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Sums of Three Squareful Numbers

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CONTENTS

We investigate the frequency of positive squareful numbers $x, y, z \leq B$ for which $x + y = z$ and present a conjecture concerning its asymptotic behavior.

1. INTRODUCTION

In this paper we examine the quantitative arithmetic of integral points on certain Campana orbifolds, following the discussions of [Abramovich 09], [Campana 05], and [Poonen 06]. Given rational points $p_i = r_i/s_i \in \mathbb{P}^1(\mathbb{Q})$ with integer multiplicities $m_i \geq 2$, for $1 \leq i \leq n$, we define the divisor $\Delta = \sum_i (1 - \frac{1}{m_i})[p_i]$. The pair $(\mathbb{P}^1, \Delta)$ defines an orbifold curve in the sense of Campana and has associated Euler characteristic

$$\chi = \chi(\mathbb{P}^1) - \deg \Delta = 2 - n + \frac{1}{m_1} + \cdots + \frac{1}{m_n}.$$

A point $r/s \in \mathbb{P}^1(\mathbb{Q})$ is said to be integral if $rs_i - sr_i$ is m_i-powerful for $1 \leq i \leq n$. Here we recall that an integer k is said to be m-powerful if $p^m \mid k$ whenever p is a prime divisor of k. We will focus our attention here on the orbifold $(\mathbb{P}^1, \Delta)$ associated with the divisor

$$\Delta = \left(1 - \frac{1}{m}\right)[0] + \left(1 - \frac{1}{m}\right)[1] + \left(1 - \frac{1}{m}\right)[\infty],$$

with Euler characteristic $\chi = -1 + \frac{3}{m}$. The density of integral points on $(\mathbb{P}^1, \Delta)$ with height at most B is captured by the counting function

$$N_{m-1}(B) = \#\big\{(x, y, z) \in \mathbb{N}^3_{\text{prim}} : x + y = z,\ x, y, z \leq B,\\ x, y, z\ m\text{-powerful}\big\},$$

where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}^3_{\text{prim}}$ denotes the set of primitive vectors in $\mathbb{N}^3$.

An old result of [Erdős and Szekeres 35] shows that there are $c_m x^{1/m} + O(x^{1/(m+1)})$ m-powerful numbers up to x for a certain constant $c_m > 0$. This leads to a basic trichotomy: we expect only finitely many integral points when $\chi < 0$, we expect $N_{m-1}(B)$ to grow at most logarithmically in B when $\chi = 0$, and we expect $N_{m-1}(B)$ to have order B^{χ} when $\chi > 0$. When $m = 3$, it is shown

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B	$N_1(B)$	$N_1(B)/(cB^{1/2})$
10^7	6562	0.774
10^8	21920	0.818
10^9	72124	0.851
10^{10}	235168	0.877
10^{11}	762580	0.900
10^{12}	2465044	0.920
10^{13}	7914884	0.934

TABLE 1. Numerical values of $N_1(B)$.

in [Nitaj 55] that $N_2(B) \gg \log B$. Our goal in this paper is to provide evidence in support of the expected order $B^{1/2}$ of $N_1(B)$ when $m = 2$.

Conjecture 1.1. *We have*

$$N_1(B) = cB^{1/2}(1 + o(1)),$$

as $B \to \infty$, *with* $c = 2.68\ldots$.

The explicit conjectured value of c is too complicated to record here, but may be found in (2–12) and (2–13). Our expression for c involves an infinite sum that converges very slowly, thereby making it difficult to evaluate numerically to high accuracy.

We may test Conjecture 1.1 by naively listing all squareful numbers up to B, and then subsequently sorting them into triples (x, y, z) that are counted by $N_1(B)$. More precisely, the algorithm loops through all squareful numbers z in increasing order, and for each z, it runs over squareful $x \in [z/2, z]$ and uses the list to verify whether $y = z - x$ is squareful. If it is, we verify whether $\gcd(x, y) = 1$ and eventually print the two corresponding points (x, y, z) and (y, x, z). The inner code of the two loops is repeated $O(s^2)$ times, where s is the number of squareful numbers involved, so that the total complexity is $O(B)$. For $B = 10^{13}$ the compilation of the list took less than two minutes on an Intel Core 2 Duo E8400 running at 3 GHz, resulting in 6840384 squareful numbers overall. The sorting algorithm required a computing time of 5587.5 minutes. In Figure 1, the values of $N_1(B)/(cB^{1/2})$ are plotted for B up to 10^{13}, where the horizontal axis runs over values of $\log_2 B$. In Table 1, we present some explicit numerical data, including the determination of the quotient $N_1(B)/(cB^{1/2})$ for large values of B. In both Figure 1 and Table 1, we took $c = 2.68$.

Any positive squareful integer k can be written uniquely as $k = x^2y^3$, with $x, y \in \mathbb{N}$ and y square-free. Using this description, we have

$$N_1(B) = \sum_{\mathbf{y} \in \mathbb{N}^3} \mu^2(y_0y_1y_2) M_{\mathbf{y}}(B), \tag{1–1}$$

where μ is the Möbius function and $M_{\mathbf{y}}(\mathrm{B})$ denotes the number of $\mathbf{x} \in \mathbb{N}^3 \cap C_{\mathbf{y}}$ such that $\gcd(x_0y_0, x_1y_1, x_2y_2) = 1$ and $x_i^2y_i^3 \le B$ for $0 \le i \le 2$. Here one is naturally led to analyze $N_1(B)$ by counting points on each conic and then summing the contribution over the $\mathbf{y}$. This is the point of view adopted by the second author [Van Valckenborgh 10], where the structure of the orbifold $(\mathbb{P}^1, \Delta)$ is generalized to a higher-dimensional analogue $(\mathbb{P}^{n-1}, \Delta)$, corresponding to a hyperplane of squareful numbers. An asymptotic formula of the expected order of magnitude is then obtained when there are $n + 1 \ge 5$ terms present in the hyperplane. In addition to this, [Van Valckenborgh 10] contains an interpretation of the leading constant in terms of local densities for the underlying quadric. We will revisit this discussion in Section 2 in order to justify the numerical value of the constant in Conjecture 1.1.

Ignoring all but the term with $\mathbf{y} = (1, 1, 1)$ in (1–1), one readily arrives at the lower bound $N_1(B) \gg B^{1/2}$, via the familiar parameterization for Pythagorean triples. Building on this observation, we will sketch a proof of the following result in Section 3.

Theorem 1.2. *We have* $N_1(B) \ge cB^{1/2}(1 + o(1))$, *where* c *is the constant in Conjecture 1.1.*

The problem of producing an upper bound of the expected order of magnitude is much more challenging. In Section 4 we shall establish the following estimate.

Theorem 1.3. *We have* $N_1(B) = O(B^{3/5+\varepsilon})$.

With more work, it ought to be possible to the term B^{ε} by a small power of a logarithm in Theorem 1.3. The proof of Theorem 1.3 involves two estimates. The first is based on fixing the $\mathbf{y}$ and counting points on the conic $C_{\mathbf{y}}$, uniformly in the coefficients. The second involves switching the roles of $\mathbf{y}$ and $\mathbf{x}$, viewing the equation as a family of plane cubics instead. For both of these, the determinant method of [Heath-Brown 02] is a key tool. The same argument has been observed by a number of mathematicians, including Valentin Blomer in private communication with the first author. In order to improve the exponent of B in Theorem 1.3, one requires a new means of treating the contribution from $\mathbf{x}, \mathbf{y}$ for which each x_i and y_i has order of magnitude $B^{1/5}$. It would be

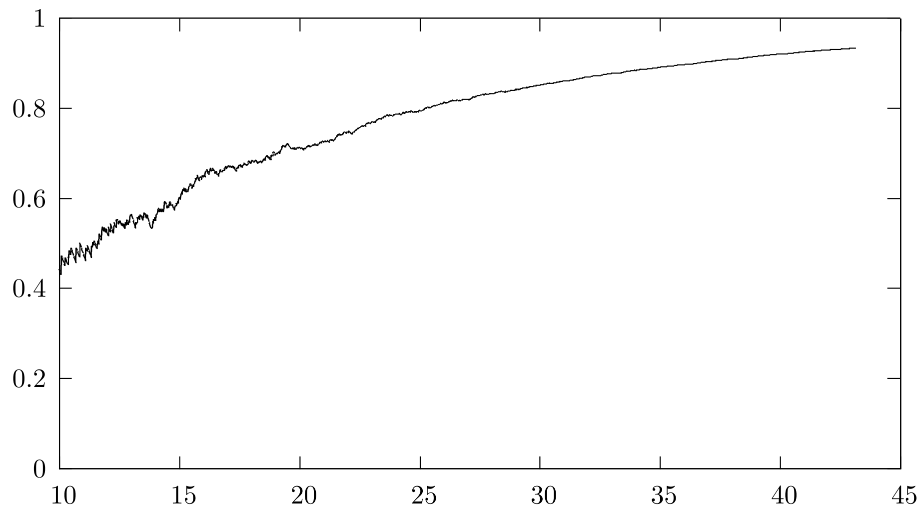

FIGURE 1. Values of $N_1(B)/(cB^{1/2})$.

desirable, for example, to have better control over the $\mathbf{y}$ that produce conics $C_{\mathbf{y}}$ containing at least one rational point of small height.

2. THE CONSTANT

Recall the expression for $N_1(B)$ in (1–1), in which $C_{\mathbf{y}}$ denotes the conic

$$x_0^2y_0^3 + x_1^2y_1^3 = x_2^2y_2^3,$$

for given $\mathbf{y} = (y_0, y_1, y_2) \in \mathbb{N}^3$. Let $H_{\mathbf{y}} : C_{\mathbf{y}}(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ denote the height function

$$[x_0, x_1, x_2] \mapsto \max\{|x_0^2y_0^3|, |x_1^2y_1^3|, |x_2^2y_2^3|\}^{1/2},$$

if $x_0, x_1, x_2 \in \mathbb{Z}$ satisfy $\gcd(x_0, x_1, x_2) = 1$. On noting that $\mathbf{x}$ and $-\mathbf{x}$ represent the same point in $\mathbb{P}^2$, we easily infer that $N_1(B)$ is approximated by the sum

$$\frac{1}{4}\sum_{\mathbf{y}\in\mathbb{N}^3} \mu^2(y_0y_1y_2)\#\{x \in C_{\mathbf{y}}(\mathbb{Q}) : H_{\mathbf{y}}(x) \leq B^{1/2},\ \gcd(x_0y_0, x_1y_1, x_2y_2) = 1\}. \tag{2–1}$$

Following the framework developed by the second author [Van Valckenborgh 10, Section 5], we are therefore led to take the value

$$c = \frac{1}{4}\sum_{\mathbf{y}\in\mathbb{N}^3} \mu^2(y_0y_1y_2)c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+) \tag{2–2}$$

in Conjecture 1.1. Here, if $C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+$ denotes the open subset of the adelic space $C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})$ carved out by the condition $\min_{0\leq i\leq 2}\{v_p(x_{i,p}y_i)\} = 0$ for each prime p, then $c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+)$ is a special case of the constant conjecturally introduced in [Peyre 95, Définition 2.5] in the broader context of Fano varieties. In particular, it follows that

$$c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+) = \alpha(C_{\mathbf{y}})\omega_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+), \tag{2–3}$$

where $\omega_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+)$ denotes the Tamagawa measure of $C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+$ associated to the height $H_{\mathbf{y}}$ and $\alpha(C_{\mathbf{y}})$ is the volume of a certain polytope contained in the cone of effective divisors.

Let $\mathbf{y} \in \mathbb{N}^3$ with $\mu^2(y_0y_1y_2) = 1$. In the present setting we have $\mathrm{Pic}(C_{\mathbf{y}}) \cong \mathbb{Z}$, and one finds, using [Peyre 95, Définition 2.4], that

$$\alpha(C_{\mathbf{y}}) = \frac{1}{2}. \tag{2–4}$$

In [Van Valckenborgh 10], which features nonsingular quadrics in $\mathbb{P}^n$ for $n \geq 4$, it is worth highlighting that the corresponding value of the constant is found to be $1/(n-1)$ using the Lefschetz hyperplane theorem. This is no longer true when one considers conics in $\mathbb{P}^2$, since the class of a hyperplane section is not a generator for the Picard group.

Turning to the Tamagawa constant, we let $S = \{\infty, 2\} \cup \{p \mid y_0y_1y_2\}$, a finite set of places. The Tamagawa measure on $C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})$ associated with the height function $H_{\mathbf{y}}$ is given by

$$\omega_{H_{\mathbf{y}}} = \lim_{s\to 1}(s-1)L_S(s, \mathrm{Pic}(\overline{C_{\mathbf{y}}})) \prod_{v\in\mathrm{Val}(\mathbb{Q})} \lambda_v^{-1}\omega_{H_{\mathbf{y}},v}, \tag{2–5}$$

where

$$\lambda_v = \begin{cases} (1-1/p)^{-1}, & \text{if } v \in \mathrm{Val}(\mathbb{Q}) - S, \\ 1, & \text{otherwise,} \end{cases} \tag{2–6}$$

and

$$L_S(s, \mathrm{Pic}(\overline{C_{\mathbf{y}}})) = \prod_{v \in \mathrm{Val}(\mathbb{Q})-S} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) \prod_{p \mid 2y_0y_1y_2} \left(1 - \frac{1}{p^s}\right).$$

Hence

$$\lim_{s \to 1}(s-1)L_S(s, \mathrm{Pic}(\overline{C_{\mathbf{y}}})) = \prod_{p \mid 2y_0y_1y_2} \left(1 - \frac{1}{p}\right). \tag{2–7}$$

In the next few sections, we will calculate the v-adic densities at the different places.

2.1. Density at the Good Places

Let p be a prime such that $p \nmid 2y_0y_1y_2$. Recall that $C_{\mathbf{y}}(\mathbb{Q}_p)^+$ is defined as the subset of points $[x_{0,p}, x_{1,p}, x_{2,p}] \in C_{\mathbf{y}}(\mathbb{Q}_p)$, with $x_{i,p} \in \mathbb{Z}_p$ and $\min_{0 \le i \le 2}\{v_p(x_{i,p})\} = 0$, for which

$$\min_{0 \le i \le 2}\{v_p(x_{i,p}y_i)\} = 0. \tag{2–8}$$

Since $p \nmid y_0y_1y_2$, this latter condition is automatically satisfied, whence $C_{\mathbf{y}}(\mathbb{Q}_p)^+ = C_{\mathbf{y}}(\mathbb{Q}_p)$. By [Peyre and Tschinkel 01, Lemmas 3.2 and 3.4] and [Peyre 95, Lemme 5.4.6], we have

$$\omega_{H_{\mathbf{y}},p}(C_{\mathbf{y}}(\mathbb{Q}_p)) = \frac{\#C_{\mathbf{y}}(\mathbb{F}_p)}{p}.$$

Since $C_{\mathbf{y}}(\mathbb{F}_p)$ is non-empty by Chevalley–Warning, we deduce that $\#C_{\mathbf{y}}(\mathbb{F}_p) = \#\mathbb{P}^1(\mathbb{F}_p) = p+1$. This implies that for the good places, we have

$$\begin{aligned}\prod_{v \in \mathrm{Val}(\mathbb{Q})-S} &\lambda_v^{-1}\omega_{H_{\mathbf{y}},v}(C_{\mathbf{y}}(\mathbb{Q}_v)^+) \\ &= \prod_{p \nmid 2y_0y_1y_2} \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right) \\ &= \frac{8}{\pi^2} \times \prod_{\substack{p \mid y_0y_1y_2 \\ p>2}} \left(1 - \frac{1}{p^2}\right)^{-1},\end{aligned} \tag{2–9}$$

since

$$\prod_{p>2}\left(1 - \frac{1}{p^2}\right) = \frac{4}{3} \times \frac{6}{\pi^2} = \frac{8}{\pi^2}.$$

2.2. Density at the Bad Places

We now suppose that p is a prime divisor of $2y_0y_1y_2$. In this case, in considering $C_{\mathbf{y}}(\mathbb{Q}_p)^+$, the condition (2–8) will no longer be satisfied trivially. Let

$$\begin{aligned}N_{\mathbf{y}}^*(p^r) = \#\Big\{\mathbf{x} \in (\mathbb{Z}/p^r\mathbb{Z})^3 - (p\mathbb{Z}/p^r\mathbb{Z})^3 :\ & y_0^3x_0^2 + y_1^3x_1^2 \equiv y_2^3x_2^2 \pmod{p^r}, \\ & \min_{0 \le i \le 2}\{v_p(x_iy_i)\} = 0\Big\}.\end{aligned}$$

Using [Peyre and Tschinkel 01, Lemmas 3.2 and 3.4] and [Peyre 95, Lemme 5.4.6], we deduce that there exists a constant of $r_0 \in \mathbb{N}$ such that

$$\omega_{H_{\mathbf{y}},p}(C_{\mathbf{y}}(\mathbb{Q}_p)^+) = \left(1 - \frac{1}{p}\right)^{-1} \times \frac{N_{\mathbf{y}}^*(p^r)}{p^{2r}}, \tag{2–10}$$

for each $r \ge r_0$. The following two results are concerned with the calculation of $N_{\mathbf{y}}^*(p^r)$ for primes $p \mid 2y_0y_1y_2$.

Lemma 2.1. *If $p \mid y_0y_1y_2$ and $p > 2$, we have*

$$\frac{N_{\mathbf{y}}^*(p^r)}{p^{2r}} = \left(1 - \frac{1}{p}\right) \times \begin{cases} \left(1 + \left(\frac{y_1y_2}{p}\right)\right), & \text{if } p \mid y_0, \\ \left(1 + \left(\frac{y_0y_2}{p}\right)\right), & \text{if } p \mid y_1, \\ \left(1 + \left(\frac{-y_0y_1}{p}\right)\right), & \text{if } p \mid y_2. \end{cases}$$

Proof. Suppose, for example, that p divides y_0. In this case, $p \nmid y_1y_2$. Modulo p, we obtain the congruence $y_1^3x_1^2 \equiv y_2^3x_2^2 \pmod{p}$. If $y_1^{-3}y_2^3$ is a square modulo p, then we can choose x_2 arbitrarily in $\mathbb{F}_p^\times$, and for each choice of x_2, there are two solutions for x_1. It follows that there are $2p(p-1)$ solutions modulo p in this case. If $y_1^{-3}y_2^3$ is not a square modulo p, then there are no solutions. We conclude that

$$N_{\mathbf{y}}^*(p) = \left(1 + \left(\frac{y_1y_2}{p}\right)\right)p(1-p).$$

Using Hensel's lemma, we deduce that $N_{\mathbf{y}}^*(p^r)$ is equal to

$$p^{2(r-1)}\left(1 + \left(\frac{y_1y_2}{p}\right)\right)p(1-p)$$

for each $r \ge 1$, which thereby completes the proof. □

Lemma 2.2. *If $r \ge 3$, we have*

$$\frac{N_{\mathbf{y}}^*(2^r)}{2^{2r}} = \begin{cases} 1, & \text{if } 2 \nmid y_0y_1y_2 \text{ and} \\ & \neg\{y_0 \equiv y_1 \equiv -y_2 \pmod 4\}, \\ 2, & \text{if } 2 \mid y_0 \text{ and } y_1 \equiv y_2 \pmod 8, \\ 2, & \text{if } 2 \mid y_1 \text{ and } y_0 \equiv y_2 \pmod 8, \\ 2, & \text{if } 2 \mid y_2 \text{ and } y_0 \equiv -y_1 \pmod 8, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows from direct calculation for the case $r = 3$. The formula for $r > 3$ follows from Hensel's lemma. $\square$

2.3. Density at the Infinite Place

It remains to consider the infinite place $v = \infty$. Let

$$D_1 = \{(y_0^3 x_0^2, y_1^3 x_1^2, y_2^3 x_2^2) \in (\mathbb{R} \cap [-1, 1])^3 : \\ y_0^3 x_0^2 + y_1^3 x_1^2 = y_2^3 x_2^2\}.$$

Using [Peyre 95, Lemme 5.4.7], we obtain

$$\omega_{H_{\mathbf{y}},\infty}(C_{\mathbf{y}}(\mathbb{R})^+) = \frac{1}{2} \times \int_{D_1} \omega_{L,\infty},$$

where

$$\omega_{L,\infty} = \frac{\mathrm{d}x_0\,\mathrm{d}x_1}{2y_2^{3/2}\sqrt{y_0^3 x_0^2 + y_1^3 x_1^2}}$$

is the Leray form. Let

$$D_2 = \{(x_0, x_1) \in (\mathbb{R} \cap [-1, 1])^2 : x_0^2 + x_1^2 \le 1\}.$$

Then it follows that

$$\begin{aligned} &\omega_{H_{\mathbf{y}},\infty}(C_{\mathbf{y}}(\mathbb{R})^+) \\ &\quad = \frac{1}{2} \times \frac{1}{(y_0 y_1 y_2)^{3/2}} \int_{D_2} \frac{1}{\sqrt{x_0^2 + x_1^2}} \mathrm{d}x_0\,\mathrm{d}x_1 \qquad (2\text{–}11) \\ &\quad = \frac{\pi}{(y_0 y_1 y_2)^{3/2}}. \end{aligned}$$

2.4. Conclusion

Recall the definition (2–5) of the Tamagawa measure, in which the convergence factors are given by (2–6). Combining (2–7), (2–9), (2–10) with Lemma 2.1 and (2–11), we deduce that $\omega_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbf{Q}})^+)$ is equal to

$$\frac{1}{(y_0 y_1 y_2)^{3/2}} \times \frac{8}{\pi} \times \sigma_{2,\mathbf{y}} \times \prod_{\substack{p \mid y_0 \\ p > 2}} \frac{\left(1 + \left(\frac{y_1 y_2}{p}\right)\right)}{\left(1 + \frac{1}{p}\right)} \\ \times \prod_{\substack{p \mid y_1 \\ p > 2}} \frac{\left(1 + \left(\frac{y_0 y_2}{p}\right)\right)}{\left(1 + \frac{1}{p}\right)} \times \prod_{\substack{p \mid y_2 \\ p > 2}} \frac{\left(1 + \left(\frac{-y_0 y_1}{p}\right)\right)}{\left(1 + \frac{1}{p}\right)},$$

where $\sigma_{2,\mathbf{y}} = \lim_{r \to \infty} 2^{-2r} N_{\mathbf{y}}^*(2^r)$ is given by Lemma 2.2. Substituting this into the definition of the conjectural constant (2–3), and combining it with (2–4), we deduce from (2–2) that

$$c = \frac{1}{\pi} \sum_{\mathbf{y} \in \mathbb{N}^3} \frac{\mu^2(y_0 y_1 y_2)}{(y_0 y_1 y_2)^{3/2}} \times \sigma_{2,\mathbf{y}} \prod_{\substack{p \mid y_0 \\ p > 2}} \frac{\left(1 + \left(\frac{y_1 y_2}{p}\right)\right)}{\left(1 + \frac{1}{p}\right)} \\ \times \prod_{\substack{p \mid y_1 \\ p > 2}} \frac{\left(1 + \left(\frac{y_0 y_2}{p}\right)\right)}{\left(1 + \frac{1}{p}\right)} \times \prod_{\substack{p \mid y_2 \\ p > 2}} \frac{\left(1 + \left(\frac{-y_0 y_1}{p}\right)\right)}{\left(1 + \frac{1}{p}\right)}. \qquad (2\text{–}12)$$

In the remainder of this section we shall attempt to simplify this expression, in order to facilitate its numerical evaluation. Writing S for the set of $\mathbf{y} \in \mathbb{N}^3$ for which $\mu^2(y_0 y_1 y_2) = 1$, we can partition S into subsets

$$S_{-1} = \{\mathbf{y} \in S : 2 \nmid y_0 y_1 y_2\}, \quad S_i = \{\mathbf{y} \in S : 2 \mid y_i\},$$

for $0 \le i \le 2$. We then split (2–12) into sums c_i over S_i, for each $-1 \le i \le 2$. To streamline the notation, we define

$$\gamma(n) = \prod_{p \mid n} \left(1 + \frac{1}{p}\right)^{-1},$$

and for $a, b \in \mathbb{N}$ with a, b square-free and $b > 1$ odd, we set $(\frac{a}{b})_* = 1$ if $(\frac{a}{p}) = 1$ for each $p \mid b$, with the convention that $(\frac{a}{1})_* = 1$.

We begin by examining c_{-1}, in which case y_0, y_1, and y_2 are all odd. We get

$$c_{-1} = \frac{1}{\pi} \sum \frac{\gamma(y_0 y_1 y_2)}{(y_0 y_1 y_2)^{3/2}} \prod_{p \mid y_0} \left(1 + \left(\frac{y_1 y_2}{p}\right)\right) \\ \times \prod_{p \mid y_1} \left(1 + \left(\frac{y_0 y_2}{p}\right)\right) \times \prod_{p \mid y_2} \left(1 + \left(\frac{-y_0 y_1}{p}\right)\right),$$

where the sum is over

$$\mathbf{y} \in S_{-1}, \ \neg\{y_0 \equiv y_1 \equiv -y_2 \pmod 4\}.$$

Substituting $d = y_0 y_1 y_2$, we obtain

$$c_{-1} = \frac{1}{\pi} \sum_{\substack{d=1 \\ 2 \nmid d}}^{\infty} \frac{\mu^2(d)\gamma(d)2^{\omega(d)}}{d^{3/2}} \times \Delta_{-1}(d),$$

where $\omega(d)$ denotes the number of distinct prime divisors of d and

$$\Delta_{-1}(d) = \#\Big\{ y_0 y_1 y_2 = d : \neg\{y_0 \equiv y_1 \equiv -y_2 \pmod 4\}, \\ \left(\frac{y_1 y_2}{y_0}\right)_* = \left(\frac{y_0 y_2}{y_1}\right)_* = \left(\frac{-y_0 y_1}{y_2}\right)_* = 1 \Big\}.$$

We next consider c_0, noting that $c_0 = c_1 = c_2$, by symmetry. If y_0 is even, we set $y_0 = 2y_0'$, where y_0' is odd. We then have that

$$c_0 = \frac{1}{\pi} \sum_{\substack{(y_0', y_1, y_2) \in S_{-1} \\ y_1 \equiv y_2 \pmod 8}} \frac{2\gamma(y_0' y_1 y_2)}{(2y_0' y_1 y_2)^{3/2}} \times \prod_{p \mid y_0'} \left(1 + \left(\frac{y_1 y_2}{p}\right)\right)$$
$$\times \prod_{p \mid y_1} \left(1 + \left(\frac{2y_0' y_2}{p}\right)\right) \times \prod_{p \mid y_2} \left(1 + \left(\frac{-2y_0' y_1}{p}\right)\right).$$

Putting $d = y_0' y_1 y_2$, we deduce as above that

$$c_0 = \frac{1}{\pi} \sum_{\substack{d=1 \\ 2 \nmid d}}^{\infty} \frac{\mu^2(d)\gamma(d) 2^{\omega(d)}}{d^{3/2}} \times \frac{\Delta_0(d)}{\sqrt{2}},$$

where now

$$\Delta_0(d) = \#\Big\{ y_0' y_1 y_2 = d : y_1 \equiv y_2 \pmod 8, \left(\frac{y_1 y_2}{y_0'}\right)_* = \left(\frac{2y_0' y_2}{y_1}\right)_* = \left(\frac{-2y_0' y_1}{y_2}\right)_* = 1 \Big\}.$$

Bringing these expressions together in (2–12), we conclude that

$$c = \frac{1}{\pi} \sum_{\substack{d=1 \\ 2 \nmid d}}^{\infty} \frac{\mu^2(d)\gamma(d) 2^{\omega(d)}}{d^{3/2}} \left(\Delta_{-1}(d) + \frac{3}{\sqrt{2}} \Delta_0(d) \right). \tag{2–13}$$

One finds by numerical computation that $c = 2.68\ldots$, as in Conjecture 1.1.

3. THE LOWER BOUND

Let $C \subset \mathbb{P}^2$ be a conic defined over $\mathbb{Q}$ and denote by $H : C(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ the exponential height function. Suppose that C is defined by a nonsingular quadratic form defined over $\mathbb{Z}$ with relatively prime coefficients all bounded in modulus by M. A number of results in the literature are directed at estimating the counting function $N_{C,H}(P) = \#\{x \in C(\mathbb{Q}) : H(x) \leq P\}$, as $P \to \infty$, with the outcome that there exist absolute constants $\delta, \psi > 0$ such that

$$N_{C,H}(P) = c_H(C(\mathbb{A}_{\mathbb{Q}}))P + O(M^{\psi} P^{1-\delta}), \tag{3–1}$$

where $c_H(C(\mathbb{A}_{\mathbb{Q}}))$ is the constant predicted in [Peyre 95]. This is a special case of work in [Franke et al. 89] on flag varieties $P \setminus G$, with G taken to be the orthogonal group in three variables. Typically, the uniformity in M is not actually recorded, but it transpires that the dependence on M is at worst polynomial.

We are now ready to establish Theorem 1.2. For any choice of $\mathbf{y}$ there are clearly $O(1)$ rational points on $C_{\mathbf{y}}$ that correspond to a solution with $x_0 x_1 x_2 = 0$. Beginning with (2–1), we deduce that

$$N_1(B) \geq \frac{1}{4} \sum_{\substack{\mathbf{y} \in \mathbb{N}^3 \\ y_0, y_1, y_2 \leq B^{\theta}}} \mu^2(y_0 y_1 y_2) N^+_{C_{\mathbf{y}}, H_{\mathbf{y}}}(B^{1/2}) + O(B^{3\theta}),$$

for any $\theta < 1/6$, where $N^+_{C_{\mathbf{y}}, H_{\mathbf{y}}}$ is defined as for $N_{C_{\mathbf{y}}, H_{\mathbf{y}}}$, but with the additional constraint that $\gcd(x_0 y_0, x_1 y_1, x_2 y_2) = 1$. Once taken in conjunction with the fact that $y_0 y_1 y_2$ is square-free and $\gcd(x_0, x_1, x_2) = 1$, we see that the coprimality condition $\gcd(x_0 y_0, x_1 y_1, x_2 y_2) = 1$ on $C_{\mathbf{y}}$ is equivalent to demanding that $\gcd(x_i, x_j, y_k) = 1$ for each permutation $\{i, j, k\} = \{0, 1, 2\}$. Using the Möbius function to remove these coprimality conditions gives

$$N^+_{C_{\mathbf{y}}, H_{\mathbf{y}}}(B^{1/2}) = \sum_{k_0 \mid y_0} \sum_{k_1 \mid y_1} \sum_{k_2 \mid y_2} \mu(k_0 k_1 k_2) N_{C_{\mathbf{k},\mathbf{y}'}, H_{\mathbf{k},\mathbf{y}'}} \left(\frac{B^{1/2}}{k_0 k_1 k_2} \right),$$

where $y_i = k_i y_i'$, $C_{\mathbf{k},\mathbf{y}'}$ is $k_0 y_0'^3 x_0^2 + k_1 y_1'^3 x_1^2 = k_2 y_2'^3 x_2^2$ and $H_{\mathbf{k},\mathbf{y}'}$ is defined as for $H_{\mathbf{y}}$ but with y_i^3 replaced by $k_i y_i'^3$. The conic $C_{\mathbf{k},\mathbf{y}'}$ has an underlying quadratic form with coefficients of size at most $B^{3\theta}$. Applying (3–1), we conclude that

$$N^+_{C_{\mathbf{y}}, H_{\mathbf{y}}}(B^{1/2}) = B^{1/2} \sum_{k_0 \mid y_0} \sum_{k_1 \mid y_1} \sum_{k_2 \mid y_2} \frac{\mu(k_0 k_1 k_2)}{k_0 k_1 k_2} \times c_{H_{\mathbf{k},\mathbf{y}'}}(C_{\mathbf{k},\mathbf{y}'}(\mathbb{A}_{\mathbb{Q}})) + O_{\varepsilon}(B^{\frac{1-\delta}{2} + 3\theta\psi + \varepsilon}),$$

for any $\varepsilon > 0$. One finds that the main term here is precisely equal to $c_{H_{\mathbf{y}}}(C_{\mathbf{y}}(\mathbb{A}_{\mathbb{Q}})^+)B^{1/2}$, in the notation of Section 2. Noting that

$$\sum_{y \leq B^{\theta}} \frac{f(y)}{y^{3/2}} = \sum_{y=1}^{\infty} \frac{f(y)}{y^{3/2}} + O_{\varepsilon}\left(B^{-\frac{\theta}{2} + \varepsilon}\right),$$

for any arithmetic function f satisfying $f(n) = O_{\varepsilon}(n^{\varepsilon})$, we deduce that

$$N_1(B) \geq cB^{1/2} + O\left(B^{3\theta}\right) + O_{\varepsilon}\left(B^{\frac{1-\delta}{2} + 3\theta(1+\psi) + \varepsilon}\right) + O_{\varepsilon}\left(B^{\frac{1-\theta}{2} + \varepsilon}\right),$$

for any $\varepsilon > 0$. We therefore conclude the proof of Theorem 1.2 by taking θ to satisfy the inequalities

$$0 < \theta < \frac{\delta}{6(1+\psi)}.$$

4. THE UPPER BOUND

The aim of this section is to prove Theorem 1.3, for which our starting point is (1–1). In order to estimate $N_1(B)$, we will view the equation in two basic ways:

as a family of conics and as a family of plane cubic curves. By [Heath-Brown 02] one can estimate rational points of bounded height on plane curves, uniformly in the coefficients of the underlying equation. We will invoke this theory through the prism of the first author's work [Browning 09, Lemma 4.10], which yields the following bound for any integer $d \geq 2$.

Lemma 4.1. *Let $\mathbf{c} \in \mathbb{Z}^3$ with $c_1 c_2 c_3 \neq 0$ and pairwise coprime coordinates. Then we have*

$$\begin{aligned}
&\#\{\mathbf{z} \in \mathbb{Z}^3 : \gcd(z_1, z_2, z_3) = 1,\ |z_i| \leq Z_i,\\
&\quad c_1 z_1^d + c_2 z_2^d + c_3 z_3^d = 0\}\\
&\ll_d \left(1 + \frac{Z_1 Z_2 Z_3}{|c_1 c_2 c_3|^{2/d}}\right)^{1/3} (c_1 c_2 c_3)^{\varepsilon}.
\end{aligned}$$

We consider the contribution $N(\mathbf{X}, \mathbf{Y})$, say, to $N_1(B)$ from $\mathbf{x}, \mathbf{y}$ such that

$$X_i \leq x_i < 2X_i, \quad Y_i \leq y_i < 2Y_i,$$

for $0 \leq i \leq 2$. Clearly $N(\mathbf{X}, \mathbf{Y}) = 0$ unless $X_i^2 Y_i^3 \leq B$ and $X_i, Y_i > 1/2$, for $0 \leq i \leq 2$. It will be convenient to set $X = X_0 X_1 X_2$ and $Y = Y_0 Y_1 Y_2$. In particular, we may henceforth assume that $X^2 Y^3 \leq B^3$. On summing over dyadic intervals, we see that

$$N_1(B) \ll \log^6 B \max_{\mathbf{X}, \mathbf{Y}} N(\mathbf{X}, \mathbf{Y}), \tag{4.1}$$

where the maximum is over $\mathbf{X}, \mathbf{Y}$ satisfying the above inequalities.

Viewing the underlying equation as a family of conics first, we take $d = 2$ in Lemma 4.1 and deduce that

$$\begin{aligned}
N(\mathbf{X}, \mathbf{Y}) &\ll \sum_{\mathbf{y}} (y_0 y_1 y_2)^{\varepsilon} \left(1 + \frac{X}{Y^3}\right)^{1/3}\\
&\ll \left(Y + X^{1/3}\right) B^{\varepsilon}.
\end{aligned}$$

Alternatively, regarding the equation as a family of cubics, we take $d = 3$ in Lemma 4.1 and obtain

$$\begin{aligned}
N(\mathbf{X}, \mathbf{Y}) &\ll \sum_{\mathbf{x}} (x_0 x_1 x_2)^{\varepsilon} \left(1 + \frac{Y}{X^{4/3}}\right)^{1/3}\\
&\ll \left(X + Y^{1/3} X^{5/9}\right) B^{\varepsilon}.
\end{aligned}$$

Bringing these two estimates together, we conclude that

$$\begin{aligned}
&N(\mathbf{X}, \mathbf{Y})\\
&\quad \ll \left(\min\{X, Y\} + \min\{Y, Y^{1/3} X^{5/9}\} + X^{1/3}\right) B^{\varepsilon}.
\end{aligned}$$

Now it is clear that $\min\{X, Y\} \leq X^{2/5} Y^{3/5} \leq B^{3/5}$ and

$$\begin{aligned}
&\min\{Y, Y^{1/3} X^{5/9}\}\\
&\quad \leq Y^{9/25} \times (Y^{1/3} X^{5/9})^{18/25} = X^{2/5} Y^{3/5} \leq B^{3/5},
\end{aligned}$$

since $X^2 Y^3 \leq B^3$. Finally, $X^{1/3} \leq B^{1/2}$. Inserting our estimate for $N(\mathbf{X}, \mathbf{Y})$ into (4.1) and redefining the choice of ε, we therefore arrive at the statement of Theorem 1.3.

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