Feigenbaum Numbers for Certain Flat-Top Families

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CONTENTS

We report on numerical results for certain families of S-unimodal maps with flat critical point. For four one-parameter families, differing in their amount of flatness, we study the Feigenbaum limits α and δ. There seems to be a finite δ and a finite α associated with each period doubling cascade in each family. Some rough numerical estimates are obtained, and our upper bound on δ is smaller than the corresponding supremum for families with nonflat critical point. One would expect that these numbers should only depend on the nature (flatness) of the maximum, and thus be constant in each family. Our data support this hypothesis for α, but are inconclusive when it comes to δ.

1. INTRODUCTION

Given the present knowledge of families of unimodals with quadratic (or, more generally, nonflat) critical point, it is natural to start looking at unimodal families with a flat critical point (the nature of the critical point being constant in each family), and investigate their metric and measure theoretical properties. In this paper we investigate the scaling properties of period doubling for certain flat-top families in a series of simple and somewhat naive numerical experiments. Still we feel that the results are of some interest, and we are not aware of any investigations parallel to the present one.

We will look at the families

$$\Phi_a(x) = \Phi_{a,q}(x) = Q\left(1 - a\varphi\left(x\right)\right),$$

where

$$Q = \left(\frac{q}{q+1}\right)^{1/q}$$

and

$$\varphi(x) = \exp\left(\frac{q+1}{q} - \frac{1}{|x|^q}\right).$$

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It is understood that $\Phi_{a,q}(0)$ is defined by continuity to equal Q, so all derivatives of $\Phi_{a,q}$ vanish at zero. Thus, $\Phi_{a,q}$ is a C^∞ flat top that behaves like $e^{-1/|x|^q}$ at the critical point $x = 0$, and Q is always the critical value.

For each $q > 0$ we get a full S-unimodal, convex family on $(-Q, Q)$ when a runs from 0 to 2. The superstable two-cycle appears at $a = 1$. Plotting the usual bifurcation diagram you obtain Figure 1. From kneading theory we know that these families go through period doubling from any primitive period length. Thus it is relevant to study the Feigenbaum limits α and δ (whose definition is recalled below). In Sections 2 and 3 we investigate these limits and obtain finite bounds. This means that Figure 1 is a finitely distorted copy of the bifurcation diagram for, say, the quadratic family. See [Collet and Eckmann 1980; Feigenbaum 1979; Melo and van Strien 1993] for background and general theory on bifurcations in unimodal families.

2. THE NUMBER δ

2.1. Background

Consider a family f_a of unimodal maps on an interval, and a set $\{I_n\}_{n=0}^\infty$ of adjacent intervals in a-space, numbered from left to right, satisfying the following conditions: f_a has a stable periodic orbit of length $2^n k$, for all $a \in I_n$; and there exists a unique $a_n \in I_n$ such that f_{a_n} has a *superstable* periodic orbit, that is, the critical point belongs to the stable periodic orbit. Then study the limit

$$\delta = \lim_{n\to\infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}. \tag{2.1}$$

For nonflat families, that is, $|f_a(x) - A| \sim |x|^r$ in a neighborhood of the critical point, the situation is as follows:

- Numerically δ always exists; it depends only on r, the order of the maximum; and it increases with r.
- Heuristically δ is the sole eigenvalue with absolute value greater than one for the derivative at the fixed point of the operator
$$T_r\colon h(x) \mapsto \alpha h\Big(h\Big(\frac{x}{\alpha}\Big)\Big),$$
acting on some suitable space of functions of the form $h(x) = A + B|x|^r +$ higher-order terms.
- The results above have been proved rigorously for $r = 1 + \varepsilon$ with ε small [Collet and Eckmann 1980], and for $r = 2$ [Lanford 1982].
- For r an even positive integer, we know from Sullivan's fundamental work [1992] that T has a fixed point. See also [Melo and van Strien 1993]. This is also proven in [Eckmann and Wittwer 1985] for m sufficiently large.
- Eckmann and Wittwer [1985] have proved that the derivative of the operator T_r at the fixed point has an eigenvalue of largest absolute value, $\Delta(r)$, which is real and positive. They obtain bounds for $\Delta(r)$, in particular
$$28.9 < \limsup_{r\to\infty} \Delta(r) < 29.6. \tag{2.2}$$
(As far as we understand it, though, the existence of this fixed point is rigorously established only in the cases mentioned above.)

For families with a truly flat segment, δ is infinite, and the length of the intervals (a_n, a_{n+1}) goes down quadraticly:

$$|(a_n, a_{n+1})| \sim |(a_{n-1}, a_n)|^2$$

for large n. This is proved for families with trapezoid shaped graphs in [Beyer and Stein 1982; Kazarinoff and Wang 1987; Wang 1994].

2.2. Numerical results

Our numerical data suggest the following result:

Result 1. *The limit* (2.1) *exists and is finite for every periodic window of the families* $\Phi_{a,q}$. *For k fixed, δ is increasing in q.* (*The flatter top, the larger* δ.) *For* $\frac{1}{4} \le q \le 1$, *we have* $22 \lessapprox \delta \lessapprox 28$.

See also Table 1 for some sharper estimates.

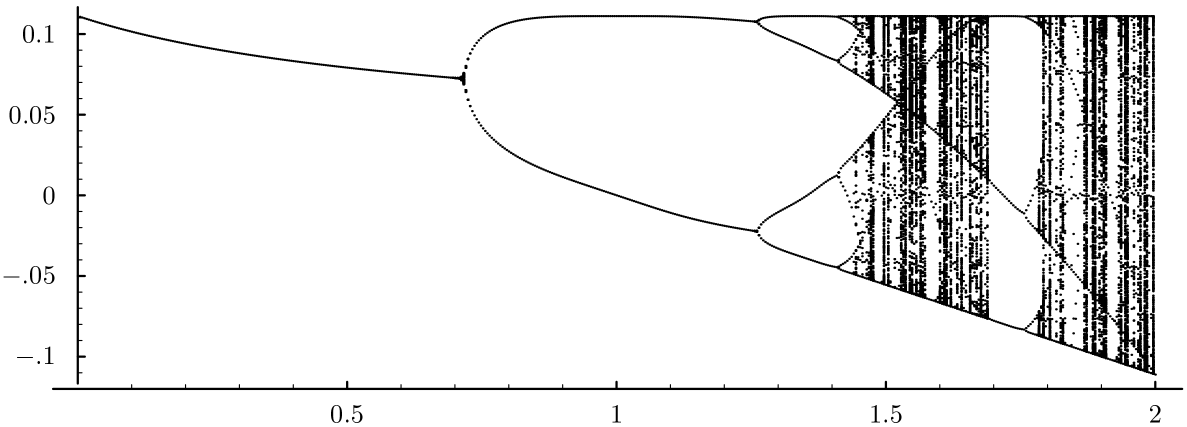

FIGURE 1. Bifurcation diagram for the family $\Phi_{a,1/2}$.

The convergence to the limit δ is much slower than in, say, the quadratic family, and we are only able to give coarse estimates of the numerical values. Especially we cannot see from our data if δ is constant over each family, as one would expect. In Table 1 we list our estimates of δ for a few values of k and q producing "nice" data. (Here k refers to the primitive period length of the periodic window under consideration.) See Section 2.3 for an account of our method and lines of thought.

q	$k=2$ I	$k=2$ II	$k=3$ I	$k=3$ II	$k=4$ I	$k=4$ II
$\frac{1}{2}$	$(24.4, 24.7)$	24.6	$(24.6, 24.7)$	24.7		24.8
$\frac{3}{4}$	$(25.2, 25.3)$	25.5	$(25.3, 25.5)$	25.4		25.8
1	$(26.1, 26.3)$	26.1	$(26.2, 26.5)$	26.3	$(26.4, 26.6)$	26.7

TABLE 1. Estimates of intervals containing δ based on the Aitken-transformed sequences (I), and estimates of δ from least square fits of exponentials (II).

Remark. Comparing (2.2) with the bounds in Result 1 and Table 1 we see that the flat tops we are considering in this paper actually have a smaller δ than maps with a high-order polynomial maximum. This is somewhat counterintuitive, since δ appears to increase with respect to flatness within the class of analytic maps and within the class of flat tops.

2.3. On Method and Accuracy

Our approach is as follows. We fix q. Then, for k fixed, we find the parameter value $a_0 = a_{0,k}$ for which $\Phi_{a,q}$ has a superstable periodic orbit of length k, with kneading sequence $\mathrm{RL}^{k-2}\mathrm{C}$. Then, recursively, we find the parameter value $a_n = a_{n,k}$ corresponding to the superstable orbit that is created after one period doubling of the orbit at a_{n-1}. We have performed calculations for $q = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and $k = 2, 3, 4, 5, 6$.

We use a bisection search algorithm, with 19-digit floating point arithmetic, and accept $a_{n,k}$ if $a = a_{n,k}$ locally minimizes $|\Phi_{a,q}^{2^n k-1}(Q)|$ on the resolution grid in the a-variable. For $q = \frac{1}{2}$ and $k = 2$, which computationally seems to be the easiest case, we have found reasonable values a_n for $n = 0, 1, \ldots, 13$, and $a_3, a_4, \ldots, a_8$ were compared with high-precision Mathematica calculations with 300 working digits. The numbers agree to 17 decimal places. For $q = 1$ and $k = 6$, the hardest case considered, we found reasonable values a_n for $n = 0, 1, \ldots, 9$.

We now compute the numbers

$$\delta_n = \delta_{n,k} = \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}.$$

We also define and compute

$$\Delta_n = \Delta_{n,k} = \frac{\delta_{n-1} - \delta_{n-2}}{\delta_n - \delta_{n-1}}.$$

Our approach to accuracy is as follows: We accept the values a_n as they are given by this algorithm. If you like, *they* are the objective of our study. This is not unreasonable in view of the high-precision results mentioned above, and the general experience that this kind of computation works well in dynamical systems numerics. Since δ_n is the quotient of the differences of successive values of a_n, roundoff errors of size 10^{-17} in a_n produce increasing absolute errors in δ_n as $n \to \infty$. Anyhow, we get δ_n to at least two decimal places for the cases considered.

n	a_n	δ_n	Δ_n
0	1		
1	1.35341891		
2	1.42786314	4.74	
3	1.43971881694707601	6.27	
4	1.44129590698672292	7.51	0.808
5	1.44147686837076865	8.71	0.967
6	1.44149522114131208	9.86	0.956
7	1.44149689902037341	10.93	0.941
8	1.44149703947566102	11.94	0.935
9	1.44149705037694717	12.88	0.931
10	1.44149705116945322	13.75	0.928
11	1.44149705122387325	14.56	0.927
12	1.44149705122742775	15.31	0.926
13	1.44149705122764988	16.00	0.925
14	1.44149705122766323	16.64	0.925

TABLE 2. Values a_n, δ_n and Δ_n for $q = \frac{1}{2}$ (recall that a_n is the parameter value for which there is a superstable cycle of order 2^{n+1}). We give fewer decimals for a_1 and a_2 since they were used as input in our main algorithm, and thus had to be computed with cruder methods. The exact value $a_0 = 1$ is obtained by hand.

Table 2 lists a_n, δ_n and Δ_n for $k = 2$ and $q = \frac{1}{2}$. The values of Δ_n suggest that $\delta_n \sim A + Be^{-Cn}$.

(We get exactly the same type of behavior for other values on q and k. Numerical difficulties increase in k, and as for q, the best-behaved cases seem to be $q = \frac{1}{2}$ and $q = \frac{3}{4}$; for q=$\frac{1}{4}$ computations are easier but convergence is slower, and for $q = 1$ convergence is faster but numerics is harder.)

The asymptotics $\delta_n \sim A + Be^{-Cn}$ indicate that the sequences $\{\delta_n\}$ converge, and for a sequence like this, converging with an approximate exponential fall-off, one can speed up convergence by applying *Aitken's δ^2-process* [Flannery et al. 1986, p. 132–133]. One considers the transformation S, also known as Shanks' transformation, mapping a sequence $\{b_n\}_{n=0}^{N}$ to the sequence $\{c_n\}_{n=1}^{N-1}$, where

$$c_n = \frac{b_{n+1}b_{n-1} - b_n^2}{b_{n+1} + b_{n-1} - 2b_n}.$$

Clearly S maps $\{A + Be^{-Cn}\}$ to the constant sequence $\{A, A, A, \ldots\}$, so it is reasonable to think that $\{\sigma_n\} := S(\{\delta_n\})$ converges, and that this limit coincides with $\lim \delta_n$. We also apply S once more and write $\{\hat{\sigma}_n\} := S(\{\sigma_n\})$. We obtain the interval estimates of Table 1 by inspection of the sequences $\{\sigma_n\}$ and $\{\hat{\sigma}_n\}$. For $q = \frac{1}{2}$ these sequences are listed in Table 3. (There are also generalized Shanks' transformations designed to take away several exponential transients in a numerically robust way [Bender et al. 1978]. In our case they do not yield any improvement in convergence.)

There is another approach as well. One can do a least square fit of $A+Be^{-Cn}$ to the sequences $\{\delta_n\}$, and read off $A = \lim_n \delta_n$. This gives small total square error ($\sim 10^{-4}$), and fairly good agreement with estimates based on the Aitken process: see Table 1.

The discrepancies between these two estimates, and the differences for fixed q but different k, may both be due to the influence of transient behavior and numerical errors.

3. THE NUMBER α

3.1. Background

Let f_a be a unimodal family, and let $\{a_n\}$ be a sequence of parameter values belonging to the same periodic window in parameter space, such that f_{a_n} has a superstable periodic orbit of length $p_n = 2^n k$, just as in Section 2.1. Then, for $n > 0$, define

$$\xi_n = f_{a_n}^{p_n/2}(0).$$

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
3	12.74	11.28	11.65	12.11	12.54
4	44.04	21.02	27.30	−118.5	3.67
5	34.83	25.22	26.99	30.92	39.13
6	28.23	26.16	26.32	27.15	28.87
7	26.46	26.10	26.60	27.43	28.55
8	25.53	24.94	24.72	24.69	24.79
9	25.06	24.91	25.02	25.24	25.56
10	24.77	24.59	24.55	24.49	23.68
11	24.64	24.65	24.64		
12	24.59	24.39			
13	24.48				
14	24.48				
7	24.47	26.16	26.35	27.17	28.90
8	24.62	24.91	24.98	25.14	25.42
9	24.22	24.94	24.83	24.92	25.01
10	24.53	24.64	24.62	25.04	24.80
11	24.57	24.60	24.55	24.94	
12	24.66	24.67			
13	24.59				

TABLE 3. The S-transformed sequences $\sigma_{n,k}$ (top) and the twice S-transformed sequences $\hat{\sigma}_{n,k}$ for $q = \frac{1}{2}$.

Thus ξ_n is halfway around the critical orbit, and is the point on the critical orbit closest to the critical point. Then study the limits

$$\alpha = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\xi_{n+1} - \xi_n}{\xi_{n+2} - \xi_{n+1}}.$$

For nonflat families, $|f_a(x) - A| \sim |x|^r$ in some neighborhood of the critical point, we have the following facts (compare Section 2.1):

- Numerically these limits exist and α is a constant depending only on r.
- $\alpha = A/g(A)$, where g is the fixed point of the operator T_r defined in Section 2.1.
- These are rigorous results in the cases $r = 2$ and $r = 1 + \varepsilon$.

3.2. Numerical results

Let $a_n = a_{n,k}$ be defined as in Section 2.3. Then define

$$\xi_n = \xi_{n,k} = \Phi_{a_n,q}^{2^{n-1}k}(0),$$

and study the limits

$$\alpha = \alpha_k = \lim_{n\to\infty} \alpha_{n,k} = \lim_{n\to\infty} \frac{\xi_{n+1,k} - \xi_{n,k}}{\xi_{n+2,k} - \xi_{n+1,k}}.$$

The sequences $\{\alpha_{n,k}\}_n$ increase monotonically (see Table 4 for two examples), and we have an a priori upper bound $\alpha_{n,k} \leq -1$. For q fixed, the sequences corresponding to $k = 2$, 3, 4, 5 and 6 seem to have the same limiting behavior. This is illustrated for $q = \frac{1}{2}$ in Figure 2, which plots the shifted sequences

$$\Sigma_k = \{\alpha_{n,k}^{1/2}\}_{n=6-k}^{14-k}.$$

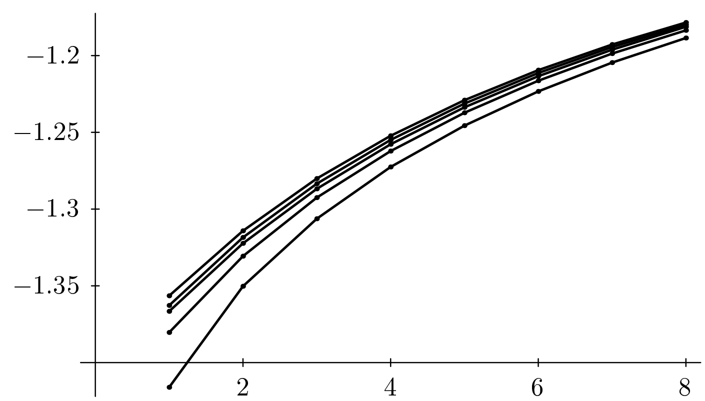

FIGURE 2. Comparing suitable shifts of the sequences $\{\alpha_n\}$ for $k = 2, 3, 4, 5, 6$ in the case $q = \frac{1}{2}$.

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\alpha_{n,2}^{1/2} =$	−1.784	−1.601	−1.497	−1.423	−1.367	−1.322	−1.287	−1.258	−1.234	−1.213	−1.196	−1.181	−1.169
$\alpha_{n,2}^{1} =$	−1.446	−1.333	−1.268	−1.222	−1.188	−1.163	−1.143	−1.127	−1.114	−1.104	−1.095		

TABLE 4. The sequences $\{\alpha_n\}$ for $k = 2$ and $q = \frac{1}{2}, 1$.

Thus we obtain the following experimental result:

Result 2. *The limits* α_k^q *exist and seem to be independent of* k. *For the cases considered*, $q = 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}$, *we have* $-1.15 \leq \alpha \leq -1$.

The fact that α seems to be independent of k supports the idea that δ is also independent of k.

4. A GENERAL CONJECTURE

We think that the following scenario is possible: Consider two unimodal families $\mathcal{F}$ and $\mathcal{G}$ consisting of the maps $F_a(x) = 1 - af(x)$ and $G_a = 1 - ag(x)$ respectively. Say that $\mathcal{F}$ and $\mathcal{G}$ are *flatness-equivalent*, written $\mathcal{F} \sim \mathcal{G}$, if for all $n > 0$ the following holds: If

$$\frac{d^k}{dx^k} f(0) = \frac{d^k}{dx^k} g(0) = 0$$

for all $k < n$, there exists a constant $C > 0$ such that

$$\frac{1}{C} < \liminf_{x \to 0} \left| \frac{(d^n f/dx^n)(x)}{(d^n g/dx^n)(x)} \right| \leq \limsup_{x \to 0} \left| \frac{(d^n f/dx^n)(x)}{(d^n g/dx^n)(x)} \right| < C.$$

Clearly $\sim$ is an equivalence relation. If $\mathcal{F}$ and $\mathcal{G}$ have polynomial top, $\mathcal{F} \sim \mathcal{G}$ just means that they are of the same polynomial order. For the maps $\Phi_{a,q}$ considered in this paper, it means equality in q. Furthermore, under this definition, the family consisting of the k-th iterate of F_a with a varying over a k period window is flatness-equivalent to the family generated by F_a, so long as f fulfills some mild regularity condition (for instance,

$$\lim_{\varepsilon \to 0} \frac{(d^k f/dx^k)^2(\varepsilon)}{(d^{k+j} f/dx^{k+j})(\varepsilon)} = 0$$

for all $k, j > 0$.) In particular, flatness equivalence classes are invariant under the doubling operator T.

Generalizing from the nonflat case, we conjecture that, for each family $\mathcal{F}$ there exist a constant $\delta = \delta(\mathcal{F})$, and $\mathcal{F} \sim \mathcal{G}$ implies $\delta(\mathcal{F}) = \delta(\mathcal{G})$. The number $\delta(\mathcal{F})$ can be interpreted as the (dominating) expanding eigenvalue of DT at the fixed point of T, operating on class($\mathcal{F}$).

At present we have no idea of how to do renormalization theory for these flat families, so we cannot give theoretical support to the results and the speculations of this paper. In the absence of theory, more extensive high precision calculations would be of interest.

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