Experimental Mathematics, 20(1):91–96, 2011

ISSN: 1058-6458 print
DOI: 10.1080/10586458.2011.544585

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Notes on the K3 Surface and the Mathieu Group M_{24}

Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa

CONTENTS

2000 AMS Subject Classification: 20D08, 14J28, 58J26
Keywords: K3 surface, elliptic genus, Mathieu groups

We point out that the elliptic genus of the K3 surface has a natural decomposition in terms of dimensions of irreducible representations of the largest Mathieu group M_{24}. The reason remains a mystery.

1. INTRODUCTION AND CONCLUSIONS

The elliptic genus of a complex D-dimensional hyper-Kähler manifold M is defined as

$$Z_{\mathrm{ell}}(\tau; z) = \mathrm{Tr}_{\mathcal{R}\times\mathcal{R}}(-1)^{F_L+F_R}\, q^{L_0}\, \bar{q}^{\bar{L}_0}\, e^{4\pi i z J^3_{0,L}}$$

in terms of the two-dimensional supersymmetric sigma model whose target space is M [Witten 87]. Since M is assumed to be hyper-Kähler, the two-dimensional theory has $\mathcal{N}=4$ superconformal algebra as its symmetry. Then L_0 and $\bar{L}_0$ are zero modes of the left- and right-moving Virasoro operators; J^3_0 is the zero mode of the third component of the affine SU(2) algebra; F_L and F_R are the left- and right-moving fermion numbers. The trace is taken over the Ramond sector of the theory. This elliptic genus is a Jacobi form of weight zero and index $D/2$.

The elliptic genus for the K3 surface was explicitly calculated in [Eguchi et al. 89] and is given by

$$Z_{\mathrm{ell}}(K3)(\tau; z) = 8\left[\left(\frac{\theta_2(\tau;z)}{\theta_2(\tau;0)}\right)^2 + \left(\frac{\theta_3(\tau;z)}{\theta_3(\tau;0)}\right)^2 + \left(\frac{\theta_4(\tau;z)}{\theta_4(\tau;0)}\right)^2\right]. \quad (1\text{–}1)$$

Here $\theta_i(\tau;z)\,(i=2,3,4)$ are Jacobi theta functions. Actually, the space of Jacobi forms of weight zero and index one is known to be one-dimensional, and thus the above result could have been guessed without explicit computation. We find that $Z_{\mathrm{ell}}(K3)(\tau; z=0)=24$ and $Z_{\mathrm{ell}}(K3)(\tau; z=1/2)=16+\mathcal{O}(q)$, and thus (1–1) reproduces the Euler number and signature of K3.

In [Eguchi et al. 89] and more recently in [Eguchi and Hikami 09], the expansion of the K3 elliptic genus in terms of the irreducible representations of the

$\mathcal{N}=4$ superconformal algebra has been discussed in detail. We first provide the data of representation theory [Eguchi and Taormina 87, Eguchi and Taormina 88]. For a rigorous mathematical exposition, see, for example, [Kac 98, Kac and Wakimoto 04].

Let us introduce the character formula of the BPS (short) representation of spin $\ell = 0$ in the Ramond sector with $(-1)^F$ insertion

$$\mathrm{ch}^{\hat{R}}_{h=\frac{1}{4},\ell=0}(\tau;z) = \frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}\mu(\tau;z), \tag{1–2}$$

$$\mu(\tau;z) = \frac{-ie^{\pi i z}}{\theta_1(\tau;z)} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi i n z}}{1-q^n e^{2\pi i z}}.$$

The BPS representation has nonvanishing index

$$\mathrm{ch}^{\hat{R}}_{h=\frac{1}{4},\ell=0}(\tau;z=0) = 1.$$

We also introduce the character of a non-BPS (long) representation with conformal dimension h:

$$q^{h-\frac{3}{8}} \frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}.$$

Then the elliptic genus is expanded as

$$Z_{\mathrm{ell}}(K3)(\tau;z) = 24\,\mathrm{ch}^{\hat{R}}_{h=\frac{1}{4},\ell=0}(\tau;z) + \Sigma(\tau)\frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}, \tag{1–3}$$

where the expansion function $\Sigma(\tau)$ is given by

$$\begin{aligned}\Sigma(\tau) &= -8\left[\mu\left(\tau;z=\frac{1}{2}\right) + \mu\left(\tau;z=\frac{1+\tau}{2}\right)\right. \\ &\quad \left. + \mu\left(\tau;z=\frac{\tau}{2}\right)\right] \\ &= -2\,q^{-1/8}\left(1-\sum_{n=1}^{\infty} A_n q^n\right).\end{aligned} \tag{1–4}$$

If one uses the relation that the non-BPS representation splits into a sum of BPS representations at the unitarity bound $h = 1/4$, then

$$q^{-\frac{1}{8}} \frac{\theta_1(\tau;z)^2}{\eta(\tau)^3} = 2\,\mathrm{ch}^{\hat{R}}_{h=\frac{1}{4},\ell=0}(\tau;z) + \mathrm{ch}^{\hat{R}}_{h=\frac{1}{4},\ell=\frac{1}{2}}(\tau;z),$$

the polar term in Σ may be eliminated, and the decomposition (1–3) can also be written as

$$\begin{aligned}Z_{\mathrm{ell}}(K3)(\tau;z) &= 20\,\mathrm{ch}^{\hat{R}}_{h=\frac{1}{4},\ell=0}(\tau;z) - 2\,\mathrm{ch}^{\hat{R}}_{h=\frac{1}{4},\ell=\frac{1}{2}}(\tau;z) \\ &\quad + 2\sum_{n=1}^{\infty} A_n q^{n-\frac{1}{8}} \frac{\theta_1(\tau;z)^2}{\eta(\tau)^3}.\end{aligned}$$

The coefficients A_n have been computed explicitly for lower orders by expanding the series (1–4) (see Table 1), and it has been conjectured that they are all positive integers [Ooguri 89, Taormina and Wendlend 89, private communication].

n	1	2	3	4	5	6	7	8	$9\cdots$
A_n	45	231	770	2277	5796	13915	30843	65550	$132825\cdots$

TABLE 1. The initial coefficients A_n of the series (1–4)

On the other hand, the asymptotic behavior of A_n at large n has recently been derived using an analogue of the Rademacher expansion of modular forms [Eguchi and Hikami 09]:

$$A_n \approx \frac{2}{\sqrt{8n-1}} e^{2\pi\sqrt{\frac{1}{2}\left(n-\frac{1}{8}\right)}}. \tag{1–5}$$

It turns out that (1–5) gives a good estimate of A_n even at smaller values of n, and this confirms the positivity of the coefficients A_n. Note that the series $\mu(\tau;z)$ (1–2) has the form of a Lerch sum (or mock theta function) and thus has a complex modular transformation law that involves Mordell's integral. In such a situation we can use the method recently developed in [Zwegers 02, Bringmann and Ono 06, Bringmann and Ono 08, Zagier 07] and construct the Poincaré–Maass series to derive the above asymptotic formula.

Table 1 contains a surprise: the first five coefficients, $A_1, \ldots, A_5$, are equal to the dimensions of the *irreducible* representations of M_{24}, the largest Mathieu group; see Section 2. The coefficients A_6 and A_7 can also be nicely decomposed as sums of dimensions:[1]

$$\begin{aligned}A_6 &= 3520 + 10395, \\ A_7 &= 10395 + 5796 + 5544 + 5313 + 2024 + 1771.\end{aligned}$$

For $n \geq 8$, it is still possible to decompose A_n into a sum of dimensions of irreducible representations of M_{24}, but the decompositions are not as unique.[2]

This observation can be compared to the famous observation in [Thompson 79], where the first few terms of the expansion coefficients of $J(q)$,

$$J(q) = \frac{1}{q} + 196884 + 21493760q^2 + \cdots, \tag{1–6}$$

could be naturally decomposed into the sum of the dimension of the irreducible representation of the monster

[1] The tentative decomposition of A_7 shown in a previous version of this paper was incorrect in view of a later study of twisted elliptic genus in [Cheng 10, Gaberdiel et al. 10]. Here it is corrected according to those papers.

[2] It may also be interesting to point out that $2, 3, 5, 7, 11, 23$ appear in the prime factorization of A_n more frequently than any other prime numbers and with certain periodicities in n. These are also prime factors of the order of M_{24}.

simple group. In [Conway and Norton 79] it is formulated in terms of an infinite-dimensional graded representation of the monster group $\bigoplus_i V_i$ such that $\dim V_i$ is the coefficient of q^i of $J(q)$, and Conway and Norton called this observation monstrous moonshine.

It was then found [Frenkel et al. 88] that this representation is naturally associated with the two-dimensional string propagating on $\mathbb{R}^{26}/\Lambda/\mathbb{Z}_2$, where Λ is the Leech lattice. See, for example, [Gannon 04] for a recent review.

In our case, the existence of a natural vector space whose graded dimension gives $\Sigma(q)$ is guaranteed by the construction: it is the Hilbert space of the two-dimensional supersymmetric conformal field theory whose target space is K3. The problem is to identify the action of M_{24} on it.[3]

The nonabelian symplectic symmetry of K3 was studied mathematically in [Mukai 88, Kondo 98]. Mukai enumerated eleven K3 surfaces that possess finite nonabelian automorphism groups. It turns out that all these groups are various subgroups of the Mathieu group M_{24}; see Section 3 for more details. Is it possible that these automorphism groups at isolated points in the moduli space of the K3 surface are enhanced to M_{24} over the whole moduli space when we consider the elliptic genus? This question is currently under study using Gepner models and matrix factorization.

As discussed in [Eguchi and Hikami 10], expansion coefficients of elliptic genera of hyper-Kähler manifolds in general have an exponential growth and are closely related to the black hole entropy. In particular, in the case of the kth symmetric product of the K3 surfaces we obtain the leading behavior

$$A_n \approx e^{2\pi\sqrt{\frac{k^2}{k+1}n-\left(\frac{k}{2(k+1)}\right)^2}},$$

which gives the entropy of the standard D1-D5 black hole $S \approx 2\pi\sqrt{kn}$ at large k ($k = Q_1Q_5$, where Q_1 and Q_5 are the numbers of D1 and D5 branes). Thus the elliptic genus of the K3 surface may be considered as describing the multiplicity of microstates of a small black hole with $Q_1 = Q_5 = 1$.

Here the situation is somewhat similar to a model of black hole described in [Witten 07], where microstates of a small black hole span the representation space of the monster group. Although the partition function of the theory is discussed, the relevant CFT is modular invariant separately in left and right sectors, and the discussion is effectively the same as considering the elliptic genus.

It will be extremely interesting to see whether the Mathieu group M_{24} in fact acts on the elliptic genus of K3.

[3] Dong and Mason pursued the analogue of monstrous moonshine in the case of M_{24}; see, for example, [Dong and Mason 94] and references therein. So far, there is no direct connection between their work and the geometry of K3.

2. APPENDIX: DATA ON M_{24}

The largest Mathieu group, M_{24}, has

$$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 244823040$$

elements. There are 26 conjugacy classes and 26 irreducible representations. The character table is given in Table 2, whose data are taken from [Math. Soc. Japan 07, Conway et al. 85]. The conjugacy class is labeled according to the convention of [Conway et al. 85]. In the character table, $e_p^{\pm}$ stands for

$$e_p^{\pm} = (\pm\sqrt{-p} - 1)/2.$$

The dimensions of the irreducible representations are, in increasing order,

1, 23, 45, 45, 231, 231, 252, 253, 483, 770, 770, 990, 990, 1035, 1035, 1035, 1265, 1771, 2024, 2277, 3312, 3520, 5313, 5796, 5544, 10395.

Here the irreducible representations of dimensions

45, 231, 770, 990, 1035

come in complex conjugate pairs. There is in addition an extra real 1035-dimensional irreducible representation.

3. APPENDIX: M_{24} AND THE CLASSICAL GEOMETRY OF K3

Here we briefly summarize the relation between the classical geometry of the K3 surface and M_{24}, first found in [Mukai 88] and elaborated in [Kondo 98].

Before proceeding, we need to recall the definition of M_{24}. Of many equivalent ways to define it, one that is most understandable to string theorists is to use an even self-dual lattice of dimension 24. Consider the root lattice of A_1 whose generator has squared length 2. Let us denote its weight lattice by A_1^*, whose generator has squared length 1/2. Take the 24-dimensional lattice ${A_1}^{24}$. This is even but not self-dual, because the dual lattice is ${A_1^*}^{24}$. An even self-dual lattice N containing ${A_1}^{24}$ will have the structure

$${A_1}^{24} \subset N \subset {A_1^*}^{24}.$$

Let $\mathcal{G} = N/{A_1}^{24}$, which is a vector subspace of ${A_1^*}^{24}/{A_1}^{24} \simeq {\mathbb{Z}_2}^{24}$. Let us represent an element of $\mathcal{G}$ by a

1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B	2B	10A	21A	21B	4A	12A
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
252	28	9	2	4	0	0	0	1	−1	−1	−1	0	0	−1	−1	0	0	0	0	12	2	0	0	4	1
253	13	10	3	1	1	1	−1	−2	0	0	0	−1	−1	0	0	1	1	1	1	−11	−1	1	1	−3	0
1771	−21	16	1	−5	0	0	−1	0	0	1	1	0	0	0	0	−1	−1	−1	7	11	1	0	0	3	0
3520	64	10	0	0	−1	−1	0	−2	0	0	0	1	1	1	1	0	0	0	−8	0	0	−1	−1	0	0
45	−3	0	0	1	e_7^+	e_7^-	−1	0	1	0	0	$-e_7^+$	$-e_7^-$	−1	−1	1	−1	1	3	5	0	e_7^-	e_7^+	−3	0
45	−3	0	0	1	e_7^-	e_7^+	−1	0	1	0	0	$-e_7^-$	$-e_7^+$	−1	−1	1	−1	1	3	5	0	e_7^+	e_7^-	−3	0
990	−18	0	0	2	e_7^+	e_7^-	0	0	0	0	0	e_7^+	e_7^-	1	1	1	−1	−2	3	−10	0	e_7^-	e_7^+	6	0
990	−18	0	0	2	e_7^-	e_7^+	0	0	0	0	0	e_7^-	e_7^+	1	1	1	−1	−2	3	−10	0	e_7^+	e_7^-	6	0
1035	−21	0	0	3	$2e_7^+$	$2e_7^-$	−1	0	1	0	0	0	0	0	0	−1	1	−1	−3	−5	0	$-e_7^-$	$-e_7^+$	3	0
1035	−21	0	0	3	$2e_7^-$	$2e_7^+$	−1	0	1	0	0	0	0	0	0	−1	1	−1	−3	−5	0	$-e_7^+$	$-e_7^-$	3	0
1035	27	0	0	−1	−1	−1	1	0	1	0	0	−1	−1	0	0	0	2	3	6	35	0	−1	−1	3	0
231	7	−3	1	−1	0	0	−1	1	0	e_{15}^+	e_{15}^-	0	0	1	1	0	0	3	0	−9	1	0	0	−1	−1
231	7	−3	1	−1	0	0	−1	1	0	e_{15}^-	e_{15}^+	0	0	1	1	0	0	3	0	−9	1	0	0	−1	−1
770	−14	5	0	−2	0	0	0	1	0	0	0	0	0	e_{23}^+	e_{23}^-	1	1	−2	−7	10	0	0	0	2	−1
770	−14	5	0	−2	0	0	0	1	0	0	0	0	0	e_{23}^-	e_{23}^+	1	1	−2	−7	10	0	0	0	2	−1
483	35	6	−2	3	0	0	−1	2	−1	1	1	0	0	0	0	0	0	3	0	3	−2	0	0	3	0
1265	49	5	0	1	−2	−2	1	1	0	0	0	0	0	0	0	0	0	−3	8	−15	0	1	1	−7	−1
2024	8	−1	−1	0	1	1	0	−1	0	−1	−1	1	1	0	0	0	0	0	8	24	−1	1	1	8	−1
2277	21	0	−3	1	2	2	−1	0	0	0	0	0	0	0	0	0	2	−3	6	−19	1	−1	−1	−3	0
3312	48	0	−3	0	1	1	0	0	1	0	0	−1	−1	0	0	0	−2	0	−6	16	1	1	1	0	0
5313	49	−15	3	−3	0	0	−1	1	0	0	0	0	0	0	0	0	0	−3	0	9	−1	0	0	1	1
5796	−28	−9	1	4	0	0	0	−1	−1	1	1	0	0	0	0	0	0	0	0	36	1	0	0	−4	−1
5544	−56	9	−1	0	0	0	0	1	0	−1	−1	0	0	1	1	0	0	0	0	24	−1	0	0	−8	1
10395	−21	0	0	−1	0	0	1	0	0	0	0	0	0	−1	−1	0	0	3	0	−45	0	0	0	3	0

TABLE 2. Character table of M_{24}.

sequence of 24 zeros and ones, and define the weight of a vector to be the number of ones in it.

The self-duality of N translates to the fact that $\mathcal{G}$ is 12-dimensional. The evenness translates to the fact that only the weight of every element of $\mathcal{G}$ is a multiple of 4. Let us further demand that only the vectors of N whose length squared is two be the roots of ${A_1}^{24}$ and not more. Then $\mathcal{G}$ does not have an element with weight 4.

These conditions fix the form of $\mathcal{G}$ uniquely, and $\mathcal{G}$ is known as the extended binary Golay code. The group M_{24} is defined as the subgroup of the permutation group S_{24} of the coordinates of ${\mathbb{Z}_2}^{24}$ that preserves $\mathcal{G}$.

The lattice N thus constructed defines a chiral CFT with $c = 24$ whose current algebra is ${A_1}^{24}$. Therefore M_{24} is the discrete symmetry of this chiral CFT.

Now let us recall that the cohomology lattice of K3,

$$\Lambda = H^*(K3, \mathbb{Z}),$$

is also an even self-dual lattice of dimension 24, but with signature $(4, 20)$. The close connection between M_{24} and the geometry of the K3 surface stems from this fact.

Take a K3 surface S, and let G be its symmetry preserving the holomorphic 2-form. Let Λ^G be the part of Λ preserved by G, and Λ_G its orthogonal complement. Then Λ_G is inside the primitive part of $H^{1,1}$, and thus it is negative definite. Using Nikulin's result, it can be shown that Λ_G is a sublattice of N. Therefore G is a subgroup of M_{24}.

The subgroup G cannot be M_{24} itself, however. The action of G on S preserves at least H^0, H^4, $H^{2,0}$,

$H^{0,2}$, and the Kähler form. Hence Λ^G is at least five-dimensional, and Λ_G is at most 19-dimensional. This implies that the action of G on N as real linear maps should at least have a five-dimensional fixed subspace. This translates to the fact that the action of G on 24 points as a subgroup of M_{24} splits them into at least five orbits.

Similarly, starting from a subgroup G of M_{24} that acts on 24 points with at least five orbits, one can construct the action of G on $H^{1,1}$. Using the global Torelli theorem, this translates to the existence of a K3 surface S whose symmetry is G.

One example is the Fermat quartic $X^4 + Y^4 + Z^4 + W^4 = 0$ in $\mathbb{CP}^3$. The symmetry is $(\mathbb{Z}_4)^2 \rtimes S_4$, with 384 elements. This is indeed a subgroup of M_{24} that decomposes 24 points into five orbits, of lengths 1, 1, 2, 4, and 16.

More examples and details of the analysis can be found in [Mukai 88, Kondo 98].

ACKNOWLEDGMENTS

The authors thank the hospitality of the Aspen Center of Physics during the workshop "Unity of String Theory," when the observation reported in this paper was made. We would like to thank S. Mukai for discussions.

T. E. is supported in part by a Grant in Aid from the Japan Ministry of Education, Culture, Sports, Science, and Technology (MEXT). H. O. is supported in part by U.S. Department of Energy grant DE-FG03-92-ER40701, the World Premier International Research Center Initiative, Grant-in-Aid for Scientific Research (C) 20540256 of MEXT, and the Humboldt Research Award. Y. T. is supported in part by NSF grant PHY-0503584, and by a Marvin L. Goldberger membership at the Institute for Advanced Study.

REFERENCES

[Bringmann and Ono 06] K. Bringmann and K. Ono. "The $f(q)$ Mock Theta Function Conjecture and Partition Ranks." *Invent. Math.* 165 (2006), 243–266.

[Bringmann and Ono 08] K. Bringmann and K. Ono. "Coefficients of Harmonic Maas Forms." Preprint, 2008.

[Cheng 10] M. C. N. Cheng. "K3 Surfaces, $\mathcal{N}=4$ Dyons, and the Mathieu Group M24." arXiv:1005.5415 [hep-th], 2010.

[Conway and Norton 79] J. H. Conway and S. P. Norton. "Monstrous Moonshine." *Bull. London Math. Soc.* 11 (1979), 308–339.

[Conway et al. 85] J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson. *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford: Clarendon Press, 1985.

[Dong and Mason 94] C. Y. Dong and G. Mason. "An Orbifold Theory of Genus Zero Associated to the Sporadic Group M_{24}." *Commun. Math. Phys.* 164 (1994) 87–104.

[Eguchi and Hikami 09] T. Eguchi and K. Hikami. "Superconformal Algebras and Mock Theta Functions 2: Rademacher Expansion for K3 Surface." *Commun. Number Theor. and Phys.* 3 (2009), 531–554.

[Eguchi and Hikami 10] T. Eguchi and K. Hikami, "$N = 4$ Superconformal Algebra and the Entropy of Hyper-Kähler Manifolds." *JHEP* 1002 (2010), 019.

[Eguchi and Taormina 87] T. Eguchi and A. Taormina. "Unitary Representations of $\mathcal{N}=4$ Superconformal Algebra." *Phys. Lett. B* 196 (1987), 75–81.

[Eguchi and Taormina 88] T. Eguchi and A. Taormina. "Character Formulas for the $\mathcal{N}=4$ Superconformal Algebra." *Phys. Lett. B* 200 (1988), 315–322.

[Eguchi et al. 89] T. Eguchi, H. Ooguri, A. Taormina, and S. K. Yang. "Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy." *Nucl. Phys. B* 315 (1989), 193–221.

[Frenkel et al. 88] I. B. Frenkel, J. Lepowsky and A. Meurman. *Vertex Operator Algebras and the Monster*, Pure and Applied Math. 134. New York: Academic Press, 1988.

[Gaberdiel et al. 10] M. R. Gaberdiel, S. Hohenegger, and R. Volpato. "Mathieu Twining Characters for K3." arXiv:1006.0221 [hep-th], 2010.

[Gannon 04] T. Gannon. "Monstrous Moonshine: The First Twenty-Five Years." arXiv:math/0402345, 2004.

[Kac 98] V. G. Kac. *Vertex Algebras for Beginners*, 2nd ed., University Lecture Series 10. Providence: American Mathematical Society, 1998.

[Kac and Wakimoto 04] V. G. Kac and M. Wakimoto. "Quantum Reduction and Representation Theory of Superconformal Algebras." *Advances in Math.* 185 (2004), 400–458.

[Kondo 98]S. Kondo. "Niemeier Lattices, Mathieu Groups and Finite Groups of Symplectic Automorphisms of K3 Surfaces." *Duke Math. Journal* 92 (1998), 593–603.

[Mukai 88] S. M. "Finite Groups of Automorphisms of K3 Surfaces and the Mathieu Group." *Invent. Math.* 94 (1988), 183–221.

[Ooguri 89] H. Ooguri. "Superconformal Symmetry and Geometry of Ricci Flat Kähler Manifolds." *Int. J. Mod. Phys. A* 4 (1989), 4303–4324.

[Thompson 79] J. G. Thompson. "Some Numerology between the Fischer–Griess Monster and the Elliptic Modular Function." *Bull. London Math. Soc.* 11 (1979), 352–353.

[Witten 87] E. Witten. "Elliptic Genera and Quantum Field Theory." *Commun. Math. Phys.* 109 (1987), 525–536.

[Witten 07] E. Witten. "Three-Dimensional Gravity Revisited." arXiv:0706.3359, 2007.

[Math. Soc. Japan 07] Mathematical Society of Japan. *Iwanami Suugaku Jiten*, 4th Japanese ed. Tokyo: Iwanami Shoten, 2007. (English translation of the 3rd Japanese edition is available as the 2nd English edition of *Encyclopedic Dictionary of Mathematics*. Cambridge: MIT press, 1987.)

[Zagier 07] D. Zagier, "Ramanujan's Mock Theta Functions and Their Applications." Séminaire Bourbaki 60ème année (2006–2007), no. 986.

[Zwegers 02] S. P. Zwegers. "Mock Theta Functions." Ph.D. thesis, Universiteit Utrecht, 2002.

Tohru Eguchi, Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502 Japan (eguchi@yukawa.kyoto-u.ac.jp)

Hirosi Ooguri, California Institute of Technology, Pasadena, CA 91125, USA (ooguri@theory.caltech.edu)

Yuji Tachikawa, School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA (yujitach@ias.edu)

Received April 21, 2010; accepted May 5, 2010.