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Some Experiments with Integral Apollonian Circle Packings

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CONTENTS

Bounded Apollonian circle packings (ACPs) are constructed by repeatedly inscribing circles into the triangular interstices of a Descartes configuration of four mutually tangent circles, one of which is internally tangent to the other three. If the original four circles have integer curvature, all of the circles in the packing will have integer curvature as well. In [Sarnak 07], Sarnak proves that there are infinitely many circles of prime curvature and infinitely many pairs of tangent circles of prime curvature in a primitive integral ACP. (A primitive integral ACP is one in which no integer greater than 1 divides the curvatures of all of the circles in the packing.) In this paper, we give a heuristic backed up by numerical data for the number of circles of prime curvature less than x and the number of "kissing primes," or *pairs* of circles of prime curvature less than x, in a primitive integral ACP. We also provide experimental evidence toward a local-to-global principle for the curvatures in a primitive integral ACP.

1. INTRODUCTION

Start with four mutually tangent circles, one of them internally tangent to the other three as in Figure 1. One can inscribe into each of the curvilinear triangles in this picture a unique circle (the uniqueness follows from an old theorem of Apollonius of Perga, ca. 200 BCE). If one continues inscribing the circles in this way, the resulting picture is called an Apollonian circle packing (ACP). A key aspect of studying such packings is to consider the radii of the circles that arise in a given ACP. However, since these radii become small very quickly, it is more convenient to study the *curvatures* of the circles, or the reciprocals of the radii. Studied in this way, ACPs possess the beautiful number-theoretic property that all of the circles in an ACP have integer curvature if the initial four have integer curvature. The number theory associated with these integral ACPs has been investigated extensively in [Graham et al. 03], [Fuchs 10], and [Kontorovich and Oh 11].

A central theorem to any of the results in these papers is Descartes's theorem, which says that the curvatures

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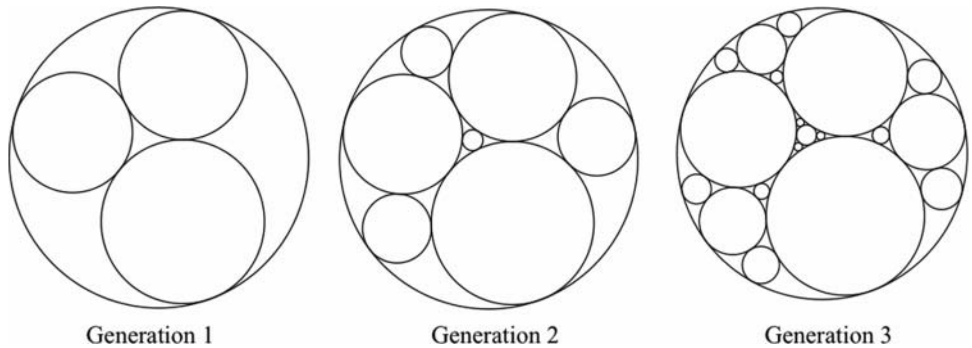

FIGURE 1. Packing circles.

(v_1, v_2, v_3, v_4) of any four mutually tangent circles satisfy what is called the Descartes equation,

$$F(v_1, v_2, v_3, v_4) \\ = 2(v_1^2 + v_2^2 + v_3^2 + v_4^2) - (v_1 + v_2 + v_3 + v_4)^2 = 0, \tag{1–1}$$

where a circle that is internally tangent to the other three is defined to have negative curvature (see [Coxeter 05] for a proof). Given this formula, we may assign to every set of four mutually tangent circles in an integral packing P a vector $\mathbf{v} \in \mathbb{Z}^4$ of the circles' curvatures. We use Descartes's equation to express any integral ACP as an orbit of a subgroup of the orthogonal group $\mathrm{O}_F(\mathbb{Z})$ acting on $\mathbf{v}$. This subgroup, called the Apollonian group, is specified in [Graham et al. 03], and we denote it by A. It is a group on the four generators

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1–2}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix},$$

derived by fixing three of the coordinates of $\mathbf{v}$ and solving $F(\mathbf{v}) = 0$ for the fourth. Note that each S_i is of order 2 and determinant -1. Also, S_i fixes all but the ith coordinate of $\mathbf{v} \in \mathbb{Z}^4$, producing a new curvature in the ith coordinate.

In their paper [Graham et al. 03], the five authors Graham, Lagarias, Mallows, Wilks, and Yan ask several fundamental questions about the curvatures in a given integer ACP, which have mostly been resolved in [Fuchs 10], [Fuchs 11], [Bourgain and Fuchs 11], and [Kontorovich and Oh 11]. In particular, they make some observations about the congruence classes of curvatures that occur in any given ACP, proving that every congruence class modulo d for integers d relatively prime to 30 should be represented in the curvatures in any given primitive ACP (this is shown for d relatively prime to 6 in [Fuchs 11]). Based on a few computer experiments, they also suggest a "strong density" conjecture, that every sufficiently large integer satisfying some congruence conditions modulo $2^a 3^b$ for some $a, b > 0$ (possibly $a = 3, b = 1$) should appear as a curvature in the packing. In [Bourgain and Fuchs 11], the authors prove a weaker conjecture than that in [Graham et al. 03] of this flavor, that the integers appearing as curvatures in a given ACP make up a positive fraction of $\mathbb{N}$. Proving the strong density conjecture would be significantly more difficult. In this paper, we use the p-adic description of the Apollonian orbit from [Fuchs 11] to formulate this conjecture precisely and provide strong experimental evidence in Section 3 in support of it. Our conjecture is specified further in the case of two different ACPs in Section 3. It is stated generally here.

Conjecture 1.1. (Local-to-global principle for ACPs.) *Let P be an integral ACP and let P_{24} be the set of residue classes modulo 24 of curvatures in P. Then there exists $X_P \in \mathbb{Z}$ such that any integer $x > X_P$ whose residue modulo 24 lies in P_{24} is in fact a curvature of a circle in P.*

We note that X above depends on the packing P under consideration. In this paper, we investigate two ACPs that we call the *bugeye* packing P_B and the *coins* packing P_C. These packings are represented by right action of the Apollonian group on $(-1, 2, 2, 3)$ and $(-11, 21, 24, 28)$, respectively (see Figure 2 for a picture). In the case of P_B, our data suggest that $X_{P_B} = 97287$, since we find no integers $97287 < x < 5 \cdot 10^8$ that violate the above conjecture. The data for P_C, however, suggest that X_{P_C} exists, but that it is greater than 10^8. Namely, there are integers $x > 10^8$ in certain residue classes in the set S_{24} that do not appear as curvatures

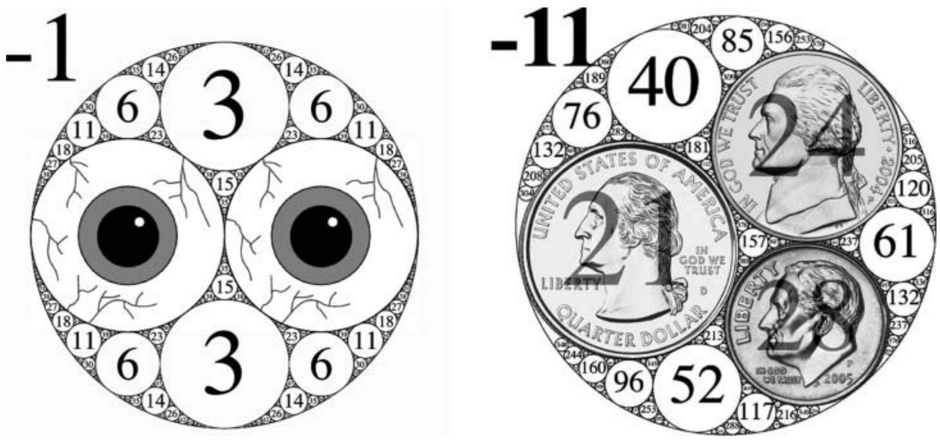

FIGURE 2. Bugeye and coins packings. The picture of the bugeye packing was made by the second author. We thank Alex Kontorovich for the picture of the coins packing.

in the packing we consider. We explain this further in Section 3.

Another interesting problem regarding ACPs is counting circles of prime curvature in a given packing. In [Sarnak 07], it is proved that there are infinitely many circles of prime curvature in any packing. In light of this, we give a heuristic in Section 2 for the weighted prime count $\psi_P(x)$:

$$\psi_P(x) = \sum_{\substack{a(C)\leq x \\ a(C)\text{ prime}}} \log\bigl(a(C)\bigr), \tag{1–3}$$

where C is a circle in the packing P and $a(C)$ is its curvature. This count is closely related to the number $\pi_P(x)$ of prime curvatures less than x in a packing P (see Remark 2.6). We confirm experimentally that our heuristic holds for the packings P_B and P_C. We note that our heuristic does not depend on the chosen packing P; in fact, it yields the correct count of prime curvatures for all of the packings we checked. We summarize all this in the following conjecture:

Conjecture 1.2. *Let $N_P(x)$ be the number of circles in a packing P of curvature less than x, and let $\psi_P(x)$ be as in (1–3). Then as $x \to \infty$,*

$$\psi_P(x) \sim L(2,\chi_4)\cdot N_P(x),$$

where $L(2,\chi_4) = 0.9159\ldots$ is the value of the Dirichlet L-series at 2 with character $\chi_4(p) = 1$ for $p \equiv 1\ (4)$ and $\chi_4(p) = -1$ for $p \equiv 3\ (4)$.

The elegant form of the constant $L(2,\chi_4)$ in this conjecture is quite striking, and it is the same as the constant in the denominator of [Graham et al. 03, Theorem 2.2] regarding the number of integer Descartes quadruples of Euclidean height at most T. However, this similarity seems to be a coincidence coming from the Euler products that are featured both in our heuristic and Graham et al.'s proof of the above-cited theorem.

It is shown in [Sarnak 11] that there are infinitely many pairs of tangent circles of prime curvature (we call these *kissing primes*). We address the question of counting kissing primes in P via the weighted sum $\psi_P^{(2)}(x)$:

$$\psi_P^{(2)}(x) = \sum_{\substack{(C,\,C')\in S \\ a(C),\,a(C')<x}} \log(a(C))\cdot\log(a(C')), \tag{1–4}$$

where S is the set of unordered pairs of tangent circles (C, C') of prime curvature in a packing P, and $a(C)$ and $a(C')$ denote their respective curvatures. In this case, it is less obvious what the relation is between $\psi_P^{(2)}(x)$ and the number $\pi_P^2(x)$ of kissing prime circles in a packing P both of whose curvatures are less than x. We therefore stick with $\psi_P^{(2)}(x)$ in our computation:

Conjecture 1.3. *Let $\psi_P^{(2)}(x)$ be as in (1–4), and let $N_P(x)$ be the number of circles in a packing P of curvature less than x. Then*

$$\psi_P^{(2)}(x) \sim c\cdot L^2(2,\chi_4)\cdot N_P(x),$$

where $N_P(x)$ is as above and $c = 1.646\ldots$ is given by

$$2\cdot\prod_{p\equiv 3\,(4)}\left(1-\frac{2}{p(p-1)^2}\right).$$

These heuristics are computed by counting primes in orbits of the Apollonian group, which is possible due to recent results of Bourgain, Gamburd, and Sarnak in [Bourgain et al. 10], as well as the recent asymptotic count in [Kontorovich and Oh 11] for the number $N_P(x)$. Our computer experiments were conducted using Java and Matlab.[1] A brief description of our algorithm and a discussion of its running time can be found in Section 4.

1.1. Arithmetic Structure of the Apollonian Group and Its Orbit

Since all of the computations and claims in this paper concern the orbit $\mathcal{O}$ of the Apollonian group A acting on a vector $\mathbf{v} \in \mathbb{Z}^4$, we recall the description of the orbit modulo d for any integer d from [Fuchs 11]. We use this description throughout Sections 2 and 3.

Theorem 1.4. (Fuchs.) *Let $\mathcal{O}$ be an orbit of A acting on a root quadruple[2] of a packing, and let $\mathcal{O}_d$ be the reduction of this orbit modulo an integer $d > 1$. Let $C = \{\mathbf{v} \neq \mathbf{0} \mid F(\mathbf{v}) = 0\}$ denote the cone without the origin, and let C_d be C over $\mathbb{Z}/d\mathbb{Z}$:*

$$C_d = \{\mathbf{v} \in \mathbb{Z}/d\mathbb{Z} \mid \mathbf{v} \not\equiv \mathbf{0}\,(d),\ F(\mathbf{v}) \equiv 0\,(d)\}.$$

Write $d = d_1 d_2$ with $(d_2, 6) = 1$ and $d_1 = 2^n 3^m$, where $n, m \geq 0$. Write $d_1 = v_1 v_2$, where $v_1 = \gcd(24, d_1)$. Then:

- (i) *The natural projection $\mathcal{O}_d \longrightarrow \mathcal{O}_{d_1} \times \mathcal{O}_{d_2}$ is surjective.*
- (ii) *Let $\pi : C_{d_1} \to C_{v_1}$ be the natural projection. Then $\mathcal{O}_{d_1} = \pi^{-1}(\mathcal{O}_{v_1})$.*
- (iii) *The natural projection $\mathcal{O}_{d_2} \longrightarrow \prod_{p^r \| d_2} \mathcal{O}_{p^r}$ is surjective and $\mathcal{O}_{p^r} = C_{p^r}$.*

This result is obtained by analyzing the reduction modulo d of the inverse image of the Apollonian group A in the spin double cover of SO_F. We note that Theorem 1.4 implies that the orbit $\mathcal{O}$ of A has multiplicative structure in reduction modulo $d = \prod_{p^r \| d} p^r$ and that it is completely characterized by its reduction modulo 24, or by $\mathcal{O}_{24}$ in our notation. This explains the dependence on P_{24} in Conjecture 1.1.

2. PRIME NUMBER THEOREMS FOR ACPS

In [Bourgain et al. 10], the authors construct an affine linear sieve that gives lower and upper bounds for prime and almost-prime points in the orbits of certain groups. In this section, we use their analysis to predict precise asymptotics on the number of prime curvatures less than x, as well as the number of pairs of tangent circles of prime curvature less than x in a given primitive Apollonian packing P. The conditions associated with the affine linear sieve for A are verified in [Bourgain et al. 10]. We recall the setup below.

Let a_n denote the number of circles of positive curvature n in a bounded packing P, and note that a_n is finite, since the number of circles of any given radius can be bounded in terms of the area of the outermost circle. We consider $1 \leq n \leq x$ and note that

$$\sum_n a_n = N_P(x),$$

where $N_P(x)$ is the number of circles of curvature less than x and is determined by the asymptotic formula in [Kontorovich and Oh 11] (see Lemma 2.2). Key to obtaining our asymptotics is computing the averages of progressions modulo d of curvatures less than x, where $d > 1$ ranges over positive square-free integers of suitable size. To this end, we define

$$X_d = \sum_{n \equiv 0\,(d)} a_n$$

and introduce a multiplicative density function $\beta(d)$ for which

$$X_d = \beta(d) \cdot N_P(x) + r(A, d), \tag{2–1}$$

where the remainder $r(A, d)$ is small according to the results in [Bourgain et al. 10].

In the case of orbits of the Apollonian group, we first define a coordinatewise function β and then relate it to the desired density function in (2–1). Let $\mathcal{O}$ be an integral orbit of A, and let $\mathcal{O}_d$ be the reduction of $\mathcal{O}$ modulo d for a square-free positive integer d. Then

$$\beta_j(d) = \frac{\#\{\mathbf{v} \in \mathcal{O}_d \mid v_j = 0\}}{\#\{\mathbf{v} \in \mathcal{O}_d\}}, \tag{2–2}$$

where v_j is the jth coordinate of $\mathbf{v}$. We recall from Theorem 1.4 that the orbit $\mathcal{O}_d$ has a multiplicative structure that carries over to the function β_j, so that

$$\beta_j(d) = \prod_{p|d} \beta_j(p).$$

Thus in order to evaluate $\beta_j(d)$ for arbitrary square-free d, we have only to determine $\beta_j(p)$ for p prime. This is summarized in the following lemma.

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[1] The programs are available at http://www.math.princeton.edu/~ksanden/ElenaKatCode.html.

[2] A root quadruple of a packing P is essentially the 4-tuple of the curvatures of the largest four circles in P. It is well defined, and its properties are discussed in [Graham et al. 03].

Lemma 2.1. *Let $d = \prod p_i$ be the prime factorization of a square-free integer $d > 1$. Then:*

(i) $\beta_j(d) = \prod \beta_j(p_i)$ *for* $1 \le j \le 4$.

(ii) *For $p \neq 2$, we have*

$$\beta_j(p) = \beta_k(p) \quad \text{for } 1 \le j, k \le 4.$$

(iii) *For any orbit $\mathcal{O}$, there exist two coordinates, i and j, such that*

$$\beta_i(2) = \beta_j(2) = 1,$$
$$\beta_k(2) = 0, \quad \text{for } k \neq i, j.$$

We say that the ith and jth coordinates are even throughout the orbit, while the other two coordinates are odd throughout the orbit.

(iv) *For $p \neq 2$, let $\beta(p) = \beta_i(p)$ for $1 \le i \le 4$. Then*

$$\beta(p) = \begin{cases} \frac{1}{p+1} & \text{for } p \equiv 1 \bmod 4, \\ \frac{p+1}{p^2+1} & \text{for } p \equiv 3 \bmod 4. \end{cases} \tag{2–3}$$

Note that given part (ii) of Lemma 2.1, our definition $\beta(p) = \beta_i(p)$ for $p \neq 2$ and $1 \le i \le 4$ in part (iv) is a natural one.

Proof. The statements in (i) and (ii) follow from Theorem 1.4. Let $\mathbf{v}$ be the root quadruple (the quadruple of the smallest curvatures) of the packing P. To prove (iii), note that any quadruple in a primitive integral ACP consists of two even and two odd curvatures (see [Sanden 09] for a discussion). Without loss of generality, assume that $\mathbf{v} = (1, 1, 0, 0) \bmod 2$, so $i = 1$ and $j = 2$ in this case. Since the Apollonian group is trivial modulo 2, we have that every vector in the orbit is of the form $(1, 1, 0, 0) \bmod 2$, so we have what we want.

To prove (iv), we use results in [Fuchs 11] and recall from Theorem 1.4 that $\mathcal{O}_p$ is the cone C_p for $p \ge 5$. Thus the numerator of $\beta(p)$ is

$$\#\{\mathbf{v} \in \mathcal{O}_p \mid v_j = 0\}$$
$$= \#\{(v_1, v_2, v_3) \in \mathbb{F}_p^3 - \{\mathbf{0}\} \mid F(v_1, v_2, v_3, 0) = 0\},$$

where F is the Descartes quadratic form and $p \ge 5$. So the numerator counts the number of nontrivial solutions to the ternary quadratic form obtained by setting one of the v_i in the Descartes form $F(\mathbf{v})$ to 0. Similarly, we have that the denominator of $\beta(p)$ is

$$\#\{\mathbf{v} \in \mathcal{O}_p\}$$
$$= \#\{(v_1, v_2, v_3, v_4) \in \mathbb{F}_p^4 - \{\mathbf{0}\} \mid F(v_1, v_2, v_3, v_4) = 0\},$$

where $p \ge 5$. So the denominator counts the number of nontrivial solutions to the Descartes form. The number of nontrivial solutions to ternary and quaternary quadratic forms over finite fields is well known (see [Cassels 78], for example). Namely,

$$\#\{(v_1, v_2, v_3, v_4) \in \mathbb{F}_p^4 - \{\mathbf{0}\} \mid F(v_1, v_2, v_3, v_4) = 0\}$$
$$= \begin{cases} p^3 + p^2 - p - 1 & \text{for } p \equiv 1 \bmod 4, \\ p^3 - p^2 + p - 1 & \text{for } p \equiv 3 \bmod 4, \end{cases} \tag{2–4}$$

for $p \ge 5$, and

$$\#\{(v_1, v_2, v_3) \in \mathbb{F}_p^3 - \{\mathbf{0}\} \mid F(v_1, v_2, v_3, 0) = 0\}$$
$$= p^2 - 1 \text{ for all odd primes } p. \tag{2–5}$$

Combining (2–4) and (2–5), we obtain the expression in (2–3) for $p \ge 5$. For $p = 3$, we compute $\mathcal{P}_p$ explicitly and find that there are two possible orbits of A modulo 3, which are illustrated via finite graphs in Figures 3 and 4. Each of these orbits consists of ten vectors $\mathbf{v} \in \mathbb{Z}^4$. In both orbits, four of the vectors $\mathbf{v}$ have $v_i = 0$ for every $1 \le i \le 4$. Thus $\beta(3) = \frac{2}{5}$ as desired. $\square$

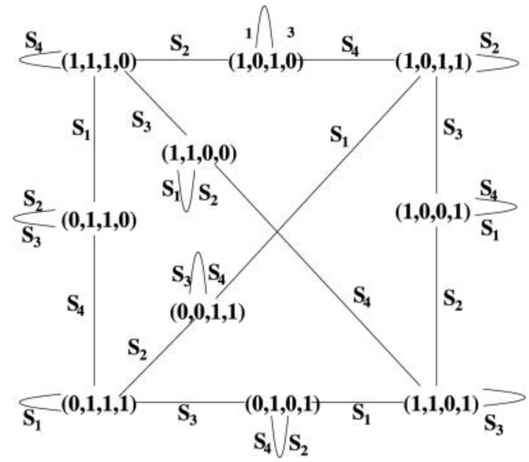

FIGURE 3. Orbit I modulo 3.

In the following two sections, we use this setup to produce a precise heuristic for the number of circles of prime curvature as well as the number of pairs of tangent circles of prime curvature less than x in a given ACP.

2.1. Predicting the Prime Number Theorem for ACPs

In order to compute the number of prime curvatures in an ACP as proposed in Conjecture 1.2, we use the setup above paired with properties of the Möbius function to pick out primes in the orbit of A (see (2–8)). We use the asymptotic in [Kontorovich and Oh 11] for the number $N_P(x)$ of curvatures less than x in a given packing P:

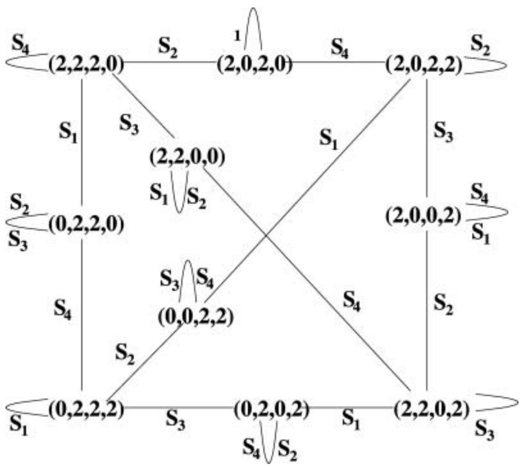

FIGURE 4. Orbit II modulo 3.

Theorem 2.2. (Kontorovich, Oh.) *Given a bounded Apollonian circle packing P, there exists a constant $c_P > 0$, which depends on the packing, such that as $x \to \infty$,*

$$N_P(x) \sim c_P \cdot x^{\delta},$$

where $\delta = 1.30568\ldots$ is the Hausdorff dimension of the limit set of A acting on hyperbolic space.

For the purpose of our computations, we will need a slightly stronger statement of Theorem 2.2, since we will sum over each coordinate of the points in the orbit of A separately. Namely, each circle in the packing is uniquely represented in the orbit $\mathcal{O}$ as a maximal coordinate of a vector $\mathbf{v}$ in $\mathbb{Z}^4$. We would like to know how many circles there are of curvature less than x that are represented in this way in the ith coordinate of a vector in the orbit. We denote this by $N_P^{(i)}(x)$:

$$N_P^{(i)}(x) = \sum_{\substack{\mathbf{v} \in \mathcal{O} \\ v_i^* \le x}} 1, \tag{2–6}$$

where v_i^* denotes the ith coordinate of $\mathbf{v} \in \mathbb{Z}^4$, which is also a maximal coordinate of $\mathbf{v}$.[3] To this end, we have the following lemma.

Lemma 2.3. *Let $N_P^{(i)}(x)$ and $N_P(x)$ be as above. Then*

$$N_P^{(1)}(x) \sim N_P^{(2)}(x) \sim N_P^{(3)}(x) \sim N_P^{(4)}(x) \sim \frac{N_P(x)}{4} \tag{2–7}$$

as x approaches infinity.

[3] It is possible that there is more than one i for which the ith coordinate is maximal.

Proof. The computation in [Kontorovich and Oh 11] of the main term in the asymptotics in Theorem 2.2 relies on the Patterson–Sullivan measure on the limit set of the Apollonian group A. In order to prove Lemma 2.3, we show that this measure is invariant under transformations on the coordinates of a vector $\mathbf{v}$ in an orbit $\mathcal{O}$ of A.

To this end, let G be the group of permutations of the coordinates $v_1, \ldots, v_4$ of a vector $\mathbf{v} \in \mathcal{O}$. The group G is finite, and its elements can be realized as 4×4 integer matrices. For example, the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

switches the first and second coordinates by left action on $\mathbf{v}^{\mathrm{T}}$. Let

$$L = (G, A)$$

be the group of 4×4 matrices generated by the Apollonian group A together with G, and note that each element of G normalizes A. For example, if M is as above, we have $M^{-1} S_1 M = S_2$, where S_1 and S_2 are as in (1–2). Similarly, any element of G switching the ith and jth coordinates of $\mathbf{v}$ conjugates S_i to S_j in this way. Thus L/A is finite, and so L is a finite extension of A. In particular, this implies that the Patterson–Sullivan measure for L is the same as for A. Since L is precisely an extension of A by the permutations of the coordinates of $\mathbf{v}$, we have that the Patterson–Sullivan measure is invariant under G. Together with Theorem 2.2 and its proof in [Kontorovich and Oh 11], this proves the lemma. $\square$

Since we are interested in counting the points in $\mathcal{O}$ for which v_i^* is prime, we sum over points for which v_i^* is 0 modulo some square-free d. It is convenient to count primes in the orbit of A with a logarithmic weight. To this end, we consider the function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^l, \\ 0 & \text{otherwise,} \end{cases}$$

for which it is well known that

$$\Lambda(n) = -\sum_{d \mid n} \mu(d) \log d, \tag{2–8}$$

where $\mu(d)$ is the Möbius function. Using this, we write down a concrete expression for the number of prime curvatures less than x in a packing P counted with a logarithmic weight:

Lemma 2.4. *Let v_i^* be the ith coordinate of a vector* $\mathbf{v}$ *in* $\mathcal{O}$ *such that v_i^* is the maximal coordinate of* $\mathbf{v}$*, and let* $\psi_P(x)$ *be is as in* (1–3)*. Then*

$$\psi_P(x) = -\sum_{1\le i\le 4} \sum_{\substack{\mathbf{v}\in\mathcal{O}\\ v_i^*\le x}} \Lambda(v_i^*) + O(x). \tag{2–9}$$

The sum in (2–9) is a count of all circles whose curvatures are *powers* of primes. Including powers of primes in our count will not affect the final answer significantly. Namely, let $N_P^{\square}(x)$ be the number of circles in a packing P whose curvatures are less than x and are perfect squares. Note that

$$N_P^{\square}(x) = O(x).$$

This is insignificant compared to the count of all curvatures in Theorem 2.2, so the sum in (2–9) is the correct one to consider. Denote by $D < N_P(x)$ the level distribution from the analysis in [Bourgain et al. 10]. That is, our moduli d are taken to be less than D. We combine Lemma 2.4, Lemma 2.3, and the expression for $\Lambda(n)$ in (2–8) to get

$$\begin{aligned}
\psi_P(x) &= -\sum_{1\le i\le 4} \sum_{\substack{\mathbf{v}\in\mathcal{O}\\ v_i^*\le x}} \sum_{d\mid v_1^*} \mu(d)\log d + O(x) \\
&= -\sum_{1\le i\le 4} \sum_{\substack{\mathbf{v}\in\mathcal{O}\\ v_i^*\le x}} \sum_{d\le D} \mu(d)\log d \sum_{v_i^*\equiv 0\ (d)} 1 \qquad (2–10)\\
&\quad -\sum_{1\le i\le 4} \sum_{\substack{\mathbf{v}\in\mathcal{O}\\ v_i^*\le x}} \sum_{d> D} \mu(d)\log d \sum_{v_i^*\equiv 0\ (d)} 1 + O(x).
\end{aligned}$$

Assuming that the Möbius function $\mu(d)$ above becomes random as d grows, the sum over $d > D$ in (2–10) is negligible, and we omit it below. We proceed by rewriting the sum over $d \le D$ in (2–10) using the density function $\beta(d)$ in (2–2). Recall that the analysis in [Bourgain et al. 10] and [Kontorovich and Oh 11] gives us that

$$\sum_{n\equiv 0\ (d)} a_n = \beta(d)\cdot c_P x^{\delta} + r(A,d),$$

where $r(A,d)$ is small on average. In particular,

$$\sum_{d\le D} r(A,d) = O(x^{\delta-\epsilon_0})$$

for some $\epsilon_0 > 0$. Paired with the assumption that μ is random, this evaluation of the remainder term implies that the expression in (2–10) is asymptotic to

$$\begin{aligned}
&-\sum_{1\le i\le 4} \frac{N_P^i(x)}{4} \sum_{d\le D} \beta_i(d)\mu(d)\log d \\
&\sim -\frac{N_P(x)}{4} \sum_{1\le i\le 4} \sum_{d\le D} \beta_i(d)\mu(d)\log d. \qquad (2–11)
\end{aligned}$$

To compute the innermost sum in the final expression above, note that

$$\begin{aligned}
&\sum_{d\le D} \beta_i(d)\mu(d)\log d \qquad (2–12)\\
&= \sum_{d>0} \beta_i(d)\mu(d)\log d - \sum_{d>D} \beta_i(d)\mu(d)\log d.
\end{aligned}$$

Assuming once again that the sum over $d > D$ is insignificant due to the conjectured randomness of the Möbius function, we have that the sum over $d \le D$ in (2–12) can be approximated by the sum over all d. With this in mind, the following lemma yields the heuristic in Conjecture 1.2.

Lemma 2.5. *Let* $\beta_i(d)$ *be as before. We have*

$$\sum_{1\le i\le 4} \sum_{d>0} \beta_i(d)\mu(d)\log d = 4\cdot L(2,\chi_4),$$

where $L(2,\chi_4) = 0.91597\ldots$ *is the value of the Dirichlet* L*-function at* 2 *with character*

$$\chi_4(p) = \begin{cases} 1 & \text{if } p\equiv 1 \bmod 4,\\ -1 & \text{if } p\equiv 3 \bmod 4.\end{cases}$$

Proof. We introduce a function

$$f_i(s) = \sum_d \beta_i(d)\mu(d)d^{-s},$$

and note that its derivative at 0 is precisely what we want:

$$f_i'(0) = -\sum_d \beta_i(d)\mu(d)\log d.$$

Since the functions β, μ, and d^s are all multiplicative, we may rewrite $f_i(s)$ as an Euler product and obtain

$$\begin{aligned}
f_i(s) &= \prod_p \left(1-\beta_i(p)p^{-s}\right)\\
&= \prod_p (1-p^{-s-1})\cdot \frac{1-\beta_i(p)p^{-s}}{1-p^{-s-1}}\\
&= \zeta^{-1}(s+1)\cdot \prod_p \frac{1-\beta_i(p)p^{-s}}{1-p^{-s-1}}\\
&= \zeta^{-1}(s+1)\cdot H_i(s),
\end{aligned}$$

where $H_i(s) = \prod_p (1 - \beta_i(p)p^{-s})(1 - p^{-s-1})^{-1}$ is holomorphic in $\Re(s) > 1/2$. Differentiating, we obtain

$$f_i'(0) = -\zeta'(1)\zeta^{-2}(1) \cdot H_i(0) + \zeta^{-1}(1) \cdot H_i'(0) = H(_i0),$$

since $-\zeta'(1)\zeta^{-2}(1) = 1$ and $\zeta^{-1}(1) = 0$. Thus it remains to compute

$$H_i(0) = \prod_p \frac{1 - \beta_i(p)}{1 - p^{-1}}.$$

From part (iii) of Lemma 2.1, we have $\beta_j(2) = \beta_k(2) = 1$ for two coordinates $1 \le j, k \le 4$. Therefore $1 - \beta_i(2) = 0$ and $H_i(0) = 0$ for $i = j, k$. For $i \ne j, k$ we have $\beta_i(2) = 0$ and

$$\begin{aligned} H_i(0) &= \frac{1}{1 - \frac{1}{2}} \cdot \prod_{p \equiv 1\,(4)} \left(1 - \frac{1}{p+1}\right) \frac{1}{1 - p^{-1}} \\ &\quad \times \prod_{p \equiv 3\,(4)} \left(1 - \frac{p+1}{p^2+1}\right) \frac{1}{1 - p^{-1}} \\ &= 2 \cdot \prod_{p \equiv 1\,(4)} \frac{p^2}{p^2 - 1} \prod_{p \equiv 3\,(4)} \frac{p^2}{p^2 + 1} \\ &= 2 \cdot L(2, \chi_4). \end{aligned}$$

Thus the sum we wish to compute is $4 \cdot L(2, \chi_4)$, as desired. □

Lemma 2.5 implies that the contribution to the sum in (2–11) of the two of the coordinates that are even throughout the orbit is 0, and the contribution of the other two coordinates is

$$\frac{N_P(x)}{4} \cdot 4 \cdot L(2, \chi_4) = N_P(x) \cdot L(2, \chi_4),$$

yielding the predicted result in Conjecture 1.2.

Remark 2.6. It is well known that $\pi_P(x) \sim \frac{\psi_P(x)}{\log x}$ as $x \to \infty$. Thus Conjecture 1.2 can also be stated in terms of $\pi_P(x)$:

$$\pi_P(x) \sim \frac{L(2, \chi_4) \cdot N_P(x)}{\log x}.$$

Since our computations rely on the multiplicativity inherent in the reduction modulo d of the Apollonian group A, our heuristic is independent of the chosen packing P in which we count prime curvatures. This is confirmed by our data: Figures 5 and 6 show the graphs of

$$y = \frac{\psi_P(x)}{N_P(x)}, \tag{2–13}$$

where $P = P_C$ and P_B are as in Section 1, and $x \le 10^8$. In both cases, (2–13) converges to $y = L(2, \chi_4)$ as predicted in Conjecture 1.2.

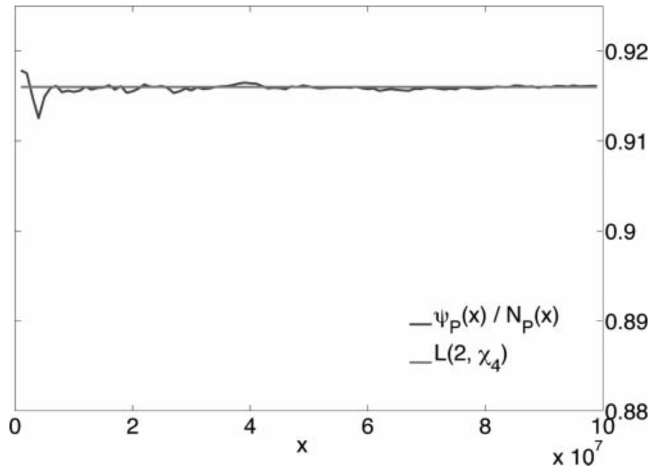

FIGURE 5. Prime number heuristic for P_C.

2.2. Predicting a Prime Number Theorem for Kissing Primes

In this section, we use the analysis in [Bourgain et al. 10], as well as the conjectured randomness of the Möbius function, to arrive at the heuristic in Conjecture 1.3 for the number of kissing primes, i.e., pairs of tangent circles both of prime curvature less than x. With the same notation as in Section 2.1, we would now like to count the points in $\mathcal{O}$ for which v_i^* and v_j are prime for some $j \ne i$, so we sum over points for which either v_i^* or v_j is 0 modulo some square-free d. To do this, we need the total number of pairs of mutually tangent circles of curvature less than x in a packing P. If $N_P(x)$ is the number of circles of curvature up to x as specified by Theorem 2.2,

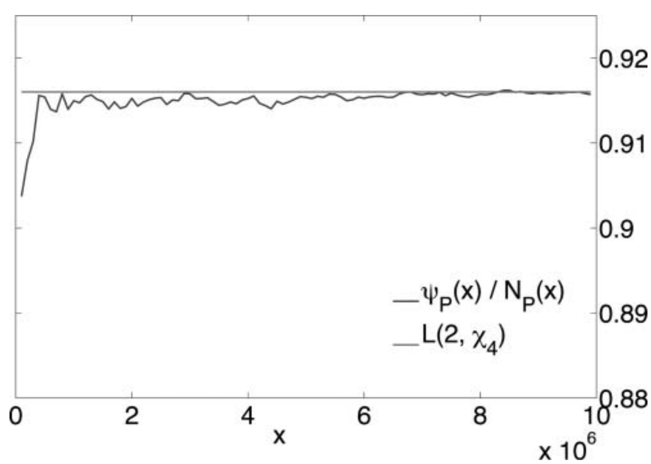

FIGURE 6. Prime number heuristic for P_B.

it is shown in [Kontorovich and Oh 11] that

$$\begin{aligned}&\#\{\text{pairs of mutually tangent circles of curvature } a \le x\}\\&\quad = 3 \cdot N_P(x) + 3.\end{aligned}$$

We again employ the function $\Lambda(n)$ in order to write down a concrete expression for the number of kissing primes less than x in a packing P.

Lemma 2.7. *Let v_i^* and v_j be two distinct coordinates of a vector $\mathbf{v}$ in $\mathcal{O}$, where v_i^* denotes the maximum coordinate of $\mathbf{v}$, and let $\psi_P^{(2)}(x)$ be as in (1–4). Then*

$$\psi_P^{(2)}(x) = \sum_{\substack{\mathbf{v}\in\mathcal{O}\\ v_i^*\le x}} \sum_{j\ne i} \Lambda(v_i^*)\Lambda(v_j) + O(x). \tag{2–14}$$

Again, the sum in (2–14) is a count of all mutually tangent pairs of circles whose curvatures are *powers* of primes, but by a similar argument to that in Section 2.1, we have that including powers of primes in our count does not affect the final answer significantly. Note that in order to evaluate (2–14), we introduce in (2–17) a function that counts points in $\mathcal{O}$ for which *two* of the coordinates are 0 modulo p. Denote by $D < x$ the level distribution from the analysis in [Bourgain et al. 10]; that is, the moduli $d > 1$ in the computations below may be taken to be less than D. We rewrite the expression in (2–14) using (2–8) and get

$$\begin{aligned}&\sum_{\substack{1\le i,j\le 4\\ i\ne j}} \sum_{\substack{\mathbf{v}\in\mathcal{O}\\ v_i^*\le x}} \left(\sum_{d_i | v_i^*} \mu(d_i)\log d_i \sum_{d_j | v_j} \mu(d_j)\log d_j \right)\\&\quad = \sum_{\substack{1\le i,j\le 4\\ i\ne j}} \sum_{\substack{\mathbf{v}\in\mathcal{O}\\ v_i^*\le x}} \left(\Sigma^- + \Sigma^+\right) + O(x),\end{aligned} \tag{2–15}$$

where

$$\begin{aligned}\Sigma^- &= \left(\sum_{d_i\le D} \mu(d_i)\log d_i \sum_{v_i^*\equiv 0\ (d_i)} 1 \right)\\&\qquad \times \left(\sum_{d_j\le D} \mu(d_j)\log d_j \sum_{v_j\equiv 0\ (d_j)} 1 \right)\end{aligned}$$

and

$$\Sigma^+ = \sum_{d_i | v_i^*} \mu(d_i)\log d_i \sum_{d_j|v_j} \mu(d_j)\log d_j - \Sigma^-.$$

As in Section 2.1, we omit Σ^+, or the terms containing $d_i > D$, in (2–15) under the assumption that μ behaves randomly for large values of d. Along with the results about the remainder term in the sieve in [Bourgain et al. 10], the expression in (2–15) is asymptotic to

$$\begin{aligned}&N_P(x) \sum_{\substack{1\le i,j\le 4\\ i\ne j}} \sum_{[d_i,d_j]\le D'} \beta_i\left(\frac{d_i}{(d_i,d_j)}\right) \beta_j\left(\frac{d_j}{(d_i,d_j)}\right)\\&\qquad \times g((d_i,d_j))\mu(d_i)\mu(d_j)\log d_i \log d_j,\end{aligned} \tag{2–16}$$

where $\beta_i(d)$ is as before, $[d_i, d_j]$ is the least common multiple of d_i and d_j, and (d_i, d_j) is their greatest common divisor. The function g above is the ratio

$$\begin{aligned}g((d_i,d_j)) &= \frac{1}{\#\{\mathbf{v}\in\mathcal{O}_{(d_i,d_j)}\}}\\&\quad \cdot \#\{\mathbf{v}\in\mathcal{O}_{(d_i,d_j)} \mid v_i \equiv 0\,((d_i,d_j)),\ v_j\equiv 0\,((d_i,d_j))\},\end{aligned} \tag{2–17}$$

where (d_i, d_j) is square-free in our case. Note that $g(d)$ is multiplicative outside of the primes 2 and 3 by Theorem 1.4, so

$$g(d) = \prod_{p|d} p,$$

and we have only to compute $g(p)$ for p prime in evaluating the sum above.

Lemma 2.8. *Let $g(p)$ be as before, where p is a prime. Then*

(i)

$$g(2) = \begin{cases} 1 & \text{if both } v_i \text{ and } v_j \text{ are even,}\\ 0 & \text{if at least one of } v_i \text{ and } v_j \text{ is odd.}\end{cases}$$

(ii)

$$g(p) = \begin{cases} \frac{1}{(p+1)^2} & \text{for } p \equiv 1 \bmod 4,\\ \frac{1}{p^2+1} & \text{for } p\equiv 3 \bmod 4.\end{cases}$$

Proof. To prove (ii), we note that Theorem 1.4 implies that the numerator of $g(p)$ is

$$\begin{aligned}&\#\{\mathbf{v}\in\mathcal{O}_d \mid v_1\equiv 0\,(d) \text{ and } v_2\equiv 0\,(d)\}\\&\quad = \#\{(v_1,v_2)\in\mathbb{F}_p^2 - \{\mathbf{0}\} \mid F(v_1,v_2,0,0) = 0\}\end{aligned}$$

for $p \ge 5$. Thus it is the number of nontrivial solutions to a binary quadratic form with determinant 0 (the Descartes form in (1–1) with two of the v_i, v_j set to 0), and so

$$\#\{\mathbf{v}\in\mathcal{O}_p \mid v_1\equiv 0 \bmod p \text{ and } v_2 \equiv 0 \bmod p\} = p-1$$

for all $p \ge 5$ (see [Cassels 78], for example). In the case $p = 3$, we observe that in both of the possible orbits of A mod 3 in Figures 3 and 4 we have $g(3) = \frac{1}{10}$, as desired. Part (i) follows from the structure of the orbit $\mathcal{O}_2$ as observed in the proof of Lemma 2.1. $\square$

Denote by $\nu(d_i, d_j)$ the greatest common divisor of d_i and d_j (we write just ν from now on and keep in mind that ν depends on d_i and d_j). Note that

$$\beta\left(\frac{[d_i, d_j]}{\nu}\right) = \beta_i\left(\frac{d_i}{\nu}\right)\beta_j\left(\frac{d_j}{\nu}\right),$$

where $\frac{d_i}{\nu}$ and $\frac{d_j}{\nu}$ are relatively prime, so we rewrite the expression in (2–16):

$$\frac{N_P(x)}{4}\sum_{\substack{1\le i,j\le 4\\ i\ne j}}\sum_{[d_i,d_j]\le D'}\beta_i\left(\frac{d_i}{\nu}\right)\beta_j\left(\frac{d_j}{\nu}\right) \times g(\nu)\mu(d_i)\mu(d_j)\log d_i \log d_j. \tag{2–18}$$

To compute the sum in (2–18), we use a similar argument to that in Section 2.1. We note that the inner sum in the expression above is equal to

$$\sum_{d_i>0}\sum_{d_j>0}\beta_i\left(\frac{d_i}{\nu}\right)\beta_j\left(\frac{d_j}{\nu}\right)g(\nu)\mu(d_i)\mu(d_j)\log d_i\log d_j - \sum_{[d_i,d_j]>D'}\beta_i\left(\frac{d_i}{\nu}\right)\beta_j\left(\frac{d_j}{\nu}\right)g(\nu)\mu(d_i)\mu(d_j)\log d_i\log d_j m, \tag{2–19}$$

where we assume that the sum over $[d_i, d_j] > D'$ is insignificant by the conjectured randomness of the Möbius function, and thus the sum over all d_i and d_j is a good heuristic for the sum in (2–18). We compute the infinite sum in the following lemma and obtain the heuristic in Conjecture 1.3.

Lemma 2.9. *Let $\beta_i(d)$, $\beta_j(d)$, and $g(\nu)$ be as before. We have*

$$\sum_{\substack{1\le i,j\le 4\\ i\ne j}}\sum_{d_i>0}\sum_{d_j>0}\beta_i\left(\frac{d_i}{\nu}\right)\beta_j\left(\frac{d_j}{\nu}\right) \times g(\nu)\mu(d_i)\mu(d_j)\log d_i\log d_j = 8\cdot L^2(2,\chi_4)\cdot\prod_{p\equiv 3\,(4)}\left(1-\frac{2}{p(p-1)^2}\right).$$

Proof. We introduce the function

$$f(s_i,s_j) = \sum_{d_i}\sum_{d_j}\beta_i\left(\frac{d_i}{\nu}\right)\beta_j\left(\frac{d_j}{\nu}\right)g(\nu)\mu(d_i)\mu(d_j)d_i^{s_i}d_j^{s_j},$$

and note that

$$\left.\frac{\partial^2 f(s_i,s_j)}{\partial s_i\partial s_j}\right|_{(0,0)} = \sum_{d_i}\sum_{d_j}\beta\left(\frac{d_i}{\nu}\right)\beta\left(\frac{d_j}{\nu}\right)g(\nu)\mu(d_i)\mu(d_j)\log d_i\log d_j,$$

which is a good heuristic for the sum in (2–18) as x tends to infinity. The difficulty in computing this is the interaction of d_i and d_j in $g(\nu)$. To this end, write $d_i = \nu e_i$ and $d_j = \nu e_j$, where $(e_i, e_j) = 1$. This gives us the following formula for $f(s_i, s_j)$:

$$\begin{aligned}
f(s_i,s_j) &= \sum_\nu g(\nu) \times \sum_{(e_i,e_j)=1}\beta_i(e_i)\beta_j(e_j)\mu(\nu e_i)\mu(\nu e_j)(\nu e_i)^{s_i}(\nu e_j)^{s_j}\\
&= \sum_\nu g(\nu)\nu^{-(s_i+s_j)} \times \sum_{e_i,e_j}\sum_{m|(e_i,e_j)}\mu(m)\beta_i(e_i)\beta_j(e_j)\mu(\nu e_i)\mu(\nu e_j)e_i^{s_i}e_j^{s_j}\\
&= \sum_\nu g(\nu)\nu^{-(s_i+s_j)}\sum_m \mu(m)\mu^2(\nu m)m^{-(s_i+s_j)} \times \sum_{\substack{(b_i,\nu m)=1\\(b_j,\nu m)=1}}\beta_i(mb_i)\beta_j(mb_j)\mu(b_i)\mu(b_j)b_i^{s_i}b_j^{s_j}\\
&= \sum_\nu g(\nu)\mu^2(\nu)\nu^{-(s_i+s_j)} \times \sum_{(m,\nu)=1}\mu(m)\beta_i(m)\beta_j(m)m^{-(s_i+s_j)}A(\nu,m,s_i,s_j),
\end{aligned}$$

where

$$\begin{aligned}
A(\nu,m,s_i,s_j) &= \sum_{\substack{(b_i,\nu m)=1\\(b_j,\nu m)=1}}\beta_i(b_i)\beta_j(b_j)\mu(b_i)\mu(b_j)b_i^{-s_i}b_j^{-s_j}\\
&= \prod_{\substack{p_i|\nu m\\ p_j|\nu m}}\left((1-\beta_i(p_i)p_i^{-s_i})(1-\beta_j(p_j)p_j^{-s_j})\right)^{-1} \times \prod_{p_i,p_j}(1-\beta_i(p_i)p_i^{-s_i})(1-\beta_j(p_j)p_j^{-s_j})\\
&= P_\nu(s_i,s_j)\cdot P_m(s_i,s_j)\cdot\zeta^{-1}(s_i+1) \times \prod_{p_i}\frac{1-\beta_i(p_i)p_i^{-s_i}}{1-p_i^{-s_i-1}}\zeta^{-1}(s_j+1)\prod_{p_j}\frac{1-\beta_j(p_j)p_j^{-s_j}}{1-p_j^{-s_j-1}}\\
&= P_\nu(s_i,s_j)\cdot P_m(s_i,s_j)\cdot\zeta^{-1}(s_i+1)\zeta^{-1}(s_j+1) \times B(s_i,s_j),
\end{aligned}$$

where

$$P_\nu(s_i,s_j)=\prod_{\substack{p_i|\nu\\ p_j|\nu}}\left((1-\beta_i(p_i)p_i^{-s_i})(1-\beta_j(p_j)p_j^{-s_j})\right)^{-1},$$

$$P_m(s_i,s_j)=\prod_{\substack{p_i|m\\ p_j|m}}\left((1-\beta_i(p_i)p_i^{-s_i})(1-\beta_j(p_j)p_j^{-s_j})\right)^{-1},$$

$$B(s_i,s_j)=\prod_{p_i}\frac{1-\beta_i(p_i)p_i^{-s_i}}{1-p_i^{-s_i-1}}\prod_{p_j}\frac{1-\beta_j(p_j)p_j^{-s_j}}{1-p_j^{-s_j-1}}.$$

We write

$$C(\nu,s_i,s_j)=g(\nu)\nu^{-(s_i+s_j)},$$
$$D(m,s_i,s_j)=\mu(m)\beta_i(m)\beta_j(m)m^{-(s_i+s_j)},$$

which gives us

$$\begin{aligned}f(s_i,s_j)=&\sum_\nu C(\nu,s_i,s_j)\cdot P_\nu(s_i,s_j)\\&\times\sum_{(m,\nu)=1}D(m,s_i,s_j)\cdot P_m(s_i,s_j)\\&\times B(s_i,s_j)\cdot\zeta^{-1}(s_i+1)\zeta^{-1}(s_j+1).\end{aligned}$$

Note that the sums over ν and m, as well as $B(s_i,s_j)$, converge and are holomorphic. We now compute the desired derivative. Write

$$\begin{aligned}G_{i,j}(\nu,m,s_i,s_j)=&\sum_\nu C(\nu,s_i,s_j)\cdot P_\nu(s_i,s_j)\\&\times\sum_{(m,\nu)=1}D(m,s_i,s_j)\cdot P_m(s_i,s_j).\end{aligned}$$

Then we have

$$\begin{aligned}\left.\frac{\partial^2 f(s_i,s_j)}{\partial s_i\partial s_j}\right|_{(0,0)}&=\zeta^{-2}(1)\left.\frac{\partial^2(G_{i,j}(\nu,m,s_i,s_j))}{\partial s_i\partial s_j}\right|_{(0,0)}\\&\quad+G_{i,j}(\nu,m,0,0)\cdot B(0,0)\left(\frac{\zeta'(1)}{\zeta^2(1)}\right)^2.\end{aligned}$$

Since $\zeta^{-2}(1)=0$, we need not compute the partial derivative of G. If $\beta_i(2)=\beta_j(2)=0$ in the expression for B, we have $B(0,0)=4\cdot L^2(2,\chi_4)$, since $\frac{\zeta'(1)}{\zeta^2(1)}=2\cdot L(2,\chi_4)$ as in Section 2.1. This holds for the sums in (2–18) over (v_i^*,v_j) and (v_j^*,v_i), where the ith and jth coordinates in our orbit are everywhere odd.

The contribution to (2–18) from the terms where v_i^* or v_j is even is 0:

Lemma 2.10. *Let $\mathcal{O}$ be an orbit of the Apollonian group. Given that the ith or jth coordinate of each vector $\mathbf{v}\in\mathcal{O}$ is even, we have*

$$G_{i,j}(\nu,m,0,0)\cdot B(0,0)=0.$$

Proof. Recall that two of the coordinates of the vectors in $\mathcal{O}$ are always even, and two are odd. We write

$$\begin{aligned}G_{i,j}&(\nu,m,0,0)\cdot B(0,0)\qquad\qquad(2\text{–}20)\\&=B(0,0)\Big(\sum_{2|\nu}C(\nu,0,0)\cdot P_\nu(0,0)\\&\qquad\times\sum_{(m,\nu)=1}D(m,0,0)\cdot P_m(0,0)\\&\qquad+\sum_{(\nu,2)=1}C(\nu,0,0)\cdot P_\nu(0,0)\\&\qquad\quad\times\sum_{(m,\nu)=1}D(m,0,0)\cdot P_m(0,0)\Big).\end{aligned}$$

Case 1: Only one of the ith and jth coordinates is odd throughout the orbit. Recall from Lemma 2.8 that $g(2)=0$ in the case that only one of the coordinates (i,j) is even throughout the orbit, and $g(2)=1$ if both coordinates are even throughout the orbit. So if only one of the coordinates (i,j) is even throughout the orbit, we have

$$B(0,0)\cdot\Big(\sum_{2|\nu}C(\nu,0,0)\cdot P_\nu(0,0)\times\sum_{(m,\nu)=1}D(m,0,0)\cdot P_m(0,0)\Big)=0.$$

Recalling that $\beta_i(p)=\beta_j(p)$ for $p>2$ from Lemma 2.1, we have

$$\begin{aligned}B(0,0)&\Big(\sum_{(\nu,2)=1}C(\nu,0,0)\cdot P_\nu(0,0)\qquad\qquad(2\text{–}21)\\&\quad\Big(\sum_{(m,\nu)=1}D(m,0,0)\cdot P_m(0,0)\Big)\Big)\\&=\prod_{p_i,p_j}\left(\frac{(1-\beta_i(p_i))(1-\beta_j(p_j))}{(1-p_i^{-1})(1-p_j^{-1})}\right)\\&\quad\times\Big(\sum_{(\nu,2)=1}C(\nu,0,0)\cdot P_\nu(0,0)\\&\quad\times\Big(\sum_{\substack{(m,\nu)=1\\(m,2)=1}}D(m,0,0)\prod_{p|m}(1-\beta_i(p))^{-2}\Big)\Big)\\&\quad+B(0,0)\cdot\Big(\sum_{(\nu,2)=1}C(\nu,0,0)\cdot P_\nu(0,0)\\&\quad\times\Big(\sum_{(2m,\nu)=1}\mu(2m)\beta_i(2m)\beta_j(2m)\cdot P_m(0,0)\Big)\Big)\\&=0+0\\&=0,\end{aligned}$$

since Lemma 2.1 implies either $1-\beta_i(2)=0$ or $1-\beta_j(2)=0$ in the first term, and either $\beta_i(2m)=0$ or $\beta_j(2m)=0$ in the second term in (2–21). We now compute the expression in (2–20) in the case that both the ith and jth coordinates are even throughout the orbit.

Case 2: The ith and jth coordinates are both even throughout the orbit. In this case, Lemma 2.8 implies that $g(2\nu) = g(\nu)$ and that $C(2\nu, 0, 0) = C(\nu, 0, 0)$ for odd ν. Also, we again have that $\beta_i(p) = \beta_j(p)$ for $p > 2$, so the first sum in (2–20) is

$$\begin{aligned} &B(0,0) \cdot \left(\sum_{2|\nu} C(\nu,0,0) \cdot P_\nu(0,0) \right. \\ &\qquad \left. \times \left(\sum_{(m,\nu)=1} D(m,0,0) \cdot P_m(0,0) \right) \right) \\ &= \left(\sum_{(\nu,2)=1} \Big(C(2\nu,0,0) \prod_{p|\nu} (1-\beta_i(p))^{-2} \Big) \right. \\ &\qquad \left. \times \left(\sum_{(m,2\nu)=1} \Big(D(m,0,0) \prod_{p|m} (1-\beta_i(p))^{-2} \Big) \right) \right) \\ &\qquad \times \prod_{p\neq 2} \left(\frac{1-\beta_i(p)}{1-p^{-1}} \right)^2 \\ &= \left(\sum_{(\nu,2)=1} \Big(C(\nu,0,0) \prod_{p|\nu} (1-\beta_i(p))^{-2} \Big) \right. \\ &\qquad \left. \times \left(\sum_{(m,2\nu)=1} \Big(D(m,0,0) \prod_{p|m} (1-\beta_i(p))^{-2} \Big) \right) \right) \\ &\qquad \times L^2(2,\chi_4). \end{aligned} \tag{2–22}$$

To compute the second sum in (2–20), we note that $\beta_i(2m) = \beta_j(2m) = \beta_i(m)$ by Lemma 2.1 and write

$$\begin{aligned} B(0,0) &\left(\sum_{(\nu,2)=1} C(\nu,0,0) \cdot P_\nu(0,0) \right. \\ &\qquad \left. \times \left(\sum_{(m,\nu)=1} D(m,0,0) \cdot P_m(0,0) \right) \right) \\ &= M_1(\nu,0,0) + M_2(\nu,0,0), \end{aligned}$$

where

$$\begin{aligned} &M_1(\nu,0,0) \\ &\quad = \left(\sum_{(\nu,2)=1} C(\nu,0,0) \cdot P_\nu(0,0) \right. \\ &\qquad \left. \times \left(\sum_{(m,2\nu)=1} \mu(m)\beta_i^2(m) \cdot \prod_{p|m} (1-\beta_i(p))^{-2} \right) \right) \\ &\qquad \times B(0,0) \end{aligned}$$

and

$$\begin{aligned} &M_2(\nu,0,0) \\ &\quad = \left(\sum_{(\nu,2)=1} C(\nu,0,0) \cdot P_\nu(0,0) \right. \\ &\qquad \left. \times \left(\sum_{(m,2\nu)=1} \mu(2m)\beta_i^2(m) \cdot \prod_{p|m} (1-\beta_i(p))^{-2} \right) \right) \\ &\qquad \times B'(0,0), \end{aligned}$$

where

$$B'(0,0) = \prod_{p\neq 2} \left(\frac{1-\beta_i(p)}{1-p^{-1}} \right)^2 .$$

Note that since $1-\beta_i(2) = 1-\beta_j(2) = 0$, we have $B(0,0) = 0$ in this case, so

$$M_1(\nu,0,0) = 0.$$

On the other hand,

$$\begin{aligned} &M_2(\nu,0,0) \\ &\quad = \left(\sum_{(\nu,2)=1} C(\nu,0,0) \cdot P_\nu(0,0) \right. \\ &\qquad \left. \left(\sum_{(m,2\nu)=1} -\mu(m)\beta_i^2(m) \cdot \prod_{p|m} (1-\beta_i(p))^{-2} \right) \right) \\ &\qquad \times B'(0,0) \\ &\quad = -\left(\sum_{(\nu,2)=1} \Big(C(\nu,0,0) \prod_{p|\nu} (1-\beta_i(p))^{-2} \Big) \right. \\ &\qquad \left. \times \left(\sum_{(m,2\nu)=1} \Big(D(m,0,0) \prod_{p|m} (1-\beta_i(p))^{-2} \Big) \right) \right) \\ &\qquad \times L^2(2,\chi_4). \end{aligned} \tag{2–23}$$

Combining (2–22) and (2–23), we have that the expression in (2–20) is 0, as desired. □

Therefore, we have that in the contributing terms of (2–18), both v_i^* and v_j are odd. This makes up two of the terms in (2–18), so we have that the sum we wish to compute is equal to

$$\begin{aligned} &2 \cdot G_{i,j}(\nu,m,0,0) \cdot 4 \cdot L^2(2,\chi_4) \\ &\quad = 8 \cdot L^2(2,\chi_4) \cdot G_{i,j}(\nu,m,0,0). \end{aligned} \tag{2–24}$$

It remains to compute $G_{i,j}(\nu,m,0,0)$ in this case. Recall that $\beta_i(2) = \beta_j(2) = 0$ in this case, and that $\beta_i(p) =$

$\beta_j(p)$ for $p > 2$. We have

$$\sum_{(m,\nu)=1} \mu(m)\beta_i^2(m) \prod_{p|m} (1-\beta_i(p))^{-2} \tag{2–25}$$
$$= \prod_p 1 - \frac{\beta_i^2(p)}{(1-\beta_i(p))^2} \prod_{p|\nu} \left(1 - \frac{\beta_i^2(p)}{(1-\beta_i(p))^2}\right)^{-1}$$
$$= \prod_{p\equiv 1\,(4)} \left(1 - \frac{1}{p^2}\right)$$
$$\times \prod_{p\equiv 3\,(4)} \left(1 - \frac{(p+1)^2}{(p^2-p)^2}\right) \prod_{p|\nu} \left(1 - \frac{\beta_i^2(p)}{(1-\beta_i(p))^2}\right)^{-1}.$$

We write

$$\sigma = \prod_{p\equiv 1\,(4)} 1 - \frac{1}{p^2} \prod_{p\equiv 3\,(4)} 1 - \frac{(p+1)^2}{(p^2-p)^2}$$

and get

$$G_{i,j}(\nu, m, 0, 0) \tag{2–26}$$
$$= \sigma \cdot \sum_\nu C(\nu,0,0) P_\nu(0,0) \cdot \prod_{p|\nu} \left(1 - \frac{\beta^2(p)}{(1-\beta(p))^2}\right)^{-1}$$
$$= \sigma \cdot \sum_\nu \prod_{p|\nu} \frac{g(p)}{(1-\beta^2(p)(1-\beta(p))^{-2})(1-\beta(p))^2}$$
$$= \sigma \cdot \prod_p 1 + \frac{g(p)}{(1-\beta^2(p)(1-\beta(p))^{-2})(1-\beta(p))^2}$$
$$= \sigma \cdot \prod_{p\equiv 1\,(4)} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p\equiv 3\,(4)} 1 + \frac{p^2+1}{p^4 - 2p^3 - 2p - 1}$$
$$= \prod_{p\equiv 3\,(4)} \left(1 - \frac{(p+1)^2}{(p^2-p)^2}\right)\left(1 + \frac{p^2+1}{p^4-2p^3-2p-1}\right).$$

Therefore our infinite sum is equal to

$$8 \cdot L^2(2,\chi_4) \cdot \prod_{p\equiv 3\,(4)} 1 - \frac{2}{p(p-1)^2}, \tag{2–27}$$

as desired. □

Conjecture 1.3 follows from our assumption that the Möbius function is random and from Lemma 2.5. Namely, we predict

$$\psi_P^{(2)}(x) \approx 8 \cdot \frac{N_P(x)}{4} \cdot L^2(2,\chi_4) \cdot \prod_{p\equiv 3\,(4)} 1 - \frac{2}{p(p-1)^2}$$
$$= c \cdot L^2(2,\chi_4),$$

where $c = 1.646\ldots$ as in Conjecture 1.3.

As with our heuristic for the number of prime curvatures less than x in a packing, this count does not depend on the packing P. Figures 7 and 8 show graphs of

$$y = \frac{\psi_P^{(2)}(x)}{N_P^{(2)}(x)}$$

for $P = P_B$ and P_C, and $x \le 10^8$. The convergence to $\alpha = c \cdot L^2(2,\chi_4)/3$ in these graphs (especially in the case of the first graph) is not as striking as that of Figures 5 and 6. This is because we are unable to compute to a large enough value of x for which our kissing prime heuristic would be most precise. This phenomenon of slower convergence in the case of k-tuples of primes for $k \ge 2$ can be observed even in the setting of the integers, and it is certainly the case here.

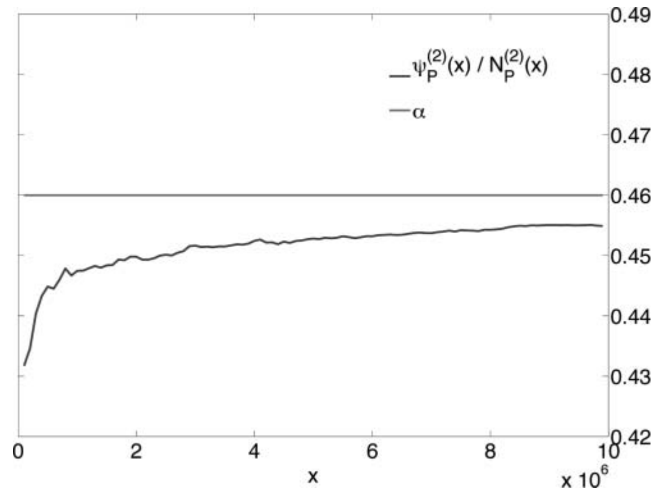

FIGURE 7. Prime number theorem for kissing primes for the packing P_B.

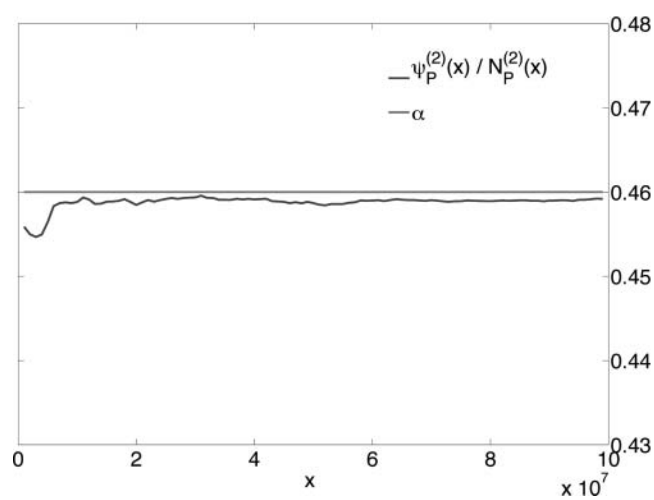

FIGURE 8. Prime number theorem for kissing primes for the packing P_C.

3. LOCAL-TO-GLOBAL PRINCIPLE FOR ACPS

In this section we present numerical evidence in support of Conjecture 1.1, which predicts a local-to-global principle for the curvatures in a given integral ACP. Since the Apollonian group A is a thin group—it is of infinite index in $\mathrm{O}_F(\mathbb{Z})$—it is remarkable that its orbit should eventually cover all the integers outside of the local obstruction modulo 24 as specified in Theorem 1.4. Proving this rigorously, however, appears to be very difficult. An analogous problem over $\mathbb{Z}$ would be to show that all large integers satisfying certain local conditions are represented by a general ternary quadratic form; this analogy is realized by fixing one of the curvatures in Descartes form and solving the problem for the resulting ternary form. While this problem was recently resolved in general in [Cogdell 03] and [Duke and Schulze-Pillot 90], even there the local-to-global principle comes in a much more complicated form, relying on congruence obstructions specified in the spin double cover. Our conjecture, which therefore has the flavor of Hilbert's 11th problem for an indefinite form, predicts a local-to-global principle of a more straightforward nature.

Our computations suggest that this conjecture is true, and we predict the value X_P in the examples we check. We consider the packings P_B and P_C introduced in Section 1. Recall that P_B corresponds to the orbit of A acting on $(-1, 2, 2, 3)$, and P_C corresponds to the orbit of A acting on $(-11, 21, 24, 28)$.

In order to explain the data we obtain in both cases, we use Theorem 1.4 to determine the congruence classes (mod 24) in the given packing. Recall that the Apollonian group A is generated by the four generators S_i in (1–2). We can view an orbit of A modulo 24 as a finite graph $\mathcal{G}_{24}$ in which each vertex corresponds to a distinct (mod 24) quadruple of curvatures, and two vertices $\mathbf{v}$ and $\mathbf{v}'$ are joined by an edge iff $S_i\mathbf{v} = \mathbf{v}'$ for some $1 \le i \le 4$.

Recall from Theorem 1.4 that for any orbit $\mathcal{O}$ of the Apollonian group,

$$\mathcal{O}_{24} = \mathcal{O}_8 \times \mathcal{O}_3, \tag{3–1}$$

so the graph $\mathcal{G}_{24}$ is completely determined by the structure of $\mathcal{O}_3$ and $\mathcal{O}_8$. There are only two possible orbits modulo 3, pictured in Figures 3 and 4. There are many more possible orbits modulo 8, and we provide the graphs for these orbits in the case of P_B and P_C in Figure 9.

Note that each vertex in $\mathcal{G}_8$ and $\mathcal{G}_3$ is connected to its neighboring vertices via all of the generators S_i. Therefore the curvatures of circles in a packing modulo 24 are equally distributed among the coordinates of the vertices in $\mathcal{G}_{24}$. Combined with Theorem 1.4, this lets us compute the ratio of curvatures in a packing that fall into a specific congruence class modulo 24. Namely, let $\mathcal{O}_{24}(P)$ be the orbit modulo 24 corresponding to a given packing P. For $\mathbf{w} \in \mathcal{O}_{24}(P)$ let w_i be the ith coordinate of $\mathbf{w}$. We define $\gamma(n, P)$ as the proportion of coordinates in $\mathcal{O}_{24}(P)$ congruent to n modulo 24. That is,

$$\gamma(n, P) = \frac{\sum_{i=1}^{4} \#\{w \in \mathcal{O}_{24}(P) \mid w_i = n\}}{4 \cdot \#\{\mathbf{w} \in \mathcal{O}_{24}(P)\}}. \tag{3–2}$$

With this notation, a packing P contains a circle of curvature congruent to n modulo 24 iff $\gamma(n, P) > 0$. Given (3–1), we express γ as follows:

$$\begin{aligned}\gamma(n, P) &= \frac{\sum_{i=1}^{4} \#\{\mathbf{w} \in \mathcal{O}_{24}(P) \mid w_i = n\}}{4 \cdot \#\{\mathbf{w} \in \mathcal{O}_{24}\}} \\ &= \frac{1}{4}\,\frac{\sum_{i=1}^{4} \#\{\mathbf{w} \in \mathcal{O}_8 \mid w_i \equiv n\ (3)\}}{\#\{\mathbf{w} \in \mathcal{O}_8\}} \\ &\quad \times \frac{\#\{w \in \mathcal{O}_3 \mid w_i \equiv n\ (8)\}}{\#\{\mathbf{w} \in \mathcal{O}_3\}}.\end{aligned} \tag{3–3}$$

The significance of γ in the case of any packing (not only the two we consider) is explained in the following lemma.

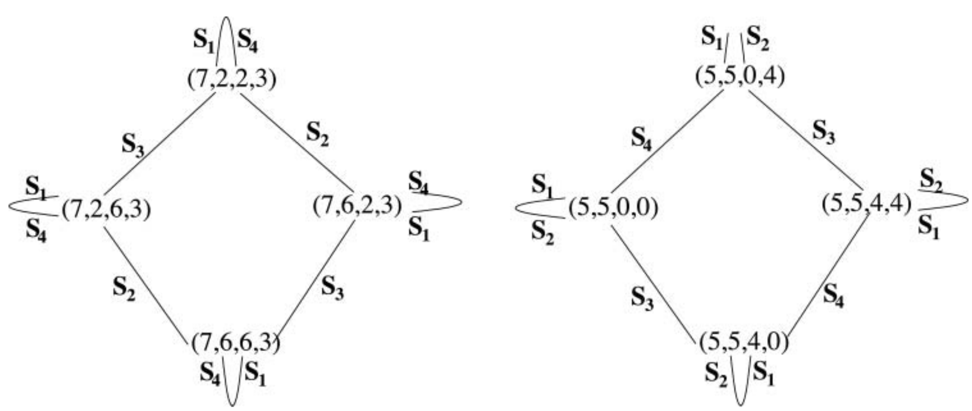

FIGURE 9. Orbits of P_B and P_C modulo 8.

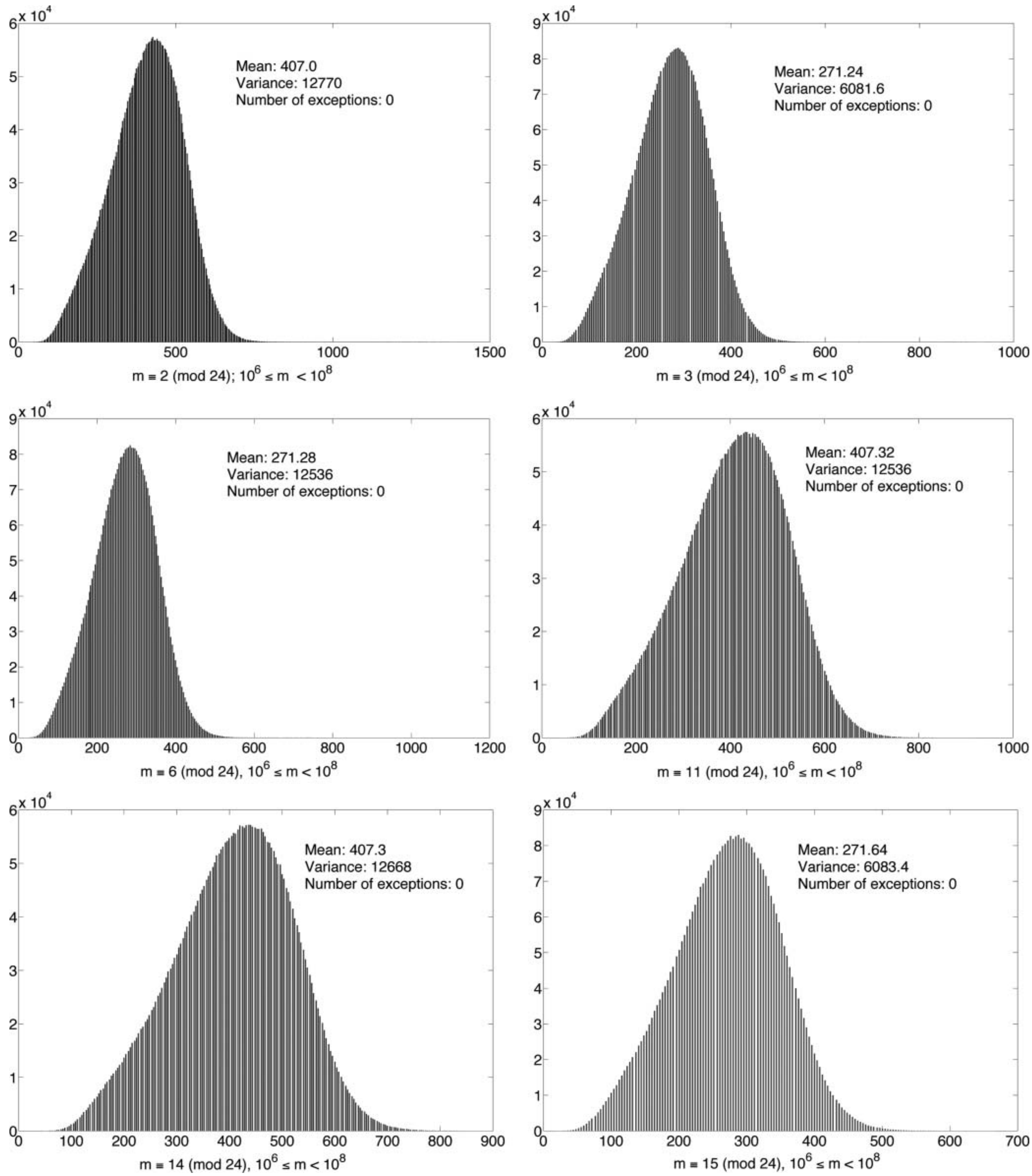

FIGURE 10. Histograms for integers occuring in P_B. *(Continued on next page)*

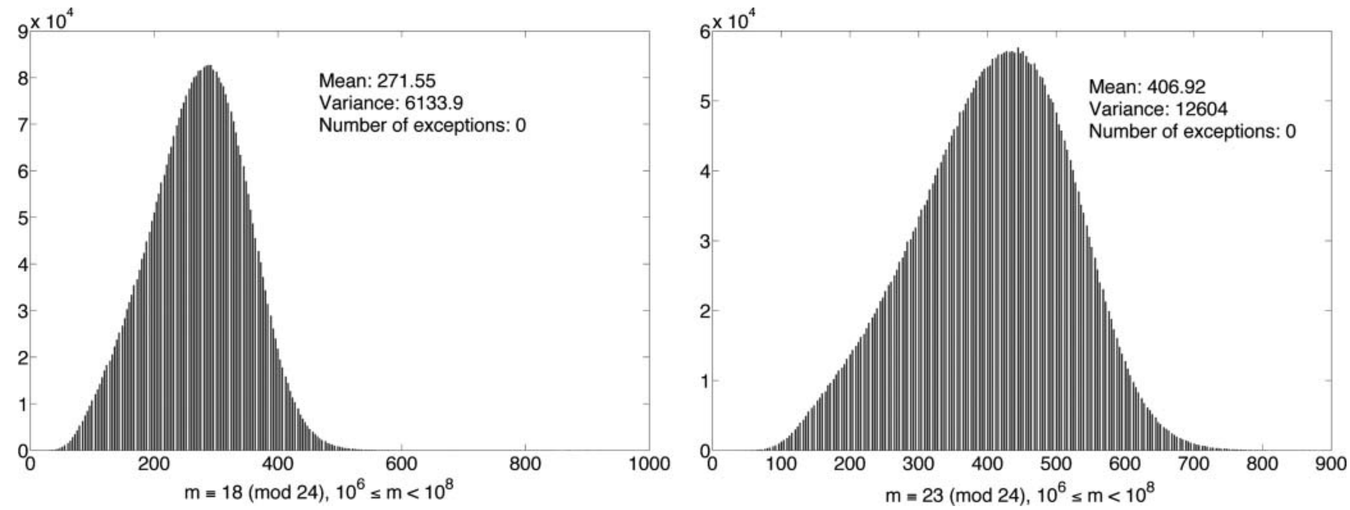

FIGURE 10. (Continued).

Lemma 3.1. *Let $N_P(x)$ be as before, let C be a circle in an integral Apollonian packing P, and let $a(C)$ be the curvature of C. Then*

$$\sum_{\substack{C \in P \\ a(C) < x \\ a(C) \equiv n\ (24)}} 1 \sim \gamma(n, P) \cdot N_P(x).$$

This follows from Theorem 1.4. Note that in general, the orbits $\mathcal{O}_8(P)$ and $\mathcal{O}_3(P)$ have respectively four and ten vertices in the corresponding finite graphs.[4] Therefore $\mathcal{G}_{24}$ always has 40 vertices, and the ratio in (3–3) is easily computed using this graph. With this in mind, we observe the following about the packing P_B.

Lemma 3.2. *Let $P_{B,24}$ denote the possible congruence classes of curvatures modulo 24 in the packing P_B, and let $N_{P_B}(x)$ be as in Theorem 2.2. Then we have the following:*

(i) $N_{P_B}(x) \sim c_{P_B} \cdot x^{\delta}$, *where* $c_{P_B} = 0.402\ldots$.

(ii) $P_{B,24} = \{2, 3, 6, 11, 14, 15, 18, 23\}$.

(iii)

$$\begin{aligned} \gamma(2, P_B) &= \frac{3}{20}, & \gamma(14, P_B) &= \frac{3}{20}, \\ \gamma(3, P_B) &= \frac{1}{10}, & \gamma(15, P_B) &= \frac{1}{10}, \\ \gamma(6, P_B) &= \frac{1}{10}, & \gamma(18, P_B) &= \frac{1}{10}, \\ \gamma(11, P_B) &= \frac{3}{20}, & \gamma(23, P_B) &= \frac{3}{20}. \end{aligned}$$

(iv) *For $10^6 < x < 5 \cdot 10^8$, let x_{24} denote $x \bmod 24$. If $x_{24} \in P_{B,24}$, then x is a curvature in the packing P_B.*

Part (iv) is an observation based solely on our computations using the algorithm described in Section 4; these are illustrated in the histograms in Figure 10. The first three parts follow from computations combined with Theorem 2.2 and Lemma 3.1.

Note that $\gamma(n, P_B) = \gamma(n + 12, P_B)$. For this particular packing, one can therefore express the local obstructions modulo 12 rather than modulo 24. Whenever this is the case for an integral ACP, we will find that there are eight congruence classes modulo 24 in the curvatures of the circles. This is observed in [Graham et al. 03], where the authors compute which integers less than 10^6 are "exceptions" for P_B; they find integers that satisfy these local conditions for P_B modulo 12 but do not occur as curvatures in the packing.[5] Our data extend the findings in [Graham et al. 03] (we consider integers up to $5 \cdot 10^8$), and show that all integers between 10^6 and 10^7 belonging to one of the congruence classes in part (ii) of Remark 3.2 appear as curvatures in the packing P_B.

The histograms depicted in Figure 10 illustrate the distribution of the frequencies with which each integer in the given range satisfying the specified congruence condition occurs as a curvature in the packing P_B. The means of the distributions of these frequencies can be computed,

[4] There are only two possible orbits modulo 3, but many more modulo 8. We examine just two such orbits here.

[5] There is a small error in the computations in [Graham et al. 03]; we have found that the integer $13806 \equiv 6\ (12)$ does not appear as a curvature in P_B. The authors' results do not reflect this.

as we do in (3–6). The variance, however, is much more difficult to predict at this time; explaining the behavior of the variance as we consider larger integers would shed more light on our local-to-global conjecture. Note that there are no exceptions to the local-to-global principle in this range whenever 0 is not a frequency represented in the histogram (i.e., each integer occurs at least once). There are several other frequencies not represented in the histograms for both P_B and P_C (these show up as gaps in the graphs), and an explanation of this aspect would be interesting.

We do the same analysis for the packing P_C. In this case, we must consider much larger integers than in the case of P_B in order to get comparable results. This can be explained partially by the fact that the constant c_P in Kontorovich and Oh's formula

$$N_P(x) \sim c_P \cdot x^\delta \tag{3–4}$$

is much smaller for the packing P_C than for the packing P_B, since the initial four circles in P_C are much larger than the initial four in P_B (See part (i) of Lemmas 3.2 and 3.3). Specifically, $c_{P_C} = 0.0176\ldots$ and $c_{P_B} = 0.402\ldots$. However, our data suggest that the proposed local-to-global principle should hold for this packing as well.

Lemma 3.3. *Let $P_{C,24}$ denote the possible congruence classes modulo 24 in the packing P_C, and let $N_{P_C}(x)$ be as in Theorem 2.2. Then we have the following:*

(i) $N_{P_C}(x) \sim c_{P_C} \cdot x^\delta$, *where* $c_{P_C} = 0.0176\ldots$.

(ii) $P_{C,24} = \{0, 4, 12, 13, 16, 21\}$.

(iii)

$$\gamma(0, P_C) = \frac{1}{10}, \quad \gamma(13, P_C) = \frac{3}{10},$$
$$\gamma(4, P_C) = \frac{3}{20}, \quad \gamma(16, P_C) = \frac{3}{20},$$
$$\gamma(12, P_C) = \frac{1}{10}, \quad \gamma(21, P_C) = \frac{4}{20}.$$

(iv) *For $10^8 < x < 5 \cdot 10^8$, let x_{24} denote x mod 24. If $x_{24} = 13$ or $x_{24} = 21$, then x is a curvature in the packing P_C.*

Again, note that part (iv) is an observation based solely on our computations, while the first three parts in Lemma 3.3 rely on Lemma 3.1 and Theorem 2.2. The histograms in Figure 11 illustrate the distribution of the frequencies with which each integer in the given range satisfying the specified congruence condition occurs as a curvature in the packing P_C. Note that as with P_B, the frequencies with which integers are represented in the packing seem to have a normal distribution. However, since the mean of this distribution is much smaller for P_C than for P_B, we find that 0 is often a frequency represented in the histograms, and so there are still some exceptions to the proposed local-to-global principle in the range we consider.

As we mentioned before, the mean in each of these histograms is easily computable: Let C denote a circle in an Apollonian packing P and let $a(C)$ denote the curvature of C. Let $I = [k, k+K]$ be an interval of length K and let $x \in I$ be an integer. Let

$$\nu(x) = \#\{C \in P \mid a(C) = x\}$$

be the number of times x is a curvature of a circle in P. For an integer $m \geq 0$, let

$$\delta(m, n) = \#\{x \in I \mid x \equiv n\,(24),\, \nu(x) = m\}.$$

Then by Lemma 3.1,

$$\sum_{\substack{x \in I \\ x \equiv n\,(24)}} \nu(x) = \sum_{m \geq 0} \delta(m, n) \cdot m. \tag{3–5}$$

The equivalence of the two sums above is easy to observe: one counts the same set of curvatures, but partitions them differently. In particular, the expression in (3–5) allows us to determine the mean of the distributions in the histograms above. Namely, denote by $x \in I$ an integer in some interval $I = [k, k+K]$ of length K. Let $1 \leq n \leq 24$, and let $\mu(n, P)$ denote the mean of the number of times $x \equiv n$ mod 24 is represented as a curvature in the packing P. Note that there are precisely $K/24$ integers congruent to n mod 24 in the interval I. Combined with (3–5), this gives us

$$\mu(n, P) \approx \frac{24 \cdot \gamma(n, P) \cdot (N_P(k+K) - N_P(k))}{K}. \tag{3–6}$$

This formula predicts the following values for the means in P_B in the range $[10^6, 10^8)$, and P_C in the range $[4 \cdot 10^8, 5 \cdot 10^8)$:

$$\begin{aligned}
\mu(2, P_B) &= \mu(11, P_B) = \mu(14, P_B) = \mu(14, P_B) \\
&= \mu(23, P_B) = 406.70\ldots, \\
\mu(3, P_B) &= \mu(6, P_B) = \mu(15, P_B) = \mu(18, P_B) \\
&= 271.13\ldots, \\
\mu(0, P_C) &= \mu(12, P_C) = 24.35\ldots, \\
\mu(4, P_C) &= \mu(16, P_C) = 36.52\ldots,
\end{aligned}$$

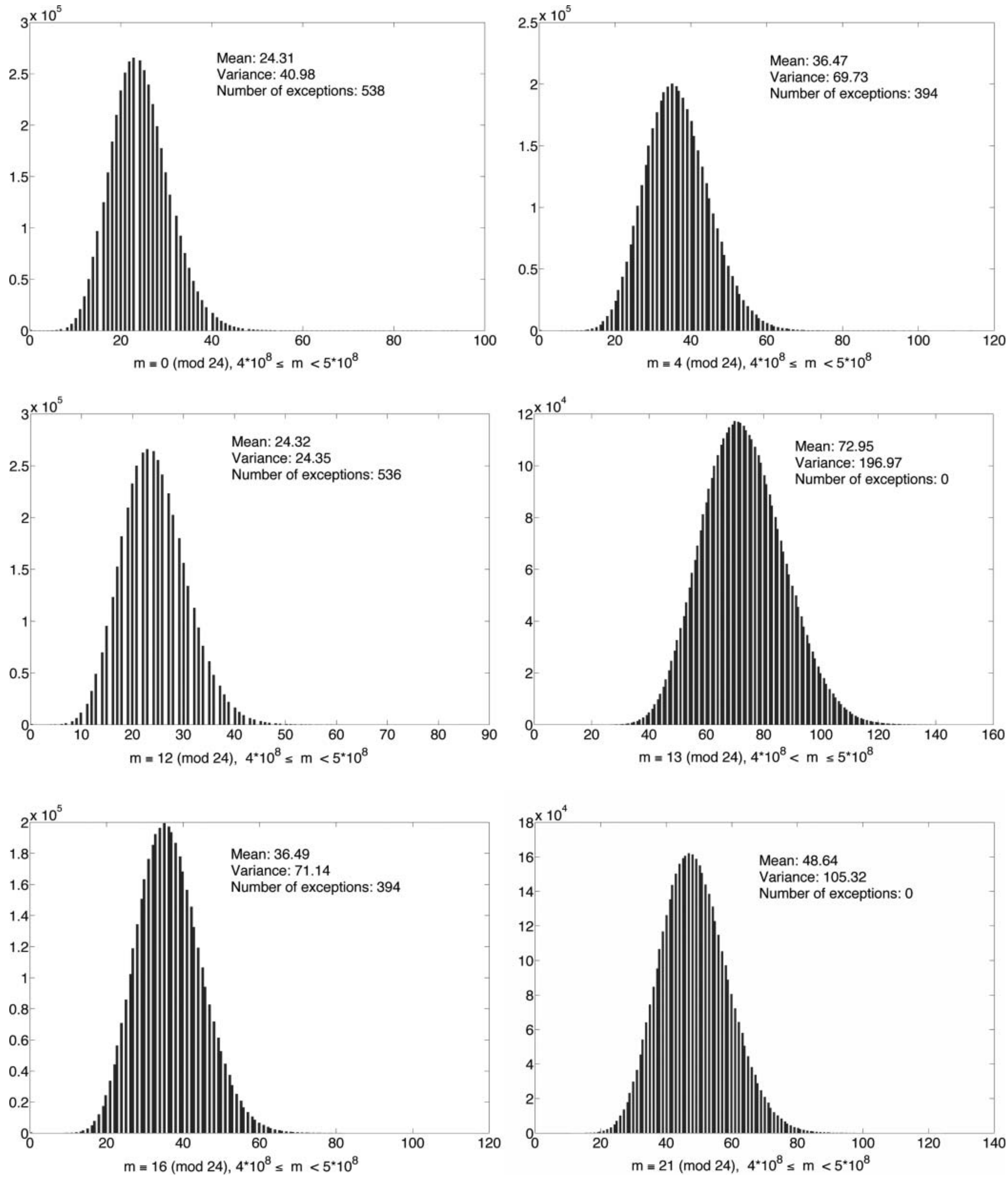

FIGURE 11. Histograms for integers occuring in P_C.

Algorithm 1 Generating all curvatures of magnitude less than x.

(1) Push the root quadruple onto the stack.

(2) Until the stack is empty, perform an iterative process:

  (a) Pop a quadruple off of the stack and generate its children.

  (b) For each child, if the new curvature created (i.e., the maximum entry of the quadruple) is less than x, then push the child onto the stack.

$$\mu(13, P_C) = 73.05\ldots,$$
$$\mu(21, P_C) = 48.70\ldots,$$

which coincides with the means observed in the histograms. This clarifies why the mean is small for packings for which the constant c_P in the formula $N_P(x) \sim c_P \cdot x^\delta$ is small, and why one needs to consider very large integers to see that the local-to-global principle for such packings should hold.

This analysis can be carried out for any ACP, and will likely yield similar results. In the direction of proving Conjecture 1.1, one might investigate how X_P depends on the given packing. Can it perhaps be expressed in terms of the constant c_P in (3–4)? One might also ask how the variance of the distributions above depends on the packing, and how it changes with the size of the integers we consider. Answering this would give further insight into the local-to-global correspondence for curvatures in integer ACPs.

4. A DESCRIPTION OF OUR ALGORITHM AND ITS RUNNING TIME

We represent an ACP by a tree of quadruples. Figure 12 shows the first two generations of the tree corresponding to P_C. To generate all curvatures of magnitude less than x, we use a LIFO (last-in–first-out) stack to generate and prune this tree. The algorithm is presented as Algorithm 1.

By pushing a quadruple onto the stack only if its maximum entry is less than x, we effectively prune the tree. Since we know that each quadruple has a larger maximum entry than its parent, we use step 2b to avoid generating branches whose quadruples are known to have entries greater than x.

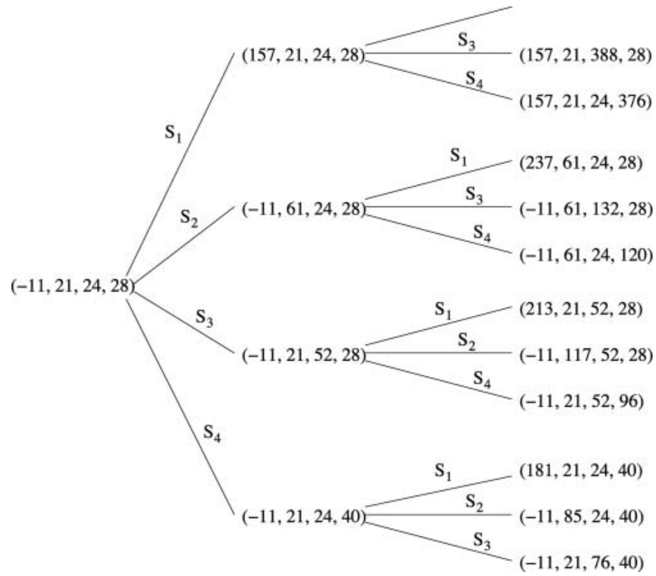

FIGURE 12. The tree of quadruples for P_C, pictured up to two generations.

Although we use the concept of a tree to *generate* curvatures, we note that the entire tree structure is not necessary to *store* such curvatures. Instead, we store the curvatures in a one-dimensional array of x elements, all initialized to zero. The ith element of the array contains the number of curvatures with magnitude i. For instance, the 24th element of the array for P_C is equal to 1, while the 25th element is equal to 0, since there are no curvatures equal to 25 in P_C.

We use these arrays to generate the histograms in Section 3. Due to Matlab's memory constraints, we limit our Matlab arrays to 10^8 entries. So to check for exceptions in the entire range $[10^6, 5 \cdot 10^8)$, we check each of the intervals $[10^6, 10^8)$, $[10^8, 2 \cdot 10^8), \ldots, [4 \cdot 10^8, 5 \cdot 10^8)$ individually. We have chosen to display the interval $[10^6, 10^8)$ in our figures in Section 3.

To count primes less than x, we simply increment a sum whenever a prime curvature is produced. To count kissing primes less than x, we increment a sum whenever a prime curvature is produced and some other member of the curvature's quadruple is prime.

It takes our algorithm $O(N_P(x))$ steps to compute $N_P(x)$, which is optimal, since each node on the tree must be visited; that is, it is not possible to skip any quadruples.

Our programs rely on Wayne and Sedgewicks's Stack data type and standard draw library [Sedgewick and Wayne 07].

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