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Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots

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CONTENTS

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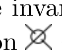
We compute many dimensions of spaces of finite type invariants of virtual knots (of several kinds) and the dimensions of the corresponding spaces of "weight systems," finding everything to be in agreement with the conjecture that "every weight system integrates."

1. "STANDARD" VIRTUAL KNOTS

For "classical" finite type invariants of ordinary knots, as defined by the schematic difference relation  $\to$  $-$  (see, e.g., [Bar-Natan 95]), it is well known that "every weight system integrates." In other words, every linear functional on chord diagrams that satisfies the 4T relation is the "top derivative" of some finite type invariant. Indeed, this simple minded statement is the main implication of the existence of the celebrated *Kontsevich integral* and of *configuration space integrals*, and it is closely related to *perturbative Chern–Simons theory* and to the theory of *Drinfel'd associators* (see overviews at [Bar-Natan and Stoimenow 97, Bar-Natan 06]).

The purpose of this note is to support the conjecture that the same is true in the context of *v-knots*, or *virtual knots* (and in fact, also in several closely related contexts). In this case, finite type invariants are defined by the schematic difference relation  $\to$  $-$  (see Section 2 for details).

We wrote a computer program[1] to compute the dimensions $\dim \mathcal{W}_n$ of spaces of weight systems (of v-knots) of various degrees, and using the *Polyak algebra* of [Goussarov et al. 00], to compute the dimensions $\dim \mathcal{V}_n/\mathcal{V}_{n-1}$ (or $\dim \mathcal{V}_{n/n-1}$, for short) of the spaces of finite type invariants (of v-knots) of various degrees (modulo invariants of lower degree). The results are displayed in Tables 1 and 2.

[1] This paper, programs, and related documentation are available online at http://www.math.toronto.edu/~drorbn/papers/v-Dims/.

n	0	1	2	3	4	5
$\dim \mathcal{W}_n$	1	0	0	1	4	17
$\dim \mathcal{V}_{n/n-1}$	1	0	0	1	4	17

TABLE 1. Dimensions for round v-knots.

n	0	1	2	3	4	5
$\dim \mathcal{W}_n$	1	0	2	7	42	246
$\dim \mathcal{V}_{n/n-1}$	1	0	2	7	42	246

TABLE 2. Dimensions for long v-knots.

Conjecture 1.1. *The pattern of equalities appearing above continues. That is, every weight system for v-knots comes from a finite type invariant of v-knots.*

In Section 3, we study several variations of the notion of virtual knots and state analogous conjectures about their finite type invariants and weight systems.

2. FINITE TYPE INVARIANTS AND WEIGHT SYSTEMS

For completeness, in this section we state all the definitions required for Conjecture 1.1, following Kauffman's original definition of virtual knots [Kauffman 99] and Goussarov, Polyak, and Viro's treatment of finite type invariants of virtual knots [Goussarov et al. 00].[2]

Definition 2.1. [Kauffman 99] A round virtual knot diagram is an immersion of an oriented circle in the plane (regarded up to planar isotopy) that has two allowed types of crossings: real and virtual, denoted by $\times$ and $\times$, modulo the real Reidemeister moves, virtual Reidemeister moves, and mixed Reidemeister 3 moves. Real Reidemeister moves can be found in Figure 1. For the virtual moves and the mixed Reidemeister 3 move, see Figure 2.

We can consider formal linear combinations of virtual knot diagrams and extend the notion of finite type invariants to virtual knots.

Definition 2.2. [Goussarov et al. 00] A semivirtual crossing is a real crossing minus a virtual crossing, while all other crossings remain the same. Given an integer n, we say that an invariant of virtual knots is of type n if it vanishes on all virtual knot diagrams with more than n semivirtual crossings, and an invariant is of finite type (or is a *finite type invariant*) if it is of type m for some finite m. The space of all type-n invariants is denoted by $\mathcal{V}_n$.

[2] An alternative notion of finite type invariants of virtual knots is given in [Kauffman 99].

In analogy to the classical case, every finite type invariant induces a "weight system."

Definition 2.3. An arrow diagram is a diagram that contains an oriented circle, called the skeleton, and arrows joining pairs of distinct points on the skeleton, modulo the relations FI, 6T, and XII (see Figure 3). The degree of a diagram is its number of arrows. The space of all arrow diagrams of degree n is denoted by $\vec{\mathcal{A}}_n$, and its dual is denoted by $\mathcal{W}_n$. A degree-n weight system is an element of $\mathcal{W}_n$.

In a similar fashion we can define long virtual knot diagrams (if we replace "circle" in Definition 2.1 by "a line") as well as the corresponding notions of finite type invariants, arrow diagrams (where the skeleton is now a line instead of a circle), and weight systems. A sample (long) arrow diagram is shown in Figure 4. As far as we know, this paper is the first time the XII-relation appears in the literature, and its derivation will be presented in Section 4.

In the case of classical knots, it is well known (see, for example, [Bar-Natan 95]) that every type-n invariant has a degree-n weight system (which satisfies the FI and 4T relations) and that its weight system determines it up to invariants of type $n-1$. The analogue of this fact for virtual knots is the following proposition, which holds both in the "round" and in the "long" cases.

Proposition 2.4. *For each positive integer n there is an injection ι from $\mathcal{V}_n/\mathcal{V}_{n-1}$ to $\mathcal{W}_n$.*

Proof. The only thing to verify is that the FI, 6T, and XII relations hold. The first two appear in [Goussarov et al. 00]. The last is discussed in Section 4. □

It remains open as to whether ι is always a surjection, or equivalently, whether "every weight system comes from an invariant." Tables 1 and 2 contain computational results that answer this question in the affirmative for small n ($1 \leq n \leq 5$).

In addition to the round and long virtual knots, we will introduce several other kinds of virtual knot theory in the next section, and for each such variation, we have

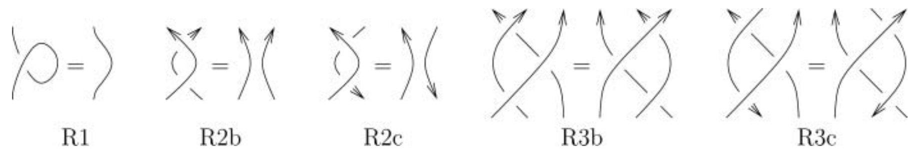


FIGURE 1. Five types of Reidemeister moves. The "b" or "braidlike" moves R2b and R3b have all strands oriented the same way; such configurations could appear in a braid. The "c" or "cyclic" moves R2c and R3c contain a planar domain whose boundary is oriented cyclically. Such configurations cannot appear within a braid. Where no orientation is indicated, we mean "any."

computational results that show that all weight systems come from finite type invariants up to degree 5, just as in the round and long cases.

3. VARIANTS

The theory of finite type invariants of ordinary knots is rather "rigid"—it is the same for round or long knots, the framed and unframed cases are not too different (in particular, a complete understanding of one is equivalent to a complete understanding of the other), and there is little else that can be tinkered with. This is not the case for virtual knot theory—round and long and framed and unframed appear to be quite different, and there are several other "parameters" that can be turned on and off at will, leading to a significant number of apparently different "virtual knot theories." In each such theory we start with a collection of virtual knot diagrams and then mod it out by some Reidemeister moves (see Figure 1). Some of the possible choices follow:

Skeleton choices: We can take the skeleton of our virtual knots to be a circle (the "round" case) or a line (the "long" case). In the case of a line, we may restrict our attention to virtual knot diagrams all of whose (real) crossings are "descending" (a crossing is *descending* if the first time it is visited along the parameterization of the knot it is visited on the "over" strand).

R23 choices: We may mod out by all R2 and R3 moves (this is the "standard" case), or only by the "braidlike" moves R2b and R3b, or we may skip R3 moves altogether and mod out only by R2b and R2c ("R2 only"). (See Figure 1.)

R1 choices: We may or may not mod out by R1 moves.

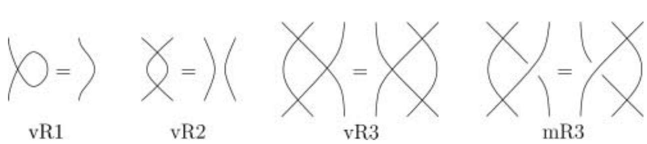


FIGURE 2. The virtual and mixed Reidemeister moves.

Other choices: The "overcrossings commute" relation is studied extensively in [Bar-Natan 11] and will not be studied here. "Flat" and "free" virtual knots are studied in [Manturov 2009a, b] and will not be studied here. "Virtual braids" are left for a future study.

Each such virtual knot theory has a notion of finite type invariants (always defined by $\otimes \to \times - \times$), and each one has a notion of "weight systems" (see Section 4). Hence the question, "does every weight system come from a finite type invariant?" makes sense in many ways. We have studied $18 = 3 \times 3 \times 2$ of these ways:

Conjecture 3.1. (18 in 1.) *For each skeleton choice ("round," "long," or "descending"), with R2 and R3 given either the "standard" or the "braidlike" or the "R2 only" treatment, with or without R1, and for every natural number n, every degree-n weight system comes from a type-n invariant.*

Using our program, we have verified the above conjecture for $n \le 5$ in all 18 cases. Below, we display the dimensions of the spaces $\mathcal{V}_{n/n-1}$ of type-n invariants modulo invariants of lower type (for each case). By our computer's hard work, the dimensions of the spaces $\mathcal{W}_n$ of weight systems are exactly the same, so they do not require a separate table. This equality of dimensions for all n is precisely the content of our conjecture. In all cases, $\dim \mathcal{V}_0 = \dim \mathcal{W}_0 = 1$, so we display the dimensions only for $n = 1, 2, 3, 4, 5$; see Table 3.

In our computations we used the Polyak algebra techniques of [Goussarov et al. 00] for the $\mathcal{V}$ spaces and straightforward linear algebra for the $\mathcal{W}$ spaces described below. The typical $n = 5$ computation involves determining the rank of a very sparse matrix with a few tens of thousands of rows and columns and takes about an hour of computer time. The main part of the program was written in Mathematica with the heavier rank computations delegated to LinBox.[3]

[3]Available online at http://www.linalg.org.

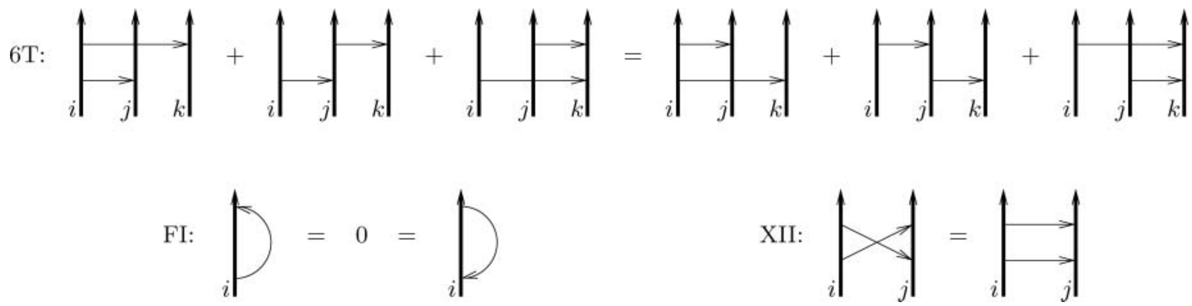


FIGURE 3. The 6T, FI, and XII relations in standard "skein" notation. Only the varying parts of the diagrams involved are shown; their skeleton pieces (labeled i, j, and k above) can be assembled along a long or a round skeleton in any way, and outside the parts shown, more arrows can be inserted.

Why bother? Why bother with such an "18 in 1" conjecture? We believe that virtual knots in general, and the question studied here on finite type invariants of virtual knots in particular, might form the correct topological framework for the study of quantum groups and the quantization of Lie bialgebras [Haviv 02, Bar-Natan 11, Etingof and Kazhdan 96]. But we are not sure yet which class of virtual knots it is that we should study. Is it the standard class, as in Section 1, or is it the one closest to Lie bialgebras, as in (c) of Table 3? Or maybe it is something else, closely related?

Thus we believe that at least some of the 18 cases in Conjecture 3.1 are deeply interesting. As for the rest (the cases involving "R2 only" or "descending v-knots," for example), these may play two kinds of roles in the future:

1. The apparently harder cases, involving all Reidemeister moves and round or long skeletons, appear quite hard. The "easier" cases may serve as "baby versions" that will force us to develop some of the techniques that we may later use while studying the harder cases.
2. We certainly hope that eventually all 18 cases (and maybe a few more) of Conjecture 3.1 will find a uniform solution. Thus the presence of so many variants of Conjecture 3.1 may serve as a further test of our understanding. Suppose we solved one of the "harder" cases. Is our solution modular enough to resolve all other cases as well?

4. ARROW DIAGRAMS AND RELATIONS

This is a short descriptive section intended only to spell out in brief, for reasons of completeness, the definitions of the spaces $\mathcal{W}_n$ of weight systems for each of the cases that we have considered. The details of how and why the spaces described below are related to finite type invariants of virtual knots can be found in [Goussarov et al. 00, Polyak 00].

$\dim \mathcal{V}_{n/n-1}$ / $\dim \mathcal{W}_n$ for ...		round v-knots	long v-knots	descending v-knots
standard R23	mod R1	$0,0,1,4,17^{(a)}$	$0,2,7,42,246^{(a)}$	$0,0,1,6,34$
	no R1	$1,1,2,7,29$	$2,5,15,67,365$	$1,1,2,8,42$
braidlike R23	mod R1	$0,0,1,4,17^{(b)}$	$0,2,7,42,246^{(b)}$	$0,0,1,6,34^{(b)}$
	no R1	$1,2,5,19,77$	$2,7,27,139,813^{(c)}$	$1,2,6,24,120^{(d)}$
R2 only	mod R1	$0,0,4,44,648$	$0,2,28,420,7808$	$0,0,2,18,174$
	no R1	$1,3,16,160,2248$	$2,10,96,1332,23880$	$1,2,9,63,570$

TABLE 3. The knots labeled (a) are the "standard" virtual knots, as in Section 1. (b) The equality of these numbers with the numbers two rows above is a bit tricky. It is not true that R1 and the braidlike R23 imply the cyclic R23. Yet at the level of arrow diagrams, FI and 6T do imply the XII relations (naming as in Section 2). Thus the equality of $\dim \mathcal{W}_n$'s is obvious, and assuming Conjecture 3.1 it implies the equality of the $\dim \mathcal{V}_{n/n-1}$'s. (c) The spaces measured in this box are dual to (long arrow diagrams)/(6T relations), and these are the spaces most closely related to Lie bialgebras [Haviv 02, Leung 08, Bar-Natan 11]. Thus in the long run this box may prove to be the most important of the variants of "virtual knots" studied here. (d) We can show that in this case $\dim \mathcal{W}_n \leq n!$ but we are missing the other inequality necessary to prove that $\dim \mathcal{W}_n = n!$.

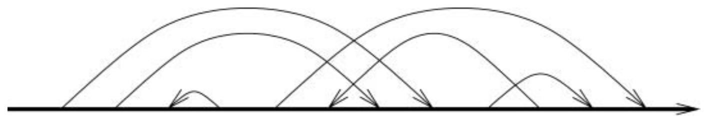


FIGURE 4. A typical arrow diagram of degree 6 (meaning, having exactly six arrows beyond the bolder "skeleton" line at the bottom).

As mentioned in Section 2, the spaces $\mathcal{W}_n$ are always the duals $\vec{\mathcal{A}}_n^\star$ of spaces $\vec{\mathcal{A}}_n$ of arrow diagrams modulo relations. For different virtual knot theories, we impose on $\vec{\mathcal{A}}_n$ different combinations of "arrow diagram relations" in Figure 3.

The spaces $\vec{\mathcal{A}}_n$ for round and long virtual knots are described in Section 2. For descending v-knots, the skeleton is again a long line, but for the diagrams in $\vec{\mathcal{A}}_n$ we allow only those whose arrows are oriented the same way as their skeleton (thus the sample diagram in Figure 4 would be excluded because two of its arrows are oriented against the orientation of the skeleton).

In the case of "standard R23," we impose both 6T and XII on $\vec{\mathcal{A}}_n$. In the case of "braidlike R23," we impose 6T but not XII. In the "R2 only" case, we impose XII but not 6T.

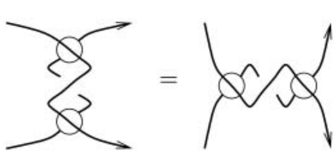
We impose the FI relation in $\vec{\mathcal{A}}_n$ if and only if we mod out by R1 at the level of v-knots.

In the case of descending v-knots, we impose 6T only if $i < j < k$ (as sites along the oriented skeleton); we impose XII only if $i < j$; and we impose only the properly oriented "left half" of FI.

The 6T and FI relations appear and are explained in [Goussarov et al. 00, Polyak 00]. For all we know, this is relation XII's maiden appearance in the literature, and thus an explanation is in order. Below are two brief derivations of XII, the first direct and elementary, and the second using the Polyak algebra. All relevant definitions are in [Goussarov et al. 00] and will not be repeated here.

4.1. A Direct Derivation of XII

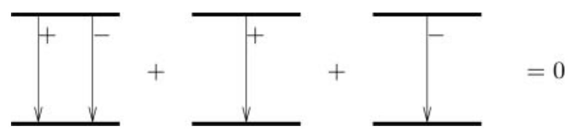
The equality



of semivirtual tangles is easy to verify directly, using the definitions of the semi-virtual crossing, , and using only (virtual moves and) R2 moves (though both braidlike and cyclic ones). But in arrow notation, this is exactly the XII relation.

4.2. A Polyak Algebra Derivation of XII

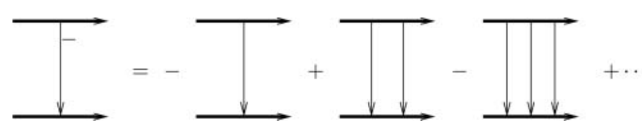
The Polyak algebra $\mathcal{P}_n$ is defined in [Goussarov et al. 00]; it is a space of "signed arrow diagrams" modulo relations that correspond to the Reidemeister moves of knot theory. The relation corresponding to the R2 move is



(1)

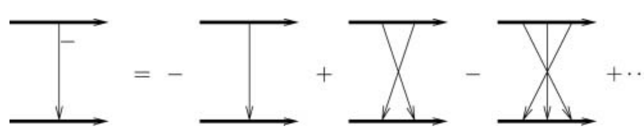
Symbolically, with a denoting the $+$ arrow and b denoting the $-$ arrow, this is $ab + a + b = 0$, or $b = -a - ab$. Solving for b in terms of a and recalling that in $\mathcal{P}_n$ we mod out by degrees higher than n, we get $b = -a + a^2 - a^3 + \cdots$ (a finite sum). Thus the negative arrows can be eliminated in $\mathcal{P}_n$ (this of course is very useful computationally, since it lowers the number of arrow diagrams that one needs to consider by a factor of about 2^n).

But in (1), the orientation of the strands is not specified, and indeed, for braidlike R2 moves these strands come out with parallel orientations, while for cyclic R2 moves they come out with opposite orientations. Thus we get two different formulas for negative arrows in terms of positive ones. The first, using parallel orientations in (1), and dropping the signs from the positive arrows, is



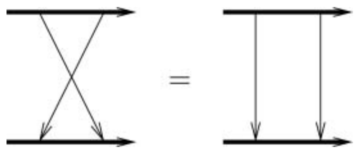
(2)

In the second such formula, using opposite orientations in (1), we flip to the right the strand that was oriented to the left at the cost of getting all the a^k terms totally twisted:



(3)

Equating these two formulas and keeping only the lowest-order terms that don't cancel, we get the XII relation:



The only benefit of the Polyak algebra derivation of XII is the following: in the computation of $\dim \mathcal{P}_n$ (which is the same as $\dim \mathcal{V}_n$) in the cases in which all R2 moves are imposed, one may restrict attention only to + arrows, but then the full right-hand sides of (2) and (3) have to be set equal, dropping only the terms of degree higher than n.

DISCLAIMER

Our computational results suggest what we believe are interesting conjectures. Yet in programming, bugs are a fact of life. An independent verification of our numbers, even without pushing beyond degree 5, would lend further support to Conjectures 1.1 and 3.1 and would be highly desirable.

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