then m_k was computed from (2–6) by PARI's numerical integration routine, Romberg quadrature. If P vanishes on the torus, then m_k was computed using

Maple V, which uses the Curtis–Clenshaw method of integration with some preliminary singularity handling. In most cases, Maple's singularity handling was not sufficient to treat the square-root singularities that occur at the endpoints of intervals where $|y(x)| = 1$ so a preliminary change of variable was made as explained above in connection with (1–16). If $A(x)$ vanishes on the circle $|x| = 1$, then of course $y(x)$ has poles at the zeros of $A(x)$ so in this case one integrates $\log|A(x)y(x)|$. Zeros on $|x| = 1$ cause no difficulty in integrating $\log|A(x)|$ since a zero at $x = x_0$ produces a term of the form $\log|e^{it} - e^{it_0}|$ for which the integral is a Clausen integral, easily handled by integration by parts.

To use the elliptic curve routines of PARI, it was first necessary to compute a Weierstrass form for E_k for each family. Starting with (2–2), we simply need to know that the Jacobian $E = J(Z)$ of the curve

$$Z: \quad y^2 = f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

is given by

$$y^2 = g(x) = x^3 + cx^2 + (bd - 4ae)x - (4ace - b^2e - ad^2).$$

If a is a square of an integer then this can be proved in an elementary way using the techniques in [Cassels 1991, Chapter 8] and one obtains a birational map from Z to E with coefficients in $\mathbb{Q}$. If a is not a square then one can obtain a birational map with coefficients in $\mathbb{Q}(\sqrt{a})$ by twisting by a, finding the Weierstrass form for the twisted equation and then twisting again by a.

A more elegant way of doing this was pointed out to me by John Cremona. According to the theory of invariants, the classical invariants of $f(x)$ are

$$I = 12ae - 3bd + c^2,$$
$$J = 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3.$$

Then $E = J(Z)$ has an equation

$$E: \quad y^2 = G(x) = x^3 - 27Ix - 27J.$$

A rational map of degree 4 from Z to E is given by a syzygy between the covariants of $f(x)$. Joe Silverman has kindly supplied the reference [Salmon 1876, pp. 187–192]. A more modern reference is [Hilbert 1993, Lecture XXII, p. 71]. It is easily shown that the two equations given for E are equivalent. A slight advantage to the first is that g and f have the same discriminant, whereas $\operatorname{discrim}(G) = 3^{12}\operatorname{discrim}(f)$.

In the Weierstrass equations presented here, if E has a rational 2-torsion point then we make this evident by choosing coordinates so that one such point is $(0, 0)$. For example, if $f(x)$ is a reciprocal polynomial, as will be the case if $P(x, y)$ is reciprocal, so $f(x) = ax^4 + bx^3 + cx^2 + bx + a$, then the above formulas give

$$\begin{aligned} g(x) &= x^3 + cx^2 + (b^2 - 4a^2)x + (2ab^2 - 4a^2c) \\ &= (x + 2a)(x^2 + (c - 2a)x + (b^2 - 2ac)), \end{aligned}$$

so E has the rational 2-torsion point $(-2a, 0)$. By shifting this to $(0, 0)$ we obtain the equation

$$E: \quad y^2 = x^3 + (c - 6a)x^2 + (8a^2 + b^2 - 4ac)x.$$

PARI's routine for computing $L(E, 2)$ requires the input of the sign ε of the functional equation. Although a method is suggested in [Batut, Bernardi, Cohen et Olivier 1995] for the determination of this sign, in our case it was simpler to compute two values for $b_E = \varepsilon NL(E, 2)/(2\pi)^2$, say b^+ and b^-, assuming the sign is $+1$ or -1 respectively. The correct sign can be recognized by observing which of b^+/m_k or b^-/m_k "is" rational. A typical entry in one of the output files (here 3.1.pos) is shown at the top of the next page.

Table 3 contains the results for the family 3.1 for $1 \le k \le 34$ (the conductor for $k = 35$ is 50811915). The discriminant here is

$$k^4(k - 2)(k + 2)(k - 6)(k + 6)$$

and a Weierstrass form is

$$y^2 = x^3 + (k^2 + 12)x^2 + 16k^2x. \qquad (2\text{–}15)$$

```
5    1155   [1, 0, 1, -4, -19]
 1.411759555382163906905864291
36.96555140788977384409893236          -33.88222932917193376574074299
 0.03819122132940356484948694122        -0.04166666666666666666666666667
26.18402777368358841500906465          -23.99999999999999999999999999
```

An entry from the file 3.1.pos, describing curve 1155H1 of [Cremona 1992]. The first line gives k, N and the coefficients $[a_1, a_2, a_3, a_4, a_6]$ of the reduced minimal model of the curve E_k. Next comes the numerical value of m_k, then b^+ and b^-. The last two lines give m_k/b^+, m_k/b^-, b^+/m_k and b^-/m_k. In this example we can confidently conjecture that $\varepsilon = -1$ and $b^-/m_k \stackrel{?}{=} -24$. The rank is $r = 1$, which is consistent with the parity conjecture $\varepsilon = (-1)^r$.

In all cases, one of b^+/m_k or b^-/m_k is an "obvious" rational, usually an integer, for those k for which the discriminant does not vanish. For the degenerate cases where the discriminant vanishes, one finds numerically that for a suitable odd Dirichlet character of conductor f that m_k/d_f is rational. The choice of f was found heuristically from the nonvanishing factor of the discriminant. For example, for the family 3.1, the discriminant vanishes for positive k if $k = 2$ or 6. For $k = 2$, the nonvanishing part of the discriminant is -2^{11} so we expect that f will be an odd power of 2 and indeed we find that $d_8/m_2 \stackrel{?}{=} 3$. For $k = 6$, the nonvanishing part of the discriminant is $6^4 \cdot 4^2 \cdot 24$ so our first guess is that $f = 24$ and indeed $m_6 \stackrel{?}{=} d_{24}/6$. We will mention more of these degenerate cases in Section 4.

In addition to the above, there is some data for the two families 3g.1 and 3g.3 with $A(x) = x^2 + x - 1$. Details on how to obtain them the relevant files will be found in the section on electronic availability on page 79. This will allow those interested to test their own conjectured formulas for m_k for these examples.

k	s	N	a_1	a_2	a_3	a_4	a_6	k	s	N	a_1	a_2	a_3	a_4	a_6
1	12	105	1	0	1	−3	1	18	−24	2880	0	0	0	−2028	34832
2		$(3m_2 = d_8,\ g = 0)$						19	5184	440895	1	0	1	−2538	48631
3	4/3	45	1	−1	0	0	−5	20	−2208	240240	0	1	0	−3136	66164
4	6	240	0	1	0	0	−12	21	1344	137655	1	−1	0	−3834	91903
5	−24	1155	1	0	1	−4	−19	22	−672	73920	0	1	0	−4641	119679
6		$(6m_6 = d_{24},\ g = 0)$						23	11520	1190595	1	0	1	−5569	158951
7	24	1365	1	0	1	−29	11	24	−864	102960	0	0	0	−6627	207074
8	24	1680	0	1	0	−56	84	25	−1920	203205	1	0	1	−7829	265331
9	48	3465	1	−1	0	−99	328	26	816	87360	0	1	0	−9185	335103
10	12	960	0	1	0	−161	639	27	864	100485	1	−1	0	−10710	428575
11	528	36465	1	0	1	−248	1361	28	32640	4084080	0	1	0	−12416	527604
12	−48	5040	0	0	0	−363	2522	29	22944	2171085	1	0	1	−14318	657371
13	−3648	285285	1	0	1	−514	4271	30	−144	20160	0	0	0	−16428	809552
14	−72	6720	0	1	0	−705	6783	31	55104	5488395	1	0	1	−18763	986681
15	768	69615	1	−1	0	−945	11200	32	8064	1007760	0	1	0	−21336	1191444
16	−192	18480	0	1	0	−1240	16148	33	−10944	1396395	1	−1	0	−24165	1450840
17	13536	1225785	1	0	1	−1599	24181	34	−960	114240	0	1	0	−27265	1722623

TABLE 3. Data for the family 3.1, defined by (2–15).

2B. Families of Nonreciprocal Polynomials, Genus 1

Now we turn to the discussion of two classes of families of nonreciprocal polynomials. Recall the discussion at the beginning of Section 2, which predicts that we should expect a formula of type E or D only if P_k satisfies conditions (A) and (G). Thus we do not expect such a formula if $k \in \operatorname{int}(K)$ but do expect such a formula if $k \in \bar{G}_\infty$.

Families coming from modular elliptic surfaces. The first class of nonreciprocal examples was suggested to me by Rodríguez Villegas as a natural generalization of the families 2.3 and 1.3 of (1–31) and (1–32). He pointed out that these two families occur in [Beauville 1982] as two of six special families of elliptic curves distinguished by possessing four singular fibres. Each family is associated with a modular group. It seems that Rodríguez Villegas' methods can treat all of these examples.

Our families 2.3 and 1.3 correspond respectively to the groups $\Gamma_0^0(6)$ and $\Gamma_0(8) \cap \Gamma_0^0(4)$. The family associated with $\Gamma_0^0(5)$ is given by [Beauville 1982]

$$X(X-Z)(Y-Z) + tYZ(X-Y) = 0. \quad (2–16)$$

Writing $X - Y = x$, $Y = y$, $Z = -1$ and $t = -k$ gives us

$$P_k(x,y) = (x+y+1)(x+1)(y+1) + kxy, \quad (2–17)$$

or

$$P_k(x,y) = (x+1)y^2 + (x^2+(k+3)x+2)y + (x+1)^2, \quad (2–18)$$

which is clearly of type (2–3). The discriminant is $k^5(k^2+11k-1)$, and a Weierstrass form is

$$Y^2 = X^3 + (k^2-6k+1)X^2 + (-8k^3+8k^2)X + 16k^4. \quad (2–19)$$

Each of the curves $P_k = 0$ has a 5-torsion point.

Plotting 10,000 points of the set K corresponding to taking x and y to be 100th roots of unity, it appears that K is a simply connected egg-shaped set with the narrow end of the egg at $k = \frac{1}{4}$ and the top of the egg at $k = -12$. (It is not difficult to prove that the intersection of K with the real axis is $[-12, \frac{1}{4}]$). Thus our conjecture would predict a formula of type E for all integers except those in $[-11, 0]$. Experiments for $|k| \le 50$ verify this expectation, as we see in Table 4.

The example (1–25) with $m(P) \stackrel{?}{=} 7b_{11}$ is the case $k = 1$ of (2–16). Here the curve $P_1 = 0$ is isomorphic to $y^2 + y = x^3 - x$, which is the modular curve $X_1(11)$. The conductor 11 also occurs for $k = -1$, where the curve is also isomorphic to $X_1(11)$ and for $k = -11$ where the curve is $X_0(11)$ (with minimal model $y^2 + y = x^3 - x^2 - 10x - 20$). However, in neither of these cases does $m(P_k)$ seem to be a rational multiple of b_{11}, consistent with the conjecture that this does not occur if $k \in \operatorname{int}(K)$. These curves have a question mark in the column for s in Table 4, indicating that no rational relation was found.

The family $\Gamma_0^0(4) \cap \Gamma(2)$ has projective equation

$$X(X^2 + 2XZ + Z^2) + tZ(X^2 - Y^2) = 0.$$

These curves all have a torsion group of order 8. A better choice of coordinates for our purposes is obtained by taking $X = x + y$, $Y = x - y$, $Z = 1$, and $t = k/4$ so

$$P_k(x,y) = (x+y)^3 + 2(x^2-y^2) + (x+y) + kxy. \quad (2–20)$$

The discriminant is $k^4(k^2-16)^2$ and a Weierstrass form is

$$y^2 = x(x+16)(x+k^2). \quad (2–21)$$

Notice that (2–20) is not of the form (2–3) since it is cubic in y rather than quadratic. Nevertheless, the discussion at the beginning of this section shows that if k is outside K then, for $|x| = 1$, $P_k(x,y) = 0$ has exactly one zero with $|y| < 1$, so we can still use Jensen's formula to compute $m(P)$ and verify that condition (B) will hold for $k \in G_\infty$. Plotting K as above, it seems that K is a roughly elliptical region with centre at the origin with major axis from -10 to 10 and minor axis from about $-8i$ to $8i$. Thus we would expect that a formula of type E for integer $|k| \ge 10$ and indeed this is what is found experimentally for $10 \le k \le 200$.

k	s	N	a_1	a_2	a_3	a_4	a_6	k	s	N	a_1	a_2	a_3	a_4	a_6
1	1/7	11	0	−1	1	0	0	−1	?	11	0	−1	1	0	0
2	1	50	1	1	1	−3	1	−2	?	38	1	1	1	0	1
3	−1	123	0	1	1	−10	10	−3	?	75	0	1	1	2	4
4	2	118	1	1	1	−25	39	−4	?	58	1	1	1	5	9
5	3	395	0	−1	1	−50	156	−5	?	155	0	−1	1	10	6
6	11	606	1	0	0	−90	324	−6	?	186	1	0	0	15	9
7	−1	175	0	−1	1	−148	748	−7	?	203	0	−1	1	20	−8
8	−3	302	1	1	1	−230	1251	−8	?	50	1	1	1	22	−9
9	4	537	0	1	1	−340	2308	−9	?	57	0	1	1	20	−32
10	25	2090	1	1	1	−485	3915	−10	?	110	1	1	1	10	−45
11	−15	2651	0	−1	1	−670	6910	−11	?	11	0	−1	1	−10	−20
12	25	1650	1	0	0	−903	10377	−12	1	66	1	0	0	−45	81
13	−16	4043	0	−1	1	−1190	16212	−13	2	325	0	−1	1	−98	378
14	−52	4886	1	1	1	−1540	22629	−14	8	574	1	1	1	−175	789
15	−36	5835	0	1	1	−1960	32764	−15	−6	885	0	1	1	−280	1684
16	−8	862	1	1	1	−2460	45949	−16	2	158	1	1	1	−420	3109
17	35	8075	0	−1	1	−3048	65808	−17	−8	1717	0	−1	1	−600	5832
18	36	3126	1	0	0	−3735	87561	−18	2	150	1	0	0	−828	9072
19	56	10811	0	−1	1	−4530	118890	−19	−14	2869	0	−1	1	−1110	14580
20	−58	6190	1	1	1	−5445	152395	−20	−18	1790	1	1	1	−1455	20725
21	−80	14091	0	1	1	−6490	199108	−21	30	4389	0	1	1	−1870	30478
22	−140	15950	1	1	1	−7678	255771	−22	40	5302	1	1	1	−2365	43251
23	60	17963	0	−1	1	−9020	332772	−23	−30	6325	0	−1	1	−2948	62568
24	−54	5034	1	0	0	−10530	415044	−24	26	1866	1	0	0	−3630	83844
25	−20	4495	0	−1	1	−12220	524056	−25	8	1745	0	−1	1	−4420	114556
26	8	806	1	1	1	−14105	638919	−26	−94	10114	1	1	1	−5330	147519
27	−15	3075	0	1	1	−16198	788134	−27	−6	1293	0	1	1	−6370	193540
28	123	15274	1	1	1	−18515	962001	−28	−60	6650	1	1	1	−7553	249471
29	−120	33611	0	−1	1	−21070	1184260	−29	−64	15109	0	−1	1	−8890	325570
30	411	36870	1	0	0	−23880	1418400	−30	216	17070	1	0	0	−10395	407025
31	−160	40331	0	−1	1	−26960	1712880	−31	80	19189	0	−1	1	−12080	515040
32	5	550	1	1	1	−30328	2020281	−32	10	1342	1	1	1	−13960	629001
33	215	47883	0	1	1	−34000	2401780	−33	−120	23925	0	1	1	−16048	777124
34	405	51986	1	1	1	−37995	2834809	−34	−230	26554	1	1	1	−18360	949849
35	−213	56315	0	−1	1	−42330	3366306	−35	−118	29365	0	−1	1	−20910	1170756
36	−100	10146	1	0	0	−47025	3921129	−36	60	5394	1	0	0	−23715	1403649
37	−240	65675	0	−1	1	−52098	4594428	−37	−4	1147	0	−1	1	−26790	1696662
38	459	70718	1	1	1	−57570	5292751	−38	360	38950	1	1	1	−30153	2002711
39	−398	76011	0	1	1	−63460	6132088	−39	192	42549	0	1	1	−33820	2382628
40	−173	20390	1	1	1	−69790	7067355	−40	100	11590	1	1	1	−37810	2814015
41	−320	87371	0	−1	1	−76580	8182440	−41	180	50389	0	−1	1	−42140	3343620
42	925	93450	1	0	0	−83853	9339057	−42	−486	54642	1	0	0	−46830	3896676
43	280	99803	0	−1	1	−91630	10706542	−43	40	11825	0	−1	1	−51898	4567948
44	405	53218	1	1	1	−99935	12118149	−44	260	31922	1	1	1	−57365	5264379

TABLE 4. Data for the family $\Gamma_0^0(5)$, defined by (2–18) and (2–19). A ? means no rational relation was found.

The final two examples are $\Gamma(3)$ with projective equation

$$A_k(X,Y,Z) = X^3 + Y^3 + Z^3 + kXYZ = 0, \quad (2\text{–}22)$$

for which we take

$$P_k(x,y) = A_k(x,y,1) = x^3 + y^3 + 1 + kxy, \quad (2\text{–}23)$$

with discriminant $(k^3+27)^3$ and Weierstrass equation

$$y^2 = x^3 - 27k^2x^2 + 216k(k^3+27)x - 432(k^3+27)^2,$$

and $\Gamma_0(9) \cap \Gamma_0^0(3)$ with equation

$$B_k(X,Y,Z) = X^2Y + Y^2Z + Z^2X + kXYZ = 0, \quad (2\text{–}24)$$

for which we take

$$Q_k(x,y) = B_k(x,y,1) = y^2 + (x^2+kx)y + x, \quad (2\text{–}25)$$

with discriminant k^3+27 and Weierstrass equation

$$y^2 = x^3 + k^2x^2 - 8kx + 16.$$

In fact (2–22) and (2–24) are 3-isogenous, the kernel of the isogeny from (2–22) to (2–24) being the torsion group of (2–22). To see this, we need only verify that

$$A_k(Y^2Z, Z^2X, X^2Y) = B_k(X^3, Y^3, Z^3).$$

This also shows that $m(P_k) = m(A_k) = m(B_k) = m(Q_k)$, so we only need to study one of the two families. We naturally choose Q_k since it is of the form (2–3). The set K corresponding to (2–23) or (2–25) is the inside of a three-cusped hypocycloid whose intersection with the real axis is $[-3,1]$. It is symmetric under rotation by $2\pi/3$. We thus expect a formula of type E to hold for all integer k except for $k \in [-3,0]$ and a formula of type D for $k=-3$ since $P_{-3}=0$ has genus 0. This has been verified numerically for $|k| \le 40$. We don't give a table of these results since we will consider in the next section a family, B.1, of polynomials equivalent to Q_{-k}.

For $k=-3$ we have

$$P_{-3} = (x+y+1)(x+\omega y+\omega^2)(x+\omega^2 y+\omega),$$

where ω is a primitive cube root of 1, so we see that

$$m(P_{-3}) = 3m(x+y+1) = 3d_3,$$

by Smyth's result (1–6). We also notice that, although $P_0 = 0$ is elliptic, we have

$$m(P_0) = m(x^3+y^3+1) = m(x+y+1) = d_3,$$

by the same result. So in this case, we do not have a formula of type E but one of type D. Since $0 \in \operatorname{int}(K)$, this is in accord with the above conjecture.

Families in generalized Weierstrass form. We can obtain an even simpler class of examples by taking

$$P(x,y) = y^2 + kxy + a_3y - x^3 - a_2x^2 - a_4x - a_6, \quad (2\text{–}26)$$

so that $P(x,y)=0$ is already in generalized Weierstrass form. With such examples, it is easy to devise experiments to test the necessity of condition (A). Depending on whether $a_6=0$ or not, the Newton polygon is a quadrilateral or a triangle. The face y^2-x^3 always has $m(y^2-x^3)=0$ but the faces $y^2+a_3y-a_6 = P(0,y)$ and $x^3+a_2x^2+a_4x+a_6 = P(x,0)$ have measure 0 only if each is the a product of a power and a cyclotomic polynomial. If $a_6=0$, then the fourth face will be of the form $\pm y^i \pm x^j$ if $P(x,0)$ and $P(0,y)$ are cyclotomic, so the condition that this face should have measure 0 is not an additional restriction. A sufficient condition for k to be in K is that

$$|k| > 2 + |a_2| + |a_3| + |a_4| + |a_6|, \quad (2\text{–}27)$$

so we expect that a formula of type E should hold for $P(x,y)$ as in (2–26) if k is an integer satisfying (2–27) and if $m(P(x,0)) = m(P(0,y)) = 0$. Of course, for fixed a_2, a_3, a_4 and a_6, we can make a more refined conjecture by computing the set K.

We test this by considering several special cases:

$$\begin{aligned} &\text{family A:}\quad P(x,y) = y^2 + kxy + by - x^3 + 1,\\ &\text{family B:}\quad P(x,y) = y^2 + kxy + by - x^3, \qquad (2\text{–}28)\\ &\text{family C:}\quad P(x,y) = y^2 + kxy - x^3 - bx,\end{aligned}$$

Here there are two parameters, but we regard b as fixed and ask for which k does a formula of type E hold. We denote the subfamily of a family F for a fixed value of b by F.b. Notice that for families A and B it suffices to take $b \geq 0$ since changing the signs of y and k has the same effect as changing the sign of b.

In family A, the face $y^2 + by + 1$ has measure 0 only if $b = 0, 1$ or 2 so P satisfies condition (A) exactly in these cases. If $|k| > b + 3$ then $k \in G_\infty$ so we expect a formula of type E for these values of k provided if $0 \leq b \leq 2$. This is what was found numerically if $b + 3 < |k| \leq 18$. On the other hand, for $b = 3$, we have $m(y^2 + 3y + 1) = \log((3 + \sqrt{5})/2)$ and we find for $6 \leq |k| \leq 19$ that no such formula holds, nor a formula involving a rational linear combination of m_k, $L'(E_k, 0)$ and $\log((3+\sqrt{5})/2)$. The reason for the small range of k considered here is that the conductor grows fairly rapidly making it difficult to compute $L'(E_k, 0)$ for large k. We can see why this occurs by noticing that the discriminant is

$$d_b(k) = k^6 + (b^3 - 36b)k^3 + (-27b^4 + 216b^2 - 432).$$

The only small b for which $d_b(k)$ is reducible is $b = 2$ for which $d_2(k) = k^3(k-4)(k^2+4k+16)$. So only when $b = 2$ do we expect a moderate growth of the conductor with $|k|$.

Notice that if we fix k and let b vary, then for large b, say $|b| > |k|+3$, the dominant term appears in the face $P(0, y) = y^2 + by + 1$. Thus, by the discussion at the beginning of this section, $m(P) = m(y^2+by+1)$, a formula of type C for $|b| > |k|+3$.

The family A is interesting in that generically the rational torsion group is trivial. This is in constrast with the examples from [Beauville 1982] discussed on page 59, where the torsion groups are nontrivial. Also the examples studied in Section 2A all have nontrivial rational 2-torsion. As we explained at the end of Section 2A, this follows from the fact that $P(x, y)$ is a reciprocal polynomial. The reason for the interest in the torsion group of $E(\mathbb{Q})$ is that the proofs of Deninger [1997] for (1–17) and Rodríguez Villegas [1996] for the family 1.3 make use of the 4-torsion points of curves in this family to construct Eisenstein–Kronecker series.

Now consider the family B. For fixed b, let $K_b = K$ denote the set of k for which $P(x, y)$ does not vanish on the torus and $G_{b,\infty}$ be the unbounded component of its complement. Then $K_b \cap \mathbb{R}$ is contained in the interval $[-b, b+2]$ so the integers in $G_{b,\infty}$ consist of $k \geq b+2$ and $k \leq -b$. The complement of K_b is connected if $b = 1$ or 2 but consists of two components if $b \geq 3$. The intersection of the bounded component with the real axis is an interval $(-b + \delta, b - 2)$, where $\delta < 1$. The sets K_b all have 3-fold rotational symmetry.

For the family B, the discriminant is $b^3(k^3 - 27b)$ so $b = 0$ does not give an elliptic curve. For $b > 0$, two faces y^2+by and $by-x^3$ have measure $\log|b|$ so our condition (A) is satisfied only if $|b| = 1$. Thus if $k \in G_{b,\infty}$ we expect a formula of type E to hold if $b = 1$ but not if $b > 1$.

If $b = 1$ and $k \geq 4$ or $k \leq -1$ with $|k| \leq 40$, then we do seem to obtain a formula of type E, as can be seen in Table 5. For $k = 3$, the discriminant vanishes and we obtain a formula of type D. In fact, as we mentioned above, the family B.1 is related to the family $\Gamma_0(9) \cap \Gamma_0^0(3)$. Indeed, if we change the signs of k and x in (2–27) and then substitute xy for y and then interchange x and y we obtain the polynomial (2–25). This shows the curves are isomorphic. Furthermore, these operations also preserve the measure of the polynomial in question, so the study of B.1 is completely equivalent to the study of the family $\Gamma_0(9)\cap\Gamma_0^0(3)$. From Table 5 we see that

$$m(y^2 - 6xy + y - x^3) \stackrel{?}{=} 3L'(E, 0),$$

where E has conductor 27. This is the example we considered in the introduction. Rodríguez Villegas' method applies to this family and E has complex multiplication so we can replace $\stackrel{?}{=}$ by $=$ in this formula.

For $2 \leq b \leq 8$, it turns out that we obtain formulas of mixed type CE for some $k \in \bar{G}_{b,\infty}$, but

k	s	N	a_1	a_2	a_3	a_4	a_6	k	s	N	a_1	a_2	a_3	a_4	a_6
								−1	1/2	14	1	0	1	−1	0
								−2	1	35	0	1	1	−1	0
3			($m = 3d_3,\ g = 0$)					−3	1	54	1	−1	0	−3	3
4	2/3	37	0	1	1	−3	1	−4	−1	91	0	1	1	−7	5
5	1/7	14	1	0	1	−11	12	−5	1/2	38	1	0	1	−16	22
6	−2	189	0	0	1	−24	45	−6	1/3	27	0	0	1	−30	63
7	2	158	1	0	1	−47	118	−7	3	370	1	0	1	−54	146
8	4	485	0	1	1	−81	255	−8	1	77	0	1	1	−89	295
9	−6	702	1	−1	0	−132	618	−9	−3	378	1	−1	0	−141	681
10	−10	973	0	1	1	−203	1048	−10	−8	1027	0	1	1	−213	1128
11	2	326	1	0	1	−300	1970	−11	−12	1358	1	0	1	−311	2080
12	2	189	0	0	1	−426	3384	−12	−15	1755	0	0	1	−438	3528
13	−18	2170	1	0	1	−589	5446	−13	2	278	1	0	1	−602	5628
14	−18	2717	0	1	1	−793	8336	−14	21	2771	0	1	1	−807	8560
15	−12	1674	1	−1	0	−1047	13305	−15	3	378	1	−1	0	−1062	13590
16	−32	4069	0	1	1	−1357	18795	−16	24	4123	0	1	1	−1373	19131
17	−30	4886	1	0	1	−1732	27588	−17	−18	2470	1	0	1	−1749	27996
18	−42	5805	0	0	1	−2178	39123	−18	57	5859	0	0	1	−2196	39609
19	−6	854	1	0	1	−2706	53940	−19	33	6886	1	0	1	−2725	54510
20	−48	7973	0	1	1	−3323	72633	−20	64	8027	0	1	1	−3343	73293
21	−6	1026	1	−1	0	−4041	99891	−21	−15	2322	1	−1	0	−4062	100668
22	96	10621	0	1	1	−4869	129160	−22	−12	2135	0	1	1	−4891	130040
23	30	6070	1	0	1	−5819	170346	−23	99	12194	1	0	1	−5842	171358
24	108	13797	0	0	1	−6900	220608	−24	−1	171	0	0	1	−6924	221760
25	96	15598	1	0	1	−8126	281242	−25	36	7826	1	0	1	−8151	282542
26	−90	17549	0	1	1	−9507	353640	−26	128	17603	0	1	1	−9533	355096
27	36	4914	1	−1	0	−11058	450348	−27	−108	19710	1	−1	0	−11085	451995
28	28	4385	0	1	1	−12791	552565	−28	−131	21979	0	1	1	−12819	554385
29	−114	24362	1	0	1	−14721	686210	−29	−9	1526	1	0	1	−14750	688240
30	−2	333	0	0	1	−16860	842625	−30	189	27027	0	0	1	−16890	844875
31	−84	14882	1	0	1	−19225	1024360	−31	144	29818	1	0	1	−19256	1026840
32	176	32741	0	1	1	−21829	1234115	−32	−240	32795	0	1	1	−21861	1236835
33	216	35910	1	−1	0	−24690	1499430	−33	−12	1998	1	−1	0	−24723	1502433
34	288	39277	0	1	1	−27823	1777056	−34	−168	39331	0	1	1	−27857	1780320
35	−12	2678	1	0	1	−31246	2123232	−35	−252	42902	1	0	1	−31281	2126802
36	−240	46629	0	0	1	−34974	2517480	−36	−297	46683	0	0	1	−35010	2521368
37	360	50626	1	0	1	−39027	2964228	−37	−54	12670	1	0	1	−39064	2968446
38	−258	54845	0	1	1	−43421	3468120	−38	−384	54899	0	1	1	−43459	3472680
39	18	3294	1	−1	0	−48177	4082193	−39	315	59346	1	−1	0	−48216	4087146
40	−432	63973	0	1	1	−53313	4720303	−40	−310	64027	0	1	1	−53353	4725623

TABLE 5. Data for the family B.1, defined by (2–28) with $b = 1$.

k	s	N	a_1	a_2	a_3	a_4	a_6	k	s	N	a_1	a_2	a_3	a_4	a_6
								−2	−3	124	0	1	0	−2	1
4	3/8	20	0	1	0	−1	0	−4	6	236	0	1	0	−9	8
6	6	324	0	0	0	−21	37	−6	9	540	0	0	0	−33	73
8	−12	916	0	1	0	−77	236	−8	−12	1132	0	1	0	−93	316
10	−18	1892	0	1	0	−198	1009	−10	−27	2108	0	1	0	−218	1169
12	−36	3348	0	0	0	−420	3313	−12	36	3564	0	0	0	−444	3601
14	54	5380	0	1	0	−786	8225	−14	−48	5596	0	1	0	−814	8673
16	−72	8084	0	1	0	−1349	18628	−16	18	1660	0	1	0	−1381	19300
18	36	3852	0	0	0	−2169	38881	−18	−36	3924	0	0	0	−2205	39853
20	144	15892	0	1	0	−3313	72304	−20	−120	16108	0	1	0	−3353	73624
22	150	21188	0	1	0	−4858	128721	−22	177	21404	0	1	0	−4902	130481
24	216	27540	0	0	0	−6888	220033	−24	270	27756	0	0	0	−6936	222337
26	−354	35044	0	1	0	−9494	352913	−26	−243	35260	0	1	0	−9546	355825
28	240	43796	0	1	0	−12777	551656	−28	384	44012	0	1	0	−12833	555296
30	−432	53892	0	0	0	−16845	841501	−30	405	54108	0	0	0	−16905	846001
32	576	65428	0	1	0	−21813	1232756	−32	−336	65644	0	1	0	−21877	1238196
34	−18	3140	0	1	0	−27806	1775425	−34	−747	78716	0	1	0	−27874	1781953
36	288	31068	0	0	0	−34956	2515537	−36	216	31140	0	0	0	−35028	2523313
38	−768	109636	0	1	0	−43402	3465841	−38	657	109852	0	1	0	−43478	3474961
40	744	127892	0	1	0	−53293	4717644	−40	1050	128108	0	1	0	−53373	4728284

TABLE 6. Data for the family B.2, defined by (2–28) with $b = 2$. Unlike the preceding tables, here the s column represents $s = L'(E_k, 0)/(m(P_k) - \frac{1}{3}\log 2)$.

only if k is arithmetically related to b in the sense that k is divisible by all prime factors of b. In this case, the formulas are of the form

$$m(y^2 + kxy + by - x^3) \stackrel{?}{=} \tfrac{1}{3}\log b + rL'(E, 0). \quad (2\text{–}29)$$

for suitable rational r. Table 6 gives the values of $s = 1/r$ for $b = 2$ and $4 \le k \le 40$ and $-40 \le k \le -2$.

For $b > 2$, it is easy to see that the face $P(0, y) = y^2 + by$ is dominant for k in the bounded component of the complement of K_b so the formula in this case is just $m(P_k) = \log b$. But notice that, even for such k, the roots $y_1(x)$ and $y_2(x)$ of $P(x, y) = 0$ satisfy $|y_1(x)| > 1 > |y_2(x)|$ since $y_1(x)y_2(x) = -x^3$, so $\nu = 1$; that is, (G)(i) holds. But $D(x) = (kx + b)^2 + 4x^3$ does not vanish in $|x| \le 1$ since $|kx + b|^2 \ge 4$ on $|x| = 1$. Thus, $\mu = 0$, not 2 as required by (G)(ii).

Notice that (2–29) is valid in particular for $k = -b$ so, changing variables slighly, we have

$$m(y^2 + kxy + ky + x^3) \stackrel{?}{=} \tfrac{1}{3}\log|k| + r_k L'(E_k, 0), \quad (2\text{–}30)$$

apparently valid for all integer $k \ne 0$. The polynomial here vanishes on the torus for all k, at the point $(-1, 1)$.

Denoting the Family B polynomial in (2–28) by $P(x, y; k, b)$ one obtains, under the change of variables $x = b^{2/3}X$, $y = bY$, the equalities

$$\begin{aligned} P(b^{2/3}X, bY; k, b) &= b^2(Y^2 + b^{-1/3}kXY + Y - X^3) \\ &= b^2 P(X, Y; b^{-1/3}k, 1). \end{aligned}$$

This shows that the corresponding elliptic curves are isomorphic over $\mathbb{C}$ and suggests that there may be a connection between

$$m(P(x, y; k, b)) \quad \text{and} \quad m(P(X, Y; b^{-1/3}k, 1)).$$

The exact relationship is not obvious unless $|b| = 1$ since the change of variable in question changes the torus $|x| = 1$, $|y| = 1$ to $|X| = |b|^{-2/3}$, $|Y| = |b|^{-1}$. However, using the method used in deriving condition (B), one can reduce the integrals over the torus to integrals around branch cuts and change variables to obtain

$$m(y^2 + b^{1/3}kxy + by - x^3) = \tfrac{1}{3}\log b + m(y^2 + kxy + y - x^3), \quad (2\text{–}31)$$

for $b \geq 1$ and sufficiently large $|k|$.

The simplest example of this relationship for integral b is

$$m(y^2+2kxy+8y-x^3) = \tfrac{1}{3}\log 8 + m(y^2+kxy+y-x^3),$$

which was discovered experimentally and led to (2–31). This is valid for $k \geq 5$ and $k \leq -4$, but is definitely not true for all k, since if $2k$ is in the bounded component of the complement of K_8, which includes all integers with $-2 \leq k \leq 3$,we have

$$m(y^2 + 2kxy + 8y - x^3) = \log 8,$$

since the face $y^2 + 8y$ dominates. On the other hand, for $k = -1$ we have

$$m(y^2 - xy + y - x^3) \stackrel{?}{=} 2b_{14}$$

(see Table 5), and for $k = 3$ we have

$$m(y^2 + 3xy + y - x^3) = 3d_3.$$

For the family C, the discriminant is equal to $b^2(k^4-64b)$. Clearly m_k is even in k since $P_k(x,y) = P_{-k}(x,-y)$. The torsion group is generically of order 2. As with the family B, there are two faces $y^2 - bx$ and $x^3 + bx$ with measure $\log|b|$ so the condition (A) holds only for $|b| = 1$. Write $K_b = K$ as above. All K_b have a 4-fold rotational symmetry. For $b > 0$, the outer boundary of K_b resembles the intersection of a pair of ellipses with major axes along the coordinate axes. For $b < 0$, the set K_b is obtained by rotating $K_{|b|}$ through $\pi/4$. To see why, let $\omega = \exp(2\pi i/8)$ and consider $P(-ix, \omega y)$. The intersection of K_b with the real axis is $[-(b+2),\, b+2]$ in case $b > 0$.

For $b = 1$, we thus expect a formula of type E for integer $|k| \geq 3$ and this is experimentally verified for $3 \leq k \leq 40$. For $b = -1$, the intersection of K_{-1} with the real axis is approximately $[-1.5, 1.5]$ so we expect a formula of type E for integer $|k| \geq 2$ and again this is experimentally verified for $2 \leq k \leq 40$. By analogy with the family B, we might expect a formula of mixed type for $|b| > 1$, at least if k satisfies the same arithmetic condition. This turns out to be correct, at least experimentally, with the correct formula apparently being

$$m(y^2 + kxy - x^3 - bx) \stackrel{?}{=} \tfrac{1}{4}\log|b| + rL'(E,0),$$

for suitable rational r, provided $k \in G_{b,\infty}$ and that the square-free part of b divides k. This has been verified for $-2 \leq b \leq 6$, $b \neq 0$ and $|b| + 2 \leq k \leq 40$.

We now return briefly to the families arising from elliptic modular surfaces, in particular to the family $\Gamma_0^0(5)$ given by (2–17) or (2–18). We can get an isomorphic curve by replacing $x+1$ by x, obtaining the polynomial $xy^2+(x^2+(k+1)x-k)y+x^2$, which can be given a more pleasant form without changing the measure by multiplying by x and changing xy to y, obtaining

$$Q_k(x,y) = y^2 + (x^2 + (k+1)x - k)y + x^3.$$

This is rather similar to the polynomial that appears in (2–30) for which we found a formula of type CE. The curve $Q_k = 0$ is isomorphic to $P_k = 0$, with P_k as in (2–17) or (2–18), and hence corresponds to the group $\Gamma_0^0(5)$. In contrast to P_k, Q_k vanishes on the torus since $Q_k(1,-1) = 0$ for all k, but recall that this is also true for the example (2–30).

The measures of two of the faces are equal to $\log|k|$ so we might expect a formula of type CE here. But notice that the coefficient of the term $(k+1)xy$ is relatively prime to k so the divisibility condition discovered in examples B and C only holds for $k = \pm 1$. So this is a good test of the conjecture that this divisiblity condition is necessary. Indeed, we find, at least for small values of $|k| > 1$ that apparently $m(Q_k)$ is not a rational combination of $\log|k|$ and $L'(E_k, 0)$.

For $k = 1$, we find that $m(Q_1) \stackrel{?}{=} 5b_{11}$, producing the example (1–24). Notice that here $P_1 = 0$ and $Q_1 = 0$ are the "same curve", but

$$m(P_1) \stackrel{?}{=} (7/5)m(Q_1).$$

To see why the example

$$Q_1 = y^2 + (x^2 + 2x - 1)y + x^3$$

works, we can embed it in the family

$$y^2 + (x^2 + kx - 1)y + x^3. \tag{2–32}$$

For the family (2–32), the set $K \cap \mathbb{R} = [-2, 2]$ so 2 is on the outer boundary of K and we expect to find a formula of type E for $k = 2$. And, as expected, we obtain formulas of type E for this family for $2 \le |k| \le 20$.

On the other hand, for $k = -1$, it seems that $m(Q_{-1})$ is not a rational multiple of b_{11}. The explanation is that the (missing) central term is not dominant. More precisely, embedding $Q_{-1} = y^2 + (x^2 + 1)y + x^3$ in the family

$$y^2 + (x^2 + kx + 1)y + x^3, \tag{2–33}$$

we find that 0 is an interior point of the set K and so our heuristics suggest that we should not expect a formula of type E for $k = 0$. Again, for the family (2–33), we find formulas of type E for $k \ge 2$ and $k \le -4$, exactly as expected since $K \cap \mathbb{R} = [-4, 2]$.

It is instructive to look at our geometric condition (G) in more detail for the family (2–33). Notice that, if $y_1(x)$ and $y_2(x)$ are the two roots of $P(x, y) = 0$ then $|y_1(x)y_2(x)| = |x^3| = 1$ for $|x| = 1$ so either both roots lie on the unit circle for $|x| = 1$ or else exactly one is outside the unit circle and one inside. Thus condition (G)(i) just barely fails for $k \in [-4, 2]$. Condition (G)(ii) in fact holds for the two values $k = 0$ and $k = 1$. The case $k = 0$ is the one occuring in the previous paragraph for which E isomorphic to the curve $X_0(11)$ of conductor 11.

For $k = 0$, there are two complex branch points in $|x| < 1$, one branch point at $x = 1$ and another real branch point in $x > 1$. If $|x| = 1$, $x \ne 1$, the two branches $y_1(x)$ and $y_2(x)$ cross the unit circle exactly when $x = \pm i$. We can number them so that $|y_1(x)| > 1$ for $x \ne 1$ on the right half of $|x| = 1$ and $|y_2(x)| > 1$ for x on the left half of $|x| = 1$. We see then that we can write

$$\begin{aligned} m(P_0) &= \frac{1}{\pi}\int_0^{\pi/2} \log|y_1| + \frac{1}{\pi}\int_{\pi/2}^{\pi} \log|y_2| \\ &= \frac{1}{\pi}\int_0^{\pi} \bigl|\log|y_1|\bigr| = A_1 + A_2, \end{aligned}$$

where A_1 is the integral over $[0, \pi/2]$ and A_2 the integral over $[\pi/2, \pi]$. We find that numerically $m(P_0) = .4056029559\ldots$, which seems not to be rationally related to b_{11}. On the other hand, the integral

$$v(P_0) = \frac{1}{2\pi}\int_{|x|=1} \log|y_1| = \frac{1}{\pi}\int_0^{\pi} \log|y_1| = A_1 - A_2$$

can be reduced to an integral of the form ω over a branch cut between the two branch points in $|x| < 1$. Thus, our earlier discussion would suggest that $v(P_0)$ should be rationally related to b_{11} and indeed we find that

$$v(P_0) = -.1521471417\ldots \stackrel{?}{=} -b_{11},$$

verified to 50 decimal place accuracy. This is in accord with our contention that in case P vanishes on the torus, it is the integral of ω around a branch cut rather than $m(P)$, which should be rationally related to $L'(E, 0)$.

Incidentally, we remark that the families (2–32) and (2–33) generically have trivial rational torsion groups except for a few small values of k. Neither of these families is of the form (2–26) because of the term x^2y. The discriminant and a Weierstrass equation for (2–32) are $k^4 - k^3 + 8k^2 - 36k + 43$ and

$$\begin{aligned} y^2 = x^3 + (k^2 - 12k + 16)x^2 - 8(k-2)(k^2 - 5k + 5)x \\ + 16(k^2 - 3k + 3)^2. \end{aligned}$$

For (2–33), they are $k^4 - k^3 - 8k^2 + 36k - 11$ and $y^2 = x^3 + (k^2-4)x^2 - 8kx + 16$.

A number of other families of the shape (2–26) have been investigated with results completely consistent with the above. The reader is welcome to download the detailed results by anonymous ftp; see the section on electronic availability on page 79. It will be interesting to see if these results can be shown to be consistent with the Bloch–Beilinson conjecture.

3. FAMILIES OF CURVES OF GENUS 2

In this section, we discuss two classes of reciprocal polynomials for which the curve

$$Z_k = \{P_k(x, y) = 0\}$$

is generically of genus 2. The particular examples to be considered again arose from our early experiments with reciprocal polynomials of small measure. In contrast to most of the genus 1 examples we considered, the polynomials $P_k(x, y)$ all vanish on the torus for all k. Thus there is no obvious analogue of the set K of Section 2. In the genus 2 case the assumption that P_k be reciprocal plays a additional role: it insures that the Jacobian $J(Z_k)$ of the curve splits into the product of two elliptic curves. Of course there are other classes of polynomials for which this would also be the case but we have not yet attempted a systematic study of such examples.

In Section 3A, we discuss the polynomials

$$P_k(x, y) = A(x)y^2 + (B(x) + kx(x+1))y + C(x), \tag{3–1}$$

where $A(x)$ is one of 1, $x \pm 1$, $x^2 + x + 1$, $x^2 + 1$ or $(x-1)^2$, $B(x)$ is one of 0 or $x^3 + 1$, and where $C(x) = x^c A(1/x)$ is chosen so that P_k is reciprocal. As in Section 2, these families of polynomials are given the names a.b, where a is 1, 2, 3, 2a, 3s, respectively for the five choices of A listed above (for the choice $x \pm 1$, we choose whichever sign makes the polynomial nontrivial), and where b is 2 or 4 for the two choices of B listed above. (The number denotes the number of nonzero coefficients in the middle coefficient of P_k). Data for these families can be obtained by anonymous ftp; see page 79.

These polynomials are a generalization of the example in (1–30). It seems that for these families, formulas of type E (or D in degenerate cases) hold for all integer k.

In Section 3B, we discuss the polynomials

$$P_k(x, y) = A(x)y^2 + B_k(x)y + C(x),$$

where $A(x)$ is one of 1, x^2+x+1, or $x^4+x^3+x^2+x+1$, where $C(x)$ is chosen so that P_k is reciprocal, and where

$$B_k(x) = x^4 + kx^3 + lx^2 + kx + 1,$$

l being chosen so that Z_k is generically of genus 2, which means in particular that $l = \pm 2k + c$ for certain choices of sign and integers c. That is, P_k is of the form

$$P_k(x, y) = A(x)y^2 + (B(x) + kx(x \pm 1)^2)y + C(x). \tag{3–2}$$

This gives 2 choices of l in case $A = 1$ and four choices in the other two cases. We denote these families by 1.5(A or B), 3.5(A to D), and 5.5(A to D). See page 79 for information on the electronic availability of data for these families.

The shape of the polynomials 3.5 is motivated by the example (1–13) but the curve defined by that polynomial is of genus 3. In contrast to the first class of examples, it seems that formulas of type E hold only for a semi-infinite interval of integers k: either $k \geq k_0$ or $k \leq k_0$ for some k_0. However, it seems that this restriction does not apply to the degenerate cases when the discriminant vanishes, which all seem to satisfy formulas of type D (interpreted in a liberal sense in one case, (3–12)).

We only consider families satisfying condition (A). Condition (G) is not appropriate here since it was defined for polynomials $P(x, y)$ that do not vanish on the torus and for which $\deg(D) \leq 4$ and neither of these conditions holds here. However, the more general condition (B) can be considered. First we discuss the branch points of $y(x)$, the solution of $P(x,y)=0$, dropping the subscript k for the moment. Since

$$Y(x)=A(x)y(x)+B(x)=\sqrt{D(x)},$$

these are simply the roots of $D(x)$ of odd order.

For both classes of examples, $D(x)$ is reciprocal. For the type (3–1), $\deg(D)=6$, and for the type (3–2), $\deg(D)=8$, but in the latter case D has a factor $(x\pm 1)^2$. So there are (generically) 6 branch points. Since D is reciprocal, these are symmetrically located relative to $|x|=1$. Let us say that points these have distribution (a,b,c) if there are a, b and c branch points outside, on and inside the unit circle $|x|=1$. So $a=c$ and hence there are 4 possible distributions: $(3,0,3)$, $(2,2,2)$, $(1,4,1)$ and $(0,6,0)$. We will find that if the distribution is $(3,0,3)$ then a formula of type E never holds. In the case of $(2,2,2)$, we always obtain a formula of type E. In the case of $(1,4,1)$, we do obtain formulas of type E for the families (3–1) but not for the families (3–2). The only examples we have of the distribution $(0,6,0)$ are degenerate cases. All this is consistent with condition (B) as we will discuss in the individual cases.

3A. The First Class of Families of Curves of Genus 2

The family 3.2, defined by

$$P_k(x,y)=(x^2+x+1)y^2+kx(x+1)y+x(x^2+x+1), \tag{3–3}$$

generalizes the example in (1–22), which is the special case $k=1$. Completing the square, we see that $P_k=0$ can be written as

$$y_1^2=k^2x^2(x+1)^2-4x(x^2+x+1)^2=D_k(x),$$

where $y_1=2(x^2+x+1)y+kx(x+1)$. Thus Z_k is hyperelliptic and generically of genus 2. The discriminant is $k^4(k^2-9)(k^2+16)^2$. When this vanishes the genus is 0 or 1. Notice that $D_k(x)$ is a reciprocal polynomial. As pointed out to me by Bjorn Poonen, it is (well) known in this case that $J(Z_k)$ is isogenous to the product of two elliptic curves $E_k\times F_k$. One substitutes

$$x=(X+1)/(X-1),\quad Y=(X-1)^3y_1,$$

and the equation reduces to one of the form $Y^2=h(X^2)$, where h is cubic. Then the two elliptic curves in question are $y^2=h(x)$ and $y^2=h^*(x)=x^3h(1/x)$ [Cassels and Flynn 1996, Chapter 14].

For the family of (3–3), the Jacobian of Z_k splits into the product of the curves

$$E_k: y^2=x^3+(k^2-24)x^2-16(k^2-9)x \tag{3–4}$$

and

$$F_k: y^2=x^3+(k^2+8)x^2+16x. \tag{3–5}$$

Generically the rational torsion groups of E_k and F_k are of orders 2 and 4 respectively. We need only consider $k\ge 0$ because of the symmetry $y\to -y$. We have verified for $1\le k\le 33$, $k\ne 3$, that

$$m_k=m(P_k)\overset{?}{=}r_kL'(E_k,0),$$

where r_k is rational—in fact, the reciprocal of an integer. For example, $m_1\overset{?}{=}\frac{1}{3}b_{34}$, $m_2\overset{?}{=}-\frac{1}{6}b_{200B}$, and $m_4\overset{?}{=}-\frac{1}{3}b_{224A}$. Clearly $m_0=0$ since P_0 is cyclotomic. In the other degenerate case $k=3$, the curve E_3 is the rational curve $y^2=x^3-15x^2$ and we find numerically that $m_3\overset{?}{=}\frac{1}{6}d_{15}$, giving a first example of the appearance of the odd Dirichlet character of conductor 15. Since Ray's method [1987] does not deal with the conductor 15, it would be desirable to find a proof of this equation.

The last example also illustrates another interesting point. Note that in this case the curve Z has genus 1, and in fact it is birationally equivalent to E_2 that is an elliptic curve of conductor 15 but here our formula is of type D, not type E. So even if $P(x,y)$ is reciprocal and Z is an elliptic curve, it is not always true that $m(P)=rL'(E,0)$. This is not in conflict with the conjecture of Section 2 since P_3 is not of the shape considered there.

To test the condition (B) for this family 3.2, we find that the distribution of the branch points is

$(2,2,2)$ if $3 < |k|$ and $(1,4,1)$ if $|k| \le 3$. In the first case, Jensen's formula immediately expresses $m(P)$ as the integral of $\log|y_1|$ over the unit circle between the two branch points on the circle so (B) holds just as in the earlier discussion of the reciprocal polynomial (page 50). For the distribution $(1,4,1)$ there are four branch points $a, b, \bar{b}, \bar{a}$ on $|x| = 1$, listed counterclockwise and with a and b in the upper half plane. Jensen's formula expresses $m(P)$ as the sum of two integrals, between a and b and $\bar{b}$ and $\bar{a}$. But these two integrals are equal so in this case we still have condition (B) satisfied.

Table 7 contains a summary of the results for the family 3.2.

			Curve E_k						Curve F_k				
k	s	N_E	a_1	a_2	a_3	a_4	a_6	N_F	a_1	a_2	a_3	a_4	a_6
1	3	34	1	0	0	−3	1	17	1	−1	1	−1	0
2	−6	200	0	1	0	−3	−2	40	0	0	0	−2	1
3		$(6m_3 = d_{15},\ g = 0 : y^2 = x^3 - 15x^2)$						15	1	1	1	−5	2
4	−3	224	0	1	0	−8	−8	32	0	0	0	−11	14
5	−6	410	1	0	0	−16	0	205	1	−1	1	−22	44
6	−12	936	0	0	0	−30	29	312	0	−1	0	−39	108
7	−60	4550	1	0	0	−53	97	455	1	−1	1	−67	226
8	36	4400	0	1	0	−88	228	80	0	0	0	−107	426
9	−24	1746	1	−1	1	−140	591	291	1	1	1	−164	740
10	−912	105560	0	1	0	−211	1014	1160	0	0	0	−242	1449
11	−228	21098	1	0	0	−308	1936	1507	1	−1	1	−346	2560
12	−72	7200	0	0	0	−435	3350	480	0	−1	0	−480	4212
13	216	24050	1	0	0	−598	5412	2405	1	−1	1	−652	6566
14	−4104	555016	0	1	0	−803	8302	2968	0	0	0	−866	9809
15	216	21690	1	−1	1	−1058	13281	3615	1	1	1	−1130	14150
16	−480	67184	0	1	0	−1368	18772	272	0	0	0	−1451	21274
17	3216	362950	1	0	0	−1743	27577	5185	1	−1	1	−1837	30756
18	1680	214200	0	0	0	−2190	39125	2040	0	−1	0	−2295	43092
19	−1356	157586	1	0	0	−2718	53956	7163	1	−1	1	−2836	58830
20	−4464	813280	0	1	0	−3336	72664	2080	0	0	0	−3467	78574
21	−624	57582	1	−1	1	−4055	99951	9597	1	1	1	−4199	102980
22	240	41800	0	1	0	−4883	129238	440	0	0	0	−5042	137801
23	−13608	1629550	1	0	0	−5833	170457	12535	1	−1	1	−6007	180686
24	264	37296	0	0	0	−6915	220754	1776	0	−1	0	−7104	232848
25	−3888	493570	1	0	0	−8141	281425	3205	1	−1	1	−8347	295594
26	69312	12000664	0	1	0	−9523	353862	17992	0	0	0	−9746	370329
27	−648	67050	1	−1	1	−11075	450627	2235	1	1	1	−11315	458552
28	1008	173600	0	1	0	−12808	552888	1120	0	0	0	−13067	574926
29	4392	646178	1	0	0	−14738	686596	24853	1	−1	1	−15016	711970
30	5904	906840	0	0	0	−16878	843077	27480	0	−1	0	−17175	872100
31	−51744	7208306	1	0	0	−19243	1024881	30287	1	−1	1	−19561	1057880
32	−6240	1055600	0	1	0	−21848	1234708	1040	0	0	0	−22187	1272026
33	−10128	1093950	1	−1	1	−24710	1500117	36465	1	1	1	−25070	1517402

TABLE 7. Data for the family 3.2, defined by (3–3). E_k and F_k are the two factors of the Jacobian, given by (3–4) and (3–5), and N_E, N_F are their conductors. The s column gives the value of $s = 1/r_k = L'(E_k, 0)/m_k$, inferred from the numerical computation to 28 decimal places.

Call an elliptic curve over $\mathbb{Q}$ *even* if it has a rational 2-torsion point and *odd* if it has no rational 2-torsion. Notice that this is a rational isogeny invariant. The curves of genus 1 arising from reciprocal polynomials that were discussed in Section 2A are all even, as we pointed out there. For the curves considered here, where

$$J(Z_k) \simeq E_k \times F_k,$$

either both factors E_k, F_k are even or both are odd since $h(x)$ has a rational zero if and only if $h^*(x)$ does.

The three curves of conductor 11 are all odd, with torsion group of order 5 or 1, so it was a challenge to find a reciprocal polynomial for which $m(P)$ is a rational multiple of b_{11}. Our first example (1–26) of such a polynomial was obtained by constructing the family 3s.4:

$$(x-1)^2y^2+(x^3+kx^2+kx+1)y+x(x-1)^2, \quad (3\text{–}6)$$

The discriminant is $(k+1)^8(k^2-11k+116)^2$ and the factors of the Jacobian are

$$E_k : y^2 = x^3 - 2(k+1)(k-3)x^2 + (k+1)^3(k-7)x + 16(k+1)^4 \quad (3\text{–}7)$$

and

$$F_k : y^2 = x^3 + (k+1)(k-7)x^2 - 32(k+1)(k-3)x + 256(k+1)^2. \quad (3\text{–}8)$$

Here the two curves E_k and F_k are both odd for all but finitely many values of k. (The exceptional values satisfy a diophantine equation that has only finitely many solutions). If $k = 7$, for example, E_7 and F_7 are both odd, having conductors 11 and 88 respectively, and, as we have already indicated in (1–26), $m_7 \stackrel{?}{=} 13b_{11}$. For $k = 0$, E_0 and F_0 each have conductor 58 but are not isogenous; they are in Cremona's classes 58A and 58B, respectively and we have $m_0 \stackrel{?}{=} -b_{58A}$. For $k = -2$, the curves E_{-2} and F_{-2} are the (odd) curves 142B and 142A, respectively, and $m_{-2} \stackrel{?}{=} -2b_{142B}$. Experimentally, we find that $m(P_k) \stackrel{?}{=} r_kL'(E_k, 0)$ for $|k| \le 20$, $k \ne -1$. Note that $m_{-1} = 0$. The data here is summarized in Table 8. The distribution of branch points for this family is $(2, 2, 2)$ for all k so the condition (B) always holds as for the family 3.2.

3B. A Second Class of Families of Curves of Genus 2

At one time, we had hoped that if $P(x, y)$ is a reciprocal polynomial satisfying the condition (A) and for which the curve $\{P(x, y) = 0\}$ of genus 1 or 2, then $m(P)$ should be a rational multiple of an appropriate $L'(E, 0)$ or $L'(\chi, -1)$. However, the class of examples to be discussed in this Section shows this is not the case and exhibits some interesting new features.

One example from this class is the family 3.5B,

$$Q_k(x, y) = (x^2 + x + 1)y^2 + (x^4 + kx^3 + (2k-4)x^2 + kx + 1)y + (x^4 + x^3 + x^2), \quad (3\text{–}9)$$

whose shape is suggested by (1–15). However, the curve defined by (1–15) has genus 3, while the middle coefficient of (3–9) has been chosen so that the curve defined by $Q_k = 0$ is generically of genus 2. Here

$$D_k(x) = (x + 1)^2(x^2 + (k - 4)x + 1) \times (x^4 + (k + 2)x^3 + (2k - 2)x^2 + (k + 2)x + 1).$$

The discriminant is

$$(k + 1)(k - 2)(k - 5)^4(k - 6)(k^2 - 4k + 20)^2.$$

The Jacobian splits into the two curves

$$E_k : y^2 = x^3 + (k^2 - 4k - 20)x^2 - 16(k - 5)(k + 1)x \quad (3\text{–}10)$$

and

$$F_k : y^2 = x^3 + (k^2-8k+20)x^2 + 16(k-5)x. \quad (3\text{–}11)$$

We find, for $6 \le k \le 35$, that $m_k \stackrel{?}{=} rL'(E_k, 0)$, but for $k < 6$, m_k does not seem rationally related to either of $L'(E_k, 0)$ or $L'(F_k, 0)$ nor to a linear combination of these and other plausible terms. For the degenerate cases $k = -1, 2,$ and 5, it seems

k	s	N_E	Curve E_k a_1	a_2	a_3	a_4	a_6	N_F	Curve F_k a_1	a_2	a_3	a_4	a_6
0	−1	58	1	−1	0	−1	1	58	1	1	1	5	9
1	−4	212	0	−1	0	−4	8	53	1	−1	1	0	0
2	−14	882	1	−1	0	−9	27	882	1	−1	1	1	39
3	−1	92	0	0	0	−1	1	184	0	−1	0	0	1
4	−6	550	1	1	0	−25	125	550	1	−1	1	−15	87
5	−20	1548	0	0	0	−39	254	387	1	−1	1	−2	2
6	−30	4214	1	−1	0	−58	454	4214	1	1	1	−43	153
7	1/13	11	0	−1	1	0	0	88	0	0	0	−4	4
8	−1	138	1	1	0	−1	1	414	1	−1	1	−92	415
9	−40	4900	0	0	0	−175	1750	1225	1	1	1	−8	6
10	−78	12826	1	1	0	−244	2534	12826	1	−1	1	−177	993
11	−10	1044	0	0	0	−21	61	2088	0	0	0	−15	23
12	−2	338	1	−1	0	−454	5812	338	1	1	1	−322	2127
13	104	13916	0	−1	0	−604	8408	3479	1	−1	1	−27	60
14	248	35550	1	−1	0	−792	11866	35550	1	−1	1	−560	5267
15	−1/2	88	0	0	0	−4	4	352	0	−1	0	−45	133
16	160	28322	1	1	0	−1306	22184	28322	1	−1	1	−930	11169
17	−12	1308	0	1	0	−20	36	981	1	−1	1	−74	262
18	−40	7942	1	−1	0	−2053	42739	7942	1	1	1	−1480	21321
19	44	6700	0	−1	0	−158	937	13400	0	0	0	−115	475
20	−204	32634	1	−1	0	−3096	75816	32634	1	−1	1	−2267	42123
−1	$m = 0$		$(g = 0 : y^2 = x^3)$						$(g = 0 : y^2 = x^3)$				
−2	−2	142	1	1	0	−1	−1	142	1	−1	1	−12	15
−3	−4	316	0	0	0	−7	−2	79	1	1	1	−2	0
−4	−2	198	1	−1	0	−18	4	198	1	−1	1	−65	209
−5	−2	196	0	−1	0	−2	1	392	0	0	0	−7	7
−6	−40	5450	1	−1	0	−67	91	5450	1	1	1	−178	831
−7	−4	396	0	0	0	−111	214	99	1	−1	1	−17	30
−8	−56	6566	1	1	0	−172	428	6566	1	−1	1	−384	2979
−9	−2/13	37	0	0	1	−1	0	296	0	−1	0	−33	85
−10	−8	978	1	0	1	−5	2	2934	1	−1	1	−722	7633
−11	−144	17900	0	−1	0	−508	3512	4475	1	−1	1	−60	192
−12	−60	11858	1	−1	0	−688	5704	11858	1	1	1	−1240	16281
−13	−36	3852	0	0	0	−57	137	7704	0	0	0	−99	379
−14	−424	78754	1	1	0	−1186	12914	78754	1	−1	1	−1995	34779
−15	280	49588	0	0	0	−1519	19894	12397	1	1	1	−155	678
−16	400	61650	1	−1	0	−1917	29241	61650	1	−1	1	−3050	65577
−17	−2	296	0	−1	0	−9	13	1184	0	0	0	−232	1360
−18	−840	184382	1	−1	0	−2944	57082	184382	1	1	1	−4477	113427
−19	−4	588	0	−1	0	−44	120	441	1	−1	1	−335	2440
−20	−76	16606	1	1	0	−4339	101069	16606	1	−1	1	−6357	196653

TABLE 8. Data for the family 3s.4, defined by (3–6). The curves E_k and F_k are given by (3–7) and (3–8).

that one has $m_2 = (4/3)d_4$, $m_5 = 2d_4$, and the unusual

$$m_{-1} \stackrel{?}{=} \tfrac{1}{3}d_7 + \tfrac{1}{6}d_{15}. \tag{3–12}$$

Incidentally, $Q_{-1} = 0$ is an elliptic curve of conductor $210 = 2.3.5.7$.

Although we cannot prove (3–12), we can prove the two formulas for m_2 and m_5. These follow from (1–8) and (2–8) and the factorizations

$$Q_2(x,y) = (y + x^2 + x + 1)((x^2 + x + 1)y + x^2)$$

and

$$Q_5(x,y) = (x^2 + x + 1)(y^2 + (x^2 + 4x + 1)y + x^2).$$

The two cases $k = 3$ and $k = 4$ illustrate another interesting phenomenon. Proceeding as in the derivation of (1–16), we find that $m(Q_k)$ is the integral of a certain $f_k(t)$ over the subset of $[0,\pi]$ where $d_k(t) = e^{-4it}D_k(e^{it}) > 0$. In case $k = 3$ or 4, and only in these cases, the set on which $d_k > 0$ consists of two intervals, $[0, t1]$, $[t2, \pi]$, say. If we let $m'(Q_k)$ and $m''(Q_k)$ be the integrals of $f_k(t)$ over these two intervals, then we find that $m'(Q_3) \stackrel{?}{=} \frac{1}{3}b_{34}$ and $m'(Q_4) \stackrel{?}{=} (-1/6)b_{200B}$, (corresponding to the curve E_k in each case), but that $m''(Q_k)$ seems unrelated to $L'(E_k, 0)$ or $L'(F_k, 0)$ in either of these cases. Some data for the family 3.5B is presented in Table 9 in the format of Table 7.

To see the relevence of condition (B) here, we look at the distribution of the branch points. For $k \le -1$ the distribution is $(3,0,3)$ and notice that we do not find a formula of type E in this range. Notice here that there are no branch points on the circle but that $D_k(x)$ has a double root at $x = -1$. So both branches of $y(x)$ are holomorphic at $x = -1$ and have $y(-1) = -1$ there. Although Jensen's formula allows one to express $m(P)$ as the integral of $\log|y_1(x)|$, over the unit circle from -1 to -1, where $y_1(x)$ is a root of $P(x,y)$ with $|y_1(x)| > 1$ on $\{|x| = 1, x \ne -1\}$, this root $y_1(x)$ is *not* a branch of the function $y(x)$. Both of these branches have an expansion $y(x) = -1 \pm c(x+1) + \cdots$ near $x = -1$ and for each $|y(x)| - 1$ changes sign as x crosses -1. Thus (B) does not hold. In essence, because there are three branch points inside and outside $|x| = 1$, if branch cuts are introduced between pairs of branch points then one of these cuts must cross the circle. (For the most symmetric arrangement of cuts, one cut crosses the circle at $x = -1$).

For $-1 \le k < 2$, and for $6 < k$, the distribution is $(2,2,2)$, in which case (B) always holds and this is exactly the interval where we do find formulas of type E.

Finally, for $2 \le k \le 6$, the distribution is $(1,4,1)$, a case where we found formulas of type E for the family 3.2. The integers in the interval in question are the two degenerate cases $k = 2$ and 6 and the two unusual cases $k = 3$ and 4 mentioned above. Here there are 4 branch points on the circle $a, b, \bar{b}, \bar{a}$ listed as for the family 3.2. In contrast to that case, however, Jensen's formula expresses $m(Q_k)$ as the sum of two integrals along $|x| = 1$, $m'(Q_k)$ between $\bar{a}$ and a and $m''(Q_k)$ between b and $\bar{b}$, where the second arc $b\bar{b}$ contains the point -1. The integral $m'(Q_k)$ is the integral between two branch points of $\log|y_1(x)|$ where $y_1(x)$ is a branch of $y(x)$. This is the integral for which $m'(Q_k) \stackrel{?}{=} r_k L'(E_k, 0)$. The other integral, however, is not of this type for the same reason as in the discussion of $(3,0,3)$: the root $y_1(x)$ being integrated is not a branch of $y(x)$ because the arc $b\bar{b}$ contains -1.

A second example is the family 1.5A with the polynomial

$$Q_k(x,y) = y^2 + (x^4 + kx^3 + 2kx^2 + kx + 1)y + x^4. \tag{3–13}$$

Here

$$D_k(x) = (x+1)^2(x^2 + (k-2)x + 1) \times (x^4 + kx^3 + (2k+2)x^2 + kx + 1),$$

and the discriminant is $k^3(k+1)(k-4)(k-8)^2$. The Jacobian splits into the two curves,

$$E_k : y^2 = x^3 + (k^2 - 4k - 8)x^2 + 16(k+1)x \tag{3–14}$$

and

$$F_k : y^2 = x^3 + (k^2 - 8k + 8)x^2 + 16x. \tag{3–15}$$

For $-50 \le k \le 4$, it seems that we obtain a formula of type E (or D in the degenerate cases) in terms of the curve E_k, as one sees from Table 13. For the degenerate case $k = 8$, we do have $m_8 \stackrel{?}{=} 4d_4$ but for no $k > 4$ does it appear that we have a formula of type E.

The distribution of the branch points is $(2, 2, 2)$ for $k < -1$ and $0 < k \le 4$ so (B) is satisfied and

			Curve E_k						Curve F_k				
k	s	N_E	a_1	a_2	a_3	a_4	a_6	N_F	a_1	a_2	a_3	a_4	a_6
0	?	200	0	1	0	−3	−2	300	0	−1	0	−13	22
1	?	34	1	0	0	−3	1	170	1	0	1	−8	6
2	$3m = 4d_4$		$(g = 0 : y^2 = x(x-12)^2)$					24	0	−1	0	−4	4
3	$3m' = b$	34	1	0	0	−3	1	102	1	1	0	−2	0
4	$-6m' = b$	200	0	1	0	−3	−2	20	0	1	0	−1	0
5	$m = 2d_4$		$(g = 0 : y^2 = x^3 - 15x^2)$						$(g = 0 : y^2 = x^3 + 5x^2)$				
6	−3	224	0	1	0	−8	−8		$(g = 0 : y^2 = x(x+4)^2)$				
7	−6	410	1	0	0	−16	0	82	1	0	1	−2	0
8	−12	936	0	0	0	−30	29	156	0	−1	0	−5	6
9	−60	4550	1	0	0	−53	97	390	1	1	0	−13	13
10	36	4400	0	1	0	−88	228	200	0	1	0	−28	48
11	−24	1746	1	−1	1	−140	591	2910	1	1	0	−52	124
12	−912	105560	0	1	0	−211	1014	2436	0	−1	0	−89	354
13	−228	21098	1	0	0	−308	1936	1918	1	0	1	−143	642
14	−72	7200	0	0	0	−435	3350	240	0	−1	0	−216	1296
15	216	24050	1	0	0	−598	5412	5550	1	1	0	−315	2025
16	−4104	555016	0	1	0	−803	8302	11660	0	1	0	−445	3468
17	216	21690	1	−1	1	−1058	13281	15906	1	1	0	−611	5565
18	−480	67184	0	1	0	−1368	18772	5304	0	−1	0	−820	9316
19	3216	362950	1	0	0	−1743	27577	55510	1	0	1	−1079	13542
20	1680	214200	0	0	0	−2190	39125	35700	0	−1	0	−1393	20482
21	−1356	157586	1	0	0	−2718	53956	11310	1	1	0	−1772	27984
22	−4464	813280	0	1	0	−3336	72664	3536	0	1	0	−2224	39636
23	−624	57582	1	−1	1	−4055	99951	46614	1	1	0	−2757	54585
24	240	41800	0	1	0	−4883	129238	1140	0	−1	0	−3381	76806
25	−13608	1629550	1	0	0	−5833	170457	103550	1	0	1	−4106	100908
26	264	37296	0	0	0	−6915	220754	31080	0	−1	0	−4940	135300
27	−3888	493570	1	0	0	−8141	281425	296142	1	1	0	−5896	171820
28	69312	12000664	0	1	0	−9523	353862	175076	0	1	0	−6985	222384
29	−648	67050	1	−1	1	−11075	450627	102810	1	1	0	−8218	283348
30	1008	173600	0	1	0	−12808	552888	1200	0	−1	0	−9608	365712
31	4392	646178	1	0	0	−14738	686596	111410	1	0	1	−11168	453306
32	5904	906840	0	0	0	−16878	843077	35724	0	−1	0	−12909	568854
33	−51744	7208306	1	0	0	−19243	1024881	41034	1	1	0	−14847	690165
34	−6240	1055600	0	1	0	−21848	1234708	105560	0	1	0	−16996	847200
35	−10128	1093950	1	−1	1	−24710	1500117	961350	1	1	0	−19370	1029600

TABLE 9. Data for the family 3.5B, defined by (3–9). The curves E_k and F_k are given by (3–10) and (3–11).

this is exactly in the case where we find a formula of type E. For $k > 4$, on the other hand, the distribution is $(3, 0, 3)$ so condition (B) is not satsified and this coincides with the k for which we find no formula of type E. (In the case $-1 \le k < 0$, the distribution is $(1, 4, 1)$, but the only integer in this interval is $k = -1$, which is a degenerate case).

All the other families of the shape (3–2) that we have examined exhibit similar behaviour. Namely, $m(P_k) \stackrel{?}{=} r_k L'(E_k, 0)$ for one of the factors of the Jacobian for all integers k in a semi-infinite interval, $k \ge k_0$ or $k \le k_0$ but apparently for no other integers except that $m(P_k)$ satisfies a formula of type D in degenerate cases. Formulas of type E always occur when the distribution of branch points is $(2, 2, 2)$ and never when the distribution is $(3, 0, 3)$.

Finally, we point out the following unexpected coincidence. An examination of Tables 7 and 9 will reveal that the families 3.2 of (3–3) and 3.5B of (3–9) have something in common. Indeed if P_k is as in (3–3) and Q_k as in (3–9), then $m(P_k) \stackrel{?}{=} m(Q_{k+2})$ for $4 \le k \le 33$, but not for $k \le 3$. Notice that E_k of (3–4) is the same as E_{k+2} of (3–10) but that F_k of (3–5) and F_{k+2} of (3–11) are different. (This is most easily checked by looking at the tables). It would be interesting to prove directly that $m(P_k) = m(Q_{k+2})$ for $k \ge 4$. This presumably should be true for all real, not just integer, $k \ge 4$.

Another coincidence of the same type relates the families 1.5A of (3–13) and 2.3 of (1–31), as one sees by an examination of Tables 2 and 13. It appears that if P_k is as in (1–31) and Q_k as in (3–13), then $m(P_{k+2}) = m(Q_{-k})$, for $k \ge 1$, but not for other values of k. Notice that in this case, the curves of the family 2.3 are of genus 1 while those of the family 1.5A are of genus 2. Again, it would be interesting to understand the reason for this behaviour.

4. DEGENERATE CASES

In this section, we collect some of the examples relevant to Chinburg's conjecture that one can realize all $d_f = L'(\chi_{-f}, -1)$ as rational multiples of measures $m(P(x, y))$ of polynomials with integer coefficients. The examples here occur as degenerate cases of the families of curves studied in the previous sections, that is, in cases where the discriminant vanishes. Thus the curves $\{P(x, y) = 0\}$ have genus either 0 or 1. We remind the reader that the symbol $\stackrel{?}{=}$ means that the formulas have only been verified to high numerical accuracy, but not proved.

As explained earlier, plausible values for f in each case were deduced by examining the nonvanishing factors in the discriminant. The values of d_f were computed in a naive way from the formula

$$d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2).$$

Since χ has period f, one only needs to compute the values of the following sums:

$$A(f, j) = \sum_{n=0}^{\infty} \frac{1}{(fn + j)^2}, \qquad (4\text{–}1)$$

for $1 \le j < f$ with j relatively prime to f. Then one simply forms

$$L(\chi_{-f}, 2) = \sum_{(j,f)=1} \chi_{-f}(j) A(f, j).$$

The series in (4–1) are slowly convergent but are easily computed by means of the Euler–Maclaurin formula (they are simply multiples of values of the Hurwitz zeta function). Fortunately, the summation routines of Maple handle this automatically and make it easy to obtain 50 decimal place accuracy. Table 11 on page 77 contains the values of d_f needed in our study as well as a few others.

Subsequently, it was realized that a more efficient way to compute $L'(\chi_{-f}, -1)$ for primitive odd χ_{-f} is to use the formula, which Grayson [1981] attributes to Bloch:

$$L'(\chi_{-f}, -1) = \frac{f}{4\pi} \sum_{m=1}^{f} \chi_{-f}(m) \mathcal{D}(\zeta_f^m).$$

			Curve E_k						Curve F_k				
k	s	N_E	a_1	a_2	a_3	a_4	a_6	N_F	a_1	a_2	a_3	a_4	a_6
1	1/2	14	1	0	1	−1	0	21	1	0	0	1	0
2	1	36	0	0	0	0	1	24	0	−1	0	1	0
3	1/2	30	1	0	1	1	2	15	1	1	1	0	0
4	1/4	20	0	1	0	4	4		$(g = 0 : y^2 = x(x-4)^2)$				
8	$m = 4d_4$		$(g = 0 : y^2 = x(x+12)^2)$						$(g = 0 : y^2 = x(x+4)^2)$				
−1	$m = 2d_3$		$(g = 0 : y^2 = x^3 - 3x^2)$					15	1	1	1	−5	2
−2	1/3	20	0	1	0	−1	0	120	0	1	0	−15	18
−3	1	66	1	0	1	−6	4	231	1	1	1	−34	62
−4	1/2	36	0	0	0	−15	22	24	0	−1	0	−64	220
−5	−1	130	1	0	1	−33	68	195	1	0	0	−110	435
−6	6	420	0	1	0	−61	164	840	0	−1	0	−175	952
−7	6	630	1	−1	0	−105	441	1155	1	1	1	−265	1550
−8	1/10	14	1	0	1	−11	12	48	0	1	0	−384	2772
−9	1	102	1	0	1	−256	1550	663	1	1	1	−539	4592
−10	2	180	0	0	0	−372	2761	840	0	−1	0	−735	7920
−11	−12	2090	1	0	1	−524	4566	3135	1	0	0	−980	11727
−12	−6	660	0	1	0	−716	7140	15	1	1	1	−80	242
−13	−12	1638	1	−1	0	−957	11637	4641	1	1	1	−1644	24972
−14	30	4004	0	1	0	−1253	16660	1848	0	1	0	−2079	35802
−15	42	4830	1	0	1	−1613	24788	6555	1	1	1	−2595	49800
−16	1	90	1	−1	1	−128	587	240	0	−1	0	−3200	70752
−17	1	170	1	0	1	−2554	49452	1785	1	0	0	−3905	93600
−18	24	2652	0	1	0	−3153	67104	3432	0	−1	0	−4719	126360
−19	−2	342	1	−1	0	−3852	92988	1311	1	1	1	−5654	161282
−20	−18	2660	0	1	0	−4660	120900	840	0	1	0	−6720	209808
−21	−42	6090	1	0	1	−5589	160336	3045	1	1	1	−7930	268502
−22	108	13860	0	0	0	−6648	208633	17160	0	−1	0	−9295	348040
−23	−84	15686	1	0	1	−7851	267074	2139	1	0	0	−10829	432840
−24	1	138	1	0	1	−576	5266	336	0	−1	0	−12544	544960
−25	6	990	1	−1	0	−10734	430740	4785	1	1	1	−14455	662900
−26	−24	4420	0	1	0	−12441	529984	26520	0	1	0	−16575	815850
−27	18	2730	1	0	1	−14344	660002	3255	1	1	1	−18920	993800
−28	2	252	0	0	0	−16455	812446	42	1	1	1	−1344	18405
−29	66	15022	1	0	1	−18791	989850	35409	1	0	0	−24344	1459935
−30	−288	33060	0	1	0	−21365	1194900	38760	0	−1	0	−27455	1760160
−31	−216	36270	1	−1	0	−24195	1454625	42315	1	1	1	−30855	2073252
−32	−2	310	1	0	0	−1706	26980	240	0	1	0	−34560	2461428
−33	−16	2706	1	0	1	−30686	2066384	50061	1	1	1	−38589	2901642
−34	324	47124	0	0	0	−34380	2453617	54264	0	−1	0	−42959	3441480
−35	246	51170	1	0	1	−38398	2892828	58695	1	0	0	−47690	4004595
−36	36	4620	0	1	0	−42756	3388644	1320	0	−1	0	−52800	4687452

TABLE 10. Data for the family 1.5A, defined by (3–13). The curves E_k and F_k are given by (3–14) and (3–15).

$$m(y^2+(x^3-4x^2-4x+1)y+x^3) \stackrel{?}{=} d_7 \tag{4–2}$$

$$m((x^2+x+1)(y^2+1)+2xy) \stackrel{?}{=} \tfrac{1}{3}d_8 \tag{4–3}$$

$$m((x^2+x+1)(y^2+x)+3x(x+1)y) \stackrel{?}{=} \tfrac{1}{6}d_{15} \tag{4–4}$$

$$m((x^2+x+1)(y^2+x^2)+(x^4-x^3-6x^2-x+1)y) \stackrel{?}{=} \tfrac{1}{3}d_7+\tfrac{1}{6}d_{15} \tag{4–5}$$

$$m((x^2+x+1)(y^2+1)+6xy) \stackrel{?}{=} \tfrac{1}{6}d_{24} \tag{4–6}$$

$$m((x^2+x+1)(y^2+x)+(x^3-4x^2-4x+1)y) \stackrel{?}{=} \tfrac{1}{18}d_{39} \tag{4–7}$$

$$m((x^4+x^3+x^2+x+1)(y^2+1)+(x^4-3x^3-6x^2-3x-1)y) \stackrel{?}{=} \tfrac{1}{30}d_{55} \tag{4–8}$$

Some formulas of type D.

Here ζ_f is a primitive fth root of unity and $\mathcal{D}(z)$ denotes the Bloch–Wigner dilogarithm. One can then take advantage of the very rapidly convergent formula of Cohen and Zagier for $\mathcal{D}(z)$ [Zagier 1991, p. 387], which is implemented in PARI. The results obtained by either method agreed to all decimal places computed.

Given a P for which one suspects a formula of type D, one can then test $m(P)$ for rational dependence on an appropriate set of d_f by using the LLL algorithm. We used PARI's "lindep2" routine for this. Indeed, for the smaller conductors, one need not make the a priori assumption that the character involved is the odd primitive character of a certain conductor but instead can test for rational linear relation between $m(P)$ and the set of $f^{3/2}A(f,j)/(4\pi)$ and deduce the appropriate character from this. For the larger conductors, this would require too high an accuracy because of the number of terms involved.

The box at the top shows some of the more interesting examples of formulas of type D that we have discovered in this way.

The families in question are 1.4, 3.1, 3.2, 3.5B, 3.1, 3.4 and 5.5A, respectively. The polynomials (4–4), (4–7) and (4–8) are the first known examples with conductors 15, 39 and 55 while (4–2) and (4–6) are simpler than Ray's examples with conductors 7 and 24. His formulas, of course, have the advantage of having been proved rigorously. Ray's construction [1987] for conductor f produces a polynomial $P_f(x,y)$ of the form (2–3) with

$$A(x)=C(x)=\Phi_f(x),$$

the minimal polynomial of the f-th roots of unity. He proves that $m(P_f)=r_f d_f$, for $f=3,4,7,8,20$ and 24, where the rational r_f is $8/7$ for $f=7$ and $(8-\chi_{-f}(2))/f$, if $f\neq 7$. As explained in [Ray 1987], $m(P_{15})$ is not a rational multiple of d_{15}. Here are his examples for $f=7,8,20$ and 24:

$$\begin{aligned}
P_7(x,y)&=\Phi_7(x)(y-1)^2+7x^2(x+1)^2y,\\
P_8(x,y)&=\Phi_8(x)(y-1)^2+8x^2y,\\
P_{20}(x,y)&=\Phi_{20}(x)(y-1)^2+20x^2(x^2-1)^2y,\\
P_{24}(x,y)&=\Phi_{24}(x)(y-1)^2+24x^2(x^2-1)^2y.
\end{aligned}$$

Here $\Phi_7(x)=(x^7-1)/(x-1)$, $\Phi_8(x)=x^4+1$, $\Phi_{20}(x)=x^8-x^6+x^4-x^2+1$ and $\Phi_{24}(x)=x^8-x^4+1$. Since the polynomials with even f are even in x, one can obtain examples of lower degree in x with the same measure by substituting x for x^2. So, for example, $P_8(\sqrt{x})=(x^2+1)(y-1)^2+8xy$, which falls into our family 3s.1 and indeed was our motivation for considering that family.

Each of Ray's examples factor into linear factors over the field $\mathbb{Q}(\sqrt{f})$ while the examples (4–2) to (4–8) do not, so it seems that new methods will be needed to prove them. It should be pointed

f	d_f
3	0.32306594721945051409363651072380639407224184078059
4	0.58312180806163756027676891293678983772813230797167
7	1.6977024570017754467712530661472615603469009152493
8	1.9171950931209540617988237536697845644585055918673
11	2.6405873587515276991795086930833344266678922692192
15	5.9943109891313472634697667581104597033919274599213
19	5.0671396778554986769345642103452579986530147923945
20	8.0289898590238510084871069628841511919082609711128
24	9.8972211917380616681360119968380589589334488983872
39	26.352191699657576015239412956951029698687366020393
55	39.878041517883214774769741622874262194032748392618

TABLE 11. Some values of $d_f = L'(\chi_{-f}, -1)$.

out that Ray's proof of $m(P_7) = (8/7)d_7$ is quite deep and involves the proof of a new dilogarithm identity.

We should also remind the reader of example (2–14), which apparently has

$$m((x^2+x-1)y^2+(x^2+5x+1)y+(-x^2+x+1)) \stackrel{?}{=} \tfrac{2}{3}\log(\frac{\sqrt{5}+1}{2}) + \tfrac{1}{6}d_{15}. \quad (4\text{–}9)$$

In this case, the polynomial does factor into linear factors over $\mathbb{Q}(\sqrt{5})$ so perhaps (4–9) is more amenable to proof. It can be reduced to a dilogarithm identity, but this identity has not yet been proved.

5. CONCLUSION

Since the same sort of remarkable numerical coincidences have been displayed in many hundreds of examples, it is fairly clear that the phenomena discovered here are real. What is lacking at the moment is a proof of most of these results. A first step would be to at least reduce the formulas to an application of the Bloch–Beilinson conjectures. Hubert Bornhorn, a doctoral student at Münster University has made some progress on this question using a motivic approach.

As we have described in Section 2, Rodríguez Villegas has used the theory of modular forms to treat the families 1.3 and 2.3 and the families arising from elliptic modular surfaces described starting on page 59, thus showing that the formulas discovered numerically would follow from the Bloch–Beilinson conjectures, except for the determination of the constants r_k. In those few cases where the curves in question have complex multiplication, his results lead to a rigorous proof of the formulas including the exact values of the r_k. It seems clear that his methods will apply to a great many, perhaps all, of the families of curves of genus 1 we considered in Section 2.

Rodríguez Villegas' methods also apply to some families of polynomials in 3 variables. These have been difficult to investigate numerically because of the difficulty of obtaining accurate numerical values for $m(P)$. Indeed, the only such formula previously known is Smyth's remarkable formula

$$m(1 + x + y + z) = 7\zeta(3)/(2\pi^2) = 14\zeta'(-2);$$

see [Boyd 1981b].

6. SOME RECENT INFORMATION

We now report on some of the progress that has been made towards some of the questions raised above in the time since a preprint of this paper was circulated.

It became clear that the condition (A) was a natural condition when we found that it had occurred before in the paper of [Cooper et al. 1994, p. 70]. There the authors consider three-manifolds M whose boundary is a single torus. They define a polynomial $A_M(x,y)$, called the A-polynomial of M, that is analogous to the Alexander polynomial of a knot or link. They prove that the faces of these polynomials are cyclotomic; i.e., that A-polynomials satisfy our condition (A). Most of the polynomials occuring in that paper are much more complicated than those we consider here, but it would be of interest to see if $m(A_M)$ has a geometric interpretation.

Independently, Hubert Bornhorn and Rodríguez Villegas have proved that the conjecture in Section 2 follows from the Bloch–Beilinson conjectures in the case that $P(x,y)$ does not vanish on the torus. In this case, the condition (A) turns out to be equivalent to the condition that some power of the symbol $\{x,y\} \in K_2(\mathcal{E})$ is in the kernel of the tame symbol. Condition (A) also implies that $m(P)$ is an integer multiple of $r\{x,y\}$ where r is the Bloch regulator defined on $K_2(\mathbb{Q}(E))$. This is exactly what is needed to apply the Bloch–Beilinson conjecture in our situation. A proof of this is sketched in [Rodríguez Villegas 1997].

We can also report some some further progress on Chinburg's conjecture discussed in Section 4. By considering polynomials $P(x,y) = A(x)y + B(x)$ that are *linear* in y rather than quadratic, we have found examples which are (numerically) rational multiples of d_f for $f = 3,4,7,8,11,15,20,24,35, 39,55$ and 84. For example, we have

$$m((x+1)^2(x^2+x+1)y + (x^2-x+1)^2) \stackrel{?}{=} \tfrac{2}{3}d_{11},$$

which is a conductor not previously found, and

$$m((x+1)^2y + (x^2+x+1)) \stackrel{?}{=} \tfrac{1}{3}d_7,$$

which is simpler than those reported above. These results turn out to be related to computations of Browkin [1989] concerning Lichtenbaum's conjecture about $K_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of an imaginary quadratic field F. In these cases it can be shown using results from [Zagier 1991] that the measures of the respective polynomials are rational multiples of the corresponding d_f. However it has not yet been proved that the rational numbers are as indicated, only that they agree with these to 50 decimal place accuracy.

We have also found a construction of irreducible polynomials giving formulas of type DE for $m(P)$. The construction exploits the fact that there is a relation between $m(P_k)$ and $m(Q_k)$ if $P_k = A(x)y^2 + B_k(x)y + C(x)$ and $Q_k = y^2 + B_k(x)y + A(x)C(x)$. Note that the curves $P_k = 0$ and $Q_k = 0$ are isomorphic, since both have the Weierstrass form $Y^2 = D_k(X)$ with $D_k = B_k^2 - 4AC$. If k is large enough so that P_k and hence Q_k does not vanish on the torus, then one has $m(Q_k) = m(P_k)$. But if the polynomials vanish on the torus then the difference $m(Q_k) - m(P_k)$ can be expressed as an integral of $(1/\pi)\log|A|$ over a subset of the unit circle. For certain values of the parameter k, it may happen that this integral can be expressed by a formula of type D. If one takes P_k to be a reciprocal family for which we expect formulas of type E, then one can obtain a formula of type DE for Q_k. For example, if we take $A = C = x^2+1$ and $B_k = kx$, we find for $k = 2$ that

$$m(Q_2) = m(P_2) + d_3 \stackrel{?}{=} b_{24} + d_3.$$

The term b_{24} thus comes from the fact that $Q_2 = 0$ is isomorphic to an elliptic curve of conductor 24 while the term d_3 comes from the way the curve $Q_2 = 0$ intersects the torus. More details of this and further examples will be presented in a future paper.

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Electronic availability

More complete tables than could be included in this paper are available online. The main directory containing the papers is ftp://math.ubc.ca/pub/boyd/mahler/ and the organization is as follows:

- Data for each genus-1 family of reciprocal polynomials (see Section 2A, page 56) can be found in the subdirectory genus1/recip, under the name a.b.pos (for $k \geq 0$) and a.b.neg (for $k < 0$, only necessary if b = 3). The header of each file contains the exact definition of P_k.
- In the same directory there are some data for the two families 3g.1 and 3g.3 (see page 58); in particular, the values of m_k to 50 decimal places, for $|k| \leq 20$, are in the files 3g.1.measures and 3g.3.measures.
- Data for each genus-1 family of reciprocal polynomials (see Section 2B, page 67) can be found in genus1/nonrecip.
- Data for the first class of genus-2 families (see Section 3A, page 68) are in the directory genus2.A, under the name a.b.pos (for $k \geq 0$) and a.b.neg (for $k < 0$, only necessary if b = 4).
- Data for the genus-2 families 1.5 and 3.5 (see Section 3B, page 70) are in the directory genus2.B. Our study of the families 5.5 is still in progress.

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