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Twisted Alexander Polynomials of Hyperbolic Knots

Nathan M. Dunfield, Stefan Friedl, and Nicholas Jackson

CONTENTS

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We study a twisted Alexander polynomial naturally associated to a hyperbolic knot in an integer homology 3-sphere via a lift of the holonomy representation to $\mathrm{SL}(2,\mathbb{C})$. It is an unambiguous symmetric Laurent polynomial whose coefficients lie in the trace field of the knot. It contains information about genus, fibering, and chirality, and moreover, is powerful enough to sometimes detect mutation.

We calculated this invariant numerically for all 313 209 hyperbolic knots in S^3 with at most 15 crossings, and found that in all cases it gave a sharp bound on the genus of the knot and determined both fibering and chirality.

We also study how such twisted Alexander polynomials vary as one moves around in an irreducible component X_0 of the $\mathrm{SL}(2,\mathbb{C})$-character variety of the knot group. We show how to understand all of these polynomials at once in terms of a polynomial whose coefficients lie in the function field of X_0. We use this to help explain some of the patterns observed for knots in S^3, and explore a potential relationship between this universal polynomial and the Culler–Shalen theory of surfaces associated to ideal points.

1. INTRODUCTION

A fundamental invariant of a knot K in an integral homology 3-sphere Y is its Alexander polynomial Δ_K. While Δ_K contains information about genus and fibering, it is determined by the maximal metabelian quotient of the fundamental group of the complement $M = Y \setminus K$, and so this topological information has clear limits. In 1990, Lin introduced the *twisted Alexander polynomial* associated to K and a representation $\alpha : \pi_1(M) \to \mathrm{GL}(n,\mathbb{F})$, where $\mathbb{F}$ is a field. These twisted Alexander polynomials also contain information about genus and fibering and have been studied by many authors (see the survey [Friedl and Vidussi 10]). Much of this work has focused on those α that factor through a finite quotient of $\pi_1(M)$, which is closely related to studying the ordinary Alexander polynomial in finite covers of M. In contrast, we study here a twisted Alexander polynomial associated to a representation coming from hyperbolic geometry.

Suppose that K is *hyperbolic*, i.e., the complement M has a complete hyperbolic metric of finite volume, and consider the associated holonomy representation $\bar{\alpha} : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$. Since $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$, there are two simple ways to get a linear representation so that we can consider the twisted Alexander polynomial: compose $\bar{\alpha}$ with the adjoint representation to get $\pi_1(M) \to \mathrm{Aut}(\mathfrak{sl}_2\mathbb{C}) \leq \mathrm{SL}(3, \mathbb{C})$, or alternatively lift $\bar{\alpha}$ to a representation $\pi_1(M) \to \mathrm{SL}(2, \mathbb{C})$. The former approach is the focus of the recent paper [Dubois and Yamaguchi 09]; the latter method is what we use here to define an invariant $\mathcal{T}_K(t) \in \mathbb{C}[t^{\pm 1}]$, called the *hyperbolic torsion polynomial*.

The hyperbolic torsion polynomial $\mathcal{T}_K$ is surprisingly little studied. To our knowledge, it has previously been looked at only for 2-bridge knots, in [Morifuji 08, Kim and Morifuji 10, Hirasawa and Murasugi 08, Silver and Williams 09]. Here we show that it contains a great deal of topological information. In fact, we show that $\mathcal{T}_K$ determines genus and fibering for all 313 209 hyperbolic knots in S^3 with at most 15 crossings, and we conjecture this to be the case for all knots in S^3.

1.1. Basic Properties

More broadly, we consider here knots in $\mathbb{Z}_2$-homology 3-spheres. The ambient manifold Y containing the knot K will always be oriented, not just be orientable, and $\mathcal{T}_K$ depends on that orientation. Following Turaev, we formulate $\mathcal{T}_K$ as a Reidemeister–Milnor torsion, since this reduces its ambiguity; in that setting, we work with the compact core of M, namely the knot exterior $X := Y \setminus \mathrm{int}(N(K))$ (see Section 2 for details). By fixing certain conventions for lifting the holonomy representation $\bar{\alpha} : \pi_1(X) \to \mathrm{PSL}(2, \mathbb{C})$ to $\alpha : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$, we construct in Section 4 a well-defined symmetric polynomial $\mathcal{T}_K \in \mathbb{C}[t^{\pm 1}]$. The first theorem summarizes its basic properties:

Theorem 1.1. *Let K be a hyperbolic knot in an oriented $\mathbb{Z}_2$-homology 3-sphere. Then $\mathcal{T}_K$ has the following properties:*

(a) *$\mathcal{T}_K$ is an unambiguous element of $\mathbb{C}[t^{\pm 1}]$ that satisfies $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$. It does not depend on an orientation of K.*

(b) *The coefficients of $\mathcal{T}_K$ lie in the trace field of K. If K has integral traces, the coefficients of $\mathcal{T}_K$ are algebraic integers.*

(c) *$\mathcal{T}_K(\xi)$ is nonzero for every root of unity ξ. In particular, $\mathcal{T}_K \neq 0$.*

(d) *If K^* denotes the mirror image of K, then $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}}_K(t)$, where the coefficients of the latter polynomial are the complex conjugates of those of $\mathcal{T}_K$.*

(e) *If K is amphichiral, then $\mathcal{T}_K$ is a real polynomial.*

(f) *The values $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are mutation-invariant.*

Moreover, $\mathcal{T}_K$ both determines and is determined by a sequence of $\mathbb{C}$-valued torsions of finite cyclic covers of X. Specifically, let X_m be the m-fold cyclic cover coming from the free abelianization of $H_1(X; \mathbb{Z})$. For the restriction α_m of α to $\pi_1(X_m)$, we consider the corresponding $\mathbb{C}$-valued torsion $\tau(X_m, \alpha_m)$. A standard argument shows that $\mathcal{T}_K$ determines all the $\tau(X_m, \alpha_m)$ (see Theorem 3.1). More interestingly, the converse holds: the torsions $\tau(X_m, \alpha_m)$ determine $\mathcal{T}_K$ (see Theorem 4.3). This latter result follows from work in [Fried 86] (see also [Hillar 05]) and [Menal-Ferrer and Porti 10].

Remark 1.2. Conjecturally, the torsions $\tau(X_m, \alpha_m)$ can be expressed as analytic torsions and as Ruelle zeta functions defined using the lengths of prime geodesics. [Ray and Singer 71, Cheeger 77, Cheeger 79, Müller 78, Park 09] for details and background material. We hope that this point of view will be helpful in the further study of $\mathcal{T}_K$.

The torsions $\tau(X_m, \alpha_m)$ are interesting invariants in their own right. For example, it is shown in [Menal-Ferrer and Porti 10] that $\tau(X_m, \alpha_m)$ is nonzero for every m. Furthermore, J. Porti showed us in a private communication (2009) that $\tau(X_1, \alpha_1) = \tau(X, \alpha) = \mathcal{T}_K(1)$ is not obviously related to hyperbolic volume. More precisely, using a variation on [Porti 97, Théorème 4.17], one can show that there exists a sequence of knots K_n whose volumes converge to a positive real number, but the numbers $\mathcal{T}_{K_n}(1)$ converge to zero. See [Porti 97, Menal-Ferrer and Porti 10, Menal-Ferrer and Porti 11] for further results.

1.2. Topological Information: Genus and Fibering

We define $x(K)$ to be the Thurston norm of a generator of $H_2(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$; if K is null-homologous in Y, then $x(K) = 2 \cdot \mathrm{genus}(K) - 1$, where $\mathrm{genus}(K)$ is the least genus of all Seifert surfaces bounding K. Also, we say that K is *fibered* if X fibers over the circle.

A key property of the ordinary Alexander polynomial Δ_K is that

$$x(K) \geq \deg(\Delta_K) - 1.$$

When K is fibered, this is an equality, and moreover, the leading coefficient of Δ_K is 1 (here we normalize Δ_K so that the leading coefficient is positive). As with any twisted Alexander/torsion polynomial, we get similar information out of $\mathcal{T}_K$:

Theorem 1.3. *Let K be a knot in an oriented $\mathbb{Z}_2$-homology sphere. Then*

$$x(K) \geq \frac{1}{2}\deg(\mathcal{T}_K).$$

If K is fibered, this is an equality and $\mathcal{T}_K$ is monic, i.e., has leading coefficient 1.

Theorem 1.3 is an immediate consequence of the definitions below and of [Friedl and Kim 06, Theorem 1.1] (for the genus bound) and of [Goda et al. 05] (for the fibered case); see also [Cha 03, Kitano and Morifuji 05, Pajitnov 07, Kitayama 09, Friedl and Kim 06] and [Friedl and Vidussi 10, Theorem 6.2].

1.3. Experimental Results

The calculations in [Cha 03, Goda et al. 05, Friedl and Kim 06]gave evidence that when one can freely choose the representation α, the twisted torsion polynomial is very successful at detecting both $x(K)$ and nonfibered knots. Moreover, [Friedl and Vidussi 01] shows that collectively, the twisted torsion polynomials of representations coming from homomorphisms to finite groups determine whether a knot is fibered. However, it is not known whether all twisted torsion polynomials together always detect $x(K)$.

Instead of considering many different representations as in the work just discussed, we focus here on a single, albeit canonical, representation. Despite this, we find that $\mathcal{T}_K$ alone is a very powerful invariant. In Section 6, we describe computations showing that the bound on $x(K)$ is sharp for all 313 209 hyperbolic knots with at most 15 crossings; in contrast, the bound from Δ_K is not sharp for 2.8% of these knots. Moreover, among such knots, $\mathcal{T}_K$ was monic only when the knot was fibered, whereas 4.0% of these knots have monic Δ_K but are not fibered. (Here we computed $\mathcal{T}_K$ numerically to a precision of 250 decimal places; see Section 6.7 for details.)

Given all this data, we are compelled to propose the following conjecture, even though on its face it feels quite implausible, given the general squishy nature of Alexander-type polynomials.

Conjecture 1.4. *For a hyperbolic knot K in S^3, the hyperbolic torsion polynomial $\mathcal{T}_K$ determines $x(K)$, or equivalently its genus. Moreover, the knot K is fibered if and only if $\mathcal{T}_K$ is monic.*

We have not done extensive experiments for knots in manifolds other than S^3, but so far, we have not encountered any examples in which $\mathcal{T}_K$ does not contain perfect genus and fibering information.

1.4. Topological Information: Chirality and Mutation

When K is amphichiral, $\mathcal{T}_K$ is a real polynomial (Theorem 1.1(e)). This turns out to be an excellent way to detect chirality. Indeed, among hyperbolic knots in S^3 with at most 15 crossings, the 353 knots for which $\mathcal{T}_K$ is real are exactly the amphichiral knots (Section 6.3).

Also, hyperbolic invariants often do not detect mutation, for example the volume [Ruberman 87], the invariant trace field [Maclachlan and Reid 03, Corollary 5.6.2], and the birationality type of the geometric component of the character variety [Cooper and Long 96, Tillmann 00, Tillmann 04]. The ordinary Alexander polynomial Δ_K is also mutation-invariant for knots in S^3. However, $x(K)$ can change under mutation, and given that $x(K)$ determines the degree of $\mathcal{T}_K$ for all 15 crossing knots, it follows that $\mathcal{T}_K$ can change under mutation; we discuss many such examples in Section 6.4. However, sometimes mutation does preserve $\mathcal{T}_K$, and we do not know of any examples of two knots with the same $\mathcal{T}_K$ that are not mutants.

1.5. Adjoint Torsion Polynomial

As we mentioned earlier, there is another natural way to obtain a torsion polynomial from the holonomy $\bar{\alpha} : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, namely by considering the adjoint representation of $\mathrm{PSL}(2,\mathbb{C})$ on its Lie algebra. The corresponding torsion polynomial was studied in [Dubois and Yamaguchi 09], partly building on [Porti 97]. We refer to this invariant here as the *adjoint torsion polynomial* and denote it by $\mathcal{T}_K^{\mathrm{adj}}$. We also numerically calculated this invariant for all knots with at most 15 crossings. In contrast to what we found for $\mathcal{T}_K$, the degree of $\mathcal{T}_K^{\mathrm{adj}}$ was not determined by the genus for 8 252 of these knots. Moreover, we found 12 knots for which the genus bound from $\mathcal{T}_K^{\mathrm{adj}}$ was not sharp even after accounting for the fact that $x(K)$ is necessarily an odd integer. The differing behaviors of these two polynomials seems very mysterious to us; understanding what is behind it might shed light on Conjecture 1.4. See Sections 5.1 and 6.6 for the details on what we found for $\mathcal{T}_K^{\mathrm{adj}}$.

1.6. Character Varieties

So far, we have focused on the twisted torsion polynomial of (a lift of) the holonomy representation of the hyperbolic structure on M. However, this representation is always part of a complex curve of representations $\pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$, and it is natural to study how the torsion polynomial changes as we vary the representation. In Sections 7 and 8, we describe how to understand all of these torsion polynomials at once, and use this to help explain some of the patterns observed in Section 6. For the special case of 2-bridge knots, how the torsion polynomial varies with the representation had previously been studied in [Morifuji 08, Kim and Morifuji 10], and our results here extend some of that work to more general knots.

Consider the character variety

$$X(K) := \mathrm{Hom}\big(\pi_1(M), \mathrm{SL}(2,\mathbb{C})\big) /\!/ \mathrm{SL}(2,\mathbb{C}),$$

which is an affine algebraic variety over $\mathbb{C}$. We show in Section 7 that each $\chi \in X(K)$ has an associated torsion polynomial $\mathcal{T}_K^{\chi}$. These $\mathcal{T}_K^{\chi}$ vary in an understandable way, in terms of a polynomial with coefficients in the ring of regular functions $\mathbb{C}[X_0]$:

Theorem 1.5. *Let X_0 be an irreducible component of $X(K)$ that contains the character of an irreducible representation. There is a unique $\mathcal{T}_K^{X_0} \in \mathbb{C}[X_0][t^{\pm 1}]$ such that for all $\chi \in X_0$, one has $\mathcal{T}_K^{\chi}(t) = \mathcal{T}_K^{X_0}(\chi)(t)$. Moreover, $\mathcal{T}_K^{X_0}$ is itself the torsion polynomial of a certain representation $\pi_1(M) \to \mathrm{SL}(2,\mathbb{F})$, and thus has all the usual properties (symmetry, genus bound, etc.).*

Corollary 1.6. *Let K be a knot in an integral homology 3-sphere. Then the following hold:*

(a) *The set $\{\chi \in X(K) \mid \deg(\mathcal{T}_K^{\chi}) = 2x(K)\}$ is Zariski open.*

(b) *The set $\{\chi \in X(K) \mid \mathcal{T}_K^{\chi}$ is monic$\}$ is Zariski closed.*

It is natural to focus on the component X_0 of $X(K)$ that contains the (lift of) the holonomy representation of the hyperbolic structure, which we call the *distinguished component*. In this case, X_0 is an algebraic curve, and we show that the following conjecture is implied by Conjecture 1.4.

Conjecture 1.7. *Let K be a hyperbolic knot in S^3, and X_0 the distinguished component of its character variety. Then $2x(K) = \deg(\mathcal{T}_K^{X_0})$, and $\mathcal{T}_K^{X_0}$ is monic if and only if K is fibered.*

At the very least, Conjecture 1.7 is true for many 2-bridge knots, as we discuss in Section 7.2. We also give several explicit examples of $\mathcal{T}_K^{X_0}$ in Section 8 and use these to explore a possible avenue for bringing the Culler–Shalen theory of surfaces associated to ideal points of $X(K)$ to bear on Conjecture 1.7.

1.7. Other Remarks and Open Problems

For simplicity, we have restricted ourselves here to the study of knots, especially those in S^3. However, we expect that many of the results and conjectures are valid for more general 3-manifolds. In the broader settings, the appropriate question is whether the twisted torsion polynomial detects the Thurston norm and fibered classes (see [Friedl and Kim 06, Friedl and Kim 08, Friedl and Vidussi 08] for more details).

We conclude this introduction with a few questions and open problems:

1. Does $\mathcal{T}_K$ determine the volume of the complement of K? Some calculational evidence is given in [Friedl and Jackson 11] and in Section 6.4 in this paper.
2. If two knots in S^3 have the same $\mathcal{T}_K$, are they necessarily mutants? See Section 6.4 for more on this.
3. Does the invariant $\mathcal{T}_K$ contain information about symmetries of the knot besides information on chirality?
4. Does there exist a hyperbolic knot with $\mathcal{T}_K(1) = 1$?
5. If $\mathcal{T}_K$ is a real polynomial, is K necessarily amphichiral?
6. For an amphichiral knot, is the top coefficient of $\mathcal{T}_K$ always positive?
7. For fibered knots, why is the *second* coefficient of $\mathcal{T}_K$ so often real? This coefficient is the sum of the eigenvalues of the monodromy acting on the twisted homology of the fiber. See Section 6.5 for more.
8. Why is $|\mathcal{T}_K(-1)| > |\mathcal{T}_K(1)|$ for 99.99% of the knots considered in Section 6.5?

2. TWISTED INVARIANTS OF 3-MANIFOLDS

In this section, we review torsions of twisted homology groups and explain how they are used to define the twisted torsion polynomial of a knot together with a representation of its fundamental group to $\mathrm{SL}(2,\mathbb{C})$.

We then summarize the basic properties of these torsion polynomials, including how to calculate them.

2.1. Torsion of Based Chain Complexes

Let C_* be a finite chain complex over a field $\mathbb{F}$. Suppose that each chain group C_i is equipped with an ordered basis c_i and that each homology group $H_i(C_*)$ is also equipped with an ordered basis h_i. Then there is an associated *torsion invariant* $\tau(C_*, c_*, h_*) \in \mathbb{F}^\times := \mathbb{F} \setminus \{0\}$, as described in the various excellent expositions [Milnor 66, Turaev 01, Turaev 02, Nicolaescu 03]. We will follow the convention of Turaev, which is the multiplicative inverse of Milnor's invariant. If the complex C_* is acyclic, then we will write $\tau(C_*, c_*) := \tau(C_*, c_*, \varnothing)$.

2.2. Twisted Homology

For the rest of this section, fix a finite CW complex X and set $\pi := \pi_1(X)$. Consider a representation $\alpha : \pi \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space over $\mathbb{F}$. We can thus view V as a left $\mathbb{Z}[\pi]$-module. To define the twisted homology groups $H_*^\alpha(X; V)$, consider the universal cover $\widetilde{X}$ of X. Regarding π as the group of deck transformations of $\widetilde{X}$ turns the cellular chain complex $C_*(\widetilde{X}) := C_*(\widetilde{X}; \mathbb{Z})$ into a left $\mathbb{Z}[\pi]$-module. We then give $C_*(\widetilde{X})$ a right $\mathbb{Z}[\pi]$-module structure via $c \cdot g := g^{-1} \cdot c$ for $c \in C_*(\widetilde{X})$ and $g \in \pi$, which allows us to consider the tensor product

$$C_*^\alpha(X; V) := C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V.$$

Now $C_*^\alpha(X; V)$ is a finite chain complex of vector spaces, and we define $H_*^\alpha(X; V)$ to be its homology.

We call two representations $\alpha : \pi \to \mathrm{GL}(V)$ and $\beta : \pi \to \mathrm{GL}(W)$ *conjugate* if there exists an isomorphism $\Psi : V \to W$ such that $\alpha(g) = \Psi^{-1} \circ \beta(g) \circ \Psi$ for all $g \in \pi$. Note that such a Ψ induces an isomorphism of $H_*^\alpha(X; V)$ with $H_*^\beta(X; W)$.

2.3. Euler Structures, Homology Orientations, and Twisted Torsion of CW Complexes

To define the twisted torsion, we first need to introduce certain additional structures on which it depends. (In our final application, most of these will come out in the wash.) First, fix an orientation of each cell of X. Then choose an ordering of the cells of X so that we can enumerate them as c_j; only the relative order of cells of the same dimension will be relevant, but it is notationally convenient to have only one subscript.

An *Euler lift* for X associates to each cell c_j of X a cell $\tilde{c}_j$ of $\widetilde{X}$ that covers it. If $\tilde{c}'_j$ is another Euler lift, then there are unique $g_j \in \pi$ such that $\tilde{c}'_j = g_j \cdot \tilde{c}_j$. We say that these two Euler lifts are *equivalent* if

$$\prod_j g_j^{(-1)^{\dim(c_j)}}$$

represents the trivial element in $H_1(X; \mathbb{Z})$. An equivalence class of Euler lifts is called an *Euler structure* on X. The set of Euler structures on X, denoted by $\mathrm{Eul}(X)$, admits a canonical free transitive action by $H_1(X; \mathbb{Z})$; see [Turaev 90, Turaev 01, Turaev 02, Friedl et al. 11] for more on these Euler structures. Finally, a *homology orientation* for X is just an orientation of the $\mathbb{R}$-vector space $H_*(X; \mathbb{R})$.

Now we can define the torsion $\tau(X, \alpha, e, \omega)$ associated to X, a representation α, an Euler structure e, and a homology orientation ω. If $H_*^\alpha(X; V) \neq 0$, we define $\tau(X, \alpha, e, \omega) := 0$, and so now assume that $H_*^\alpha(X; V) = 0$. Up to sign, the torsion we seek will be that of the twisted cellular chain complex $C_*^\alpha(X; V)$ with respect to the following ordered basis. Let $\{v_k\}$ be any basis of V, and $\{\tilde{c}_j\}$ any Euler lift representing e. Order the basis $\{\tilde{c}_j \otimes v_k\}$ for $C_*^\alpha(X; V)$ lexicographically, i.e., $\tilde{c}_j \otimes v_k < \tilde{c}_{j'} \otimes v_{k'}$ if either $j < j'$ or both $j = j'$ and $k < k'$. We thus have a based acyclic complex $C_*^\alpha(X; V)$, and we can consider

$$\tau(C_*^\alpha(X; V), c_* \otimes v_*) \in \mathbb{F}^\times.$$

When $\dim(V)$ is even, this torsion is in fact independent of all the choices involved, but when $\dim(V)$ is odd, we need to augment it as follows to remove a sign ambiguity.

Pick an ordered basis h_i for $H_*(X; \mathbb{R})$ representing our homology orientation ω. Since we have ordered the cells of X, we can consider the torsion

$$\tau\big(C_*(X; \mathbb{R}), c_*, h_*\big) \in \mathbb{R}^\times.$$

We define $\beta_i(X) = \sum_{k=0}^{i} \dim\big(H_k(X; \mathbb{R})\big)$ and $\gamma_i(X) = \sum_{k=0}^{i} \dim\big(C_k(X; \mathbb{R})\big)$, and then set $N(X) = \sum_i \beta_i(X) \cdot \gamma_i(X)$. Following [Friedl et al. 11], which generalizes the ideas in [Turaev 86, Turaev 90], we now define

$$\begin{aligned} &\tau(X, \alpha, e, \omega) \\ &\quad := (-1)^{N(X)\cdot\dim(V)} \cdot \tau\big(C_*^\alpha(X; V), c_* \otimes v_*\big) \\ &\qquad \cdot \operatorname{sign}\Big(\tau\big(C_*(X; \mathbb{R}), c_*, h_*\big)\Big)^{\dim(V)}. \end{aligned}$$

A straightforward calculation using the basic properties of torsion shows that this invariant does not depend on any of the choices involved, i.e., it is independent of the ordering and orientation of the cells of X, the choice of representatives for e and ω, and the particular basis for V. Similar elementary arguments prove the following lemma. Here $-\omega$ denotes the opposite homology orientation to ω, and note that $(\det \circ \alpha) : \pi \to \mathbb{F}$ factors through $H_1(X;\mathbb{Z})$.

Lemma 2.1. *If β is conjugate to α, then given $h \in H_1(X;\mathbb{Z})$ and $\epsilon \in \{-1, 1\}$, one has*

$$\tau(X, \beta, h \cdot e, \epsilon \cdot \omega) = \epsilon^{\dim(V)} \cdot \big((\det \circ \alpha)(h)\big)^{-1} \cdot \tau(X, \alpha, e, \omega).$$

2.4. Twisted Torsion of 3-Manifolds

Let N be a 3-manifold whose boundary is empty or consists of tori. We first recall some facts about Spin^c-structures on N; see [Turaev 02, Section XI.1] for details. The set $\mathrm{Spin}^c(N)$ of such structures admits a free and transitive action by $H_1(N;\mathbb{Z})$. Moreover, there exists a map $c_1 : \mathrm{Spin}^c(N) \to H_1(N;\mathbb{Z})$ that has the property that $c_1(h \cdot \mathfrak{s}) = 2h + c_1(\mathfrak{s})$ for every $h \in H_1(N;\mathbb{Z})$ and $\mathfrak{s} \in \mathrm{Spin}^c(N)$.

Now consider a triangulation X of N. By [Turaev 02, Section XI], there exists a canonical bijection $\mathrm{Spin}^c(N) \to \mathrm{Eul}(X)$ that is equivariant with respect to the actions by $H_1(N;\mathbb{Z}) = H_1(X;\mathbb{Z})$. Given a representation $\alpha : \pi_1(N) \to \mathrm{GL}(V)$, an element $\mathfrak{s} \in \mathrm{Spin}^c(N)$, and a homology orientation ω for N, we define

$$\tau(N, \alpha, \mathfrak{s}, \omega) := \tau(X, \alpha, e, \omega),$$

where e is the Euler structure on X corresponding to $\mathfrak{s}$. It follows from [Turaev 86, Turaev 90] that $\tau(N, \alpha, \mathfrak{s}, \omega)$ is independent of the choice of triangulation and hence is well defined. See also [Friedl et al. 11] for more details about $\tau(N, \alpha, \mathfrak{s}, \omega)$.

2.5. Twisted Torsion Polynomial of a Knot

Let K be a knot in a rational homology 3-sphere Y. Throughout, we write $X_K := Y \setminus \mathrm{int}(N(K))$ for the knot exterior, which is a compact manifold with torus boundary. We define an *orientation* of K to be a choice of oriented meridian μ_K; if Y is oriented, instead of just orientable, this is equivalent to the usual notion. Suppose now that K is oriented. Let $\pi_K := \pi_1(X_K)$ and take $\phi_K : \pi_K \to \mathbb{Z}$ to be the unique epimorphism such that $\phi(\mu_K) > 0$. There is a canonical homology orientation ω_K for X_K as follows: take a point as a basis for $H_0(X_K;\mathbb{R})$ and take $\{\mu_K\}$ as the basis for $H_1(X_K;\mathbb{R})$. We will drop K from such notation if the knot is understood from the context.

For a representation $\alpha : \pi \to \mathrm{GL}(n, R)$ over a commutative domain R, we define a torsion polynomial as follows. Consider the left $\mathbb{Z}[\pi]$-module structure on $R^n \otimes_R R[t^{\pm 1}] \cong R[t^{\pm 1}]^n$ given by

$$g \cdot (v \otimes p) := \big(\alpha(g) \cdot v\big) \otimes \big(t^{\phi(g)} p\big)$$

for $g \in \pi$ and $v \otimes p \in R^n \otimes_R R[t^{\pm 1}]$. Put differently, we get a representation $\alpha \otimes \phi : \pi \to \mathrm{GL}(n, R[t^{\pm 1}])$. We denote by $Q(t)$ the field of fractions of $R[t^{\pm 1}]$. The representation $\alpha \otimes \phi$ allows us to view $R[t^{\pm 1}]^n$ and $Q(t)^n$ as left $\mathbb{Z}[\pi]$-modules. Given $\mathfrak{s} \in \mathrm{Spin}^c(X)$, we define

$$\tau(K, \alpha, \mathfrak{s}) := \tau(X_K, \alpha \otimes \phi, \mathfrak{s}, \omega_K) \in Q(t)$$

to be the *twisted torsion polynomial of the oriented knot* K corresponding to the representation α and the Spin^c-structure $\mathfrak{s}$. Calling $\tau(K, \alpha, \mathfrak{s})$ a polynomial even though it is defined as a rational function is reasonable given Theorem 2.4 below.

Remark 2.2. The study of twisted polynomial invariants of knots was introduced in [Lin 01]. The invariant $\tau(K, \alpha, \mathfrak{s})$ can be viewed as a refined version of the twisted Alexander polynomial of a knot and of Wada's invariant. We refer to [Wada 94, Kitano 96, Friedl and Vidussi 10] for more on twisted invariants of knots and 3-manifolds.

2.6. The SL(2, $\mathbb{F}$) Torsion Polynomial of a Knot

Our focus in this paper is on 2-dimensional representations, and we now give a variant of $\tau(K, \alpha, \mathfrak{s})$ that does not depend on $\mathfrak{s}$ or the orientation of K. Specifically, for an *unoriented* knot K in a $\mathbb{Q}$HS and a representation $\alpha : \pi \to \mathrm{SL}(2, \mathbb{F})$, we define

$$\mathcal{T}_K^\alpha := t^{\phi(c_1(\mathfrak{s}))} \cdot \tau(K, \alpha, \mathfrak{s}) \qquad (2\text{–}1)$$

for every $\mathfrak{s} \in \mathrm{Spin}^c(X)$ and choice of oriented meridian μ and prove the following result.

Theorem 2.3. *For $\alpha : \pi \to \mathrm{SL}(2, \mathbb{F})$, the invariant $\mathcal{T}_K^\alpha$ is a well-defined element of $\mathbb{F}(t)$ that is symmetric, i.e., $\mathcal{T}_K^\alpha(t^{-1}) = \mathcal{T}_K^\alpha(t)$.*

We will call $\mathcal{T}_K^\alpha \in \mathbb{F}(t)$ the *twisted torsion polynomial associated to K and α.*

Proof. That the right-hand side of (2–1) is independent of the choice of $\mathfrak{s}$ follows easily from Lemma 2.1 using the observation that $\det((\alpha \otimes \phi)(g)) = t^{2\phi(g)}$ for $g \in \pi$ and the properties of c_1 given in Section 2.4.

The choice of meridian μ affects the right-hand side of (2–1) in two ways: it is used to define the homology orientation ω and the homomorphism $\phi : \pi \to \mathbb{Z}$. The first doesn't matter, by Lemma 2.1, but switching ϕ to $-\phi$ is equivalent to replacing t with t^{-1}. Hence being independent of the choice of meridian is equivalent to the final claim that $\mathcal{T}_K^\alpha$ is symmetric.

Now any $\mathrm{SL}(2,\mathbb{F})$-representation preserves the bilinear form on $\mathbb{F}^2$ given by $(v,w)\mapsto \det(v\,w)$. Using this observation, it is shown in [Friedl et al. 11, Theorem 7.3], generalizing [Hillman et al. 10, Corollary 3.4] and building on work in [Turaev 86, Turaev 90], that in our context, we have

$$\tau(K,\alpha,\mathfrak{s})\left(t^{-1}\right) = t^{2\phi(c_1(\mathfrak{s}))}\cdot \tau(K,\alpha,\mathfrak{s}),$$

which establishes the symmetry $\mathcal{T}_K^\alpha$ and hence the theorem. □

While in general, $\mathcal{T}_K^\alpha$ is a rational function, it is frequently a Laurent polynomial or can be computed in terms of the ordinary Alexander polynomial Δ_K.

Theorem 2.4. *Let K be a knot in a $\mathbb{Q}\,$HS and let $\alpha : \pi_K \to \mathrm{SL}(2,\mathbb{F})$ be a representation.*

(a) *If α is irreducible, then $\mathcal{T}_K^\alpha$ lies in $\mathbb{F}[t^{\pm 1}]$.*

(b) *If α is reducible, then $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$, where β is the diagonal part of α, i.e., a diagonal representation such that $\mathrm{tr}\big(\beta(g)\big) = \mathrm{tr}\big(\alpha(g)\big)$ for all $g\in\pi$.*

(c) *If α is reducible and factors through*

$$H_1(X_K;\mathbb{Z})/(\text{torsion}),$$

then

$$\mathcal{T}_K^\alpha(t) = \frac{\Delta_K(zt)\cdot\Delta_K(z^{-1}t)}{t-(z+z^{-1})+t^{-1}},$$

where z, z^{-1} are the eigenvalues of $\alpha(\mu_K)$, and Δ_K is the symmetrized Alexander polynomial.

When the ambient manifold Y is a $\mathbb{Z}$HS, then the torsion polynomial of any reducible representation α to $\mathrm{SL}(2,\mathbb{F})$ can be computed by combining (b) and (c); when $H_1(Y;\mathbb{Z})$ is finite but nontrivial, then $\mathcal{T}_K^\alpha$ is the product of the torsion polynomials of two 1-dimensional representations, but it may not be directly related to Δ_K.

Proof. Part (a) is due to [Kitano and Morifuji 05] and is seen as follows. Since α is irreducible, there is a $g\in[\pi,\pi]$ such that $\alpha(g)$ does not have trace 2 (see, e.g., [Culler and Shalen 83, Lemma 1.5.1] or the first part of the proof of [Kitano and Morifuji 05, Theorem 1.1]). Then take a presentation of π for which g is a generator and apply Proposition 2.5 below with $x_i = g$; since $\phi(g) = 0$ and $\mathrm{tr}\big(\alpha(g)\big)\neq 2$, the denominator in (2–2) lies in $\mathbb{F}^\times$, and hence $\mathcal{T}_K^\alpha$ is in $\mathbb{F}[t^{\pm1}]$.

For part (b), conjugate α so that it is upper diagonal:

$$\alpha(g) = \begin{pmatrix} a(g) & b(g)\\ 0 & a(g)^{-1}\end{pmatrix} \quad \text{for all } g\in\pi.$$

The diagonal part of α is the representation β given by

$$g\mapsto \begin{pmatrix} a(g) & 0\\ 0 & a(g)^{-1}\end{pmatrix}.$$

It is easy to see, for instance using (2–2), that $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$. Finally, part (c) follows from a straightforward calculation with (2–2); see, e.g., [Turaev 01, Turaev 02]. □

2.7. Calculation of Torsion Polynomials Using Fox Calculus

Suppose we are given a knot K in a $\mathbb{Q}\,$HS and a representation $\alpha:\pi_1(X_K)\to \mathrm{SL}(2,\mathbb{F})$. In this section, we give a simple method for computing $\mathcal{T}_K^\alpha$. As usual, we write $\pi := \pi_1(X_K)$ and $\phi = \phi_K$. We can extend the group homomorphism $\alpha\otimes\phi : \pi\to \mathrm{GL}(2,\mathbb{F}[t^{\pm1}])$ to a ring homomorphism $\mathbb{Z}[\pi]\to M(2,\mathbb{F}[t^{\pm1}])$, which we also denote by $\alpha\otimes\phi$. Given a $k\times l$ matrix $A=(a_{ij})$ over $\mathbb{Z}[\pi]$, we denote by $(\alpha\otimes\phi)(A)$ the $2k\times 2l$ matrix obtained from A by replacing each entry a_{ij} by the 2×2 matrix $(\alpha\otimes\phi)(a_{ij})$.

Now let $F=\langle x_1,\dots,x_n\rangle$ be the free group on n generators. By [Fox 53, Fox 54, Crowell and Fox 63] and also [Harvey 05, Section 6], there exists for each x_i a *Fox derivative*

$$\frac{\partial}{\partial x_i} : F\to\mathbb{Z}[F]$$

with the following two properties:

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij} \quad\text{and}\quad \frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i} \quad \text{for all } u,v\in F.$$

We also need the involution of $\mathbb{Z}[F]$ that sends $g\in F$ to g^{-1} and respects addition (this is not an algebra automorphism, since it induces an antihomomorphism for multiplication). We denote the image of $a\in\mathbb{Z}[F]$ under this map by $\bar a$, and if A is a matrix over $\mathbb{Z}[F]$, then $\overline{A}$ denotes the result of applying this map to each entry.

The following allows for the efficient calculation of $\mathcal{T}_K^\alpha$, since π always has such a presentation (e.g., if K is a knot in S^3, one can use a Wirtinger presentation).

Proposition 2.5. *Let K be a knot in a $\mathbb{Q}\,$HS, and $\langle x_1,\dots,x_n \mid r_1,\dots,r_{n-1}\rangle$ a presentation of π_K of deficiency one. Let A be the $n\times(n-1)$ matrix with entries $a_{ij} = \frac{\partial r_j}{\partial x_i}$. Fix a generator x_i and consider the matrix A_i obtained from A by deleting the ith row. Then* *there exists an $l \in \mathbb{Z}$ such that for every even-dimensional representation $\alpha : \pi \to \mathrm{GL}(V)$, one has*

$$\mathcal{T}_K^\alpha(t) = t^l \cdot \frac{\det\left((\alpha \otimes \phi)\left(\bar{A}_i\right)\right)}{\det\left((\alpha \otimes \phi)\left(\bar{x}_i - 1\right)\right)} \tag{2–2}$$

whenever the denominator is nonzero.

The same formula also holds, up to a sign, when $\dim(V)$ is odd. An easy way to ensure a nonzero denominator in (2–2) is to choose an x_i such that $\phi(x_i) \neq 0$; then $\det\left((\alpha \otimes \phi)\left(\bar{x}_i - 1\right)\right)$ is essentially the characteristic polynomial of $\alpha(x_i)^{-1}$ and hence nonzero.

Remark 2.6. Wada's invariant (see [Wada 94]) is defined to be

$$\frac{\det\left((\alpha \otimes \phi)\left(A_i^t\right)\right)}{\det\left((\alpha \otimes \phi)\left(x_i - 1\right)\right)}.$$

In [Friedl and Vidussi 10, p. 53], it is erroneously claimed that up to multiplication by a power of t, the torsion polynomial $\mathcal{T}_K^\alpha$ agrees with Wada's invariant. Since there seems to be some confusion in the literature regarding the precise relationship between twisted torsion and Wada's invariant, we discuss it in detail in Section 2.8. In that section, we will also see that for representations into $\mathrm{SL}(2, \mathbb{F})$, Wada's invariant does in fact agree with $\mathcal{T}_K^\alpha(t)$. In particular, the invariant studied in [Kim and Morifuji 10] agrees with $\mathcal{T}_K^\alpha(t)$.

Proposition 2.5 is an immediate consequence of the following.

Proposition 2.7. *Let K, π, A be as above. For each generator x_i, there is an $\mathfrak{s} \in \mathrm{Spin}^c(X_K)$ such that for every even-dimensional representation $\beta : \pi \to \mathrm{GL}(V)$, one has*

$$\tau(X_K, \beta, \mathfrak{s}) = \frac{\det\left(\beta(\bar{A}_i)\right)}{\det\left(\beta(\bar{x}_i - 1)\right)} \tag{2–3}$$

whenever the denominator is nonzero.

The homology orientation ω is suppressed in (2–3) because by Lemma 2.1, it doesn't affect τ, since $\dim(V)$ is even.

Proof of Proposition 2.7. Let X be the canonical 2-complex corresponding to the presentation of π, i.e., X has one cell of dimension zero, n cells of dimension one, and $n-1$ cells of dimension two. Since the Whitehead group of π is trivial [Waldhausen 78], it follows that X is simple-homotopy equivalent to every other CW decomposition of X_K; in particular, it is simple-homotopy equivalent to a triangulation. By standard results (see, e.g., [Turaev 01, Section 8]), we can now use X to calculate the torsion of X_K.

Consider the Euler structure e for X that is given by picking an arbitrary lift of the vertex of X to the universal cover $\widetilde{X}$ and then taking the lift of each x_i that starts at this base point. Reading out the words r_j in $x_1, \dots, x_n$ starting at the base point gives a canonical lift for the 2-cells corresponding to the relators. With respect to this basing, the chain complex $C_*(\widetilde{X})$ is isomorphic to the chain complex

$$0 \to \mathbb{Z}[\pi]^{n-1} \xrightarrow{\partial_2} \mathbb{Z}[\pi]^n \xrightarrow{\partial_1} \mathbb{Z}[\pi] \to 0.$$

The bases of $C_2(\widetilde{X})$ and $C_1(\widetilde{X})$ are abusively denoted by $\{r_j\}$ and $\{x_i\}$, and the basis of $C_0(\widetilde{X})$ is the lifted base point b. Thus

$$\partial_2(r_j) = \sum_i \frac{\partial r_j}{\partial x_i} x_i = \sum_i a_{ij} x_i \quad \text{and} \quad \partial_1(x_i) = (x_i - 1)b.$$

Now fix a basis $\{v_k\}$ for V. If we then view elements $v \in V$ as vertical vectors and $\beta(g)$ as a matrix, the left $\mathbb{Z}[\pi]$-module structure on V is given by $g \cdot v = \beta(g)v$. Thus in the complex $C_*^\beta(X; V) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V$, we have

$$\begin{aligned}\partial_2(r_j \otimes v_k) &= \sum_i (a_{ij} x_i \otimes v_k) = \sum_i (x_i \cdot \bar{a}_{ij} \otimes v_k) \\ &= \sum_i (x_i \otimes \bar{a}_{ij} \cdot v_k) = \sum_i (x_i \otimes \beta(\bar{a}_{ij}) v_k).\end{aligned}$$

Thus with the basis-ordering conventions of Section 2.3, the twisted chain complex $C_*^\beta(X; V)$ is given by

$$0 \to V^{n-1} \xrightarrow{\beta(\bar{A})} V^n \xrightarrow{(\beta(\bar{x}_1 - 1), \dots, \beta(\bar{x}_n - 1))} V \to 0, \tag{2–4}$$

where as usual, matrices act on the left of vertical vectors.

From now on, we assume that $\det(\beta(\bar{x}_i - 1)) \neq 0$, since otherwise, there is nothing to prove. First, consider the case that $\det\left(\beta(\bar{A}_i)\right) = 0$. We claim in this case that $C_*^\beta(X; V)$ is not acyclic, and thus (2–3) holds by the definition of τ. Consider any $v \in V^{n-1}$ that is in the kernel of $\bar{A}_i$; because $\beta(\bar{x}_i - 1)$ is nonsingular, the fact that $\partial^2 = 0$ forces v to be in the kernel of $\bar{A}$. Hence $H_2^\beta(X; V) \neq 0$, as needed.

When instead, $\det\left(\beta(\bar{A}_i)\right) \neq 0$, then both boundary maps in (2–4) have full rank, and hence the complex is acyclic. Following [Turaev 01, Section 2.2], we can use a suitable matrix τ-chain to compute the desired torsion. Specifically, [Turaev 01, Theorem 2.2] gives

$$\tau(X, \beta, e) = \frac{\det\left(\beta(\bar{A}_i)\right)}{\det\left(\beta(\bar{x}_i - 1)\right)}. \tag{2–5}$$

Here we are using that $\dim(V)$ is even, which forces the sign discussed in [Turaev 01, Remark 2.4] to be positive. Also, the convention of [Turaev 01] is to record a basis as the rows of a matrix, whereas we use the columns; this is irrelevant, since the determinant is transpose-invariant. Given (2–5), if we take $\mathfrak{s}$ be the Spin^c-structure corresponding to e, we have established (2–3). □

2.8. Connection to Wada's Invariant

We now explain why the formula (2–2) differs from the one used to define Wada's invariant [Wada 94], and how Wada's invariant also arises as a torsion of a suitable chain complex. To start, suppose we have a representation $\beta : \pi \to \mathrm{GL}(d, \mathbb{F})$, where as usual, π is the fundamental group of a knot exterior. The representation β makes $V := \mathbb{F}^d$ into both a left and a right $\mathbb{Z}[\pi]$-module. The left module ${}_\beta V$ is defined by $g \cdot v := \beta(g)v$, where $v \in V$ is viewed as a column vector, and the right module V_β is defined by $v \cdot g := v\beta(g)$, where now $v \in V$ is viewed as a row vector.

Given a left $\mathbb{Z}[\pi]$-module W, we denote by W^{op} the right $\mathbb{Z}[\pi]$-module given by $w \cdot f := \overline{f} \cdot w$. Similarly, we can define a left module W^{op} for a given right $\mathbb{Z}[\pi]$-module W. In Section 2, we started with the left modules $C_*(\widetilde{X})$ and ${}_\beta V$ and used the chain complex

$$C_*^\beta(\widetilde{X}, \mathbb{F}^d) := C_*(\widetilde{X})^{\mathrm{op}} \otimes_{\mathbb{Z}[\pi]} {}_\beta V$$

in defining the torsion.

One could instead consider the chain complex

$$V_\beta \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}).$$

Here, if $\{v_k\}$ is a basis for V and $\{\tilde{c}_j\}$ is a $\mathbb{Z}[\pi]$-basis for $C_*(\widetilde{X})$, then we endow $V_\beta \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})$ with the basis $\{v_k \otimes \tilde{c}_j\}$ ordered reverse lexicographically, i.e., $v_k \otimes \tilde{c}_j < v_{k'} \otimes \tilde{c}_{j'}$ if either $j < j'$ or both $j = j'$ and $k < k'$.

Suppose now we want to compute the torsion of $V_\beta \otimes C_*(\widetilde{X})$ using the setup of the proof of Theorem 2.7. Then we have

$$\begin{aligned} \partial_2(v_k \otimes r_j) &= \sum_i (v_k \otimes a_{ij} x_i) = \sum_i (v_k \otimes a_{ij} \cdot x_i) \\ &= \sum_i (v_k \cdot a_{ij} \otimes x_i) = \sum_i (v_k \, \beta(\alpha_{ij}) \otimes x_i). \end{aligned}$$

Since we are focusing on a right module V_β, it is natural to write the matrices for the boundary maps in $V_\beta \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})$ as matrices that act on the right of row vectors. With these conventions, one gets the chain complex

$$0 \to V^{n-1} \xrightarrow{\beta(A^t)} V^n \xrightarrow{(\beta(x_1 - 1), \ldots, \beta(x_n - 1))^t} V \to 0,$$

where here A^t denotes the transpose of A, and so A^t is an $(n-1) \times n$ matrix over $\mathbb{Z}[\pi]$. As in the proof of Theorem 2.7, in the generic case, [Turaev 01, Theorem 2.2] gives that

$$\tau\left(V_\beta \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})\right) = \frac{\det\left(\beta(A_i^t)\right)}{\det\left(\beta(x_i - 1)\right)}.$$

Up to the sign of the denominator, this is precisely the formula for Wada's invariant given in [Wada 94].

It is important to note here that $\beta(A^t)$ is not necessarily the same as $\left(\beta(A)\right)^t$, and hence Wada's invariant may differ from our $\tau(X, \beta)$. However, note that there exists a canonical isomorphism

$$\begin{gathered} V_\beta \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}) \to C_*(\widetilde{X})^{\mathrm{op}} \otimes_{\mathbb{Z}[\pi]} \left(V_\beta\right)^{\mathrm{op}} \\ v \otimes \sigma \mapsto \sigma \otimes v \end{gathered}$$

that moreover, respects the ordered bases. Thus these chain complexes have the same torsion invariant. It is easy to see that the left module $(V_\beta)^{\mathrm{op}}$ is isomorphic to ${}_{\beta^*}V$, where $\beta^* : \pi \to \mathrm{GL}(d, \mathbb{F})$ is the representation given by $\beta^*(g) := \left(\beta(g)^{-1}\right)^t$. Thus Wada's invariant for β is our torsion $\tau(X, \beta^*)$.

Our focus in this paper is on β of the form $\alpha \otimes \phi$, where $\alpha : \pi \to \mathrm{SL}(2, \mathbb{F})$ and $\phi : \pi \to \mathbb{Z}$ is the usual epimorphism. Note that α^* is conjugate to α (see, e.g., [Hillman et al. 10]), and hence $(\alpha \otimes \phi)^*$ is conjugate to $\alpha^* \otimes (-\phi)$. Since we argued in Section 2.6 that $\mathcal{T}^\alpha$ is independent of the choice of ϕ, it follows that in this case, our $\mathcal{T}^\alpha$ is exactly Wada's invariant for α.

3. TWISTED TORSION OF CYCLIC COVERS

As usual, let K be a knot in a $\mathbb{Q}$HS with exterior X and fundamental group π. For an irreducible representation $\alpha : \pi \to \mathrm{SL}(2, \mathbb{C})$, in this section we relate the torsion polynomial $\mathcal{T}_K^\alpha$ to a sequence of $\mathbb{C}$-valued torsions of finite cyclic covers of X. We show that the latter determines the former, and will use this connection in Section 4 to prove nonvanishing of the hyperbolic torsion polynomial.

To start, pick an orientation of K to fix the homomorphism $\phi : \pi \to \mathbb{Z}$. For each $m \in \mathbb{N}$, we denote by X_m the m-fold cyclic cover corresponding to $\pi_m := \phi^{-1}(m\mathbb{Z})$. We denote by α_m the restriction of α to $\pi_m = \pi_1(X_m)$. Since the dimension is even and the image of α_m lies in $\mathrm{SL}(2, \mathbb{C})$, it follows from Lemma 2.1 that the torsion $\tau(X_m, \alpha, \mathfrak{s}, \omega) \in \mathbb{C}$ does not depend on the choice of Spin^c-structure or homology orientation; therefore, we denote it by $\tau(X_m, \alpha_m)$. We also let $\boldsymbol{\mu}_m$ be the set of all mth roots of unity in $\mathbb{C}$. The first result of this section is the following (see [Dubois and Yamaguchi 09, Corollary 27] for a related result).

Theorem 3.1. *Let K be a knot in a $\mathbb{Q}\,\mathrm{HS}$ with exterior X and fundamental group π. Let $\alpha : \pi \to \mathrm{SL}(2,\mathbb{C})$ be an irreducible representation. Then for every $m \in \mathbb{N}$, we have*

$$\prod_{\zeta \in \boldsymbol{\mu}_m} \mathcal{T}_K^{\alpha}(\zeta) = \tau(X_m, \alpha_m).$$

Note here that since α is irreducible, the torsion polynomial $\mathcal{T}_K^{\alpha}$ is in $\mathbb{C}[t^{\pm 1}]$ by Theorem 2.4(a), and so $\mathcal{T}_K^{\alpha}(\xi)$ is well defined for every $\xi \in \mathbb{C}^{\times}$. Our proof of Theorem 3.1 is inspired by an argument from [Turaev 86, Section 1.9]. Combining Theorem 3.1 with a (generalization of) a result of [Fried 86], we will prove the following.

Theorem 3.2. *If $\tau(X_m, \alpha_m)$ is nonzero for every $m \in \mathbb{N}$, then the $\tau(X_m, \alpha_m)$ determine $\mathcal{T}_K^{\alpha}(t) \in \mathbb{C}(t)$.*

To state the key lemmas, we first need some notation. We denote by γ_m the representation $\pi \to \mathrm{GL}\big(\mathbb{C}[\mathbb{Z}_m]\big)$ that is the composite of the epimorphism $\pi \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_m$ with the regular representation of $\mathbb{Z}_m$ on $\mathbb{C}[\mathbb{Z}_m]$. Given $\xi \in \mathbb{C}^{\times}$, we denote by λ_ξ the representation $\pi \to \mathrm{GL}(1,\mathbb{C})$ that sends $g \in \pi$ to $\xi^{\phi(g)}$. We first prove Theorem 3.1 assuming the following lemmas.

Lemma 3.3. $\tau(X_m, \alpha_m) = \tau(X, \alpha \otimes \gamma_m)$.

Lemma 3.4. *For every $\xi \in \mathbb{C}^{\times}$ and $\mathfrak{s} \in \mathrm{Spin}^c(X)$, we have*

$$\tau(X, \alpha \otimes \phi, \mathfrak{s})(\xi) = \tau(X, \alpha \otimes \lambda_\xi, \mathfrak{s}). \tag{3–1}$$

Proof of Theorem 3.1. Using Lemma 3.3 and the fact that γ_m and $\bigoplus_{\zeta \in \boldsymbol{\mu}_m} \lambda_\zeta$ are conjugate representations of π, we have

$$\tau(X_m, \alpha_m) = \tau(X, \alpha \otimes \gamma_m) = \prod_{\zeta \in \boldsymbol{\mu}_m} \tau(X, \alpha \otimes \lambda_\zeta, \mathfrak{s}).$$

Note here that while the other terms do not depend on $\mathfrak{s}$, those in the product at the right do, since $\alpha \otimes \lambda_\zeta$ is no longer a special linear representation. We now apply Lemma 3.4 to find that

$$\begin{aligned} &\tau(X_m, \alpha_m) \\ &\quad = \prod_{\zeta \in \boldsymbol{\mu}_m} \tau(X, \alpha \otimes \lambda_\zeta, \mathfrak{s}) = \prod_{\zeta \in \boldsymbol{\mu}_m} \tau(X, \alpha \otimes \phi, \mathfrak{s})(\zeta) \\ &\quad = \prod_{\zeta \in \boldsymbol{\mu}_m} (\zeta)^{-\phi(c_1(\mathfrak{s}))} \mathcal{T}_K^{\alpha}(\zeta) = \prod_{\zeta \in \boldsymbol{\mu}_m} \mathcal{T}_K^{\alpha}(\zeta), \end{aligned}$$

where the last two equalities follow from (2–1) and the fact that $\prod \zeta = 1$. $\square$

Proof of Lemma 3.3. The idea is that for suitable choices, one gets an isomorphism

$$C_*^{\alpha_m}(X_m; V) \to C_*^{\gamma_m \otimes \alpha}\big(X; \mathbb{C}[\mathbb{Z}_m] \otimes_{\mathbb{C}} V\big)$$

as *based* chain complexes over $\mathbb{C}$, and hence their torsions are the same.

Fix a triangulation for X with an ordering c_j of its cells, as well as an Euler lift $c_j \mapsto \tilde{c}_j$ of the cells to the universal cover $\widetilde{X}$. Let $\phi_m : \pi \to \mathbb{Z}_m$ be the epimorphism whose kernel is $\pi_m = \pi_1(X_m)$, and fix $g \in \pi$, where $\bar{g} = \phi_m(g)$ generates $\mathbb{Z}_m$.

Consider the triangulation of X_m that is pulled back from that of X, and let c'_j be the cell in X_m that is the image of $\tilde{c}_j$ under $\widetilde{X} \to X_m$. Then each cell of X_m has a unique expression as $g^k \cdot c'_j$ for k in $\{0, \dots, k-1\}$, where here g^k acts on X_m as a deck transformation.

We order these cells so that $g^k \cdot c'_j < g^{k'} \cdot c'_{j'}$ if $j < j'$ or both $j = j'$ and $k < k'$. In computing torsion, we shall use the Euler lift $g^k \cdot c'_j \mapsto g^k \cdot \tilde{c}_j$ for X_m.

Let V denote $\mathbb{C}^2$ with the π-module structure given by α, and let $\{v_1, v_2\}$ be an ordered basis for V. Consider the map

$$f : C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_m]} V \to C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} \big(\mathbb{C}[\mathbb{Z}_m] \otimes_{\mathbb{C}} V\big) \tag{3–2}$$

induced by $\tilde{c} \otimes v \mapsto \tilde{c} \otimes (1 \otimes v)$; this is well defined, since for $h \in \pi_m$, we have

$$\begin{aligned} f\big((\tilde{c} \cdot h) \otimes v\big) &= (\tilde{c} \cdot h) \otimes (1 \otimes v) = \tilde{c} \otimes \big(h \cdot (1 \otimes v)\big) \\ &= \tilde{c} \otimes \big((h \cdot 1) \otimes (h \cdot v)\big) = \tilde{c} \otimes \big(1 \otimes (h \cdot v)\big) \\ &= f\big(\tilde{c} \otimes (h \cdot v)\big), \end{aligned}$$

where we have used $h \in \pi_m$ to see that $h \cdot 1 = 1$ in $\mathbb{C}[\mathbb{Z}_m]$. Clearly, f is a chain map of complexes of $\mathbb{C}$-vector spaces, and it is an isomorphism, since it sends the elements of the basis $\{(g^k \cdot \tilde{c}_j) \otimes v_\ell\}$ to those of the basis $\{\tilde{c}_k \otimes (\bar{g}^{-k} \otimes g^{-k} \cdot v_\ell)\}$. Now choose $\{v_{k,\ell} = \bar{g}^{-k} \otimes g^{-k} \cdot v_\ell\}$ as our basis for $\mathbb{C}[\mathbb{Z}_m] \otimes_{\mathbb{C}} V$ and order them by $v_{k,\ell} < v_{k',\ell'}$ if $k < k'$ or both $k = k'$ and $\ell < \ell'$. Then with the ordered bases used in Section 2.3, the map f in (3–2) is an isomorphism of based chain complexes. In particular, the complexes have the same torsion, which proves the lemma. $\square$

Proof of Lemma 3.4. Since for every $a \in \mathbb{Z}[\pi]$, we have $(\alpha \otimes \phi)(a)(\xi) = \alpha \otimes \lambda_\xi(a)$, the result should follow by computing both sides of (3–1) with Proposition 2.7. The only issue is that we need to ensure the nonvanishing of the denominators in (2–3) for both $\alpha\otimes\phi$ and $\alpha\otimes\lambda_\xi$. Since α is irreducible, we can choose $g\in[\pi,\pi]$ so that $\operatorname{tr}(\alpha(g))\neq 2$ (see, e.g., [Culler and Shalen 83, Lemma 1.5.1]). Notice then that

$$\alpha\otimes\phi\left(g^{-1}-1\right)=\alpha\otimes\lambda_\xi\left(g^{-1}-1\right)=\alpha\left(g^{-1}-1\right),$$

and since $\operatorname{tr}(\alpha(g))\neq 2$, we have $\det\left(\alpha(g^{-1}-1)\right)\neq 0$. Hence if we take a suitable presentation of π where g is a generator, then we can apply Proposition 2.7 with $x_i=g$ to both $\alpha\otimes\phi$ and $\alpha\otimes\lambda_\xi$ and so prove the lemma. □

We turn now to the proof of Theorem 3.2, which says that typically the torsions $\tau(X_m,\alpha_m)$ collectively determine $\mathcal{T}_K^\alpha$ (by Theorem 3.1, the hypothesis that $\tau(X_m,\alpha_m)\neq 0$ for all m is equivalent to no root of $\mathcal{T}_K^\alpha$ being a root of unity). A polynomial p in $\mathbb{C}[t]$ of degree d is *palindromic* if $p(t)=t^d p(1/t)$, or equivalently if its coefficients satisfy $a_k=a_{d-k}$ for $0\le k\le d$. For a polynomial $p\in\mathbb{C}[t]$ and $m\in\mathbb{N}$, we denote by $r_m(p)$ the resultant of t^m-1 and p, i.e.,

$$\begin{aligned}r_n(p)&=\operatorname{Res}(p,t^m-1)=(-1)^{md}\operatorname{Res}(t^m-1,p)\\&=(-1)^{md}\prod_{\zeta\in\boldsymbol{\mu}_m}p(\zeta),\end{aligned}$$

where here d is the degree of p. The following theorem was proved in [Fried 86] for $p\in\mathbb{R}[t]$ and generalized in [Hillar 05] to the case of $\mathbb{C}[t]$.

Theorem 3.5. *Suppose p and q are palindromic polynomials in $\mathbb{C}[t]$. If $r_m(p)=r_m(q)\neq 0$ for all $m\in\mathbb{N}$, then $p=q$.*

Theorem 3.2 now follows easily from Theorems 3.1 and 3.5 and the symmetry of $\mathcal{T}_K^\alpha$ shown in Theorem 2.3.

Remark 3.6. We just saw that under mild assumptions, the torsions $\tau(X_m,\alpha_m)$ of cyclic covers determine the $\mathbb{C}(t)$-valued torsion polynomial $\mathcal{T}_K^a$. It would be very interesting if one could directly read off the degree and the top coefficient of $\mathcal{T}_K^\alpha$ from the $\tau(X_m,\alpha_m)$. See [Hillar and Levine 07] for some of what is known about recovering a palindromic polynomial p from the sequence $r_m(p)$; in particular, when p is monic and of even degree d, Sturmfels and Zworski conjecture that one needs to know $r_m(p)$ only for $m\le d/2+1$ to recover p.

4. TORSION POLYNOMIALS OF HYPERBOLIC KNOTS

Let K be a hyperbolic knot in an oriented $\mathbb{Z}_2$-homology sphere Y. In this section, we define the hyperbolic torsion polynomial $\mathcal{T}_K$ associated to a certain preferred lift to $\mathrm{SL}(2,\mathbb{C})$ of the holonomy representation of its hyperbolic structure.

4.1. The Discrete and Faithful SL(2, ℂ) Representations

As usual, we write $\pi=\pi_K:=\pi_1(X_K)$, and let $\mu\in\pi$ be a meridian for K. The orientation of μ, or equivalently of K, will not matter in this section, but fix one so that $\phi:\pi\to\mathbb{Z}$ is determined.

From now on, assume that $M=Y\setminus K\cong\operatorname{int}(X)$ has a complete hyperbolic structure. The manifold M inherits an orientation from Y, and so its universal cover $\widetilde{M}$ can be identified with $\mathbb{H}^3$ by an orientation-preserving isometry. This identification is unique up to the action of $\operatorname{Isom}^+(\mathbb{H}^3)=\mathrm{PSL}(2,\mathbb{C})$, and the action of π on $\widetilde{M}=\mathbb{H}^3$ gives the *holonomy representation* $\bar\alpha:\pi\to\mathrm{PSL}(2,\mathbb{C})$, which is unique up to conjugation.

Remark 4.1. By Mostow–Prasad rigidity, the complete hyperbolic structure on M is unique. Thus $\bar\alpha$ is determined, up to conjugacy, solely by the knot K and the orientation of the ambient manifold Y. A subtle point is that there are actually two conjugacy classes of discrete faithful representations $\pi_K\to\mathrm{PSL}(2,\mathbb{C})$; the other one corresponds to reversing the orientation of Y (not K) or equivalently complex-conjugating the entries of the image matrices.

To define a torsion polynomial, we want a representation into $\mathrm{SL}(2,\mathbb{C})$ rather than $\mathrm{PSL}(2,\mathbb{C})$. Thurston proved that $\bar\alpha$ always lifts to a representation $\alpha:\pi\to\mathrm{SL}(2,\mathbb{C})$; see [Thurston 97] and [Shalen 02, Section 1.6] for details. In fact, there are exactly two such lifts, the other being $g\mapsto(-1)^{\phi(g)}\alpha(g)$; the point is that any other lift has the form $g\mapsto\epsilon(g)\alpha(g)$ for some homomorphism $\epsilon:\pi\to\{\pm1\}$, i.e., some element of $H^1(M;\mathbb{Z}_2)=\mathbb{Z}_2$. Now $\bar\alpha(\mu)$ is parabolic, and so $\operatorname{tr}\left(\alpha(\mu)\right)=\pm2$. Since Y is a $\mathbb{Z}_2$HS, we know that $\phi(\mu)$ is odd; hence there is a lift α such that $\operatorname{tr}\left(\alpha(\mu)\right)=2$; arbitrarily, we focus on that lift and call it the *distinguished representation*. This representation is determined, up to conjugacy, solely by K (sans orientation). We explain below the simple change that results if we instead required the trace to be -2.

4.2. The Hyperbolic Torsion Polynomial

For a hyperbolic knot K in an oriented $\mathbb{Z}_2$HS, we define the *hyperbolic torsion polynomial* to be

$$\mathcal{T}_K(t) := \mathcal{T}_K^{\alpha}(t),$$

where $\alpha : \pi \to \mathrm{SL}(2, \mathbb{C})$ is the distinguished representation. Before proving Theorem 1.1, which summarizes basic properties of $\mathcal{T}_K(t)$, we give a few definitions. The *trace field* $\mathbb{F}_K$ of K is the field obtained by adjoining to $\mathbb{Q}$ the elements $\mathrm{tr}\big(\alpha(g)\big)$ for all $g \in \pi$; this is a finite extension of $\mathbb{Q}$ and an important number-theoretic invariant of the hyperbolic structure on M; see [Maclachlan and Reid 03] for more. We say that K has *integral traces* if every $\mathrm{tr}\big(\alpha(g)\big)$ is an algebraic integer (this is necessarily the case if M does not contain a closed essential surface; see, e.g., [Maclachlan and Reid 03, Theorem 5.2.2]). Also, we denote by K^* the result of switching the orientation of the ambient manifold Y; we call K^* the *mirror image* of K. We call K *amphichiral* if Y has an orientation-reversing self-homeomorphism that takes K to itself; equivalently, $K = K^*$ in the category of knots in oriented 3-manifolds.

Theorem 4.2. *Let K be a hyperbolic knot in an oriented $\mathbb{Z}_2$-homology 3-sphere. Then $\mathcal{T}_K$ has the following properties:*

(a) *$\mathcal{T}_K$ is an unambiguous element of $\mathbb{C}[t^{\pm 1}]$ that satisfies $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$. It does not depend on an orientation of K.*

(b) *The coefficients of $\mathcal{T}_K$ lie in the trace field of K. If K has integral traces, the coefficients of $\mathcal{T}_K$ are algebraic integers.*

(c) *$\mathcal{T}_K(\xi)$ is nonzero for every root of unity ξ.*

(d) *If K^* denotes the mirror image of K, then $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}}_K(t)$, where the coefficients of the latter polynomial are the complex conjugates of those of $\mathcal{T}_K$.*

(e) *If K is amphichiral, then $\mathcal{T}_K$ is a real polynomial.*

(f) *The values $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are mutation-invariant.*

For the special case of 2-bridge knots and $\xi = \pm 1$, assertion (c) is also a consequence of [Hirasawa and Murasugi 08] and [Silver and Williams 09].

Proof. Since the distinguished representation α is irreducible, part (a) follows from Theorems 2.3 and 2.4(a).

Next, since M has a cusp, by [Neumann and Reid 92, Lemma 2.6], we can conjugate α so that its image lies in $\mathrm{SL}(2, \mathbb{F}_K)$, where $\mathbb{F}_K$ is the trace field; hence $\mathcal{T}_K \in \mathbb{F}_K[t^{\pm 1}]$, proving the first part of (b). To see the other part, first using [Maclachlan and Reid 03, Theorem 5.2.4], we can conjugate α so that $\alpha(\pi) \subset \mathrm{SL}(2, \mathcal{O}_{\mathbb{K}})$, where here $\mathcal{O}_{\mathbb{K}}$ is the ring of algebraic integers in some number field $\mathbb{K}$ (which might be a proper extension of $\mathbb{F}_K$). We now compute $\mathcal{T}_K^{\alpha}$ by applying Proposition 2.5 to a presentation of π in which μ is our preferred generator. Since $\alpha(\mu)$ is parabolic with trace 2, the denominator in (2–2) is

$$p(t) := \det\big((\alpha \otimes \phi)(\mu^{-1} - 1)\big) = (t^k - 1)^2,$$

where $k = -\phi(\mu) \neq 0$. Thus by (2–2), we know that $p(t) \cdot \mathcal{T}_K^{\alpha}$ is in $\mathcal{O}_K[t^{\pm 1}]$. Then since $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic, the leading coefficient of $\mathcal{T}_K^{\alpha}$ must be integral. An easy inductive argument now shows that all the other coefficients are also integral, proving part (b).

The proof of (c) uses Theorem 3.1, and we handle all mth roots of unity at once. In the notation of Section 3, we have

$$\prod_{\zeta \in \boldsymbol{\mu}_m} \mathcal{T}_K(\zeta) = \tau(X_m, \alpha_m). \tag{4–1}$$

By [Menal-Ferrer and Porti 10, Theorem 0.4], which builds on [Raghunathan 65], we have that $H_*^{\alpha_m}(X_m; \mathbb{C}^2) = 0$, or equivalently, $\tau(X_m, \alpha_m)$ is nonzero. Thus by (4–1), we must have $\mathcal{T}_K(\zeta) \neq 0$ for every mth root of unity, establishing part (c).

For (d), the distinguished representation for the mirror knot K^* is $\bar{\alpha} : \pi \to \mathrm{SL}(2, \mathbb{C})$, where each $\bar{\alpha}(g)$ is the matrix that is the complex conjugate of $\alpha(g)$. Since our choice of orientation for the meridian μ was arbitrary, we can use the same ϕ in calculating both $\mathcal{T}_K$ and $\mathcal{T}_{K^*}$. Thus we have

$$\begin{aligned}\mathcal{T}_{K^*}(t) &= t^{\phi(c_1(\mathfrak{s}))}\tau(X, \bar{\alpha} \otimes \phi) = t^{\phi(c_1(\mathfrak{s}))}\overline{\tau(X_k, \alpha \otimes \phi)} \\ &= \overline{\mathcal{T}}_K(t),\end{aligned}$$

proving (d). Next, claim (e) follows immediately from (d). Finally, claim (f) is a recent result of [Menal-Ferrer and Porti 11]. $\square$

As in Section 3, we now consider the $\mathbb{C}$-valued torsions $\tau(X_m, \alpha_m)$ of finite cyclic covers of X_K. Somewhat surprisingly, these determine $\mathcal{T}_K$:

Theorem 4.3. *Let K be a hyperbolic knot in a $\mathbb{Z}_2$HS with distinguished representation $\alpha : \pi_K \to \mathrm{SL}(2, \mathbb{C})$. Then $\mathcal{T}_K$ is determined by the torsions $\tau(X_m, \alpha_m) \in \mathbb{C}$.*

Proof. As discussed in the proof of Theorem 1.1(c), every $\tau(X_m, \alpha_m)$ in nonzero, so the result is immediate from Theorem 3.2. $\square$

Remark 4.4. In choosing our distinguished representation, we arbitrarily chose the lift $\alpha : \pi \to \mathrm{SL}(2, \mathbb{C})$, where $\mathrm{tr}(\alpha(\mu)) = 2$. As discussed, the other lift β is given by $g \mapsto (-1)^{\phi(g)}\alpha(g)$. Note that given $g \in \pi$, we have

$$\big((\beta \otimes \phi)(g)\big)(t) = \beta(g) \cdot t^{\phi(g)} = \alpha(g) \cdot (-1)^{\phi(g)} \cdot t^{\phi(g)} \\ = \alpha(g) \cdot (-t)^{\phi(g)} = \big((\alpha \otimes \phi)(g)\big)(-t).$$

It follows from Proposition 2.5 that $\mathcal{T}_K^{\beta}(t) = \mathcal{T}_K^{\alpha}(-t) = \mathcal{T}_K(-t)$. Put differently, using β instead of α would simply replace t by $-t$.

Remark 4.5. When Y is not a $\mathbb{Z}_2$HS, the choice of lift α of the holonomy representation can have a more dramatic effect on $\mathcal{T}^{\alpha}$. For example, consider the manifold m130 in the notation of [Callahan et al. 99, Culler et al. 12]. This manifold is a twice-punctured genus-1 surface bundle over the circle, and since $H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_8$, there are four distinct lifts of the holonomy representation. Two of these lifts give $\mathcal{T}_K^{\alpha} = \left(t^2 + t^{-2}\right) - 2i$, and the other two give

$$\mathcal{T}_K^{\alpha} = \left(t^2 + t^{-2}\right) + \sqrt{-8 - 8i}\left(t^1 + t^{-1}\right) - 6i$$

for the two distinct square roots of $-8 - 8i$. In particular, the fields generated by the coefficients are different; only the latter two give the whole trace field.

5. EXAMPLE: THE CONWAY AND KINOSHITA–TERASAKA KNOTS

The Conway and Kinoshita–Terasaka knots are a famous pair of mutant knots both of which have trivial Alexander polynomial. Despite their close relationship, they have different genera. Thus they are a natural place to start our exploration of $\mathcal{T}_K$, and we devote this section to examining them in detail.

The Conway knot C is the mirror of the knot 11n34 in the numbering of [Hoste and Thistlethwaite 99, Hoste et al. 98]. The program Snap [Goodman and Neumann 12, Coulson et al. 00] finds that the trace field $\mathbb{F}$ of the hyperbolic structure on the exterior of C is the extension of $\mathbb{Q}$ obtained by adjoining the root $\theta \approx 0.1233737 - 0.5213097i$ of

$$p(x) = x^{11} - x^{10} + 3x^9 - 4x^8 + 5x^7 - 8x^6 + 8x^5 - 5x^4 \\ + 6x^3 - 5x^2 + 2x - 1.$$

Snap also finds the explicit holonomy representation $\pi_C \to \mathrm{SL}(2, \mathbb{F})$, and one can directly apply Proposition 2.5 to compute $\mathcal{T}_C$. If we set

$$\eta = \frac{1}{53}\big(20\theta^{10} + 9\theta^9 + 28\theta^8 + 3\theta^7 + \theta^6 + 19\theta^5 + 10\theta^4 \\ + 47\theta^3 + 6\theta + 1\big),$$

then $\left\{\eta, \theta, \theta^2, \ldots, \theta^{10}\right\}$ is an integral basis for $\mathcal{O}_{\mathbb{F}}$, and we obtain the result shown in Figure 1.

The Kinoshita–Terasaka knot is the mirror of 11n42. Its trace field is the same as for the Conway knot (since $[\mathbb{F} : \mathbb{Q}]$ is odd, the trace field is also the invariant trace field, which is mutation-invariant), and one obtains the result shown in Figure 2.

From the above, we see that $\mathcal{T}_K$ is not invariant under mutation. Since C and KT have genera 3 and 2 respectively and $\deg(\mathcal{T}_C) = 10$ and $\deg(\mathcal{T}_{KT}) = 6$, we see that Conjecture 1.4 holds for both knots. Also note that the coefficients of these polynomials are not real, certifying the fact that both knots are chiral.

Remark 5.1. It was shown in [Friedl and Kim 06, Section 5] that twisted Alexander polynomials corresponding to representations over finite fields detect the genus of all knots with at most twelve crossings. For example, for

$$\begin{aligned}
\mathcal{T}_C(t) = {} & \left(-79\theta^{10} - 35\theta^9 - 111\theta^8 - 11\theta^7 - 4\theta^6 - 71\theta^5 - 38\theta^4 - 187\theta^3 - 2\theta^2 - 24\theta + 206\eta\right)\left(t^5 + t^{-5}\right) \\
& + \left(257\theta^{10} + 114\theta^9 + 361\theta^8 + 36\theta^7 + 13\theta^6 + 232\theta^5 + 124\theta^4 + 608\theta^3 + 6\theta^2 + 78\theta - 671\eta\right)\left(t^4 + t^{-4}\right) \\
& + \left(-372\theta^{10} - 165\theta^9 - 523\theta^8 - 51\theta^7 - 21\theta^6 - 334\theta^5 - 183\theta^4 - 877\theta^3 - 11\theta^2 - 111\theta + 972\eta\right)\left(t^3 + t^{-3}\right) \\
& + \left(373\theta^{10} + 162\theta^9 + 528\theta^8 + 40\theta^7 + 33\theta^6 + 312\theta^5 + 200\theta^4 + 866\theta^3 + 24\theta^2 + 99\theta - 968\eta\right)\left(t^2 + t^{-2}\right) \\
& + \left(-303\theta^{10} - 115\theta^9 - 445\theta^8 + 14\theta^7 - 75\theta^6 - 152\theta^5 - 227\theta^4 - 649\theta^3 - 73\theta^2 - 29\theta + 749\eta\right)\left(t^1 + t^{-1}\right) \\
& + \left(116\theta^{10} + 14\theta^9 + 200\theta^8 - 88\theta^7 + 116\theta^6 - 122\theta^5 + 204\theta^4 + 146\theta^3 + 124\theta^2 - 78\theta - 220\eta\right) \\
\approx {} & \left(4.89524 + 0.09920i\right)\left(t^5 + t^{-5}\right) + \left(-15.68571 - 0.29761i\right)\left(t^4 + t^{-4}\right) + \left(23.10363 - 0.07842i\right)\left(t^3 + t^{-3}\right) \\
& + \left(-26.94164 + 4.84509i\right)\left(t^2 + t^{-2}\right) + \left(38.38349 - 24.49426i\right)\left(t^1 + t^{-1}\right) + \left(-43.32401 + 44.08061i\right).
\end{aligned}$$

FIGURE 1. The hyperbolic torsion polynomial of the Conway knot.

$$\begin{aligned}\mathcal{T}_{KT}(t) = &\left(-55\theta^{10} - 24\theta^9 - 78\theta^8 - 6\theta^7 - 5\theta^6 - 45\theta^5 - 29\theta^4 - 128\theta^3 - 5\theta^2 - 15\theta + 142\eta\right)\left(t^3 + t^{-3}\right) \\ &+ \left(293\theta^{10} + 126\theta^9 + 416\theta^8 + 28\theta^7 + 29\theta^6 + 236\theta^5 + 160\theta^4 + 678\theta^3 + 24\theta^2 + 75\theta - 756\eta\right)\left(t^2 + t^{-2}\right) \\ &+ \left(-699\theta^{10} - 291\theta^9 - 1001\theta^8 - 42\theta^7 - 95\theta^6 - 512\theta^5 - 419\theta^4 - 1585\theta^3 - 81\theta^2 - 149\theta + 1785\eta\right)\left(t^1 + t^{-1}\right) \\ &+ \left(790\theta^{10} + 314\theta^9 + 1146\theta^8 + 8\theta^7 + 150\theta^6 + 494\theta^5 + 532\theta^4 + 1738\theta^3 + 136\theta^2 + 126\theta - 1986\eta\right) \\ \approx &\left(4.41793 - 0.37603i\right)\left(t^3 + t^{-3}\right) + \left(-22.94164 + 4.84509i\right)\left(t^2 + t^{-2}\right) + \left(61.96443 - 24.09744i\right)\left(t^1 + t^{-1}\right) \\ &+ \left(-82.69542 + 43.48539i\right).\end{aligned}$$

FIGURE 2. The hyperbolic torsion polynomial of the Kinoshita-Terasaka knot.

the Conway knot, there is a representation $\alpha : \pi_1(X_C) \to \mathrm{GL}(4, \mathbb{F}_{13})$ such that the corresponding torsion polynomial $\mathcal{T}_C^\alpha \in \mathbb{F}_{13}[t^{\pm 1}]$ has degree 14, and hence

$$x(C) \geq \frac{1}{4}\deg\left(\mathcal{T}_C^\alpha\right) = 3.5.$$

In particular, this shows that $x(C) = 5$, since $x(C) = 2\,\mathrm{genus}(C) - 1$ is an odd integer.

The calculation using the discrete and faithful $\mathrm{SL}(2, \mathbb{C})$ representation is arguably more satisfactory, since it gives the equality

$$x(C) = \frac{1}{2}\deg(\mathcal{T}_C)$$

on the nose, and not just after rounding up to odd integers. Interestingly, we have not found an example for which this rounding trick applies to $\mathcal{T}_K$; at least for knots with at most 15 crossings, one always has $x(K) = \deg(\mathcal{T}_K)/2$ (see Section 6).

5.1. The Adjoint Representation

For an oriented hyperbolic knot K with distinguished representation $\alpha : \pi_1(X_K) \to \mathrm{SL}(2, \mathbb{C})$, we now consider the adjoint representation

$$\begin{aligned}\alpha_{\mathrm{adj}} : \pi_1(X_K) &\to \mathrm{Aut}\big(\mathfrak{sl}(2, \mathbb{C})\big) \\ g &\mapsto A \mapsto \alpha(g) A\, \alpha(g)^{-1}\end{aligned}$$

associated to α. It is well known that this representation is also faithful and irreducible. Using sign-refined torsion and the orientation on K, one gets an invariant $\mathcal{T}_K^{\mathrm{adj}} \in \mathbb{C}[t^{\pm 1}]$ that is well defined up to multiplication by an element of the form t^k. We refer to [Dubois and Yamaguchi 09] for details on this construction and for further information on $\mathcal{T}_K^{\mathrm{adj}}$; one thing that is shown there is that $\mathcal{T}_K^{\mathrm{adj}}(t) = -\mathcal{T}_K^{\mathrm{adj}}(t^{-1})$ up to a power of t, and so $\mathcal{T}_K^{\mathrm{adj}}$ has odd degree.

For the Conway knot we calculate that

$$\begin{aligned}\mathcal{T}_C^{\mathrm{adj}}(t) \approx &\left(-0.2788 + 16.4072i\right)\left(t^{13} - 1\right) \\ &+ \left(-3.9858 - 20.1706i\right)\left(t^{12} - t\right) \\ &+ \left(-4.2204 - 60.5497i\right)\left(t^{11} - t^2\right) \\ &+ \left(52.0953 + 134.5013i\right)\left(t^{10} - t^3\right) \\ &+ \left(-147.7856 - 46.07448i\right)\left(t^9 - t^4\right) \\ &+ \left(897.2087 + 62.3265i\right)\left(t^8 - t^5\right) \\ &+ \left(-2465.8556 - 1308.0110i\right)\left(t^7 - t^6\right),\end{aligned}$$

and for the Kinoshita-Terasaka knot we obtained

$$\begin{aligned}\mathcal{T}_{KT}^{\mathrm{adj}}(t) \approx &\left(-0.7378 + 12.4047i\right)\left(t^7 - 1\right) \\ &+ \left(29.9408 - 56.5548i\right)\left(t^6 - t\right) \\ &+ \left(-655.7823 - 173.0400i\right)\left(t^5 - t^2\right) \\ &+ \left(2056.7509 + 1678.4875i\right)\left(t^4 - t^3\right).\end{aligned}$$

Since $\dim\big(\mathfrak{sl}(2, \mathbb{C})\big) = 3$, it follows from [Friedl and Kim 06, Theorem 1.1] that

$$x(K) \geq \frac{1}{3}\deg\left(\mathcal{T}_K^{\mathrm{adj}}(t)\right) \tag{5–1}$$

and hence

$$x(C) \geq \frac{13}{3} \quad \text{and} \quad x(KT) \geq \frac{7}{3}. \tag{5–2}$$

Thus using that $x(K)$ is an integer, we get $x(C) \geq 5$ and $x(KT) \geq 3$, which are sharp. Intriguingly, in contrast to the case of $\mathcal{T}_K$, one does not have equality in (5–1) and (5–2) for these two knots. Below, in Section 6.6, we describe some knots for which $\mathcal{T}_K^{\mathrm{adj}}$ fails to give a sharp bound on $x(K)$ even after one takes into account that $x(K)$ is an odd integer.

6. KNOTS WITH AT MOST 15 CROSSINGS

There are 313 231 prime knots with 15 or fewer crossings [Hoste et al. 98], of which all but 22 are hyperbolic. For each of these hyperbolic knots, we computed a high-precision numerical approximation to $\mathcal{T}_K$ (see Section 6.7 for details), and this section is devoted to describing the various properties and patterns we found.

6.1. Genus

The genus bound from $\mathcal{T}_K$ given in Theorem 1.3 is sharp for all 313 209 hyperbolic knots with 15 or fewer crossings; that is, $x(K) = \deg(\mathcal{T}_K)/2$ for all these knots. In contrast, the ordinary Alexander polynomial fails to detect the genus for 8 834 of these knots, which is 2.8% of the total.

We showed that the genus bound from $\mathcal{T}_K$ is sharp using the following techniques to give upper bounds on the genus. First, for the alternating knots (36% of the total), the genus is simply determined by the Alexander polynomial [Murasugi 58, Crowell 59]. For the nonalternating knots, we first did 0-surgery on the knot K to get a closed 3-manifold N; by [Gabai 87], the genus of K is the same as that of the simplest homologically nontrivial surface in N. We then applied the method of [Dunfield and Ramakrishnan 10, Section 6.7] to a triangulation of N to quickly find a homologically nontrivial surface. Since this surface need not be of minimal genus, when necessary we randomized the triangulation of N until we found a surface whose genus matched the lower bound from $\mathcal{T}_K$.

6.2. Fibering

We also found that $\mathcal{T}_K$ gives a sharp obstruction to fibering for all hyperbolic knots with at most 15 crossings. In particular, the 118 252 hyperbolic knots for which $\mathcal{T}_K$ is monic are all fibered. In contrast, while the ordinary Alexander polynomial always certifies nonfibering for alternating knots [Murasugi 63, Gabai 86], among the 201 702 nonalternating knots there are 7 972, or 4.0%, whose Alexander polynomials are monic but do not fiber.

To confirm fibering when $\mathcal{T}_K$ is monic, we used a slight generalization of the method of [Dunfield and Ramakrishnan 10, Section 6.11]. Again by [Gabai 87], it is equivalent to show that the 0-surgery N is fibered. Starting with the minimal-genus surface S found as above, we split N open along S, and tried to simplify a presentation for the fundamental group of $N \setminus S$ until it was obviously that of a surface group. If it was, then it followed that $N \setminus S = S \times I$ and N was fibered. The difference with [Dunfield and Ramakrishnan 10] is that we allowed S to be a general normal surface instead of the restricted class of [Dunfield and Ramakrishnan 10, Figure 6.13]. We handled this by splitting the manifold open along S and triangulating the result using Regina [Burton 09].

6.3. Chirality

For hyperbolic knots with at most 15 crossings, we found that a knot is amphichiral if and only if $\mathcal{T}_K$ has real coefficients. In particular, there are 353 such knots with $\mathcal{T}_K$ real, and SnapPy [Culler et al. 12] easily confirms that they are all amphichiral. (This matches the count of amphichiral knots from [Hoste et al. 98, Table A1].)

In contrast, the numbers $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ do not always detect chirality. For example, the chiral knot $10_{153} = 10n10$ has $\mathcal{T}_K(1) = 4$, and $10_{157} = 10n42$ has $\mathcal{T}_K(-1) = 576$. Moreover, the knot 14a506 has both $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ real. (This last claim was checked to the higher precision of 10 000 decimal places.)

6.4. Knots with the Same $\mathcal{T}_K$

While we saw in Section 5 that $\mathcal{T}_K$ is not mutation-invariant, there are still pairs of knots with the same $\mathcal{T}_K$. In particular, among knots with at most 15 crossings, there are 2 739 groups of more than one knot that share the same $\mathcal{T}_K$, namely 2 700 pairs and 39 triples. Here we do not distinguish between a knot and its mirror image, and having the same $\mathcal{T}_K$ means that the coefficients agree to 5 000 decimal places. Stoimenow found that there are 34 349 groups of mutant knots among those with at most 15 crossings, involving some 77 680 distinct knots [Stoimenow 12]. Thus there are many examples in which mutation changes $\mathcal{T}_K$. However, all of the examples we found of knots with the same $\mathcal{T}_K$ are in fact mutants.

As mentioned, it is shown in [Menal-Ferrer and Porti 11] that the evaluations $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are mutation-invariant. We found 38 pairs of nonmutant knots with the same $\mathcal{T}_K(1)$ and the same $\mathcal{T}_K(-1)$. Suggestively, several of these pairs (including the five pairs shown in [Dunfield et al. 10, Figure 3.9]; see also [Stoimenow and Tanaka 09, Tables 2 and 3]) are known to be genus-2 mutants. We also found a triple of mutually nonmutant knots $\{10a121, 12a1202, 12n706\}$ where $\mathcal{T}_K(+1) = -4$, and a similar sextet

$$\{10n10, 12n881, 13n592, 13n2126, 15n9378, 15n22014\}$$

where $\mathcal{T}_K(+1) = 4$; however, within these groups, the value $\mathcal{T}_K(-1)$ did not agree.

6.5. Other Patterns

We found two other intriguing patterns that we are unable to explain. The first is that the second-highest coefficient of $\mathcal{T}_K$ is often real for fibered knots. In particular, this is the case for 53.1% (62 763 of 118 252) of the fibered knots in this sample. In contrast, the second coefficient is real for only 0.2% (364 of 194 957) of nonfibered knots. (Arguably, the right comparison is with the leading coefficient of $\mathcal{T}_K$ for nonfibered knots; even fewer (0.05%) of these are real.) For fibered knots, the twisted homology of the universal cyclic cover can be identified with that of the fiber; hence the action of a generator of the deck group on this homology of the cover can be thought of as the action of the monodromy of the bundle on the twisted homology of the fiber. The second coefficient of $\mathcal{T}_K$ is then just the sum of the eigenvalues of this monodromy, but it is unclear why this should often be a real number.

The second observation is that $|\mathcal{T}_K(-1)| > |\mathcal{T}_K(1)|$ for all but 22 (less than 0.01%) of these knots. The exceptions are nonalternating, and all but one (15n151121) is fibered.

6.6. Adjoint Polynomial

As discussed in Section 5.1, a torsion polynomial $\mathcal{T}_K^{\mathrm{adj}}$ constructed by composing the holonomy representation with the adjoint representation of $\mathrm{PSL}(2,\mathbb{C})$ on its Lie algebra was studied in [Dubois and Yamaguchi 09]. We also numerically calculated this invariant for all knots with at most 15 crossings. In contrast to what we found for $\mathcal{T}_K$, there was not always an equality in the bound of (5–1) and (5–2). In fact, some 8 252 of these knots had $x(K) > \frac{1}{3}\deg(\mathcal{T}_K^{\mathrm{adj}})$. All such knots were nonalternating, and were among the 8 834 knots for which Δ_K fails to give a sharp bound on $x(K)$. However, using the trick from Section 5.1 that $x(K)$ is an odd integer, we showed that the bound on $x(K)$ from $\mathcal{T}_K^{\mathrm{adj}}$ is effectively sharp in all but 12 cases. The 12 knots for which $\mathcal{T}_K^{\mathrm{adj}}$ fails to determine the genus are as follows: there are seven knots where $x(K) = 7$ (i.e., genus 4) but $\deg(\mathcal{T}_K^{\mathrm{adj}}) = 15$, namely

$$\{15n75595, 15n75615, 15n75858, 15n75883, 15n75948, 15n99458, 15n112466\},$$

and five knots with $x(K) = 9$ (i.e., genus 5) but $\deg(\mathcal{T}_K^{\mathrm{adj}}) = 21$, namely

$$\{15n59545, 15n62671, 15n68947, 15n109077, 15n85615\}.$$

In these 12 cases, we computed $\mathcal{T}_K^{\mathrm{adj}}$ to the higher accuracy of 10 000 decimal places.

Intriguingly, the polynomial $\mathcal{T}_K^{\mathrm{adj}}$ did better at providing an obstruction to fibering; just as for $\mathcal{T}_K$, it was monic only for those knots in the sample that are actually fibered.

6.7. Computational Details

The complete software used for these computations, as well as a table of $\mathcal{T}_K$ for all these knots, is available at http://dunfield.info/torsion. The software runs within Sage,[1] and makes use of SnapPy [Culler et al. 12] and t3m [Culler and Dunfield 10]. It finds very high precision solutions to the gluing equations, in the manner of Snap [Goodman and Neumann 12, Coulson et al. 00], and extracts from this a high-precision approximation to the distinguished representation. Except as noted above, we did all computations with 250 decimal places of precision. Even at this accuracy, $\mathcal{T}_K$ is fast to compute for these knots, taking only a couple of seconds each on a late-2010 high-end desktop computer. However, to save space, only 40 digits were saved in the final table.

To guard against error, two of the authors independently wrote programs that computed $\mathcal{T}_K$, and the outputs of these programs were then compared for all nonalternating knots with 14 crossings.

7. TWISTED TORSION AND THE CHARACTER VARIETY OF A KNOT

As usual, consider a hyperbolic knot K in a $\mathbb{Z}_2$HS, and let $\pi := \pi_1(X_K)$. So far, we have focused on the torsion polynomial of the distinguished representation $\alpha : \pi \to \mathrm{SL}(2,\mathbb{C})$ coming from the hyperbolic structure. However, this representation is always part of a complex curve of representations $\pi \to \mathrm{SL}(2,\mathbb{C})$, and it is natural to ask whether there is additional topological information in the torsion polynomials of these other representations. In this section, we describe how to understand all of these torsion polynomials at once, and use this to help explain some of the patterns observed in Section 6. For the special case of 2-bridge knots, it had previously been studied in [Morifuji 08, Kim and Morifuji 10] how the torsion polynomial varies with the representation, and we extend here some of those results to more general knots.

To state our results, we must first review some basics about character varieties; throughout, see the classic paper [Culler and Shalen 83] or the survey [Shalen 02] for details. Consider the *representation variety* $R(K) := \mathrm{Hom}\big(\pi, \mathrm{SL}(2,\mathbb{C})\big)$, which is an affine algebraic variety over $\mathbb{C}$. The group $\mathrm{SL}(2,\mathbb{C})$ acts on $R(K)$ by conjugating each representation; the algebro-geometric quotient $X(K) := R(K) /\!/ \mathrm{SL}(2,\mathbb{C})$ is called the *character variety*. More concretely, $X(K)$ is the set

[1] Available at http://www.sagemath.org.

of characters of representations $\alpha \in R(K)$, i.e., functions $\chi_\alpha : \pi \to \mathbb{C}$ of the form $\chi_\alpha(g) = \operatorname{tr}(\alpha(g))$ for $g \in \pi$. When α is irreducible, the preimage of χ_α under the projection $R(K) \to X(K)$ is just all conjugates of α, but distinct conjugacy classes of reducible representations can sometimes have the same character. Still, it makes sense to call a character irreducible or reducible depending on which kind of representations it comes from.

The character variety $X(K)$ is also an affine algebraic variety over $\mathbb{C}$; its coordinate ring $\mathbb{C}[X(K)]$, which consists of all regular functions on $X(K)$, is simply the subring $\mathbb{C}[R(K)]^{\mathrm{SL}(2,\mathbb{C})}$ of regular functions on $R(K)$ that are invariant under conjugation. We start by showing that it makes sense to define a torsion polynomial $\mathcal{T}_K^\chi$ for $\chi \in X(K)$ via $\mathcal{T}_K^\chi := \mathcal{T}_K^\alpha$ for every α with $\chi_\alpha = \chi$.

Lemma 7.1. *If $\alpha, \beta \in R(K)$ have the same character, then $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$.*

Proof. As discussed, if α is irreducible, then β must be conjugate to α; hence they have the same torsion polynomial. If instead α is reducible, then Theorem 2.4(b) shows that $\mathcal{T}_K^\alpha$ depends only on the diagonal part of α, which can be recovered from its character. Since β must also be reducible and has the same character as α, we again get $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$. $\square$

An irreducible component X_0 of $X(K)$ has $\dim_{\mathbb{C}}(X_0) \geq 1$, since the exterior of K has boundary a torus. There are two possibilities for X_0: either it consists entirely of reducible characters, or it contains an irreducible character. In the latter case, it turns out that irreducible characters are Zariski open in X_0, and every character in X_0 is that of a representation with nonabelian image. Since the torsion polynomials of reducible representations are boring (see Theorem 2.4 and the discussion immediately following), we focus on those components containing an irreducible character. We denote the union of all such components by $X(K)^{\mathrm{irr}}$; equivalently, $X(K)^{\mathrm{irr}}$ is the Zariski closure of the set of irreducible characters.

It is natural to ask how $\mathcal{T}_K^\chi$ varies as a function of χ. We have obtained the following result.

Theorem 7.2. *Let X_0 be an irreducible component of $X(K)^{\mathrm{irr}}$. There is a unique $\mathcal{T}_K^{X_0} \in \mathbb{C}[X_0][t^{\pm 1}]$ such that for all $\chi \in X_0$, one has $\mathcal{T}_K^\chi(t) = \mathcal{T}_K^{X_0}(\chi)(t)$. Moreover, $\mathcal{T}_K^{X_0}$ is itself the torsion polynomial of a certain representation $\pi \to \mathrm{SL}(2, \mathbb{F})$ and thus has all the usual properties (symmetry, genus bound, etc.).*

We give several explicit examples of $\mathcal{T}_K^{X_0}$ later, in Section 8. The following result is immediate from Theorem 1.5.

Corollary 7.3. *Let X_0 be an irreducible component of $X(K)^{\mathrm{irr}}$. Then*

(a) *For all $\chi \in X_0$, we have $\deg(\mathcal{T}_K^\chi) \leq \deg\left(\mathcal{T}_K^{X_0}\right)$ with equality on a nonempty Zariski-open subset.*

(b) *If $\mathcal{T}_K^{X_0}$ is monic, then $\mathcal{T}_K^\chi$ is monic for all $\chi \in X_0$. Otherwise, $\mathcal{T}_K^\chi$ is monic only on a proper Zariski-closed subset.*

In particular, when X_0 is a curve, the genus bound and fibering obstruction given by $\mathcal{T}_K^\chi$ are the same for all $\chi \in X_0$ except on a finite set where $\mathcal{T}_K^\chi$ provides weaker information. We can also repackage Corollary 7.3 as a uniform statement on all of $X(K)$.

Corollary 7.4. *Let K be a knot in an integral homology 3-sphere. Then*

(a) *The set $\{\chi \in X(K) \mid \deg(\mathcal{T}_K^\chi) = 2x(K)\}$ is Zariski open.*

(b) *The set $\{\chi \in X(K) \mid \mathcal{T}_K^\chi$ is monic $\}$ is Zariski closed.*

Proof. It suffices to consider the intersections of these sets with each irreducible component X_0 of $X(M)$. If X_0 consists solely of reducible characters, the result is immediate from Theorem 2.4(c). Otherwise, it follows from Corollary 7.3 combined with the fact that $\deg(\mathcal{T}_K^{X_0}) \leq 2x(K)$. $\square$

We now turn to the proof of Theorem 1.5.

Proof of Theorem 1.5. By [Culler and Shalen 83, Proposition 1.4.4], there is an irreducible component R_0 of $R(K)$ such that the projection $R_0 \to X(K)$ surjects onto X_0. Consider the tautological representation

$$\rho_{\mathrm{taut}} : \pi \to \mathrm{SL}\big(2, \mathbb{C}[R_0]\big),$$

which sends $g \in \pi$ to the matrix $\rho_{\mathrm{taut}}(g)$ of regular functions on R_0 so that

$$\rho_{\mathrm{taut}}(g)(\alpha) = \alpha(g) \quad \text{for all } \alpha \in R_0.$$

Since R_0 is irreducible, $\mathbb{C}[R_0]$ is an integral domain. Thus we can consider its field of fractions, i.e., the field of rational functions $\mathbb{C}(R_0)$. Working over $\mathbb{C}(R_0)$, there is an associated torsion polynomial $\mathcal{T}_K^{\mathrm{taut}}$ that is in $\mathbb{C}(R_0)[t^{\pm 1}]$, since ρ_{taut} is irreducible. From Lemma 2.5, it is clear that for every $\alpha \in R_0$, we have $\mathcal{T}_K^{\alpha}(t) = \mathcal{T}_K^{\mathrm{taut}}(\alpha)(t)$ in $\mathbb{C}[t^{\pm 1}]$. Hence the coefficients of $\mathcal{T}_K^{\mathrm{taut}}$ have well-defined values at every point $\alpha \in R_0$, and so lie in $\mathbb{C}[R_0]$. Now, since the torsion polynomial is invariant under conjugation, each coefficient of $\mathcal{T}_K^{\mathrm{taut}}$ lies in $\mathbb{C}[X_0] = \mathbb{C}[R_0]^{\mathrm{SL}(2,\mathbb{C})}$, and hence $\mathcal{T}_K^{\mathrm{taut}}$ descends to an element of $\mathbb{C}[X_0][t^{\pm 1}]$, which is the $\mathcal{T}_K^{X_0}$ we seek. □

7.1. The Distinguished Component

It is natural to focus on the component X_0 of $X(M)$ that contains the distinguished representation. In this case, X_0 is an algebraic curve, and we refer to it as the *distinguished component*. By Corollary 7.3, the following conjecture that $\mathcal{T}_K^{X_0}$ detects both the genus and fibering of K is implied by Conjecture 1.4.

Conjecture 7.5. *Let K be a hyperbolic knot in S^3, and X_0 the distinguished component of its character variety. Then $2x(K) = \deg(\mathcal{T}_K^{X_0})$, and $\mathcal{T}_K^{X_0}$ is monic if and only if K is fibered.*

As we explain in Section 7.2, this conjecture is true for many 2-bridge knots.

One pattern in Section 6 is that $\mathcal{T}_K$ never gave worse topological information than the ordinary Alexander polynomial Δ_K. In certain circumstances, Corollary 7.3 allows us to relate Δ_K to $\mathcal{T}_K$, as we now discuss. First, we can sometimes show that $\mathcal{T}_K^{X_0}$ must contain at least as much topological information as Δ_K.

Lemma 7.6. *Let K be a knot in a $\mathbb{Z}$HS. Suppose X_0 is a component of $X(K)^{\mathrm{irr}}$ that contains a reducible character. Then $\deg(\mathcal{T}_K^{X_0}) \geq 2\deg(\Delta_K) - 2$, and if Δ_K is nonmonic, then so is $\mathcal{T}_K^{X_0}$.*

Proof. Let α be a reducible representation whose character lies in X_0. By Theorem 2.4(c), the torsion polynomial $\mathcal{T}_K^{\alpha}$ has degree $2\deg(\Delta_K) - 2$, and its leading coefficient is the square of that of $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$. The result now follows from Corollary 7.3. □

Now consider the distinguished representation α and distinguished component $X_0 \subset X(K)$. We say that α is *sufficiently generic* if $\deg(\mathcal{T}_K) = \deg(\mathcal{T}_K^{X_0})$ and $\mathcal{T}_K$ is monic only if $\mathcal{T}_K^{X_0}$ is. Corollary 7.3 suggests that most knots will have sufficiently generic distinguished representations; however, because the distinguished character takes on only values that are algebraic numbers, there seems to be no a priori reason why this must always be the case. Regardless, our intuition is that the hypothesis of this next proposition holds quite often.

Proposition 7.7. *Let K be a knot in a $\mathbb{Z}$HS whose distinguished representation is sufficiently generic and whose distinguished component of $X(M)$ contains a reducible character. Then $\deg(\mathcal{T}_K) \geq 2\deg(\Delta_K) - 2$, and if Δ_K is nonmonic, then so is $\mathcal{T}_K$.*

7.2. 2-Bridge Knots

For 2-bridge knots in S^3, the torsion polynomial as a function on $X(M)^{\mathrm{irr}}$ was studied in [Kim and Morifuji 10]. Since 2-bridge knots are alternating, the ordinary Alexander polynomial Δ_K determines the genus and whether K fibers [Murasugi 63, Crowell 59, Murasugi 63, Gabai 86]. However, as mentioned, there seems to be no a priori reason that the same must be true for $\mathcal{T}_K^{X_0}$. We now sketch what is known about this special case, starting with two results from [Kim and Morifuji 10].

Theorem 7.8. [Kim and Morifuji 10, Theorem 4.2] *Let K be a hyperbolic 2-bridge knot. Then there exists a component $X_0 \subset X(K)^{\mathrm{irr}}$ such that $2x(K) = \deg(\mathcal{T}_K^{X_0})$ and $\mathcal{T}_K^{X_0}$ is monic if and only if K is fibered. In particular, if $X(K)^{\mathrm{irr}}$ is irreducible, then Conjecture 1.7 holds for K.*

Proof. Since Δ_K detects the genus, it is nonconstant and so has a nontrivial root. For every knot in a $\mathbb{Z}$HS, a root of Δ_K gives rise to a reducible representation $\pi_K \to \mathrm{SL}(2,\mathbb{C})$ with nonabelian image. In the case of 2-bridge knots, the character of any such representation belongs to a component $X_0 \subset X(M)^{\mathrm{irr}}$ (see [Hilden et al. 95, Remark 1.9 and Corollary 2.9], originating in [Burde 90, Proposition 2.3] and the comment following it), and Lemma 7.6 now finishes the proof. □

Theorem 7.9. [Kim and Morifuji 10, Lemma 4.8 and Theorem 4.9] *Let $K = K(p,q)$ be a hyperbolic 2-bridge knot, and let c be the leading coefficient of Δ_K. Suppose there exists a prime divisor ℓ of p such that if $c \neq \pm 1$, then the reduction of c modulo ℓ is not in $\{-1, 0, 1\}$. Then Conjecture 1.7 holds for K.*

Proof sketch. Let X_0 be any component of $X(K)^{\mathrm{irr}}$. First, one shows that X_0 contains a character χ such that $\mathrm{tr}(\mu_K) = 0$. In [Kim and Morifuji 10, Lemma 4.6], this is shown using the particular structure of π_K, and it also follows from the following more general fact.

Claim 7.10. *Let K be a knot in S^3 whose exterior contains no closed essential surface. If X_0 is a component of $X(M)$, then given $a \in \mathbb{C}$, there is $\chi \in X_0$ such that $\mathrm{tr}(\mu_K) = a$.*

Two-bridge knots satisfy the hypothesis of Claim 7.10 by [Hatcher and Thurston 85], and the proof of the claim is straightforward from the Culler–Shalen theory of surfaces associated to ideal points of X_0. Specifically, on the smooth projective model of X_0, the rational function $\mathrm{tr}(\mu_K)$ takes on the value a somewhere, and if this were at an ideal point, the associated essential surface would have either to be closed or have meridian boundary; the latter situation also implies the existence of a closed essential surface by [Culler et al. 87, Theorem 2.0.3].

The representation corresponding to a χ such that $\mathrm{tr}(\mu_K) = 0$ is irreducible but has metabelian image, and in the 2-bridge case one can use this to calculate $\mathcal{T}_K^\chi$ explicitly. In particular, in [Kim and Morifuji 10], the authors find that, provided there exists a prime ℓ as in the hypothesis, the polynomial $\mathcal{T}_K^\chi$ is nonmonic and $\deg \mathcal{T}_K^\chi = 2x(K)$. We then apply Corollary 7.3 to see that Conjecture 1.7 holds. □

Another interesting class of characters in $X(M)^{\mathrm{irr}}$ comprises those of representations for which μ_K is parabolic (e.g., the distinguished representation); such parabolic representations must occur on every component X_0 by Claim 7.10. For the 3 830 nonfibered 2-bridge knots with $q < p \le 287$, we numerically computed $\mathcal{T}_K^\chi$ for all such parabolic characters, using a precision of 150 decimal places. In every case, the polynomial $\mathcal{T}_K^\chi$ was nonmonic and gave a sharp genus bound. Since 2-bridge knots contain no closed essential surfaces, every component of $X(M)$ is a curve. Thus for all 2-bridge knots with $p \le 287$, there are only finitely many $\chi \in X(M)$ such that $\mathcal{T}_K^\chi$ is monic or $\deg(\mathcal{T}_K^\chi) < 2\deg(\Delta_K) - 2$, as conjectured in [Kim and Morifuji 10].

8. CHARACTER VARIETY EXAMPLES

As with many things related to the character variety, while $\mathcal{T}_K^{X_0}$ is a very natural concept, actually computing it can be difficult. Here, we content ourselves with finding $\mathcal{T}_K^{X_0}$ for three of the simplest examples. In each case, there is only one natural choice for X_0, and moreover, it is isomorphic to $\mathbb{C} \setminus \{\text{finite set}\}$. Thus X_0 is rational, and $\mathbb{C}(X_0)$ is just rational functions in one variable, which makes it easy to express the answer. We do one fibered example and two that are nonfibered; in all cases, the simplest Seifert surface has genus 1.

8.1. Example: m003

We start with the sibling M of the figure-8 complement, which is one of the two orientable cusped hyperbolic 3-manifolds of minimal volume. The manifold is m003 in the SnapPea census [Callahan et al. 99, Culler et al. 12], and is the once-punctured torus bundle over the circle with monodromy $\left(\begin{smallmatrix}-2 & 1\\ 1 & -1\end{smallmatrix}\right)$. Its homology is $H_1(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, and it is, for instance, the complement of a null-homologous knot in $L(5,1)$. After randomizing the triangulation a bit, SnapPy gives the following presentation:

$$\pi := \pi_1(M) = \langle a, b \mid bab^3aba^{-2} = 1 \rangle.$$

We will view $X(\pi)$ as a subvariety of $X(\langle a,b\rangle)$, where $\langle a,b\rangle$ is the free group on $\{a,b\}$. Now $X(\langle a,b\rangle) \cong \mathbb{C}^3$, where the coordinates are $(x,y,z) = (\mathrm{tr}(a), \mathrm{tr}(b), \mathrm{tr}(ab))$; this is because the trace of every word $w \in \langle a,b\rangle$ can be expressed in terms of these coordinates using the fundamental relation $\mathrm{tr}(UV) = \mathrm{tr}(U)\mathrm{tr}(V) - \mathrm{tr}(UV^{-1})$ for $U, V \in \mathrm{SL}(2,\mathbb{C})$. Since π is defined by the single relator $R = bab^3aba^{-2}$, the character variety $X(\pi)$ is cut out by the polynomials corresponding to $\mathrm{tr}(R) = 2$, $\mathrm{tr}([a,R]) = 2$, and $\mathrm{tr}([b,R]) = 2$. Using Gröbner bases in SAGE to decompose $X(\pi)$ into irreducible components over $\mathbb{Q}$, we found a unique component X_0 that contains an irreducible character, i.e., contains a point such that $\mathrm{tr}([a,b]) \neq 2$. Explicitly, the ideal of X_0 is $\langle yz - x - z, xz + 1\rangle$, and hence X_0 can be bijectively parameterized by

$$f : \mathbb{C} \setminus \{0\} \to X_0, \quad \text{where } f(u) = (u, 1 - u^2, -1/u).$$

To compute $\mathcal{T}_K^{X_0}$, we consider the curve $R_0 \subset R(\pi)$ lying above X_0 consisting of representations of the form

$$\rho(a) = \begin{pmatrix} u & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} 0 & v \\ -v^{-1} & 1 - u^2 \end{pmatrix},$$

where $v + v^{-1} = u^{-1}$. Such representations are parameterized by $v \in \mathbb{C} \setminus \{0\}$, and hence $\mathbb{C}(R_0) \cong \mathbb{C}(v)$, and we have an explicit $\pi \to \mathrm{GL}(2, \mathbb{C}(v))$, which is the restriction of the tautological representation. Using Lemma 2.5, we find that the torsion polynomial of this representation is

$$t - \frac{2(v^4 + v^2 + 1)}{v^3 + v} + t^{-1}.$$

Substituting in $v = \pm\left(1 - \sqrt{-4u^2 + 1}\right)/2u$ to eliminate v gives the final answer

$$\mathcal{T}_K^{X_0}(t) = t + \frac{2(u^2 - 1)}{u} + t^{-1}.$$

8.2. Example: m006

The census manifold $M = \mathrm{m006}$ can also be described as $5/2$ surgery on one component of the Whitehead link

L. (Here our conventions are such that $+1$ surgery on either component of L gives the trefoil knot, whereas -1 surgery gives the figure-8 knot.) Thus M is, for instance, the complement of a null-homologous knot in the lens space $L(5,2)$, and again $H_1(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$. Using spun-normal surfaces, it is easy to check via [Culler and Dunfield 10] that there is a Seifert surface in M that has genus 1 with one boundary component.

SnapPy gives the presentation

$$\pi := \pi_1(M) = \left\langle a, b \;\middle|\; b^2abab^2a^{-2} = 1 \right\rangle.$$

Changing the generators to $a_{\text{new}} = a^{-1}$ and $b_{\text{new}} = ab$ rewrites this as

$$\pi = \left\langle a, b \;\middle|\; a^3bab^3ab = 1 \right\rangle.$$

Using the same setup as in the previous example, we find a single component X_0 containing an irreducible character. The ideal of X_0 is $\langle x - y, y^2z - z - 1\rangle$, and hence we can bijectively parameterize X_0 by

$$f : \mathbb{C} \setminus \{1, -1\} \to X_0, \quad \text{where } f(u) = \left(u, u, (u^2-1)^{-1}\right). \tag{8–1}$$

Considering the curve of representations given by

$$\rho(a) = \begin{pmatrix} v & 1 \\ 0 & v^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} v^{-1} & 0 \\ \frac{3-2u^2}{u^2-1} & v \end{pmatrix},$$

where $v + v^{-1} = u$, and again directly applying Lemma 2.5 and eliminating v gives

$$\mathcal{T}_K^{X_0}(t) = \frac{2u^2-1}{u^2-1}\left(t + t^{-1}\right) + \frac{2u^3}{u^2-1}.$$

8.3. m037

The census manifold $M = \mathrm{m037}$ has $H_1(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, and so is not a knot in a $\mathbb{Z}/2$-homology sphere. However, this makes no difference in this character variety context. Again, using spun-normal surfaces, one easily checks that there is a Seifert surface in M that has genus 1 with one boundary component. Now $\pi = \left\langle a, b \;\middle|\; a^3ba^2ba^3b^{-2} = 1\right\rangle$, and this time, there are two components of $X(\pi)$ containing irreducible characters. However, one of these consists entirely of metabelian representations that factor through the epimorphism $\pi \to \mathbb{Z}/2 * \mathbb{Z}/2 = \left\langle c, d \;\middle|\; c^2 = d^2 = 1\right\rangle$ given by $a \mapsto c$ and $b \mapsto d$. Focusing on the other component X_0, it turns out that the ideal is $\langle xz - 2y, 4y^2 - z^2 - 4\rangle$, and so we can parameterize X_0 by

$$f : \mathbb{C} \setminus \{-2, 0, 2\} \to X_0,$$

where

$$f(u) = \left(\frac{u^2+4}{4u}, \frac{u^2+4}{u^2-4}, \frac{8u}{u^2-4}\right),$$

and then calculate

$$\mathcal{T}_K^{X_0}(t) = \frac{(u+2)^4}{16u^2}\left(t + t^{-1}\right) + \frac{(u+2)\left(u^4 + 4u^3 - 8u^2 + 16u + 16\right)}{8\,(u-2)u^2}.$$

8.4. The Role of Ideal Points

A key part of the Culler–Shalen theory is the association of an essential surface in the manifold M to each ideal point of a curve $X_0 \subset X(M)$. The details can be found, for example, in [Shalen 02], but in brief, consider the smooth projective model $\widehat{X}_0$ with its rational map $\widehat{X}_0 \to X_0$. Now $\widehat{X}_0$ is a smooth Riemann surface, and the finitely many points where $\widehat{X}_0 \to X_0$ is undefined are called the *ideal points* of X_0. To each such point x, there is an associated nontrivial action of $\pi := \pi_1(M)$ on a simplicial tree T_x. One then constructs a surface S in M dual to this action, which can be taken to be essential (i.e., incompressible, boundary incompressible, and not boundary parallel). Since minimal complexity Seifert surfaces often arise from an ideal point of some X_0, a very natural idea is thus to try to use such an ideal point x to say something about $\mathcal{T}_K^{X_0}$. Moreover, provided that $X_0 \subset X(M)^{\text{irr}}$, a surface associated to an ideal point is never a fiber or semifiber, which suggests that one might hope to prove nonmonotonicity of $\mathcal{T}_K^{X_0}$ by examining $\mathcal{T}_K^{X_0}(x)$. Thus, we now compute what happens to $\mathcal{T}_K^{X_0}$ at such ideal points in our two nonfibered examples m006 and m037. (Aside: It is known that even for knots in S^3, not all boundary slopes need arise from ideal points [Chesebro and Tillmann 07], so it is probably too much to expect that there is always an ideal point that gives a Seifert surface.)

8.5. Ideal Points of m006

If we view the parameterization (8–1) above as a rational map from $P^1(\mathbb{C}) \to X_0$, we have ideal points corresponding to $u \in \{-1, 1, \infty\}$. To calculate the boundary slopes of the surfaces associated to each of these, we consider the trace functions of SnapPy's preferred basis μ, λ for

$\pi_1(\partial M)$. In our presentation for π, we calculate

$$\begin{aligned}\operatorname{tr}(\mu) &= \operatorname{tr}(a^2bab) = x\left(z^2 - z - 1\right)\\ &= -\frac{u\left(u^4 - u^2 - 1\right)}{(u-1)^2(u+1)^2},\\ \operatorname{tr}(\lambda) &= \operatorname{tr}(a^3ba) = -x^3 - xz + 2x\\ &= -\frac{u\left(u^4 - 3\,u^2 + 3\right)}{(u-1)(u+1)},\\ \operatorname{tr}(\mu\lambda) &= x^4 - 2x^2 - z + 1 = \frac{\left(u^2-2\right)\left(u^4 - u^2 + 1\right)}{(u-1)(u+1)}.\end{aligned}$$

Now consider an ideal point x with associated surface S, and pick a simple closed curve γ on $\pi_1(\partial M)$. Then the number of times γ intersects ∂S is twice the order of the pole of $\operatorname{tr}(\gamma)$ at x (here if $\operatorname{tr}(\gamma)$ has a zero of order m at x, this counts as a pole of order 0, not one of order $-m$). The above formulas thus show that the points $u = 1$ and $u = -1$ give surfaces with boundary slope $\mu\lambda^2$, whereas $u = \infty$ gives one with boundary slope $\mu^3\lambda^{-1}$. The latter is the homological longitude, and since there is only one spun-normal surface with that boundary slope and each choice of spinning direction, it follows that the surface associated to $\xi = \infty$ must be the minimal-genus Seifert surface. Thus we are interested in

$$\mathcal{T}_K^{X_0}(u=\infty)(t) = 2\left(t + t^{-1}\right) + (\text{simple pole})t^0.$$

8.6. Ideal Points of m037

This time, we have four ideal points corresponding to $u = -2, 2, 0, \infty$. We obtain

$$\begin{aligned}\operatorname{tr}(\mu) &= \operatorname{tr}(a^2ba^3) = \frac{u^8 - 48u^6 + 96u^4 - 768u^2 + 256}{64(u-2)(u+2)u^3},\\ \operatorname{tr}(\lambda) &= \operatorname{tr}(a^{-1}ba^3b)\\ &= -\frac{1}{256\,(u-2)^2\,(u+2)^2u^4}\\ &\quad\times\left(u^{12} - 72u^{10} + 1264u^8 - 12032u^6 + 20224u^4\right.\\ &\qquad\left.- 18432u^2 + 4096\right),\\ \operatorname{tr}(\mu\lambda) &= -\frac{u^8 - 16u^6 + 352u^4 - 256u^2 + 256}{4(u-2)^3(u+2)^3u}.\end{aligned}$$

Hence $\{2, -2\}$ give surfaces with boundary slope $\mu^2\lambda^{-1}$, and $\{0, \infty\}$ give surfaces with boundary slope $\mu^4\lambda^3$. In fact, the homological longitude is $\mu^2\lambda^{-1}$, and again using spun-normal surfaces, one easily checks that surfaces associated to $\{2, -2\}$ are the minimal-genus Seifert surface. Thus, we care about

$$\mathcal{T}_K^{X_0}(u=2)(t) = 4\left(t + t^{-1}\right) + (\text{simple pole})t^0$$

and

$$\mathcal{T}_K^{X_0}(u=-2)(t) = 0.$$

8.7. General Picture for Ideal Points

Based on the preceding examples and a heuristic calculation for tunnel-number-one manifolds, we conjecture the following:

Conjecture 8.1. *Let K be a knot in a rational homology 3-sphere, and X_0 a component of $X(K)^{\text{irr}}$. Suppose x is an ideal point of X_0 that gives a Seifert surface (hence K is nonfibered). Then the leading coefficient of $\mathcal{T}_K^{X_0}$ has a finite value at x.*

Unfortunately, Conjecture 8.1 does not seem particularly promising as an attack on Conjecture 1.7 for distinguishing fibered versus nonfibered cases. Moreover, in terms of looking at such Seifert ideal points to show that $\mathcal{T}_K^{X_0}$ determines the genus, the second ideal point $u = -2$ in Section 8.6 where $\mathcal{T}_K^{X_0}$ vanishes is not a promising sign.

However, in trying to use an ideal point x of X_0 that gives a Seifert surface to understand $\mathcal{T}_K^{X_0}$, it may be wrong to focus on just the value of $\mathcal{T}_K^{X_0}$ at x. After all, there is no representation of π corresponding to x. Rather, as in the construction of the surface associated to x, perhaps one should view x as giving a valuation on $\mathbb{C}(R_0)$, where R_0 is a component of $R(M)$ surjecting onto X_0.

If we unwind the definition of the associated surface and its properties, we are left with the following abstract situation. There is a field $\mathbb{F}$ with an additive valuation $v : \mathbb{F}^\times \to \mathbb{Z}$ with a representation $\rho : \pi \to \mathrm{SL}(2, \mathbb{F})$ such that for each $\gamma \in \pi$, we have $v(\operatorname{tr}(\gamma)) \geq |\phi(\gamma)|$, where $\phi : \pi \to \mathbb{Z}$ is the usual free abelianization homomorphism.

This alone is not enough, because even in the fibered case, one always has such a setup by looking at an ideal point of a component of $X(M)$ consisting of reducible representations. Thus it seems that the key to such an approach must be to exploit the fact that since X_0 contains an irreducible character, there is a $\gamma \in \pi$ with $\phi(\gamma) = 0$, yet $v(\operatorname{tr}(\gamma))$ is arbitrarily large.

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Nathan M. Dunfield, Department of Mathematics, University of Illinois, 1409 W. Green Street, Urbana, IL 61801, USA (nathan@dunfield.info)

Stefan Friedl, Mathematisches Institut, Raum 020, Weyertal 86-90, Universität zu Köln, 50931 Köln, Germany (sfriedl@gmail.com)

Nicholas Jackson, Mathematics Institute, Zeeman Building, University of Warwick, Coventry, CV4 7AL, UK (nicholas.jackson@warwick.ac.uk)