Experimental Mathematics, 20(4):487–505, 2011

ISSN: 1058-6458 print / 1944-950X online
DOI: 10.1080/10586458.2011.594671

On the Integral Cohomology of Bianchi Groups

Mehmet Haluk Şengün

CONTENTS

2000 AMS Subject Classification: Primary 11F41, 11F75; Secondary 22E40
Keywords: Bianchi groups, torsion homology, cuspidal cohomology

Extensive and systematic machine computations are carried out to investigate the integral cohomology of the Euclidean Bianchi groups and their congruence subgroups. The collected data give insight into several aspects, including the asymptotic behavior of the torsion in the first homology. Along with the experimental work, some basic properties of the integral cohomology are recorded with an eye toward the liftability issue of Hecke eigenvalue systems.

1. INTRODUCTION

Bianchi groups are groups of the form $\mathrm{PSL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of an imaginary quadratic field. First studied by L. Bianchi in 1892 [Bianchi 92], these groups form an important class of arithmetic Kleinian groups. In fact, it is well known that any noncocompact arithmetic Kleinian group, after conjugation, is commensurable with a Bianchi group.

In this paper, I will be interested in the cohomology of Bianchi groups with certain $\mathcal{O}$-module coefficients. These cohomology groups are fundamental to the study of Bianchi modular forms, that is, modular forms (for GL_2) over an imaginary quadratic field. In contrast to their analogues over totally real fields, i.e., Hilbert modular forms, the arithmetic of Bianchi modular forms is little understood. One of the features that obstruct the application of standard methods is the torsion in the cohomology of Bianchi groups.

The first computations of torsion appeared in [Elstrodt et al. 81], in which the abelianizations $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$ ($\simeq H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})$) of congruence subgroups $\Gamma_0(\mathfrak{p})$ for prime ideals $\mathfrak{p}$ of $\mathbb{Z}[i]$ of residue degree 1 and norm ≤ 400 were computed. In the same paper, numerical evidence suggesting a connection between some of the 2-torsion classes and certain S_3-extensions of $\mathbb{Q}(i)$ was exhibited. Later, [Figueiredo 99] provided several more examples that suggested the same kind of connection between certain 3-torsion classes and certain $\mathrm{SL}_2(\mathbb{F}_3)$-extensions of the fields $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3$. Recently, further examples have been given in [Şengün 11a], and also in [Torrey 11], where the author formulates and tests an analogue of the strong modularity conjecture of [Serre 87], which has since been proved.

In [Grunewald and Schwermer 93], the conjugacy classes of small-index subgroups of several Bianchi groups were determined. In particular, the authors computed the abelianizations of a large number of small-index (≤ 12) subgroups. Depending on the data they collected and the data in [Elstrodt et al. 81], they speculated that for any finite-index subgroup Γ of a Bianchi group G, if a prime p appears as an exponent for an element of Γ^{ab}, then p should be smaller than half the index of Γ in G.

In the unpublished thesis [Taylor 88], the existence of p^m-torsion classes in $H_1(\Gamma_1(Np^r), E_{k,\ell}(\mathcal{O}))$ was proved (notation will be explained later) with $k \neq \ell$ under special circumstances. In the unpublished thesis [Priplata 00], the author numerically investigated the torsion for some Bianchi groups, mostly for coefficient modules $E_{k,\ell}$ where $k \neq \ell$. It is worth remarking that the (co)homology of Bianchi groups is related to cuspidal Bianchi modular forms only in the parallel-weight case, that is, in the case $k = \ell$.

In this paper, I report on my extensive systematic computations of the integral (co)homology of Bianchi groups. More specifically, I worked with the groups $\mathrm{PSL}(\mathcal{O}_d)$ and $\mathrm{PGL}_2(\mathcal{O}_d)$ with $-d = 1, 2, 3, 7, 11$ and also with their congruence subgroups. Motivated by the possibility of congruences between cuspidal Bianchi modular forms and torsion classes, and also the arithmetic connections mentioned above, I limited myself to the (co)homology with parallel weights. The data I collected make possible the following assertions:

- They show that H^2_{cusp} can have sporadically large torsion part with very little torsion-free part; in particular, the speculations of Grunewald and Schwermer mentioned above are false.
- They support a recent conjecture of Long, Maclachlan, and Reid [Long et al. 06] on the existence of certain families of rational homology spheres.
- They strongly suggest that an analogue of the very recent result of [Bergeron and Venkatesh 10] on the asymptotic behavior of the torsion in the homology of cocompact arithmetic congruence lattices in $\mathrm{SL}_2(\mathbb{C})$ holds for Bianchi groups.

Along with the computational work, I record some basic properties of the integral cohomology with an eye toward the liftability issue of Hecke eigenvalue systems.

2. THE MODULES

Given a commutative a ring R, let $E_k(R)$ denote the space of homogeneous degree k polynomials in two variables over R. Note that $\{x^{k-i}y^i : 0 \leq i \leq k\}$ is an R-basis of $E_k(R)$.

For a polynomial $P(X,Y)$ in $E_k(R)$ and a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{M}_2(R)$, we have the right action

$$\left(P \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(X,Y) = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\right) = P\left(aX + bY, cX + dY\right).$$

Let $\mathcal{O}$ be the ring of integers of an imaginary quadratic field. Consider the $\mathrm{M}_2(\mathcal{O})$-module

$$E_{k,\ell}(\mathcal{O}) := E_k(\mathcal{O}) \otimes_{\mathcal{O}} \overline{E_\ell(\mathcal{O})}.$$

Here the overline on the second factor indicates that action on the second factor is twisted with complex conjugation. Note that we should insist that $k+\ell$ be even so that $-\mathrm{Id}$ acts trivially and thus $\mathrm{PSL}(2,\mathcal{O})$ acts on it as well.

It is useful to remark that

$$E_{k,\ell}(\mathcal{O}) \simeq \mathrm{Sym}^k(\mathcal{O}^2) \otimes_{\mathcal{O}} \overline{\mathrm{Sym}}^\ell(\mathcal{O}^2)$$

as $\mathrm{M}_2(\mathcal{O})$-modules, where $\mathrm{Sym}^i(\mathcal{O}^2)$ is the ith symmetric power of the standard representation of $\mathrm{M}_2(\mathcal{O})$ on $\mathcal{O}^2$. Here the overline on the second factor means that the action is twisted with complex conjugation.

Let π be a prime element of $\mathcal{O}$ over a rational prime p. Put κ_π for its residue field. We put

$$E_{k,l}(\kappa_\pi) := E_{k,\ell}(\mathcal{O}) \otimes_{\mathcal{O}} \kappa_\pi.$$

If p splits in $\mathcal{O}$, then

$$E_{k,l}(\kappa_\pi) \simeq E_k(\kappa_\pi) \otimes E_\ell(\kappa_{\bar\pi}).$$

Thus $\mathrm{PSL}_2(\mathcal{O})$ acts on it by reduction modulo π on the first factor and by reduction modulo $\bar\pi$ on the second. If p is inert in $\mathcal{O}$, then

$$E_{k,l}(\kappa_\pi) \simeq E_k(\kappa_\pi) \otimes E_\ell(\kappa_\pi)^\sigma.$$

Here $\mathrm{PSL}_2(\mathcal{O})$ acts by reduction modulo π. The action on the second factor is twisted by the nontrivial automorphism σ of κ_π.

Finally, when p is ramified in $\mathcal{O}$, we have

$$E_{k,l}(\kappa_\pi) \simeq E_k(\kappa_\pi) \otimes E_\ell(\kappa_\pi).$$

Here the action of $\mathrm{PSL}_2(\mathcal{O})$ is via reduction modulo π and is the same on both factors.

A result of [Brauer and Nesbitt 41] tells us that in the inert and split cases, the $\mathrm{PSL}_2(\mathcal{O})$-modules $E_{k,\ell}(\kappa_\pi)$ are irreducible only when $0 \leq k, \ell \leq p-1$. In the ramified case, $E_{k,l}(\kappa_\pi)$ is never an irreducible $\mathrm{PSL}_2(\mathcal{O})$-module unless $k = 0 \leq \ell \leq p-1$ or $\ell = 0 \leq k \leq p-1$. For more on the structure of these modules, we refer to reader to [Şengün and Türkelli 09].

The following will be used later.

Proposition 2.1. *Let $\mathcal{O}$ be the ring of integers of an imaginary quadratic field. Let $k \geq \ell$ and put $R = \mathcal{O}[\frac{1}{k!}]$. Then there is a $\mathrm{PSL}_2(\mathcal{O})$-equivariant perfect pairing*

$$E_{k,\ell}(R) \times E_{k,\ell}(R) \to R.$$

It is well known (see, e.g., [Wiese 07, Lemma 2.4]) that there is a perfect pairing on $\mathrm{Sym}^n(R^2)$ coming from the determinant pairing on R^2 whenever $n!$ is invertible in the ring R. The proposition follows by taking the product of the two pairings associated with the two factors of $E_{k,\ell}$. For an explicit description of this pairing, see [Berger 08, Section 2.4]. As a corollary, we see that the modules $E_{k,\ell}(R)$ are self-dual.

3. THE COHOMOLOGY

In this section I will investigate the integral cohomology of Bianchi groups. My treatment is heavily influenced by [Hida 81, Wang 94, Serre 70, Wiese 07].

Let K be an imaginary quadratic field. Let $\mathcal{O}$ be its ring of integers. Let G be the associated Bianchi group. Let Γ be a finite-index subgroup of G. In this paper, we will focus on the $\mathcal{O}$-modules

$$H^i(\Gamma, E_{k,l}(\mathcal{O})), \quad i = 1, 2.$$

It is well known that these are finitely generated.

Definition 3.1. Let $\pi \in \mathcal{O}$ be a prime element over the rational prime p. Assume that $H^i(\Gamma, E_{k,\ell}(\mathcal{O}))$ has π-torsion, i.e., that it contains a nonzero class c such that $\pi \cdot c = 0$. We say that π is a *large torsion* if $k, \ell < p$. Otherwise, we say that π is a *small torsion*.

Proposition 3.2. *Let π be prime element of $\mathcal{O}$ over the rational prime p. Put κ_π for its residue field. Let Γ be a torsion-free finite-index subgroup of the Bianchi group G.*

(a) *If Γ surjects onto $\mathrm{PSL}_2(\kappa_\pi)$ and π is unramified, then $H^1(\Gamma, E_{k,\ell}(\mathcal{O}))$ has no large π-torsion.*

(b) *If Γ surjects onto $\mathrm{PSL}_2(\kappa_\pi)$ and π is ramified, then $H^1(\Gamma, E_{k,\ell}(\mathcal{O}))$ has no π-torsion if and only if $k = 0 \leq \ell \leq p-1$ or $\ell \leq 0 \leq k \leq p-1$.*

(c) *The obstruction to the lifting of a class in $H^1(\Gamma, E_{k,\ell}(\kappa_\pi))$ to $H^1(\Gamma, E_{k,\ell}(\mathcal{O}))$ is the π-torsion in $H^2(\Gamma, E_{k,\ell}(\mathcal{O}))$.*

(d) *$H^2(\Gamma, E_{k,\ell}(\mathcal{O})) \otimes \kappa_\pi \simeq H^2(\Gamma, E_{k,\ell}(\kappa_\pi))$ for every k, ℓ.*

Proof. In the following, let us put $E = E_{k,\ell}$. Consider the short exact sequence

$$0 \to E(\mathcal{O}) \xrightarrow{\cdot\pi} E(\mathcal{O}) \to E(\kappa_\pi) \to 0,$$

where $\cdot\pi$ is the multiplication-by-π map.

The associated long exact sequence gives the short exact sequence

$$0 \to H^i(\Gamma, E(\mathcal{O})) \otimes \kappa_\pi \to H^i(\Gamma, E(\kappa_\pi)) \\ \to H^{i+1}(\Gamma, E(\mathcal{O}))[\pi] \to 0$$

for $i \geq 0$. Here $H^j(\Gamma, E(\mathcal{O}))[\pi]$ denotes the kernel of the map induced by $\cdot\pi$.

Putting $i = 0$, we get

$$E(\kappa_\pi)^\Gamma \simeq H^1(\Gamma, E(\mathcal{O}))[\pi].$$

Now (a) and (b) follow via the irreducibility discussions of the previous section. For $i = 1$, we get

$$0 \to H^1(\Gamma, E(\mathcal{O})) \otimes \kappa_\pi \to H^1(\Gamma, E(\kappa_\pi)) \\ \to H^2(\Gamma, E(\mathcal{O}))[\pi] \to 0,$$

which explains the claim (c). It is known that the virtual cohomological dimension of a Bianchi group is 2. Setting $i = 2$, we get

$$H^2(\Gamma, E(\mathcal{O})) \otimes \kappa_\pi \simeq H^2(\Gamma, E(\kappa_\pi)),$$

finishing the proof. □

Each cohomology space comes equipped with a commuting family $\mathbb{T}$ of Hecke operators acting on it; see [Şengün and Türkelli 09]. An eigenvalue system with values in a ring R is a ring homomorphism $\Phi : \mathbb{T} \to R$. We say that an eigenvalue system Φ occurs in an $R\mathbb{T}$-module A if there is a nonzero element $a \in A$ such that $Ta = \Phi(T)a$ for all T in $\mathbb{T}$. Using the lifting theorem [Ash and Stevens 86, Proposition 1.2.2], we can lift an eigenvalue system occurring in $H^2(\Gamma, E(\kappa_\pi))$ to one occurring in $H^2(\Gamma, E(\mathcal{R}))$, where $\mathcal{R}$ is some finite extension of the completion of $\mathcal{O}$ at π. The possible p-torsion in $H^2(\Gamma, E(\mathcal{O}))$ obstructs us from applying the lifting theorem to lift eigenvalue systems occurring in $H^1(\Gamma, E(\kappa_\pi))$.

3.1. Cuspidal Cohomology

There is a subspace of the cohomology that is of special interest due to the fact that it can be identified with cuspidal Bianchi modular forms.

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of class number h_K with ring of integers $\mathcal{O} = \mathcal{O}_d$. Let $\mathbb{P}$ denote the projective line over K and $G = \mathrm{PSL}_2(\mathcal{O})$. The group $\mathrm{PSL}_2(K)$ acts naturally on K^2 and thus on $\mathbb{P}$. It is well known that the cardinality $|\mathbb{P}/G|$ of the set $\mathbb{P}/G$ of G-orbits of $\mathbb{P}$ is equal to h_K. Hence $|\mathbb{P}/\Gamma|$ is finite for any finite-index subgroup Γ of G. We will call the elements of $\mathbb{P}/\Gamma$ the *cusps* of Γ.

For every $D \in \mathbb{P}$, let B_D be the Borel subgroup of G defined by the (setwise) stabilizer of D in G. Then the pointwise stabilizer of D in G is the unipotent radical U_D of the Borel subgroup B_D. Let Γ be a finite-index subgroup of G, and D_c a representative for a cusp c of Γ. Define

$$\Gamma_c := B_{D_c} \cap \Gamma.$$

If Γ_c is torsion-free (this is automatic if Γ is itself torsion-free or $-d \neq 1, 3$), then $\Gamma_c = U_{D_c} \cap \Gamma$ and Γ_c is free abelian of rank two (see [Serre 70, p. 507]). The group

$$U(\Gamma) := \bigoplus_{c \in \mathbb{P}/\Gamma} \Gamma_c$$

is independent, up to isomorphism, of the choice of representatives taken for the cusps of Γ.

Let E be a Γ-module. Consider the long exact sequence of relative group cohomology for the pair $(\Gamma, U(\Gamma))$

$$\cdots \to H_c^{i-1}(\Gamma, E) \to H^i(\Gamma, E) \to H^i(U(\Gamma), E) \to \cdots,$$

where $H_c^n(\Gamma, E) := H^n(\Gamma; U(\Gamma), E)$ and the third arrow is given by the restriction maps.

Definition 3.3. The *cuspidal cohomology* $H^i_{\mathrm{cusp}}(\Gamma, E)$ is defined as the image of the cohomology with compact support in $H^i(\Gamma, E)$, or equivalently as the kernel of the restriction map $H^i(\Gamma, E) \to H^i(U(\Gamma), E)$.

Remark 3.4. Let $S_k(\Gamma)$ denote the space of cuspidal Bianchi modular forms with level Γ and weight k. In [Harder 87] the so-called Eichler–Shimura–Harder isomorphism

$$S_k(\Gamma) \simeq H^1_{\mathrm{cusp}}(\Gamma, E_{k,k}(\mathbb{C})) \simeq H^2_{\mathrm{cusp}}(\Gamma, E_{k,k}(\mathbb{C}))$$

of Hecke modules is proved. Note that the second isomorphism is an instance of a duality result saying that if F is a field in which 6 is invertible, then

$$H^1_{\mathrm{cusp}}(\Gamma, E(F))^\vee \simeq H^2_{\mathrm{cusp}}(\Gamma, E(F)^\vee)$$

as Hecke modules, where $-^\vee$ denotes the dual; see [Ash and Stevens 86, Lemma 1.4.3].

Deep results from [Borel and Wallach 80, Section II] imply that whenever $k \neq \ell$, the cuspidal cohomology $H^i_{\mathrm{cusp}}(\Gamma, E_{k,\ell}(\mathbb{C}))$ vanishes. It is important to remark that this is no longer true when the module $E_{k,\ell}$ is not over a field of characteristic 0. In particular, $H^i_{\mathrm{cusp}}(\Gamma, E_{k,\ell}(\mathcal{O}))$ is completely torsion when $k \neq \ell$.

Proposition 3.5. *Let Γ be a finite-index subgroup of the Bianchi group $\mathrm{PSL}_2(\mathcal{O})$. Assume either that Γ is torsion-free or that $-d \neq 1, 3$. Then $H^2(U(\Gamma), E_{k,\ell}(\mathcal{O}))$ has no large torsion.*

Proof. It is enough to prove the claim for a single cusp c of Γ, that is, for $H^2(\Gamma_c, E)$. So fix a cusp c and Γ_c. Let $E = E_{k,\ell}$ and $t = \max\{k, \ell\}$. Put $R = \mathcal{O}[\frac{1}{t!}]$. Composition of the cup product and the perfect pairing of Proposition 2.1 gives us a pairing

$$\begin{array}{c} H^0(\Gamma_c, E(R)) \times H^2(\Gamma_c, E(R)) \\ \Big\downarrow \cup \\ H^2(\Gamma_c, E(R) \otimes_R E(R)) \xrightarrow{(\cdot,\cdot)} H^2(\Gamma_c, R) \simeq R \end{array}$$

That $H^2(\Gamma_c, R) \simeq R$ can be shown as follows. Recall that Γ_c is free abelian with two generators, say a, u. It is known (see [Mac Lane 63, p. 188]) that the tensor product of the two resolutions

$$0 \longrightarrow R[\langle a\rangle] \xrightarrow{1-a} R[\langle a\rangle] \xrightarrow{\varepsilon} R \longrightarrow 0,$$

$$0 \longrightarrow R[\langle u\rangle] \xrightarrow{1-u} R[\langle u\rangle] \xrightarrow{\varepsilon} R \longrightarrow 0,$$

where ε is the usual augmentation map, gives a resolution of Γ_c. One sees from this resolution that the second cohomology of Γ_c with any (right) R-module M can be described as

$$H^2(\Gamma_c, M) \simeq M/\left(M(1-a) + M(1-u)\right).$$

In the case of a trivial module R, it follows immediately that $H^2(\Gamma_c, R) \simeq R$.

The above pairing gives that

$$H^2(\Gamma_c, E(R)) \simeq H^0(\Gamma_c, E(R))^\vee.$$

Clearly $H^0(\Gamma_c, E(R)) \simeq E(R)^{\Gamma_c}$ is torsion-free. This implies that its dual, and hence $H^2(\Gamma_c, E(R))$, is torsion-free. The claim that there can be only small torsion in $H^2(\Gamma_c, E(\mathcal{O}))$ now follows, since $R = \mathcal{O}[\frac{1}{t!}]$. $\square$

As a corollary we see that the cuspidal part of H^2 is responsible for the possible large torsion. The referee brought to my attention that the analogue of the above result for $H^1(U(\Gamma), E_{k,k}(\mathcal{O}))$ was proven in [Urban 95, Proposition 2.4.1]. This result similarly implies that the possible large torsion in $H^2_c(\Gamma, E_{k,k}(\mathcal{O}))$ comes from $H^2_{\text{cusp}}(\Gamma, E_{k,k}(\mathcal{O}))$ as well.

Proposition 3.6. *Let Γ be a torsion-free finite-index subgroup of the Bianchi group $\mathrm{PSL}_2(\mathcal{O})$. Let π be a prime element of $\mathcal{O}$ over the rationa prime p, and put κ_π for the residue field of the ideal generated by π. Then*

$$H^2_{cusp}(\Gamma, E_{k,\ell}(\mathcal{O})) \otimes \kappa_\pi \simeq H^2_{cusp}(\Gamma, E_{k,\ell}(\kappa_\pi))$$

for every $k, \ell < p$."

Proof. Put $E = E_{k,\ell}(\mathcal{O})$. Now consider the commutative diagram

$$\begin{tikzcd} & H^2(\Gamma, E) \arrow[r, "\cdot\pi"] \arrow[d] & H^2(\Gamma, E) \arrow[r] \arrow[d] & H^2(\Gamma, E(\kappa_\pi)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^2(U(\Gamma), E) \arrow[r, "\cdot\pi"] & H^2(U(\Gamma), E) \arrow[r] & H^2(U(\Gamma), E(\kappa_\pi)) & \end{tikzcd}$$

Here the vertical maps are given by the usual restriction maps.

The horizontal lines are exact. The exactness of the first line comes from Proposition 3.2(d). The exactness of the second line amounts to Proposition 3.5.

Observe that the cokernel of the restriction map $H^2(\Gamma, E) \to H^2(U(\Gamma), E)$ is isomorphic to $H^3_{\text{cusp}}(\Gamma, E) \subset H^3(\Gamma, E)$. Since the virtual cohomological dimension of a Bianchi group is 2 and Γ is torsion-free, we have $H^3(\Gamma, E) = 0$. Now the claim follows by the snake lemma. $\square$

Let us end this section with the following observation on lifting eigenvalue systems.

Proposition 3.7. *Let $\pi \in \mathcal{O}$ be a prime element over the rational prime $p > 3$. Let Γ be a torsion-free finite-index subgroup of $\mathrm{PSL}_2(\mathcal{O})$. Let Φ be an eigenvalue system occurring in $H^1_{\text{cusp}}(\Gamma, E_{k,\ell}(\kappa_\pi))$ with $k, \ell < p$ and $\kappa_\pi = \mathcal{O}/(\pi)$. If Φ does not lift to $H^1_{\text{cusp}}(\Gamma, E_{k,\ell}(\mathcal{R}))$ for any finite extension $\mathcal{R}$ of the completion $\mathcal{O}_\pi$ of $\mathcal{O}$ at π, then there is a π-torsion eigenclass $c \in H^2_{\text{cusp}}(\Gamma, E_{k,\ell}(\mathcal{O}))$ realizing a lift of Φ.*

Proof. Since $p > 3$, by the duality result mentioned in Remark 3.4, we deduce that $\Phi^\vee$ lives in $H^2_{\text{cusp}}(\Gamma, E_{k,\ell}(\kappa_\pi))$. Note that our coefficient modules are self-dual. Since $k, \ell < p$, using Proposition 3.6 and the lifting theorem of Ash and Stevens mentioned after Proposition 3.2, we infer that there is an eigenvalue system Ψ living in $H^2_{\text{cusp}}(\Gamma, E_{k,\ell}(\mathcal{R}))$ lifting $\Phi^\vee$ for some finite extension $\mathcal{R}$ of $\mathcal{O}_\pi$. If Ψ is not realized by a torsion eigenclass $c \in H^2_{\text{cusp}}(\Gamma, E_{k,\ell}(\mathcal{R}))$, then we can realize Ψ in $H^2_{\text{cusp}}(\Gamma, E_{k,\ell}(L))$, where L is the field of fractions of $\mathcal{R}$. By duality again, $\Psi^\vee$ occurs in $H^1_{\text{cusp}}(\Gamma, E_{k,\ell}(L))$. Since $\Psi^\vee$ has integral values, it can be realized in $H^1_{\text{cusp}}(\Gamma, E_{k,\ell}(\mathcal{R}))$. Clearly $\Psi^\vee$ is a lift of Φ, and this contradicts our starting assumption of nonliftability. To finish, observe that $H^i_{\text{cusp}}(\Gamma, E_{k,\ell}(\mathcal{R})) \simeq H^i_{\text{cusp}}(\Gamma, E_{k,\ell}(\mathcal{O})) \otimes_{\mathcal{O}} \mathcal{R}$. $\square$

4. FIRST COHOMOLOGY

I will now describe a method, first observed in [Fox 53], that allows us to compute H^1 of any finitely presented group with coefficients in a finite-dimensional module. It is well known that Bianchi groups are finitely presented. Presentations for many Bianchi groups are in the literature; see, for example, [Finis et al. 10].

Let me illustrate the method through an example. A formal exposition is contained in [Finis et al. 10]. Let $w = \sqrt{-2}$ and $G = \mathrm{PSL}_2(\mathbb{Z}[w])$. It is known that

$$\begin{aligned} G =& \langle A, B, U \mid B^2 = (AB)^3 = [A, U] \\ &= (BU^2BU^{-1})^2 = 1 \rangle, \end{aligned}$$

where A, B, U can be realized respectively as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}.$$

Let E be any G-module. Given any cocycle $f : G \to E$, any value $f(X)$ can be expressed linearly in terms of the images $f(A), f(B), f(U)$ of the generators of G, e.g.,

$$f(ABU) = f(A) \cdot BU + f(B) \cdot U + f(U).$$

Moreover, $f(A), f(B), f(U)$ satisfy the linear equations coming from the relations of the presentation. For example,

$$B^2 = 1 \Longrightarrow f(B)(B + 1) = 0.$$

Conversely, any pair $(x, y, z) \in E^3$ satisfying the linear equations coming from the presentation gives a cocycle uniquely. Thus the space of cocycles can be seen as the kernel of the matrix corresponding to this linear system. One obtains the coboundaries similarly and hence computes $H^1(G, E)$ as the quotient of the two spaces.

Note that to compute with a finite-index subgroup Γ of G, it is not practical to apply the method to a presentation of Γ (which can be derived from that of G once the coset representatives are known). It is best to use Shapiro's lemma and compute $H^1(G, \mathrm{Coind}^G_\Gamma(E))$.

n	norms of elementary divisors	primes	rank
0			0
1	[4]	(2)	1
2	[2, 16]	(2)	0
3	[2, 2, 4]	(2)	1
4	[2, 2, 2, 8, 1152]	(2,3)	0
5	[2, 2, 2, 2, 4, 4]	(2)	2
6	[2, 2, 2, 2, 2, 2, 8, 8, 800]	(2,5)	0
7	[2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4]	(2)	3
8	[2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 8, 8, 32, 225792]	(2,3,7)	0
9	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4]	(2)	3
10	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 16, 288]	(2,3)	1

TABLE 1. Data for $H^1(\mathrm{PSL}_2(\mathcal{O}_1), E_{n,n}(\mathcal{O}_1))$.

4.1. Data on the Integral First Cohomology

I have implemented the above algorithm in Magma [Bosma et al. 97] for the five Euclidean imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11$. In the following, let $\mathcal{O}_d$ denote the corresponding ring of integers.

By the theory of modules over principal ideal domains, we know that our $\mathcal{O}$-module $H^1(\Gamma, E(\mathcal{O}))$ has a decomposition

$$H^1(\Gamma, E(\mathcal{O})) \simeq \mathcal{O}/(a_1) \oplus \cdots \oplus \mathcal{O}/(a_m) \oplus \mathcal{O}^r$$

with $a_i \neq 0, 1$ and $a_i \mid a_{i+1}$. The a_i are called elementary divisors and are unique up to multiplication by units. The exponent r is called the rank.

In Tables 1 through 3, I report on some of my computations. Observe that the torsion is always "small," as proved in Proposition 3.2. The only exception to this is the ramifying prime, which always appears in the torsion. We show the rank in a separate column, since it provides a means to check our work against the dimension computations of [Finis et al. 10].

5. SECOND COHOMOLOGY

The main method I employ for computing the second cohomology is based on reduction theory as used in [Schwermer and Vogtmann 83]. The cohomological dimension of Bianchi groups is 2, and the symmetric space they act on, namely the hyperbolic 3-space $\mathbb{H} \simeq \mathbb{C} \times \mathbb{R}^+$, is 3-dimensional. Reduction theory gives us a contractible 2-dimensional CW-complex inside $\mathbb{H}$ that is a deformation retract for the action of the Bianchi group. Moreover, the cellular action of the Bianchi group on the CW-complex is cocompact. This makes the CW-complex a suitable tool for cohomological computations.

I will continue to focus on the Euclidean imaginary quadratic fields. The reduction theory for Bianchi groups has been worked out for these fields in [Mendoza 79] and [Flöge 83]. See also [Brunner et al. 85, Rahm and Fuchs 11].

For an overview of Mendoza's construction, I refer readers to [Schwermer and Vogtmann 83]. I will exhibit the method for the case of the Bianchi group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[w])$ with $w = \sqrt{-2}$.

Let $\mathcal{C}$ be the 2-dimensional CW-complex constructed by Mendoza for Γ. Then a fundamental cellular domain

n	norms of elementary divisors	primes	rank
1	[8]	(2)	1
2	[2, 32]	(2)	1
3	[2, 2, 8]	(2)	2
4	[2, 2, 2, 8, 1152]	(2,3)	1
5	[2, 2, 2, 2, 8, 8]	(2)	3
6	[2, 2, 2, 2, 2, 2, 8, 8, 7200]	(2,3,5)	2
7	[2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8]	(2)	4
8	[2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 8, 8, 32, 225792]	(2,3,7)	2
9	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 8]	(2)	5
10	[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 8, 8, 8, 8, 32, 288]	(2,3)	3

TABLE 2. Data for $H^1(\mathrm{PSL}_2(\mathcal{O}_2), E_{n,n}(\mathcal{O}_2))$.

n	norms of elementary divisors	primes	rank
0			0
1	[3]	(3)	0
2	[3]	(3)	1
3	[3, 108]	(2,3)	0
4	[3, 3, 12]	(2,3)	0
5	[3, 3, 12]	(2,3)	1
6	[3, 3, 3, 3, 10800]	(2,3,5)	1
7	[3, 3, 3, 3, 3, 12]	(2,3)	1
8	[3, 3, 3, 3, 3, 12, 2352]	(2,3,7)	1
9	[3, 3, 3, 3, 3, 3, 108, 972]	(2,3)	1

TABLE 3. Data for $H^1(\mathrm{PSL}_2(\mathcal{O}_3), E_{n,n}(\mathcal{O}_3))$.

$\mathcal{F}$ for the action of Γ on $\mathcal{C}$ is given by the area on the unit hemisphere centered at the origin of $\mathbb{H}$ above the rectangle in $\mathbb{C} \times \{0\}$ with vertices $(\pm\frac{w}{2}, 0)$ and $(\frac{1}{2} \pm \frac{w}{2}, 0)$.

Let

$$a := \begin{pmatrix} 1 & w \\ w & -1 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad c := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The stabilizers of the edges (1-cells) and the vertices (0-cells) of $\mathcal{F}$ are shown in Figure 1.

The horizontal edges are identified by the element $g = \left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)$, that is, $gP_1P_2 = P_4P_3$. Thus the quotient by Γ is a cylinder. Moreover, the stabilizer of the whole rectangle (2-cell) is trivial.

From these combinatorial data, one can compute the (co)homology. One way to do this is to feed the data into the equivariant cohomology spectral sequence

$$E_1^{p,q}(M) = \bigoplus_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma, M) \Longrightarrow H^{p+q}(\Gamma, M),$$

where M is any $\mathbb{Z}\Gamma$-module and Σ_p is a set of representatives of all the Γ-orbits of the p-cells of $\mathcal{C}$. See [Brown 94, p. 164] for a description. The homological version of this spectral sequence has been used in [Schwermer and Vogtmann 83, Rahm and Fuchs 11]. See

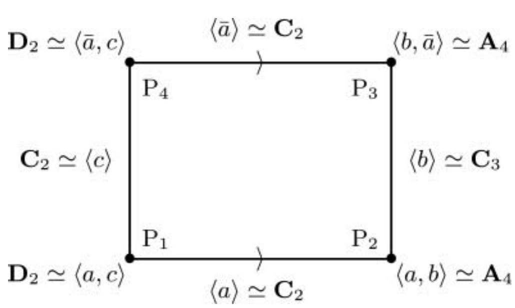

FIGURE 1. The stabilizers of the edges and the vertices of $\mathcal{F}$ for $\mathrm{PSL}_2(\mathcal{O}_2)$.

[Yasaki 08, Section 10] for another method to extract the same information.

Let Γ_i, Γ_{ij} stand for the stabilizers of the vertex P_i and the edge between P_i and P_j respectively. Let M be a right Γ-module over $\mathbb{Z}[w][\frac{1}{6}]$. Since primes above 2 and 3 are inverted, the cohomology of the (finite) stabilizers vanish in degree greater than 0. Hence, we have $E_1^{p,q}(M) = 0$ for all $q > 0$. Therefore, the spectral sequence is concentrated on the horizontal axis $q = 0$, and the cohomology of the cochain complex

$$E_1^{0,0} \xrightarrow{d_1^{0,0}} E_1^{1,0}(M) \xrightarrow{d_1^{1,0}} E_1^{2,0}(M)$$

gives $H^*(\Gamma, M)$, that is,

$$\begin{aligned} H^0(\Gamma, M) &= \ker(d_1^{0,0}), \\ H^1(\Gamma, M) &= \ker(d_1^{1,0}) / \operatorname{im}(d_1^{0,0}), \\ H^2(\Gamma, M) &= M / \operatorname{im}(d_1^{1,0}). \end{aligned}$$

Now with the appropriate substitutions, the cochain complex reads

$$\bigoplus_{\text{vertex}\, i} H^0(\Gamma_i, M) \xrightarrow{d_1^{0,0}} \bigoplus_{\text{edge}\, ij} H^0(\Gamma_{ij}, M) \xrightarrow{d_1^{1,0}} H^0(\langle \mathrm{Id} \rangle, M).$$

Here $\langle \mathrm{Id} \rangle$ is the trivial stabilizer of the 2-cell $\mathcal{F}$.

To compute H^2 explicitly, it remains to describe the differential $d_1^{1,0}$. One can choose the orientation on $\mathcal{F}$ so that the differential map becomes as follows:

$$M^{\Gamma_1} \oplus M^{\Gamma_2} \xrightarrow{d_1^{0,0}} M^{\Gamma_{12}} \oplus M^{\Gamma_{23}} \oplus M^{\Gamma_{41}} \xrightarrow{d_1^{1,0}} M,$$

where

$$d_1^{1,0}(m_{12}, m_{23}, m_{41}) = m_{12} + m_{23} + m_{41} - m_{12} \cdot g^{-1}.$$

The information on the fundamental 2-cell for the groups $\mathrm{PSL}_2(\mathcal{O}_d)$ with $-d = 1, 2, 3, 7, 11$ is included in the article [Schwermer and Vogtmann 83], so I do not repeat it here. The same information for the groups $\mathrm{PGL}_2(\mathcal{O}_d)$ with $-d = 1, 2, 3, 7, 11$ is not included in that article, and one needs to go to the above-mentioned thesis of Mendoza (although information for a few of the groups is contained in [Brunner et al. 85] as well), which is hard to access from outside Germany. So I will now describe the information on these groups in pictorial form as above.

$\mathrm{PGL}_2(\mathcal{O}_1)$: Put $i = \sqrt{-1}$. Let

$$a := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad c := \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$

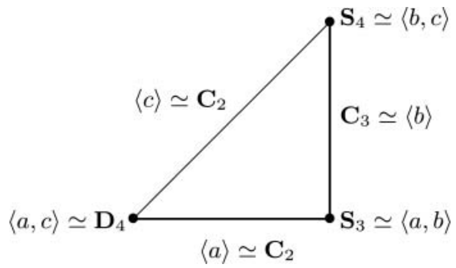

FIGURE 2. The stabilizers of the edges and the vertices of $\mathcal{F}$ for $\mathrm{PGL}_2(\mathcal{O}_1)$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in Figure 2. There are no identifications and the stabilizer of the triangle (2-cell) is trivial.

$\mathrm{PGL}_2(\mathcal{O}_2)$: Put $w = \sqrt{-2}$. Let

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$c := \begin{pmatrix} w & 1 \\ 1 & 0 \end{pmatrix}, \quad d := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in Figure 3. There are no identifications and the stabilizer of the rectangle (2-cell) is trivial.

$\mathrm{PGL}_2(\mathcal{O}_3)$: Put $w = \frac{1+\sqrt{-3}}{2}$. Let

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad c := \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}.$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in Figure 4. There are no identifications and the stabilizer of the triangle (2-cell) is trivial.

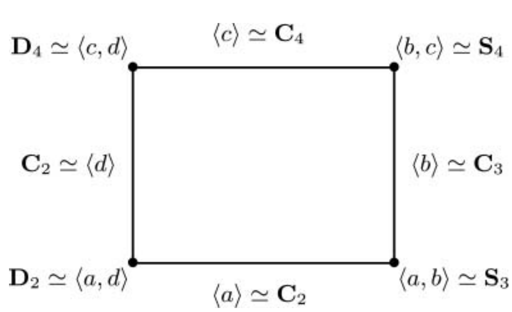

FIGURE 3. The stabilizers of the edges and the vertices of $\mathcal{F}$ for $\mathrm{PGL}_2(\mathcal{O}_2)$.

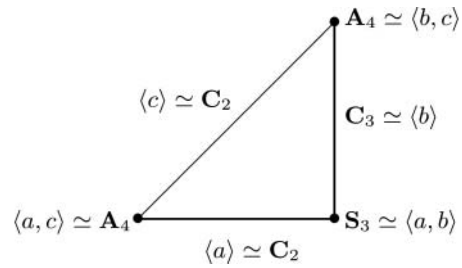

FIGURE 4. The stabilizers of the edges and the vertices of $\mathcal{F}$ for $\mathrm{PGL}_2(\mathcal{O}_3)$.

$\mathrm{PGL}_2(\mathcal{O}_7)$: Put $w = \frac{1+\sqrt{-7}}{2}$. Let

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$c := \begin{pmatrix} 1 & -w \\ \overline{w} & -1 \end{pmatrix}, \quad d := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in Figure 5. The two adjacent short edges on the top are identified via $g = \left(\begin{smallmatrix} 1 & -w \\ 0 & -1 \end{smallmatrix}\right)$, which fixes the vertex between them. Thus these two edges are oppositely oriented, and the stabilizer of the vertex between them is $\mathbf{D}_2 \simeq \langle c, g\rangle$. Again the stabilizer of the whole 2-cell is trivial.

$\mathrm{PGL}_2(\mathcal{O}_{11})$: Put $w = \frac{1+\sqrt{-11}}{2}$. Let

$$a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$c := \begin{pmatrix} 1 & -w \\ \overline{w} & -2 \end{pmatrix}, \quad d := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The stabilizers of the edges and the vertices of $\mathcal{F}$ are shown in Figure 6. The two adjacent short edges on the

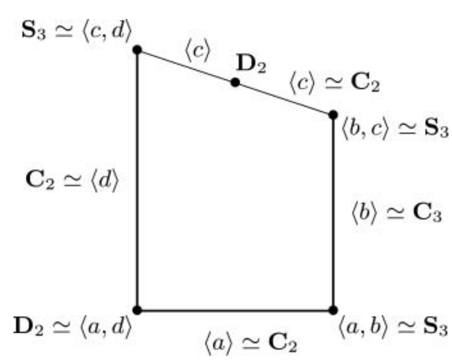

FIGURE 5. The stabilizers of the edges and the vertices of $\mathcal{F}$ for $\mathrm{PGL}_2(\mathcal{O}_7)$.

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3]	1
4	[2, 3]	1
5	[2]	2
6	[2, 3, 5]	1
7	[2, 3, 7]	3
8	[2, 3, 5, 7]	1
9	[2, 3]	3
10	[2, 3, 5, 7]	2
11	[2, 3, 5, 11]	4
12	[2, 3, 5, 7, 11]	1
13	[2, 3, 5]	5
14	[2, 3, 5, 7, 11, 13]	2
15	[2, 3, 5, 7]	5
16	[2, 3, 5, 7, 11, 13]	2
17	[2, 3, 5, 7]	6
18	[2, 3, 5, 7, 11, 13, 17, **19**, **23**]	2
19	[2, 3, 5, 7, 13, 19]	7
20	[2, 3, 5, 7, 11, 13, 17, 19, **409**, **6997**]	2
21	[2, 3, 5, 7, **59**]	7
22	[2, 3, 5, 7, 11, 13, 17, 19, **13707791**]	3
23	[2, 3, 5, 7, 11, **23**, **113**]	8
24	[2, 3, 5, 7, 11, 13, 17, 19, 23, **1033**, **4457**, **18743**]	2
25	[2, 3, 5, 7, 11, 13, 17, **1523**]	9

TABLE 4. The primes that divide the size of the torsion part of $H^2(\mathrm{PSL}_2(\mathcal{O}_1), E_{n,n}(\mathcal{O}_1))$.

top are identified via $g = \left(\begin{smallmatrix} 1 & -w \\ 0 & -1 \end{smallmatrix}\right)$, which fixes the vertex between them. Thus these two edges are oppositely oriented, and the stabilizer of the vertex between them is $\mathbf{S}_3 \simeq \langle c, g\rangle$. Again the stabilizer of the whole 2-cell is trivial.

5.1. Data on the Integral Second Cohomology: Level 1

I have implemented the above algorithm in Magma. I have not inverted the primes above 2 and 3, and thus the computations may not give correct data on 2, 3-torsion. In Tables 4 through 13, I give a complete list of the primes that appear in the torsion part of the second cohomology of both PSL and PGL. The large primes are highlighted in boldface.

The data imply that if (p) ramifies in $\mathcal{O}$, then there is p-torsion in the integral second cohomology (except in the case $k = \ell = 0$), but I have not been able to prove that this always obtains.

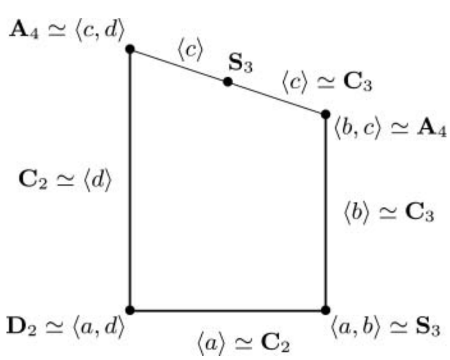

FIGURE 6. The stabilizers of the edges and the vertices of $\mathcal{F}$ for $\mathrm{PGL}_2(\mathcal{O}_{11})$.

5.2. Higher Level

Now let us focus on the second cohomology of congruence subgroups. For computational considerations, we will focus on the subgroups of the type

$$\Gamma_0(\mathfrak{a}) : \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : c \equiv 0 \mod \mathfrak{a} \right\}.$$

Here G is the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$, and $\mathfrak{a}$ is an ideal of $\mathcal{O}_d$, which is called the *level*.

5.2.1. Trivial Weight: Torsion.

To compute the second cohomology with trivial weight, the approach employed in [Elstrodt et al. 81] is more

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3]	1
4	[2]	1
5	[2]	2
6	[2, 3, 5]	1
7	[2, 3, 7]	2
8	[2, 3, 5, 7]	1
9	[2, 3]	3
10	[2, 3, 5, 7]	2
11	[2, 3, 11]	3
12	[2, 3, 5, 11]	1
13	[2, 3, 5]	4
14	[2, 3, 5, 7, 11, 13]	2
15	[2, 3, 5, 7]	4
16	[2, 3, 5, 7, 13]	2
17	[2, 3, 5, 7]	5
18	[2, 3, 5, 7, 11, 17]	2
19	[2, 3, 5, 19]	5
20	[2, 3, 5, 7, 13, 17, 19, **409**]	2
21	[2, 3, 5, 7]	6
22	[2, 3, 5, 7, 11, 19]	3
23	[2, 3, 5, 7, 11, 23]	6
24	[2, 3, 5, 7, 11, 13, 17, 23, **1033**]	2
25	[2, 3, 5, 7, 11, 17]	7
26	[2, 3, 5, 7, 11, 13, 19, 23, **157**, **683**]	3
27	[2, 3, 5, 7]	7
28	[2, 3, 5, 7, 11, 13, 17, **664197637**]	3
29	[2, 3, 5, 7, 11, 13, **89**]	8
30	[2, 3, 5, 7, 11, 13, 19, 23, 29, **211**, **36312691**]	3

TABLE 5. The primes that divide the size of the torsion part of $H^2(\mathrm{PGL}_2(\mathcal{O}_1), E_{n,n}(\mathcal{O}_1))$.

efficient than the reduction-theory approach used above. The idea is to compute the abelianization of the congruence subgroup Γ using a (finite) presentation for the Bianchi group G and the knowledge of the permutation action of the generators of G on a set of coset representatives of Γ in G.

The relationship between first homology and second cohomology is given by the Lefschetz duality. Let R be any module in which 6 is invertible. Then for any $R[\Gamma]$-module E, we have

$$H_1(\Gamma, E) \simeq H_c^2(\Gamma, E),$$

where the right-hand side is the cohomology with compact support; see Section 2.

We are interested only in the case of the trivial coefficients $R = \mathbb{Z}[1/6]$. We need to study the exact sequence

$$H^1(\Gamma, R) \to H^1(U(\Gamma), R) \to H_c^2(\Gamma, R) \to H^2_{\mathrm{cusp}}(\Gamma, R) \to 0.$$

Assume that K has class number one and that $\Gamma = \Gamma_0(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of residue degree one. It is easy to see that Γ has only two cusps $\{0, \infty\}$, that is, $|\Gamma \setminus \mathbb{P}| = 2$, and the two classes are represented by the elements $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. If $-d \neq 1, 3$, then the stabilizers of the cusps are free abelian of rank 2, and by [Serre 70, Corollaire 3, p. 517], the image of $H^1(\Gamma, R)$ has rank 2 in $H^1(U(\Gamma), R)$, which has rank 4. For $-d = 1, 3$, the situation is complicated by the existence of torsion in the stabilizers of cusps. In this case, let $\Gamma_c^+ := \Gamma \cap U_{D_c}$ for each cusp c of Γ in the terminology of Section 3.1. Then

$$\Gamma_c/\Gamma_c^+ \simeq \mu := \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \right\rangle,$$

where ε is a generator of the roots of unity in $\mathcal{O}_d$. The inflation–restriction sequence then gives

$$H^1(\Gamma_c, R) \simeq H^0(\mu, H^1(\Gamma_c^+, R)).$$

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3]	2
4	[2, 3]	1
5	[2, 3, 5]	3
6	[2, 3, 5]	2
7	[2, 3, 5, 7]	4
8	[2, 3, 5, 7]	2
9	[2, 3, 5, 7, **31**]	5
10	[2, 3, 5, 7]	3
11	[2, 3, 5, 7, 11]	6
12	[2, 3, 5, 7, 11, **37**]	3
13	[2, 3, 5, 7, 11, 13, **547**]	7
14	[2, 3, 5, 7, 11, 13, **439**, **110281**]	4
15	[2, 3, 5, 7, 11, 13, **61**, **163**]	8

TABLE 6. The primes that divide the size of the torsion part of $H^2(\mathrm{PSL}_2(\mathcal{O}_2), E_{n,n}(\mathcal{O}_2))$.

Now let us directly show that the latter is trivial. Without loss of generality, assume that $c = \infty$ and thus that $\Gamma_c^+ := (\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}) \cap \Gamma$. The action of μ on $H^1(\Gamma_c^+, R)$ is given as $(\tau f)(x) = f(\tau x \tau^{-1})\tau = f(\tau x \tau^{-1})$ for every $\tau \in \mu$ and every 1-cocycle $f : \Gamma_c^+ \to R$. Take $\varepsilon \in \mathcal{O}_d^*$ such that $\varepsilon^2 = -1$ and put $\tau = (\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix})$. Then for any element $x \in \Gamma_c^+$, we have $\tau x \tau^{-1} = x^{-1}$. The condition $f(x) = f(x^{-1})$ for every x forces the 1-cocycle f to be the trivial cocycle $f = 0$. We are done.

n	primes	rank
1	[ ]	1
2	[3]	1
3	[3]	1
4	[2, 3]	1
5	[2, 3, 5]	1
6	[2, 3, 5]	2
7	[2, 3, 5]	2
8	[2, 3, 7]	1
9	[2, 3, 7]	2
10	[2, 3, 5]	3
11	[2, 3, 5, 11]	2
12	[2, 3, 5, 7, 11]	2
13	[2, 3, 5, 11]	3
14	[2, 3, 5, 7, 13]	3
15	[2, 3, 5, 7, 13]	3
16	[2, 3, 5, 7, 11]	3
17	[2, 3, 5, 7, 17]	3
18	[2, 3, 5, 7, 13, 17]	4
19	[2, 3, 5, 7, 11, 17, **61**]	4
20	[2, 3, 5, 7, 19]	3
21	[2, 3, 5, 7, 13, 19, **151**]	4
22	[2, 3, 5, 7, 11, 17]	5
23	[2, 3, 5, 7, 11, 23, **103**]	4
24	[2, 3, 5, 7, 11, 13, 17, 19, 23, **53**]	4
25	[2, 3, 5, 7, 11, 17, 23, **29**, **947**]	5
26	[2, 3, 5, 7, 11, 13]	5

TABLE 8. The primes that divide the size of the torsion part of $H^2(\mathrm{PSL}_2(\mathcal{O}_3), E_{n,n}(\mathcal{O}_3))$.

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2]	2
4	[2, 3]	1
5	[2, 5]	3
6	[2, 3, 5]	1
7	[2, 3, 7]	4
8	[2, 3, 5, 7]	1
9	[2, 3]	5
10	[2, 3, 5, 7]	2
11	[2, 3, 5]	6
12	[2, 3, 5, 7, 11, **37**]	1
13	[2, 3, 5, 13]	7
14	[2, 3, 5, 7, 11, 13, **110281**]	2
15	[2, 3, 5, 7]	8
16	[2, 3, 5, 7, 11, 13, **1671337**]	2
17	[2, 3, 5, 7, **103**]	9
18	[2, 3, 5, 7, 11, 13, 17, **3812807473**]	2
19	[2, 3, 5, 7, **907**]	10
20	[2, 3, 5, 7, 11, 13, 17, 19, **3511**, **879556698451244053**]	2

TABLE 7. The primes that divide the size of the torsion part of $H^2(\mathrm{PGL}_2(\mathcal{O}_2), E_{n,n}(\mathcal{O}_2))$.

n	primes	rank
1	[ ]	1
2	[ ]	1
3	[2, 3]	1
4	[2, 3]	1
5	[2, 3, 5]	1
6	[2, 3, 5]	1
7	[2, 3]	2
8	[2, 3, 7]	1
9	[2, 3]	2
10	[2, 3, 5]	2
11	[2, 3, 5, 11]	2
12	[2, 3, 5, 7, 11]	1
13	[2, 3]	3
14	[2, 3, 5, 7, 13]	2
15	[2, 3, 5]	3
16	[2, 3, 5, 7, 11]	2
17	[2, 3, 5, 17]	3
18	[2, 3, 5, 7, 13, 17]	2
19	[2, 3, 5, 7]	4
20	[2, 3, 5, 7, 19]	2
21	[2, 3, 5, 7]	4
22	[2, 3, 5, 7, 11, 17]	3
23	[2, 3, 5, 11, 23]	4
24	[2, 3, 5, 7, 11, 13, 19, 23, **53**]	2
25	[2, 3, 5, 7]	5
26	[2, 3, 5, 7, 11, 13]	3
27	[2, 3, 5, 7, 11]	5
28	[2, 3, 5, 7, 11, 13, 17, 23]	3
29	[2, 3, 5, 7, 29]	5

TABLE 9. The primes that divide the size of the torsion part of $H^2(\mathrm{PGL}_2(\mathcal{O}_3), E_{n,n}(\mathcal{O}_3))$.

An alternative way is to use the geometric approach, which I mostly avoided in this paper. Then our discussion here follows from the observation that the cross section of a cusp is an orbifold with underlying manifold a sphere in the cases $-d = 1, 3$ and is a torus in the remaining cases.

I wrote programs in MAGMA to compute $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$ for the Euclidean $\mathcal{O}_d$, that is, for $-d = 1, 2, 3, 7, 11$. It is easily seen that the torsion get very large very quickly compared to the norm of the level of the congruence subgroup. Recall from the introduction that it was speculated in [Grunewald and Schwermer 93] that no p-torsion appearing in the abelianization of a finite-index subgroup Γ is greater then half of the index of Γ inside the Bianchi group. In the case of $\Gamma_0(\mathfrak{p})$, Grunewald and Schwermer's speculation says that any p-torsion that appears in $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$ should satisfy

$$p \leq \frac{\mathbf{N}\mathfrak{p} + 1}{2},$$

where $\mathbf{N}\mathfrak{p}$ is the norm of the prime ideal $\mathfrak{p}$.

n	primes	rank
1	[ ]	1
2	[2, 7]	1
3	[2, 3, 7]	1
4	[2, 3, 7]	2
5	[2, 3, 5, 7]	2
6	[2, 3, 5, 7]	2
7	[2, 3, 5, 7]	3
8	[2, 3, 5, 7]	3
9	[2, 3, 5, 7]	3
10	[2, 3, 5, 7]	4
11	[2, 3, 5, 7, 11]	4
12	[2, 3, 5, 7, 11, **127**]	6
13	[2, 3, 5, 7, 11, 13, **31**]	5
14	[2, 3, 5, 7, 11, 13, **73**]	5
15	[2, 3, 5, 7, 11, 13, **271**, **431**]	5
16	[2, 3, 5, 7, 11, 13]	6
17	[2, 3, 5, 7, 11, 13, 17, **37**, **67**, **89**, **101**, **277**]	6
18	[2, 3, 5, 7, 11, 13, 17, **43**, **457**, **2069**, **3323**]	6

TABLE 10. The primes that divide the size of the torsion part of $H^2(\mathrm{PSL}_2(\mathcal{O}_7), E_{n,n}(\mathcal{O}_7))$.

The smallest primes that witness the falsity of this speculation for our five Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$ are listed in Table 14.

My computations of $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$ agree perfectly with those of [Elstrodt et al. 81] mentioned in the introduction where they overlap. It is easily observed that the primes in the torsion grow to astronomical sizes even within the range $\mathbf{N}\mathfrak{p} \leq 5000$. In Table 15 is a sample of the primes that appear in the torsion of $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$ with $4900 \leq \mathbf{N}\mathfrak{p} \leq 5000$ for the five Euclidean imaginary quadratic fields.[1]

5.2.2. Trivial Weight: Rank.

In this section I will report on the rank of $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{a}), \mathcal{O})$. This rank is clearly equal to the dimension of $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{a}), \mathbb{C})$, and thus its nonvanishing is conjecturally connected to abelian varieties of GL_2 type over imaginary quadratic fields (see [Cremona 84, Elstrodt et al. 81, Şengün 11b]). Moreover, the vanishing of this rank in certain cases is equivalent to the existence of rational homology spheres (see [Long et al. 06]). I report here on my computations related to these two aspects.

As explained in the previous subsection, the rank of $\Gamma_0(\mathfrak{a})^{\mathrm{ab}}$ is related to the rank r of $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{a}), \mathcal{O})$. More precisely, when $\mathfrak{p}$ is a prime ideal of residue degree one,

[1] The complete list is available on my website (http://www.uni-due.de/hm0074).

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2, 3, 7]	1
4	[2, 3, 7]	1
5	[2, 5, 7]	2
6	[2, 3, 5, 7]	1
7	[2, 3, 7]	3
8	[2, 3, 5, 7]	1
9	[2, 3, 7]	3
10	[2, 3, 5, 7]	2
11	[2, 3, 5, 7]	4
12	[2, 3, 5, 7, 11, **127**]	1
13	[2, 3, 5, 7, 13]	5
14	[2, 3, 5, 7, 11, 13, **73**]	2
15	[2, 3, 5, 7, **431**]	5
16	[2, 3, 5, 7, 11, 13]	2
17	[2, 3, 5, 7, 17, **37**]	6
18	[2, 3, 5, 7, 11, 13, 17, **43**, **457**, **2069**, **3323**]	2
19	[2, 3, 5, 7, 13, 19, **311**]	7
20	[2, 3, 5, 7, 11, 13, 17, 19, **42197**, **12272815271**]	2

TABLE 11. The primes that divide the size of the torsion part of $H^2(\mathrm{PGL}_2(\mathcal{O}_7), E_{n,n}(\mathcal{O}_7))$.

our discussion above shows that

$$r = \mathrm{rank}(\Gamma_0(\mathfrak{p})^{\mathrm{ab}}), \quad \text{for } -d = 1, 3$$

and

$$r + 2 = \mathrm{rank}(\Gamma_0(\mathfrak{p})^{\mathrm{ab}}), \quad \text{for } -d \neq 1, 3.$$

I have computed the rank of $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$ for prime ideals $\mathfrak{p}$ of residue degree one and norm up to 45000 for $-d = 1, 3$, 30000 for $-d = 2$, and 21000 for $-d = 7, 11$. I report on the distribution of prime levels according to the ranks in Table 16, where I use $N_r(x)$ to denote the number of primes of residue degree 1 with norm less than $x \cdot 1000$ and such that $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{a}), \mathcal{O})$ has rank r. It is curious that for all five $\mathcal{O}_d$, approximately 90% of the time the rank was 0. Note that in [Finis et al. 10], the authors used an efficient method that works with finite fields to approximate the ranks up to norm 60000 for $-d = 1$. My computations for $-d = 1$ agree with theirs where they overlap, except that the values $12113, 12373$ are missing from their Table 10 and 12941 is missing from their Table 11.

It is believed that there are infinitely many prime ideals $\mathfrak{p}$ of residue degree 1 such that $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{C})) = 0$. On the other hand, in analogy with the conjecture that there are infinitely many elliptic curves over $\mathbb{Q}$ with prime conductor (see [Brumer and Silverman 96, p. 97], it is reasonable to expect that there are infinitely many prime ideals $\mathfrak{p}$ of residue degree 1 and $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{C})) \neq 0$. In [Finis et al. 10], in analogy with the distribution questions for elliptic curves (see [Brumer and McGuinness 90]), the following question was posed (stated here in a slightly more general form).

Question 5.1. Let $\mathcal{O}_d$ be the ring of integers of an imaginary quadratic number field. Is there a constant C_d such

n	primes	rank
1	[ ]	1
2	[2]	2
3	[2, 3, 11]	2
4	[2, 3, 11]	2
5	[2, 3, 5, 11]	3
6	[2, 3, 5, 11]	4
7	[2, 3, 5, 7, 11]	4
8	[2, 3, 5, 7, 11]	4
9	[2, 3, 5, 7, 11, **23**]	5
10	[2, 3, 5, 7, 11]	8
11	[2, 3, 5, 7, 11, **37**]	6
12	[2, 3, 5, 7, 11]	6
13	[2, 3, 5, 7, 11, 13, **43**, **19973**]	7
14	[2, 3, 5, 7, 11, 13]	8
15	[2, 3, 5, 7, 11, 13, **31**, **47**, **1409**, **30817**]	8
16	[2, 3, 5, 7, 11, 13, 17, 19, **41**, **281**]	8

TABLE 12. The primes that divide the size of the torsion part of $H^2(\mathrm{PSL}_2(\mathcal{O}_{11}), E_{n,n}(\mathcal{O}_{11}))$.

n	primes	rank
1	[ ]	1
2	[2]	1
3	[2]	2
4	[2, 3, 11]	1
5	[2, 11]	3
6	[2, 3, 5, 11]	1
7	[2, 3, 7, 11]	4
8	[2, 3, 5, 7, 11]	1
9	[2, 3, 11]	5
10	[2, 3, 5, 7, 11]	4
11	[2, 3, 5, 11]	6
12	[2, 3, 5, 7, 11]	1
13	[2, 3, 5, 11, 13]	7
14	[2, 3, 5, 7, 11, 13]	2
15	[2, 3, 5, 7, 11, **47**]	8
16	[2, 3, 5, 7, 11, 13, **41**, **281**]	2
17	[2, 3, 5, 7, 11, 17, **67**]	9
18	[2, 3, 5, 7, 11, 13, 17, **449**, **20147**, **201797**]	2

TABLE 13. The primes that divide the size of the torsion part of $H^2(\mathrm{PGL}_2(\mathcal{O}_{11}), E_{n,n}(\mathcal{O}_{11}))$.

that the asymptotic relation

$$\sum_{\mathfrak{p}, \mathbf{N}\mathfrak{p} \le x} \dim H^1_{\text{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{C}) \sim C_d \frac{x^{5/6}}{\log x}$$

holds as x goes to infinity, where the sum ranges over prime ideals $\mathfrak{p} \triangleleft \mathcal{O}_d$ of residue degree 1?

Let us put $L_d(x) := \sum_{\mathfrak{p}, \mathbf{N}\mathfrak{p} \le x} \dim H^1_{\text{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{C})$ for the ring $\mathcal{O}_d$ and $R(x) := x^{5/6}/\log x$. Table 17 compares the two functions $L(x)$ and $R(x)$ within the range of my computations.

In [Calegari and Dunfield 06], the authors constructed a family of commensurable arithmetic rational homology 3-spheres, that is, commensurable arithmetic Kleinian groups Γ such that $H_1(\Gamma \setminus \mathbb{H}, \mathbb{Q}) \simeq H_1(\Gamma, \mathbb{Q}) = 0$. In [Long et al. 06], the authors asked whether there are infinitely many commensurability classes of arithmetic rational homology 3-spheres. In the same paper, they posed the following two conjectures (they are slightly rephrased in an equivalent form that fits better with this paper).

Conjecture 5.2.

(1) *There exist infinitely many pairs of prime ideals* $\{\mathfrak{p}_1, \mathfrak{p}_2\} \subset \mathbb{Z}[i]$ *such that*

$$H^2_{\text{cusp}}(\Gamma_0(\mathfrak{p}_1\mathfrak{p}_2), \mathbb{Q}) = 0.$$

(2) *Let* $\mathfrak{p} = (1+i)$. *There are infinitely many prime ideals* $\mathfrak{q} \subset \mathbb{Z}[i]$ *with* $\mathbf{N}\mathfrak{q} = 1 \bmod 12$ *such that*

$$H^2_{\text{cusp}}(\Gamma_0(\mathfrak{p}\mathfrak{q}), \mathbb{Q}) = 0.$$

If the second conjecture holds, then using the Jacquet–Langlands correspondence, one gets (see [Long et al. 06, p. 29]) a positive answer to the question of Long et al. stated above.

I computed the ranks of $\Gamma_0(\mathfrak{p}\mathfrak{q})^{\text{ab}}$, where $\mathfrak{p} = (1+i)$ and $\mathfrak{q} \subset \mathbb{Z}[i]$ are prime with $\mathbf{N}\mathfrak{q} = 1 \bmod 12$ of norm $\le$ 14850. There are 423 such prime ideals $\mathfrak{q}$, 245 of them satisfying the desired property that

$$H^2_{\text{cusp}}(\Gamma_0(\mathfrak{p}\mathfrak{q}), \mathbb{Q}) = \Gamma_0(\mathfrak{p}\mathfrak{q})^{\text{ab}} \otimes \mathbb{Q} = 0.$$

The uniform distribution of the primes with vanishing rank supports the second conjecture.

5.3. Asymptotics of Torsion

Very recently, significant results were obtained [Müller 10, Bergeron and Venkatesh 10] that relate the asymptotic behavior of the size of the torsion in the homology of certain cocompact lattices in $\mathrm{SL}_2(\mathbb{C})$ to that of the volume of the associated 3-folds. The following is a special case of the main result of Bergeron and Venkatesh.

Theorem 5.3. *Let* Γ *be an arithmetic subgroup of* $\mathrm{SL}_2(\mathbb{C})$ *and* $\{\Gamma_n\}$ *be a tower of congruence subgroups of* Γ *such that* $\bigcap_n \Gamma_n = \{1\}$. *Let* X *denote the hyperbolic 3-fold associated to* Γ *with volume vol(*X*). Then*

$$\lim_{n\to\infty} \frac{\log |H_1(\Gamma_n, E_{k,\ell})_{\text{tor}}|}{[\Gamma : \Gamma_n]} = \frac{1}{6\pi} \cdot c_{k,\ell} \cdot \mathrm{vol}(X), \quad k \neq \ell,$$

d	Norm of $\mathfrak{p}$	rank of $\Gamma_0(\mathfrak{p})^{\text{ab}}$	prime torsion of $\Gamma_0(\mathfrak{p})^{\text{ab}}$
1	401	0	[2, 5, 41, 271]
2	193	2	[2, 3, 23, 251]
3	937	0	[2, 3, 13, 599]
7	137	2	[2, 17, 83]
11	103	2	[2, 3, 17, 19, 71]

TABLE 14. Smallest counterexamples to Grunewald–Schwermer.

$\mathbf{N}\mathfrak{p}$	some of the primes that divide the size of the torsion of $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$
	$\mathrm{PSL}(\mathcal{O}_1)$
4909	[2, 3, 7, 13, 409, 10691, 22871, 29423, 56980673, 71143433]
4933	[2, 3, 37, 101, 137, 577, 947, 21169, 194981]
4937	[2, 7, 37, 617, 10859, 108893, 4408530403, 157824962047]
4957	[2, 3, 7, 13, 31, 59, 14347, 3051863, 9405667, 23132267]
4969	[2, 3, 23, 71, 373, 191299, 39861006443, 8672729371087]
4973	[2, 11, 13, 47, 71, 113, 127, 331, 6317, 7949, 39023, 628801, 2995319]
4993	[2, 3, 5, 7, 11, 13, 101, 173, 798569, 5995036891, 18513420749]
	$\mathrm{PSL}(\mathcal{O}_2)$
4931	[. . . , 3772418780827, 67462419379713541, 4425411062257370822320521779]
4937	[. . . , 1889149903, 7397090738497, 880941232181841675673769]
4969	[. . . , 2728733329370698225919458399, 11452559584740094034878819578826O381871]
4987	[. . . , 1354882997352809, 167973141926075800477, 1092106383035778134156629]
4993	[. . . , 15997185593, 146336789672061572439301187, 484401755474381467446262O193]
	$\mathrm{PSL}(\mathcal{O}_3)$
4903	[3, 7, 19, 29, 37, 43, 61, 137, 191, 733]
4909	[2, 3, 7, 13, 19, 47, 67, 409, 1409]
4933	[2, 3, 5, 137, 173, 383, 719, 1451, 100057]
4951	[3, 5, 7, 11, 271, 3797, 6696049]
4957	[2, 3, 5, 7, 23, 43, 59, 233, 823, 62207]
4969	[2, 3, 5, 7, 23, 181, 2591, 516336433]
4987	[2, 3, 11, 71, 277, 619, 21977, 1971691]
4993	[2, 3, 11, 13, 29, 727, 4153, 27127]
4999	[2, 3, 7, 17, 29, 41, 83, 38593, 179623]
	$\mathrm{PSL}(\mathcal{O}_7)$
4909	[. . . , 3354447021713, 666100957349057134013, 1336355737543020209509З]
4937	[. . . , 836083247742263, 600017487726483699711, 1344885261548364695671]
4943	[. . . , 94861335404089, 157213239530981, 345644733766517, 714087340201211]
4951	[. . . , 42137202713,11756096619570265637, 4774583154593351З537]
4957	[. . . , 6803766726937001299, 21088956680308937473, 34130091188757085391]
4967	[. . . , 42061245937, 3414861551033731, 385786872173747641]
4993	[. . . , 16112554517, 22230923149, 47405513059, 17179435084786759]
4999	[. . . , 47183940647, 47747826462797, 17672551376413817076181731254111б531]
	$\mathrm{PSL}(\mathcal{O}_{11})$
4909	[. . . , 4916027001531847941150З7, 31607539487402198903985237411069250З1]
4931	[. . . , 59242366654994144915737, 39715305737753649310745751408277З]
4933	[. . . , 4715915801312220993010O9, 75335725443953423041625З]
4937	[. . . , 774606120056702384410790118960699805738139]
4943	[. . . , 496859062013858727411, 7533150099701393721041, 1806172579157695730540919793]
4951	[. . . , 32561299447966536475490232836221, 5758585827071565173844533348539O1]
4973	[. . . , 66807933418297145З623, 22233561207174526986764400647 17]
4987	[. . . , 26685596532560442049106969671, 121708009502005164710374726093]
4999	[. . . , 3527099799815465200483594259770849462007841043363584 7]

TABLE 15. A modest sample of large torsion occurring in $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$.

where $E_{k,\ell}$ denotes the standard Γ-lattice inside $E_{k,\ell}(\mathbb{C})$. Here $c_{k,\ell}$ is a positive rational number that depends only on k, ℓ.

Bianchi groups and their congruence subgroups are outside the scope of these results, since they are not compact. It would be interesting to investigate numerically whether similar asymptotic relations hold for them as well.

I will compare the size of the torsion in the first homology and the volume of the associated 3-folds. Since it is computationally costly to increase the weight, I will concentrate on increasing the level.

	$d=-1$		$d=-2$		$d=-3$		$d=-7$		$d=-11$	
r	$N_r(45)$	%	$N_r(30)$	%	$N_r(45)$	%	$N_r(21)$	%	$N_r(21)$	%
0	2061	88.8	1480	91.8	2033	87.4	1054	89.5	1056	89.7
1	177	7.6	94	5.83	184	7.91	89	7.6	96	8.15
2	66	2.8	31	1.92	82	3.52	29	2.5	22	1.90
3	10	0.4	5	0.31	21	0.91	4	0.3	2	0.17
4	4	0.2	1	0.07	5	0.22	0	0	0	0
5	1	0.04	0	0	0	0	1	0	0	0
6	1	0.04	1	0.07	0	0	0	0.1	1	0.08
7	1	0.04	0	0	1	0.04	0	0	0	0
≥ 8	0	0	0	0	0	0	0	0	0	0
> 0	260	11.2	132	8.2	293	12.6	123	10.5	121	10.3

TABLE 16. Distribution of dimension of $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{C})$.

Let $\Gamma_0(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}$ with residue degree one. Let $H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{\mathrm{tor}}$ denote the torsion part of the first homology of $\Gamma_0(\mathfrak{p})$ with coefficients in $\mathbb{Z}$. Let $\mathrm{vol}(\Gamma_0(\mathfrak{p}) \setminus \mathbb{H})$ denote the volume of the 3-fold $\Gamma_0(\mathfrak{p}) \setminus \mathbb{H}$, where $\mathbb{H}$ is hyperbolic 3-space. In light of the result of Bergeron–Venkatesh, the following is an interesting question.

Question 5.4. With the notation of the above paragraph, is there a constant C, independent of d, such that the asymptotic relation

$$\log|H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{tor}| \sim C \cdot \mathrm{vol}(\Gamma_0(\mathfrak{p}) \setminus \mathbb{H})$$

holds as the norm of the ideals $\mathfrak{p} \lhd \mathcal{O}_d$ that are prime with residue degree one tends to infinity?

To investigate the question computationally, we need to approximate the volumes first. Using the well-known formula (see [Grunewald and Kühnlein 98])

$$\mathbb{V}_d := \mathrm{vol}(\mathrm{PSL}_2(\mathcal{O}_d) \setminus \mathbb{H}) = \frac{|\triangle_d|^{3/2}}{4\pi^2}\, \zeta_{K_d}(2),$$

where $\triangle_d$ is the discriminant of the field K_d and ζ_{K_d} is the Dedekind zeta function of K_d, we get

$$\begin{aligned}
\mathbb{V}_1 &\approx 0.30532186472573967168486783831 1,\\
\mathbb{V}_2 &\approx 1.00384100334119813727236488577,\\
\mathbb{V}_3 &\approx 0.16915693440160893750353375904 6,\\
\mathbb{V}_7 &\approx 0.88891492781635326359890415420 2,\\
\mathbb{V}_{11} &\approx 1.38260830790264587367165334450.
\end{aligned}$$

Now for $\mathfrak{p} \lhd \mathcal{O}_d$ prime of residue degree one over the rational prime p, we have

$$\mathrm{vol}(\Gamma_0(\mathfrak{p}) \setminus \mathbb{H}) = (p+1) \cdot \mathbb{V}_d.$$

I have collected data on the ratio of $\log|H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{\mathrm{tor}}|$ to $\mathrm{vol}(\Gamma_0(\mathfrak{p}) \setminus \mathbb{H})$ in the case of

x	$R(x)/L_1(x)$	$R(x)/L_2(x)$	$R(x)/L_3(x)$	$R(x)/L_7(x)$	$R(x)/L_{11}(x)$
3000	1.793	3.654	1.827	2.294	2.099
6000	1.650	3.172	1.540	2.489	2.101
9000	1.720	3.334	1.435	2.462	2.281
12000	1.828	3.608	1.534	2.617	2.495
15000	1.927	3.610	1.524	2.662	2.731
18000	1.950	3.292	1.482	2.457	2.678
21000	1.912	3.114	1.575	2.464	2.642
24000	1.801	2.993	1.543	-	-
27000	1.782	3.000	1.594	-	-
30000	1.781	2.884	1.591	-	-
33000	1.830	-	1.632	-	-
36000	1.831	-	1.627	-	-
39000	1.825	-	1.612	-	-
42000	1.885	-	1.628	-	-
45000	1.887	-	1.607	-	-

TABLE 17. Data related to the asymptotics of nonvanishing of cuspidal cohomology.

$\mathcal{O}_1$		$\mathcal{O}_2$		$\mathcal{O}_3$		$\mathcal{O}_7$		$\mathcal{O}_{11}$	
$\mathbf{N}\mathfrak{p}$	$T_\mathfrak{p}/V_\mathfrak{p}$	$\mathbf{N}\mathfrak{p}$	$T_\mathfrak{p}/V_\mathfrak{p}$	$\mathbf{N}\mathfrak{p}$	$T_\mathfrak{p}/V_\mathfrak{p}$	$\mathbf{N}\mathfrak{p}$	$T_\mathfrak{p}/V_\mathfrak{p}$	$\mathbf{N}\mathfrak{p}$	$T_\mathfrak{p}/V_\mathfrak{p}$
44533	0.05342	27817	0.05338	44533	0.05288	20549	0.05337	19583	0.05368
44537	0.05391	27827	0.05247	44563	0.05288	20563	0.05414	19603	0.05324
44549	0.05250	27851	0.05300	44587	0.05559	20693	0.05411	19661	0.05212
44617	0.05467	27883	0.05463	44617	0.05279	20707	0.05410	19699	0.05244
44621	0.05509	27947	0.05282	44623	0.05352	20717	0.05297	19717	0.05331
44633	0.05390	27953	0.05221	44641	0.05558	20731	0.05269	19727	0.05327
44641	0.05317	27961	0.05439	44647	0.05581	20743	0.05353	19739	0.05346
44657	0.05203	28001	0.05342	44683	0.05509	20749	0.05172	19759	0.05385
44701	0.05520	28019	0.05161	44701	0.05791	20759	0.05121	19793	0.05410
44729	0.05351	28027	0.05359	44773	0.05280	20771	0.05204	19801	0.05296
44741	0.05355	28051	0.05231	44797	0.05357	20773	0.05411	19853	0.05157
44753	0.05533	28057	0.05238	44809	0.05239	20857	0.05258	19867	0.05442
44773	0.05604	28081	0.05214	44839	0.05606	20897	0.05290	19889	0.05311
44777	0.05573	28097	0.05198	44851	0.05300	20899	0.05470	19891	0.05352
44789	0.05172	28099	0.05353	44887	0.05332	20903	0.05326	19913	0.05324
44797	0.05480	28123	0.05271	44893	0.05427	20939	0.05348	19919	0.05191
44809	0.05220	28163	0.05233	44917	0.05308	20959	0.05395	19937	0.05389
44893	0.05476	28201	0.05140	44953	0.05433	20981	0.05243	19963	0.05383
44909	0.05227	28211	0.05325	44959	0.05292	20983	0.05425	19979	0.05266
44917	0.05281	28219	0.05185	44971	0.05547	21001	0.04985	19991	0.05346
44953	0.05441	28283	0.05312	44983	0.05481	21011	0.05473	20021	0.05318

TABLE 18. the ratio the size of the torsion to the volume as level grows.

the five Euclidean $\mathcal{O}_d$. If we ignore the first 500 entries in each case, the average ratios read

$$0.054291, \quad 0.053140, \quad 0.055386, \quad 0.053206, \quad 0.053131$$

respectively for $-d = 1, 2, 3, 7, 11$. The range of my computations were up to norm

$$45000, \quad 30000, \quad 45000, \quad 21000, \quad 21000$$

respectively. It is very significant that the ratio is very close to

$$\frac{1}{6\pi} \approx 0.05305164769729844525629458779 08,$$

which is the constant for the Lie group $\mathrm{SL}_2(\mathbb{C})$ that appears in the above result of Bergeron–Venkatesh. A small sample is given in Table 18, where I use the convention $T_\mathfrak{p} := \log|H_1(\Gamma_0(\mathfrak{p}), \mathbb{Z})_{\mathrm{tor}}|$ and $V_\mathfrak{p} := \mathrm{vol}(\Gamma_0(\mathfrak{p}) \setminus \mathbb{H})$. The complete data are available at my website.

ACKNOWLEDGMENTS

I began this work toward the end of a postdoctoral fellowship at the Arithmetic Geometry Research Group of the University of Duisburg–Essen under SFB/TR 45 of the Deutsche Forschungsgemeinschaft. I am grateful for the wonderful ambience and hospitality provided by the Mathematics Department. I gratefully acknowledge that all the computations were carried out on the computers of the Institute for Experimental Mathematics, Essen. I thank the excellent Hausdorff Institute for Mathematics in Bonn, where I completed the rest of this work as a visitor. I would like to thank Gebhard Böckle and Gabor Wiese for enlightening discussions and many comments and corrections on this work. My correspondence with Nicolas Bergeron and Akshay Venkatesh was very encouraging and gave me more directions for experimentation. I am grateful to Nansen Petrosyan for his help with the spectral sequences. Finally, I thank J.-P. Serre and the referees, whose numerous comments improved the paper.

DEDICATION

After the first draft of this work was completed, Fritz Grunewald, whom I consider my second PhD adviser, tragically passed away. Over the years, I benefited greatly from his invaluable guidance and generous support. I dedicate this paper to his memory with admiration, gratitude, and love.

REFERENCES

[Ash and Stevens 86] A. Ash and G. Stevens. "Cohomology of Arithmetic Groups and Congruences between Systems of Hecke Eigenvalues." *J. Reine Angew. Math.* 365 (1986), 192–220.

[Berger 08] T. Berger. "Denominators of Eisenstein Cohomology Classes for GL_2 over Imaginary Quadratic Fields." *Manuscripta Math.* 125:4 (2008), 427–470.

[Bergeron and Venkatesh 10] N. Bergeron and A. Venkatesh. "The Asymptotic Growth of Torsion Homology for Arithmetic Groups." Preprint, 2010.

[Bianchi 92] L. Bianchi. "Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginari." *Math. Ann.* 40:3 (1892), 332–412.

[Borel and Wallach 80] A. Borel and N. Wallach. *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*. Princeton: Princeton University Press, 1980.

[Bosma et al. 97] W. Bosma, J. Cannon, and C. Playoust. "The Magma Algebra System. I. The User Language." *J. Symbolic Comput.* 24:3-4 (1997), 235–265.

[Brauer and Nesbitt 41] R. Brauer and C. Nesbitt. "On the Modular Characters of Groups." *Ann. of Math. (2)* 42 (1941), 556–590.

[Brown 94] K. S. Brown. *Cohomology of Groups*. New York: Springer, 1994.

[Brumer and McGuinness 90] A. Brumer and O. McGuinness. "The Behavior of the Mordell–Weil Group of Elliptic Curves." *Bulletin of the AMS* 23 (1990), 375–382.

[Brumer and Silverman 96] A. Brumer and J. Silverman. "The Number of Elliptic Curves over $\mathbb{Q}$ with Conductor N." *Manuscripta Math.* 91 (1996), 95–102.

[Brunner et al. 85] A. M. Brunner, Y. Lee, and N. J. Wielenberg. "Polyhedral Groups and Graph Amalgamation Products." *Topology Appl.* 20:3 (1985), 289–304.

[Calegari and Dunfield 06] F. Calegari and N. Dunfield. "Automorphic Forms and Rational Homology 3-Spheres." *Geom. Topol.* 10 (2006), 295–329.

[Cremona 84] J. E. Cremona. "Hyperbolic Tessellations, Modular Symbols, and Elliptic Curves over Complex Quadratic Fields." *Compositio Math.* 51:3 (1984), 275–324.

[Elstrodt et al. 81] J. Elstrodt, F. Grunewald, and J. Mennicke. "PSL(2) over Imaginary Quadratic Integers." In *Arithmetic Conference (Metz, 1981)*, Astrisque 94, pp. 43–60. Paris: Soc. Math. France, 1982.

[Fox 53] R. Fox. "Free Differential Calculus. II. Derivation in the Free Group Ring." *Ann. of Math. (2)* 57 (1953), 547–560.

[Finis et al. 10] T. Finis, F. Grunewald, and P. Tirao. "Cohomology of Lattices in $\mathrm{SL}_2(\mathbb{C})$." *Exp. Math.* 19 (2010), 29–63.

[Figueiredo 99] L. M. Figueiredo. "Serre's Conjecture for Imaginary Quadratic Fields." *Compositio Math.* 118:1 (1999), 103–122.

[Flöge 83] D. Flöge. "Zur Struktur der PSL_2 über einigen imaginär-quadratischen Zahlringen." *Math. Z.* 183:2 (1983), 255–279.

[Grunewald and Kühnlein 98] F. Grunewald and S. Kühnlein. "On the Proof of Humbert's Volume Formula." *Manuscripta Math.* 95:4 (1998), 431–436.

[Grunewald and Schwermer 93] F. Grunewald and J. Schwermer. "Subgroups of Bianchi Groups and Arithmetic Quotients of Hyperbolic 3-Space." *Trans. Amer. Math. Soc.* 335:1 (1993), 47–78.

[Harder 87] G. Harder. "Eisenstein Cohomology of Arithmetic Groups. The Case GL_2." *Invent. Math.* 89:1 (1987), 37–118.

[Hida 81] H. Hida. "Congruence of Cusp Forms and Special Values of Their Zeta Functions." *Invent. Math.* 63:2 (1981), 225–261.

[Long et al. 06] D. Long, C. Maclachlan, and A. Reid. "Arithmetic Fuchsian Groups of Genus Zero." *Pure Appl. Math. Q.* 2:2, part 2 (2006), 569–599.

[Mac Lane 63] S. Mac Lane. *Homology*. Berlin: Springer, 1963.

[Mendoza 79] E. R. Mendoza. "Cohomology of PGL_2 over Imaginary Quadratic Integers." PhD thesis, Universität Bonn Mathematisches Institut, 1979.

[Müller 10] W. Müller. "The Asymptotics of the Ray–Singer Analytic Torsion of Hyperbolic 3-Manifolds." Preprint, 2010.

[Priplata 00] C. Priplata. "Cohomology and Homology of PSL_2 over Imaginary Quadratic Integers with General

Coefficients $M_{n,m}(\mathcal{O}_K)$ and Hecke Eigenvalues." PhD. thesis, Heinrich Heine Universität, Düsseldorf, 2000.

[Rahm and Fuchs 11] A. Rahm and M. Fuchs. "The Integral Homology of PSL_2 over Imaginary Quadratic Integers of Class Number Two." *J. of Pure and Applied Algebra* 215 (2011), 1443–1472.

[Serre 70] J.-P. Serre. "Le problème des groupes de congruence pour SL_2." *Ann. of Math. (2)* 92 (1970), 489–527.

[Serre 87] J.-P.Serre. "Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$." *Duke Math. J.* 54 (1987), 179–230.

[Şengün 11a] M.H.Şengün. A numerical study of mod p Bianchi modular forms and Galois representations. Preprint, 2011a.

[Şengün 11b] M. H. Şengün. "Hyperelliptic Curves of Genus 2 and Bianchi Modular Forms." In preparation, 2011b.

[Şengün and Türkelli 09] M. H. Şengün and S. Türkelli. "Weight Reduction for mod ℓ Bianchi Modular Forms." *J. Number Theory* 129 (2009), 2010–2019.

[Schwermer and Vogtmann 83] J. Schwermer and K. Vogtmann. "The Integral Homology of SL_2 and PSL_2 of Euclidean Imaginary Quadratic Integers." *Comment. Math. Helv.* 58:4 (1983), 573–598.

[Taylor 88] R. Taylor. "On Congruences between Modular Forms." PhD. thesis, Princeton University, 1988.

[Torrey 11] R. Torrey. "On Serre's Conjecture over Imaginary Quadratic Fields." Preprint, 2011.

[Urban 95] E. Urban. "Formes automorphes cuspidales pour GL_2 sur un corps quadratique imaginaire. Valeurs spéciales de fonctions L et congruences." *Compositio Math.* 99:3 (1995), 283–324.

[Vogtmann 85] K. Vogtmann. "Rational Homology of Bianchi Groups." *Math. Ann.* 272:3 (1985), 399–419.

[Wang 94] X. D. Wang. "The Hecke Operators on the Cusp Forms of $\Gamma_0(N)$ with Nebentype." In *On Artin's Conjecture for Odd 2-Dimensional Representations*, Lecture Notes in Mathematics 1585, pp. 95–108. Berlin: Springer, 1994.

[Wiese 07] G. Wiese. "On the Faithfulness of Parabolic Cohomology as a Hecke Module over a Finite Field." *J. Reine Angew. Math.* 606 (2007), 79–103.

[Yasaki 08] D. Yasaki. "An Explicit Spine for the Picard Modular Group over the Gaussian Integers." *J. Number Theory* 128:1 (2008), 207–234.

Mehmet Haluk Şengün, Departament d'Algebra i Geometria, Facultat de Matemátiques, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, Barcelona, Spain (mehmet.sengun@uni-due.de)

Received June 15, 2010; accepted October 15, 2010.