The 2-adic Valuation of Stirling Numbers

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CONTENTS

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We analyze properties of the 2-adic valuations of $S(n,k)$, the Stirling numbers of the second kind. For fixed $k \in \mathbb{N}$, a conjectured pattern for the valuation is provided in terms of the dyadic format of n. This conjecture is established for $k = 5$.

1. INTRODUCTION

Divisibility properties of integer sequences have long been objects of interest. In contemporary language these are expressed in terms of p-adic valuations: given a prime p and a positive integer m, there exist unique integers a, n, with a not divisible by p and $n \geq 0$, such that $m = ap^n$. The number n is called the *p-adic valuation* of m. We write $n = \nu_p(m)$. Thus, $\nu_p(m)$ is the highest power of p that divides m. The graph in Figure 1 shows the function $\nu_2(m)$. Here and elsewhere in this paper we connect successive points in the graph in order to visually convey the rises and drops of the sequence.

A celebrated example is due to Legendre [Legendre 30], who established

$$\nu_p(m!) = \frac{m - s_p(m)}{p-1}.$$

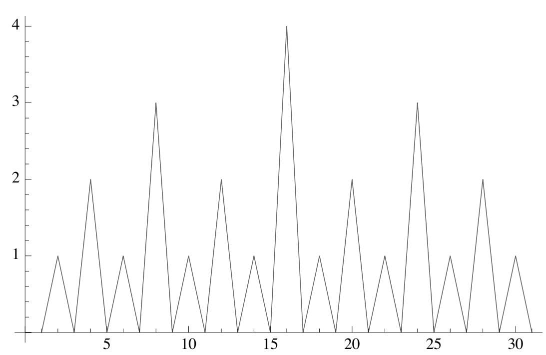

FIGURE 1. The 2-adic valuation of m.

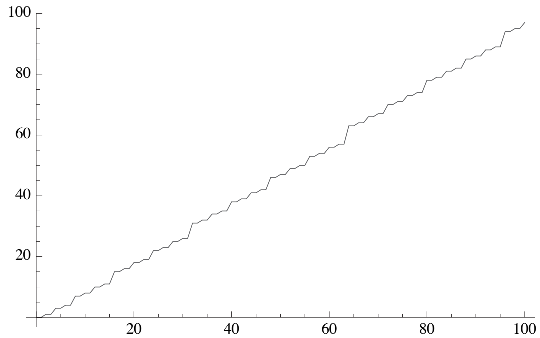

FIGURE 2. The 2-adic valuation of $m!$.

Here $s_p(m)$ is the sum of the base-p digits of m. In particular,

$$\nu_2(m!) = m - s_2(m). \tag{1–1}$$

The reader will find in [Graham et al. 94] details about this identity. Figure 2 shows the graph of $\nu_2(m!)$, exhibiting its linear growth with some oscillatory behavior. If $m = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + \cdots + a_r \cdot 2^r$, with $a_j \in \{0, 1\}$, so that $2^r \leq m \leq 2^{r+1}$, then $s_2(m) = O(\log_2(m))$ and we also have

$$\lim_{m\to\infty} \frac{\nu_2(m!)}{m} = 1.$$

Figure 3 shows the error term $s_2(m) = m - \nu_2(m!)$.

Legendre's result (1–1) provides an elementary proof of Kummer's identity

$$\nu_2\left(\binom{m}{k}\right) = s_2(k) + s_2(m-k) - s_2(m).$$

Not many explicit identities of this type are known.

The function ν_p is extended to $\mathbb{Q}$ by defining $\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b)$. The *p-adic metric* is then given by

$$|r|_p := p^{-\nu_p(r)}.$$

It satisfies the ultrametric inequality

$$|r_1 + r_1|_p \leq \max\{\, |r_1|_p, |r_2|_p \,\}. \tag{1–2}$$

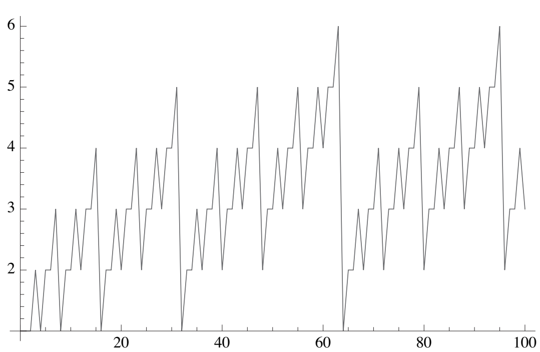

FIGURE 3. The error $m - \nu_2(m!)$.

The completion of $\mathbb{Q}$ under this metric, denoted by $\mathbb{Q}_p$, is the field of p-adic numbers. The set $\mathbb{Z}_p := \{\, x \in \mathbb{Q}_p : |x|_p \leq 1 \,\}$ is the ring of p-adic integers.

Our interest in 2-adic valuations began with the sequence

$$b_{l,m} := \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}, \tag{1–3}$$

for $m \in \mathbb{N}$ and $0 \leq l \leq m$. This sequence appears in the evaluation of the definite integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}.$$

In [Boros and Moll 99], it was shown that the polynomial

$$P_m(a) := 2^{-2m} \sum_{l=0}^{m} b_{l,m} a^l$$

satisfies

$$P_m(a) = 2^{m+3/2}(a+1)^{m+1/2} N_{0,4}(a;m)/\pi.$$

The reader will find in [Boros and Moll 04] more details on this integral.

The results on the 2-adic valuations of $b_{l,m}$ are expressed in terms of

$$A_{l,m} := \frac{l!m!}{2^{m-l}} b_{l,m}. \tag{1–4}$$

The coefficients $A_{l,m}$ can also be written as

$$A_{l,m} = \alpha_l(m) \prod_{k=1}^{m}(4k-1) - \beta_l(m) \prod_{k=1}^{m}(4k+1),$$

for some polynomials α_l, β_l, with integer coefficients and of degree l and $l-1$, respectively. The next remarkable property was conjectured in [Boros et al. 01] and established by J. Little in [Little 05].

Theorem 1.1. *All the zeros of $\alpha_l(m)$ and $\beta_l(m)$ lie on the vertical line* $\operatorname{Re} m = -\frac{1}{2}$.

The next theorem [Amdeberhan et al. 08] gives 2-adic properties of $A_{l,m}$.

Theorem 1.2. *The 2-adic valuation of $A_{l,m}$ satisfies*

$$\nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l,$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the Pochhammer symbol.

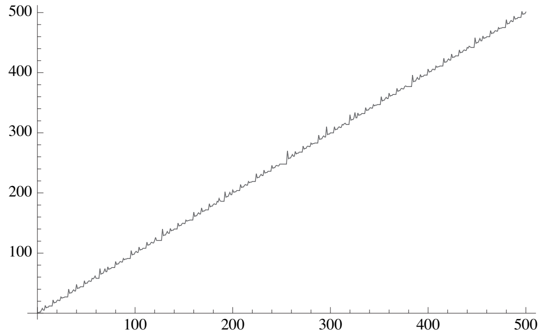

FIGURE 4. The 2-adic valuation of $C_1(n)$.

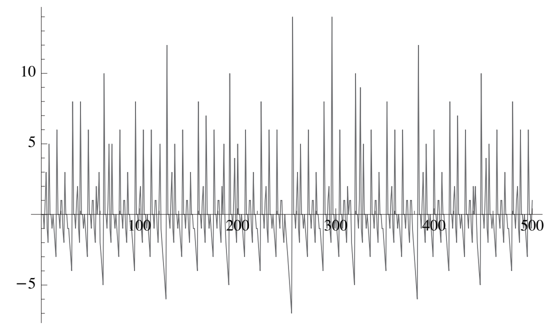

FIGURE 5. The error $\nu_2(C_1(n)) - n$.

The formula

$$(a)_k = \frac{(a+k-1)!}{(a-1)!}$$

and Legendre's identity (1–1) yield the following expression for $\nu_2(A_{l,m})$.

Corollary 1.3. *The* 2*-adic valuation of* $A_{l,m}$ *is given by*

$$\nu_2(A_{l,m}) = 3l - s_2(m+l) + s_2(m-l).$$

There are many other examples of 2-adic valuations considered in the literature. Henri Cohen [Cohen 99] has discussed the sum[1]

$$C_k(n) := \sum_{j=1}^{n} \frac{2^j}{j^k}.$$

These are the partial sums of the polylogarithmic series

$$\mathrm{Li}_k(x) := \sum_{j=1}^{\infty} \frac{x^j}{j^k}.$$

The series converges in $\mathbb{Q}_2$, provided that $\nu_2(x) \geq 1$. Cohen proves that

$$\nu_2(C_1(2^m)) = 2^m + 2m - 4, \text{ for } m \geq 4,$$

and

$$\nu_2(C_2(2^m)) = 2^m + m - 1, \text{ for } m \geq 4.$$

The graph in Figure 4 shows the linear growth of $\nu_2(C_1(n))$, and Figure 5 presents the error term $\nu_2(C_1(n)) - n$.

In this paper we analyze the 2-adic valuation of the Stirling numbers of the second kind $S(n,k)$, defined for

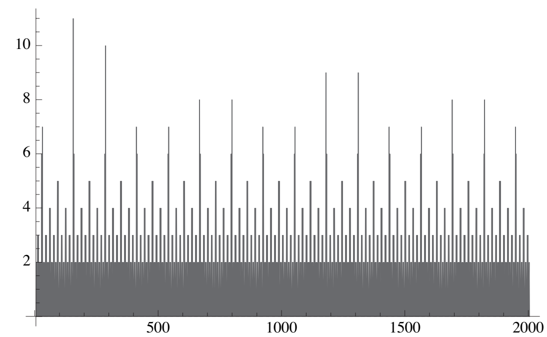

FIGURE 6. The data for $S(n, 5)$.

$n \in \mathbb{N}$ and $0 \leq k \leq n$ as the number of ways to partition a set of n elements into exactly k nonempty subsets. Figures 6 to 8 show the function $\nu_2(S(n,k))$ for fixed k. These graphs indicate the complexity of the problem considered here.

Section 7 gives a larger selection of such pictures. In this paper we describe an algorithm that leads to a first description of the function $\nu_2(S(n,k))$ as depicted in the figures.

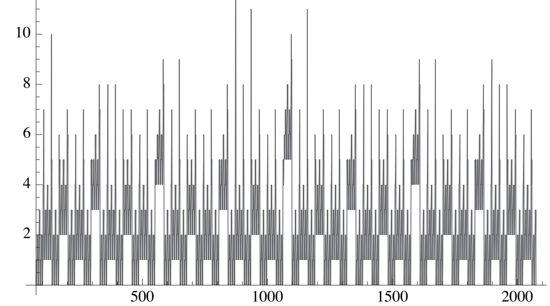

FIGURE 7. The data for $S(n, 75)$.

[1]Cohen uses the notation $s_k(n)$, employed here in a different context.

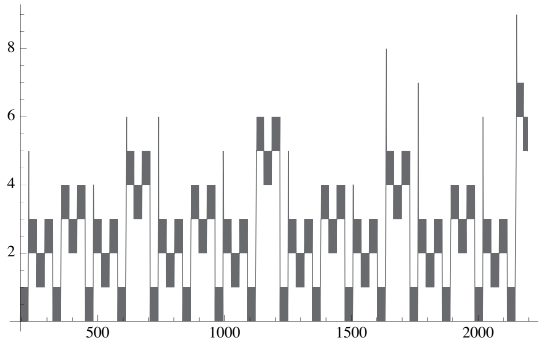

FIGURE 8. The data for $S(n, 195)$.

2. DYADIC m-LEVELS AND CONSTANT CLASSES FOR STIRLING NUMBERS

In this section we introduce the concept of m-level that will be used in the description of our main conjecture, Conjecture 2.4.

Definition 2.1. Let $k \in \mathbb{N}$ be fixed and $m \in \mathbb{N}$. Then for $0 \leq j < 2^m$ define

$$C_{m,j} := \{ 2^m i + j : i \in \mathbb{N} \}.$$

The first value of the index $i \in \mathbb{N}$ in the definition of $C_{m,j}$ is the smallest one that yields $2^m i + j \geq k$. For example, for $k = 5$ and $m = 6$, we have

$$C_{6,28} = \{ 2^6 i + 28 : i \geq 0 \}.$$

We use the notation

$$\nu_2(C_{m,j}) = \{ \nu_2 \left(S(2^m i + j, k) \right) : i \in \mathbb{N} \}.$$

The classes $C_{m,j}$ form a partition of $\mathbb{N}$ into classes modulo 2^m. For example, for $m = 2$, we have the four classes

$$C_{2,0} = \{ 2^2 i : i \in \mathbb{N} \},$$
$$C_{2,1} = \{ 2^2 i + 1 : i \in \mathbb{N} \},$$
$$C_{2,2} = \{ 2^2 i + 2 : i \in \mathbb{N} \},$$
$$C_{2,3} = \{ 2^2 i + 3 : i \in \mathbb{N} \}.$$

The class $C_{m,j}$ is called *constant* if $\nu_2(C_{m,j})$ consists of a single value. This single value is called the *constant of the class* $C_{m,j}$.

For example, Corollary 4.2 shows that

$$\nu_2(S(4i + 1, 5)) = 0,$$

independently of i. Therefore, the class $C_{2,1}$ is constant. Similarly, $C_{2,2}$ is constant with $\nu_2(C_{2,2}) = 0$.

Remark 2.2. Observe that the constant class and its constant value depend on the index k. This has been omitted in the notation for the class.

We now introduce inductively the concept of *m-level*. For $m = 1$, the 1-level consists of the two classes

$$C_{1,0} = \{ 2i : i \in \mathbb{N} \} \text{ and } C_{1,1} = \{ 2i + 1 : i \in \mathbb{N} \},$$

that is, the even and odd integers. Assume that the $(m - 1)$-level has been defined and that it consists of the s classes

$$C_{m-1,i_1}, \ C_{m-1,i_2}, \ \ldots, \ C_{m-1,i_s}.$$

Each class C_{m-1,i_j} splits into two classes modulo 2^m, namely, C_{m,i_j} and $C_{m,i_j+2^{m-1}}$. The m-level is formed by the nonconstant classes modulo 2^m.

Example 2.3. We describe the case of Stirling numbers $S(n, 10)$. Start with the fact that the 4-level consists of the classes $C_{4,7}$, $C_{4,8}$, $C_{4,9}$, and $C_{4,14}$. These split into the eight classes

$$C_{5,7}, \ C_{5,23}, \ C_{5,8}, \ C_{5,24}, \ C_{5,9}, \ C_{5,25}, \ C_{5,14}, \ C_{5,30},$$

modulo 32. Then one checks that $C_{5,23}$, $C_{5,24}$, $C_{5,25}$, and $C_{5,30}$ are all constant (with constant value 2 for each of them). The other four classes form the 5-level:

$$\{ C_{5,7}, \ C_{5,8}, \ C_{5,9}, \ C_{5,14} \}.$$

We are now ready to state our main conjecture.

Conjecture 2.4. *Let $k \in \mathbb{N}$ be fixed. Then we conjecture that*

(a) *there exist a level $m_0(k)$ and an integer $\mu(k)$ such that for any $m \geq m_0(k)$, the number of nonconstant classes of level m is $\mu(k)$, independently of m;*

(b) *moreover, for each $m \geq m_0(k)$, each of the $\mu(k)$ nonconstant classes splits into one constant and one nonconstant subclass. The latter generates the next level set.*

Example 2.5. The conjecture is illustrated for $k = 11$. We claim that $m_0(11) = 3$ and $\mu(11) = 4$. The prediction is that for levels $m \geq 3$, we have four nonconstant classes. Indeed, the classes $C_{2,0}$, $C_{2,1}$, $C_{2,2}$, $C_{2,3}$, have nonconstant 2-adic valuation. Thus, every class in the 2-level splits according to the diagram. To compute the next step, we observe that

$$\nu_2(C_{3,3}) = \nu_2(C_{3,5}) = \{0\},$$
$$\nu_2(C_{3,4}) = \nu_2(C_{3,6}) = \{1\},$$

so there are four constant classes. The remaining four classes $C_{3,0}$, $C_{3,1}$, $C_{3,2}$, and $C_{3,7}$ form the 3-level. Observe that each of the four classes from the 2-level splits into a constant class and a class that forms part of the 3-level.

This process continues. At the next step, the classes of the 3-level split in two, giving a total of eight classes modulo 2^4. For example, $C_{3,2}$ splits into $C_{4,2}$ and $C_{4,10}$. The conjecture states that *exactly* one of these classes has constant 2-adic valuation. Indeed, the class $C_{4,2}$ satisfies $\nu_2(C_{4,2}) \equiv 2$, and $\nu_2(C_{4,10})$ is not constant.

Example 2.6. Figure 9 illustrates this process in the case $k = 7$. The first row of the figure shows the classes at level 2. The class $C_{2,0}$ has constant valuation $\nu_2(C_{2,0}) = 2$, and the class $C_{2,3}$ satisfies $\nu_2(C_{2,3}) = 0$. The remaining two classes, namely $C_{2,1}$ and $C_{2,3}$, form the second level, whose members split into the pairs $\{C_{3,1}, C_{3,5}\}$ and $\{C_{3,2}, C_{3,6}\}$. In each pair we find a class of constant valuation and one, nonconstant, that will be split to proceed with the diagram. The diagram shows that $m_0(7) = 2$ and $\mu(7) = 2$.

Example 2.7. A case with a twist is $k = 13$. Level 3 has eight classes, and only three of them are constant (one expects half of them to be so). The five remaining classes split into ten classes, with six of them constant. At the next splitting, that is, at level 5, we return to the expected count with eight classes, half of which are nonconstant. Thus, in this case, we have $m_0(13) = 5$ and $\mu(13) = 4$.

Elementary Formulas. Throughout the paper we will use several elementary properties of $S(n,k)$:

- relation to Pochhammer:

$$x^n = \sum_{k=0}^{n} S(n,k)(x-k+1)_k;$$

- an explicit formula:

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n; \qquad (2\text{–}1)$$

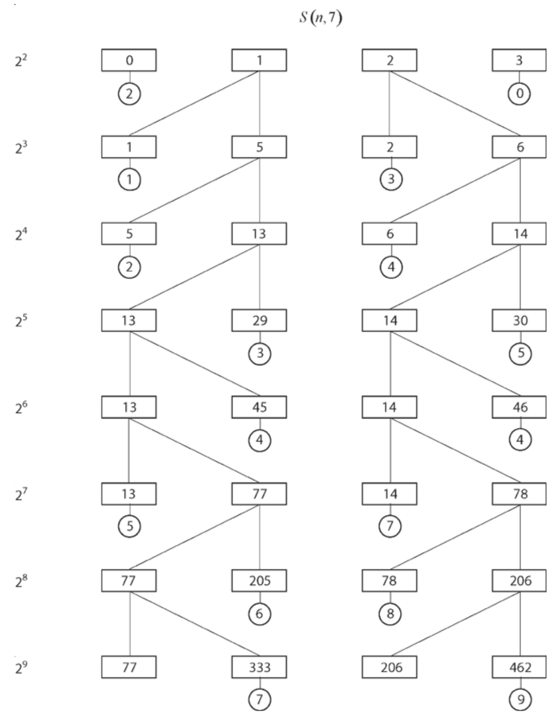

FIGURE 9. The splitting for $k = 7$.

- the generating function

$$\frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-kx)} = \sum_{n=1}^{\infty} S(n,k)x^n;$$

- the recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k). \qquad (2\text{–}2)$$

Lengyel [Lengyel 94] conjectured, and De Wannemacker [De Wannemacker 05] proved, a special case of the 2-adic valuation of $S(n,k)$:

$$\nu_2\left(S(2^n,k)\right) = s_2(k) - 1, \qquad (2\text{–}3)$$

independently of n. Here $s_2(k)$ is the sum of the binary digits of k. A numerical experiment suggests that

$$\nu_2\left(S(2^n+1,k+1)\right) = s_2(k) - 1 \qquad (2\text{–}4)$$

is a companion of (2–3). In the general case, De Wannemacker [De Wannemacker 07] established the inequality

$$\nu_2\left(S(n,k)\right) \geq s_2(k) - s_2(n), \quad 0 \leq k \leq n. \qquad (2\text{–}5)$$

The difference in (2–5) is more regular if $k-1$ is close to a power of 2. Figure 10 shows the (irregular) case $k = 101$, and Figure 11 shows the smoother case $k = 129$.

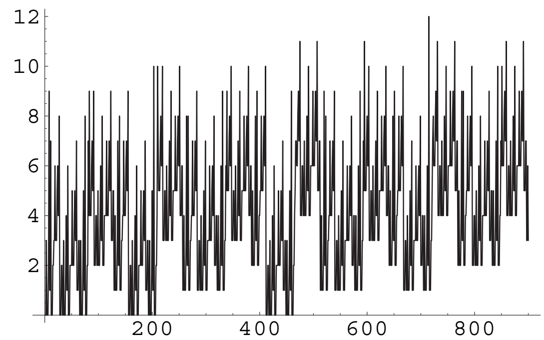

FIGURE 10. De Wannemacker difference for $k = 101$.

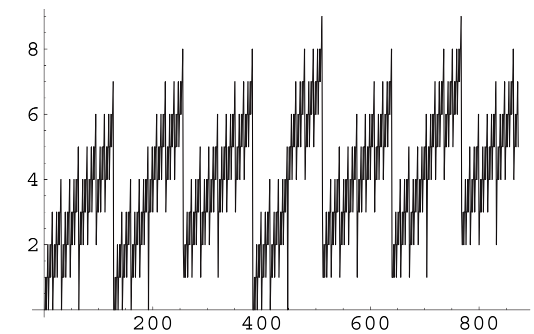

FIGURE 11. De Wannemacker difference for $k = 129$.

3. THE ELEMENTARY CASES $1 \leq k \leq 4$

This section presents, for the sake of completeness, the 2-adic valuation of $S(n,k)$ for $1 \leq k \leq 4$. The arguments are all elementary.

Lemma 3.1. *The Stirling numbers of order* 1 *are given by* $S(n,1) = 1$, *for all* $n \in \mathbb{N}$. *Therefore*

$$\nu_2(S(n,1)) = 0.$$

Proof: There is a unique way to partition a set of n elements into one nonempty set: take them all. □

Lemma 3.2. *The Stirling numbers of order* 2 *are given by* $S(n,2) = 2^n - 1$, *for all* $n \in \mathbb{N}$. *Therefore*

$$\nu_2(S(n,2)) = 0.$$

Proof: The formula for $S(n,2)$ comes from (2–1). It can also be established by induction. Using the recurrence (2–2) and Lemma 3.1, we have

$$\begin{aligned} S(n,2) &= S(n-1,1) + 2S(n-1,2) \\ &= 1 + 2(2^{n-1} - 1) = 2^n - 1, \end{aligned}$$

completing the proof. □

Lemma 3.3. *The Stirling numbers of order* 3 *are given by*

$$S(n,3) = \frac{1}{2}(3^{n-1} - 2^n + 1).$$

Moreover,

$$\nu_2(S(n,3)) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof: The expression for $S(n,3)$ comes from (2–1). An inductive proof also follows directly from the recurrence (2–2),

$$S(n,3) = S(n-1,2) + 3S(n-1,3), \qquad (3\text{–}1)$$

and Lemma 3.2. To prove the expression for $\nu_2(S(n,3))$ we iterate the recurrence and obtain

$$2^n - 1 = S(n,3) - \sum_{k=1}^{N-1} 3^k(2^{n-k} - 1) - 3^N S(n-N,3),$$

and with $N = n-1$ we have

$$S(n,3) = 2^n - 1 - \sum_{k=1}^{n-2} 3^k(2^{n-k} - 1).$$

If n is odd, then $S(n,3)$ is odd and $\nu_2(S(n,3)) = 0$.

For n even, the recurrence (3–1) yields

$$S(n,3) = 2^{n-1} + 3 \cdot 2^{n-2} - 4 + 3^2 S(n-2,3). \qquad (3\text{–}2)$$

As an inductive step, assume that $S(n-2,3) = 2T_{n-2}$, with T_{n-2} odd. Then (3–2) yields

$$\frac{1}{2}S(n,3) = 2^{n-2} + 3 \cdot 2^{n-3} + 3^2 T_{n-2} - 2,$$

and we conclude that $S(n,3)/2$ is an odd integer. Therefore $\nu_2(S(n,3)) = 1$ as claimed. □

We now present a second proof of this result using elementary properties of the valuation ν_2. In particular, we use the ultrametric inequality

$$\nu_2(x_1 + x_2) \geq \min \{ \nu_2(x_1), \nu_2(x_2) \}. \qquad (3\text{–}3)$$

The inequality is strict unless $\nu(x_1) = \nu_2(x_2)$. This inequality is equivalent to (1–2).

Second proof of Lemma 3.3: The powers of 3 modulo 8 satisfy

$$3^m + 1 \equiv 2 + (-1)^{m+1} \bmod 8,$$

because $3^{2k} \equiv 1 \bmod 8$. Therefore $3^m + 1 = 8t + 3 + (-1)^{m+1}$, for some $t \in \mathbb{Z}$. Now

$$\nu_2(8t) = 3 + \nu_2(t) > \nu_2(3 + (-1)^{m+1}),$$

and the ultrametric inequality (3–3) yields

$$\nu_2(3^m+1) = \nu_2(3+(-1)^{m+1}) = \begin{cases} 2 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases} \tag{3–4}$$

The Stirling numbers $S(n,3)$ are given by

$$2S(n,3) = 3^{n-1} + 1 - 2^n$$

and $\nu_2(2^n) = n > 2 \geq \nu_2(3^{n-1}+1)$. We conclude that

$$\nu_2(S(n,3)) = \nu_2(3^{n-1}+1-2^n) - 1 = \nu_2(3^{n-1}+1) - 1.$$

The result now follows from (3–4). □

We now discuss the Stirling number of order 4.

Lemma 3.4. *The Stirling numbers of order* 4 *are given by*

$$S(n,4) = \frac{1}{6}(4^{n-1} - 3^n - 3 \cdot 2^{n+1} - 1).$$

Moreover,

$$\nu_2(S(n,4)) = \begin{cases} 1 & \textit{if } n \textit{ is odd,} \\ 0 & \textit{if } n \textit{ is even.} \end{cases}$$

That is, $\nu_2(S(n,4)) = 1 - \nu_2(S(n,3))$.

Proof: The expression for $S(n,4)$ comes from (2–1). To establish the formula for $\nu_2(S(n,4))$ we use the recurrence (2–2) in the case $k=4$:

$$S(n,4) = S(n-1,3) + 4S(n-1,4). \tag{3–5}$$

For n even, the value $S(n-1,3)$ is odd, so that $S(n,4)$ is odd and $\nu_2(S(n,4)) = 0$. For n odd, $S(n,4)$ is even, since $S(n-1,3)$ is even. The recurrence (3–5) is now written as

$$\frac{1}{2}S(n,4) = \frac{1}{2}S(n-1,3) + 2S(n-1,4).$$

The value $\nu_2(S(n-1,3)) = 1$ shows that the right-hand side is odd, yielding $\nu_2(S(n,4)) = 1$. □

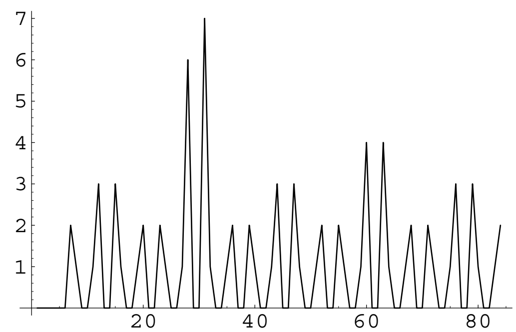

FIGURE 12. The 2-adic valuation of $S(n,5)$.

4. THE STIRLING NUMBERS OF ORDER 5

The elementary cases discussed in the previous section are the only ones for which the 2-adic valuation $\nu_2(S(n,k))$ is easy to compute. The first nontrivial case occurs when $k = 5$. The graph in Figure 12 gives $\nu_2(S(n,5))$, and we now explore its properties.

The 1-level consists of the two classes

$$\{ C_{1,0},\ C_{1,1} \}.$$

These two classes split into $\{ C_{2,0}, C_{2,1}, C_{2,2}, C_{2,3} \}$ modulo 4. The parity of $S(n,5)$ determines two of them.

Lemma 4.1. *The Stirling numbers $S(n,5)$ are given by*

$$S(n,5) = \frac{1}{24}(5^{n-1} - 4^n + 2 \cdot 3^n - 2^{n+1} + 1).$$

They satisfy

$$S(n,5) \equiv \begin{cases} 1 \bmod 2 & \textit{if } n \equiv 1 \textit{ or } 2 \bmod 4, \\ 0 \bmod 2 & \textit{if } n \equiv 3 \textit{ or } 0 \bmod 4. \end{cases}$$

Proof: The explicit formula (2–1) yields the expression for $S(n,5)$. The recurrence

$$S(n,5) = S(n-1,4) + 5S(n-1,5)$$

and the parity

$$S(n,4) \equiv \begin{cases} 1 \bmod 2 & \text{if } n \equiv 0 \bmod 2, \\ 0 \bmod 2 & \text{if } n \equiv 1 \bmod 2, \end{cases}$$

give the result by induction. □

Corollary 4.2. *The Stirling numbers $S(n,5)$ satisfy*

$$\nu_2(S(4n+1,5)) = \nu_2(S(4n+2,5)) = 0, \textit{ for all } n \in \mathbb{N}.$$

The corollary states that the classes $C_{2,1}$ and $C_{2,2}$ are constant, so the 2-level is

$$\{ C_{2,0}, C_{2,3} \}.$$

This confirms part of the main conjecture; here $m_0 = 3$ in view of $2^2 < 5 \leq 2^3$, and the first level at which we find constant classes is $m_0 - 1 = 2$.

Remark 4.3. Corollary 4.2 reduces the discussion of $\nu_2(S(n,5))$ to the indices $n \equiv 0$ or $3 \bmod 4$. These two branches can be treated in parallel. Introduce the notation

$$q_n := \nu_2(S(n,5)),$$

and consider the table of values

$$X := \{ q_{4i}, q_{4i+3} : i \geq 2 \}.$$

This begins

$$X = \{ 1,1,3,3,1,1,2,2,1,1,\mathbf{6},\mathbf{7},1,1,\ldots \},$$

and after a while it continues as

$$X = \{ \ldots,1,1,2,2,1,1,\mathbf{11},\mathbf{6},1,1,2,2,\ldots \}.$$

We observe that $q_{4i} = q_{4i+3}$ for most indices.

Definition 4.4. The index i is called *exceptional* if $q_{4i} \neq q_{4i+3}$.

The first exceptional index is $i = 7$, where $q_{28} = 6 \neq q_{31} = 7$. The list of exceptional indices continues as follows: $\{ 7, 39, 71, 103, \ldots \}$.

Conjecture 4.5. *The set of exceptional indices is* $\{ 32j + 7 : j \geq 1 \}$.

We now consider the class

$$C_{2,0} := \{ q_{4i} = \nu_2(S(4i),5) : i \geq 2 \},$$

where we have omitted the first term $S(4,5) = 0$. The class $C_{2,0}$ begins

$$C_{2,0} = \{ 1,3,1,2,1,6,1,2,1,3,1,2,1,4,1,2,1,3,1,2,\ldots \},$$

and it splits according to the parity of the index i into

$$C_{3,4} = \{ q_{8i+4} : i \geq 1 \} \quad \text{and} \quad C_{3,0} = \{ q_{8i} : i \geq 1 \}.$$

The data suggest that $C_{3,0}$ is constant. This is easy to check.

Proposition 4.6. *The Stirling numbers of order* 5 *satisfy*

$$\nu_2(S(8i,5)) = 1, \text{ for all } i \geq 1.$$

Proof: We analyze the identity

$$24S(8i,5) = 5^{8i-1} - 4^{8i} + 2 \cdot 3^{8i} - 2^{8i+1} + 1$$

modulo 32. Using $5^8 \equiv 1$ and $5^7 \equiv 13$, we obtain $5^{8i-1} \equiv 13$. Also, $4^{8i} \equiv 2^{8i+1} \bmod 0$. Finally, $3^{8i} \equiv 81^{2i} \equiv 17^{2i} \equiv 1$. Therefore

$$5^{8i-1} - 4^{8i} + 2 \cdot 3^{8i} - 2^{8i+1} + 1 \equiv 16 \bmod 32.$$

We obtain that $24S(8i,5) = 32t + 16$ for some $t \in \mathbb{N}$, and this yields $3S(8i,5) = 2(2t+1)$. Therefore $\nu_2(S(8i,5)) = 1$. □

We now consider the class $C_{3,4}$.

Proposition 4.7. *The Stirling numbers of order* 5 *satisfy*

$$\nu_2(S(8i+4,5)) \geq 2, \text{ for all } i \geq 1.$$

Proof: We analyze the identity

$$24S(8i+4,5) = 5^{8i+3} - 4^{8i+4} + 2 \cdot 3^{8i+4} - 2^{8i+5} + 1$$

modulo 32. Using $5^8 \equiv 1$, $5^3 \equiv 29$, $3^8 \equiv 1$, $3^4 \equiv 17$, and $2^4 \equiv 16$ modulo 32, we obtain

$$24S(8i+4,5) \equiv 0 \bmod 32.$$

Therefore $24S(8i+4,5) = 32t$ for some $t \in \mathbb{N}$, and this yields $\nu_2(S(8i+4,5) \geq 2$. □

Note 4.8. In [Lengyel 94] it is established that

$$\nu_2(k!S(n,k)) = k - 1,$$

for $n = a2^q$, a odd, and $q \geq k-2$. In the special case $k = 5$ this yields $\nu_2(S(n,5)) = 1$ for $n = a2^q$ and $q \geq 3$. These values of n have the form $n = 8a \cdot 2^{q-3}$, so this is included in Proposition 4.6.

Remark 4.9. A similar argument yields

$$\nu_2(S(8i+3,5)) = 1 \quad \text{and} \quad \nu_2(S(8i+7,5)) \geq 2.$$

We conclude that the 3-level is

$$\{ C_{3,4}, C_{3,7} \}.$$

This confirms the main conjecture: each of the classes of the 2-level produces a constant class and a second one in the 3-level.

We now consider the class $C_{3,4}$ and its splitting as $C_{4,4}$ and $C_{4,12}$. The data for $C_{3,4}$ begin

$$C_{3,4} = \{3,2,6,2,3,2,4,2,3,2,5,2,3,2,4,2,3,2,11, \\ 2,3,2,\dots\}.$$

This suggests that the values with even index are all 2. This can be verified.

Proposition 4.10. *The Stirling numbers of order* 5 *satisfy*

$$\nu_2(S(16i+4,5)) = 2, \text{ for all } \geq 1.$$

Proof: We analyze the identity

$$24S(16i+4,5) = 5^{16i+3} - 4^{16i+4} + 2\cdot 3^{16i+4} - 2^{16i+5} + 1$$

modulo 64. Using $5^{16} \equiv 1, 5^3 \equiv 61, 3^{16} \equiv 1$, and $3^4 \equiv 17$, we obtain

$$5^{16i+3} - 4^{16i+4} + 2\cdot 3^{16i+4} - 2^{16i+5} + 1 \equiv 32 \mod 64.$$

Therefore $24S(16i+4,5) = 64t+32$ for some $t \in \mathbb{N}$. This gives $3S(16i+4,5) = 4(2t+1)$, and it follows that $\nu_2(S(16i+4,5)) = 2$. □

Note 4.11. A similar argument shows that

$$\nu_2(S(16i+12,5)) \geq 3$$

and also

$$\nu_2(S(16i+7,5)) = 2 \quad \text{and} \quad \nu_2(S(16i+15,5)) \geq 3.$$

Therefore the 4-level is $\{C_{4,12}, C_{4,15}\}$.

This splitting process of the classes can be continued, and according to our main conjecture, the number of elements in the m-level is always constant. To prove the statement similar to Propositions 4.6 and 4.10 requires us to analyze the congruence

$$24S(2^m i+j,5) \equiv 5^{2^m i+j-1} - 4^{2^m i+j} + 2\cdot 3^{2^m i+j} \\ - 2^{2^m i+j+1} + 1 \bmod 2^{m+2}.$$

This can be done for specific choices of j, namely those giving the indices at the m-level. At the moment we cannot predict which values of j will appear at the m-level. In the next section we present a proof of this conjecture for the special case $k=5$.

Problem 4.12. Is there a combinatorial mechanism that enables us to make such a binary choice for each m-level split class?

In [Lundell 78], the Stirling-like numbers

$$T_p(n,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

are studied, where the prime p is fixed and the index j is omitted in the sum if it is divisible by p. Clarke [Clarke 95] conjectured that

$$\nu_p(k!S(n,k)) = \nu_p(T(n,k)). \tag{4–1}$$

From this conjecture he derives an expression for $\nu_2(S(n,5))$ in terms of the zeros of the form $f_{0,5}(x) = 5 + 10\cdot 3^x + 5^x$ in the ring of 2-adic integers $\mathbb{Z}_2$.

Theorem 4.13. *Let u_0 and u_1 be the 2-adic zeros of the function $f_{0,5}$. Then, under the assumption that conjecture* (4–1) *holds, we have*

$$\nu_2(S(n,5)) = \begin{cases} -1 + \nu_2(n-u_0) & \text{if } n \text{ is even}, \\ -1 + \nu_2(n-u_1) & \text{if } n \text{ is odd}. \end{cases}$$

Here u_0 is the unique zero of $f_{0,5}$ that satisfies $u_0 \in 2\mathbb{Z}_2$, and u_1 is the other zero of $f_{0,5}$; it satisfies $u_1 \in 1+2\mathbb{Z}_2$.

Clarke [Clarke 95] also obtained similar expressions for $\nu_2(S(n,6))$ and $\nu_2(S(n,7))$ in terms of zeros of the functions

$$f_{0,6} = -6 - 20\cdot 3^x - 6\cdot 5^x$$

and

$$f_{0,7} = 7 + 35\cdot 3^x + 21\cdot 5^x + 7^x.$$

5. PROOF OF THE MAIN CONJECTURE FOR $k = 5$

The goal of this section is to prove the main conjecture in the case $k=5$. The parameter m_0 is 3 in view of $2^2 < 5 \leq 2^3$. In the previous section we verified that $m_0 - 1 = 2$ is the first level for constant classes. We now prove this splitting of classes.

Theorem 5.1. *Assume $m \geq m_0$. Then the m-level consists of exactly two split classes: $C_{m,j}$ and $C_{m,j+2^{m-1}}$. They satisfy*

$$\nu_2(C_{m,j}) > m-3 \quad \text{and} \quad \nu_2(C_{m,j+2^{m-1}}) > m-3.$$

Then exactly one, call it C^1, satisfies

$$\nu_2(C^1) = \{m-2\},$$

and the other one, call it C^2, satisfies

$$\nu_2(C^2) > m-2.$$

The proof of this theorem requires several elementary results on 2-adic valuations.

Lemma 5.2. *For $m \in \mathbb{N}$ we have $\nu_2\left(5^{2^m}-1\right) = m+2$.*

Proof: Start at $m=1$ with $\nu_2(24)=3$. The inductive step uses

$$5^{2^{m+1}}-1 = (5^{2^m}-1)\cdot(5^{2^m}+1).$$

Now $5^k+1 \equiv 2 \bmod 4$, so that $5^{2^m}+1 = 2\alpha_1$ with α_1 odd. Thus

$$\begin{aligned}\nu_2(5^{2^{m+1}}-1) &= \nu_2(5^{2^m}-1)+\nu_2(5^{2^m}+1)\\ &= (m+2)+1 = m+3,\end{aligned}$$

completing the proof. □

The same type of argument produces the next lemmas.

Lemma 5.3. *For $m \in \mathbb{N}$, we have $\nu_2(3^{2^m}-1) = m+2$.*

Lemma 5.4. *For $m \in \mathbb{N}$, we have $\nu_2(5^{2^m}-3^{2^m}) = m+3$.*

Proof: The inductive step uses

$$5^{2^{m+1}}-3^{2^{m+1}} = \left(5^{2^m}-3^{2^m}\right)\times\left(\left(5^{2^m}-1\right)+\left(3^{2^m}+1\right)\right).$$

Therefore $\nu_2(5^{2^m}-1) = m+2$ and $3^{2^m} \equiv 1 \bmod 4$, whence $\nu_2(3^{2^m}+1)=1$. We conclude that

$$\nu_2\left(\left(5^{2^m}-1\right)+\left(3^{2^m}+1\right)\right) = \min\{\, m+2, 1\,\} = 1.$$

We obtain

$$\nu_2(5^{2^{m+1}}-3^{2^{m+1}}) = m+4,$$

and this concludes the inductive step. □

The recurrence (2–2) for the Stirling numbers $S(n,5)$ is

$$S(n,5) = 5S(n-1,5)+S(n-1,4).$$

Iterating this result yields the next lemma.

Lemma 5.5. *Let $t \in \mathbb{N}$. Then*

$$S(n,5)-5^tS(n-t,5) = \sum_{j=0}^{t-1} 5^j S(n-j-1,4).$$

Proof of Theorem 5.1: We have already checked the conjecture for the 2-level. The inductive hypothesis states that the $(m-1)$-level survivor has the form

$$C_{m,k} = \{\, \nu_2(S(2^m n+k,5)) : n \geq 1\,\}$$

and that $\nu_2(S(2^m n+k,5)) > m-2$. At the next level this class splits into the two classes

$$C_{m+1,k} = \{\, \nu_2(S(2^{m+1}n+k,5)) : n \geq 1\,\}$$

and

$$C_{m+1,k+2^m} = \{\, \nu_2(S(2^{m+1}n+k+2^m,5)) : n \geq 1\,\},$$

and every element of each of these two classes is greater than or equal to $m-1$. □

We now prove that one of these classes reduces to the singleton $\{\, m-1\,\}$ and that every element in the other class is strictly greater than $m-1$.

The first step is to use Lemma 5.5 to compare the values of $S(2^{m+1}n+k,5)$ and $S(2^{m+1}n+k+2^m,5)$. Define

$$M = 2^m-1 \text{ and } N = 2^{m+1}n+k;$$

then we have

$$\begin{aligned}&S(2^{m+1}n+k+2^m,5)-5^{2^m}S(2^{m+1}n+k,5)\\ &\quad= \sum_{j=0}^{M} 5^{M-j}S(N+j,4).\end{aligned}$$

Proposition 5.6. *With the notation as above,*

$$\nu_2\Big(\sum_{j=0}^{M} 5^{M-j}S(N+j,4)\Big) = m-1.$$

Proof: The explicit formula (2–1) yields

$$6S(n,4) = 4^{n-1}+3\cdot 2^{n-1}-3^n-1.$$

Thus

$$\begin{aligned}&6\sum_{j=0}^{M} 5^{M-j}S(N+j,4)\\ &\quad= 4^{N-1}\left(5^{M+1}-4^{M+1}\right)+2^{N-1}\left(5^{M+1}-2^{M+1}\right)\\ &\qquad -3^N\times\frac{1}{2}\left(5^{M+1}-3^{M+1}\right)-\frac{1}{4}\left(5^{M+1}-1\right).\end{aligned}$$

The results in Lemmas 5.2, 5.3, and 5.4 yield

$$\begin{aligned}&6\sum_{j=0}^{M} 5^{M-j}S(N+j,4)\\ &\quad= 4^{N-1}\alpha_1+2^{N-1}\alpha_2-3^N\cdot 2^{m+2}\alpha_3-2^m\alpha_4\end{aligned}$$

with α_j odd integers. Write this as

$$\begin{aligned}&6\sum_{j=0}^{M} 5^{M-j}S(N+j,4)\\ &\quad= 2^{N-1}\big(2^{N-1}\alpha_1+\alpha_2\big)-2^m\left(4\alpha_3 3^N+1\right)\\ &\quad\equiv T_1+T_2.\end{aligned}$$

Then $\nu_2(T_1) = N-1 > m = \nu_2(T_2)$, and we obtain

$$\nu_2\Big(\sum_{j=0}^{M} 5^{M-j} S(N+j,4)\Big) = m-1.$$

We conclude that

$$S(2^{m+1}n+k+2^m,5) - 5^{2^m} S(2^{m+1}n+k,5) = 2^{m-1}\alpha_5,$$

with α_5 odd. Define

$$X := 2^{-m+1} S(2^{m+1}n+k+2^m,5),$$
$$Y := 2^{-m+1} S(2^{m+1}n+k,5).$$

Then X and Y are integers and $X - Y \equiv 1 \bmod 2$, so that they have opposite parity. If X is even and Y is odd, we obtain

$$\nu_2\left(S(2^{m+1}n+k+2^m,5)\right) > m-1$$

and

$$\nu_2\left(S(2^{m+1}n+k,5)\right) = m-1.$$

The case X odd and Y even is similar. This completes the proof. □

6. SOME APPROXIMATIONS

In this section we present some approximations to the function $\nu_2(S(n,5))$. These approximations were derived empirically, and they support our belief that 2-adic valuations of Stirling numbers can be well approximated by simple integer combinations of the most basic 2-adic valuations, that is, of the integers.

For each prime p, define

$$\lambda_p(m) = \frac{1}{2}\left(1 - (-1)^{m \bmod p}\right).$$

First approximation. Define

$$f_1(m) := \left\lfloor \frac{m+1}{2} \right\rfloor + 112\lambda_2(m) + 50\lambda_2(m+1).$$

Then $\nu_2(S(m,5))$ and $\nu_2(f_1(m))$ agree for most values. The first time they differ is at $m = 156$, where

$$\nu_2(S(156,5)) - \nu_2(f_1(156)) = 4.$$

The first few indices for which $\nu_2(S(m,5)) \neq \nu_2(f_1(m))$ are $\{156, 287, 412, 668, 799, \dots\}$.

Conjecture 6.1. *Define*

$$x_1(m) = 156 + 125\left\lfloor \frac{4m}{3} \right\rfloor + 6\left\lfloor \frac{2m+1}{3} \right\rfloor$$

and

$$I_1 = \{\, x_1(m) : m \geq 0 \,\}. \tag{6–1}$$

Then $\nu_2(S(m,5)) = \nu_2(f_1(m))$ unless $m \in I_1$.

The parity of the exceptions in I_1 is easy to establish: every third element is odd, and the even indices of I_1 are in the arithmetic progression $256m + 156$.

Second approximation. We now describe a new approximation to the error

$$\mathrm{err}_2(m,5) := \nu_2(S(m,5)) - \nu_2(f_1(m)).$$

Define

$$m_3(m) := (m+2) \bmod 3,$$
$$\alpha_m := \lambda_3(m+2)\,(1+\lambda_3(m)) + \lambda_2(m+1)\lambda_3(m).$$

Now define

$$f_2(m) = \binom{2m_3}{m_3}\left\lfloor \frac{m+2}{3} \right\rfloor + 208\lambda_3(m+1) + 27\lambda_2(m)\lambda_3(m).$$

The next conjecture improves the prediction of Conjecture 6.1.

Conjecture 6.2. *Define*

$$\mathrm{err}_2(x_1(m)) := \nu_2(S(x_1(m),5) - (-1)^{\alpha_m}\nu_2(f_2(m))$$

and

$$x_2(m) = 109 + 107\left\lfloor \frac{4m+2}{3} \right\rfloor + 85\left\lfloor \frac{4m+1}{3} \right\rfloor.$$

Finally, let $I_2 = \{\, x_2(m) : m \geq 0 \,\}$. Then $\mathrm{err}_2(m) = 0$ unless $m \in I_2$.

There is single class per level, which we write as

$$C_{m,j} = \{\, q_{2^m i + j} : i \in \mathbb{N} \,\}, \tag{6–2}$$

where $j = j(m)$ is the index that corresponds to the nonconstant class at the m-level. The first few examples are listed below:

$$C_{2,4} = \{\, q_{4i+4} : i \in \mathbb{N} \,\},$$
$$C_{3,4} = \{\, q_{8i+4} : i \in \mathbb{N} \,\},$$
$$C_{4,12} = \{\, q_{16i-4} : i \in \mathbb{N} \,\},$$
$$C_{5,28} = \{\, q_{32i-4} : i \in \mathbb{N} \,\},$$
$$C_{6,28} = \{\, q_{64i-36} : i \in \mathbb{N} \,\},$$
$$C_{7,156} = \{\, q_{128i-100} : i \in \mathbb{N} \,\},$$
$$C_{8,156} = \{\, q_{256i-100} : i \in \mathbb{N} \,\},$$
$$C_{9,156} = \{\, q_{512i-356} : i \in \mathbb{N} \,\},$$
$$C_{10,156} = \{\, q_{1024i-868} : i \in \mathbb{N} \,\}.$$

We have observed a connection between the indices $j(m)$ and the set of exceptional indices I_1 in (6–1).

Conjecture 6.3. *Construct a list of numbers* $\{\, c_i : i \in \mathbb{N} \,\}$ *according to the following rules. Let* $c_1 = 8$ *(the first index in the class* $C_{2,4}$*), and then define* c_j *as the first value on* $C_{m,j}$ *that is strictly bigger than* c_{j-1}*. The set* C *begins*

$$C = \{\, 8,\ 12,\ 28, 60,\ 92,\ 156,\ 412,\ 668,\ 1180,\ \ldots\}.$$

Then, starting at 156*, the number* c_i *belongs to* I_1*.*

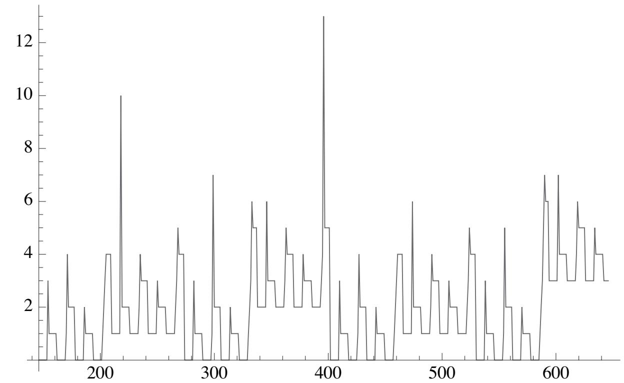

FIGURE 15. The data for $S(n, 146)$.

7. A SAMPLE OF PICTURES

In this section we present, in Figures 13 to 23, data that illustrate the wide variety of behavior for the 2-adic valuation of Stirling numbers $S(n, k)$.

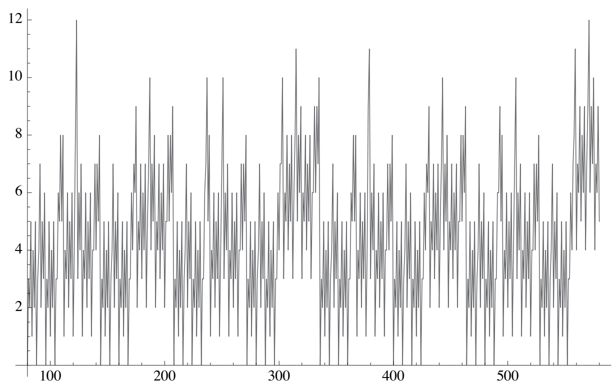

FIGURE 13. The data for $S(n, 80)$.

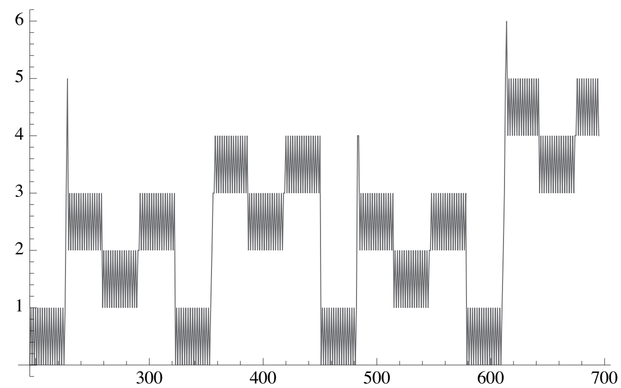

FIGURE 16. The data for $S(n, 195)$.

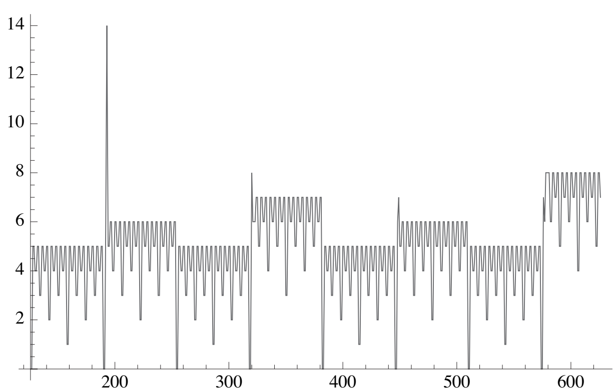

FIGURE 14. The data for $S(n, 126)$.

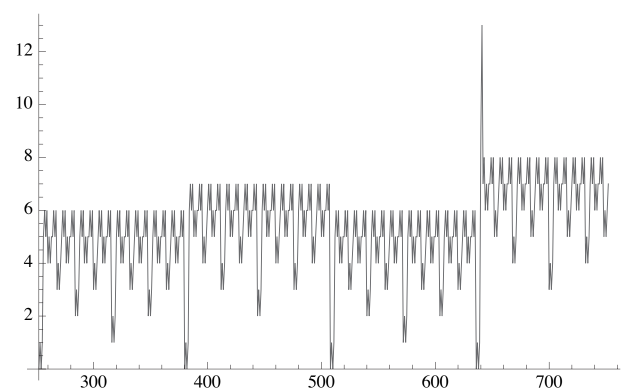

FIGURE 17. The data for $S(n, 252)$.

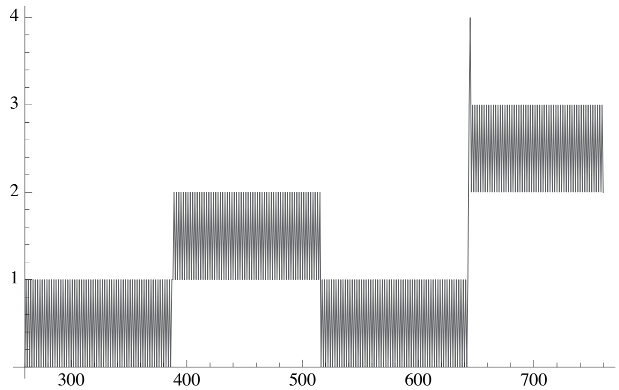

FIGURE 18. The data for $S(n, 260)$.

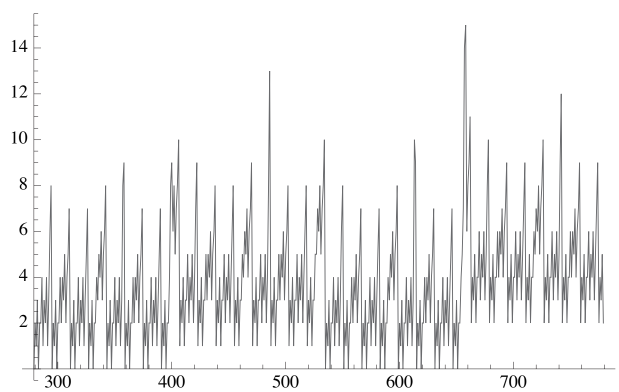

FIGURE 19. The data for $S(n, 279)$.

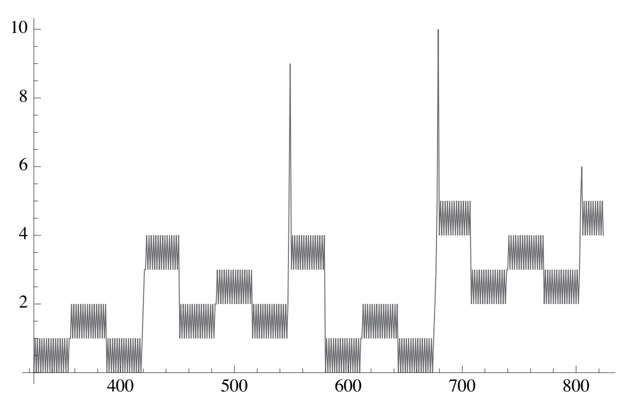

FIGURE 20. The data for $S(n, 324)$.

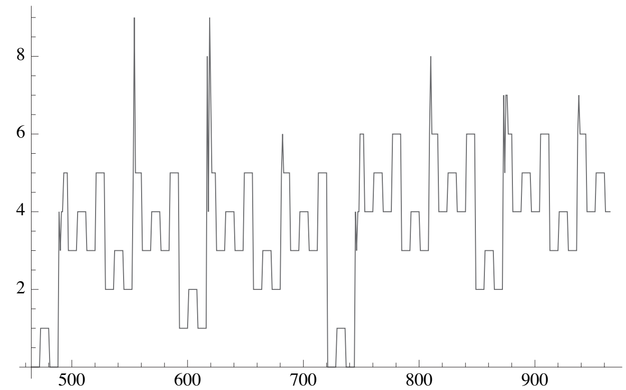

FIGURE 21. The data for $S(n, 465)$.

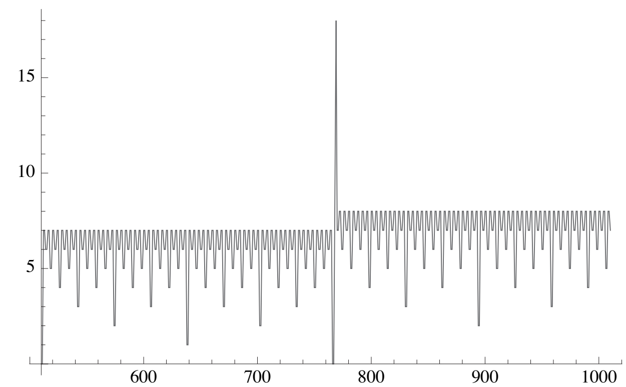

FIGURE 22. The data for $S(n, 510)$.

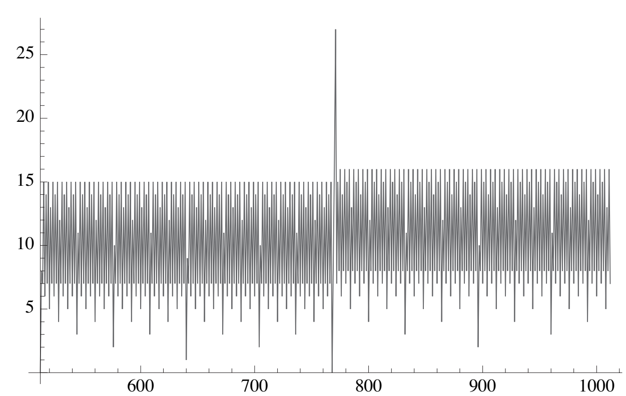

FIGURE 23. The data for $S(n, 512)$.

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